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ELLIPTIC CENTRAL CHARACTERS AND BLOCKS OF FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

PAVEL I. ETINGOF AND ADRIANO A. MOURA

For Igor Frenkel, on the occasion of his 50th birthday

Abstract. The category of finite dimensional (type 1) representations of a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is not semisimple. However, as any abelian category with finite-length objects, it admits a unique decomposition in a direct sum of indecomposable subcategories (blocks). We define the elliptic central character of a finite dimensional (type 1) representation of $U_q(\widehat{\mathfrak{g}})$ and show that the block decomposition of this category is parametrized by these elliptic central characters.

Introduction

In this paper we describe the block decomposition of the category of finite dimensional representations of a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, where $|q| < 1$. Namely, we find that the blocks are parametrized by elliptic central characters, which are certain elliptic functions attached to irreducible representations.

The plan of the paper is as follows.

In Section 1, we recall the basics about blocks in abelian categories.

In Section 2, we recall the definition of $U_q(\widehat{\mathfrak{g}})$ and the basic facts about its finite dimensional representations.

In Section 3, we define the elliptic central character of finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$. Namely, we show that if X, Y are such representations and X is irreducible, then the operator $R^{2,1}_{Y,X}(z^{-1})R_{X,Y}(z) : X \otimes Y \to X \otimes Y$ (where R is the R-matrix) is of the form $1 \otimes \xi_X(z)|_Y$, where $\xi_X(z)$ is an endomorphism of the identity functor with coefficients in the field of elliptic functions. If Y is also irreducible, then $\xi_X(z)|_Y$ is a scalar. The collection of these scalars for all irreducible X is called the elliptic central character of Y. (Thus, the elliptic central character plays the role of non-existent nontrivial central elements of $U_q(\widehat{\mathfrak{g}})$.) Our main result is

Theorem 1. *Blocks in the category of finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$ consist of representations whose simple constituents have a given elliptic central character.*

The rest of the paper is devoted to the proof of Theorem 1. This proof is based on a result of Chari and Kashiwara on the cyclicity of tensor products of fundamental

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representations, and a lengthy case-by-case analysis. It would be interesting to obtain a uniform proof.

In Section 4, we prove Theorem 1 for type A.

In Section 5, we recall the Drinfeld realization of $U_q(\widehat{\mathfrak{g}})$, and explain how to compute elliptic central characters of fundamental representations.

In Section 6, we prove Theorem 1 for types B-G. This is much more technically challenging than type A (for types E and F we relied on a computer to perform some calculations). The proof for type E_8 is discussed in Appendix A.

In Appendix B, we list the formulas used in Section 6 to compute the elliptic central characters; some of them were obtained using a computer.

We note that although elliptic central characters were used in this paper for a particular purpose (to classify blocks of finite dimensional representations), they may be used in other problems about $U_q(\widehat{\mathfrak{g}})$. For example, as is shown below, they can sometimes be used to decide when an irreducible finite dimensional representation occurs in the tensor product of two others. Therefore, we feel that elliptic central characters are worthy of further study. In particular, it would be interesting to study their connections with other objects in representation theory of $U_q(\widehat{\mathfrak{g}})$, such as minimal affinizations and q-characters.

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1. Block decomposition of an abelian category

Let us recall the basics about blocks in abelian categories. This material is standard, and we give it for the reader's convenience.

Let $\mathcal{C}$ be an abelian category, in which every object has finite length. In this case, it is well known that any object is uniquely representable as a direct sum of indecomposable objects.

Definition 1.1. Two indecomposable objects X_1, X_2 of $\mathcal{C}$ are *linked* if there is no splitting of $\mathcal{C}$ in a direct sum of two abelian categories, $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, such that $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$.

It is easy to see that linking is an equivalence relation.

Proposition 1.1. *The category $\mathcal{C}$ admits a unique decomposition into a direct sum of indecomposable abelain categories: $\mathcal{C} = \bigoplus_{\alpha \in I} \mathcal{C}_\alpha$.*

Proof. Let I be the set of equivalence classes of linked indecomposable objects, and for $\alpha \in I$ let $\mathcal{C}_\alpha$ be the subcategory of $\mathcal{C}$, consisting of direct sums of objects from α. By the uniqueness of the decomposition into indecomposables, we have $\mathcal{C} = \bigoplus_{\alpha \in I} \mathcal{C}_\alpha$. Furthermore, the categories $\mathcal{C}_\alpha$ are indecomposable. Indeed, if $\mathcal{C}_\alpha = \mathcal{C}_\alpha^1 \oplus \mathcal{C}_\alpha^2$ is a nontrivial decomposition, then any indecomposables $X_1 \in \mathcal{C}_\alpha^1$, $X_2 \in \mathcal{C}_\alpha^2$ are not linked, a contradiction.

The uniqueness of the decomposition is obvious. □

Definition 1.2. The subcategories $\mathcal{C}_\alpha$ are called the *blocks* of $\mathcal{C}$, and the decomposition of Proposition 1.1 is called the *block decomposition* of $\mathcal{C}$.

Recall that for any $X \in \mathcal{C}$, one can uniquely specify the simple objects (with multiplicities) which occur as constituents in X (the Jordan-Holder Theorem).

The following trivial lemma will be used below.

Lemma 1.2.

(a) *Two simple objects are linked if they occur as constituents of the same indecomposable object.*

(b) *Two indecomposable objects are linked if they have some linked simple constituents.*

Proof. Clear. □

2. Quantum affine algebras and the category $\mathcal{C}$

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. The simple roots will be denoted $\alpha_1, \dots, \alpha_n$, the fundamental weights $\omega_1, \dots, \omega_n$ and the invariant bilinear form $\langle , \rangle$ is normalized so that $\langle \theta, \theta \rangle = 2$ for the maximal root $\theta = \sum \theta_i \alpha_i$. The Cartan matrix is $C = (c_{ij})$, $i, j \in I = \{1, \dots, n\}$. Set $r^\vee$ to be the maximal number of edges connecting two vertices in the Dynkin diagram of $\mathfrak{g}$ and define a renormalized form

$$(\cdot, \cdot) = r^\vee \langle \cdot, \cdot \rangle$$

Then let

$$d_i = \frac{(\alpha_i, \alpha_i)}{2} \tag{2.1}$$

and set $D = \operatorname{diag}(d_i)$, so that $B = DC$ is symmetric.

We consider $\widehat{\mathfrak{g}}$, the loop (or affine) algebra associated to $\mathfrak{g}$ (we do not consider central extension). The matrices B, C, D can be extended to their affine counterparts $\hat{B}, \hat{C}, \hat{D}$, of size $n+1$. Their entries will be denoted by b_{ij}, c_{ij}, d_i as before, but with i, j running from 0 to n. For $q \in \mathbb{C}^*$ (not a root of 1), the Hopf algebra $U_q(\widehat{\mathfrak{g}})$ is the algebra generated by $k_i^{\pm 1}, x_i^{\pm}, i = 0, \dots, n$, with relations

$$\begin{gathered} k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i, \qquad k_0 \prod_{i=1}^n k_i^{\theta_i} = 1, \\ k_i x_j^{\pm} k_i^{-1} = q_i^{\pm c_{ij}} x_j^{\pm}, \\ [x_i^+, x_j^+] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{m=0}^{1-c_{ij}} \begin{bmatrix} 1 - c_{ij} \\ m \end{bmatrix}_{q_i} (x_i^{\pm})^m \, x_j^{\pm} \, (x_i^{\pm})^{1-c_{ij}-m} = 0, \quad i \neq j. \end{gathered} \tag{2.2}$$

Here $q_i = q^{d_i}$ and $\begin{bmatrix} r \\ m \end{bmatrix}_q = \frac{[r]_q!}{[m]_q![r-m]_q!}$ where $[m]_q = \frac{q^m - q^{-m}}{q-q^{-1}}$ and $[m]_q! = [m]_q[m-1]_q \dots [1]_q$. The coalgebra structure and the antipode are given by

$$\begin{gathered}\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \qquad \Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+, \\ \Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-, \\ \varepsilon(k_i) = 1, \qquad \varepsilon(x_i^{\pm}) = 0, \\ S(k_i) = k_i^{-1}, \qquad S(x_i^+) = -x_i^+ k_i^{-1}, \qquad S(x_i^-) = -k_i x_i^-. \end{gathered} \tag{2.3}$$

$U_q(\widehat{\mathfrak{g}})$ has (in an appropriate sense [12]) a universal R-matrix that we denote by $\mathcal{R}$.

Given a representation Y of $U_q(\widehat{\mathfrak{g}})$ and $\lambda \in \mathfrak{h}^*$, we set

$$Y[\lambda] = \{v \in Y; k_i v = q^{(\lambda, \alpha_i)} v\}.$$

We will consider the category $\mathcal{C}$ of finite dimensional (type **1**) representations of $U_q(\widehat{\mathfrak{g}})$. It consists of finite dimensional representations with weight decomposition

$$Y = \bigoplus_{\lambda} Y[\lambda]$$

for λ in the weight lattice of $\mathfrak{g}$.

Let $\mathbf{Gr}(\mathcal{C})$ be the Grothendiek ring of $\mathcal{C}$. We will need the following theorem [16].

Theorem 2.1. $\mathbf{Gr}(\mathcal{C})$ *is a commutative ring.*

3. The elliptic central character

Recall that if A is a finite dimensional algebra, then blocks in the category A-mod are parametrized by characters of the center, $\chi : Z(A) \to \mathbb{C}$.

Of course, the algebra $U_q(\widehat{\mathfrak{g}})$ is not finite dimensional, and has a trivial center. Nevertheless, we will define a nontrivial analog of the notion of a central character (the *elliptic central character*) for $U_q(\widehat{\mathfrak{g}})$, which will allow us to compute the block decomposition of the category of its finite dimensional representations.

For $z \in \mathbb{C}^*$, let D_z be the automorphism of $U_q(\widehat{\mathfrak{g}})$ given by $D_z(x_0^+) = z x_0^+$, $D_z(x_0^-) = z^{-1} x_0^-$ and the identity on the other generators. Given $X \in \mathcal{C}$ we can consider the family of shifted representations $X(z)$ obtained from X by composing with D_z.

Let $X, Y \in \mathcal{C}$ and $\mathcal{R}|_{X(z)\otimes Y}$ be denoted by $R_{X,Y}(z)$. $R_{X,Y}(z)$ is a meromorphic function of z regular at 0 [21, 13]. Define

$$\eta_{X,Y}(z) = R^{21}_{Y,X}(z^{-1}) R_{X,Y}(z) \in \mathrm{End}_{U_q(\widehat{\mathfrak{g}})}(X(z) \otimes Y) \tag{3.1}$$

where $R^{21}_{Y,X}(z^{-1}) = (P\mathcal{R}P)|_{Y(z^{-1})\otimes X}$ and P is the flip map. Then $\eta_{X,Y}$ is an elliptic function of z with period $q^{2r^\vee h^\vee}$ [21], i.e., a meromorphic function on the elliptic curve

$$E = \frac{\mathbb{C}^*}{q^{2r^\vee h^\vee \mathbb{Z}}}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

Let Id_E denote the identity functor of the category $\mathcal{C}_E := \mathcal{C} \otimes_{\mathbb{C}} \mathbb{C}(E)$, where $\mathbb{C}(E)$ is the field of meromorphic functions on E.

Proposition 3.1. *If X is irreducible, there exists an element $\xi_X \in \mathrm{End}_{U_q(\widehat{\mathfrak{g}})}(\mathrm{Id}_E)$, such that $\eta_{X,Y}(z) = 1 \otimes \xi_X(z)|_Y$ for almost all $z \in \mathbb{C}^*$.*

Proposition 3.1 follows from the following lemma.

Lemma 3.2. *Let $Y \in \mathcal{C}$. Given a simple object $X \in \mathcal{C}$, the map $\xi \mapsto 1 \otimes \xi$, defines an isomorphism* $\mathrm{End}_{U_q(\widehat{\mathfrak{g}})}(Y) \cong \mathrm{End}_{U_q(\widehat{\mathfrak{g}})}(X(z) \otimes Y)$ *for almost all $z \in \mathbb{C}^*$.*

Proof. We have

$$\begin{aligned}\mathrm{End}_{U_q(\widehat{\mathfrak{g}})}(X(z) \otimes Y) &\cong \mathrm{Hom}_{U_q(\widehat{\mathfrak{g}})}(Y, {}^*X(z) \otimes X(z) \otimes Y)\\ &\cong \mathrm{Hom}_{U_q(\widehat{\mathfrak{g}})}(Y \otimes {}^*Y, ({}^*X \otimes X)(z)).\end{aligned}$$

Let $Z_1, \dots, Z_n$ be the nontrivial constituents of a composition series of ${}^*X \otimes X$. Then, for almost all z, none of the $Z_i(z)$ occurs as a constituent in $Y \otimes {}^*Y$ (as $Z_i(z)$ are pairwise non-isomorphic for fixed i and $z \in \mathbb{C}^*$) and, consequently, the image of any morphism $f : Y \otimes {}^*Y \to ({}^*X \otimes X)(z)$ has only trivial constituents.

It is easy to show that $\mathrm{Ext}^1_{U_q(\widehat{\mathfrak{g}})}(\mathbb{C}, \mathbb{C}) = 0$. Thus, the image of f is trivial, i.e., either zero or 1-dimensional (since X is simple). The lemma is proved. □

Corollary 3.3. *For almost all $z, u, w \in \mathbb{C}^*$ we have:*

(a) *If Y is irreducible, then $\xi_X(z)|_Y$ is a scalar operator, and $\xi_X(z)|_Y = \xi_Y(z^{-1})|_X$.*
(b) $\xi_X(z)|_{Y_1 \otimes Y_2} = \xi_X(z)|_{Y_1} \otimes \xi_X(z)|_{Y_2}$. *In particular, if Y_i are irreducible, and Y is a subquotient in $Y_1 \otimes Y_2$, then $\xi_X(z)|_Y = \xi_X(z)|_{Y_1}\xi_X(z)|_{Y_2} \in \mathbb{C}$. Similarly, if an irreducible X is a subquotient of the tensor product of two irreducibles X_1, X_2, then $\xi_X = \xi_{X_1}\xi_{X_2} = \xi_{X_2}\xi_{X_1}$.*
(c) $\xi_X(z)|_{Y^*} = ((\xi_X(z)|_Y)^{-1})^*$.
(d) $\xi_X(z)|_{Y(u)} = \xi_X(\frac{z}{u})|_Y$ *and* $\xi_{X(w)}(z) = \xi_X(zw)$.

Proof. The first statement immediately follows from Lemma 3.2. The second follows from the fusion laws for the R-matrix. In particular, $\xi_X(z)|_Y \otimes \xi_X(z)|_{Y^*} = \xi_X(z)|_{Y \otimes Y^*} = 1$. The last equality follows since the trivial representation occurs in $Y \otimes Y^*$. This proves the third statement. The fourth statement is a consequence of the fact that the same is true for $R_{X,Y}(z)$. □

Definition 3.1. Let $\mathcal{I} \subset \mathrm{Ob}(\mathcal{C})$ be the set of isomorphism classes of simple objects. An **elliptic central character** in $\mathcal{C}$ is a map $\chi : \mathcal{I} \to \mathbb{C}(E)$ such that the following holds for almost all $z, w \in \mathbb{C}^*$:

(a) $\chi_{X(w)}(z) = \chi_X(zw)$.
(b) If X is a subquotient of $X_1 \otimes X_2$, for $X, X_i \in \mathcal{I}$, then $\chi_X = \chi_{X_1}\chi_{X_2}$.
(c) $\chi_{\mathbb{C}} = 1$.

Here χ_X is the evaluation of χ at the simple object X.

Given an elliptic central character χ, set

$$\mathcal{C}_\chi = \{Y \in \mathcal{C}; \xi_X(z)|_Z = \chi_X(z) \text{ for all } Z \in \mathcal{I}, Z \text{ is a constituent of } Y\}$$

(it is possible that $\mathcal{C}_\chi = 0$). If $Y \in \mathcal{C}_\chi$, we will say that the elliptic central character of Y is χ.

It immediately follows that $\mathcal{C}$ is the direct sum of all $\mathcal{C}_\chi$.

We are ready to state our main result.

Theorem 3.4. *The categories $\mathcal{C}_\chi$ are indecomposable. In other words, they are the blocks of $\mathcal{C}$.*

The set of χ for which $\mathcal{C}_\chi$ is nonzero can be explicitly described. This description will be clear from the proof. It will also be clear from the proof that if elliptic central characters of two representations coincide up to scaling, then they coincide; so an elliptic central character of a representation is completely determined by its divisor of zeros and poles on the elliptic curve E.

4. The $\mathfrak{sl}_{n+1}$ case

To prove Theorem 3.4 for all $\mathfrak{g}$ we will need to use another realization of $U_q(\widehat{\mathfrak{g}})$ using Drinfeld "loop-like" generators. However, due to the existence of Jimbo's algebra homomorphism $U_q(\widehat{\mathfrak{g}}) \to U_q(\mathfrak{g})$ when $\mathfrak{g}$ is $\mathfrak{sl}_{n+1}$ [19, 12], we can already proceed with the proof in this case.

We have $r^\vee = 1$, $h^\vee = n+1$ (but we will keep the notation $h^\vee$ to maintain similarities with types B-G). Let $V = \mathbb{C}^{n+1}$ be the vector representation of $U_q(\widehat{\mathfrak{sl}}_{n+1})$. The following proposition is well known (see [5, 8] for example).

Proposition 4.1. *Any irreducible object of $\mathcal{C}$ is a subquotient of a tensor product of the form $V(z_1) \otimes \cdots \otimes V(z_m)$.*

Corollary 4.2. *Any elliptic central character χ is determined by χ_V.*

The following is a consequence of the results in [20, 3].

Proposition 4.3. *For a tensor product $V(z_1) \otimes \cdots \otimes V(z_m)$ to be cyclic on the highest weight vector (hence, indecomposable) it suffices that $z_j/z_k \neq q^2$ for $j < k$.*

Proposition 4.4. *Let $z_1, \ldots, z_m$ be a sequence satisfying the condition in Proposition 4.3 and $s \in S_m$. Then there exists a map of $U_q(\widehat{\mathfrak{g}})$-modules $V(z_1) \otimes \cdots \otimes V(z_m) \to V(z_{s(1)}) \otimes \cdots \otimes V(z_{s(m)})$. Furthermore, if $z_{s(1)}, \ldots, z_{s(m)}$ also satisfies the condition, then the corresponding tensor products are isomorphic.*

Proof. It is enough to prove it for transpositions. In this case, the map is given by the action of $P\bar{R}_{V,V}(z_j/z_{j+1})$, where $\bar{R}$ is the normalized R-matrix of Proposition 5.3 below, and P is the flip map. The R-matrix $\bar{R}_{V,V}(z)$ has singularities at $z = q^{\pm 2}$. Therefore, if $z_{s(1)}, \ldots, z_{s(m)}$ satisfies the condition in Proposition 4.3, the transposed factors satisfy $z_j/z_k \neq q^{\pm 2}$, and the map becomes an isomorphism. □

Definition 4.1. We say that a sequence $z_1, \ldots, z_m$ is non-resonant if it satisfies the condition of Proposition 4.3.

Since any sequence can be arranged in a non-resonant order, we shall denote by $Y(z_1, \ldots, z_m)$ any of the corresponding isomorphic indecomposable tensor products obtained from $z_1, \ldots, z_m$.

It follows from Lemma 1.2 and Propositions 4.1, 4.3 and 4.4, that to prove Theorem 3.4 for $U_q(\widehat{\mathfrak{sl}}_{n+1})$, it is enough to show that if $Y(z_1, \ldots, z_m)$ and $Y(w_1, \ldots, w_k)$ have the same elliptic central character, then they are linked.

Lemma 4.5. *The trivial representation is contained in $Y(w, q^2 w, \ldots, q^{2(h^\vee - 1)} w)$, for any $w \in \mathbb{C}^*$.*

Proof. It is contained in $Y(w, q^2 w, \ldots, q^{2(h^\vee - 1)} w)$ as "the top quantum exterior power" of $V(w)$. □

Lemma 4.6. *For any $w \in \mathbb{C}^*$, $Y(z_1, \dots, z_m)$ is linked to $Y(z_1, \dots, z_m, w, q^2 w, \dots, q^{2(h^\vee-1)}w)$. In particular, $Y(z_1, \dots, z_m)$ is linked to $Y(z_1, \dots, z_{j-1}, z_j q^{2h^\vee}, z_{j+1}, \dots, z_m)$.*

Proof. It follows immediately from Lemma 4.5 and Proposition 4.4 since we have a map

$$Y(z_1, \dots, z_m, w, q^2 w, \dots, q^{2(h^\vee-1)}w) \to Y(z_1, \dots, z_m) \otimes Y(w, q^2 w, \dots, q^{2(h^\vee-1)}w).$$

For the second statement, let Y_1, Y_2 be simple constituents in each of the considered tensor products respectively. Then both Y_i are subquotients of $Y(z_1, \dots, z_j, q^2 z_j, \dots, q^{2h^\vee} z_j, z_{j+1}, \dots, z_m)$. □

In light of Lemma 4.6 it remains to show that if $Y(z_1, \dots, z_m)$ and $Y(w_1, \dots, w_k)$ have the same elliptic central character, then the sequence $(z_1, ..., z_m)$ can be obtained from $(w_1, ..., w_k)$ by permutations and by adding and removing sequences $(z, q^2 z, ..., q^{2(h^\vee-1)}z)$ (which includes the transformations $z_j \to z_j q^{2h^\vee}$).

To do this, we will write down the formula for $\xi_V(z)|_V$ and analyze its singularity structure. Drinfeld realization will be used to do this in the other cases, but for $\mathfrak{sl}_{n+1}$ one can compute it in a more "naive" way, as was (essentially) done in [23]. We have

$$\xi_V(z)|_V = q^{\frac{2(h^\vee-1)}{h^\vee}} \prod_{j=0}^{\infty} \varrho(q^{2jh^\vee} z)\varrho(q^{2jh^\vee} z^{-1}) \tag{4.1}$$

where

$$\varrho(z) = \frac{(1-z)(1-zq^{2h^\vee})}{(1-zq^2)(1-zq^{2(h^\vee-1)})}. \tag{4.2}$$

Then, the structure of zeros and poles of $\xi_V(z)|_V$ on E is given by the following pictures

$$\begin{array}{cccccc} \bullet^{1} & \bullet^{q^2} & \bullet^{q^4} & \dots & \bullet^{q^{2(h^\vee-2)}} & \bullet^{q^{2(h^\vee-1)}} \\ 2 & -1 & 0 & \dots & 0 & -1 \end{array} \qquad \text{for } n \geq 2$$

and

$$\begin{array}{cc} \bullet^{1} & \bullet^{q^2} \\ 2 & -2 \end{array} \qquad \text{for } n = 1$$

where positive numbers stand for zeros (of that order) and negative for poles.

The fact that the trivial representation is contained in $Y(w, q^2 w, \dots, q^{2(h^\vee-1)}w)$ is reflected in the relation

$$\prod_{s=0}^{h^\vee-1} \xi_V(zw^{-1}q^{-2s}) = 1. \tag{4.3}$$

To prove our claim, we need to show that any multiplicative relation between $\xi_V(zu)$, $u \in \mathbb{C}^*$ is a combination of relations of the form (4.3). For this, it suffices to show that the functions $\xi_V(z), \xi_V(zq^{-2}),...,\xi_V(zq^{-2(h^\vee-2)})$ are multiplicatively independent (for $\mathfrak{sl}_2$ this is clear).

To do this, we will rephrase the problem in a linear algebra setting. Consider the group $\mathbb{Z}^{h^\vee-1}$. To the function $\xi_V(zq^{-2s})|_V$, $0 \leq s \leq h^\vee - 2$ assign a vector v_s

in $\mathbb{Z}^{h^\vee-1}$ given by

$$\begin{aligned} v_0 &= (2,-1,0,0,\dots,0),\\ v_{h^\vee-2} &= (0,0,\dots,0,-1,2),\\ v_s &= (0,\dots,0,-1,2,-1,0,\dots,0) \qquad \text{for } 0<s<h^\vee-2 \end{aligned}$$

where the 2 is the sth entry, if we label them from 0 to $h^\vee - 2$. The entries of these vectors are the orders of the singularities of $\xi_V(zq^{-2s})$ on the sequence $1, q^2, ..., q^{2(h^\vee-2)}$. Then we are left to show that the vectors v_s are linearly independent. This is equivalent to showing that the matrix T_n, whose rows are the vectors v_s, has a nonvanishing determinant. But one easily sees that $\det T_n = 2\det T_{n-1} - \det T_{n-2}$ and use induction to get $\det T_n = n+1$ (in fact, T_n is the Cartan matrix of type A_n).

This proves Theorem 3.4 for $\mathfrak{sl}_{n+1}$.

5. Drinfeld realization

To compute the elliptic central characters for the other $\mathfrak{g}$ we will need to use a suitable formula for $\mathcal{R}$. This formula is found in [22] and we will use the version in [16] involving the Drinfeld realization [11] of $U_q(\widehat{\mathfrak{g}})$ in terms of "loop-like" generators. In fact, we will use the following (slightly different) realization proposed in [2].

Theorem 5.1. *$U_q(\widehat{\mathfrak{g}})$ is isomorphic to the algebra with generators $k_i^{\pm 1}$, $h_{i,l}$, $x^{\pm}_{i,r}$, where $i \in I$, $l \in \mathbb{Z}\backslash\{0\}$ and $r \in \mathbb{Z}$ with defining relations*

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i k_j = k_j k_i, \qquad k_i h_{j,r} = h_{j,r} k_i,$$

$$k_i x_{j,r} \pm k_i^{-1} = q_i^{\pm b_{ij}} x^{\pm}_{j,r}, \qquad [h_{i,r}, x^{\pm}_{j,s}] = \pm\frac{1}{r}[rc_{ij}]_{q^i}\, x_{j,r+s},$$

$$x^{\pm}_{i,r+1} x^{\pm}_{j,s} - q_i^{\pm c_{ij}} x^{\pm}_{j,s} x^{\pm}_{i,r+1} = q_i^{\pm c_{ij}} x^{\pm}_{i,r} x^{\pm}_{j,s+1} - x^{\pm}_{j,s+1} x_{i,r}, \tag{5.1}$$

$$[x^+_{i,r}, x^-_{j,s}] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\left(\phi^+_{i,r+s} - \phi^-_{i,r+s}\right)$$

$$\sum_{s\in S_m}\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x^{\pm}_{i,r_{s(1)}} \dots x^{\pm}_{i,r_{s(k)}} x^{\pm}_{j,s}\, x^{\pm}_{i,r_{s(k+1)}} \dots x^{\pm}_{i,r_{s(m)}} = 0 \quad \text{if } i \neq j$$

where $r_1, \dots, r_m \in \mathbb{Z}$, $m = 1 - c_{ij}$, S_m is the symmetric group on m symbols and $\phi^{\pm}_{i,r}$ are given by the identity of the power series

$$\sum_{r=0}^{\infty} \phi^{\pm}_{i,r} u^{\pm r} = k_i^{\pm 1} \exp\Big(\pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{\pm s}\Big). \tag{5.2}$$

Remark. In fact this version follows the notation in [16]. But observe that the comultiplication in [16] differs from ours. They are connected by the automorphism of $U_q(\widehat{\mathfrak{g}})$ sending k_i to k_i^{-1}, q to q^{-1} and keeping fixed the other generators. This will give rise to some differences in the signs. We use the original definitions of [10, 18] which also coincide with [6, 7] and [23, 13] (these last two are the reasons for our choice). But we remark that the definition in [16] has been used also in [3, 14] and many other most recent works.

Using the "loop-like" generators, to each fundamental weight ω_i, one may associate a family of (shifted) *fundamental representations* $V_i(z)$. In general, the fundamental representations are not irreducible as $U_q(\mathfrak{g})$-modules, but we still have the following theorem [6, 7].

Theorem 5.2. *Any irreducible representation of $U_q(\widehat{\mathfrak{g}})$ is isomorphic to a subquotient of a tensor product of shifted fundamental representations.*

Remark. Using the Drinfeld realization, Chari and Pressley (see, e.g., [7]) defined the concept of an affinization of V_λ for any finite dimensional $U_q(\mathfrak{g})$-module V_λ with highest weight λ. By definition, an irreducible affinization of V_λ is a $U_q(\widehat{\mathfrak{g}})$-representation V isomorphic to one of the form $V_\lambda \oplus \bigoplus_{\mu<\lambda} V_\mu$ as a representation of $U_q(\mathfrak{g})$. We refer to [7] for more details. We also mention one fact that will be used later. Namely, if $\lambda = \sum \lambda_i\omega_i$, then all irreducible affinizations of V_λ are obtained as subquotients of tensor products of the form $\bigotimes_i^n (\bigotimes_{j=1}^{\lambda_i} V_i(z_{ij}))$.

The following proposition [17, 12] is our first tool to compute $\xi_X(z)|_Y$.

Proposition 5.3. *Let X, Y be irreducible representations of $U_q(\widehat{\mathfrak{g}})$. Then*

$$R_{X,Y}(z) = f_{X,Y}(z)\bar{R}_{X,Y}(z) \tag{5.3}$$

where $f_{X,Y}$ is a scalar meromorphic function in $\mathbb{C}$, regular at 0 with $f_{X,Y}(0) \neq 0$, and the matrix elements of $\bar{R}_{X,Y}(z)$ are rational functions of z regular at 0 and such that $\bar{R}_{X,Y}(z)(x_0 \otimes y_0) = x_0 \otimes y_0$. Here x_0, y_0 are the highest weight vectors of X and Y as $U_q(\mathfrak{g})$-modules. If $|q| < 1$, $f_{X,Y}$ can be represented as

$$f_{X,Y}(z) = q^{(\lambda,\mu)} \prod_{j=0}^{\infty} \varrho_{X,Y}(q^{2jr^\vee h^\vee} z)$$

where λ and μ are the highest weights of X and Y and $\varrho_{X,Y}$ is a rational function such that $\varrho_{X,Y}(0) = 1$. Furthermore, $\bar{R}$ is unitary:

$$\bar{R}^{21}_{Y,X}(z^{-1})\bar{R}_{X,Y}(z) = 1. \tag{5.4}$$

Corollary 5.4. *If X, Y are irreducible, $\xi_X(z)|_Y$ is the scalar operator given by $f_{X,Y}(z)f_{X,Y}(z^{-1})$.*

It was proved in [14] that if $\bar{R}_{V_i,V_j}(z)$ is not regular at z_0, then z_0 must belong to the set $\mathcal{P} = \{q^k; 2 \leq k \leq r^\vee h^\vee, k \in \mathbb{Z}\}$ and, if $\bar{R}_{V_i,V_j}(z_0)$ is not invertible, then $z_0 \in \mathcal{P}^{-1} = \{q^{-k}; 2 \leq k \leq r^\vee h^\vee, k \in \mathbb{Z}\}$.

Corollary 5.5. *Let $\varrho_{V_i,V_j}(z)$ be the function of Proposition 5.3 and $\mathcal{P}_{ij}$ the subset of $\mathcal{P}$ where $\varrho_{V_i,V_j}(z)$ has a pole. Then $\bar{R}_{V_i,V_j}(z)$ is not invertible exactly on $\mathcal{P}_{ij}^{-1}$.*

Proof. The function $\varrho_{X,Y}(z)$ is characterized by the following equation [12]:

$$((\bar{R}_{X,Y}(z))^{-1})^{t_1} = \varrho_{X,Y}(z)((\bar{R}_{X,Y}(q^{2r^\vee h^\vee} z))^{t_1})^{-1} \tag{5.5}$$

where $(\sum a_j \otimes b_j)^{t_1} = \sum a_j^* \otimes b_j$. Recall that $V_j(z)^* \cong V_{j^*}(q^{r^\vee h^\vee})$ where V_{j^*} is the fundamental representation of $U_q(\mathfrak{g})$ dual to V_j. Then, if $z_0 \in P_{ij}^{\pm 1}$ and $\bar{R}_{V_i,V_j}(z_0)$ is invertible, both $((\bar{R}_{V_i,V_j}(z))^{-1})^{t_1}$ and $((\bar{R}_{V_i,V_j}(q^{2r^\vee h^\vee} z))^{t_1})^{\pm 1}$ are regular at z_0. Hence, so is $\varrho_{V_i,V_j}(z)$. Conversely, if $\bar{R}_{V_i,V_j}(z_0)$ is not invertible, then $((\bar{R}_{V_i,V_j}(z))^{-1})^{t_1}$ has a pole at z_0, but $((\bar{R}_{V_i,V_j}(q^{2r^\vee h^\vee} z))^{t_1})^{-1}$ is still regular. Thus, $\rho_{X,Y}$ must have a pole at z_0. □

The following corollary will be useful later. Let $\mathcal{S}_{ij} = \mathcal{P}_{ij} \cup \mathcal{P}_{ij}^{-1}$.

Corollary 5.6. *Suppose that $V_r(q^p)$ is a subrepresentation of $V_i \otimes V_j(q^l)$. Then, for any $m = 1, \ldots, n$, we have $\mathcal{S}_{mr} q^p \subset \mathcal{S}_{mi} \cup \mathcal{S}_{mj} q^l$.*

Proof. Given m and $z \in \mathbb{C}$, consider the inclusion

$$V_m(z) \otimes V_r(q^p) \hookrightarrow V_m(z) \otimes V_i \otimes V_j(q^l).$$

By the fusion laws for the universal R-matrix, the singularities of $\bar{R}_{V_m, V_i \otimes V_j(q^l)}(z)$ (poles and points where it is not invertible), must be contained in $\mathcal{S}_{mi} \cup \mathcal{S}_{mj} q^l$. On the other hand, let $\bar{R}^r_{m,ij}(z)$ denote the restriction of $\bar{R}_{V_m, V_i \otimes V_j(q^l)}(z)$ to $V_m(z) \otimes V_r(q^p)$. Then $g(z) \bar{R}^r_{m,ij}(z) = \bar{R}_{V_m, V_r(q^p)}(z)$, for some rational function $g(z)$. Suppose that, at z_0, $\bar{R}_{V_m, V_r(q^p)}(z)$ is not invertible, but $\bar{R}^r_{m,ij}(z_0)$ is defined and invertible. We conclude that $g(z_0) = 0$. But this would imply that $\bar{R}_{V_m, V_r(q^p)}(z_0) = 0$, which is impossible by the normalization of $\bar{R}_{V_m, V_r(q^p)}(z)$. Since is $z \mapsto z^{-1}$ is a bijection $\mathcal{P}_{mr} \to \mathcal{P}_{mr}^{-1}$, we conclude that $\bar{R}^r_{m,ij}(z_0)$ has a singularity whenever $\bar{R}_{V_m, V_r(q^p)}(z)$ has a pole. □

Now we need a tool to calculate $f_{X,Y}$ for fundamental X and Y. It is the formula for $\mathcal{R}$ found in [22]. The method we will describe now was developed for general irreducibles X, Y in [16]. So our computation is a specialization of those in [16]. First we define the matrices $B(q), D(q)$ and $M(q)$ to be $b_{ij}(q) = [b_{ij}]_q$, $d_{ij}(q) = \delta_{ij}[d_i]_q$, for $i, j = 1, \ldots, n$ and $M(q) = D(q)\tilde{B}(q)D(q)$ where $\tilde{B}(q) = B(q)^{-1}$.

Theorem 5.7 ([22])**.** *The universal R-matrix $\mathcal{R}$ of $U_q(\widehat{\mathfrak{g}})$ can be represented in the form*

$$\mathcal{R} = \mathcal{R}^+ \breve{\mathcal{R}} \mathcal{R}^- \mathcal{R}^0 \tag{5.6}$$

where $\mathcal{R}^\pm \in U_q(\widehat{\mathfrak{n}}_\pm) \otimes U_q(\widehat{\mathfrak{n}}_\mp)$, $\mathcal{R}^0(x \otimes y) = q^{(\lambda,\mu)} x \otimes y$ if x, y have weight λ and μ respectively, and the "imaginary" R-matrix $\breve{\mathcal{R}}$ is given by

$$\breve{\mathcal{R}} = \exp\Big((q - q^{-1}) \sum_{k>0,i,j} \frac{k}{[k]_q} \tilde{b}_{ij}(q^k) h_{i,k} \otimes h_{j,-k}\Big). \tag{5.7}$$

Then we can compute $f_{X,Y}$ by calculating the action of $\mathcal{R}$ on the tensor product of the corresponding highest weight vectors. $\mathcal{R}^\pm$ will act as the identity while $\mathcal{R}^0$ will contribute a constant. Hence, the essential information is contained in $\breve{\mathcal{R}}$. Let v_i be the highest weight vector of $V_i(z)$. The action of $h_{j,k}$ on v_i is given by [16],

$$h_{j,k} v_i = \delta_{ij} \frac{(q_j^k - q_j^{-k}) z^k}{(q - q^{-1})k}\, v_j. \tag{5.8}$$

Consequently,

$$\breve{\mathcal{R}}(v_i \otimes v_j) = \exp\Big(\sum_{k>0} m_{ij}(q^k)(q^k - q^{-k}) \frac{z^k}{k}\Big) v_i \otimes v_j = q^{-(\omega_i,\omega_j)} f_{V_i,V_j}(z) v_i \otimes v_j. \tag{5.9}$$

The singularity structure for $f_{V_i,V_j}(z)$ can then be read from the matrix $M(q)$ in the following way. The term $m_{ij}(q^k)(q^k - q^{-k})\frac{z^k}{k}$ will be of the form $\frac{\pi(q^k)}{1-q^{pk}} \frac{z^k}{k}$, where $\pi(q)$ is a Laurent polynomial in q (symmetric under $q \mapsto q^{-1}$) and $p = 2r^\vee h^\vee$. Then, each monomial $\pm q^m$ in $\pi(q)$ will contribute with a factor $\prod_{l\geq 0}(1 - q^m z q^{pl})^{\mp 1}$

to $f_{V_i,V_j}(z)$ (recalling that $\exp(-\sum_k \frac{y^k}{k}) = \exp(\log(1-y)) = 1-y$ and that $\frac{1}{1-x} = \sum_l x^l$).

Remark. The matrices $M(q)$ are listed in Appendix B. For the classical algebras they were listed in [15] (with a few misprints). For types E and F we used the software Mathematica to compute $M(q)$.

6. Proof of Theorem 3.4: The remaining cases

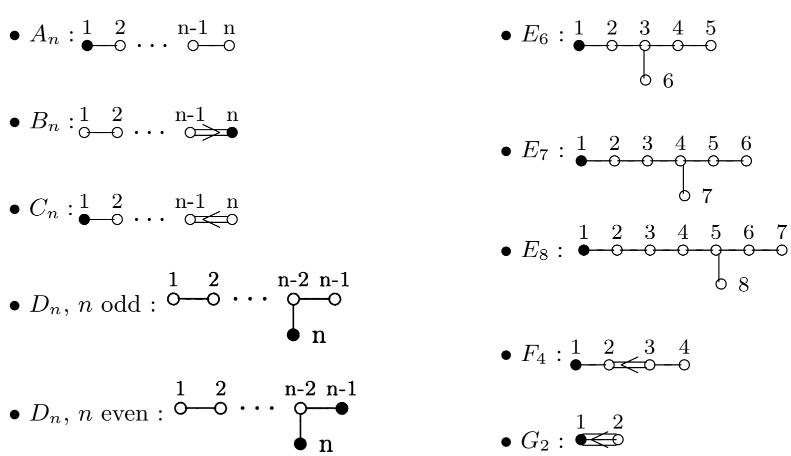
In this section we will use the notation $\xi_{ij}(z) = \xi_{V_i}(z)|_{V_j}$.

We state the following version of Proposition 4.1.

Theorem 6.1. *Every irreducible object of $\mathcal{C}$ is a subquotient of a tensor product of the form $V_{i_1}(z_1)\otimes\cdots\otimes V_{i_m}(z_m)$ where i_j run through the indices of the black nodes of the Dynkin diagram of $\mathfrak{g}$ in Table 1.*

In light of Theorem 5.2, it is enough to prove it for fundamental representations. It follows from [8]. In fact, it is proved only for classical $\mathfrak{g}$ there, but the remaining cases follow from these considering diagram subalgebras, as was pointed out to us by the referee (see [4] for explanations on how to use diagram subalgebras arguments). However, elliptic central characters can be used to prove it directly. For $\mathfrak{e}_6$ the same kind of computations was done in [4]. We will prove it in the remaining cases together with the proof of Theorem 3.4. (We note that as far as we know, in the cases $\mathfrak{f}_4$ and $\mathfrak{g}_2$, these computations were also done by Chari and Pressley in 1991 when preparing the paper [4]; however, they were not written in the paper, since the result was not of interest at that time.)

Table 1.



Corollary 6.2. *If $\mathfrak{g}$ is not of type D_n for n even, then any elliptic central character χ is determined by χ_V, where $V = V_b$ and b is the index of the black node in Table 1. For D_n, when n is even, χ is determined by its value on the two half spin representations V_{n-1} and V_n.*

Combining the results in [20, 3] with Corollary 5.5, we get the following stronger version of Proposition 4.3, which is crucial in our proof of Theorem 3.4.

Theorem 6.3. *For a tensor product of fundamental representations $V_{k_1}(z_1)\otimes\cdots\otimes V_{k_l}(z_l)$ to be cyclic on the highest weight vector (hence indecomposable), it suffices that*

$$\frac{z_r}{z_s}\neq q^{2d_{k_s}+p}\quad for\quad r<s\,,\quad p\geq 0\quad and\quad 2\leq 2k_s+p\leq r^\vee h^\vee.$$

In other words, it suffices that $\bar{R}_{V_{k_r},V_{k_s}}(\frac{z_r}{z_s})$ be regular for $r<s$.

Then we can prove the corresponding version of Proposition 4.4 in a similar way and define $Y(z_1,\dots,z_m)$ analogously to Definition 4.1. For D_n, when n is even, we let the half spin representations be denoted by $V_\pm$ and define $Y_+(z_1,\dots,z_m)$ to be any of the isomorphic indecomposable tensor products obtained from $z_1,\dots,z_m$ using only shifts of V_+. Similarly, we define $Y_-(w_1,\dots,w_l)$ using V_-. Then we can define $Y(z_1,\dots,z_m|w_1,\dots,w_l)$ to be any non-resonant (i.e., satisfying the cyclicity condition on the highest weight vector) reordering of $Y_+(z_1,\dots,z_m)\otimes Y_-(w_1,\dots,w_l)$.

Therefore, once we have computed the singularity structure of ξ_{ij}, the proof goes, case by case, in a similar way as it did for type A_n. Namely, the proof consists of two steps.

Step 1. We prove a version of Lemma 4.6 stating the basic linking relations.

Step 2. To conclude that, if $Y(z_1,\dots,z_m)$ and $Y(w_1,\dots,w_l)$ have the same elliptic central character, then they are linked, we find all multiplicative relations between the considered $\xi_{ij}(zq^{-2s})$ (by computing the kernel of a certain integer matrix over $\mathbb{Z}$) and check that they correspond to the linking relations of the lemma.

6.1. **Type B_n.** For B_n we have $r^\vee=2, h^\vee=2n-1$. The black node corresponds to the spin representation V_n. The singularity arrangement for $\xi_{nn}(z)$ is

$$\begin{array}{ccccccccccc}
\bullet^{1} & \bullet^{q^2} & \bullet^{q^4} & \dots & \bullet^{q^{2h^\vee-2}} & \bullet^{q^{2h^\vee}} & \bullet^{q^{2h^\vee+2}} & \dots & \bullet^{q^{4h^\vee-4}} & \bullet^{q^{4h^\vee-2}} \\
2 & -1 & 1 & \dots & 1 & -2 & 1 & \dots & 1 & -1
\end{array}$$

where the dots mean that the sequence goes on like

$$\begin{array}{cccccc}
\dots & \bullet^{q^{2(k-2)}} & \bullet^{q^{2(k-1)}} & \bullet^{q^{2k}} & \bullet^{q^{2(k+1)}} & \dots \\
\dots & 1 & -1 & 1 & -1 & \dots
\end{array}$$

except at 1 and $q^{2h^\vee}$.

The corresponding version of Lemma 4.6 is

Lemma 6.4. *For any $w\in\mathbb{C}^*$, $Y(z_1,\dots,z_m)$ is linked to $Y(z_1,\dots,z_m,w,q^{r^\vee h^\vee}w)$. In particular, $Y(z_1,\dots,z_m)$ is linked to $Y(z_1,\dots,z_{j-1},z_jq^{2r^\vee h^\vee},z_{j+1},\dots,z_m)$.*

Proof. It is proved analogously to Lemma 4.6 and follows from the fact that $\mathbb{C}$ occurs as a constituent of $Y(w,q^{r^\vee h^\vee}w)=V(w)\otimes V(q^{r^\vee h^\vee}w)$. □

For the last part of the proof (i.e., the proof that there is no relations between $\xi_{nn}(zu)$ other than given by Lemma 6.4), we consider the assignment

$$\xi_{nn}(zq^{-2s})\mapsto v_s:=((-1)^s,(-1)^{s+1},\dots,-1,2,-1,\dots,(-1)^s)\in\mathbb{Z}^{h^\vee}$$

for $s=0,\dots,h^\vee-1$. The entries of these vectors are the orders of the singularities of $\xi_{nn}(zq^{-2s})$ on the sequence $1,q^2,...,q^{2h^\vee-2}$.

Remark. Observe that these vectors contain information only about the "first half" of the singularity structure. Since V_n is "self dual", the "second half" is obtained from the first one by a change of signs.

The corresponding matrix T_n is the $h^\vee \times h^\vee$-matrix of the form

$$T_n = \begin{pmatrix} b & a & -a & \dots & a & -a \\ a & b & a & \ddots & -a & a \\ -a & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a & -a & \ddots & a & b & a \\ -a & a & \dots & -a & a & b \end{pmatrix}$$

where $a = -1$ and $b = 2$. Then $\det T_n = (a+b)^{h^\vee - 1}(b - (h^\vee - 1)a) = h^\vee + 1 = 2n$. Thus, the rows of this matrix are linearly independent, and hence there is no additional relations, as desired.

6.2. **Type C_n.** Here $r^\vee = 2, h^\vee = n+1$ and, for $1 \le i \le j \le n$, the zeros and poles arrangement of $\xi_{ij}(z)$ is then given by

$$\begin{array}{cccccccc} \bullet q^{|j-i|} & \bullet q^{j+i} & \bullet q^{2h^\vee-(j+i)} & \bullet q^{2h^\vee-|j-i|} & \bullet q^{2h^\vee+|j-i|} & \bullet q^{2h^\vee+j+i} & \bullet q^{4h^\vee-(j+i)} & \bullet q^{4h^\vee-|j-i|} \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{array}$$

The black node in Table 1 corresponds to the natural vector representation $V = V_1$. Therefore, we can restrict ourselves to analysing the singularities of ξ_{11},

$$\begin{array}{cccccc} \bullet 1 & \bullet q^{2} & \bullet q^{2h^\vee-2} & \bullet q^{2h^\vee} & \bullet q^{2h^\vee+2} & \bullet q^{4h^\vee-2} \\ 2 & -1 & 1 & -2 & 1 & -1 \end{array}$$

Lemma 6.4 remains valid as in the B_n case. To perform the second step of the proof, we assign

$$\begin{array}{lcl} \xi_{11}(z) & \mapsto & v_0 := (2,-1,0,0,\dots,0,0,1) \\ \xi_{11}(zq^{-2}) & \mapsto & v_1 := (-1,2,-1,0,0,\dots,0,0) \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_{11}(zq^{-2(h^\vee-2)}) & \mapsto & v_{h^\vee-2} := (0,0,\dots,0,0,-1,2,-1) \\ \xi_{11}(zq^{-2(h^\vee-1)}) & \mapsto & v_{h^\vee-1} := (1,0,0,\dots,0,0,-1,2) \end{array}$$

in $\mathbb{Z}^{h^\vee}$. Again, the entries of these vectors are the orders of the singularities of $\xi_{11}(zq^{-2s})$ on the sequence $1, q^2, ..., q^{2h^\vee-2}$. Here $\det T_n = 4$ (use the first column to compute it from the type A_n case). So again we have no additional relations, and Theorem 3.4 is proved.

6.3. **Type D_n.** For D_n we have $r^\vee = 1, h^\vee = 2(n-1)$. We begin with the case of even n. We have to consider the two half spin representations that we denote $V_\pm$. The singularity structure for $\xi_{ij}(z)$ is given by

$$\xi_{+-} = \xi_{-+} : \quad \begin{array}{cccccccccc} \bullet 1 & \bullet q^{2} & \bullet q^{4} & \dots & \bullet q^{h^\vee-2} & \bullet q^{h^\vee} & \bullet q^{h^\vee+2} & \dots & \bullet q^{2h^\vee-4} & \bullet q^{2h^\vee-2} \\ 0 & 1 & -1 & \dots & -1 & 0 & -1 & \dots & -1 & 1 \end{array}$$

$$\xi_{++} = \xi_{--} : \quad \begin{array}{cccccccccc} \bullet 1 & \bullet q^{2} & \bullet q^{4} & \dots & \bullet q^{h^\vee-2} & \bullet q^{h^\vee} & \bullet q^{h^\vee+2} & \dots & \bullet q^{2h^\vee-4} & \bullet q^{2h^\vee-2} \\ 2 & -1 & 1 & \dots & 1 & -2 & 1 & \dots & 1 & -1 \end{array}$$

The dots here mean the same thing as in the B_n case.

Recall we have defined indecomposable representations $Y(z_1, \dots, z_m|w_1, \dots, w_l)$. The elliptic central character of $Y = Y(z_1, \dots, z_m|w_1, \dots, w_l)$ is determined by the pair

$$(\xi_{V_+}(z)|_Y, \xi_{V_-}(z)|_Y) = \Big(\prod_{i,j=1}^{m,l} \xi_{++}(\frac{z}{z_i})\xi_{+-}(\frac{z}{w_j}), \prod_{i,j=1}^{m,l} \xi_{-+}(\frac{z}{z_i})\xi_{--}(\frac{z}{w_j})\Big) \in \mathbb{C}(E) \times \mathbb{C}(E). \tag{6.1}$$

Lemma 6.5.

(a) *For any $u \in \mathbb{C}^*$, $Y(z_1, \dots, z_m|w_1, \dots, w_l)$ is linked to $Y(z_1, \dots, z_m, u, q^{h^\vee}u|w_1, \dots, w_l)$ and to $Y(z_1, \dots, z_m|w_1, \dots, w_l, u, q^{h^\vee}u)$. In particular, we conclude that $Y(z_1, \dots, z_m|w_1, \dots, w_l)$ is linked to $Y(z_1, \dots, z_iq^{2h^\vee}, \dots, z_m|w_1, \dots, w_l)$ and to $Y(z_1, \dots, z_m|w_1, \dots, w_jq^{2h^\vee}, \dots, w_l)$.*

(b) *$Y(z_1, \dots, z_m, u, uq^2|w_1, \dots, w_l, uq^{h^\vee}, uq^{2+h^\vee})$ is linked to $Y(z_1, \dots, z_m|w_1, \dots, w_l)$.*

Proof. The first is proved exactly as in the B_n case since $V_\pm^* \cong V_\pm(q^{h^\vee})$. The second follows from the fact that $V_+ \otimes V_+(q^2)$ and $V_- \otimes V_-(q^2)$ have $V_{n-2}(q)$ as a subrepresentation [4] and because $V_{n-2}^* \cong V_{n-2}(q^{h^\vee})$. □

Remark. Adding or removing sequences of the form $(u, uq^2|uq^{h^\vee}, uq^{2+h^\vee})$, it is easy to show that $Y(z_1, \dots, z_m, u, uq^{2(2k-1)}|w_1, \dots, w_l, uq^{h^\vee}, uq^{2(2k-1)+h^\vee})$ is linked to $Y(z_1, \dots, z_m|w_1, \dots, w_l)$, for $k = 1, \dots, (n-2)/2 = (\frac{h^\vee}{2} - 1)/2$.

Now we have to show that the multiplicative relations between $\xi_{++}(zq^{-2s})$, $\xi_{--}(zq^{-2s})$ and $\xi_{\pm\mp}(zq^{-2s})$, for $s = 0, \dots, \frac{h^\vee}{2} - 1$, are expressed via the transformations of Lemma 6.5. Since $\xi_{++}(z) = \xi_{--}(z)$ and $\xi_{\pm\mp}(z) = \xi_{\mp\pm}(z)$, we are left to check the relations between $\xi_{++}(zq^{-2s})$ and $\xi_{+-}(zq^{-2s})$. Consider the group $\mathbb{Z}^{\frac{h^\vee}{2}}$ and set

$$\xi_{++}(q^{-2s}z) \mapsto v_s := ((-1)^s, (-1)^{s+1}, \dots, -1, 2, -1, \dots, (-1)^s), \tag{6.2}$$

$$\xi_{+-}(q^{-2s}z) \mapsto w_s := ((-1)^{s+1}, (-1)^s, \dots, 1, 0, 1, \dots, (-1)^{s+1}). \tag{6.3}$$

Proceeding as in the B_n case, we can show that the $\xi_{++}(zq^{-2s})$, are multiplicatively independent (the corresponding matrix T_n is of the same form). The same is true for the $\xi_{+-}(zq^{-2s})$ (using the same arguments). Then, keeping (6.1) in mind, we are left to find coefficients a_s, a'_s such that

$$\sum_{s=0}^{\frac{h^\vee}{2}-1} a_s v_s = \sum_{s=0}^{\frac{h^\vee}{2}-1} a'_s w_s \quad \text{and} \quad \sum_{s=0}^{\frac{h^\vee}{2}-1} a'_s v_s = \sum_{s=0}^{\frac{h^\vee}{2}-1} a_s w_s. \tag{6.4}$$

But it is easy to see that we must have $a_s = a'_s$ and, consequently, that $\sum_s (-1)^s a_s = 0$. We have the following basis of solutions

$$(a_0, a_1, \dots, a_{l-1}, a_l, a_{l+1}, \dots, a_{\frac{h^\vee}{2}-1}) = (1, 0, \dots, 0, (-1)^{l+1}, 0, \dots, 0). \tag{6.5}$$

But these are exactly the transformations in the remark after Lemma 6.5.

In the odd case the elliptic central character is determined by its value on one of the half spin representations, V_n. The singularity structure for $\xi_{nn}(z)$ is given by the picture

$$\begin{array}{cccccccccc} \bullet^{1} & \bullet^{q^2} & \bullet^{q^4} & \dots & \bullet^{q^{h^\vee-2}} & \bullet^{q^{h^\vee}} & \bullet^{q^{h^\vee+2}} & \dots & \bullet^{q^{2h^\vee-4}} & \bullet^{q^{2h^\vee-2}} \\ 2 & -1 & 1 & \dots & -1 & 0 & -1 & \dots & 1 & -1 \end{array}$$

Lemma 6.6. *$Y(z_1,\dots,z_m,u,uq^2,uq^{h^\vee},uq^{h^\vee+2})$ is linked to $Y(z_1,\dots,z_m)$. Hence $Y(z_1,\dots,z_m)$ is linked to $Y(z_1,\dots,z_{j-1},z_jq^{2h^\vee},z_{j+1},\dots,z_m)$.*

Proof. Exactly as the proof of part (b) of Lemma 6.5. □

Now we have to check that the relations between $\xi_{nn}(zq^{2s})$, for $0\le s<h^\vee$, are expressed by the transformations of Lemma 6.6. We consider the group $\mathbb{Z}^{h^\vee}$ and the vectors

$$v_k=((-1)^k,(-1)^{k+1},\dots,-1,2,-1,\dots,(-1)^{k+1}), \tag{6.6}$$

$$w_k=((-1)^k,(-1)^{k+1},\dots,-1,0,-1,\dots,(-1)^{k+1}) \tag{6.7}$$

in $\mathbb{Z}^{h^\vee/2}$, for $k=0,\dots,\frac{h^\vee}{2}-1=n-2$, and assign to $\xi_{nn}(zq^{-2s})$ the vector $u_s\in\mathbb{Z}^{h^\vee}$ given by $u_s:=(v_s,w_s)$, if $s<\frac{h^\vee}{2}$, or by $u_s:=(w_{s-\frac{h^\vee}{2}},v_{s-\frac{h^\vee}{2}})$, if $s\ge\frac{h^\vee}{2}$. The corresponding matrix T_n is the $h^\vee\times h^\vee$-matrix of the form

$$T_n=\begin{pmatrix}\bar T_n & \dot T_n\\ \dot T_n & \bar T_n\end{pmatrix}$$

where the rows of $\bar T_n$ are the vectors v_k and the rows of $\dot T_n$ are the w_k. By the discussion of the even case we know that the sets $\{v_k\}$ and $\{w_k\}$ are linearly independent. In fact, one can show that the rank of T_n is $\frac{h^\vee}{2}+1=n$. The linear relations between the two sets of vectors are similar to that of the even case and have the form

$$u_0+u_{\frac{h^\vee}{2}}+(-1)^{k+1}(u_k+u_{\frac{h^\vee}{2}+k})=0,\qquad k=1,\dots,\frac{h^\vee}{2}-1. \tag{6.8}$$

Similarly, to the remark after Lemma 6.5, one shows that these relations are all obtained from the one in Lemma 6.6 (here we use $V_n^*\cong V_{n-1}(q^{h^\vee})$).

6.4. **Type E_n.** $\bullet\, E_6$:

The dual Coxeter number is $h^\vee=12$ and $r^\vee=1$. The black node corresponds to one of the 27-dimensional fundamental representations. The singularity structure for $\xi_{11}(z)$ is

$$\begin{array}{cccccccccccc} \bullet^{1} & \bullet^{q^2} & \bullet^{q^4} & \bullet^{q^6} & \bullet^{q^8} & \bullet^{q^{10}} & \bullet^{q^{12}} & \bullet^{q^{14}} & \bullet^{q^{16}} & \bullet^{q^{18}} & \bullet^{q^{20}} & \bullet^{q^{22}} \\ 2 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \end{array}$$

Lemma 6.7.

(a) *$Y(z_1,\dots,z_m,w,wq^8,wq^{16})$ is linked to $Y(z_1,\dots,z_m)$. In particular, $Y(z_1,\dots,z_m)$ is linked to $Y(z_1,\dots,z_{j-1},z_jq^{24},z_{j+1},\dots,z_m)$.*
(b) *$Y(z_1,\dots,z_m,w,wq^2,wq^{10},wq^{12})$ is linked to $Y(z_1,\dots,z_m,wq^6)$.*

Proof. Item (a) follows since $V_5(q^4)$ is a subrepresentation of $V_1\otimes V_1(q^8)$ [4] and because $V_1^*\cong V_5(q^{12})$. For item (b) we will use a sequence of subrepresentations, all of them computed in [4]. By part (a), $V_1(q^2)\otimes V_1(q^{10})$ has $V_5(q^6)$ as a subrepresentation and, consequently, $V_1\otimes V_1(q^2)\otimes V_1(q^{10})\otimes V_1(q^{12})$ has $V_1\otimes V_5(q^6)\otimes V_1(q^{12})$ as a subrepresentation. Now, $V_6(q^3)$ is a subrepresentation of $V_1\otimes V_5(q^6)$ and we get that $V_1 \otimes V_6(q^9)$ is a subrepresentation of $V_1 \otimes V_5(q^6) \otimes V_1(q^{12})$. Finally, we get $V_1(q^6)$ as a subrepresentation of $V_1 \otimes V_6(q^9)$. □

Remark. All these subrepresentation relations can be obtained studying tensor product decompositions and the singularity structure of the elliptic central character. This will be the procedure to prove Theorem 6.1 for the remaining $\mathfrak{g}$.

The combinatorics part is in the group $\mathbb{Z}^{12}$. We consider the vectors $v_0, \dots, v_{11}$ obtained from

$$v_0 = (2, -1, 0, 1, -1, 0, 0, 0, -1, 1, 0, -1)$$

by cyclic permutation of the coordinates, corresponding to $\xi_{11}(zq^{-2s})$, where the entries of v_s have the usual meaning. The rank of the (12×12)-matrix T_n thus obtained is 6. We first eliminate the linear relations related to item (a) of Lemma 6.7. They are

$$v_k + v_{k+4} + v_{k+8} = 0 \tag{6.9}$$

for $k = 0, 1, 2, 3$. Removing $v_8, \dots, v_{11}$ from our set of vectors, we check that the remaining vectors $v_0, \dots, v_7$ satisfy the linear relations

$$v_{k+3} = v_k + v_{k+1} + v_{k+5} + v_{k+6} \tag{6.10}$$

for $k = 0, 1$. This is exactly part (b) of Lemma 6.7. Since the vectors $v_0, \dots, v_5$ are linearly independent, the proof is complete.

Remark. The rank and the kernel of the corresponding matrix T_n for $\mathfrak{e}_n$ and $\mathfrak{f}_4$ were computed using the computer software *Mathematica*.

• E_7: Here $h^\vee = 18$. The black node corresponds to the affinization of the 56-dimensional fundamental representation. We list the singularities of ξ_{11} and ξ_{16}. These are the two we need to prove Theorem 3.4. To prove Theorem 6.1 we will need to analyse other $\xi_{1j}(z)$, but they can be read from the matrices $M(q)$ in the appendix.

ξ_{11}:	$\bullet^{1}$	$\bullet^{q^2}$	$\bullet^{q^8}$	$\bullet^{q^{10}}$	$\bullet^{q^{16}}$	$\bullet^{q^{18}}$	$\bullet^{q^{20}}$	$\bullet^{q^{26}}$	$\bullet^{q^{28}}$	$\bullet^{q^{34}}$
	2	−1	1	−1	1	−2	1	−1	1	−1

ξ_{16}:	$\bullet^{q^5}$	$\bullet^{q^7}$	$\bullet^{q^{11}}$	$\bullet^{q^{13}}$	$\bullet^{q^{23}}$	$\bullet^{q^{25}}$	$\bullet^{q^{29}}$	$\bullet^{q^{31}}$
	1	−1	1	−1	−1	1	−1	1

We begin by proving Theorem 6.1. We denote the finite dimensional representation of $U_q(\mathfrak{e}_7)$ with highest weight λ by V_λ^f. It is known that V_1^f is affinizable, i.e., the $U_q(\widehat{\mathfrak{e}}_7)$ fundamental representation V_1 is isomorphic to V_1^f as a $U_q(\mathfrak{e}_7)$-module. Now

$$V_1^f \otimes V_1^f \cong V_{2\omega_1}^f \oplus V_2^f \oplus V_6^f \oplus \mathbb{C} \tag{6.11}$$

Using Corollary 5.5 we see that the normalized R-matrix is not invertible at q^{-2}, q^{-10} and q^{-18}. The associated elliptic central characters for these cases are, respectively, the same as the elliptic central characters as $V_2(q), V_6(q^5)$ and $\mathbb{C}$, respectively. In fact, these are subrepresentations: by (6.11) and the associated elliptic central character, the kernel of $P\bar{R}_{V_1,V_1}(q^{-2})$ must be of the form $V_2(q^{1+36k})$ for some $k \in \mathbb{Z}$. To conclude that $k = 0$, one uses Corollary 5.6 with $i = j = 1, r = 2$ and $m = 1$. The story for $V_6(q^5)$ is exactly the same. The affinization of V_6^f (which corresponds to the adjoint representation of $\mathfrak{e}_7$) is isomorphic to $V_6^f \oplus \mathbb{C}$ and thus we have

$$V_6 \otimes V_6(w) \cong V_{2\omega_6}^f \oplus V_5^f \oplus V_2^f \oplus 3V_6^f \oplus 2\mathbb{C}.$$

The R-matrix is non-invertible when $w = q^2, q^8, q^{12}, q^{18}$. We proceed as before to check that $V_5(q)$ is a subrepresentation of $V_6 \otimes V_6(q^2)$. This is the only new fundamental representation we get here. Then we go to

$$V_1 \otimes V_6(w) \cong V_{\omega_1+\omega_6}^f \oplus V_7^f \oplus 2V_1$$

and the R-matrix is non-invertible at $w = q^7, q^{13}$. The new fundamental representation we get is $V_7(q^4)$ as a subrepresentation of $V_1 \otimes V_6(q^7)$.

Remark. For the next tensor product we will need the following important observations. All the possible affinizations of $V_{\omega_1+\omega_6}^f$ (denoted by $V_{\omega_1+\omega_6}[w](u)$) occur as (shifts) of subquotients of $V_1(u) \otimes V_6(wu)$. Therefore, all possible $\xi_{V_1}(z)|_{V_{\omega_1+\omega_6}[w](u)}$ are of the form $\xi_{11}(z/u)\xi_{16}(z/wu)$.

We now study the tensor product $V_1 \otimes V_2(w)$. The affinization of V_2^f is isomorphic to $V_2^f \oplus V_6^f \oplus \mathbb{C}$. We have

$$V_1 \otimes V_2(w) \cong V_{\omega_1+\omega_2}^f \oplus V_3^f \oplus 2V_{\omega_1+\omega_6}^f \oplus 2V_7^f \oplus 3V_1^f.$$

The values of w where $V_1 \otimes V_2(w)$ has a subrepresentation not containing the highest weight component are q^3, q^{11} and q^{17} and, at these points, the corresponding elliptic central characters are those of $V_3(q^2), V_7(q^8)$ and $V_1(q^{16})$. By the last remark we know that all possible elliptic central characters of $V_{\omega_1+\omega_6}[w](u)$ at V_1 are of the form $\xi_{11}(z/u)\xi_{16}(z/wu)$. One checks now that this will never produce the elliptic central character of $V_3(x)$ for any x and, therefore, we do get $V_3(q^2)$ as a subrepresentation. To complete the proof of Theorem 6.1 we need to get V_4. This is done as before using

$$V_1^f \otimes V_3^f \cong V_{\omega_1+\omega_3}^f \oplus V_4^f \oplus V_{\omega_2+\omega_6}^f \oplus V_{\omega_1+\omega_7}^f \oplus V_5^f \oplus V_2^f.$$

Thus $V_4(q^3)$ occurs as subrepresentation of $V_1 \otimes V_3(q^4)$.

Remark. The tensor product decompositions for $\mathfrak{e}_7$ and $\mathfrak{e}_8$ were computed using the computer package *LiE* (http://wwwmathlabo.univ-poitiers.fr/ maavl/LiE).

Now we proceed with the proof of Theorem 3.4.

Lemma 6.8.

(a) *For any $w \in \mathbb{C}^*$, $Y(z_1, \dots, z_m)$ is linked to $Y(z_1, \dots, z_m, w, wq^{18})$. In particular, $Y(z_1, \dots, z_m)$ is linked to $Y(z_1, \dots, z_{j-1}, z_j q^{36}, z_{j+1}, \dots, z_m)$.*

(b) *$Y(z_1, \dots, z_m, w, wq^2, wq^{12}, wq^{14}, wq^{24}, wq^{26})$ is linked to $Y(z_1, \dots, z_m)$.*

Proof. Part (a) is clear from $V_1^* \cong V_1(q^{18})$. Let us prove (b). Using that $V_6(q^5)$ is a subrepresentation of $V_1 \otimes V_1(q^{10})$ we find that $V_1 \otimes V_6(q^7) \otimes V_6(q^{19})$ is a subrepresentation of $V_1 \otimes V_1(q^2) \otimes V_1(q^{12}) \otimes V_1(q^{14}) \otimes V_1(q^{24})$. Now use that $V_6(q^6)$ is a subrepresentation of $V_6 \otimes V_6(q^{12})$ to get that $V_1 \otimes V_6(q^{13})$ is a subrepresentation of $V_1 \otimes V_6(q^7) \otimes V_6(q^{19})$. Since $V_1(q^8)$ is a subrepresentation of $V_1 \otimes V_6(q^{13})$ and $V_1(q^{26}) \cong V_1(q^8)^*$, we have $\mathbb{C} \subset V_1 \otimes V_1(q^2) \otimes V_1(q^{12}) \otimes V_1(q^{14}) \otimes V_1(q^{24}) \otimes V_1(q^{26})$, so we are done. ∎

We now consider the group $\mathbb{Z}^9$ and the vectors $v_0, \ldots, v_8$ corresponding to $\xi_{11}(zq^{-2s})$ as usual (the entries are the orders of the singularities of $\xi_{11}(zq^{-2s})$ on the sequence $1, q^2, ..., q^{16}$). The matrix T_n has rank 7 and the nontrivial linear relations are

$$v_k + v_{k+1} + v_{k+6} + v_{k+7} - v_{k+3} - v_{k+4} = 0 \qquad (6.12)$$

for $k = 0, 1$. These relations are implemented by part (b) of Lemma 6.8, so Theorem 3.4 is proved.

• E_8: For $\mathfrak{e}_8$ $h^\vee = 30$ and $r^\vee = 1$. The black node corresponds to the affinization of the adjoint representation. The singularity structures for ξ_{11} and ξ_{17} are respectively

$\bullet^{1}$	$\bullet^{q^2}$	$\bullet^{q^{10}}$	$\bullet^{q^{12}}$	$\bullet^{q^{18}}$	$\bullet^{q^{20}}$	$\bullet^{q^{28}}$	$\bullet^{q^{30}}$	$\bullet^{q^{32}}$	$\bullet^{q^{40}}$	$\bullet^{q^{42}}$	$\bullet^{q^{48}}$	$\bullet^{q^{50}}$	$\bullet^{q^{58}}$
2	−1	1	−1	1	−1	1	−2	1	−1	1	−1	1	−1

$\bullet^{q^6}$	$\bullet^{q^8}$	$\bullet^{q^{12}}$	$\bullet^{q^{14}}$	$\bullet^{q^{16}}$	$\bullet^{q^{18}}$	$\bullet^{q^{22}}$	$\bullet^{q^{24}}$	$\bullet^{q^{36}}$	$\bullet^{q^{38}}$	$\bullet^{q^{42}}$	$\bullet^{q^{44}}$	$\bullet^{q^{46}}$	$\bullet^{q^{48}}$	$\bullet^{q^{52}}$	$\bullet^{q^{54}}$
1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1

Theorem 6.1 is proved as in the $\mathfrak{e}_7$ case using the following relations:

$$\begin{array}{cc} V_2(q) \subset V_1 \otimes V_1(q^2), & V_7(q^6) \subset V_1 \otimes V_1(q^{12}), \\ V_1(q^{10}) \subset V_1 \otimes V_1(q^{20}), & V_8(q^5) \subset V_1 \otimes V_7(q^8), \\ V_6(q) \subset V_7 \otimes V_7(q^2), & V_3(q^2) \subset V_1 \otimes V_2(q^3), \\ V_4(q^3) \subset V_1 \otimes V_3(q^4), & V_5(q^4) \subset V_1 \otimes V_4(q^5). \end{array} \qquad (6.13)$$

The tensor product decomposition for fundamental representations of $\mathfrak{e}_8$ involves much longer expressions than those we had to deal in the $\mathfrak{e}_7$ case, so we will not write them down completely. We will just write two steps, since we will need them to prove Theorem 3.4. We sketch the remaining steps in the appendix. The first step is the tensor product $V_1 \otimes V_1(w)$. Since $V_1 \cong V_1^f \oplus \mathbb{C}$, we have

$$V_1 \otimes V_1(w) \cong V_{2\omega_1}^f \oplus V_2^f \oplus V_7^f \oplus 3V_1^f \oplus 2\mathbb{C}.$$

From this we prove the first line of (6.13) and check that the elliptic central character (at V_1) of the affinizations $V_{2\omega_1}[w](u)$ coincides with that of a fundamental representation only when $w \in \{q^{\pm 2}, q^{\pm 12}, q^{\pm 20}\}$ (modulo q^{60}), when it has the elliptic central character of $V_2(q^{\pm 1}), V_7(q^{\pm 6})$ or $V_1(q^{\pm 10})$, respectively. The proof that these are really subrepresentations when $w \in \{q^2, q^{12}, q^{20}\}$, is analogous to the one we did for $\mathfrak{e}_7$.

In the second step we consider $V_1 \otimes V_7(w)$. Since $V_7 \cong V_7^f \oplus V_1^f \oplus \mathbb{C}$, we have

$$V_1 \otimes V_7(w) \cong V_{\omega_1+\omega_7}^f \oplus V_8^f \oplus V_{2\omega_1}^f \oplus 2V_2^f \oplus 3V_7^f \oplus 4V_1^f \oplus 2\mathbb{C}.$$

The normalized R-matrix is not invertible at $w = q^8, q^{14}, q^{18}, q^{24}$ and the corresponding elliptic central characters coincide with the ones of $V_8(q^5), V_2(q^9), V_7(q^{12})$ and $V_1(q^{18})$, respectively. Since all other components do not have the elliptic central character of V_8, we conclude that we really get an affinization of V_8^f as a subrepresentation, and it must be $V_8(q^5)$. We will come back to this tensor product in the proof of Lemma 6.9 below.

Lemma 6.9.

(a) *For any* $w \in \mathbb{C}^*$, $Y(z_1, \dots, z_m)$ *is linked to* $Y(z_1, \dots, z_m, w, wq^{30})$. *In particular,* $Y(z_1, \dots, z_m)$ *is linked to* $Y(z_1, \dots, z_{j-1}, z_j q^{60}, z_{j+1}, \dots, z_m)$.

(b) $Y(z_1, \dots, z_m, w, wq^{20}, wq^{40})$ *is linked to* $Y(z_1, \dots, z_m)$.

(c) $Y(z_1, \dots, z_m, w, wq^{12}, wq^{24}, wq^{36}, wq^{48})$ *is linked to* $Y(z_1, \dots, z_m)$.

Proof. Part (a) is clear from $V_1^* \cong V_1(q^{30})$. Part (b) follows since $V_1(q^{10})$ is a subrepresentation of $V_1 \otimes V_1(q^{20})$. Part (c) is more delicate. First $V_7(q^6) \subset V_1 \otimes V_1(q^{12})$ and, consequently, $V_1 \otimes V_7(q^{18}) \otimes V_7(q^{42})$ is a subrepresentation of $V_1 \otimes V_1(q^{12}) \otimes V_1(q^{24}) \otimes V_1(q^{36}) \otimes V_1(q^{48})$. Let us introduce the notation $Y(z_1, \dots, z_m | u_1, \dots, u_l)$ meaning a non-resonant reordering of $V_1(z_1) \otimes \cdots \otimes V_1(z_m) \otimes V_7(u_1) \otimes \cdots \otimes V_7(u_l)$. Then we have shown that $Y(z_1, \dots z_m, w | wq^{18}, wq^{42})$ is a subrepresentation of $Y(z_1, \dots, z_m, w, wq^{12}, wq^{24}, wq^{36}, wq^{48})$.

Go back to the tensor product $V_1 \otimes V_7(w)$. We cannot conclude whether we have $V_7(q^{12})$ as a subrepresentation of $V_1 \otimes V_7(q^{18})$. If it was true, the proof would be completed, as $V_7^* \cong V_7(q^{30})$. If it is not true, then we must have, as a subrepresentation, an irreducible affinization of $V_{2\omega_1}^f$ with the elliptic central character of $V_7(q^{12})$. From the analysis of $V_1 \otimes V_1(w)$ we know that this affinization must be a quotient of the form

$$V_1(q^6) \otimes V_1(q^{18+60j}) \twoheadrightarrow V_{2\omega_1}[q^{12+60j}](q^6).$$

Let $Y(\overrightarrow{z}, (w|wq^{18}), wq^{42})$, where $\overrightarrow{z} = (z_1, \dots, z_m)$, denote the corresponding subrepresentation of the original $Y(\overrightarrow{z}, w, wq^{12}, wq^{24}, wq^{36}, wq^{48})$. We have shown

$$\begin{aligned} Y(\overrightarrow{z}, wq^6, wq^{18+60j} | wq^{42}) &\twoheadrightarrow Y(\overrightarrow{z}, (w|wq^{18}), wq^{42}) \\ &\hookrightarrow Y(\overrightarrow{z}, w, wq^{12}, wq^{24}, wq^{36}, wq^{48}). \end{aligned}$$

We complete the proof with the following diagram:

$$Y(\overrightarrow{z}) \hookrightarrow Y(\overrightarrow{z} | wq^{12}, wq^{42}) \hookrightarrow Y(\overrightarrow{z}, wq^6, wq^{18} | wq^{42}) \sim Y(\overrightarrow{z}, wq^6, wq^{18+60j} | wq^{42}).$$

The first inclusion follows from $V_7^* \cong V_7(q^{30})$ and the second since $V_7(q^{12}) \hookrightarrow V_1(q^6) \otimes V_1(q^{18})$. The symbol $\sim$ denotes a linking relation and is immediate from part (a). $\square$

As usual, now we define vectors $v_0, \dots, v_{14}$ in $\mathbb{Z}^{15}$, encoding the orders of the singularities of $\xi_{11}(zq^{-2s})$ on the sequence $1, q^2, ..., q^{28}$. The corresponding matrix T_n has rank 8. We find the following (linearly independent) relations

$$v_j + v_{j+10} - v_{j+5} = 0 \qquad \text{for } j = 0, 1, 2, 3, 4, \tag{6.14}$$

$$v_k + v_{k+6} + v_{k+12} - v_{k+3} - v_{k+9} = 0 \qquad \text{for } k = 0, 1. \tag{6.15}$$

Relation (6.14) is part (b) of Lemma 6.9, while (6.15) is part (c).

6.5. F_4. For $\mathfrak{f}_4$ we have $r^\vee = 2$ and $h^\vee = 9$. The black node corresponds to V_1, the affinization of the 26-dimensional representation of $\mathfrak{f}_4$. We list the singularity structure only for $\xi_{11}(z)$.

$\bullet^{1}$	$\bullet^{q^2}$	$\bullet^{q^4}$	$\bullet^{q^6}$	$\bullet^{q^8}$	$\bullet^{q^{10}}$	$\bullet^{q^{12}}$	$\bullet^{q^{14}}$	$\bullet^{q^{16}}$	$\bullet^{q^{18}}$	$\bullet^{q^{20}}$	$\bullet^{q^{22}}$	$\bullet^{q^{24}}$	$\bullet^{q^{26}}$	$\bullet^{q^{28}}$	$\bullet^{q^{30}}$	$\bullet^{q^{32}}$	$\bullet^{q^{34}}$
2	−1	0	1	−1	1	−1	0	1	−2	1	0	−1	1	−1	1	0	−1

V_1^f is affinizable and we have

$$V_1 \otimes V_1(w) \cong V_{2\omega_1}^f \oplus V_2^f \oplus V_4^f \oplus V_1^f \oplus \mathbb{C}.$$

By Corollary 5.5, $\bar{R}_{V_1,V_1}(z)$ is not invertible at $z = q^{-2}, q^{-8}, q^{-12}, q^{-18}$. Arguing as in the $\mathfrak{e}_7$ case we prove that the subrepresentations of $V_1 \otimes V_1(w)$, for $w = q^2, q^8, q^{12}, q^{18}$ are, respectively, $V_2(q), V_4(q^4), V_1(q^6)$ and $\mathbb{C}$. The affinization of the adjoint representation V_4 is isomorphic to $V_4^f \oplus \mathbb{C}$ and we have

$$V_4 \otimes V_4(w) \cong V_{2\omega_4}^f \oplus V_3^f \oplus V_{2\omega_1}^f \oplus 3V_4^f \oplus 2\mathbb{C}.$$

As before we get that $V_3(q^2)$ is a subrepresentation of $V_4 \otimes V_4(q^4)$ and Theorem 6.1 is proved.

To prove Theorem 3.4 we state the following:

Lemma 6.10.

(a) *For any $w \in \mathbb{C}^*$, $Y(z_1, \ldots, z_m)$ is linked to $Y(z_1, \ldots, z_m, w, wq^{18})$. In particular, $Y(z_1, \ldots, z_m)$ is linked to $Y(z_1, \ldots, z_{j-1}, z_j q^{36}, z_{j+1}, \ldots, z_m)$.*

(b) *$Y(z_1, \ldots, z_m, w, wq^{12}, wq^{24})$ is linked to $Y(z_1, \ldots, z_m)$.*

Proof. Part (a) follows from $V_1^* \cong V_1(q^{18})$. Since $V_1(q^6)$ is a subrepresentation of $V_1 \otimes V_1(q^{12})$, part (b) is proved. □

We define $v_0, \ldots, v_8 \in \mathbb{Z}^9$ corresponding to $\xi_{11}(zq^{-2s})$. The rank of T_n is 6. The linear relations give exactly part (b) of Lemma 6.10.

$$v_k + v_{k+6} - v_{k+3} = 0 \qquad \text{for } k = 0, 1, 2. \tag{6.16}$$

6.6. G_2**.** In this case $r^\vee = 3$ and $h^\vee = 4$. The zeros and poles are given by

$\xi_{11}(z)$:	1	q^2	q^4	q^8	q^{10}	q^{12}	q^{14}	q^{16}	q^{20}	q^{22}
	2	−1	1	−1	1	−2	1	−1	1	−1

$\xi_{12}(z)$:	q	q^5	q^7	q^{11}	q^{13}	q^{17}	q^{19}	q^{23}
	1	1	−1	−1	−1	−1	1	1

The black node corresponds to V_1, the affinization of the 7-dimensional representation of $\mathfrak{g}_2$. V_1 is affinizable and

$$V_1 \otimes V_1(w) \cong V_{2\omega_1}^f \oplus V_2^f \oplus V_1^f \oplus \mathbb{C}.$$

Subrepresentations are obtained at $w = q^2, q^8, q^{12}$ and they are $V_2(q), V_1(q^4)$ and $\mathbb{C}$, respectively. This proves Theorem 6.1.

The usual lemma is immediate.

Lemma 6.11.

(a) *For any $w \in \mathbb{C}^*$, $Y(z_1, \ldots, z_m)$ is linked to $Y(z_1, \ldots, z_m, w, wq^{12})$. In particular, $Y(z_1, \ldots, z_m)$ is linked to $Y(z_1, \ldots, z_{j-1}, z_j q^{24}, z_{j+1}, \ldots, z_m)$.*

(b) *$Y(z_1, \ldots, z_m, w, wq^8, wq^{16})$ is linked to $Y(z_1, \ldots, z_m)$.*

Then we consider the vectors encoding the singularity structures of $\xi_{11}(zq^{-2s})$ in the sequence $1, q^2, \ldots, q^{10}$:

$$v_0 = (2, -1, 1, 0, -1, 1), \quad v_1 = (-1, 2, -1, 1, 0, -1), \quad v_2 = (1, -1, 2, -1, 1, 0),$$
$$v_3 = (0, 1, -1, 2, -1, 1), \quad v_4 = (-1, 0, 1, -1, 2, -1), \quad v_5 = (1, -1, 0, 1, -1, 2),$$

and observe that $v_{2+k} = v_k + v_{4+k}$, for $k = 0, 1$, which is part (b) of the lemma. One easily checks that $v_0, \ldots, v_3$ are linearly independent to complete the proof.

Appendix A. Sketch of the Proof of Theorem 6.1 for E_8

The idea of the proof is the following (recall the definition of the set $\mathcal{P}_{ij}$ in Corollary 5.5).

1. Compute the $U_q(\mathfrak{g})$-tensor product decomposition of $V_i \otimes V_j(w)$, for already obtained V_i, V_j, and check that a new V_k^f occurs.

2. Compute all possible elliptic central characters (ECC) of $V_i \otimes V_j(w)$ at V_1 and check that it will coincide with the ECC of the new V_k for some $w \in P_{ij}$.

3. Check that all affinizations of the remaining components never have the ECC of V_k to conclude that $V_k(wq^s)$ is indeed a subrepresentation of $V_i \otimes V_j(w)$, for some $s \in \mathbb{Z}$.

It becomes clear that we need the list of the singularity structures for ξ_{ij}. They can be obtained directly from the matrix $M(q)$ given in the section below. The following lemma is helpful.

Lemma A.1. *$V_i \otimes V_j(w)$ has ECC of some $V_k(u)$ if and only if w belongs (modulo q^{60}) to $\mathcal{P}_{ij}^{\pm 1}$.*

Proof. First observe that the singularities of ξ_{ij} have order ± 1 or ± 2, and that if order ± 2 occurs, it occurs only once and, in that case, $i = j$. Hence, for $\xi_{ii}(z)\xi_{ij}(z/w)$, coincide with some $\xi_{ik}(zu)$, the zero of order 2 in $\xi_{ii}(z)$ must be combined with a pole of $\xi_{ij}(z/w)$. □

Using the periodicity properties of ξ_{ij}, we see that to find all possible "fundamental" ECC of $V_i \otimes V_j(w)$, we just need to take $w \in P_{ij}$. In Table 2 we give this list for the pairs (i, j) that we will need:

Table 2.

Pair	$\mathcal{P}_{ij}$	ECC of
(1,1)	$\{q^2, q^{12}, q^{20}, q^{30}\}$	$V_2, V_7, V_1, \mathbb{C}$
(1,2)	$\{q^3, q^{13}, q^{21}, q^{29}\}$	V_3, V_8, V_7, V_1
(1,3)	$\{q^4, q^{14}, q^{22}, q^{28}\}$	V_4, V_6, V_8, V_2
(1,4)	$\{q^5, q^{23}, q^{27}\}$	V_5, V_6, V_3
(1,6)	$\{q^9, q^{19}, q^{25}\}$	V_4, V_3, V_8
(1,7)	$\{q^8, q^{14}, q^{18}, q^{24}\}$	V_8, V_2, V_7, V_1
(1,8)	$\{q^7, q^{11}, q^{17}, q^{21}, q^{25}\}$	V_6, V_3, V_8, V_2, V_7
(2,2)	$\{q^4, q^{12}, q^{14}, q^{20}, q^{22}, q^{30}\}$	$V_4, V_6, V_3, V_2, V_7, \mathbb{C}$
(2,7)	$\{q^9, q^{19}, q^{25}\}$	V_6, V_2, V_1
(2,8)	$\{q^{16}, q^{25}\}$	V_3, V_1
(3,7)	$\{q^{16}, q^{25}\}$	V_4, V_6, V_8, V_2
(7,7)	$\{q^2, q^8, q^{14}, q^{20}, q^{24}, q^{30}\}$	$V_6, V_8, V_2, V_7, V_1, \mathbb{C}$
(7,8)	$\{q^5, q^{23}, q^{27}\}$	V_4, V_7, V_1

Recall that we began studying $V_1 \otimes V_1(w)$ and obtained V_2 and V_7 as subrepresentations. Then we considered $V_1 \otimes V_7(w)$ and obtained V_8 as a subrepresentation. It is clear that multiplicities do not affect our arguments, so we will list the components and will add a sign "$\oplus$mult" at the end to indicate that the remaining factors have already been listed. We continue with

$$V_7 \otimes V_7(w) \cong V_{2\omega_7}^f \oplus V_6^f \oplus V_{\omega_1+\omega_7}^f \oplus V_8^f \oplus V_{2\omega_1}^f \oplus V_2^f \oplus V_7^f \oplus V_1^f \oplus \mathbb{C} \oplus \text{mult},$$

$$V_1 \otimes V_2(w) \cong V_{\omega_1+\omega_2}^f \oplus V_3^f \oplus V_{\omega_1+\omega_7}^f \oplus V_8^f \oplus V_{2\omega_1}^f \oplus V_2^f \oplus V_7^f \oplus V_1^f \oplus \mathbb{C} \oplus \text{mult}.$$

Looking at Table 2 we see that affinizations of $V_{\omega_1+\omega_7}$ and $V_{2\omega_1}$ never have ECC of V_6 nor of V_3 (up to shift). Therefore, the kernel of $\bar{R}_{V_7,V_7}(q^2)$ must be an affinization of V_6^f, while the kernel of $\bar{R}_{V_1,V_2}(q^3)$ must be an affinization of V_3^f. We will not use the knowledge of the "$U_q(\mathfrak{g})$-tail" of V_3 (i.e., the representations of $U_q(\mathfrak{g})$ added to V_3^f to obtain V_3). So, for the next tensor product, we will assume that V_3 has the longest possible "tail". Then we would have

$$\begin{aligned} V_1 \otimes V_3(w) \cong\ & V^f_{\omega_1+\omega_3} \oplus V_4^f \oplus V^f_{\omega_2+\omega_7} \oplus V^f_{\omega_1+\omega_8} \oplus V_6^f \oplus V^f_{\omega_1+\omega_2} \oplus V_3^f \oplus V^f_{\omega_1+\omega_7} \\ & \oplus V_8^f \oplus V_2^f \oplus V^f_{2\omega_1} \oplus V_7^f \oplus V_1^f \oplus V^f_{2\omega_1+\omega_7} \oplus V^f_{2\omega_7} \oplus V^f_{3\omega_1} \oplus \mathbb{C} \oplus \text{mult.} \end{aligned}$$

From Table 2 we see that, at $w = q^4$, this tensor product has the ECC of V_4 and that none of the other components with height-2 highest weight can have this ECC (we use the definition height$(\lambda) = \sum \lambda_i$, for a dominant weight $\lambda = \sum \lambda_i \omega_i$). We still need to check that this is also true for the height-3 highest weight components. The ECC at V_1 for these components are of the form $\xi_{1i}(z/w_1)\xi_{1j}(z/w_2)\xi_{1k}(z/w_3)$. We used the computer software *Mathematica* to check that such products (for the i, j, k we need) will never coincide with $\xi_{14}(u)$. Therefore, we must have an affinization of V_4 as a subrepresentation of $V_1 \otimes V_3(q^4)$. Again assume that V_4 has the longest possible "tail". Then

$$\begin{aligned} V_1 \otimes V_4(w) \cong\ & (V_1^f \otimes V_4^f) \oplus (V_1^f \otimes V^f_{\omega_2+\omega_7}) \oplus (V_1^f \otimes V^f_{\omega_1+\omega_8}) \oplus (V_1^f \otimes V_6^f) \\ & \oplus (V_1^f \otimes V^f_{\omega_1+\omega_2}) \oplus (V_1^f \otimes V_3^f) \oplus (V_1^f \otimes V^f_{\omega_1+\omega_7}) \oplus (V_1^f \otimes V_8^f) \\ & \oplus (V_1^f \otimes V_2^f) \oplus (V_1^f \otimes V^f_{2\omega_1}) \oplus (V_1^f \otimes V_7^f) \oplus (V_1^f \otimes V_1^f) \\ & \oplus (V_1^f \otimes V^f_{2\omega_1+\omega_7}) \oplus (V_1^f \otimes V^f_{2\omega_7}) \oplus (V_1^f \otimes V^f_{3\omega_1}) \oplus V_1^f \oplus V_4 \oplus \text{mult.} \end{aligned}$$

Using Table 2 and the computer one checks that all terms after $V_1^f \otimes V_4^f$ never have ECC of V_5. Now

$$\begin{aligned} V_1^f \otimes V_4^f \cong\ & V^f_{\omega_1+\omega_4} \oplus V_5^f \oplus V^f_{\omega_3+\omega_7} \oplus V^f_{\omega_2+\omega_8} \oplus V^f_{\omega_1+\omega_6} \oplus V^f_{\omega_7+\omega_8} \oplus V^f_{\omega_1+\omega_3} \\ & \oplus V_4^f \oplus V^f_{\omega_2+\omega_7} \oplus V^f_{\omega_1+\omega_8} \oplus V_6^f \oplus V_3^f. \end{aligned}$$

Use Table 2 again to conclude that we obtain an affinization of V_5^f as a subrepresentation of $V_1 \otimes V_4(q^5)$.

Appendix B. The Matrices $M(q)$

• A_n:

$$m_{ij}(q) = \frac{(q^i - q^{-i})(q^{h^\vee - j} - q^{-(h^\vee - j)})}{(q - q^{-1})(q^{h^\vee} - q^{-h^\vee})} = -q^{h^\vee} \frac{(q^i - q^{-i})(q^{h^\vee - j} - q^{-(h^\vee - j)})}{(q - q^{-1})(1 - q^{2h^\vee})}$$

for $1 \le i \le j \le n$.

• B_n: For $i \le j < n$,

$$m_{ij}(q) = \frac{(q^{2i} - q^{-2i})(q^{h^\vee - 2j} + q^{-(h^\vee - 2j)})(q + q^{-1})}{(q^2 - q^{-2})(q^{h^\vee} + q^{-h^\vee})}$$

$$= -q^{2h^\vee} \frac{(q^{h^\vee} - q^{-h^\vee})(q^{2i} - q^{-2i})(q^{h^\vee - 2j} + q^{-(h^\vee - 2j)})}{(q - q^{-1})(1 - q^{4h^\vee})},$$

$$m_{in}(q) = \frac{(q^{2i} - q^{-2i})(q + q^{-1})}{(q^2 - q^{-2})(q^{h^\vee} + q^{-h^\vee})} = -q^{2h^\vee} \frac{(q^{h^\vee} - q^{-h^\vee})(q^{2i} - q^{-2i})}{(q - q^{-1})(1 - q^{4h^\vee})},$$

$$m_{nn}(q) = \frac{(q^{h^\vee + 1} - q^{-(h^\vee + 1)})}{(q^2 - q^{-2})(q^{h^\vee} + q^{-h^\vee})} = -q^{2h^\vee} \frac{(q^{h^\vee} - q^{-h^\vee})(\sum_{k=0}^{h^\vee} (-1)^k q^{h^\vee - 2k})}{(q - q^{-1})(1 - q^{4h^\vee})}.$$

• C_n:

$$m_{ij}(q) = \frac{(q^i - q^{-i})(q^{h^\vee - j} + q^{-(h^\vee - j)})}{(q - q^{-1})(q^{h^\vee} + q^{-h^\vee})}$$

$$= -q^{2h^\vee} \frac{(q^{h^\vee} - q^{-h^\vee})(q^i - q^{-i})(q^{h^\vee - j} + q^{-(h^\vee - j)})}{(q - q^{-1})(1 - q^{4h^\vee})}.$$

for $1 \le i \le j \le n$.

• D_n: For $i, j < n - 1$,

$$m_{ij}(q) = \frac{(q^i - q^{-i})(q^{n-1-j} + q^{-(n-1-j)})}{(q - q^{-1})(q^{n-1} + q^{-(n-1)})}$$

$$= -q^{h^\vee} \frac{(q^{h^\vee/2} - q^{-h^\vee/2})(q^i - q^{-i})(q^{h^\vee/2 - j} + q^{-(h^\vee/2 - j)})}{(q - q^{-1})(1 - q^{2h^\vee})},$$

$$m_{i\, n-1} = m_{in} = \frac{(q^i - q^{-i})}{(q - q^{-1})(q^{n-1} + q^{-(n-1)})}$$

$$= -q^{h^\vee} \frac{(q^{h^\vee/2} - q^{-h^\vee/2})(q^i - q^{-i})}{(q - q^{-1})(1 - q^{2h^\vee})},$$

$$m_{n-1 n} = \frac{(q^{n-2} - q^{-(n-2)})}{(q - q^{-1})(q + q^{-1})(q^{n-1} + q^{-(n-1)})}$$

$$= -q^{h^\vee} \frac{(q^{h^\vee/2} - q^{-h^\vee/2})(q^{h^\vee/2 - 1} - q^{-(h^\vee/2 - 1)})}{(q - q^{-1})(q + q^{-1})(1 - q^{2h^\vee})},$$

$$m_{n-1\, n-1} = m_{nn} = \frac{(q^n - q^{-n})}{(q - q^{-1})(q + q^{-1})(q^{n-1} + q^{-(n-1)})}$$

$$= -q^{h^\vee} \frac{(q^{h^\vee/2} - q^{-h^\vee/2})(q^{h^\vee/2 + 1} - q^{-(h^\vee/2 + 1)})}{(q - q^{-1})(q + q^{-1})(1 - q^{2h^\vee})}.$$

• E_6:

$$\det B(q) = q^6 + q^4 - 1 + q^{-4} + q^{-6}$$

$$\frac{1}{\det B(q)} = \frac{-q^5 + 2q^7 - 2q^9 + q^{11} + q^{13} - 2q^{15} + 2q^{17} - q^{19}}{(q - q^{-1})(1 - q^{24})}.$$

The entries of $M(q)$ are of the form

$$m_{ij}(q) = \frac{n_{ij}(q)}{p(q)} \qquad \text{where} \qquad p(q) = (q - q^{-1})(1 - q^{24}).$$

We now list the numerators n_{ij}:

$$
\begin{aligned}
n_{11}(q) &= n_{55}(q) = -1 + q^2 - q^6 + q^8 + q^{16} - q^{18} + q^{22} - q^{24},\\
n_{12}(q) &= n_{21}(q) = n_{45}(q) = n_{54}(q) = -q + q^3 - q^5 + q^9 + q^{15} - q^{19} + q^{21} - q^{23},\\
n_{13}(q) &= n_{31}(q) = n_{35}(q) = n_{53}(q) = n_{26}(q) = n_{62}(q) = n_{46}(q)\\
&= n_{64}(q) = -q^2 + q^{10} + q^{14} - q^{22},\\
n_{14}(q) &= n_{41}(q) = n_{25}(q) = n_{52}(q) = -q^3 + q^7 - q^9 + q^{11} + q^{13} - q^{15} + q^{17} - q^{21},\\
n_{15}(q) &= n_{51}(q) = -q^4 + q^6 - q^{10} + 2q^{12} - q^{14} + q^{18} - q^{20},\\
n_{16}(q) &= n_{61}(q) = n_{56}(q) = n_{65}(q) = -q^3 + q^5 - q^7 + q^9 + q^{15} - q^{17} + q^{19} - q^{21},\\
n_{22}(q) &= n_{44}(q) = -1 - q^6 + q^8 + q^{10} + q^{14} + q^{16} - q^{18} - q^{24},\\
n_{23}(q) &= n_{32}(q) = n_{34}(q) = n_{43}(q) = -q - q^3 + q^9 + q^{11} + q^{13} + q^{15} - q^{21} - q^{23},\\
n_{24}(q) &= n_{42}(q) = -q^2 - q^4 + q^6 + 2q^{12} + q^{18} - q^{20} - q^{22},\\
n_{33}(q) &= -1 - q^2 - q^4 + q^8 + q^{10} + 2q^{12} + q^{14} + q^{16} - q^{20} - q^{22} - q^{24},\\
n_{36}(q) &= n_{63}(q) = -q - q^5 + q^7 + q^{11} + q^{13} + q^{17} - q^{19} - q^{23},\\
n_{66}(q) &= -1 + q^2 - q^4 + q^8 - q^{10} + 2q^{12} - q^{14} + q^{16} - q^{20} + q^{22} - q^{24}.
\end{aligned}
$$

• E_7:

$$
\det B(q) = q^7 + q^5 - q - q^{-1} + q^{-5} + q^{-7},
$$
$$
\frac{1}{\det B(q)} = \frac{-q^6 + 2q^8 - 2q^{10} + q^{12} + q^{24} - 2q^{26} + 2q^{28} + q^{34} - q^{36}}{(q - q^{-1})(1 - q^{36})}.
$$

The numerators of $m_{ij}(q)$ are:

$$
\begin{aligned}
n_{11}(q) &= -1 + q^2 - q^8 + q^{10} - q^{16} + 2q^{18} - q^{20} + q^{26} - q^{28} + q^{34} - q^{36},\\
n_{12}(q) &= n_{21}(q) = -q + q^3 - q^7 + q^{11} - q^{15} + q^{17} + q^{19} - q^{21} + q^{25} - q^{29} + q^{33} - q^{35},\\
n_{13}(q) &= n_{31}(q) = -q^2 + q^4 - q^6 + q^{12} - q^{14} + q^{16} + q^{20} - q^{22} + q^{24} - q^{30} + q^{32} - q^{34},\\
n_{14}(q) &= n_{41}(q) = n_{27}(q) = n_{72}(q) = n_{36}(q) = n_{63}(q) = -q^3 + q^{15} + q^{21} - q^{33},\\
n_{15}(q) &= n_{51}(q) = n_{26}(q) = n_{26}(q) = -q^4 + q^8 - q^{10} + q^{14} + q^{22} - q^{26} + q^{28} - q^{32},\\
n_{16}(q) &= n_{61}(q) = -q^5 + q^7 - q^{11} + q^{13} + q^{23} - q^{25} + q^{29} - q^{31},\\
n_{17}(q) &= n_{71}(q) = -q^4 + q^6 - q^8 + q^{10} - q^{12} + q^{14} + q^{22} - q^{24} + q^{26} - q^{28} + q^{30} - q^{32},\\
n_{22}(q) &= -1 + q^4 - q^6 - q^8 + q^{10} + q^{12} - q^{14} + 2q^{18} - q^{22}\\
&\quad + q^{24} + q^{26} - q^{28} - q^{30} + q^{32} - q^{36},\\
n_{23}(q) &= n_{32}(q) = -q - q^7 + q^{11} + q^{17} + q^{19} + q^{25} - q^{29} - q^{35},\\
n_{24}(q) &= n_{42}(q) = n_{35}(q) = n_{53}(q) = -q^2 - q^4 + q^{14} + q^{16} + q^{20} + q^{22} - q^{32} - q^{34},\\
n_{25}(q) &= n_{52}(q) = -q^3 - q^5 + q^7 - q^{11} + q^{13} + q^{15} + q^{21} + q^{23} - q^{25} + q^{29} - q^{31} - q^{33},\\
n_{33}(q) &= -1 - q^4 - q^8 + q^{10} + q^{14} + 2q^{18} + q^{22} + q^{26} - q^{28} - q^{32} - q^{36},\\
n_{34}(q) &= n_{43}(q) = -q - q^3 - q^5 + q^{13} + q^{15} + q^{17} + q^{19} + q^{21} + q^{23} - q^{31} - q^{33} - q^{35},\\
n_{37}(q) &= n_{73}(q) = -q^2 - q^6 + q^8 - q^{10} + q^{12} + q^{16} + q^{20} + q^{24} - q^{26} + q^{28} - q^{30} - q^{34},
\end{aligned}
$$

$$\begin{aligned}
n_{44}(q) &= -1 - q^2 - q^4 - q^6 + q^{12} + q^{14} + q^{16} + 2q^{18} + q^{20} + q^{22} \\
&\quad + q^{24} - q^{30} - q^{32} - q^{34} - q^{36}, \\
n_{45}(q) &= n_{54}(q) = -q - q^3 - q^7 + q^{11} + q^{15} + q^{17} + q^{19} + q^{21} + q^{25} - q^{29} - q^{33} - q^{35}, \\
n_{46}(q) &= n_{64}(q) = n_{57}(q) = n_{75}(q) = -q^2 - q^8 + q^{10} + q^{16} + q^{20} + q^{26} - q^{28} - q^{34}, \\
n_{47}(q) &= n_{74}(q) = -q - q^5 + q^{13} + q^{17} + q^{19} + q^{23} - q^{31} - q^{35}, \\
n_{55}(q) &= -1 - q^6 + q^{12} + 2q^{18} + q^{24} - q^{30} - q^{36}, \\
n_{56}(q) &= n_{65}(q) = -q + q^3 - q^5 + q^{13} - q^{15} + q^{17} + q^{19} - q^{21} + q^{23} - q^{31} + q^{33} - q^{35}, \\
n_{66}(q) &= -1 + q^2 - q^6 + q^8 - q^{10} + q^{12} - q^{16} + 2q^{18} - q^{20} \\
&\quad + q^{24} - q^{26} + q^{28} - q^{30} + q^{34} - q^{36}, \\
n_{67}(q) &= n_{76}(q) = -q^3 + q^5 - q^7 + q^{11} - q^{13} + q^{15} + q^{21} - q^{23} + q^{25} - q^{29} + q^{31} - q^{33}, \\
n_{77}(q) &= -1 + q^2 - q^4 + q^{14} - q^{16} + 2q^{18} - q^{20} + q^{22} - q^{32} + q^{34} - q^{36}.
\end{aligned}$$

• E_8:

$$\det B(q) = q^8 + q^6 - q^2 - 1 - q^{-2} + q^{-6} + q^{-8}, \qquad \frac{1}{\det B(q)} = \frac{p(q)}{(q - q^{-1})(1 - q^{60})}$$

where

$$\begin{aligned}
p(q) =& - q^7 + 2q^9 - 2q^11 + q^{13} - q^{17} + 2q^{19} - 2q^{21} + q^{23} \\
&+ q^{37} - 2q^{39} + 2q^{41} - 2q^{43} + q^{47} - 2q^{49} + 2q^{51} - q^{53}.
\end{aligned}$$

The numerators of $m_{ij}(q)$ are

$$\begin{aligned}
n_{11} &= -1 + q^2 - q^{10} + q^{12} - q^{18} + q^{20} - q^{28} + 2q^{30} - q^{32} + q^{40} \\
&\quad - q^{42} + q^{48} - q^{50} + q^{58} - q^{60}, \\
n_{12} &= -q + q^3 - q^9 + q^{13} - q^{17} + q^{21} - q^{27} + q^{29} + q^{31} - q^{33} \\
&\quad + q^{39} - q^{43} + q^{47} - q^{51} + q^{57} - q^{59}, \\
n_{13} &= -q^2 + q^4 - q^8 + q^{14} - q^{16} + q^{22} - q^{26} + q^{28} + q^{32} - q^{34} \\
&\quad + q^{38} - q^{44} + q^{46} - q^{52} + q^{56} - q^{58}, \\
n_{14} &= n_{78} = -q^3 + q^5 - q^7 + q^{23} - q^{25} + q^{27} + q^{33} - q^{35} + q^{37} - q^{53} + q^{55} - q^{57}, \\
n_{15} &= n_{28} = n_{37} = -q^4 - q^{14} + q^{16} + q^{26} + q^{34} + q^{44} - q^{46} - q^{56}, \\
n_{16} &= n_{27} = -q^5 + q^9 - q^{11} + q^{19} - q^{21} + q^{25} + q^{35} - q^{39} + q^{41} - q^{49} + q^{51} - q^{55}, \\
n_{17} &= -q^6 + q^8 - q^{12} + q^{14} - q^{16} + q^{18} - q^{22} + q^{24} + q^{36} - q^{38} \\
&\quad + q^{42} - q^{44} + q^{46} - q^{48} + q^{52} - q^{54}, \\
n_{18} &= -q^5 + q^7 - q^9 + q^{11} - q^{13} + q^{17} - q^{19} + q^{21} - q^{23} + q^{25} \\
&\quad + q^{35} - q^{37} + q^{39} - q^{41} + q^{43} - q^{47} + q^{49} - q^{51} + q^{53} - q^{55},
\end{aligned}$$

$$n_{22} = -1 + q^4 - q^8 - q^{10} + q^{12} + q^{14} - q^{16} - q^{18} + q^{20} + q^{22} - q^{26} + 2q^{30} - q^{34} + q^{38} + q^{40} - q^{42} - q^{44} + q^{46} + q^{48} - q^{50} - q^{52} + q^{56} - q^{60},$$

$$n_{23} = -q + q^5 - q^7 - q^9 + q^{13} - q^{17} + q^{21} + q^{23} - q^{25} + q^{29} + q^{31} - q^{35} + q^{37} + q^{39} - q^{43} + q^{47} - q^{51} - q^{53} + q^{55} - q^{59},$$

$$n_{24} = n_{57} = n_{68} = -q^2 - q^8 + q^{22} + q^{28} + q^{32} + q^{38} - q^{52} - q^{58},$$

$$n_{25} = n_{36} = -q^3 - q^5 - q^{13} + q^{17} + q^{25} + q^{27} + q^{33} + q^{35} + q^{43} - q^{47} - q^{55} - q^{57},$$

$$n_{26} = -q^4 - q^6 + q^8 - q^{12} + q^{18} - q^{22} + q^{24} + q^{26} + q^{34} + q^{36} - q^{38} + q^{42} - q^{48} + q^{52} - q^{54} - q^{56},$$

$$n_{33} = -1 - q^8 - q^{10} + q^{12} - q^{18} + q^{22} + 2q^{30} + q^{38} + q^{40} - q^{42} + q^{48} - q^{50} - q^{52} - q^{60},$$

$$n_{34} = n_{58} = -q - q^5 - q^9 + q^{21} + q^{25} + q^{29} + q^{31} + q^{35} + q^{39} - q^{51} - q^{55} - q^{59},$$

$$n_{35} = -q^2 - q^4 - q^6 - q^{12} + q^{18} + q^{24} + q^{26} + q^{28} + q^{32} + q^{34} + q^{36} + q^{42} - q^{48} - q^{54} - q^{56} - q^{58},$$

$$n_{38} = -q^3 - q^7 + q^9 - q^{11} + q^{19} - q^{21} + q^{23} + q^{27} + q^{33} + q^{37} - q^{39} + q^{41} - q^{49} + q^{51} - q^{53} - q^{57},$$

$$n_{44} = -1 - q^4 - q^6 - q^{10} + q^{20} + q^{24} + q^{26} + 2q^{30} + q^{34} + q^{36} + q^{40} - q^{50} - q^{54} - q^{56} - q^{60},$$

$$n_{45} = -q - q^3 - q^5 - q^7 - q^{11} + q^{19} + q^{23} + q^{25} + q^{27} + q^{29} + q^{31} + q^{33} + q^{35} + q^{37} + q^{41} - q^{49} - q^{53} - q^{55} - q^{57} - q^{59},$$

$$n_{46} = -q^2 - q^4 - q^8 - q^{14} + q^{16} + q^{22} + q^{26} + q^{28} + q^{32} + q^{34} + q^{38} + q^{44} - q^{46} - q^{52} - q^{56} - q^{58},$$

$$n_{47} = -q^3 - q^9 + q^{11} - q^{13} + q^{17} - q^{19} + q^{21} + q^{27} + q^{33} + q^{39} - q^{41} + q^{43} - q^{47} + q^{49} - q^{51} - q^{57},$$

$$n_{48} = -q^2 - q^6 - q^{12} + q^{14} - q^{16} + q^{18} + q^{24} + q^{28} + q^{32} + q^{36} + q^{42} - q^{44} + q^{46} - q^{48} - q^{54} - q^{58},$$

$$n_{55} = -1 - q^2 - q^4 - q^6 - q^8 - q^{10} + q^{20} + q^{22} + q^{24} + q^{26} + q^{28} + 2q^{30} + q^{32} + q^{34} + q^{36} + q^{38} + q^{40} - q^{50} - q^{52} - q^{54} - q^{56} - q^{58} - q^{60},$$

$$n_{56} = -q - q^3 - q^7 - q^9 + q^{21} + q^{23} + q^{27} + q^{29} + q^{31} + q^{33} + q^{37} + q^{39} - q^{51} - q^{53} - q^{57} - q^{59}$$

$$n_{66} = -1 - q^6 - q^{10} + q^{14} + q^{16} + q^{20} + q^{24} + 2q^{30} + q^{36} + q^{40} - q^{44} + q^{46} - q^{50} - q^{54} - q^{60},$$

$$n_{67} = -q + q^3 - q^5 - q^{11} + q^{13} - q^{17} + q^{19} + q^{25} - q^{27} + q^{29} + q^{31} - q^{33} + q^{35} + q^{41} - q^{43} + q^{47} - q^{49} - q^{55} + q^{57} - q^{59},$$

$$n_{77} = -1 + q^2 - q^6 + q^8 - q^{10} + q^{14} - q^{16} + q^{20} - q^{22} + q^{24} - q^{28} + 2q^{30} - q^{32} + q^{36} - q^{38} + q^{40} - q^{44} + q^{46} - q^{50} + q^{52} - q^{54} + q^{58} - q^{60},$$

$$n_{88} = -1 + q^2 - q^4 - q^{10} + q^{12} - q^{14} + q^{16} - q^{18} + q^{20} + q^{26} - q^{28} + 2q^{30} - q^{32} + q^{34} + q^{40} - q^{42} + q^{44} - q^{46} + q^{48} - q^{50} - q^{56} + q^{58} - q^{60}.$$

• F_4:

$$B(q) = \begin{pmatrix} [2]_q & -1 & 0 & 0 \\ -1 & [2]_q & -[2]_q & 0 \\ 0 & -[2]_q & [4]_q & -[2]_q \\ 0 & 0 & -[2]_q & [4]_q \end{pmatrix},$$

$$\det B(q) = q^8 + 2q^6 + q^4 - q^2 - 2 - q^{-2} + q^{-4} + 2q^{-6} + q^{-8},$$

$$\frac{1}{\det B(q)} = \frac{-q^6 + 2q^8 - 2q^{10} + q^{12} + q^{24} - 2q^{26} + 2q^{28} - q^{30}}{(q+q^{-1})(q-q^{-1})(1-q^{36})},$$

and the entries of $M(q)$ are for $m_{ij}(q) = \frac{n_{ij}(q)}{(q-q^{-1})(1-q^{36})}$, where the numerators are

$$\begin{aligned}
n_{11}(q) &= -1 + q^2 - q^6 + q^8 - q^{10} + q^{12} - q^{16} + 2q^{18} - q^{20} + q^{24} \\
&\quad - q^{26} + q^{28} - q^{30} + q^{34} - q^{36}, \\
n_{12}(q) &= n_{21}(q) = -q + q^3 - q^5 + q^{13} - q^{15} + q^{17} + q^{19} - q^{21} + q^{23} - q^{31} + q^{33} - q^{35}, \\
n_{13}(q) &= n_{31}(q) = -q^2 - q^8 + q^{10} + q^{16} + q^{20} + q^{26} - q^{28} - q^{34}, \\
n_{14}(q) &= n_{41}(q) = -q^4 + q^8 - q^{10} + q^{14} + q^{22} - q^{26} + q^{28} - q^{32}, \\
n_{22}(q) &= -1 - q^6 + q^{12} + 2q^{18} + q^{24} - q^{30} - q^{36}, \\
n_{23}(q) &= n_{32}(q) = -q - q^3 - q^7 + q^{11} + q^{15} + q^{17} + q^{19} + q^{21} + q^{25} - q^{29} - q^{33} - q^{35}, \\
n_{24}(q) &= n_{42}(q) = -q^3 - q^5 + q^7 - q^{11} + q^{13} + q^{15} + q^{21} + q^{23} - q^{25} + q^{29} - q^{31} - q^{33}, \\
n_{33}(q) &= -1 - q^2 - q^4 - q^6 + q^{12} + q^{14} + q^{16} + 2q^{18} + q^{20} + q^{22} \\
&\quad + q^{24} - q^{30} - q^{32} - q^{34} - q^{36}, \\
n_{34}(q) &= n_{43}(q) = -q^2 - q^4 + q^{14} + q^{16} + q^{20} + q^{22} - q^{32} - q^{34}, \\
n_{44}(q) &= -1 + q^4 - q^6 - q^8 + q^{10} + q^{12} - q^{14} + 2q^{18} - q^{22} + q^{24} \\
&\quad + q^{26} - q^{28} - q^{30} + q^{32} - q^{36}.
\end{aligned}$$

• G_2:

$$B(q) = \begin{pmatrix} [2]_q & -[3]_q \\ -[3]_q & [6]_q \end{pmatrix}, \qquad D(q) = \begin{pmatrix} 1 & 0 \\ 0 & [3]_q \end{pmatrix},$$

$$\det B(q) = q^6 + q^4 - 1 + q^{-4} + q^{-6},$$

$$\frac{1}{\det B(q)} = -\frac{(q-q^{-1})(q^7 + q^{11} + q^{13} + q^{17})}{(1-q^{24})} = \frac{q^6 - q^8 + q^{10} - q^{14} + q^{16} - q^{18}}{(1-q^{24})},$$

$$M(q) = \frac{1}{\det B(q)} \begin{pmatrix} [6]_q & [3]_q^2 \\ [3]_q^2 & [2]_q[3]_q^2 \end{pmatrix} = \frac{1}{\det B(q)} \begin{pmatrix} [6]_q & [3]_q^2 \\ [3]_q^2 & (q+q^{-1})[3]_q^2 \end{pmatrix}.$$

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Massachussets Institute of Technology, 77 Massachussets Ave., Room 2-176, Cambridge, Massachusetts 02139

E-mail address: etingof@math.mit.edu

IMECC/UNICAMP, Caixa Postal: 6065, CEP: 13083-970, Campinas SP Brazil

E-mail address: adrianoam@ime.unicamp.br