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CATEGORY $\mathcal{O}$: QUIVERS AND ENDOMORPHISM RINGS OF PROJECTIVES

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ABSTRACT. We describe an algorithm for computing quivers of category $\mathcal{O}$ of a finite dimensional semisimple Lie algebra. The main tool for this is Soergel's description of the endomorphism ring of the antidominant indecomposable projective module of a regular block as an algebra of coinvariants. We give explicit calculations for root systems of rank 1 and 2 for regular and singular blocks and also quivers for regular blocks for type A_3.

The main result in this paper is a necessary and sufficient condition for an endomorphism ring of an indecomposable projective object of $\mathcal{O}$ to be commutative. We give also an explicit formula for the socle of a projective object with a short proof using Soergel's functor $\mathbb{V}$ and finish with a generalization of this functor to Harish-Chandra bimodules and parabolic versions of category $\mathcal{O}$.

1. Introduction

For a finite dimensional semisimple Lie algebra $\mathfrak{g}$ with Borel and Cartan subalgebras $\mathfrak{b}$ and $\mathfrak{h}$ resp., we consider the so-called category $\mathcal{O}$ (originally defined by [BGG]). This category decomposes into blocks, where each block has as objects certain $\mathfrak{g}$-modules with a fixed general central character. Every block is quasi-hereditary in the sense of [CPS] and can be considered as a category of modules over a finite dimensional algebra. The work of Soergel (see, e.g., [So1]) can be used to find an explicit algorithm which computes the corresponding quiver describing each block.

Soergel's key result is the description of the endomorphism ring of the antidominant indecomposable projective module as the algebra of coinvariants. For a regular block this algebra is just a quotient of the algebra of regular functions on the Cartan subalgebra of our semisimple Lie algebra, which depends only on the Cartan subalgebra and on the Weyl group. Soergel defined a functor from a fixed block of category $\mathcal{O}$ into the category of finite dimensional modules over this algebra of coinvariants. This functor is fully faithful on projectives. Therefore it is possible to describe homomorphisms between indecomposable projectives of $\mathcal{O}$ as homomorphisms between finite dimensional modules over a commutative finite dimensional algebra.

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Having pointed out these results of Soergel we want to explain how we can obtain the describing quiver of a block. We give an algorithm for the computation of such quivers.

We have explicitly computed quivers for regular and singular (integral) blocks for all root systems of rank one and two. Finally, we give the quiver for regular integral blocks for the root system A_3 with an explicit representation corresponding to the projective Verma module which gives an explicit insight into its submodule structure. The motivation for this was to understand the relation between the submodule lattice of projective Verma modules and the primitive ideals of the universal enveloping algebra in a very explicit way. A more theoretical approach to this can be found in [St]. The calculations were done using Maple[1], GAP[2] and C.

As the main result of this paper, we will prove that the ring of endomorphisms of an indecomposable projective object in an integral block of $\mathcal{O}$ is commutative if and only if the projective Verma module in this block occurs with multiplicity one in a Verma flag of the projective in question. That this condition is necessary can be also proved by some deformation arguments developed in [So2]. (For a proof see [Jau].) We give a more elementary proof for this fact and also show that this condition is sufficient, illustrated by some explicit examples.

We give an explicit formula for the socle of a projective module in an integral block. A special case of this is the socle of the projective-injective module in such a block, which can be found in [Ir1]. More generally we describe the socle of projective Harish-Chandra modules with a fixed generalized central character from the right. This implies the faithfulness on such projectives for a generalization of Soergel's combinatorial functor for Harish-Chandra bimodules.

In the last section we describe a generalization of Soergel's functor to parabolic versions of $\mathcal{O}$.

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Notations. Tensor products and dimensions are always meant to be as vector spaces over the complex numbers if not otherwise stated.

2. Category $\mathcal{O}$: Definitions and the main properties

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with a chosen Borel and a fixed Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. The corresponding universal enveloping algebras are denoted by $\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{b})$, etc.

We consider the category $\mathcal{O}$ defined as

$$\mathcal{O} := \left\{ M \in \mathfrak{g}\text{-mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated as a } \mathcal{U}(\mathfrak{g})\text{-module,} \\ M \text{ is locally finite for } \mathfrak{n}, \\ \mathfrak{h} \text{ acts diagonally on } M, \end{array} \right\}$$

[1]Maple V is a registered trademark of Waterloo Maple, Inc.

[2]GAP—Groups, Algorithms, and Programming. Lehrstuhl D für Mathematik, RWTH, Aachen, Germany, 1995.

where the second condition means that $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{n}) \cdot m < \infty$ for all $m \in M$, and the last says that $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$ denotes the μ-weight space of M.

Many results about this category can be found, for example, in [BGG, Ja1, Ja2]. We want to list a few of these properties needed in the following without giving proofs.

The category $\mathcal{O}$ decomposes into a direct sum of full subcategories $\mathcal{O}_\chi$ indexed by central characters χ of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$. Let $S = S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$ be the symmetric algebra over $\mathfrak{h}$ considered as regular functions on $\mathfrak{h}^*$, together with the dot-action of the Weyl group W, defined as $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$, where ρ is the half-sum of positive roots. Let $\mathcal{Z} = \mathcal{Z}(\mathcal{U})$ be the center of $\mathcal{U}$. Using the so-called Harish-Chandra isomorphism (see, e.g., [Ja1, Satz 1.5], [Di, Theorem 7.4.5]) $\mathcal{Z} \to S^{W \cdot}$ and the fact that S is integral over $S^{W \cdot}$ ([Di, Theorem 7.4.8]) we get an isomorphism $\xi : \mathfrak{h}^*/(W \cdot) \to \operatorname{Max} \mathcal{Z}$. Here $\operatorname{Max} \mathcal{Z}$ denotes the set of maximal ideals in $\mathcal{Z}$. This yields the following decomposition into blocks

$$\mathcal{O} = \bigoplus_{\chi \in \operatorname{Max} \mathcal{Z}} \mathcal{O}_\chi = \bigoplus_{\lambda \in \mathfrak{h}^*/(W \cdot)} \mathcal{O}_\lambda,$$

where $\mathcal{O}_\chi$ denotes the subcategory of $\mathcal{O}$ consisting of all objects killed by some power of χ. It denotes the same block as $\mathcal{O}_\lambda$ if $\xi(\lambda) = \chi$.

$\mathcal{O}_\lambda$ is called a *regular block* of category $\mathcal{O}$ iff λ is regular, that is, if $\lambda + \rho$ is not zero at any coroot $\check{\alpha}$ belonging to $\mathfrak{b}$.

For all $\lambda \in \mathfrak{h}^*$ we have a standard module, the Verma module $M(\lambda) = \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ denotes the irreducible $\mathfrak{h}$-module with weight λ enlarged by the trivial action to a module over the Borel subalgebra. This Verma module is a highest weight module of highest weight λ and has central character $\xi(\lambda)$. We denote by $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$.

Theorem 2.1.

(1) Every object in $\mathcal{O}$ has finite length.
(2) $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(M, N) < \infty$ *for all* $M, N \in \operatorname{Ob}(\mathcal{O})$.
(3) There is a one-one correspondence

$$\begin{array}{ccl} \mathfrak{h}^* & \stackrel{1:1}{\longleftrightarrow} & \{\text{simples of } \mathcal{O} \quad (\text{up to isomorphism})\} \\ \lambda & \longmapsto & L(\lambda) \end{array}$$

There are only a finite number of simple modules in each block. More precisely, the simples of $\mathcal{O}_\lambda$ are in bijection with W/W_λ, where $W_\lambda = \{w \in W | w \cdot \lambda = \lambda\}$ denotes the stabilizer of λ under the dot-action of W. The map is given by

$$\begin{array}{ccl} W/W_\lambda & \stackrel{1:1}{\longleftrightarrow} & \left\{ \begin{array}{l} \textit{simples of } \mathcal{O}_\lambda \\ \textit{(up to isomorphism)} \end{array} \right\} \\ x & \longmapsto & L(x \cdot \lambda), \textit{ the simple module with highest weight } x \cdot \lambda. \end{array}$$

(4) For every $\lambda \in \mathfrak{h}^*$ there is (up to isomorphism) a unique indecomposable projective module $P(\lambda) \in \mathcal{O}$ surjecting onto $L(\lambda)$. Therefore there are enough projective objects in $\mathcal{O}$.

For each block $\mathcal{O}_\lambda$ we get the following bijection of sets:

$$W/W_\lambda \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{l} \textit{indecomposable projectives of } \mathcal{O}_\lambda \\ \textit{(up to isomorphism)} \end{array} \right\}$$

$$x \quad \longmapsto \quad P(x \cdot \lambda), \text{ the projective cover of } L(x \cdot \lambda).$$

(5) *$P(\lambda)$ has a Verma flag for every $\lambda \in \mathfrak{h}^*$, and the multiplicities $(P(\lambda) : M(\mu))$ are given by the reciprocity formula $(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)]$, where the last expression denotes the multiplicity of $L(\lambda)$ in a composition series of $M(\mu)$.*

(6) *The Verma module $M(\lambda)$ is projective iff λ is dominant, that is, $\langle \lambda+\rho, \check{\alpha}\rangle \notin \{-1, -2, \ldots\}$ for all positive coroots $\check{\alpha}$.*

(7) *The Verma module $M(\lambda)$ is simple iff λ is antidominant, that is $\langle \lambda+\rho, \check{\alpha}\rangle \notin \{1, 2, \ldots\}$ for all positive coroots $\check{\alpha}$.*

Proof. See [Ja1, Kapitel 1] and [Ja2, Kapitel 4]. □

2.1. **Category $\mathcal{O}$ as a module category over a finite dimensional algebra.** If we fix $\lambda \in \mathfrak{h}^*$, we get an equivalence of categories of the block $\mathcal{O}_\lambda$ with a certain category of finite dimensional modules over a finite dimensional algebra in the following way:

For R a ring, we denote by mof-R the category of finitely generated right R-modules. Take $P := \bigoplus_{x \in W/W_\lambda} P(x \cdot \lambda)$ the direct sum over all indecomposable projectives in $\mathcal{O}_\lambda$. This is a projective generator and we get an equivalence of categories (see [Ba, Theorem 1.3])

$$\begin{aligned} e_\lambda : \mathcal{O}_\lambda \quad &\longrightarrow \quad \text{mof-}\operatorname{End}_{\mathfrak{g}}(P) \\ M \quad &\longmapsto \quad \operatorname{Hom}_{\mathfrak{g}}(P, M). \end{aligned} \tag{2.1}$$

In this way we can consider each block as a category of finitely generated modules over a quasi-hereditary algebra or as a highest weight category (in the sense of [CPS]).

3. Category $\mathcal{O}$ and quivers

We start with the definition of a (finite) quiver defined over $\mathbb{C}$.

Definition 3.1. A (finite) quiver Q(V,E) over $\mathbb{C}$ is an oriented graph, where V is the (finite) set of vertices and E is the (finite) set of arrows between vertices.

We define the "source"-map $\mathbf{s}$ and the "end"-map $\mathbf{e} : E \to V$ such that if $f \in E$ is an arrow from the vertex i to the vertex j we set $\mathbf{s}(f) = i$ and $\mathbf{e}(f) = j$.

A path (of length t) in the quiver $Q(V, E)$ is an ordered sequence of arrows $p = f_t \ldots\ f_2 f_1$ with $\mathbf{e}(f_i) = \mathbf{s}(f_{i+1})$ for $1 \leq i < t$.

We call a complex artin algebra A *basic*, if it is a direct sum of pairwise non-isomorphic indecomposable projective modules. Given a finite dimensional basic algebra the corresponding quiver can be constructed in the following way: We have a bijection between the simple A-modules (up to isomorphism) and the vertices of such a quiver belonging to A. The number of arrows from the vertex i to vertex j is given by $\dim_{\mathbb{C}} \operatorname{Ext}(S(j), S(i))$ of the corresponding simple modules $S(i)$ and $S(j)$. This number is the same as the dimension of the vector space $\operatorname{Hom}_A\big(P(i), \operatorname{rad}(P(j))/\operatorname{rad}(\operatorname{rad}(P(j)))\big)$, where $P(i)$ is the projective cover of the simple module corresponding to the vertex i. If we also take for each vertex the corresponding trivial path we obtain by concatenating paths the so-called "path algebra". This algebra can be described by generators and relations. The relations for this path algebra are often just called "relations of the quiver". Every basic finite dimensional algebra over an algebraically closed field is isomorphic to the path algebra with certain relations of some quiver. (For more details see, for example, [ARS]).

Using the equivalence of categories (2.1) we can associate to each block of $\mathcal{O}$ a quiver describing the whole block.

4. Computation of quivers using the algebra of coinvariants

Theoretically, a quiver corresponding to a certain block of category $\mathcal{O}$ is given by a projective generator. But such a projective generator is not easy to handle and therefore not very useful for explicit calculations. What is needed is a better description of the homomorphisms between projectives. The first step can be done by describing the endomorphisms of the indecomposable projective module belonging to the longest element of the Weyl group.

Fix $\lambda \in \mathfrak{h}^*$ dominant and integral. The center $\mathcal{Z}$ of the universal enveloping algebra $\mathcal{U}$ yields by multiplication a map $\mathcal{Z} \to \mathrm{End}_{\mathfrak{g}}(P(w_0^\lambda \cdot \lambda))$, where w_0^λ denotes the longest element of W/W_λ. On the other hand, we have a map $\mathcal{Z} \to S^{W\cdot} \to S/(S_+^W)$ by composing the Harish-Chandra-isomorphism and the natural projection. Here S_+ denotes the maximal ideal of S consisting of all regular functions vanishing at zero and (S_+^W) is the ideal generated by polynomials without a constant term, invariant under the (usual!) action of the Weyl group. The key result for computing quivers of category $\mathcal{O}$ is that, for $\lambda = \rho$, both of these maps are surjective and have the same kernel. We therefore get the following very beautiful theorem:

Theorem 4.1. ([So1, Endomorphismensatz], *with a nice proof for regular* λ *by* [Be].) *Let* $\lambda \in \mathfrak{h}^*$ *be dominant and integral and* W_λ *its stabilizer under the dot-action of the Weyl group. Let* w_0^λ *be the longest element of* W/W_λ*. Then there is an isomorphism of algebras*

$$\mathrm{End}_{\mathfrak{g}}(P(w_0^\lambda \cdot \lambda)) \cong (S/(S_+^W))^{W_\lambda}.$$

Remark. The algebra $(S/(S_+^W))$ is the so-called "algebra of coinvariants" and its dimension (as complex vector space) is just the order of the Weyl group (see [Bo]). In the following we denote it by C and its invariants C^{W_λ} by C^λ. This algebra is commutative, so we can consider right C-modules also as left C-modules.

Using this description of the endomorphisms of this "big" projective module we can describe the homomorphisms between the other indecomposable projective modules of a certain block using the following.

Theorem 4.2 ([So1, Struktursatz 9])**.** *Let* $\lambda \in \mathfrak{h}^*$ *be dominant and integral. The exact functor*

$$\begin{array}{rccl} \mathbb{V}_\lambda : & \mathcal{O}_\lambda & \to & C^\lambda\text{-mof} \\ & M & \mapsto & \mathrm{Hom}_{\mathfrak{g}}(P(w_0^\lambda \cdot \lambda), M) \end{array}$$

is fully faithful on projectives.

In other words, for $x, y \in W/W_\lambda$*, there is an isomorphism of vector spaces*

$$\mathrm{Hom}_{\mathfrak{g}}(P(x \cdot \lambda), P(y \cdot \lambda)) \cong \mathrm{Hom}_{C^\lambda}(\mathbb{V}_\lambda P(x \cdot \lambda), \mathbb{V}_\lambda P(y \cdot \lambda)).$$

Remark. More precisely, the Struktursatz of [So1] says that the functor $\mathbb{V}_\lambda$ induces even an isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}(M,P) \cong \operatorname{Hom}_{C^\lambda}(\mathbb{V}_\lambda M, \mathbb{V}_\lambda P)$$

for arbitrary M in $\mathcal{O}_\lambda$ and any projective module P in $\mathcal{O}_\lambda$.

If we use translation functors, we can describe the right-hand side in a more manageable way for computing quivers.

Let $\mu, \lambda \in \mathfrak{h}^*$ such that $\lambda - \mu$ is an integral weight. Let $E(\mu - \lambda)$ be the finite dimensional irreducible $\mathfrak{g}$-module with extremal weight $\mu - \lambda$. Then we define the translation functor (which is an irreducible projective functor in the sense of [BG])

$$\begin{aligned} T_\lambda^\mu \quad : \quad & \mathcal{O}_\lambda \longrightarrow \mathcal{O}_\mu \\ & M \mapsto \mathrm{pr}_\mu(E(\mu-\lambda)\otimes M), \end{aligned}$$

where pr_μ denotes the projection to the block $\mathcal{O}_\mu$. If we choose λ regular and μ singular with $W_\mu = \{1, s\}$ for a simple reflection s, we can define the composition $\theta_s = T_\mu^\lambda T_\lambda^\mu$ as a functor on $\mathcal{O}_\lambda$. For $x \in W$ with $x = s_r \cdots s_3 s_2 s_1$ a reduced expression and λ dominant, the module $P(x\cdot\lambda)$ is isomorphic to a direct summand of $\theta_{s_1}\cdots\theta_{s_r}M(\lambda)$. Even more, it is the unique indecomposable direct summand of $M(\lambda)$ not isomorphic to some $P(y\cdot\lambda)$ with $y < x$ (more details can be found, e.g., in [Ja2], [BG]).

The combinatorial description of our projective modules can be given by the following.

Theorem 4.3 ([So1, Theorem 10])**.** *Let $\lambda \in \mathfrak{h}^*$ be regular and let s be a simple reflection. Denote by C^s the invariants of C under the action of s. There is a natural equivalence of functors $\mathcal{O}_\lambda \to C$-mof*

$$\mathbb{V}_\lambda\theta_s \cong C \otimes_{C^s} \mathbb{V}_\lambda.$$

Corollary 4.4. *Let $x = s_r \ldots s_3 s_2 s_1$ be a reduced expression of $x \in W$. Then the module $\mathbb{V}_\lambda P(x\cdot\lambda)$ is isomorphic to a direct summand of $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_3}} C \cdots C \otimes_{C^{s_r}} \mathbb{C}$.*

Remark. The theorem is also true for singular λ when replacing C by C^λ.

The corollary makes it possible to compute quivers for category $\mathcal{O}$. We will give some examples.

5. Examples: Lie algebras of rank 1 and 2

In this section we will describe quivers of blocks $\mathcal{O}_\lambda$, where λ is integral and regular and $\mathfrak{g}$ has a root system of rank at most two. Before giving these examples we want to explain more precisely the method we use to calculate them. The algorithm we use consists of five steps:

(1) *Compute a basis of C and describe the algebra structure in this basis.*

We identify S with a polynomial ring. After calculating generators of S_+^W we use Groebner bases (see, e.g., [AL, Theorem 2.1.6]) to find a basis of C.

(2) *For $x \in W$ and $x = s_r \ldots s_3 s_2 s_1$, a reduced expression, find a basis of $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_3}} C \cdots \otimes_{C^{s_r}} \mathbb{C}$ and express the C-action in this basis.*

It is not very hard to give a basis where all elements have the form $x_1 \otimes x_2 \otimes \cdots \otimes x_r$ with $x_i \in C$ a monomial of degree 0 or 1. The operation of C could more easily be described by using the results about tensor products of smaller length, but needs some computer capacity.

(3) *Find a basis of the desired direct summand $\mathbb{V}_\lambda P(x \cdot \lambda)$ and describe the C-module structure using the results of the first step.*

This is the most difficult part, because we have no algorithm which can do this in general. Therefore it depends on the situation how we can find the desired direct summand. It makes it more manageable if the desired submodule is generated by the element $1 \otimes 1 \otimes \cdots \otimes 1$. This is always the case for root systems of rank 2. For higher ranks one has to use ad hoc methods (see example A_3).

(4) *Take the previous results to compute all homomorphisms between any two indecomposable projectives in $\mathcal{O}_\lambda$.*

This can be done by Theorem 4.2. In the previous steps the C-modules $\mathbb{V}P(x \cdot \lambda)$ were completely described. Hence it is possible to determine all C-linear maps between them. It is simply solving a system of linear equations to find which $\mathbb{C}$-linear maps are in fact C-linear. The problem here is that this step needs a great deal of computer capacity.

(5) *Find elements of the homomorphism spaces which represent the arrows of a quiver.*

Composing maps between our indecomposable modules we can decide which morphisms must be represented by a path of length longer than one. The other ones represent the arrows.

(6) *Find the relations by computing all possible compositions of linear maps represented by the arrows.*

This step is not very difficult and can by solved by a state of the art computer program. The calculation loop stops as soon as the dimension of the resulting path algebra is correct.

Remark and Notation. Using the famous conjecture of Kazhdan-Lusztig ([KL, Conjecture 1.5], proved in [BB], [BK]) one can verify the number of arrows which are given as certain coefficients of the corresponding Kazhdan-Lusztig polynomials (see [BGS, Proposition 3.6.1] with $\mathfrak{q} = \mathfrak{b}$ or the explicit formula [Ir1, Corollary 3]). The existence of a contravariant duality d on $\mathcal{O}$ which fixes simple objects (see, e.g., [Ja2, 4.10, 4.11]) implies that for all vertices i and j of quivers describing a block of $\mathcal{O}$ the number of arrows from i to j is the same as the number of arrows the other way around. In the following we use the symbol $\leftrightarrow$ instead of $\leftrightarrows$. The symbol $(i \to j)$ denotes the arrow from i to j, if there is only one such arrow.

5.1. **Regular blocks.** To calculate a basis of the coinvariant algebra we use the description of the regular functions on $\mathfrak{h}$ as a polynomial ring in dim $\mathfrak{h}$ variables identifying the indeterminants with the simple coroots.

5.1.1. *The case A_1.* In this case we have two indecomposable projective objects in an integral regular block. One can see at once, that $C \cong \mathbb{C}[X]/(X^2)$, which has as basis the images of 1 and X under the canonical projection. On the other side we have $\mathbb{V}M(\lambda) \cong \mathbb{C}$, the trivial module. All homomorphisms between projectives are therefore given by C-linear maps between C and $\mathbb{C}$. We get easily a basis for the homomorphisms between these modules:

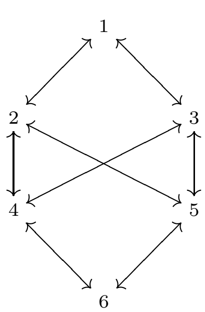
$\mathbb{C} \to \mathbb{C}$	id	
$\mathbb{C} \to C$	$p(1 \to 2)$:	$1 \mapsto X$
$C \to \mathbb{C}$	$p(2 \to 1)$:	$1 \mapsto 1$ $X \mapsto 0$
$C \to C$	id	$1 \mapsto X$ $X \mapsto 0$

We observe that $p(1 \to 2) \circ p(2 \to 1)$ is just the last map. Therefore we get the following quiver with relations (where the first vertex corresponds to the projective Verma module $M(\lambda)$ and the second vertex corresponds to the indecomposable projective-injective module):

$$1 \longleftrightarrow 2 \qquad \text{relations: } (2 \to 1) \circ (1 \to 2) = 0.$$

To make it more readable we write instead of expressions like $(2 \to 1) \circ (1 \to 2)$ just $(1 \to 2 \to 1)$.

5.1.2. *The case A_2.* In this case the algebra of coinvariants is isomorphic to $\mathbb{C}[X, Y]/\langle X^2 + XY + Y^2, 2XY^2 + Y^3, Y^4\rangle$. Using the theory of Groebner bases we can find as basis the images of 1, X, Y, X^2, Y^2, X^3 of the canonical projection. Identifying X and Y with the coroots $\check{\alpha}$ and $\check{\beta}$ resp., we get a basis of modules which are of the form $C \otimes_{C^{s_1}} C \otimes_{C^{s_2}} C \otimes_{C^{s_3}} C \cdots C \otimes_{C^{s_r}} \mathbb{C}$. We have for example a basis of $C \otimes_{C^{s_\alpha}} \mathbb{C}$ given by the vectors $1 \otimes 1$ and $X \otimes 1$ (and the equality $Y \otimes 1 = -\frac{1}{2} X \otimes 1$). For the root system A_2 we can skip step 3 of our algorithm. The reason for this is Theorem 4.3 and the fact that the projectives occurring after translating the dominant Verma module through the walls at most twice are still indecomposable, which can be verified by comparing the characters of the translated Verma module in question and the indecomposable projective modules. After all calculations we get the following quiver with relations (where $(i \to j)$ denotes the arrow from i to j):



$(1 \to 2 \to 1) = 0$
$(1 \to 3 \to 1) = 0$
$(2 \to 4 \to 2) = 0$
$(3 \to 5 \to 3) = 0$
$(1 \to 3 \to 4) = (1 \to 2 \to 4)$
$(1 \to 3 \to 5) = -\frac{1}{2}(1 \to 2 \to 5)$
$(2 \to 4 \to 3) = -2(2 \to 1 \to 3)$
$(2 \to 5 \to 2) = -4(2 \to 1 \to 2)$
$(2 \to 5 \to 3) = 4(2 \to 1 \to 3)$
$(2 \to 5 \to 6) = 2(2 \to 4 \to 6)$

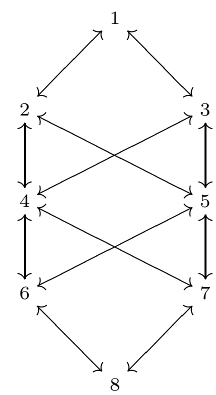
$(4 \to 6 \to 5) = \frac{3}{2}(4 \to 2 \to 5) + \frac{3}{2}(4 \to 3 \to 5)$
$(5 \to 6 \to 4) = -\frac{3}{2}(5 \to 2 \to 4) - \frac{3}{2}(5 \to 3 \to 4)$

$(3 \to 4 \to 2) = -2(3 \to 1 \to 2)$
$(3 \to 4 \to 3) = -4(3 \to 1 \to 3)$
$(3 \to 5 \to 2) = 4(3 \to 1 \to 2)$
$(3 \to 5 \to 6) = -(3 \to 4 \to 6)$
$(4 \to 3 \to 1) = (4 \to 2 \to 1)$
$(4 \to 6 \to 4) = -3(4 \to 2 \to 4)$
$(5 \to 3 \to 1) = -\frac{1}{2}(5 \to 2 \to 1)$
$(5 \to 6 \to 5) = -\frac{3}{2}(5 \to 3 \to 5)$
$(6 \to 5 \to 2) = -2(6 \to 4 \to 2)$
$(6 \to 5 \to 3) = (6 \to 4 \to 3)$

The shape of the relations. In the previous example and in all the following ones it is possible to avoid rational coefficients and choose generators such that only integral coefficients occur. There might be perhaps a representation theoretical reason for that. The path algebra might be considered as a $\mathbb{Z}$-graded algebra, generated by its one element of degree 0 and its arrows, all of degree 1. The results of [BGS] imply that the relations can be choosen such that they are compatible with the grading.

We now give quivers for type B_2 and G_2, since they turned out to be useful for explicit calculations such as finding socle and radical filtrations of given representations.

5.1.3. *The case B_2.* In this case we have $C[X,Y] \cong \mathbb{C}/\langle Y^4, Y^2+2XY+2X^2\rangle$, which has as basis the images of the elements $1, X, Y, XY, Y^2, XY^2, Y^3, XY^3$ under the canonical projection. We get the following quiver with relations:

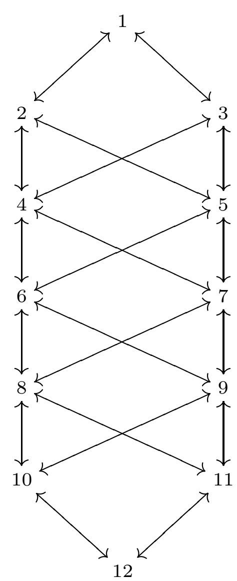


$$\begin{aligned}
&(1\to 2\to 1)=0\\
&(1\to 3\to 1)=0\\
&(2\to 5\to 2)=0\\
&(3\to 4\to 3)=0\\
&(1\to 3\to 4)=(1\to 2\to 4)\\
&(1\to 3\to 5)=-2(1\to 2\to 5)\\
&(2\to 4\to 2)=2(2\to 1\to 2)\\
&(2\to 4\to 3)=(2\to 1\to 3)\\
&(2\to 4\to 6)=(2\to 5\to 6)\\
&(2\to 4\to 7)=\tfrac{1}{2}(2\to 5\to 7)\\
&(2\to 5\to 3)=-2(2\to 1\to 3)\\
&(3\to 4\to 2)=-(3\to 1\to 2)\\
&(3\to 4\to 6)=-\tfrac{1}{2}(3\to 5\to 6)\\
&(3\to 4\to 7)=-\tfrac{1}{4}(3\to 5\to 7)\\
&(3\to 5\to 2)=-2(3\to 1\to 2)\\
&(3\to 5\to 3)=-4(3\to 1\to 3)
\end{aligned}$$

$$\begin{aligned}
&(4\to 3\to 1)=-(4\to 2\to 1)\\
&(4\to 6\to 4)=-2(4\to 2\to 4)\\
&(4\to 7\to 4)=2(4\to 3\to 4)\\
&(4\to 7\to 8)=-(4\to 6\to 8)\\
&(5\to 3\to 1)=-2(5\to 2\to 1)\\
&(5\to 6\to 5)=(5\to 2\to 5)\\
&(5\to 7\to 5)=-2(5\to 3\to 5)\\
&(5\to 7\to 8)=-2(5\to 6\to 8)\\
&(6\to 4\to 2)=2(6\to 5\to 2)\\
&(6\to 4\to 3)=(6\to 5\to 3)\\
&(6\to 8\to 6)=-2(6\to 5\to 6)\\
&(7\to 4\to 2)=-\tfrac{1}{2}(7\to 5\to 2)\\
&(7\to 4\to 3)=-\tfrac{1}{4}(7\to 5\to 3)\\
&(7\to 8\to 7)=2(7\to 4\to 7)\\
&(8\to 7\to 4)=-(8\to 6\to 4)\\
&(8\to 7\to 5)=4(8\to 6\to 5)
\end{aligned}$$

$$\begin{aligned}
&(4\to 6\to 5)=-(4\to 2\to 5)-(4\to 3\to 5)\\
&(4\to 7\to 5)=-4(4\to 2\to 5)-2(4\to 3\to 5)\\
&(5\to 6\to 4)=-2(5\to 2\to 4)+2(5\to 3\to 4)\\
&(5\to 7\to 4)=4(5\to 2\to 4)-2(5\to 3\to 4)\\
&(6\to 8\to 7)=-(6\to 5\to 7)+(6\to 4\to 7)\\
&(7\to 8\to 6)=\tfrac{1}{2}(7\to 5\to 6)+(7\to 4\to 6)
\end{aligned}$$

5.1.4. *The case G_2.* In this case we have $C \cong \mathbb{C}[X,Y]/\langle Y^2+3XY+3X^2, Y^6\rangle$, which has as basis the images of $1, X, Y, XY, Y^2, XY^2, Y^3, XY^3, Y^4, XY^4, Y^5, XY^5$ under the canonical projection. We get the following quiver with relations:



$$\begin{aligned}
&(1\to 2\to 1)=0\\
&(1\to 3\to 1)=0\\
&(2\to 5\to 2)=0\\
&(3\to 4\to 3)=0\\
&(1\to 3\to 4)=(1\to 2\to 4)\\
&(1\to 3\to 5)=-2(1\to 2\to 5)\\
&(2\to 4\to 2)=-3(2\to 1\to 2)\\
&(2\to 4\to 3)=(2\to 1\to 3)\\
&(2\to 4\to 6)=-2(2\to 5\to 6)\\
&(2\to 4\to 7)=\tfrac{1}{2}(2\to 5\to 7)\\
&(2\to 5\to 3)=-2(2\to 1\to 3)\\
&(3\to 4\to 2)=(3\to 1\to 2)\\
&(3\to 4\to 6)=(3\to 5\to 6)\\
&(3\to 4\to 7)=-\tfrac{1}{4}(3\to 5\to 7)\\
&(3\to 5\to 2)=-2(3\to 1\to 2)\\
&(3\to 5\to 3)=-4(3\to 1\to 3)\\
&(4\to 3\to 1)=(4\to 2\to 1)\\
&(4\to 7\to 4)=2(4\to 3\to 4)\\
&(4\to 7\to 8)=-\tfrac{1}{12}(4\to 6\to 8)\\
&(4\to 7\to 9)=-\tfrac{1}{3}(4\to 6\to 9)\\
&(5\to 3\to 1)=-2(5\to 2\to 1)\\
&(5\to 6\to 5)=2(5\to 2\to 5)\\
&(5\to 7\to 8)=\tfrac{1}{3}(5\to 6\to 8)
\end{aligned}$$

$$\begin{aligned}
&(5\to 7\to 9)=\tfrac{4}{3}(5\to 6\to 9)\\
&(6\to 4\to 2)=4(6\to 5\to 2)\\
&(6\to 4\to 3)=-2(6\to 5\to 3)\\
&(6\to 8\to 10)=4(6\to 9\to 10)\\
&(6\to 8\to 11)=(6\to 9\to 11)\\
&(6\to 8\to 6)=4(6\to 4\to 6)\\
&(6\to 9\to 6)=2(6\to 5\to 6)\\
&(7\to 4\to 2)=\tfrac{1}{4}(7\to 5\to 2)\\
&(7\to 4\to 3)=-\tfrac{1}{8}(7\to 5\to 3)\\
&(7\to 8\to 10)=(7\to 9\to 10)\\
&(7\to 8\to 11)=\tfrac{1}{4}(7\to 9\to 11)\\
&(7\to 8\to 7)=2(7\to 4\to 7)\\
&(7\to 9\to 7)=-4(7\to 5\to 7)\\
&(8\to 11\to 12)=-\tfrac{1}{6}(8\to 10\to 12)\\
&(8\to 11\to 8)=2(8\to 7\to 8)\\
&(8\to 7\to 4)=\tfrac{1}{32}(8\to 6\to 4)\\
&(8\to 7\to 5)=\tfrac{1}{2}(8\to 6\to 5)\\
&(9\to 10\to 9)=2(9\to 6\to 9)\\
&(9\to 11\to 12)=-\tfrac{2}{3}(9\to 10\to 12)\\
&(9\to 7\to 4)=2(9\to 6\to 4)\\
&(9\to 7\to 5)=32(9\to 6\to 5)\\
&(10\to 8\to 6)=128(10\to 9\to 6)\\
&(10\to 8\to 7)=2(10\to 9\to 7)
\end{aligned}$$

$(10 \to 12 \to 10) = -\frac{9}{4}(10 \to 9 \to 10)$
$(11 \to 8 \to 6) = \frac{1}{2}(11 \to 9 \to 6)$
$(11 \to 8 \to 7) = \frac{1}{128}(11 \to 9 \to 7)$
$(11 \to 12 \to 11) = \frac{9}{4}(11 \to 8 \to 11)$
$(12 \to 11 \to 8) = \frac{1}{16}(12 \to 10 \to 8)$
$(12 \to 11 \to 9) = 16(12 \to 10 \to 9)$
$(4 \to 6 \to 4) = 32(4 \to 2 \to 4) + 48(4 \to 3 \to 4)$
$(4 \to 6 \to 5) = -4(4 \to 2 \to 5) + 6(4 \to 3 \to 5)$
$(4 \to 7 \to 5) = 8(4 \to 2 \to 5) - 4(4 \to 3 \to 5)$
$(5 \to 6 \to 4) = 8(5 \to 2 \to 4) - 12(5 \to 3 \to 4)$
$(5 \to 7 \to 4) = 4(5 \to 2 \to 4) - 2(5 \to 3 \to 4)$
$(5 \to 7 \to 5) = -16(5 \to 2 \to 5) - 8(5 \to 3 \to 5)$
$(10 \to 12 \to 11) = -\frac{9}{8}(10 \to 9 \to 11) - \frac{9}{64}(10 \to 8 \to 11)$
$(11 \to 12 \to 10) = \frac{3}{64}(11 \to 9 \to 10) + \frac{3}{8}(11 \to 8 \to 10)$
$(6 \to 8 \to 7) = 4(6 \to 5 \to 7) + (6 \to 4 \to 7)$
$(6 \to 9 \to 7) = 8(6 \to 5 \to 7) + 8(6 \to 4 \to 7)$
$(7 \to 8 \to 6) = \frac{8}{3}(7 \to 5 \to 6) - \frac{8}{3}(7 \to 4 \to 6)$
$(7 \to 9 \to 6) = \frac{1}{3}(7 \to 5 \to 6) - \frac{4}{3}(7 \to 4 \to 6)$
$(8 \to 10 \to 8) = -2(8 \to 6 \to 8) + 96(8 \to 7 \to 8)$
$(8 \to 10 \to 9) = \frac{1}{4}(8 \to 6 \to 9) + 3(8 \to 7 \to 9)$
$(8 \to 11 \to 9) = 2(8 \to 6 \to 9) + 8(8 \to 7 \to 9)$
$(9 \to 10 \to 8) = 8(9 \to 6 \to 8) + 6(9 \to 7 \to 8)$
$(9 \to 11 \to 8) = (9 \to 6 \to 8) + \frac{1}{4}(9 \to 7 \to 8)$
$(9 \to 11 \to 9) = 32(9 \to 6 \to 9) - 2(9 \to 7 \to 9)$

5.2. **Singular blocks.** In this section we want to give quivers for singular blocks. In the most singular case, where $\lambda = -\rho$, the situation is very easy, because there is only one Verma module which is a simple projective module. So, in rank two case, we restrict ourselves to studying only the situation where λ is exactly on one wall.

5.2.1. *The case A_2.* Suppose λ is singular with respect to the coroot X. We can choose as basis for C^λ, the images of $\{1, a, a^2\}$, with $a = X + 2Y$. We have three simples in this block and get the following quiver:

$$\begin{tikzcd} & 1 \arrow[dl, leftrightarrow] \\ 2 \arrow[dr, leftrightarrow] & \\ & 3 \end{tikzcd}$$

and the relations

$$(1 \to 2 \to 1) = 0$$
$$(2 \to 1 \to 2) = (2 \to 3 \to 2)$$

(the simple number 1 belongs to the dominant weight and number 3 to the antidominant weight).

5.2.2. *The number of arrows and Kazhdan-Lusztig polynomials.* We can also in the singular case get the number of arrows using Kazhdan-Lusztig polynomials, but not as directly as in the regular case. First of all we have to describe our chosen block by a Koszul ring. Let λ be dominant and integral and denote by $\mathcal{S}_\lambda$ the set of all simple reflections stabilizing λ under the dot-action. Let W_λ be the stabilizer of λ and W^λ be the set of longest representatives of the cosets W/W_λ. Let $\mathfrak{g}_\lambda$ be the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to S_λ.

Let $\mathcal{O}^{\mathfrak{g}_\lambda}$ denote the corresponding parabolic category $\mathcal{O}$, i.e., the full subcategory of $\mathcal{O}_0$ whose objects are locally $\mathfrak{g}_\lambda$-finite. The simple objects of $\mathcal{O}^{\mathfrak{g}_\lambda}$ are the $L(x^{-1}w_0 \cdot 0)$ for $x \in W^\lambda$. We write $L^\lambda(x)$ for $L^\lambda(x^{-1}w_0 \cdot 0)$. Denote by $P^\lambda(x)$ the projective cover of $L(x^{-1}w_0 \cdot 0)$ in $\mathcal{O}^{\mathfrak{g}_\lambda}$. Put $A_\lambda = \mathrm{End}_{\mathcal{O}_\lambda}(\bigoplus_{x \in W^\lambda} P(x \cdot \lambda))$ and $A^\lambda = \mathrm{End}_{\mathcal{O}^{\mathfrak{g}_\lambda}}(\bigoplus_{x \in W^\lambda} P^\lambda(x))$.

By the main theorem of Beilinson, Ginzburg and Soergel ([BGS, Theorem 1.1.3]), there are isomorphisms of finite dimensional complex algebras

$$(5.1)\quad A_\lambda = \mathrm{End}_{\mathcal{O}_\lambda}(\bigoplus_{x \in W^\lambda} P(x \cdot \lambda)) \quad \cong \quad \mathrm{Ext}^\bullet_{\mathcal{O}^{\mathfrak{g}_\lambda}}(\bigoplus_{x \in W^\lambda} L^\lambda(x), \bigoplus_{x \in W^\lambda} L^\lambda(x)).$$
$$A^\lambda = \mathrm{End}_{\mathcal{O}^{\mathfrak{g}_\lambda}}(\bigoplus_{x \in W^\lambda} P^\lambda(x)) \quad \cong \quad \mathrm{Ext}^\bullet_{\mathcal{O}_\lambda}(\bigoplus_{x \in W^\lambda} L(x \cdot \lambda), \bigoplus_{x \in W^\lambda} L(x \cdot \lambda)).$$

Moreover, A_λ inherits the structure of a graded algebra from the natural grading of the Ext-algebra and with this grading A_λ becomes Koszul. The ring A^λ is then the Koszul dual of A_λ. The isomorphisms can be chosen such that the obvious idempotents $\mathbf{1}_x$ for $x \in W^\lambda$ correspond. In particular, extensions of simples in a singular block can be described by endomorphisms of projectives in a parabolic category $\mathcal{O}$.

Recall the equivalence of categories $e_\lambda : \mathcal{O}_\lambda \cong \text{mof-}A_\lambda$ and (with the same proof) one has $\mathcal{O}^{\mathfrak{g}_\lambda} \cong \text{mof-}A^\lambda$. Under these equivalences $A^0_\lambda \mathbf{1}_x$ (where A^0_λ denotes the part of degree zero) corresponds to $L(x \cdot \lambda)$ and $(A^\lambda)^0 \mathbf{1}_x$ corresponds to $L(x^{-1}w_0 \cdot 0)$.

Denote by gmof-A_λ the category of finite dimensional graded right $A_\lambda-$modules. There is the grading forgetting functor $f_\lambda : \text{gmof-}A_\lambda \to \text{mof-}A_\lambda \cong \mathcal{O}_\lambda$. *A lift* of an object M of $\mathcal{O}_\lambda$ (respectively $M \in \text{Ob}(\text{mof-}A_\lambda)$) is an object $\tilde{M}$ of gmof-A_λ such that $f_\lambda(\tilde{M}) \cong e_\lambda(M)$ (respectively $f_\lambda(\tilde{M}) \cong M$). If $L \in \text{Ob}(\text{mof-}A_\lambda)$ is simple, then any lift $\tilde{L}$ will be simple and concentrated in a single degree. Let L^x_λ denote the lift of $A^0_\lambda \mathbf{1}_x$ which is concentrated in degree zero. Then every simple object in gmof-A_λ is given by $L^\lambda_x\langle i\rangle$ for some x and i. Here $\langle i\rangle$ is the shift operator, satisfying $M\langle i\rangle_k = M_{k-i}$ for any graded module M. The projective cover P^x_λ of L^x_λ in A_λ-gmof is a lift of $A_\lambda \mathbf{1}_x$ and $P(x \cdot \lambda)$, respectively.

To get lifts of our Verma modules we can consider them as projective covers in the truncated subcategory of $\mathcal{O}_\lambda$. For $x \in W^\lambda$ this subcategory is the full subcategory consisting of all modules having all simple composition factors of the form $L(y \cdot \lambda)$ with $y \not< x$ in the Bruhat ordering. The Verma module $M(x \cdot \lambda)$ corresponds under the equivalence (2.1) of section 2.1 to A_λ/I, where I is the ideal generated by all idempotents $\mathbf{1}_y$ with $y < x$. Because I is homogeneous, A_λ/I has a natural grading and hence we get the required lift. (For details see [BGS, 3.11].) Moreover, this lift is unique up to isomorphism and a grading shift (see [St, Lemma 4.3.5]).

We denote by $\mathcal{D}(\mathcal{A})$ the derived category of an abelian category $\mathcal{A}$. For a complex (X^i, δ) we denote by $X[n]$ the shifted complex $(X[n])^i = X^{n+i}$ with differentials $(-1)^n\delta$. We get the following isomorphisms of vector spaces:

$$\begin{aligned}
\text{Ext}^i_{\mathcal{O}_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda)) &\cong \text{Ext}^i_{A_\lambda}(A^0_\lambda \mathbf{1}_x, A^0_\lambda \mathbf{1}_y) \\
&\cong \text{Hom}_{\mathcal{D}(\text{mof-}A_\lambda)}(A^0_\lambda \mathbf{1}_x, A^0_\lambda \mathbf{1}_x[i]) \\
&\cong \text{Hom}_{\mathcal{D}(\text{gmof-}A_\lambda)}(L^x_\lambda, L^y_\lambda[i]\langle i\rangle).
\end{aligned}$$

The second isomorphism can be found in [KS, I, Exerc.17]. The last is given by the Koszul condition ([BGS, Proposition 2.1.3]). To get Kazhdan-Lusztig polynomials into the picture we have to use a key result of Beilinson, Ginzburg and Soergel ([BGS, Theorems 1.2.6 and 3.11.1]) which gives an equivalence $\mathcal{D}^b(A_\lambda\text{-gmof}) \cong \mathcal{D}^b(A^\lambda\text{-gmof})$ of triangulated categories, where $\mathcal{D}^b$ denotes the bounded derived category. This equivalence yields finally the isomorphism

$$\text{Ext}^i_{\mathcal{O}_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda)) \cong \text{Hom}_{\mathcal{D}(\text{gmof-}A^\lambda)}(P^\lambda_y\langle i\rangle, P^\lambda_x)$$

where $P^\lambda_x = A^\lambda \mathbf{1}_x$ denotes the lift of the projective cover of $L^\lambda(x)$ under the grading forgetting functor $f^\lambda : A^\lambda\text{-gmof} \to \mathcal{O}^\lambda$, such that P^λ_x is the projective cover of L^λ_x, which is the lift of $L^\lambda(x)$ sitting in degree zero.

This last equation gives a formula for the desired dimension of extensions of simples. Recall that the number of arrows from the vertex corresponding to the simple module $L(y \cdot \lambda)$ to the vertex corresponding to the simple module $L(x \cdot \lambda)$ is the dimension of $\text{Ext}^1_{\mathcal{O}_\lambda}(L(x \cdot \lambda), L(y \cdot \lambda))$. We can thus calculate the number of arrows of our quivers using multiplicities of simples in a decomposition series of projectives in the following way:

$$\begin{aligned} \dim \operatorname{Ext}^i_{\mathcal{O}_\lambda}(L(x\cdot\lambda), L(y\cdot\lambda)) &= [P^\lambda_x : L^\lambda_y\langle i\rangle] \\ &= \sum_{z\in W^\lambda, j\in\mathbb{Z}} \big(P^\lambda_x : M^\lambda_z\langle j\rangle\big)[M^\lambda_z\langle j\rangle : L^\lambda_y\langle i\rangle]. \end{aligned} \tag{5.2}$$

The projectives P^λ_x have a flag with subquotients isomorphic to some lifted Verma modules of $\mathcal{O}^{\mathfrak{g}_\lambda}$, which we denote by $M^\lambda_z\langle j\rangle$. In the equation above $(P^\lambda_x : M^\lambda_z\langle j\rangle)$ denotes the corresponding multiplicity. The last step is now to formulate these multiplicities in terms of (parabolic) Kazhdan-Lusztig polynomials. The definition of these polynomials can be found in [KL]; our special case is treated in Theorem 3.11.4 of [BGS], i.e., $\big(P^\lambda_x : M^\lambda_z\langle i\rangle\big) = [M^\lambda_z : L^\lambda_x\langle i\rangle] = n^\lambda_{z,x}\langle i\rangle$. (The latter denotes the coefficient of v^i in the corresponding Kazhdan-Lusztig polynomial $n^\lambda_{z,x}\in\mathbb{Z}[v]$ in the notation of [So3]). Hence we get the following formula involving parabolic Kazhdan-Lusztig polynomials to compute the number of arrows:

$$\dim \operatorname{Ext}^i_{\mathcal{O}_\lambda}(L(x\cdot\lambda), L(y\cdot\lambda)) = \sum_{z\in W^\lambda, j\in\mathbb{N}} n^\lambda_{z,x}\langle j\rangle\, n^\lambda_{z,y}\langle i-j\rangle. \tag{5.3}$$

Let us calculate multiplicities for a singular block for the root system A_2. We have three Verma modules. We order them by their highest weights, such that the projective one is the first one. The polynomials are just the parabolic Kazhdan-Lusztig polynomials $n_{x,y}$ in the notation of [So3], where x, y are representatives of minimal length of $W_\lambda\backslash W$ and given by the following matrix:

$$B(v) := \begin{pmatrix} 1 & v & 0 \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the numbers of arrows in our quiver are given (using formula (5.3)) by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This means that the total number of arrows is just four as in the picture in the last section.

5.2.3. *The case B_2.* Let λ be integral, lying on the wall belonging to the long root. Let $a = Y + 2X \in \mathbb{C}[X,Y]$. Then C^λ has as basis the images of $\{1, a, a^2, a^3\}$ under the canonical projection with the identification introduced in the last chapter. If λ is on the other wall, take only $a = X + Y$. With a few (not very hard) calculations we get in both situations the following quiver:

$$\begin{tikzcd} & 1 \arrow[dr, leftrightarrow] & \\ 3 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & 2 \\ & 4 & \end{tikzcd}$$

with relations
$(1\to 2\to 1) = 0$
$(2\to 1\to 2) = (2\to 3\to 2)$
$(3\to 2\to 3) = (3\to 4\to 3)$

5.2.4. *The case G_2.* For λ on the wall belonging to the long root we have as basis of C^λ the images of $1, a, a^2, a^3, a^4, a^5$ where $a = 2X + Y$. In the other possible situation we can choose $a = \frac{3}{2}X + Y$. We can calculate the following quiver:

$$\begin{tikzcd}
2 \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & 1 \\
4 \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & 3 \\
6 \arrow[r, leftrightarrow] & 5
\end{tikzcd}$$

with relations
$(1 \to 2 \to 1) = 0$
$(2 \to 1 \to 2) = (2 \to 3 \to 2)$
$(3 \to 2 \to 3) = (3 \to 4 \to 3)$
$(4 \to 3 \to 4) = (4 \to 5 \to 4)$
$(5 \to 4 \to 5) = (5 \to 6 \to 5)$

6. A more complicated example: The case A3

In this case we have $C \cong \mathbb{C}[x, y, z]/I$ where I is the ideal

$$\begin{aligned}
I = \langle & z^4y^2 + z^5y, xz^4 + 2z^4y + z^5, z^4 + 2z^3y + 3y^2z^2 + 2y^3z + y^4, \\
& 2y^3z^2 + 3y^2z^3 + 3z^4y + z^5, 5z^3x + 10z^3y + 3z^4 + 10z^2xy + 10y^2z^2, z^6, \\
& 4xy + 4yz + 3x^2 + 4y^2 + 3z^2 + 2xz, 5z^2x + 7z^2y + 3z^3 + 5xyz + 5xy^2 + 3y^2z + 2y^3\rangle.
\end{aligned}$$

The Weyl group is isomorphic to the symmetric group S_4, so it has 24 elements. We can choose as (Groebner) basis of C the images of the elements

$$\begin{gathered}
1, x, y, z, xy, xz, y^2, yz, z^2, xyz, xz^2, y^3, y^2z, yz^2, \\
z^3, xz^3, yz^5, y^3z, y^2z^2, yz^3, z^4, y^2z^3, yz^4, z^5
\end{gathered}$$

under the canonical projection.

6.1. **A quiver for regular blocks.** If we denote by 1, 2, 3 the three simple reflections, we have

$$\begin{aligned}
W = \{&0, 3, 2, 1, 12, 23, 32, 13, 21, 123, 312, 232, 121, 231, 321, \\
&2312, 1232, 1231, 1321, 2321, 12132, 32312, 12321, 123121\}.
\end{aligned} \tag{6.1}$$

For this quiver the calculations are very hard. We have also the problem, that, in general, there exists no algorithm to find our desired direct summands (see step 3 of the algorithm). For A3, there are two indecomposable projectives, in which the dominant Verma module occurs twice. The corresponding elements of the Weyl group are 2312 and 12321. In these cases (see Lemma 7.3) we have no single generator of the direct summand, which is needed in our algorithm to find a basis of the desired direct summand without any problems.

On the other hand, we can use Kazhdan-Lusztig polynomials to compute the dimensions of the C-modules corresponding to these projectives. Fortunately, after successive translations through the walls of the reflections 2,1,3,2, our dominant Verma module still stays indecomposable, because it has the same Verma flag as the desired indecomposable module, namely of length $2^4 = 16$. Therefore it has to be indecomposable and we don't have to look for a direct summand. So there is no problem with step 3.

Using Kazhdan-Lusztig polynomials we are able to compute the length of a Verma flag of the projective indecomposable $P(12321 \cdot 0)$ corresponding to 12321. The result is 24, hence $\dim \mathbb{V}P(12321 \cdot 0) = 24$. On the other hand, we get

$\dim \mathbb{V}P(1232\cdot 0) = 12$. Therefore $P(12321\cdot 0) \cong \theta_1 P(1232\cdot 0)$ holds. Hence we easily get a basis of $\mathbb{V}P(12321\cdot 0)$ using a basis of $P(1232\cdot 0)$. And so we are done. An explicit quiver can be found in the Appendix.

7. Indecomposable projectives: Commutativity of their endomorphism rings

Given an indecomposable projective object P, we want to describe its endomorphism ring. In the extremal case, when P is a Verma module, the situation is easy. The "other" extremal case is covered by Theorem 4.1. In both cases we get a commutative ring. Using the functor $\mathbb{V}$ we can give a necessary and sufficient condition for the commutativity of endomorphism rings of indecomposable projectives objects.

Theorem 7.1. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral and $P = P(x\cdot\lambda) \in \mathcal{O}_\lambda$ be indecomposable and projective. Then the following statements are equivalent:*

(1) $\mathrm{End}_{\mathfrak{g}}(P)$ is commutative.
(2) $(P : M(\lambda)) = 1$.
(3) There is a surjection $\mathcal{Z} \twoheadrightarrow \mathrm{End}_{\mathfrak{g}}(P)$.

Before proving the theorem we give some examples for λ regular.

Examples 7.2. a) If $\mathfrak{g} = \mathfrak{sl}(2)$, we have $\mathrm{End}_{\mathfrak{g}}(P(0)) \cong \mathbb{C}$ and $\mathrm{End}_{\mathfrak{g}}(P(s\cdot 0)) \cong C \cong \mathbb{C}[X]/(X^2)$. Obviously both are commutative.

b) Let $\mathfrak{g} = \mathfrak{sl}(3)$. Recall that we have $C \cong \mathbb{C}[X,Y]/\langle X^2+XY+Y^2, 2XY^2+Y^3, Y^4\rangle$ and there is a $\mathbb{C}$-basis given by the images of the elements $1, X, Y, X^2, Y^2, X^3$ under the canonical projection. Some easy calculations yield the following rings of endomorphisms:

$$\begin{aligned}
\mathrm{End}(P(0)) &\cong C/\langle X, Y\rangle,\\
\mathrm{End}(P(s_\alpha\cdot 0)) &\cong \mathrm{End}(P(s_\beta\cdot 0)) \cong C/\langle Y, X^2\rangle,\\
\mathrm{End}(P(s_\beta s_\alpha\cdot 0)) &\cong \mathrm{End}(P(s_\alpha s_\beta\cdot 0)) \cong C/\langle Y^2, X^3\rangle.
\end{aligned}$$

The endomorphism rings in the previous examples are all commutative. Moreover, they are quotients of the algebra of coinvariants. The reason for this is the following general result.

Lemma 7.3. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral and $x \in W/W_\lambda$. Let* min *denote the cardinality of a minimal system of generators of $\mathbb{V}P(x\cdot\lambda)$. Then the equality*

$$\min = (P(x\cdot\lambda) : M(\lambda))$$

holds.

Proof. Let λ be regular. Let $M \in C$-mof be a finite dimensional C-module. Consider the following canonical C-morphisms:

$$M \cong C \otimes_C M \stackrel{pr\otimes id}{\longrightarrow} C/C^+ \otimes_C M \cong M/C^+M.$$

The composition is surjective and $C/C^+ \cong \mathbb{C}$. Choosing preimages of a fixed $\mathbb{C}$-basis of the image M/C^+M gives a minimal system of generators of M as a

C-module. Hence the following equalities hold:

$$\begin{aligned}\min &= \dim \mathbb{V}P(x\cdot\lambda)/C^+\mathbb{V}P(x\cdot\lambda) = \dim \operatorname{Hom}_{\mathbb{C}}(\mathbb{V}P(x\cdot\lambda)/C^+\mathbb{V}P(x\cdot\lambda), \mathbb{C})\\ &= \dim \operatorname{Hom}_C(\mathbb{V}P(x\cdot\lambda)/C^+\mathbb{V}P(x\cdot\lambda), C/C^+)\\ &= \dim \operatorname{Hom}_C(\mathbb{V}P(x\cdot\lambda), C/C^+)\\ &= \dim \operatorname{Hom}_{\mathfrak{g}}(P(x\cdot\lambda), M(\lambda)) = [M(\lambda) : L(x\cdot\lambda)] = (P(x\cdot\lambda) : M(\lambda)).\end{aligned}$$

For singular λ the argument is just the same, if C is replaced by C^λ. □

The last lemma should show why (2) implies (1) in the theorem. We need some conventions to prove the theorem:

Convention 7.4. We consider $S = S(\mathfrak{h})$ as an even-graded algebra, so $S = \bigoplus_{i\in\mathbb{N}} S^{2i}$. We also assume that $S^2 = \mathfrak{h}$ holds. The grading is compatible with the action of the Weyl group, so the algebra C inherits a grading. Inductively the module $\mathbb{V}P(x\cdot\lambda)$ becomes a graded C-module for all $x \in W$ and $\lambda \in \mathfrak{h}^*$. Details can be found in [So1]. By convention the module $\mathbb{V}P(x\cdot\lambda)$ should be considered as a graded C-module with highest degree $l(x)$.

Proof of Theorem 7.1. We prove the theorem only for λ regular, but the arguments are the same for singular weights.

- 3) $\Rightarrow$ 1): This is obvious.
- 2) $\Rightarrow$ 3): By Lemma 7.3 the module $\mathbb{V}P(x\cdot\lambda)$ and therefore also $\operatorname{End}_{\mathfrak{g}}(P(x\cdot\lambda))$ is a quotient of C. Hence the latter is also a quotient of $\mathcal{Z}$.
- 1) $\Rightarrow$ 2): We assume that there are higher multiplicities. The theorem will be proved by constructing two morphisms g and f which do not commute.

Construction of $\psi \in \operatorname{End}_C(\mathbb{V}P)$ *with the property* $\operatorname{im}(\psi) \not\subseteq \mathbb{V}\ker\chi_\lambda P$. Let $P = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_r \supset N_{r+1} = \{0\}$ be a Verma flag; that is, a filtration such that $N_i/N_{i+1} \cong M(\lambda_i)$ where $\lambda_i \in \mathfrak{h}^*$ and $0 \le i \le r$. This filtration induces by restriction a "quasi-Verma flag" on $\ker\chi_\lambda P$, which is a filtration all of whose subquotients are isomorphic to submodules of Verma modules. Let $\widetilde{M}(\lambda_i) \subseteq M(\lambda_i)$ denote the corresponding subquotients and $J := \{i \mid \widetilde{M}(\lambda_i) \neq 0\} \subset \{0, 1, \ldots, r\}$. We thus have the following equalities in the Grothendieck group of $\mathcal{O}$:

$$[P] = \sum_{i=0}^{r}[M(\lambda_i)] \qquad \text{and} \qquad [\ker\chi_\lambda P] = \sum_{i\in J}[\widetilde{M}(\lambda_i)].$$

By the arguments in the proof of Lemma 7.3 we know the codimension of $\mathbb{V}\ker\chi_\lambda P$ in $\mathbb{V}P$, namely the multiplicity $(P : M(\lambda))$, and hence $|J| = r - (P : M(\lambda))$.

Let $s \in \{0, 1, \ldots, r\}$, $s \notin J$ such that $\lambda_s \neq x\cdot\lambda$. It is well known (see, e.g., [Di]), that there is an inclusion $M(x\cdot\lambda) \hookrightarrow M(\lambda_s)$. The projectivity of P yields a nontrivial morphism $\phi_s : P \to N_s$, hence an endomorphism of P. By construction $\operatorname{im}\mathbb{V}\phi_s \not\subseteq \mathbb{V}\ker\chi_\lambda P$ and $\phi_s \notin \mathbb{C}\operatorname{id}$, because $\lambda_s \neq x\cdot\lambda$. All these morphisms $\mathbb{V}\phi_s$ have the property that $\mathbb{V}\phi_s(e_1) \notin \mathbb{V}\ker\chi_\lambda P$, where $e_1 \in \mathbb{V}P$ has minimal degree $-m = -l(x)$.

We choose $\psi \notin \mathbb{C}\operatorname{id}$ homogeneous of minimal degree, such that $\psi(e_1) \notin \mathbb{V}\ker\chi_\lambda P$. Let $\tilde{e_2} := \psi(e_1)$ be of degree $-n$.

We claim, that there is a $d \in C$ such that $\psi + d\cdot\operatorname{id}$ does not commute with a certain map f.

The definition of d. Let $h \in \mathbb{V}P$ be of maximal possible degree $m = l(x)$. The vector space of polynomials of degree m is generated by the m-th-power of degree 1 polynomials in $S(\mathfrak{h})$ (see [Hu]). Hence there is an element $e \in \mathfrak{h}$ such that $e^m e_1 = h$. Therefore there are homogeneous elements d, $a \in C$ of degree $m-n$ and $m+n$, respectively, such that $a(de_1 + \psi(e_1)) = h$. If there is a $c \in C$ such that $c\psi(e_1) = h$, we can choose $a = c$ and $d = 0$. Otherwise, take $a = e^{\frac{m+n}{2}}$ and $d = e^{\frac{m-n}{2}}$.

The definition of f. Let $e_2 := de_1 + \psi(e_1)$ and complete e_1, e_2 to a minimal system of generators $e_1, e_2, \cdots, e_r$ of $\mathbb{V}P$. Define $f \in \mathrm{End}_C(\mathbb{V}P)$ by

$$\begin{array}{rcl} f : e_1 & \mapsto & ae_1 \\ e_i & \mapsto & 0, \text{ if } i \in \{2, \cdots, r\} \end{array}$$

For checking that the map is well defined assume $ce_1 = 0$ for $c \in C$. Then $f(ce_1) = cae_1 = ace_1 = 0$. Let $c \in C$ be homogeneous and let $ce_1 = \sum_{i=2}^r \alpha_i e_i$, where $\alpha_i \in C$. By definition of the e_i's it follows $\deg(c) > m-n$, hence $\deg(f(ce_1)) = \deg(cae_1) > (m-n) + (m+n) - m = m$. So $(f(ce_1)) = 0$, since the highest degree of $\mathbb{V}P$ is m.

The endomorphisms $g = d \cdot \mathrm{id} + \psi$ and f do not commute. By definition we have

$$f \circ (d \cdot \mathrm{id} + \psi)(e_1) = 0.$$

On the other hand,

$$(d \cdot \mathrm{id} + \psi) \circ f(e_1) = h.$$

Therefore we have proved that, if $(P : M(\lambda)) > 1$, then there are $f, g \in \mathrm{End}_{\mathfrak{g}}(P)$, such that $f \circ g \neq g \circ f$. The theorem follows. $\square$

Remark 7.5.
a) It is not very difficult to find a basis of $\mathbb{V}P(x \cdot \lambda)$: Let $P(x \cdot \lambda) = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = \{0\}$ be a Verma flag, such that $M_i/M_{i+1} \cong M(\lambda_i)$ for $1 \leq i \leq r$. Let $0 \neq f_{\lambda_i} \in \mathrm{Hom}_{\mathfrak{g}}(P(w_o \cdot \lambda), M(\lambda_i))$. (This map is unique up to a scalar.) The canonical map from M_i to $M(\lambda_i)$ has a lift $h_i \in \mathbb{V}P$ via f_{λ_i}. Then $\{h_i\}_{1 \leq i \leq r}$ is a basis of $\mathbb{V}P(x \cdot \lambda)$.

   To see this, assume $\sum_{i=1}^r c_i h_i = 0$. Let $x \in P(w_o \cdot \lambda)$ be such that $h_1(x) \in M_1 \setminus M_2$. Evaluating the sum at the point x yields $c_1 = 0$, as the images of h_i for $1 < i \leq r$ are contained in M_2. Inductively $c_i = 0$ holds for all i. Comparing the dimensions we are done.

b) A basis of the socle of $P(x \cdot \lambda)$ is given by

$$s_i : P(w_o \cdot \lambda) \twoheadrightarrow L(w_o \cdot \lambda) \hookrightarrow M(\lambda) \overset{\mathrm{incl}_i}{\hookrightarrow} \bigoplus_{[P(x\cdot\lambda):M(\lambda)]} M(\lambda) \hookrightarrow P(x \cdot \lambda),$$

   where incl_i denotes the inclusion in the i-th direct summand (see Theorem 8.1 below).

c) If all multiplicities are at most one, the module $\mathbb{V}P(x \cdot \lambda)$ is generated (as a C-module) by h_1.

The injectivity of Soergel's structure theorem (although it is not proven in this way) relies on the fact that the socle of a projective module in $\mathcal{O}$ is a direct sum of simples corresponding to the longest element in the Weyl group. On the other hand, we can determine these socles explicitly using the structure theorem.

8. The socle of a projective object

Theorem 8.1. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Denote by w_o the longest element of the Weyl group. Then the socle of a projective module in $\mathcal{O}_\lambda$ is given by the formula*

$$\operatorname{soc} P(x \cdot \lambda) = \bigoplus_{i=1}^{(P(x\cdot\lambda):M(\lambda))} L(w_o \cdot \lambda).$$

Proof. The socle contains only simples corresponding to the longest element of the Weyl group. Otherwise, we would get

$$0 \neq \dim \operatorname{Hom}_{\mathfrak{g}}(L(y\cdot\lambda), P(x\cdot\lambda)) \cong \operatorname{Hom}_C(\mathbb{V}L(y\cdot\lambda), \mathbb{V}P(x\cdot\lambda)) \cong \operatorname{Hom}_C(0, \mathbb{V}P(x\cdot\lambda)),$$

because $\mathbb{V}$ annihilates all simples without maximal Gelfand-Kirillov dimension. Another way to see this is that because $P(x\cdot\lambda)$ has a Verma flag, $X_{-\alpha} \in \mathfrak{g}_\alpha$ acts freely on $P(x \cdot \lambda)$ for all simple roots α. So it is the same for the socle and therefore the socle has to be a direct sum of simple Verma modules.

Let $L := L(w_o \cdot \lambda) \subseteq \operatorname{soc} P(x \cdot \lambda)$ be the simple Verma module in $\mathcal{O}_\lambda$. Consider the exact sequence $L \hookrightarrow M(\lambda) \twoheadrightarrow M(\lambda)/L$. This yields an exact sequence

$$\operatorname{Hom}_{\mathfrak{g}}(M(\lambda)/L, P(x \cdot \lambda)) \hookrightarrow \operatorname{Hom}_{\mathfrak{g}}(M(\lambda), P(x \cdot \lambda)) \xrightarrow{f} \operatorname{Hom}_{\mathfrak{g}}(L, P(x \cdot \lambda)).$$

The module $P(x \cdot \lambda)$ is projective and $(M(\lambda) : L) = 1$, so the remark after Theorem 4.2 gives

$$0 = \operatorname{Hom}_C(\mathbb{V}(M(\lambda)/L), \mathbb{V}P(x \cdot \lambda)) = \operatorname{Hom}_{\mathfrak{g}}(M(\lambda)/L, P(x \cdot \lambda)).$$

Hence f is an injection. From the equalities (which use the remark after Theorem 4.2)

$$\begin{aligned} \dim \operatorname{Hom}_{\mathfrak{g}}(M(\lambda), P(x \cdot \lambda)) &= \dim \operatorname{Hom}_C(\mathbb{C}, \mathbb{V}P(x \cdot \lambda)) \\ &= \dim \operatorname{Hom}_C(\mathbb{V}L, \mathbb{V}P(x \cdot \lambda)) \\ &= \dim \operatorname{Hom}_{\mathfrak{g}}(L, P(x \cdot \lambda)) \end{aligned}$$

it follows that $\operatorname{Hom}_{\mathfrak{g}}(L, P(x \cdot \lambda)) = \operatorname{Hom}_{\mathfrak{g}}(M(\lambda), P(x \cdot \lambda))$. Therefore the theorem is proved. □

9. Generalization to Harish-Chandra bimodules

Recall that for a $\mathcal{U}(\mathfrak{g})$-bimodule M, the adjoint action of $\mathfrak{g}$ is defined by $x.m = xm - mx$ for any $x \in \mathfrak{g}$ and $m \in M$. We consider the category $\mathcal{H}$ of *Harish-Chandra bimodules*, a full subcategory of the category of all $\mathcal{U}(\mathfrak{g})$-bimodules. By definition, the objects are all bimodules of finite length which are locally finite with respect to the adjoint action of $\mathfrak{g}$. For results and description of this category see, e.g., [BG], [Ja2], [So2].

To generalize our previous theorem to Harish-Chandra bimodules we have to describe the Harish-Chandra bimodules which are projective in the full subcategory consisting of modules with certain generalized central character from the right. Let I be an ideal of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ with finite $\mathcal{Z}$-codimension, i.e., $\dim_{\mathbb{C}}(\mathcal{Z}/\mathcal{Z} \cap I)$ is finite. Denote by $\mathcal{H}^I$ the full subcategory of $\mathcal{H}$ consisting of all objects which are annihilated by I from the right-hand side. The projectives in $\mathcal{H}^I$ are described by the following:

Lemma 9.1. *The projective objects in $\mathcal{H}^I$ are the direct summands of modules of the form $E \otimes \mathcal{U}/I$, where E is a finite dimensional $\mathfrak{g}$-module with trivial right $\mathfrak{g}$-action; in particular, $\mathcal{H}^I$ has enough projectives.*

Proof. This is mutatis mutandis [Ja2, Satz 6.14 c)]; for details see [St]. □

For dominant weights λ and μ we denote by ${}^{\infty}_{\mu}\mathcal{H}^{n}_{\lambda}$ and ${}^{\infty}_{\mu}\mathcal{H}^{\infty}_{\lambda}$ respectively, the full subcategories of $\mathcal{H}$ given by the following set of objects $\{M \in \mathcal{H} \mid \ker\chi_\mu^m M = 0 = M\ker\chi_\lambda^n, \quad m \gg 0\}$ and $\{M \in \mathcal{H} \mid \ker\chi_\mu^m M = 0 = M\ker\chi_\lambda^m, \quad m \gg 0\}$. For M, $N \in \mathcal{O}$ the space $\mathrm{Hom}_{\mathbb{C}}(M,N)$ has a natural $\mathcal{U}$-bimodule structure given by $ufv(m) = u(f(vm))$ for u, $v \in \mathcal{U}$, $m \in M$ and $f \in \mathrm{Hom}_{\mathbb{C}}(M,N)$. We denote by $\mathcal{L}(M,N)$ the largest locally finite submodule for the adjoint action. This is an object in $\mathcal{H}$ ([Di, Proposition 1.7.9]). We get as a generalization of Theorem 8.1:

Theorem 9.2. *Let λ, μ be integral dominant weights. Let $X \in {}^{\infty}_{\mu}\mathcal{H}^{n}_{\lambda}$ be projective for some fixed integer n. Then, the socle of X is a direct sum of modules of the form $\mathcal{L}(M(\lambda), L(w_o \cdot \mu))$ (i.e., copies of the simple object with maximal Gelfand-Kirillov dimension).*

Proof. Note, that any simple object in ${}^{\infty}_{\mu}\mathcal{H}^{n}_{\lambda}$ is of the form $\mathcal{L}(M(\lambda), L(w \cdot \mu))$ for some $w \in W$. This object has maximal Gelfand-Kirillov dimension, if and only if $L(w \cdot \mu)$ has as well. The latter is exactly the case if $w \cdot \mu = w_o \cdot \mu$ (see [Ja2, 10.12, 8.15 and 9.1]).

Let $\mathfrak{m} = \ker\chi_\lambda$ and consider the filtration of $\mathcal{Z}$-modules

$$\mathcal{Z}/\mathfrak{m}^n \supset \mathfrak{m}/\mathfrak{m}^n \supset \cdots \supset \mathfrak{m}^{n-1}/\mathfrak{m}^n \supset \{0\}$$

with semisimple subquotients. The universal enveloping algebra is a free $\mathcal{Z}$-module, even a free left $\mathcal{Z} \otimes \mathcal{U}(\mathfrak{n}_-)$-module (see [MS, Lemma 5.7] or [Ko, Theorem 0.13]). Applying the (exact) functor $\mathcal{U}\otimes_{\mathcal{Z}}\bullet$ to the filtration above gives rise to a filtration of $\mathcal{U}\otimes_{\mathcal{Z}}\mathcal{Z}/\mathfrak{m}^n = \mathcal{U}/\mathcal{U}(\ker\chi_\lambda)^n$ starting with $M := \mathcal{U}\otimes_{\mathcal{Z}}\mathfrak{m}^{n-1}/\mathfrak{m}^n \cong \bigoplus \mathcal{U}/\mathcal{U}(\ker\chi_\lambda)$, where the direct sum has $\dim_{\mathbb{C}}(\mathfrak{m}^{n-1}/\mathfrak{m}^n)$ many summands. Moreover, by construction, this submodule contains all elements annihilated by $\mathfrak{m} = (\ker\chi_\lambda)$. In particular, it contains the socle of $\mathcal{U}/\mathcal{U}(\ker\chi_\lambda)^n$. Obviously, $M \in {}^{\infty}_{\lambda}\mathcal{H}^{1}_{\lambda}$. This category is equivalent to a certain subcategory of $\mathcal{O}$ (via the functor $\mathcal{T}_{M(\lambda)}$ in the notation of [Ja2]) such that M corresponds to a direct sum of Verma modules $M(\lambda)$. Hence, the socle of M, and therefore also of $\mathcal{U}\otimes_{\mathcal{Z}}\mathcal{Z}/\mathfrak{m}^n$, consists only of simple modules with maximal Gelfand-Kirillov dimension. Since this property is still valid after tensoring with some finite dimensional $\mathfrak{g}$-module E (see [Ja2, 8.13]) and taking direct summands, the statement of the theorem follows by the previous description of the projective objects (Lemma 9.1). □

The previous result implies the faithfulness on projectives of the "generalized" functor $\mathbb{V}$ for Harish-Chandra modules defined in [So2].

For λ, $\mu \in \mathfrak{h}^*$ dominant and integral, let ${}_\lambda L_\mu = \mathcal{L}(M(\lambda), L(w_o \cdot \mu))$ be the unique simple module with maximal Gelfand-Kirillov dimension in its block. Let $P^n = {}_\lambda P^n_\mu$ be its projective cover in ${}^{\infty}_{\mu}\mathcal{H}^{n}_{\lambda}$. We choose compatible surjections $\mathrm{p}_{\mathrm{m,n}} : P^m \twoheadrightarrow P^n$ for $m \geq n$ which define a projective system. The generalized functor $\mathbb{V} = \mathbb{V}_{\lambda,\mu}$: ${}^{\infty}_{\mu}\mathcal{H}^{\infty}_{\lambda} \longrightarrow \mathbb{C}-\mathrm{mof}$ is then defined as $\mathbb{V}X := \varinjlim \mathrm{Hom}_{\mathcal{H}}(P^n, X)$. The functor is exact ([La, S.171]) and annihilates, by definition, all simple objects except ${}_\lambda L_\mu$ having maximal Gelfand-Kirillov dimension in its block, moreover $\dim_{\mathbb{C}} \mathbb{V}({}_\lambda L_\mu) = 1$. Via the action of the center $\mathcal{Z}$ on X, we can consider $\mathbb{V}_{\lambda,\mu}$ as a functor into the category of finitely generated $\mathcal{Z}\otimes\mathcal{Z}$-modules. The last theorem now implies the faithfulness of $\mathbb{V}$ on projective modules in ${}_{\mu}^{\infty}\mathcal{H}_{\lambda}^{n}$:

Corollary 9.3. *Let λ, μ be integral dominant weights. Let X_1, $X_2 \in {}_{\mu}^{\infty}\mathcal{H}_{\lambda}^{n}$ be projective for some fixed integer n. Then $\mathbb{V} = \mathbb{V}_{\lambda,\mu}$ induces an inclusion*

$$\mathrm{Hom}_{\mathcal{H}}(X_1, X_2) \hookrightarrow \mathrm{Hom}_{\mathcal{Z}\otimes\mathcal{Z}}(\mathbb{V}X_1, \mathbb{V}X_2).$$

Proof. The socle of any projective object contains only simple composition factors which are not annihilated by $\mathbb{V}$ and $\operatorname{im}\mathbb{V}f \cong \mathbb{V}\operatorname{im} f$ for any $f \in \mathrm{Hom}_{\mathcal{H}}(X_1, X_2)$. □

It is not completely clear whether the functor is in fact fully faithful in general (except for regular λ (see [So2] and [St]), or in the case $\lambda = \mu = -\rho$ with direct calculations).

10. Generalization to the parabolic situation

Let $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Let $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ denote the full subcategory of $\mathcal{O}_{\lambda}$ whose objects are locally $\mathfrak{p}$-finite. (In the previous sections we used the notation $\mathcal{O}^{\mathfrak{p}}$ instead of $\mathcal{O}_0^{\mathfrak{p}}$!) Let $Q_{\lambda}^{\mathfrak{p}}$ be the direct sum of all indecomposable projective-injective modules in an integral block $\mathcal{O}_{\lambda}^{\mathfrak{p}}$. We can consider the following generalization of Soergel's functor:

$$\begin{array}{rccc} \mathbb{V}_{\lambda}^{\mathfrak{p}}: & \mathcal{O}_{\lambda}^{\mathfrak{p}} & \longrightarrow & \text{mof-}\mathrm{End}_{\mathfrak{g}}(Q_{\lambda}^{\mathfrak{p}}) \\ & M & \longmapsto & \mathrm{Hom}_{\mathfrak{g}}(Q_{\lambda}^{\mathfrak{p}}, M) \end{array}$$

It has the following properties similar to the original functor:

Theorem 10.1. *Let $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ be a parabolic subalgebra. The following holds for any (dominant) integral weight λ:*

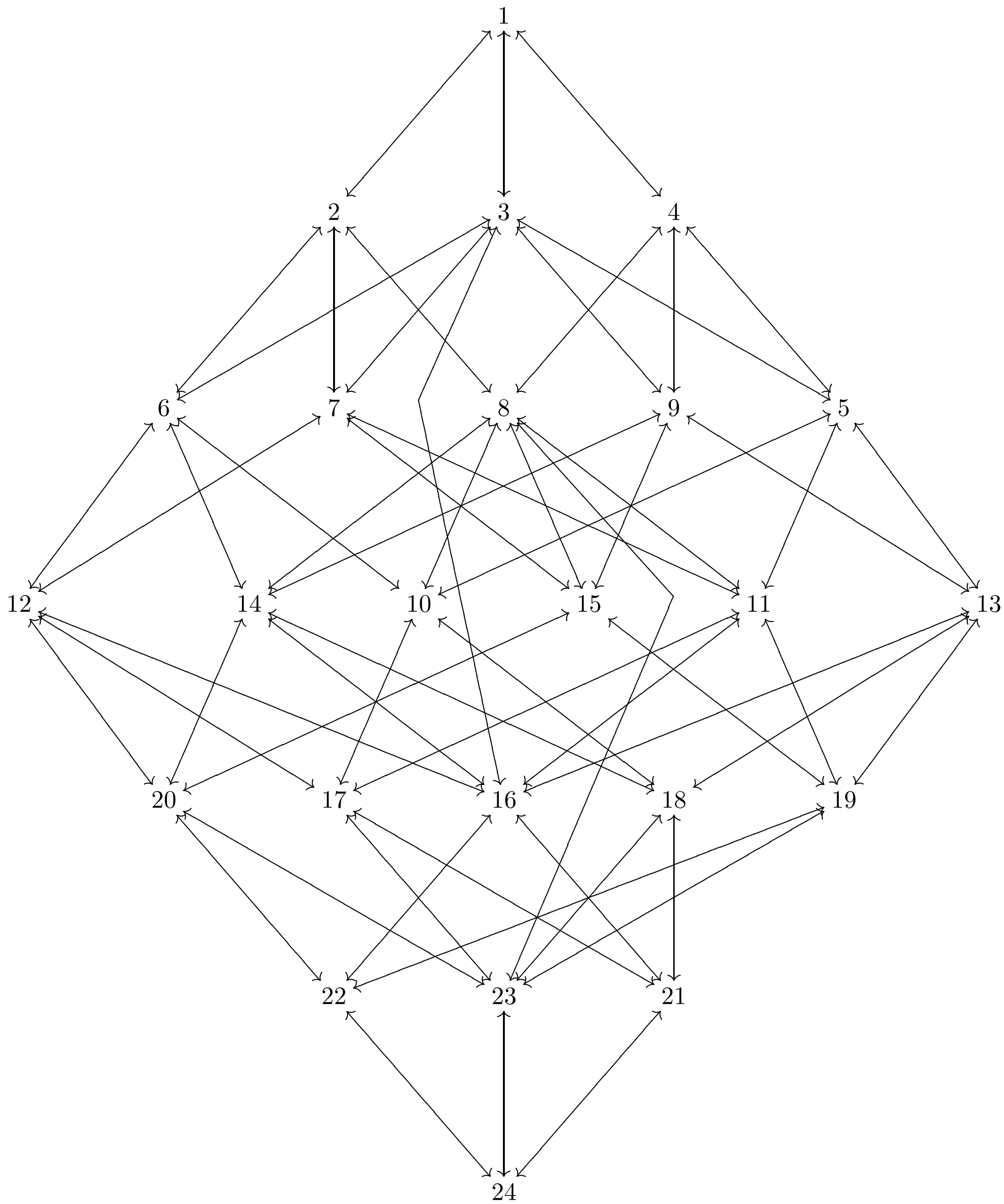
1. *The functor $\mathbb{V} = \mathbb{V}_{\lambda}^{\mathfrak{p}}$ is exact. For any simple object $L \in \mathcal{O}_{\lambda}^{\mathfrak{p}}$ we have $\mathbb{V}(L) = 0$ except if L is of maximal Gelfand-Kirillov dimension. In that case* $\dim_{\mathbb{C}} \mathbb{V}(L) = 1$.
2. *The functor is fully faithful on projective objects, i.e., for any P_1, $P_2 \in \mathcal{O}_{\lambda}^{\mathfrak{p}}$ projective, it induces an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(P_1, P_2) \cong \mathrm{Hom}_{\mathrm{End}_{\mathfrak{g}}(Q_{\mathfrak{p}})}(\mathbb{V}P_1, \mathbb{V}P_2).$$

Proof. The functor is exact by definition and annihilates exactly the simple modules which do not occur in the head of $Q_{\lambda}^{\mathfrak{p}}$. On the other hand, the simple modules occuring in the head of $Q_{\lambda}^{\mathfrak{p}}$ are exactly the ones with maximal Gelfand-Kirillov dimension ([Ir2, 4.3 and Addendum]. The first part of the theorem follows. Let P be a minimal projective generator of $\mathcal{O}_{\lambda}^{\mathfrak{p}}$ and let A be its endomorphism ring with subring $B = \mathrm{End}_{\mathfrak{g}}(Q_{\lambda}^{\mathfrak{p}})$. For the second statement it is enough to prove that A satisfies the double *centralizer property* (see [KSX]) $A \cong \mathrm{End}_{\mathrm{End}_B}\left(\mathrm{Hom}_{\mathfrak{g}}(P, Q_{\lambda}^{\mathfrak{p}})\right)$. That A satisfies this property follows directly from [T, (7.1)] or [KSX, Theorem 2.10]. (Note that the assumptions, in the formulation of [KSX, Theorem 2.10], are satisfied: Since all projective objects in $\mathcal{O}^{\mathfrak{p}}$ have a Verma flag, Irving's result [Ir2, 4.3] implies that $\mathrm{Hom}_{\mathfrak{g}}(P, Q_{\lambda}^{\mathfrak{p}})$ is a minimal faithful right ideal in A. The assumption on the dominant dimension is obviously true.) □

Appendix: A quiver describing an integral regular block for type A_3

The corresponding simple and projective objects are numbered in the order as listed in (6.1). Finally, we get the following quiver with a lot of arrows and very many relations:



$(1 \to 2 \to 1) = 0$
$(1 \to 3 \to 1) = 0$
$(1 \to 4 \to 1) = 0$
$(21 \to 16 \to 3) = 0$
$(22 \to 16 \to 3) = 0$
$(24 \to 23 \to 8) = 0$
$(2 \to 7 \to 2) = 0$
$(2 \to 8 \to 2) = 0$
$(3 \to 16 \to 21) = 0$
$(3 \to 16 \to 22) = 0$
$(3 \to 16 \to 3) = 0$
$(3 \to 6 \to 3) = 0$
$(3 \to 9 \to 3) = 0$
$(4 \to 5 \to 4) = 0$
$(4 \to 8 \to 4) = 0$
$(5 \to 10 \to 5) = 0$
$(6 \to 14 \to 6) = 0$
$(7 \to 15 \to 7) = 0$
$(8 \to 11 \to 8) = 0$
$(8 \to 23 \to 24) = 0$
$(8 \to 23 \to 8) = 0$
$(9 \to 14 \to 9) = 0$
$(12 \to 16 \to 13) = 0$
$(12 \to 20 \to 12) = 0$
$(13 \to 16 \to 12) = 0$
$(13 \to 18 \to 13) = 0$
$(14 \to 16 \to 14) = 0$
$(17 \to 23 \to 17) = 0$
$(19 \to 23 \to 19) = 0$
$(1 \to 3 \to 6) = -(1 \to 2 \to 6)$
$(1 \to 3 \to 7) = (1 \to 2 \to 7)$
$(1 \to 4 \to 5) = (1 \to 3 \to 5)$
$(1 \to 4 \to 8) = (1 \to 2 \to 8)$
$(1 \to 4 \to 9) = -2(1 \to 3 \to 9)$
$(2 \to 6 \to 2) = (2 \to 1 \to 2)$
$(2 \to 6 \to 3) = -(2 \to 1 \to 3)$
$(2 \to 7 \to 12) = -(2 \to 6 \to 12)$
$(2 \to 7 \to 3) = (2 \to 1 \to 3)$
$(2 \to 8 \to 10) = -(2 \to 6 \to 10)$
$(2 \to 8 \to 11) = 2(2 \to 7 \to 11)$
$(2 \to 8 \to 14) = -2(2 \to 6 \to 14)$
$(2 \to 8 \to 15) = -2(2 \to 7 \to 15)$
$(2 \to 8 \to 4) = (2 \to 1 \to 4)$
$(3 \to 16 \to 11) = 1/2(3 \to 5 \to 11)$
$(3 \to 16 \to 12) = 2(3 \to 6 \to 12)$
$(3 \to 16 \to 13) = 2(3 \to 5 \to 13)$
$(3 \to 16 \to 14) = (3 \to 6 \to 14)$
$(3 \to 5 \to 3) = 2(3 \to 1 \to 3)$
$(3 \to 5 \to 4) = (3 \to 1 \to 4)$
$(3 \to 6 \to 10) = (3 \to 5 \to 10)$
$(3 \to 6 \to 2) = (3 \to 1 \to 2)$
$(3 \to 7 \to 11) = 1/2(3 \to 5 \to 11)$
$(3 \to 7 \to 12) = (3 \to 6 \to 12)$
$(3 \to 7 \to 2) = (3 \to 1 \to 2)$
$(3 \to 7 \to 3) = -(3 \to 1 \to 3)$
$(3 \to 9 \to 13) = (3 \to 5 \to 13)$
$(3 \to 9 \to 14) = 2(3 \to 6 \to 14)$
$(3 \to 9 \to 15) = -2(3 \to 7 \to 15)$
$(3 \to 9 \to 4) = (3 \to 1 \to 4)$
$(4 \to 5 \to 3) = (4 \to 1 \to 3)$
$(4 \to 8 \to 10) = (4 \to 5 \to 10)$
$(4 \to 8 \to 11) = (4 \to 5 \to 11)$
$(4 \to 8 \to 2) = (4 \to 1 \to 2)$
$(4 \to 9 \to 13) = -2(4 \to 5 \to 13)$
$(4 \to 9 \to 14) = -2(4 \to 8 \to 14)$
$(4 \to 9 \to 15) = -2(4 \to 8 \to 15)$
$(4 \to 9 \to 3) = -2(4 \to 1 \to 3)$
$(4 \to 9 \to 4) = -(4 \to 1 \to 4)$
$(5 \to 10 \to 6) = (5 \to 3 \to 6)$
$(5 \to 10 \to 8) = (5 \to 4 \to 8)$
$(5 \to 11 \to 17) = -4(5 \to 10 \to 17)$
$(5 \to 11 \to 5) = 4(5 \to 4 \to 5)$
$(5 \to 11 \to 7) = 2(5 \to 3 \to 7)$
$(5 \to 11 \to 8) = -4(5 \to 4 \to 8)$
$(5 \to 13 \to 18) = 2(5 \to 10 \to 18)$
$(5 \to 13 \to 19) = -1/2(5 \to 11 \to 19)$
$(5 \to 13 \to 5) = 2(5 \to 4 \to 5)$
$(5 \to 4 \to 1) = (5 \to 3 \to 1)$
$(6 \to 10 \to 5) = (6 \to 3 \to 5)$
$(6 \to 10 \to 8) = (6 \to 2 \to 8)$
$(6 \to 12 \to 17) = (6 \to 10 \to 17)$
$(6 \to 12 \to 6) = (6 \to 3 \to 6)$
$(6 \to 14 \to 18) = -(6 \to 10 \to 18)$
$(6 \to 14 \to 20) = 2(6 \to 12 \to 20)$
$(6 \to 14 \to 8) = -2(6 \to 2 \to 8)$
$(6 \to 14 \to 9) = -2(6 \to 3 \to 9)$
$(6 \to 3 \to 1) = (6 \to 2 \to 1)$
$(7 \to 11 \to 5) = -2(7 \to 3 \to 5)$
$(7 \to 11 \to 7) = 2(7 \to 2 \to 7)$
$(7 \to 11 \to 8) = -2(7 \to 2 \to 8)$
$(7 \to 12 \to 17) = -1/2(7 \to 11 \to 17)$
$(7 \to 12 \to 7) = (7 \to 2 \to 7)$
$(7 \to 15 \to 19) = -(7 \to 11 \to 19)$
$(7 \to 15 \to 20) = -2(7 \to 12 \to 20)$
$(7 \to 15 \to 8) = -2(7 \to 2 \to 8)$
$(7 \to 15 \to 9) = (7 \to 3 \to 9)$
$(7 \to 3 \to 1) = (7 \to 2 \to 1)$
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$(8 \to 10 \to 6) = -(8 \to 2 \to 6)$
$(8 \to 10 \to 8) = -(8 \to 4 \to 8)$
$(8 \to 11 \to 17) = -4(8 \to 10 \to 17)$
$(8 \to 11 \to 5) = -4(8 \to 4 \to 5)$
$(8 \to 11 \to 7) = 2(8 \to 2 \to 7)$
$(8 \to 14 \to 16) = (8 \to 11 \to 16)$
$(8 \to 14 \to 18) = -2(8 \to 10 \to 18)$
$(8 \to 14 \to 6) = -2(8 \to 2 \to 6)$
$(8 \to 14 \to 9) = 2(8 \to 4 \to 9)$
$(8 \to 15 \to 19) = (8 \to 11 \to 19)$
$(8 \to 15 \to 20) = (8 \to 14 \to 20)$
$(8 \to 15 \to 7) = -2(8 \to 2 \to 7)$
$(8 \to 15 \to 8) = 2(8 \to 2 \to 8)$
$(8 \to 15 \to 9) = (8 \to 4 \to 9)$
$(8 \to 23 \to 17) = 2(8 \to 10 \to 17)$
$(8 \to 23 \to 18) = -(8 \to 10 \to 18)$
$(8 \to 23 \to 19) = -1/2(8 \to 11 \to 19)$
$(8 \to 23 \to 20) = -(8 \to 14 \to 20)$
$(8 \to 4 \to 1) = (8 \to 2 \to 1)$
$(9 \to 13 \to 9) = -2(9 \to 3 \to 9)$
$(9 \to 14 \to 18) = -(9 \to 13 \to 18)$
$(9 \to 14 \to 6) = 2(9 \to 3 \to 6)$
$(9 \to 14 \to 8) = -4(9 \to 4 \to 8)$
$(9 \to 15 \to 19) = -2(9 \to 13 \to 19)$
$(9 \to 15 \to 20) = (9 \to 14 \to 20)$
$(9 \to 15 \to 7) = -2(9 \to 3 \to 7)$
$(9 \to 15 \to 8) = 4(9 \to 4 \to 8)$
$(9 \to 4 \to 1) = (9 \to 3 \to 1)$
$(10 \to 17 \to 10) = (10 \to 5 \to 10)$
$(10 \to 17 \to 12) = (10 \to 6 \to 12)$
$(10 \to 18 \to 10) = -2(10 \to 8 \to 10)$
$(10 \to 18 \to 13) = -2(10 \to 5 \to 13)$
$(10 \to 18 \to 21) = -2(10 \to 17 \to 21)$
$(10 \to 6 \to 3) = (10 \to 5 \to 3)$
$(10 \to 8 \to 2) = (10 \to 6 \to 2)$
$(10 \to 8 \to 4) = (10 \to 5 \to 4)$
$(11 \to 16 \to 12) = (11 \to 7 \to 12)$
$(11 \to 16 \to 13) = (11 \to 5 \to 13)$
$(11 \to 16 \to 14) = -1/4(11 \to 8 \to 14)$
$(11 \to 16 \to 3) = -1/2(11 \to 5 \to 3)$
$(11 \to 17 \to 11) = (11 \to 8 \to 11)$
$(11 \to 17 \to 12) = -2(11 \to 7 \to 12)$
$(11 \to 17 \to 21) = (11 \to 16 \to 21)$
$(11 \to 19 \to 11) = 2(11 \to 8 \to 11)$
$(11 \to 19 \to 13) = -2(11 \to 5 \to 13)$
$(11 \to 19 \to 22) = -4(11 \to 16 \to 22)$
$(11 \to 7 \to 3) = -1/2(11 \to 5 \to 3)$
$(11 \to 8 \to 2) = -2(11 \to 7 \to 2)$
$(11 \to 8 \to 4) = (11 \to 5 \to 4)$
$(12 \to 16 \to 11) = (12 \to 7 \to 11)$
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$(12 \to 16 \to 14) = (12 \to 6 \to 14)$
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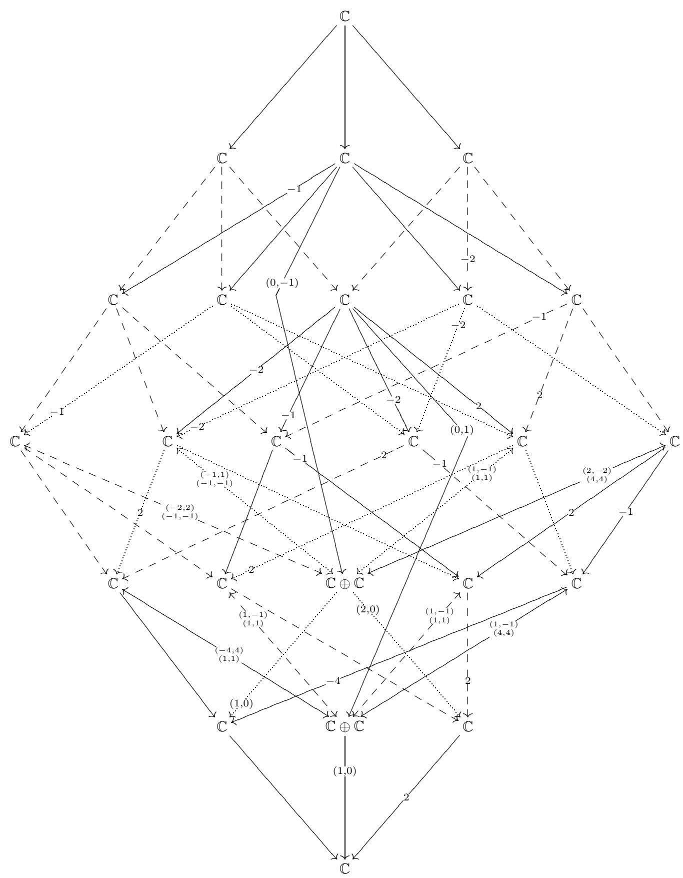
$(12 \to 17 \to 21) = -1/2(12 \to 16 \to 21)$
$(12 \to 20 \to 14) = (12 \to 6 \to 14)$
$(12 \to 20 \to 15) = (12 \to 7 \to 15)$
$(12 \to 20 \to 22) = -(12 \to 16 \to 22)$
$(12 \to 20 \to 23) = (12 \to 17 \to 23)$
$(12 \to 7 \to 2) = (12 \to 6 \to 2)$
$(12 \to 7 \to 3) = (12 \to 6 \to 3)$
$(13 \to 16 \to 11) = (13 \to 5 \to 11)$
$(13 \to 16 \to 13) = 2(13 \to 9 \to 13)$
$(13 \to 16 \to 14) = 1/2(13 \to 9 \to 14)$
$(13 \to 16 \to 3) = -2(13 \to 5 \to 3)$
$(13 \to 18 \to 10) = -2(13 \to 5 \to 10)$

$(13 \to 18 \to 14) = 1/2(13 \to 9 \to 14)$
$(13 \to 18 \to 21) = 1/2(13 \to 16 \to 21)$
$(13 \to 19 \to 11) = 2(13 \to 5 \to 11)$
$(13 \to 19 \to 13) = 2(13 \to 5 \to 13)$
$(13 \to 19 \to 15) = -(13 \to 9 \to 15)$
$(13 \to 19 \to 22) = (13 \to 16 \to 22)$
$(13 \to 19 \to 23) = -2(13 \to 18 \to 23)$
$(13 \to 9 \to 3) = 2(13 \to 5 \to 3)$
$(13 \to 9 \to 4) = 2(13 \to 5 \to 4)$
$(14 \to 16 \to 11) = -1/4(14 \to 8 \to 11)$
$(14 \to 16 \to 12) = (14 \to 6 \to 12)$
$(14 \to 16 \to 13) = (14 \to 9 \to 13)$
$(14 \to 16 \to 3) = (14 \to 6 \to 3)$
$(14 \to 18 \to 13) = -(14 \to 9 \to 13)$
$(14 \to 18 \to 21) = -(14 \to 16 \to 21)$
$(14 \to 20 \to 12) = 2(14 \to 6 \to 12)$
$(14 \to 20 \to 22) = -2(14 \to 16 \to 22)$
$(14 \to 8 \to 2) = -2(14 \to 6 \to 2)$
$(14 \to 9 \to 3) = -2(14 \to 6 \to 3)$
$(14 \to 9 \to 4) = (14 \to 8 \to 4)$
$(15 \to 19 \to 13) = -4(15 \to 9 \to 13)$
$(15 \to 19 \to 15) = -4(15 \to 7 \to 15)$
$(15 \to 20 \to 12) = -2(15 \to 7 \to 12)$
$(15 \to 20 \to 15) = -1/2(15 \to 8 \to 15)$
$(15 \to 20 \to 22) = 1/2(15 \to 19 \to 22)$
$(15 \to 8 \to 2) = -2(15 \to 7 \to 2)$
$(15 \to 9 \to 3) = (15 \to 7 \to 3)$
$(15 \to 9 \to 4) = -1/2(15 \to 8 \to 4)$
$(16 \to 14 \to 8) = (16 \to 11 \to 8)$
$(16 \to 22 \to 24) = 1/4(16 \to 21 \to 24)$
$(17 \to 11 \to 5) = -4(17 \to 10 \to 5)$
$(17 \to 11 \to 8) = -4(17 \to 10 \to 8)$
$(17 \to 12 \to 6) = (17 \to 10 \to 6)$
$(17 \to 12 \to 7) = 1/2(17 \to 11 \to 7)$
$(17 \to 21 \to 18) = -2(17 \to 10 \to 18)$
$(17 \to 23 \to 18) = 2(17 \to 10 \to 18)$
$(17 \to 23 \to 19) = (17 \to 11 \to 19)$
$(17 \to 23 \to 20) = 4(17 \to 12 \to 20)$
$(17 \to 23 \to 24) = -3/10(17 \to 21 \to 24)$
$(17 \to 23 \to 8) = -2(17 \to 10 \to 8)$
$(18 \to 13 \to 5) = 2(18 \to 10 \to 5)$
$(18 \to 14 \to 6) = -(18 \to 10 \to 6)$
$(18 \to 14 \to 8) = 2(18 \to 10 \to 8)$
$(18 \to 14 \to 9) = 1/2(18 \to 13 \to 9)$
$(18 \to 21 \to 17) = -2(18 \to 10 \to 17)$
$(18 \to 21 \to 18) = -(18 \to 13 \to 18)$
$(18 \to 23 \to 17) = 2(18 \to 10 \to 17)$
$(18 \to 23 \to 19) = -(18 \to 13 \to 19)$
$(18 \to 23 \to 20) = 2(18 \to 14 \to 20)$
$(18 \to 23 \to 24) = -3/20(18 \to 21 \to 24)$
$(18 \to 23 \to 8) = (18 \to 10 \to 8)$
$(19 \to 13 \to 5) = 1/2(19 \to 11 \to 5)$
$(19 \to 15 \to 7) = (19 \to 11 \to 7)$
$(19 \to 15 \to 8) = (19 \to 11 \to 8)$
$(19 \to 15 \to 9) = -1/2(19 \to 13 \to 9)$
$(19 \to 22 \to 20) = 2(19 \to 15 \to 20)$
$(19 \to 23 \to 17) = -(19 \to 11 \to 17)$
$(19 \to 23 \to 18) = -(19 \to 13 \to 18)$
$(19 \to 23 \to 20) = 2(19 \to 15 \to 20)$
$(19 \to 23 \to 24) = 3/5(19 \to 22 \to 24)$
$(19 \to 23 \to 8) = -1/2(19 \to 11 \to 8)$
$(20 \to 14 \to 6) = (20 \to 12 \to 6)$
$(20 \to 15 \to 7) = (20 \to 12 \to 7)$
$(20 \to 15 \to 8) = (20 \to 14 \to 8)$
$(20 \to 15 \to 9) = -1/2(20 \to 14 \to 9)$
$(20 \to 22 \to 19) = 2(20 \to 15 \to 19)$
$(20 \to 22 \to 20) = -4(20 \to 12 \to 20)$
$(20 \to 23 \to 17) = -2(20 \to 12 \to 17)$
$(20 \to 23 \to 18) = -2(20 \to 14 \to 18)$
$(20 \to 23 \to 19) = -2(20 \to 15 \to 19)$
$(20 \to 23 \to 24) = -3/5(20 \to 22 \to 24)$
$(20 \to 23 \to 8) = (20 \to 14 \to 8)$
$(21 \to 17 \to 11) = -(21 \to 16 \to 11)$
$(21 \to 17 \to 12) = -1/2(21 \to 16 \to 12)$
$(21 \to 18 \to 10) = -2(21 \to 17 \to 10)$
$(21 \to 18 \to 13) = -1/2(21 \to 16 \to 13)$
$(21 \to 18 \to 14) = -(21 \to 16 \to 14)$
$(21 \to 24 \to 21) = -40(21 \to 18 \to 21)$
$(21 \to 24 \to 22) = -10(21 \to 16 \to 22)$
$(22 \to 19 \to 11) = -4(22 \to 16 \to 11)$
$(22 \to 19 \to 13) = -(22 \to 16 \to 13)$
$(22 \to 20 \to 12) = -2(22 \to 16 \to 12)$
$(22 \to 20 \to 14) = -2(22 \to 16 \to 14)$
$(22 \to 20 \to 15) = 1/2(22 \to 19 \to 15)$
$(22 \to 24 \to 21) = -10(22 \to 16 \to 21)$
$(22 \to 24 \to 22) = 5/2(22 \to 20 \to 22)$
$(23 \to 19 \to 13) = -2(23 \to 18 \to 13)$
$(23 \to 20 \to 12) = -2(23 \to 17 \to 12)$
$(24 \to 22 \to 16) = 1/4(24 \to 21 \to 16)$
$(24 \to 23 \to 17) = -1/2(24 \to 21 \to 17)$
$(24 \to 23 \to 18) = -1/4(24 \to 21 \to 18)$
$(24 \to 23 \to 19) = (24 \to 22 \to 19)$
$(24 \to 23 \to 20) = (24 \to 22 \to 20)$
$(5 \to 13 \to 16) = -2(5 \to 3 \to 16) + 2(5 \to 11 \to 16)$
$(5 \to 13 \to 9) = -2(5 \to 3 \to 9) - 2(5 \to 4 \to 9)$
$(6 \to 10 \to 6) = -2(6 \to 2 \to 6) + 2(6 \to 3 \to 6)$
$(6 \to 12 \to 7) = (6 \to 2 \to 7) + (6 \to 3 \to 7)$
$(6 \to 14 \to 16) = -(6 \to 3 \to 16) + (6 \to 12 \to 16)$
$(7 \to 12 \to 16) = (7 \to 3 \to 16) + (7 \to 11 \to 16)$
$(7 \to 12 \to 6) = -(7 \to 2 \to 6) + (7 \to 3 \to 6)$
$(8 \to 14 \to 8) = -2(8 \to 2 \to 8) + 4(8 \to 4 \to 8)$
$(9 \to 13 \to 5) = -(9 \to 3 \to 5) + 2(9 \to 4 \to 5)$
$(9 \to 14 \to 16) = (9 \to 3 \to 16) + 1/2(9 \to 13 \to 16)$
$(9 \to 15 \to 9) = -(9 \to 3 \to 9) - (9 \to 4 \to 9)$
$(10 \to 17 \to 11) = (10 \to 5 \to 11) + (10 \to 8 \to 11)$
$(10 \to 18 \to 14) = -(10 \to 6 \to 14) + (10 \to 8 \to 14)$
$(10 \to 18 \to 23) = -2(10 \to 8 \to 23) - (10 \to 17 \to 23)$
$(11 \to 16 \to 11) = 1/4(11 \to 5 \to 11) + 1/2(11 \to 7 \to 11)$
$(11 \to 17 \to 10) = (11 \to 5 \to 10) + (11 \to 8 \to 10)$
$(11 \to 19 \to 15) = 4(11 \to 7 \to 15) - 2(11 \to 8 \to 15)$
$(11 \to 19 \to 23) = 4(11 \to 8 \to 23) + 2(11 \to 17 \to 23)$
$(14 \to 18 \to 10) = -(14 \to 6 \to 10) - (14 \to 8 \to 10)$
$(14 \to 18 \to 14) = (14 \to 6 \to 14) - 1/2(14 \to 9 \to 14)$
$(14 \to 20 \to 14) = (14 \to 6 \to 14) - 1/2(14 \to 9 \to 14)$
$(14 \to 20 \to 15) = 1/2(14 \to 8 \to 15) + 1/2(14 \to 9 \to 15)$
$(14 \to 20 \to 23) = 2(14 \to 8 \to 23) + 2(14 \to 18 \to 23)$
$(15 \to 19 \to 11) = -4(15 \to 7 \to 11) - 2(15 \to 8 \to 11)$
$(15 \to 20 \to 14) = 1/2(15 \to 8 \to 14) - (15 \to 9 \to 14)$
$(15 \to 20 \to 23) = -2(15 \to 8 \to 23) - 1/2(15 \to 19 \to 23)$
$(16 \to 12 \to 7) = (16 \to 3 \to 7) + (16 \to 11 \to 7)$
$(16 \to 13 \to 5) = 2(16 \to 3 \to 5) + 2(16 \to 11 \to 5)$
$(16 \to 14 \to 6) = -(16 \to 3 \to 6) + (16 \to 12 \to 6)$
$(16 \to 14 \to 9) = -(16 \to 3 \to 9) + 1/4(16 \to 13 \to 9)$
$(16 \to 21 \to 16) = 4(16 \to 12 \to 16) - 8(16 \to 14 \to 16)$
$(16 \to 21 \to 17) = -2(16 \to 11 \to 17) - 2(16 \to 12 \to 17)$
$(16 \to 21 \to 18) = -(16 \to 13 \to 18) - 4(16 \to 14 \to 18)$
$(16 \to 22 \to 16) = -1/2(16 \to 13 \to 16) + 4(16 \to 14 \to 16)$
$(16 \to 22 \to 19) = 2(16 \to 11 \to 19) + (16 \to 13 \to 19)$
$(16 \to 22 \to 20) = 4(16 \to 12 \to 20) - 4(16 \to 14 \to 20)$
$(17 \to 21 \to 16) = 2(17 \to 11 \to 16) - 2(17 \to 12 \to 16)$
$(17 \to 21 \to 17) = 2(17 \to 10 \to 17) + 1/2(17 \to 11 \to 17)$
$(18 \to 21 \to 16) = (18 \to 13 \to 16) - 4(18 \to 14 \to 16)$
$(18 \to 23 \to 18) = -(18 \to 10 \to 18) + 1/2(18 \to 13 \to 18)$
$(19 \to 22 \to 16) = 2(19 \to 11 \to 16) - (19 \to 13 \to 16)$
$(19 \to 22 \to 19) = (19 \to 11 \to 19) + (19 \to 15 \to 19)$
$(20 \to 22 \to 16) = 2(20 \to 12 \to 16) - 4(20 \to 14 \to 16)$
$(20 \to 23 \to 20) = 4(20 \to 12 \to 20) - 4(20 \to 15 \to 20)$
$(21 \to 24 \to 23) = 20(21 \to 17 \to 23) - 20(21 \to 18 \to 23)$
$(22 \to 24 \to 23) = -5/2(22 \to 19 \to 23) - 5/2(22 \to 20 \to 23)$
$(23 \to 18 \to 10) = 2(23 \to 8 \to 10) - (23 \to 17 \to 10)$
$(23 \to 19 \to 11) = 4(23 \to 8 \to 11) - 2(23 \to 17 \to 11)$
$(23 \to 20 \to 14) = -2(23 \to 8 \to 14) - 2(23 \to 18 \to 14)$
$(23 \to 20 \to 15) = 2(23 \to 8 \to 15) + 1/2(23 \to 19 \to 15)$
$(23 \to 24 \to 21) = 12(23 \to 17 \to 21) - 12(23 \to 18 \to 21)$
$(23 \to 24 \to 22) = -3/2(23 \to 19 \to 22) + 3/2(23 \to 20 \to 22)$
$(23 \to 24 \to 23) = 3(23 \to 17 \to 23) - 3/2(23 \to 19 \to 23)$

The dominant Verma module as a representation of the quiver. *A representation of a quiver* consists of a collection of vector spaces and linear maps arranged in the same shape as the quiver, i.e., with a vector space for each vertex and a linear map for each arrow. If the quiver has relations, than the corresponding linear maps should also fulfill these relations.

Let A be a basic algebra. Then there is an equivalence between the category of A-modules with finite length and the representations of the quiver corresponding to A (see [ARS, Theorem 1.5]). Consider the case where $A = \mathrm{End}_{\mathfrak{g}}(P)$ and recall the



equivalence of categories (2.1) in section 2.1. We can describe the objects in $\mathcal{O}$ by representations of certain quivers. The picture above shows a representation of the quiver corresponding to the dominant Verma module. We omit the arrows belonging to the zero map. If not otherwise stated, the map is the identity. Otherwise, we write down the map as a matrix or (in case of one-dimensional vector spaces) just as a number. If there is a double arrow signified by two maps, the upper one corresponds to the arrow pointing up.

The picture can for example give an explicit description of submodules which are not sums of Verma submodules (see the remark in [Ja1, page 4]).

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