REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 7, Pages 374–403 (September 10, 2003)
S 1088-4165(03)00204-8

CHARACTER SHEAVES ON DISCONNECTED GROUPS, I

G. LUSZTIG

ABSTRACT. In this paper we begin the study of character sheaves on a not necessarily connected reductive algebraic group G. One of the themes of this paper is the construction of a decomposition of G into finitely many strata and of a family of local systems on each stratum.

INTRODUCTION

Our aim in this series of papers is to develop a theory of character sheaves on a not necessarily connected reductive algebraic group G. In the case of connected groups such a theory appeared in [L2] and [L3]. An extension to disconnected groups has been sketched in [L4] without proofs; here we try to give a fuller and more precise treatment and to supply the proofs that were missing in [L4]. The main object of the theory, the character sheaves of G, are certain simple perverse sheaves on G, equivariant with respect to the conjugation action of the identity component of G. At least for connected G (over a finite field) the character sheaves are intimately related with the characters of irreducible representations of the group of rational points of G and such a relationship is also expected in the disconnected case. The theory of character sheaves on G is also crucial for the classification of "unipotent representations" of simple p-adic groups [L5], and here one is forced to allow G to be disconnected if one wants to include p-adic groups that are not inner forms of split groups.

The present paper tries to extend parts of [L2, §1-§4] from the connected case to the general case. We develop enough background so that we are able to define (see 6.7) the notion of "admissible complex" on G, one of the two incarnations of the character sheaves of G. One of the themes of this paper is the construction of a decomposition of G into finitely many strata (generalizing a construction in [L2, 3.1]); see §3. Each stratum is a locally closed, irreducible, smooth subvariety of G. Each stratum is a union of G^0-conjugacy classes of fixed dimension; more precisely, the centralizers of two points in the same stratum have G^0-conjugate identity components. Also, the closure of any stratum is a union of strata. Each stratum carries some natural local systems which extend to intersection cohomology complexes on the closure, which we also describe by a direct image construction, using the dimension estimates in §4.

Acknowledgments. I wish to thank F. Digne and J. Michel for some useful comments on [L4].

Received by the editors May 14, 2003.
2000 *Mathematics Subject Classification.* Primary 20G99.
This work was supported in part by the National Science Foundation.

©2003 American Mathematical Society

Contents

1. Preliminaries on reductive groups

1.1. We fix an algebraically closed field $\mathbf{k}$. All algebraic varieties are assumed to be over $\mathbf{k}$. All algebraic groups are assumed to be affine.

We shall use the following notation. If H is a group, the centre of H is denoted by $\mathcal{Z}_H$; if H' is a subgroup of H, let $N_H(H') = \{h \in H; hH'h^{-1} = H'\}$. If, in addition, H'' is a subgroup of H, let $Z_{H'}(H'') = \{h' \in H'; h'h'' = h''h' \quad \forall h'' \in H''\}$; if $h \in H$, let $Z_{H'}(h) = \{h' \in H'; h'h = hh'\}$. If H is an algebraic group, we denote by H^0 the identity component of H and we set $H_{ss} = H/\mathcal{Z}^0_{H^0}$; for $h \in H$ we denote by h_s (resp. h_u) the semisimple (resp. unipotent) part of h, so that $h = h_s h_u = h_u h_s$. If X is a subset of H we set $X_s = \{h_s; h \in X\}$. The unipotent radical of H (assumed to be connected) is denoted by U_H.

We fix an algebraic group G such that G^0 is reductive. (We then say that G is reductive.) Let $\mathfrak{g} = \text{Lie } G$.

1.2. Let T be a torus and let $f : T \to T$ be an automorphism of finite order with fixed point set T^f. We show that

(a) *the homomorphism* $(T^f)^0 \times T \to T, (t, x) \mapsto xtf(x)^{-1}$ *is surjective.*

This can be reduced to an analogous statement about a finite dimensional $\mathbf{Q}$-vector space V and a linear map $\phi : V \to V$ of finite order: the linear map $\text{Ker}(\phi - 1) \times V \to V, (w, v) \mapsto w + v - \phi(v)$ is surjective. Alternatively, according to [B, 11.6] the homomorphism $T^f \times T \to T, (t, x) \mapsto xtf(x)^{-1}$ is surjective. Then automatically the restriction $(T^f \times T)^0 \to T^0$ is surjective and (a) holds.

1.3. Let $g \in G$. We show that

(a) *the homomorphism* $(\mathcal{Z}^0_{G^0} \cap Z_G(g))^0 \times \mathcal{Z}^0_{G^0} \to \mathcal{Z}^0_{G^0}, (t, x) \mapsto xtgx^{-1}g^{-1}$ *is surjective.*

$\text{Ad}(g) : \mathcal{Z}^0_{G^0} \to \mathcal{Z}^0_{G^0}$ is of finite order since some power of g is in G^0. Therefore, (a) is a special case of 1.2(a).

1.4. Let $g \in G$. Then g normalizes some Borel of G^0; see [St, 7.2]. Following [St, 9], we say that g is *quasi-semisimple* if there exist a Borel B of G^0 and a maximal torus T of B such that $gBg^{-1} = B, gTg^{-1} = T$. If g is semisimple, then it is quasi-semisimple; see [St, 7.5, 7.6]. More generally, by an argument similar to that in [St, 7.6], we see that

(a) *if g is semisimple and P is a parabolic of G^0 such that $gPg^{-1} = P$, then there exists a Levi L of P such that $gLg^{-1} = L$.*

Here are some further results.

(b) *If g is semisimple or, more generally, quasi-semisimple, then $Z_G(g)$ is reductive. Moreover, if B is a Borel of G^0 such that $gBg^{-1} = B$, then $B \cap Z_G(g)^0$ is a Borel of $Z_G(g)^0$.* See [Sp, II, 1.17, 2.21].

(c) *g is quasi-semisimple if and only if g_u is quasi-semisimple in the reductive group $Z_G(g_s)$.* See [Sp, II, 2.22].

(d) *If g is quasi-semisimple and T_1 is a maximal torus of $Z_G(g)^0$, then there is a unique maximal torus T of G^0 such that $T_1 \subset T$. See* [Sp, II, 1.15]. *(We have necessarily that $T = Z_{G^0}(T_1)$ and $T_1 = (T \cap Z_G(g))^0$.)*

(e) *g is quasi-semisimple if and only if the G^0-conjugacy class of g is closed in G.* See [Sp, II, 1,.15].

1.5. Assume that $s \in G$ is semisimple. Let T_1 be a maximal torus of $Z_G(s)^0$. Clearly, some power of s is in $\mathcal{Z}_{Z_G(s)^0}$ hence in T_1 (since $Z_G(s)^0$ is reductive). Thus, the subgroup $\langle s\rangle T_1$ generated by s and T_1 is a closed diagonalizable subgroup of G with identity component T_1. For any character $\alpha : \langle s\rangle T_1 \to \mathbf{k}^*$ let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; \mathrm{Ad}(a)x = \alpha(a)x \quad \forall a \in \langle s\rangle T_1\}$. Then $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$. Let R be the set of all α such that $\alpha \neq 1, \mathfrak{g}_\alpha \neq 0$. For $\alpha \in R$ we have necessarily that $\dim \mathfrak{g}_a = 1$. (Compare [L5, 6.18].) We have $\mathfrak{g}_1 = \mathrm{Lie}\, T$ where $T = Z_{G^0}(T_1)$.

1.6. Let $\bar{E}(G^0)$ be the set of all pairs (B,T) where B is a Borel of G^0 and T is a maximal torus of B. The group $\mathrm{Aut}(G^0)$ of automorphisms of G^0 acts naturally on $\bar{E}(G^0)$. It is known that to G^0 one can associate canonically an algebraic variety $E(G^0)$ whose points are called "épinglages" with the following properties.

(i) There is a natural action of $\mathrm{Aut}(G^0)$ on $E(G^0)$ which restricts to a free transitive action of the group of inner automorphisms of G^0 on $E(G^0)$.

(ii) There is a natural $\mathrm{Aut}(G^0)$-equivariant map $p : E(G^0) \to \bar{E}(G^0)$.

1.7. Let $g \in G$. Assume that $\mathrm{Ad}(g)e = e$ where $e \in E(G^0)$. Let $(B,T) = p(e)$. Then $gBg^{-1} = B, gTg^{-1} = T$ hence g is quasi-semisimple. In particular, $Z_G(g)$ is reductive. The following results are known.

(a) $Z_B(g)^0$ is a Borel of $Z_G(g)^0$.

(b) $\mathcal{Z}^0_{Z_G(g)^0} = (\mathcal{Z}_{G^0} \cap Z_G(g))^0$;

(c) *$P \mapsto Z_P(g)^0$ is a bijection between the set of parabolics of G^0 that contain B and are normalized by g and the set of parabolics of $Z_G(g)^0$ that contain $Z_B(g)^0$; moreover, if L is a Levi of P, then $Z_L(g)^0$ is a Levi of $Z_P(g)^0$.*

1.8. Assume that $u \in G$ is unipotent and $(B,T) \in \bar{E}(G^0)$ is such that $uBu^{-1} = B, uTu^{-1} = T$. We show that

(a) *there exists $e \in E(G^0)$ such that $p(e) = (B,T), \mathrm{Ad}(u)e = e$.*

Let $E' = \{e \in E(G^0); p(e) = (B,T)\}$. Then T acts transitively on E'. Moreover, $\mathrm{Ad}(u)E' = E'$. Hence the subgroup $\langle u\rangle T$ generated by u and T acts on E'. Since $u \in N_G T$ and $(N_G T)^0 = T$, some power of u belongs to T. Hence $\langle u\rangle T$ is a closed subgroup of G with identity component T. Let $e_0 \in E'$. We have $\mathrm{Ad}(u)e_0 = \mathrm{Ad}(t^{-1})e_0$ for some $t \in T$. Thus, tu belongs to the stabilizer of e_0 in $\langle u\rangle T$, a closed subgroup of $\langle u\rangle T$. Then the unipotent part $(tu)_u$ also belongs to this stabilizer that is, $\mathrm{Ad}((tu)_u)e_0 = e_0$. The image of $(tu)_s$ in the unipotent group $(\langle u\rangle T)/T$ must be 1. Hence $(tu)_u$ has the same image as tu or as u. Thus, $(tu)_u = t'u$ where $t' \in T$ and we have $\mathrm{Ad}(t'u)e_0 = e_0$ with $t'u$ unipotent. By 1.2(a) with $f : T \to T, f(x) = uxu^{-1}$ (of finite order), we have $t' = t_2 t_1 u t_1^{-1} u^{-1}$ for some $t_1, t_2 \in T$ with $ut_2 = t_2 u$. Then $t'u = t_2 t_1 u t_1^{-1}$. Since t_2 is semisimple and it commutes with $t_1 u t_1^{-1}$ which is unipotent, we see that $t_1 u t_1^{-1} = (t'u)_u = t'u$. Thus $\mathrm{Ad}(t_1 u t_1^{-1})e_0 = e_0$ hence $\mathrm{Ad}(u)e = e$ where $e = \mathrm{Ad}(t_1)^{-1}e_0$. This proves (a).

1.9. Let D be a connected component of G which contains some unipotent element of G. Then

(a) *D contains a unique closed unipotent G^0-conjugacy class; this is the set of unipotent, quasi-semisimple elements in D.* See [Sp, II, 2.21].

1.10. Let P be a parabolic of G^0 and let L be a Levi of P. Let $g \in N_G L \cap N_G P$. We show that

(a) $Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0) = L$.

Let $L' = Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0)$. Then L' is a reductive, connected subgroup of G^0 and $L \subset L'$. Moreover, $P \cap L'$ is a parabolic subgroup of L' with Levi L. We have $gL'g^{-1} = L'$, $g(P \cap L')g^{-1} = P \cap L'$. If (a) is true for $N_G L', P \cap L', L, g$ instead of G, L, P, g, then we would have $Z_{L'}((\mathcal{Z}_L \cap Z_L(g))^0) = L$. Hence $L' = L$. Thus, to prove (a), we may assume that $L' = G^0$ and we must show that $L = G^0$. Replacing g by a left L-translate, we may assume that there exists $(B_1, T) \in \bar{E}(L)$ such that $gB_1g^{-1} = B_1, gTg^{-1} = T$. Since g normalizes U_P, it also normalizes the Borel $B = B_1 U_P$ of G^0. Let $e \in E(G^0)$ be such that $p(e) = (B, T)$. Then $p(\mathrm{Ad}(g)e) = (B, T)$. We can find $g_0 \in G^0$ such that $\mathrm{Ad}(g)e = \mathrm{Ad}(g_0)e$ hence $(B, T) = (g_0 B g_0^{-1}, g_0 T g_0^{-1})$ and $g_0 \in T$, $\mathrm{Ad}(g_0^{-1}g)e = e$. Replacing g by $g_0^{-1}g$ we may assume, in addition, that $\mathrm{Ad}(g)e = e$. Then automatically $\mathrm{Ad}(g)$ fixes an épinglage of L which lies over $(B_1, T) \in \bar{E}(L)$. By 1.7(b) for (g, G) and (g, L), we have

$$(\mathcal{Z}_{G^0} \cap Z_G(g))^0 = \mathcal{Z}^0_{Z_{G^0}(g)^0}, (\mathcal{Z}_L \cap Z_L(g))^0 = \mathcal{Z}^0_{Z_L(g)^0}.$$

Since $L' = G^0$, we have $(\mathcal{Z}_L \cap Z_L(g))^0 \subset (\mathcal{Z}_{G^0} \cap Z_G(g))^0$, hence $\mathcal{Z}^0_{Z_L(g)^0} \subset \mathcal{Z}^0_{Z_{G^0}(g)^0}$. Since $Z_L(g)^0$ is a Levi of a parabolic of $Z_{G^0}(g)^0$ (see 1.7(c)), we have $Z_L(g)^0 = Z_{Z_{G^0}(g)^0}(\mathcal{Z}^0_{Z_L(g)^0})$ hence $Z_L(g)^0 = Z_{G^0}(g)^0$. It follows that $Z_P(g)^0 = Z_{G^0}(g)^0$. Using 1.7(c) we deduce that $P = G^0$ hence $L = G^0$. This proves (a).

1.11. Let P be a parabolic of G^0. Let $s \in N_G P$ be semisimple. Then

(a) *$U_P \cap Z_G(s)$ is connected.*

Since s normalizes U_P, this follows from [B, 9.8].

1.12. Let P be a parabolic of G^0. Let $s \in N_G P$ be semisimple. Choose a Levi L of P such that $s \in N_G L$. (See 1.4(a).) Let $Q = Z_G(s)^0 \cap P$. We show that

(a) *Q is a parabolic of $Z_G(s)^0$ with Levi $Z_L(s)^0 = Z_G(s)^0 \cap L$ and $U_Q = U_P \cap Z_G(s)$;*

(b) *$Z_L(s)^0$ and $Z_G(s)^0$ have a common maximal torus and $\mathcal{Z}_{Z_G(s)^0} \subset \mathcal{Z}_{Z_L(s)^0}$.*

We prove (a). By 1.4(b) (for G or for $N_G L$), $Z_G(s)$ and $Z_{N_G L}(s)$ are reductive. Since the image of s in the reductive group $N_G P/U_P$ is semisimple, it normalizes some Borel of P/U_P; taking inverse image under $P \to P/U_P$ we see that s normalizes some Borel B of P. Using 1.4(b) we see that $B \cap Z_G(s)^0$ is a Borel of $Z_G(s)^0$. Since $B \cap Z_G(s)^0 \subset Q \subset Z_G(s)^0$, we see that Q is a parabolic of $Z_G(s)^0$.

Now $U_P \cap Z_G(s) = U_P \cap Z_G(s)^0$ (see 1.11) is a normal unipotent subgroup of Q, hence it is contained in U_Q. Since L has finite index in $N_G L$, $Z_L(s)$ has finite index in $Z_{N_G L}(s)$ hence $Z_L(s)$ is reductive. Let $y \in Q$. Then $y \in P$, hence we can write uniquely $y = xu$ where $x \in L, u \in U_P$. Since $sys^{-1} = y$ we have $y = (sxs^{-1})(sus^{-1})$ and $sxs^{-1} \in L$, $sus^{-1} \in U_P$. By uniqueness we have $sxs^{-1} = x, sus^{-1} = u$. Thus, $x \in Z_L(s)$, $u \in U_P \cap Z_G(s)$. The map $y \mapsto x$ is a morphism of algebraic groups $f : Q \to Z_L(s)$. Since Q is connected, $f(Q)$ is contained in $Z_L(s)^0$. We see that $Q = Z_L(s)^0(U_P \cap Z_G(s))$. Clearly, $Z_L(s)^0 \cap (U_P \cap Z_G(s)) = \{1\}$. Since

$Z_L(s)^0$ is reductive and $U_P \cap Z_G(s) \subset U_Q$, we have $U_P \cap Z_G(s) = U_Q$. Clearly, $Z_L(s)^0 \subset Z_G(s)^0 \cap L$. Conversely, let $x \in Z_G(s)^0 \cap L$. Then $x \in Q$, hence $x = x'x''$ with $x' \in Z_L(s)^0, x'' \in U_Q$ and $x'^{-1}x = x'' \in (Z_G(s)^0 \cap L) \cap U_Q = \{1\}$. Hence $x = x'$ and $Z_L(s)^0 = Z_G(s)^0 \cap L$. This proves (a).

We prove (b). Let T_1 be a maximal torus of $Z_L(s)^0$. Let T_2 be a maximal torus of $Z_G(s)^0$ containing T_1. By 1.4(d) (for $N_G L$ instead of G), $\tilde{T} := Z_L(T_1)$ is a maximal torus of L. By 1.4(d) (for G), $T := Z_{G^0}(T_2)$ is a maximal torus of G^0. Now $(\mathcal{Z}_L \cap Z_L(s))^0$ is contained in $\mathcal{Z}_{Z_L(s)^0}$, hence is contained in T_1. Hence $Z_{G^0}(T_1) \subset Z_{G^0}((\mathcal{Z}_L \cap Z_L(s))^0) = L$ where the last equality comes from 1.10(a). Since $T \subset Z_{G^0}(T_1)$, it follows that $T \subset L$ and since $T_1 \subset T$ and T is commutative, we have $T \subset Z_L(T_1) = \tilde{T}$. Since $\tilde{T}, T$ are maximal tori of G^0, we must have $\tilde{T} = T$. By 1.4(d) we have $T_1 = Z_{\tilde{T}}(s)^0$, $T_2 = Z_T(s)^0$. It follows that $T_1 = T_2$. This proves the first assertion of (b). Since $Z_G(s)^0$ is reductive, its centre is contained in any maximal torus of $Z_G(s)^0$, in particular, in T_2. Since $T_1 = T_2$ and $T_1 \subset Z_L(s)^0$, we see that $\mathcal{Z}_{Z_G(s)^0} \subset Z_L(s)^0$ and (b) follows.

1.13. Let P be a parabolic of G^0. Let $s \in N_G P$ be semisimple. Assume that $Z_G(s)^0 \subset P$. We show that

(a) *there is a unique Levi L of P such that $Z_G(s)^0 \subset L$. We have $s \in N_G L$.*

Let T_0 be a maximal torus of $Z_G(s)^0$. Let T be a maximal torus of P containing T_0. By 1.4(d), $Z_{G^0}(T_0)$ is a maximal torus of G^0. It contains T, hence it is equal to T. Clearly, $Z_{G^0}(T_0)$ is normalized by s; hence $sTs^{-1} = T$. Let L be the unique Levi of P such that $T \subset L$. Now sLs^{-1} is a Levi of $sPs^{-1} = P$ containing $sTs^{-1} = T$. By uniqueness, we have $sLs^{-1} = L$. By 1.12(a), $U_P \cap Z_G(s)^0$ equals the unipotent radical of $Z_P(s)^0 = Z_G(s)^0$ hence it is 1. This implies, by 1.12(a), that $Z_P(s)^0$ is a parabolic of $Z_G(s)^0$ with Levi $Z_L(s)^0$ and with unipotent radical $\{1\}$. Hence $Z_P(s)^0 = Z_G(s)^0 = Z_L(s)^0$. In particular, $Z_G(s)^0 \subset L$. This proves the existence of L. Assume now that L' is another Levi of P such that $Z_G(s)^0 \subset L'$. Since $T_0 \subset Z_G(s)^0$, we have $T_0 \subset L'$. Hence T_0 is contained in a maximal torus T' of L'. Now T' is also a maximal torus of G^0. Since T_0 is contained in a unique maximal torus of G^0 (see 1.4(d)) we have $T = T'$. Thus, L, L' are Levi subgroups of P containing a common maximal torus of P. It follows that $L = L'$. By the uniqueness of L, we have $L = sLs^{-1}$. This proves (a).

1.14. Let $g \in G$ be quasi-semisimple and let T_1 be a maximal torus of $Z_G(g)^0$. Let $\mathcal{N} = \{n \in G^0; ngT_1n^{-1} = gT_1\}$. We show that

(a) $\mathcal{N}^0 = T_1$, *hence $\mathcal{N}/T_1$ is finite;*

(b) *any element of gT_1 is quasi-semisimple;*

(c) *any quasi-semisimple element g' in gG^0 is G^0-conjugate to some element in gT_1;*

(d) *two elements $g', g'' \in gT_1$ are in the same G^0-conjugacy class if and only if they are in the same $\mathcal{N}/T_1$-orbit on gT_1 for the $\mathcal{N}/T_1$-action induced by the conjugation action of $\mathcal{N}$ on gT_1.*

(Closely related results for nonconnected compact Lie groups appear in [D].)

We prove (a). Let T be the unique maximal torus of G^0 such that $T_1 \subset T$. We have $T_1 = (T \cap Z_G(g))^0$. (See 1.4(d).) If $n \in \mathcal{N}$, then $ngn^{-1} = g\tau$ with $\tau \in T_1$ and $gT_1 = ngT_1n^{-1} = g\tau nT_1n^{-1}$, hence $nT_1n^{-1} = T_1$. Thus, $\mathcal{N} \subset N_{G^0}(T_1)$. Since $\mathcal{N}^0 \subset N_{G^0}(T_1)$, we must have $\mathcal{N}^0 \subset Z_{G^0}(T_1) = T$ by a standard rigidity argument. If $n \in \mathcal{N}^0$, then $ngn^{-1} = gt_n$ with $t_n \in T_1$; since $n \in Z_{G^0}(T_1)$, we see that $n \mapsto t_n$ is a morphism of algebraic groups $f : \mathcal{N}^0 \to T_1$. Clearly, we can find $k \ge 1$ such that g^k is in $\mathcal{Z}_{Z_G(s)^0}$, hence in T_1 (since $Z_G(s)^0$ is reductive). Then $g^k = ng^kn^{-1} = (ngn^{-1})^k = (gt_n)^k = g^kt_n^k$, hence $t_n^k = 1$ for all $n \in \mathcal{N}^0$. Thus, $f(\mathcal{N}^0)$ is contained in a finite subgroup of T_1; being connected, it is $\{1\}$. Thus $\mathcal{N}^0 \subset T \cap Z_G(g)$, hence $\mathcal{N}^0 \subset (T \cap Z_G(g))^0 = T_1$. The inclusion $T_1 \subset \mathcal{N}^0$ is obvious and (a) follows.

We prove (b). The conjugation action of T_1 on the variety of all Borels of G^0 that are normalized by g must have a fixed point since this variety is projective. Thus there exists a Borel B of G^0 such that $T_1 \subset B, gBg^{-1} = B$. Now T_1 is contained in some maximal torus of B, which is necessarily T, by the definition of T. Since gTg^{-1} is a maximal torus of G that contains T, we have $gTg^{-1} = T$ (again by the definition of T). If $t \in T_1$, then t normalizes both B and T since $t \in T$. Hence gt normalizes B and T. Thus, gt is quasi-semisimple.

We prove (c). Let T_1' be a maximal torus in $Z_G(g')^0$. As in the proof of (b) we can find a Borel B' of G^0 and a maximal torus T' of B' such that $T_1' \subset T'$ and $g'B'g'^{-1} = B', g'T'g'^{-1} = T'$. We can find $h \in G^0$ such that $hB'h^{-1} = B, hT'h^{-1} = T$ (with B, T as above). Let $g'' = hg'h^{-1}$. Then $g''Bg''^{-1} = B, g''Tg''^{-1} = T$. We also have $gBg^{-1} = B, gTg^{-1} = T$. We have $g'' = gy$ where $y \in G^0$ satisfies $yBy^{-1} = B, yTy^{-1} = T$. It follows that $y \in T$. Since a power of g is in T_1, $\mathrm{Ad}(g) : T \to T$ has finite order. Using 1.2(a) we can write $y = g^{-1}y_2gy_2^{-1}y_1$ with $y_2 \in T, y_1 \in (T \cap Z_G(g))^0 = T_1$. Then $gy = y_2gy_1y_2^{-1}$. We see that gy is T-conjugate (hence G^0-conjugate) to gy_1. Hence g' is G^0-conjugate to $gy_1 \in gT_1$. This proves (c).

We prove (d). Let $g', g'' \in gT_1$ be such that $g'' = xg'x^{-1}$ where $x \in G^0$. We have $g'' = gt$ with $t \in T_1$. Clearly, T_1 is a maximal torus of $Z_G(g')^0$ and a maximal torus of $Z_G(g'')^0$. Then $x^{-1}T_1x$ is a maximal torus of $Z_G(x^{-1}g''x)^0 = Z_G(g')^0$. The maximal tori $T_1, x^{-1}T_1x$ of $Z_G(g')^0$ are conjugate in $Z_G(g')^0$, that is, there exists $z \in Z_G(g')^0$ such that $zx^{-1}T_1xz^{-1} = T_1$. Let $n = zx^{-1}$. We have $nT_1n^{-1} = T_1$ and $n^{-1}g'n = xz^{-1}g'zx^{-1} = xg'x^{-1} = g''$. We have $n^{-1}gT_1n = n^{-1}g'T_1n = g''n^{-1}T_1n = g''T_1 = gT_1$ so that $n^{-1} \in \mathcal{N}$. This proves (c).

1.15. We shall need the following result.

(a) *The number of unipotent G-conjugacy classes of G is finite. The number of unipotent G^0-conjugacy classes of G is finite.*

These two statements are clearly equivalent. In the case where $G = G^0$ is connected, (a) is proved in [L1]. The author handled also the general case by a method similar to that in [L1]. See [Sp, I, 4.1].

1.16. Let $\chi \in \mathrm{Hom}(\mathbf{k}^*, G^0)$. For any $k \in \mathbf{Z}$ we set $\mathfrak{g}_k = \{x \in \mathfrak{g}; \mathrm{Ad}(\chi(a))x = a^kx \quad \forall a \in \mathbf{k}^*\}$. Then $\sum_{k \ge 0} \mathfrak{g}_k = \mathrm{Lie}\ P_\chi$ for a well-defined parabolic P_χ of G^0. We have $\sum_{k>0} \mathfrak{g}_k = \mathrm{Lie}\ U_{P_\chi}$.

1.17. Let Q be a parabolic of G^0. Let $g \in N_GQ$. We show that

(a) *there exists $u \in U_Q$ and $\chi \in \mathrm{Hom}(\mathbf{k}^*, G^0)$ such that $g\chi(a)g^{-1} = u\chi(a)u^{-1}$ for all $a \in \mathbf{k}^*$ and $P_\chi = Q$.*

Let $\pi : Q \to Q/U_Q$ be the obvious map. As one easily checks, one can find $\chi' \in \mathrm{Hom}(\mathbf{k}^*, G^0)$ such that $\chi'(\mathbf{k}^*) \subset \pi^{-1}(\mathcal{Z}^0_{Q/U_Q})$ and $P_{\chi'} = Q$. Let T be a maximal torus of $\pi^{-1}(\mathcal{Z}^0_{Q/U_Q})$ that contains $\chi'(\mathbf{k}^*)$. We can find $n \ge 1$ such that $g^n \in Q$. For $j \in [0, n-1]$, g^jTg^{-j} is a maximal torus of $\pi^{-1}(\mathcal{Z}^0_{Q/U_Q})$, a connected solvable group with unipotent radical U_Q. Hence we can find $u_j \in U_Q$ such that $g^jTg^{-j} = u_jTu_j^{-1}$. Define $\chi_j \in \mathrm{Hom}(\mathbf{k}^*, T)$ by $\chi_j(a) = u_j^{-1}g^j\chi'(a)g^{-j}u_j$ and $\chi \in \mathrm{Hom}(\mathbf{k}^*, T)$ by $\chi(a) = \chi_0(a)\chi_1(a)\dots\chi_{n-1}(a)$. Define an automorphism $f : \mathcal{Z}^0_{Q/U_Q} \to \mathcal{Z}^0_{Q/U_Q}$ by $f(\pi(x)) = \pi(gxg^{-1})$ for all $x \in \pi^{-1}(\mathcal{Z}^0_{Q/U_Q})$. For $a \in \mathbf{k}^*$ we have $\pi(\chi(a)) = \pi(\chi'(a))f(\pi(\chi'(a)))f^2(\pi(\chi'(a)))\dots f^{n-1}(\pi(\chi'(a)))$. Since $f^n = 1$ it follows that $f(\pi(\chi(a))) = \pi(\chi(a))$, that is, $\pi(g\chi(a)g^{-1}) = \pi(\chi(a))$. By a standard argument, if $\lambda, \lambda' \in \mathrm{Hom}(\mathbf{k}^*, Q)$ are such that $\pi(\lambda(a)) = \pi(\lambda'(a))$ for all $a \in \mathbf{k}^*$, then there exists $u \in U_Q$ such that $\lambda(a) = u\lambda'(a)u^{-1}$ for all a. Thus, there exists $u \in U_Q$ such that $g\chi(a)g^{-1} = u\chi(a)u^{-1}$ for all a. We have $P_{\chi_j} = u_j^{-1}g^jP_{\chi'}g^{-j}u_j = u_j^{-1}g^jQg^{-j}u_j = u_j^{-1}Qu_j = Q$. Hence the $\mathbf{k}^*$-action $a \mapsto \mathrm{Ad}(\chi_j(a))$ has ≥ 0 weights on Lie U_Q and < 0 weights on $\mathfrak{g}/\mathrm{Lie}\, Q$. Since these actions (for $j = 0, 1, \dots, n-1$) commute with each other, it follows that the $\mathbf{k}^*$-action $a \mapsto \mathrm{Ad}(\chi(a)) = \mathrm{Ad}(\chi_0(a))\mathrm{Ad}(\chi_1(a))\dots\mathrm{Ad}(\chi_{n-1}(a))$ has ≥ 0 weights on Lie U_Q and < 0 weights on $\mathfrak{g}/\mathrm{Lie}\, Q$. Hence $P_\chi = Q$. This proves (a).

1.18. Let $g \in G$. Let Q be a parabolic subgroup of $Z_G(g_s)^0$ such that $g_uQg_u^{-1} = Q$. We show that

(a) *there exists a parabolic P of G^0 such that $P \cap Z_G(g_s)^0 = Q$ and $gPg^{-1} = P$.*
By 1.17, we can find $\chi : \mathbf{k}^* \to Z_G(g_s)^0$ such that P_χ (relative to $Z_G(g_s)^0$) is Q and there exists $u \in U_Q$ such that $g_u\chi(a)g_u^{-1} = u\chi(a)u^{-1}$ for all $a \in \mathbf{k}^*$. Since $g_s\chi(a)g_s^{-1} = \chi(a)$ for all a, we have $g\chi(a)g^{-1} = u\chi(a)u^{-1}$ for all a. Define $\tilde{\chi} : \mathbf{k}^* \to G^0, \tilde{\chi}' : \mathbf{k}^* \to G^0$ by $\tilde{\chi}(a) = \chi(a)$, $\tilde{\chi}'(a) = g\tilde{\chi}(a)g^{-1} = u\tilde{\chi}(a)u^{-1}$ for all a. Let $P = P_{\tilde{\chi}}$ (relative to G). From the definition we have Lie $P \cap$ Lie $Z_G(g_s)^0 =$ Lie Q. Hence $P \cap Z_G(g_s)^0 = Q$. We have $gPg^{-1} = P_{\tilde{\chi}'} = uPu^{-1} = P$ since $u \in U_Q \subset Q \subset P$. This proves (a).

1.19. Let T, T' be tori and let $f \in \mathrm{Hom}(T', T)$ be surjective. Let $\tau : T \to T, \tau' : T' \to T'$ be automorphisms of finite order with fixed point sets $T^\tau, T'^{\tau'}$ such that $f\tau' = \tau f$. We show that

(a) *f restricts to a surjective homomorphism $(T'^{\tau'})^0 \to (T^\tau)^0$.*
This can be reduced to the analogous statement where T, T' are replaced by their groups of co-characters tensored by $\mathbf{Q}$. In that case we use the fact that an automorphism of finite order of a finite dimensional $\mathbf{Q}$-vector space is semisimple.

1.20. Let $\pi : G \to G_{ss}$ be the canonical map. Let $a \in G$ be semisimple. Let $\phi : Z_G(a)^0 \to Z_{G_{ss}}(\pi(a))^0$ be the homomorphism induced by π. We show that

(a) *ϕ is surjective and* $\mathrm{Ker}(\phi) \subset \mathcal{Z}^0_{Z_G(a)^0}$.

(b) *ϕ induces a surjective homomorphism* $\mathcal{Z}^0_{Z_G(a)^0} \to \mathcal{Z}^0_{Z_{G_{ss}}(\pi(a))^0}$.

Let I be the image of ϕ. Let T be a maximal torus of $Z_{G_{ss}}(\pi(a))^0$. Then $T' = \pi^{-1}(T)$ is a torus in G^0 and $\mathrm{Ad}(a)(T') = T'$. Moreover, since $Z_{G^0}T'$ has finite index in N_GT', we see that there exists an integer $n \ge 1$ such that $\mathrm{Ad}(a)^n(t') = t'$ for all $t' \in T'$. Let $T'' = \{t' \in T'; \mathrm{Ad}(a)(t') = t'\}^0$. The obvious homomorphism $T' \to T$ restricts to a homomorphism $T'' \to T'$ which is surjective, by 1.19(a). Since $T'' \subset Z_G(a)^0$, we see that $T' \subset I$. Thus, I contains the union of all maximal tori of $Z_{G_{ss}}(\pi(a))^0$, which is dense in $Z_{G_{ss}}(\pi(a))^0$, since $Z_{G_{ss}}(\pi(a))^0$ is reductive, connected. Thus, I is dense in $Z_{G_{ss}}(\pi(a))^0$. It is clearly closed, hence $I = Z_{G_{ss}}(\pi(a))^0$. This proves the first assertion of (a). The second assertion of (a) is obvious. Now

(b) is a special case of the following general statement. Let $H \to H'$ be a surjective homomorphism of connected reductive groups whose kernel is contained in the centre of H. Then the induced homomorphism $\mathcal{Z}^0_H \to \mathcal{Z}^0_{H'}$ is surjective.

1.21. If D is a connected component of G, we set

(a) ${}^D\mathcal{Z}_{G^0} = \mathcal{Z}_{G^0} \cap Z_G(g)$

where $g \in D$. (This does not depend on the choice of g.) We write ${}^D\mathcal{Z}^0_{G^0}$ instead of $({}^D\mathcal{Z}_{G^0})^0$. Now let X be a subset of D stable under G^0-conjugacy. We show that

(b) if ${}^D\mathcal{Z}^0_{G^0}X \subset X$, then $\mathcal{Z}^0_{G^0}X \subset X$.

(The converse is obvious.) Let $z \in \mathcal{Z}^0_{G^0}, g \in X$. We must show that $zg \in X$. Clearly, some power of g is in G^0 hence some power of $\mathrm{Ad}(g) : \mathcal{Z}^0_{G^0} \to \mathcal{Z}^0_{G^0}$ is 1. Using 1.2(a), we can write $z = txgx^{-1}g^{-1}$ with $t \in (\mathcal{Z}^0_{G^0} \cap Z_G(g))^0 = {}^D\mathcal{Z}^0_{G^0}, x \in \mathcal{Z}^0_{G^0}$. Then $zg = txgx^{-1} \in txXx^{-1} = tX = X$. This proves (b).

Consider the $\mathcal{Z}^0_{G^0} \times G^0$-action

(c) $(z, x) : y \mapsto xzyx^{-1}$

on G or D. This restricts to a ${}^D\mathcal{Z}^0_{G^0} \times G^0$-action on G. From (b) we see that:

(d) *The action (c) of $\mathcal{Z}^0_{G^0} \times G^0$ on D and its restriction to ${}^D\mathcal{Z}^0_{G^0} \times G^0$ have exactly the same orbits, that is, any orbit for one action is an orbit for the other action.*

1.22. Let C be an orbit of the $\mathcal{Z}^0_{G^0} \times G^0$-action 1.21(c) on G. Let D be the connected component of G that contains C. Let $C' = \{y \in D; y_s \in C_s\}$. We show that

(a) *C_s and C' are closed in G and $y \mapsto y_s$ is a morphism $C' \to C_s$;*

(b) if $\bar{C}$ is the closure of C in G and $h \in \bar{C}$, then there exists $h' \in C$ such that $h_s = h'_s$ and $h'^{-1}h \in Z_G(h_s)^0$.

We prove (a). Let $\sigma \in C$. By 1.21(d), we have $C = \{xz\sigma x^{-1}; x \in G^0, z \in {}^D\mathcal{Z}_{G^0}\}$. For x, z as above we have $(xz\sigma x^{-1})_s = xz\sigma_s x^{-1}$ since $z = z_s, z\sigma = \sigma z$. Thus, $C_s = \{xz\sigma_s x^{-1}; x \in G^0, z \in {}^D\mathcal{Z}^0_{G^0}\}$, so that C_s is an orbit for the action 1.21(c) of ${}^D\mathcal{Z}^0_{G^0} \times G^0$ on G. We may assume that G is generated by D. Then ${}^D\mathcal{Z}^0_{G^0}$ is a closed normal subgroup of G and we may form $G' = G/{}^D\mathcal{Z}^0_{G^0}$. Let $\pi : G \to G'$ be the obvious homomorphism. Then $\pi(C_s)$ is a semisimple G'^0-conjugacy class in G' hence $\pi(C_s)$ is closed in G' by 1.4(e). Let $G' \subset GL_n(\mathbf{k})$ be an imbedding of algebraic groups with $n \geq 1$. Let Y be the (semisimple) class in $GL_n(\mathbf{k})$ that contains $\pi(C_s)$. Let $Y' = \{h \in GL_n(\mathbf{k}); h_s \in Y\}$. It is well known that Y' is closed in $GL_n(\mathbf{k})$ and $Y' \to Y, g \mapsto g_s$ is a morphism of varieties. Hence $Y' \cap G'$ is closed in G' and $\rho : Y' \cap G' \to Y \cap G', g \mapsto g_s$ is a morphism of varieties. Let $\mathcal{X} = \{g \in \pi(D); g_s \in \pi(C_s)\}$. Since $\pi(C_s)$ is closed in $Y \cap G'$ and $\mathcal{X} = \rho^{-1}(\pi(C_s)) \cap \pi(D)$, we see that $\mathcal{X}$ is closed in $Y' \cap G'$. Next we note that $C' = \pi^{-1}(\mathcal{X})$. (The inclusion $C' \subset \pi^{-1}(\mathcal{X})$ is obvious; the reverse inclusion follows from the equality ${}^D\mathcal{Z}^0_{G^0}C_s = C_s$.) We see that C' is closed in D. Let $a \in C_s$. Let H be the isotropy group of a in ${}^D\mathcal{Z}^0_{G^0} \times G^0$ (which acts transitively on C_s). Let $R = \{g \in D; g_s = a\}$. The ${}^D\mathcal{Z}^0_{G^0} \times G^0$-action on D restricts to an action on C' and this restricts to an H-action on R which induces an isomorphism of algebraic varieties $({}^D\mathcal{Z}^0_{G^0} \times G^0) \times_H R \xrightarrow{\sim} C'$. Via this isomorphism and the isomorphism $({}^D\mathcal{Z}^0_{G^0} \times G^0) \times_H \{a\} \xrightarrow{\sim} C_s$, the map $C' \to C_s, g \mapsto g_s$ becomes the morphism $({}^D\mathcal{Z}^0_{G^0} \times G^0) \times_H R \to ({}^D\mathcal{Z}^0_{G^0} \times G^0) \times_H \{a\}$ induced by the obvious map $R \to \{a\}$. It follows that $C' \to C_s, g \mapsto g_s$ is a morphism of algebraic varieties. This proves (a).

We prove (b). We have $\bar{C} \subset D$. From (a) we see that $g \mapsto g_s$ is a morphism $\bar{C} \to C_s$. This morphism is equivariant with respect to the actions 1.21(c) of ${}^D\mathcal{Z}^0_{G^0} \times G^0$ (transitive on C_s). Since C is a dense subset of $\bar{C}$ invariant under this action, it follows that for any $a \in C_s$, the intersection of C with $\bar{C}_a = \{x \in \bar{C}; x_s = a\}$ is dense in $\bar{C}_a$. Take $a = h_s$. Then any connected component c of $\bar{C}_a$ contains some point of $C \cap \bar{C}_a$. Let c be the connected component containing h. Let $h' \in C \cap c$. Then $h'_s = a$. Since $\bar{C}_a \subset Z_G(a)$ we have $c \subset Z_G(a)$. More precisely, c is contained in a connected component of $Z_G(a)$. Thus $h'^{-1}h \in Z_G(a)^0$. This proves (b).

1.23. Let $\mathbf{c}$ be a G^0-conjugacy class in G. Let D be the connected component of G that contains $\mathbf{c}$. Let $\Gamma = \{z \in {}^D\mathcal{Z}^0_{G^0}; z\mathbf{c} = \mathbf{c}\}$. We show that

(a) Γ *is finite.*

Let $c \in \mathbf{c}$. If $z \in \Gamma$, then $zc = cz = gcg^{-1}$; hence $zc_s = c_s z = gc_s g^{-1}$ for some $g \in G^0$. Let T_1 be a maximal torus of $Z_G(c_s)^0$. We have ${}^D\mathcal{Z}^0_{G^0} \subset T_1$. By 1.4(a) and 1.4(d), the G^0-conjugacy class of c_s intersects c_sT_1 in a finite set; that is, $F := \{hc_sh^{-1}; h \in G^0\} \cap c_sT_1$ is finite. Since $c_s\Gamma \subset F$, we see that $c_s\Gamma$ is finite and (a) follows.

From (a) we deduce that, for any subgroup τ of ${}^D\mathcal{Z}^0_{G^0}$ we have

(b) $\dim(\tau\mathbf{c}) = \dim\tau + \dim\mathbf{c}$.

1.24. Let $\mathbf{c}, \mathbf{c}'$ be two G^0-conjugacy classes in G contained in the same connected component D of G. Let δ, δ' be two subtori in $\zeta := {}^D\mathcal{Z}^0_{G^0}$. We show that

(a) $\delta\mathbf{c} \cap \delta'\mathbf{c}'$ *is a finite union of subsets of the form* $(\delta \cap \delta')\mathbf{c}''$ *where* $\mathbf{c}''$ *are* G^0*-conjugacy classes in* D.

We may assume that $\delta\mathbf{c} \cap \delta'\mathbf{c} \neq \emptyset$. Let $x \in \delta\mathbf{c} \cap \delta'\mathbf{c}'$. Let $\mathbf{c}_1$ be the G^0-conjugacy class of x. Then $\delta\mathbf{c} = \delta\mathbf{c}_1, \delta'\mathbf{c}' = \delta'\mathbf{c}_1$. Thus we may assume that $\mathbf{c} = \mathbf{c}'$. Let Γ be as in 1.23. Then $\phi : \zeta \times \mathbf{c} \to \zeta\mathbf{c}, (a, c) \mapsto ac$ is a principal covering with group Γ (which acts by $z : (a, c) \mapsto (az, c)$). Since $\phi(\delta \times \mathbf{c}) = \delta\mathbf{c}, \phi(\delta' \times \mathbf{c}) = \delta'\mathbf{c}$, we have $\phi^{-1}(\delta\mathbf{c}) = \bigcup_{z\in\Gamma}(\delta z \times \mathbf{c})$, $\phi^{-1}(\delta'\mathbf{c}) = \bigcup_{z\in\Gamma}\gamma(\delta' z \times \mathbf{c})$, and

$$\phi^{-1}(\delta\mathbf{c} \cap \delta'\mathbf{c}) = \bigcup_{z\in\Gamma}(\delta z \times \mathbf{c}) \cap \bigcup_{z'\in\Gamma}(\delta' z' \times \mathbf{c})$$
$$= \bigcup_{z,z'\in\Gamma}(\delta z \times \mathbf{c}) \cap (\delta' z' \times \mathbf{c}) = \bigcup_{z,z'\in\Gamma}((\delta z \cap \delta' z') \times \mathbf{c}).$$

Hence

$$\delta\mathbf{c} \cap \delta'\mathbf{c} = \phi(\bigcup_{z,z'\in\Gamma}((\delta z \cap \delta' z') \times \mathbf{c})) = \bigcup_{z,z'\in\Gamma}((\delta z \cap \delta' z')\mathbf{c}).$$

Now $\delta z \cap \delta' z'$ is either empty or of the form $(\delta \cap \delta')z_1$ with $z_1 \in \zeta$. Since Γ is finite (see 1.23) we see that $\delta\mathbf{c} \cap \delta'\mathbf{c} = \cup_{z_1}(\delta \cap \delta')z_1\mathbf{c}$ where z_1 runs over a finite subset of ζ. (For any such z_1, $z_1\mathbf{c}$ is a G^0-orbit in D.)

1.25. Let P' (resp. P'') be a parabolic of G^0 with Levi L' (resp. L''). Assume that L', L'' contain a common maximal torus. Then $P' \cap P''$ is a connected group with Levi $L' \cap L''$. We show that

(a) *any* $g \in N_GP' \cap N_GP''$ *can be written uniquely in the form* $g = z\omega$ *where* $z \in N_GL' \cap N_GP' \cap N_GL'' \cap N_GP'', \omega \in U_{P'\cap P''}$. *Thus,* $N_GP' \cap N_GP''$ *is a semidirect product* $(N_GL' \cap N_GP' \cap N_GL'' \cap N_GP'')U_{P'\cap P''}$.

Since g^{-1} normalizes $P' \cap P''$, we see that $g^{-1}(L' \cap L'')g$ is a Levi of $P' \cap P''$; hence it equals $\omega^{-1}(L' \cap L'')\omega$ for some $\omega \in U_{P\cap P'}$. Then $z := g\omega^{-1}$ normalizes $L' \cap L''$ and also P' and P''. Now L' is the unique Levi of P' that contains

$L' \cap L''$. Clearly, $zL'z^{-1}$ is again a Levi of P' that contains $L' \cap L''$. Hence $zL'z^{-1} = L'$ so that $z \in N_G L' \cap N_G P'$. Similarly, $z \in N_G L'' \cap N_G P''$. Thus, $z \in N_G L' \cap N_G P' \cap N_G L'' \cap N_G P''$ and $g = z\omega$, as required. To prove uniqueness, it is enough to show that $(N_G L' \cap N_G P' \cap N_G L'' \cap N_G P'') \cap U_{P' \cap P''} = \{1\}$. If $u \in U_{P' \cap P''}$ normalizes L' and L'' then, using $u \in P'$, we have $u \in L'$ and similarly $u \in L''$ hence $u \in L' \cap L''$; but $L' \cap L''$, being a Levi of $P' \cap P''$, has trivial intersection with $U_{P' \cap P''}$ hence $u = 1$.

We have

(b) $U_{P' \cap P''} = (L' \cap U_{P''})(U_{P'} \cap P'')$ *(semidirect product) and also* $U_{P' \cap P''} = (L'' \cap U_{P'})(U_{P''} \cap P')$ *(semidirect product).*

1.26. Let P be a parabolic of G^0 with Levi L. Then

(a) *any* $g \in N_G P$ *can be written uniquely in the form* $g = z\omega$ *where* $z \in N_G L \cap N_G P, \omega \in U_P$. *Thus,* $N_G P$ *is a semidirect product* $(N_G L \cap N_G P)U_P$.

This is a special case of 1.25(a) with $P' = P'' = P, L' = L'' = L$.

1.27. Let H' be an algebraic group, let H be a closed subgroup of H' and let $\mathbf{c}$ be a semisimple H'^0-conjugacy class in H'. Then

(a) $H \cap \mathbf{c}$ *is a finite union of (semisimple)* H^0*-conjugacy classes in* H.

We can regard H' as a closed subgroup of $GL_n(\mathbf{k})$ for some $n \geq 1$. Let $\mathbf{c}_1$ be the conjugacy class in $GL_n(\mathbf{k})$ that contains $\mathbf{c}$. It is enough to prove (a) with $\mathbf{c}$ replaced by $\mathbf{c}_1$. Thus we may assume that $H' = GL_n(\mathbf{k})$. Let D be a connected component of H that contains some semisimple elements. We can find a closed diagonalizable subgroup T_D of H such that any H^0-conjugacy class in D meets T_D. (We pick a semisimple element $s \in D$ and a maximal torus T_1 in $Z_H(s)^0$. We can take T_D to be the subgroup generated by T_1 and s. The fact that this has the required properties can be deduced from the analogous property of the reductive group H/U_{H^0}; see 1.14(c).) It is enough to prove that $T_D \cap \mathbf{c}$ is finite. Now T_D is contained in a maximal torus T of $GL_n(\mathbf{k})$ and it is enough to show that $T \cap \mathbf{c}$ is finite. But this is a well-known property of $GL_n(\mathbf{k})$.

2. Isolated elements of G

2.1. Let $g \in G$. Let

$$T(g) = (\mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g_u))^0 = (\mathcal{Z}^0_{Z_G(g_s)^0} \cap Z_G(g_u))^0$$
$$= (\mathcal{Z}_{Z_{G^0}(g_s)^0} \cap Z_{G^0}(g_u))^0 = (\mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g))^0$$

(a torus, since $Z_G(g_s)^0$ is reductive). We have $T(g) \subset Z_G(g)$. We sometimes write $T_G(g)$ instead of $T(g)$. Clearly,

$$T(xgx^{-1}) = xT(g)x^{-1} \text{ for any } x \in G.$$

Let

$$L(g) = Z_{G^0}(T(g)), \hat{L}(g) = N_G(L(g)).$$

Then

(a) $L(g)$ *is the Levi of a parabolic* P *of* G^0 *such that* $gPg^{-1} = P$. Indeed, we can find $\chi \in \mathrm{Hom}(\mathbf{k}^*, G^0)$ such that $\chi(\mathbf{k}^*) \subset T(g)$ and $L(g) = Z_{G^0}(\chi(\mathbf{k}^*))$. Then $P = P_\chi$; see 1.16, is as required.

From the definition we have

(b) $Z_G(g_s)^0 \subset L(g)$.

The next result shows that $L(g)$ is characterized by being minimal with the properties (a) and (b).

(c) *Let Q be a parabolic of G^0 with Levi L such that $g \in N_G L \cap N_G Q$ and $Z_G(g_s)^0 \subset L$. Then $L(g) \subset L$.*

Since $Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0) = L$ (see 1.10(a)), it is enough to show that $Z_{G^0}(T(g)) \subset Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0)$ or that $(\mathcal{Z}_L \cap Z_L(g))^0 \subset T(g)$ or that $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset T(g)$. Clearly, $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset Z_G(g_s)$, hence $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset Z_G(g_s)^0$. Since $Z_G(g_s)^0 \subset L$, we have $\mathcal{Z}_L \cap Z_G(g_s)^0 \subset \mathcal{Z}_{Z_G(g_s)^0}$; hence $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset \mathcal{Z}_{Z_G(g_s)^0}$. Clearly, $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset Z_G(g)$, hence $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset \mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g)$ and $(\mathcal{Z}_L^0 \cap Z_G(g))^0 \subset (\mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g))^0 = T(g)$, as required.

Since $T(g) \subset Z_G(g)$, we have $g \in \hat{L}(g)$; hence $T_{\hat{L}(g)}(g)$ is defined. We have

(d) $T_{\hat{L}(g)}(g) = T(g)$.

Since $Z_{\hat{L}(g)}(g_u) = Z_G(g_u) \cap \hat{L}(g)$ and $\mathcal{Z}_{Z_{\hat{L}(g)}(g_s)^0} \subset \hat{L}(g)$, we have $T_{\hat{L}(g)}(g) = (\mathcal{Z}_{Z_{\hat{L}(g)}(g_s)^0} \cap Z_{\hat{L}(g)}(g_u))^0 = (\mathcal{Z}_{Z_{\hat{L}(g)}(g_s)^0} \cap Z_G(g_u))^0$. From (b) we have $Z_G(g_s)^0 \subset L(g) \subset \hat{L}(g)$, hence $Z_{\hat{L}(g)}(g_s)^0 = Z_G(g_s)^0$ and $T_{\hat{L}(g)}(g) = (\mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g_u))^0 = T(g)$. This proves (d).

We shall need the following result.

(e) *Let $g, g' \in G$ be such that $g_s = g'_s$ and $g'^{-1}g \in Z_G(g_s)^0$. Then $T(g) = T(g')$.*

We must show that $(\mathcal{Z}^0_{Z_G(g_s)^0} \cap Z_G(g))^0 = (\mathcal{Z}^0_{Z_G(g_s)^0} \cap Z_G(g'))^0$. It is enough to show that, for $x \in \mathcal{Z}^0_{Z_G(g_s)^0}$, the conditions $xg = gx$ and $xg' = g'x$ are equivalent. Since x commutes with any element of $Z_G(g_s)^0$, it commutes with $g'^{-1}g$. This proves (e).

2.2. We show that the following five conditions for $g \in G$ are equivalent:

(i) $L(g) = G^0$;

(ii) $T(g) \subset \mathcal{Z}_{G^0}$;

(iii) $T(g) = {}^D\mathcal{Z}^0_{G^0}$ *where D is the connected component of G containing g;*

(iv) *there is no proper parabolic P of G^0 with Levi L such that $g \in N_G L \cap N_G P, Z_G(g_s)^0 \subset L$;*

(v) *there is no proper parabolic P of G^0 such that $g \in N_G P, Z_G(g_s)^0 \subset P$.*

Indeed, it is clear that (iii) $\Longrightarrow$ (ii)$\leftrightarrow$(i). Assume now that (ii) holds. Then any element of $T(g)$ commutes with any element of G^0; since $T(g) \subset Z_G(g)$, we have $T(g) \subset \mathcal{Z}_{G^0} \cap Z_G(g)$. Since $T(g)$ is connected, we have $T(g) \subset (\mathcal{Z}_{G^0} \cap Z_G(g))^0$. The reverse inclusion is obvious. We see that (iii) holds. Thus, the equivalence of (i), (ii), and (iii) is established. The equivalence of (i) and (iv) follows from 2.1(c). It remains to prove the equivalence of (iv) and (v). It is enough to prove the following statement.

If P is a parabolic of G^0 such that $g \in N_G P, Z_G(g_s)^0 \subset P$, then there exists a Levi of P that contains $Z_G(g_s)^0$ and is normalized by g.

By 1.13(a) there is a unique Levi L of P such that $Z_G(g_s)^0 \subset L$. Now gLg^{-1} is a Levi of $gPg^{-1} = P$ containing $gZ_G(g_s)^0 g^{-1} = Z_G(g_s)^0$. By uniqueness, we have $gLg^{-1} = L$. This completes the prove of equivalence of (i)–(v).

We say that g is *isolated* (in G) if it satisfies the equivalent conditions (i)–(v).

If $g \in G$ and G' is a closed subgroup of G containing g and G^0, then, clearly,

(a) g is isolated in G if and only if g is isolated in G'.

By 2.1(d) we have for $g \in G$:

$$Z_{\hat{L}(g)}(T_{\hat{L}(g)}(g))^0 = Z_{\hat{L}(g)}(T(g))^0 = (\hat{L}(g) \cap Z_G(T(g))^0(\hat{L}(g))^0;$$

hence

(b) *g is isolated in $\hat{L}(g)$.*

2.3. Let $\pi : G \to G_{ss}$ be the obvious map. Let $g \in G$. We show that

(a) *g is isolated in G if and only if $\pi(g)$ is isolated in G_{ss}. Equivalently, the set of isolated elements in G is the inverse image under π of the set of isolated elements in G_{ss}.*

Using the criterion 2.2(v) we see that it is enough to show that conditions (i) and (ii) below are equivalent:

(i) there exists a proper parabolic P of G such that $Z_G(g_s)^0 \subset P, gPg^{-1} = P$;

(ii) there exists a proper parabolic $\bar{P}$ of G_{ss} such that $Z_{G_{ss}}(\pi(g)_s)^0 \subset \bar{P}$, $\pi(g)\bar{P}\pi(g)^{-1} = \bar{P}$.

Assume that (ii) holds. Let $\bar{P}$ be as in (ii). Then $P = \pi^{-1}(\bar{P})$ is as in (i). Conversely, assume that (i) holds. Let P be as in (i). Then $\bar{P} = \pi(P)$ is as in (ii) since π induces a surjection $Z_G(g_s)^0 \to Z_{G_{ss}}(\pi(g)_s)^0$ (see 1.20(a)).

From (a) we see that

(b) *the set of isolated elements in G is stable under left or right translation by $\mathcal{Z}^0_{G^0}$.*

2.4. Let $g \in G$. By 1.9(a) (applied to $Z_G(g_s)$ instead of G) the set of unipotent elements in $Z_G(g_s)^0 g_u$ that are quasi-semisimple in $Z_G(g_s)$ is a single $Z_G(g_s)^0$-conjugacy class. Let u be an element of this set. According to 1.4(c), $g_s u$ is quasi-semisimple in G. Thus there exist a Borel of G^0 and a maximal torus of it, both normalized by $g_s u$; these are automatically normalized by u, hence u is quasi-semisimple in G. Let $H = Z_G(u)$, a reductive group, by 1.4(b). We have $g_s \in H$. We show that

(a) *$\mathcal{Z}_{Z_H(g_s)^0}$ and its subgroup $\mathcal{Z}_{Z_G(g_s)^0} \cap H$ have the same identity component.*

Let $\tilde{G} = Z_G(g_s)$. Let $\tilde{H} = Z_{\tilde{G}}(u)$ (a reductive group by 1.4(b)). We must prove that $\mathcal{Z}_{\tilde{H}^0}$ and its subgroup $\mathcal{Z}(\tilde{G}^0) \cap \tilde{H}$ have the same identity component. This follows from 1.7(b) applied to $\tilde{G}, u$ instead of G, u. (Note that 1.7(b) is applicable by 1.8(a).)

Next we show that

(b) $(\mathcal{Z}_{G^0} \cap Z_G(g))^0 = (\mathcal{Z}_{H^0} \cap Z_G(g))^0$.

By 1.7(b) (with g replaced by u), we have $\mathcal{Z}^0_{H^0} = (\mathcal{Z}_{G^0} \cap H)^0$. (Note that 1.7(b) is applicable by 1.8(a).) Since $Z_G(g) \subset H$, the left-hand side of (b) is

$$(\mathcal{Z}_{G^0} \cap H \cap Z_G(g))^0 = ((\mathcal{Z}_{G^0} \cap H)^0 \cap Z_G(g))^0 = (\mathcal{Z}^0_{H^0} \cap Z_G(g))^0$$

and this equals the right-hand side of (b). This proves (b).

Lemma 2.5. *In the setup of 2.4, g is isolated in G if and only if g_s is isolated in H.*

We have

$$T_H(g_s) = (\mathcal{Z}_{Z_H(g_s)^0} \cap Z_H(g_s))^0 = \mathcal{Z}^0_{Z_H(g_s)},$$

$$\mathcal{Z}_{Z_G(g_s)^0} \cap H = \mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g_u)$$

$$= \mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g_s) \cap Z_G(g_u) = \mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g),$$

since $u \in g_u Z_G(g_s)^0$. Hence $T(g) = (\mathcal{Z}_{Z_G(g_s)^0} \cap H)^0$. Using 2.4(a) we deduce that $T(g) = T_H(g_s)$. The condition that g is isolated in G is that $T(g) = (\mathcal{Z}_{G^0} \cap Z_G(g))^0$. The condition that g_s is isolated in H is that $T_H(g_s) = (\mathcal{Z}_{H^0} \cap Z_H(g_s))^0$, that is,

$T_H(g_s) = (\mathcal{Z}_{G^0} \cap Z_G(g))^0$ (see 2.4(b)). These two conditions are equivalent since $T(g) = T_H(g_s)$. The lemma is proved.

Lemma 2.6. *Assume that $\mathcal{Z}^0_{G^0} = \{1\}$. Then the set of semisimple elements that are isolated in G is a union of finitely many G^0-conjugacy classes.*

We fix a connected component D of G which contains some semisimple element. Let s be a semisimple element in D. Let T_1 be a maximal torus of $Z_G(s)^0$. By 1.14(c), any semisimple element in D is G^0-conjugate to some element in sT_1. Hence it is enough to show sT_1 contains only finitely many elements that are isolated in G. There exist finitely many closed connected subgroups $H_1, H_2, \dots, H_n$ of G^0 such that for any $s' \in sT_1$, $Z_G(s')^0$ is one of $H_1, H_2, \dots, H_n$. (Indeed, with the notation of 1.5, Lie $Z_G(s')^0$ is spanned by Lie T_1 and by some of the lines $\mathfrak{g}_\alpha, \alpha \in R$.) We have $sT_1 = \bigsqcup_{i\in[1,n]} X_i$ where $X_i = \{s' \in sT_1; Z^0_G(s') = H_i\}$. If $s' \in X_i$, the condition that s' is isolated in G is that $\mathcal{Z}_{H_i}$ is finite. Thus, either all elements in X_i are isolated in G or none are isolated. We may assume that X_i is nonempty and consists of isolated elements if and only if $i \in [1, k]$. Here $k \le n$. We fix $s'_i \in X_i$ for each $i \in [1, k]$. If $s' \in X_i$, then $s' \in Z_G(H_i)$ and $s'_i \in Z_G(H_i)$; hence $s'^{-1}_i s' \in Z_G(H_i)$. Now $s'^{-1}_i s' \in T_1 \subset H_i$. Hence $s'^{-1}_i s' \in \mathcal{Z}_{H_i}$. Thus, the set of elements of sT_1 that are isolated in G is contained in the finite set $\bigcup_{i\in[1,k]} s'_i \mathcal{Z}_{H_i}$. The lemma is proved.

Lemma 2.7. *The action 1.21(c) of $\mathcal{Z}^0_{G^0} \times G^0$ on G leaves stable the set of isolated elements in G and has only finitely many orbits there.*

From 2.3(a) we see that the first assertion of the lemma holds and that, to prove the second assertion, it is enough to show that the conjugation action of G^0_{ss} on the set of isolated elements in G_{ss} has only finitely many orbits there. Thus we may assume that $\mathcal{Z}^0_{G^0} = \{1\}$. Let D be a connected component of G. Let Y be the set of all elements of D that are isolated in G. Let $\tilde{Y}$ be the set of all pairs (g, u) where $g \in D$, u is a unipotent element in $Z_G(g_s)^0 g_u$ that is quasi-semisimple in $Z_G(g_s)$ and g_s is isolated in $Z_G(u)$. Let $\rho : \tilde{Y} \to Y$ be the first projection. (This is well defined and surjective by Lemma 2.5.) Let $(g_0, u_0) \in \tilde{Y}$. Let $H = Z_G(u_0)$ (a reductive group, by 1.4(b)). Using 1.7(b) (which is applicable in view of 1.8(a)) we see that $\mathcal{Z}^0_{H^0} = \{1\}$. Applying Lemma 2.6 to H instead of G, we can find isolated semisimple elements $s_1, s_2, \dots, s_n$ in H such that any isolated semisimple element in H is H^0-conjugate to some s_i. By 1.15(a), for any $i \in [1, n]$ we can find unipotent elements $v_{i1}, v_{i2}, \dots, v_{ip_i}$ in $Z_G(s_i)$ such that any unipotent element in $Z_G(s_i)$ is $Z_G(s_i)^0$-conjugate to some v_{ij}. It is enough to show that

(a) *Let $\mathcal{O}$ be an orbit for the G^0-action on $\tilde{Y}$ given by conjugation on both coordinates. Then $(s_i v_{ij}, u') \in \mathcal{O}$ for some $i \in [1, n], j \in [1, p_i]$ and some u'.*

(Indeed, since ρ is surjective, (a) would imply that any G^0-conjugacy class in Y contains $s_i v_{ij}$ for some $i \in [1, n], j \in [1, p_i]$.) Let $(g, u) \in \mathcal{O}$. Now $uG^0 = g_u G^0 = (g_0)_u G^0 = u_0 G^0$ (since $gG^0 = g_0 G^0 = D$). Thus u, u_0 are unipotent, quasi-semisimple in G, in the same connected component of G, hence $u_0 = huh^{-1}$ for some $h \in G^0$. (See 1.9(a).) We have $(hgh^{-1}, huh^{-1}) \in \mathcal{O}$. Setting $g' = hgh^{-1}$ we have $(g', u_0) \in \mathcal{O}$. Since g'_s is an isolated semisimple element in H, we can find $h' \in H^0$ such that $h' g'_s h'^{-1} = s_i$ with $i \in [1, n]$. We have $(h' g' h'^{-1}, h' u_0 h'^{-1}) \in \mathcal{O}$. Now $h' u_0 h'^{-1} = u_0$, the semisimple part of $h' g' h'^{-1}$ is $h' g'_s h^{-1} = s_i$ and the unipotent part of $h' g' h'^{-1}$ is a unipotent element $\tilde{v}_i$ in $Z_G(s_i)$. Thus we have $(s_i \tilde{v}_i, u_0) \in \mathcal{O}$.

We can find $h'' \in Z_G(s_i)^0$ such that $h''\tilde{v}_i h''^{-1} = v_{ij}$ with $j \in [1, p_i]$. We have $(s_i v_{ij}, h'' u_0 h''^{-1}) = (h'' s_i \tilde{v}_i h''^{-1}, h'' u_0 h''^{-1}) \in \mathcal{O}$. This proves (a). The lemma is proved.

Lemma 2.8. *The set of isolated elements of G is closed in G.*

Using 2.3(a) we see that if the lemma holds for G_{ss}, then it holds for G. Hence we may assume that $\mathcal{Z}^0_{G^0} = \{1\}$. Let G' be the set of isolated elements of G. Since G' is a union of finitely many G^0-conjugacy classes (see Lemma 2.7), we see that G'_s is a union of finitely many semisimple G^0-conjugacy classes $E^1, E^2, \dots, E^n$. For $i \in [1, n]$ let G'^i be the inverse image of E^i under $G' \to G'_s, g \mapsto g_s$. It is enough to prove that G'^i is closed in G for any $i \in [1, n]$. Let $E'^i = \{g \in G; g_s \in E^i\}$. By 1.22(a), E^i, E'^i are closed in G and $\pi : E'^i \to E^i, g \mapsto g_s$ is a morphism. Note that π commutes with the conjugation action of G^0 on E^i, E'^i and that action is transitive on E^i. Hence a G^0-stable subset of E'^i is closed if and only if its intersection with $\pi^{-1}(s)$ is closed in G for some/any $s \in E^i$. Thus, to prove that G'^i is closed in E'^i (hence in G) it is enough to prove that, if $s \in E^i$, then $\{g \in G'^i; g_s = s\} = \{g \in G'; g_s = s\}$ is closed in G. Let $\tau = \mathcal{Z}^0_{Z_G(s)^0}$. We must show that $\{su; u \in G, u \text{ unipotent}, us = su, (\tau \cap Z_G(u))^0 = \{1\}\}$ is closed in G or that $\{u \in Z_G(s); u \text{ unipotent}, (\tau \cap Z_G(u))^0 = \{1\}\}$ is closed in $Z_G(s)$. Since the set of unipotent elements in $Z_G(s)$ is closed in $Z_G(s)$, it is enough to show that $\{g \in Z_G(s); (\tau \cap Z_G(g))^0 = \{1\}\}$ is closed in $Z_G(s)$. Let X be a connected component of $Z_G(s)$. It is enough to show that $X_0 := \{g \in X; (\tau \cap Z_G(g))^0 = \{1\}\}$ is closed in X. Now $\tau \cap Z_G(g)$ depends only on the connected component of g in $Z_G(s)$. Thus, either $X_0 = X$ or $X_0 = \emptyset$. In both cases, X_0 is closed in X. The lemma is proved.

3. A stratification of G

3.1. For $g, g' \in G$ we write $g \sim g'$ if $g'g^{-1} \in T(g) = T(g')$. This is an equivalence relation on G. For $x \in G^0, g, g' \in G$ with $g \sim g'$ we have $xgx^{-1} \sim xg'x^{-1}$ since $xT(g)x^{-1} = T(xgx^{-1})$. Hence the relation on G given by $g \asymp g'$ if $xgx^{-1} \sim g'$ for some $x \in G^0$ is an equivalence relation. The equivalence classes on G for $\asymp$ are called the *strata* of G. Each stratum of G is contained in a connected component of G and is stable under conjugation by G^0.

Lemma 3.2. *Let $\pi : G \to G_{ss}$ be the obvious homomorphism. Let $g, g' \in G$. We have $g \asymp g'$ (in G) if and only if $\pi(g) \asymp \pi(g')$ (in G_{ss}). Thus, the strata of G are exactly the inverse images under π of the strata of G_{ss} and each stratum of G is stable under left or right translation by $\mathcal{Z}^0_{G^0}$.*

Let $g \in G$. By 1.20(b), π induces a surjective homomorphism of tori $\mathcal{Z}^0_{Z_G(g_s)^0} \to \mathcal{Z}^0_{Z_{G_{ss}}(\pi(g)_s)^0}$. This is compatible with the automorphisms

$$\mathrm{Ad}(g_u) : \mathcal{Z}^0_{Z_G(g_s)^0} \to \mathcal{Z}^0_{Z_G(g_s)^0}, \mathrm{Ad}(\pi(g)_u) : \mathcal{Z}^0_{Z_{G_{ss}}(\pi(g)_s)^0} \to \mathcal{Z}^0_{Z_G(g_s)^0},$$

which have finite order. (Some power of g_u belongs to $Z_G(g_s)^0$.) Applying 1.19(a) we see that π restricts to a surjective homomorphism

$$(\mathcal{Z}^0_{Z_G(g_s)^0} \cap Z_G(g_u))^0 \to (\mathcal{Z}^0_{Z_{G_{ss}}(\pi(g)_s)^0} \cap Z_{G_{ss}}(\pi(g)_u))^0.$$

Thus, we have

(a) $\pi(T_G(g)) = T_{G_{ss}}(\pi(g))$.

We have $T_G(g) \subset ((T_G(g)\mathcal{Z}^0_{G^0}) \cap Z_G(g))^0 = T_G(g)(\mathcal{Z}^0_{G^0} \cap Z_G(g))^0 = T_G(g)$. Hence

(b) $T_G(g) = ((T_G(g)\mathcal{Z}^0_{G^0}) \cap Z_G(g))^0$.

If $z'' \in (\mathcal{Z}_{G^0} \cap Z_G(g))^0$, we have

(c) $T_G(z''g) = T_G(g)$ and $z''g \sim g$.

The first assertion follows from 2.1(e) since $(z''g)_s = g_s$ and $(z''g)^{-1}g \in Z_G(g_s)^0$. The second assertion follows from the first since $z''gg^{-1} = z'' \in T(g)$.

If $z \in \mathcal{Z}^0_{G^0}$, then

(d) $T_G(zg) = T_G(g)$ and $zg \asymp g'$.

By 1.2(a) we have $z = z'z''gz'^{-1}g^{-1}$ for some $z' \in \mathcal{Z}^0_{G^0}, z'' \in (\mathcal{Z}^0_{G^0} \cap Z_G(g))^0$. Hence

$$T_G(zg) = T_G(z'z''gz'^{-1}) = z'T_G(z''g)z'^{-1} = z'T_G(g)z'^{-1} = T_G(g)$$

where the third equality comes from (c). We also have $zg = z'z''gz'^{-1} \asymp z''g \asymp g$. (See (c).) This proves (d).

Next, for $g, g' \in G$,

(e) *we have* $\pi(g) \sim \pi(g')$ *(in* G_{ss}*) if and only if* $zg \sim g'$ *(in* G*) for some* $z \in \mathcal{Z}^0_{G^0}$.

Assume that $z \in \mathcal{Z}^0_{G^0}$ and $zg \sim g'$, that is, $g'g^{-1}z^{-1} \in T_G(zg) = T_G(g')$. Applying π and using (a), we obtain $\pi(g')\pi(g)^{-1} \in T_{G_{ss}}(\pi(g)) = T_{G_{ss}}(\pi(g'))$. Thus, $\pi(g) \sim \pi(g')$ in G_{ss}.

Conversely, assume that $\pi(g) \sim \pi(g')$ in G_{ss}, that is, $\pi(g')\pi(g)^{-1} \in T_{G_{ss}}(\pi(g)) = T_{G_{ss}}(\pi(g'))$. Using (a), we deduce that $g'g^{-1} \in T_G(g)\mathcal{Z}^0_{G^0} = T_G(g')\mathcal{Z}^0_{G^0}$. Thus $g' = tzg$ where $t \in T_G(g), z \in \mathcal{Z}^0_{G^0}$. Using (b), we have

$$\begin{aligned} T_G(g') &= ((T_G(g')\mathcal{Z}^0_{G^0}) \cap Z_G(g'))^0 = ((T_G(g)\mathcal{Z}^0_{G^0}) \cap Z_G(tzg))^0 \\ &= ((T_G(g)\mathcal{Z}^0_{G^0}) \cap Z_G(g))^0 = T_G(g) = T_G(zg). \end{aligned}$$

(The third equality holds since tz belongs to the commutative group $T_G(g)\mathcal{Z}^0_{G^0}$ and the fifth equality comes from (d).) Thus, $T_G(g') = T_G(zg)$. Now $g'(zg)^{-1} = t \in T_G(g) = T_G(zg)$. Hence $g' \sim zg$. This proves (e).

The lemma follows from (d) and (e).

3.3. The set of isolated elements of G is a union of strata of G. (Assume that $g \in G$ is isolated and $g' \in G$ is in the stratum of g. We must show that g' is isolated. We may assume that $g' \sim g$. Then $T(g') = T(g)$. By assumption, $T(g) \subset \mathcal{Z}^0_{G^0}$. Hence $T'(g) \subset \mathcal{Z}^0_{G^0}$ and g' is isolated.) The strata of G that consist of isolated elements are called *isolated strata*. We show that

(a) *any isolated stratum of* G *is a single orbit for the action 1.21(c) of* $\mathcal{Z}^0_{G^0} \times G^0$ *on* G.

In view of 2.3(a) and Lemma 3.2, it is enough to consider the case where $\mathcal{Z}^0_{G^0} = \{1\}$. In this case we must show that any isolated stratum C of G is a single G^0-conjugacy class in G. Let $g, g' \in C$. We have $g' = hzgh^{-1}$ where $h \in G^0, z \in T(g)$. Since g is isolated, we have $T(g) = \{1\}$. Hence $g' = hgh^{-1}$. This proves (a).

Lemma 3.4. *Let* C *be a stratum of* G. *If* $g, g' \in C$, *then* $Z_G(g)^0, Z_G(g')^0$ *are* G^0*-conjugate.*

We may assume that $g \sim g'$. We have $g' = tg$ where $t \in T(g)$. If $x \in Z_G(g)^0$, then $x \in Z_G(g_s)^0$ and $t \in \mathcal{Z}_{Z_G(g_s)^0}$; hence $xt = tx$. We have also $xg = gx$, hence $xtg = tgx$. Thus, $x \in Z_G(g')$. We see that $Z_G(g)^0 \subset Z_G(g')$ so that $Z_G(g)^0 \subset Z_G(g')^0$. Interchanging the roles of g, g' we obtain $Z_G(g')^0 \subset Z_G(g)^0$, hence $Z_G(g)^0 = Z_G(g')^0$. The lemma is proved.

3.5. Let $\mathbf{A}$ be the set of all pairs (L, S) where L is a Levi of some parabolic of G^0 and S is an isolated stratum of $N_G L$ with the following property: there exists a parabolic P of G^0 with Levi L such that $S \subset N_G P$. Now G^0 acts on $\mathbf{A}$ by conjugation. Let $G^0 \backslash \mathbf{A}$ be the set of orbits of this action.

Lemma 3.6. *Let C be a stratum of G. For $g \in C$ let $L = L(g)$ and let S be the stratum of $N_G L$ that contains g. Then $(L, S) \in \mathbf{A}$ and $C \mapsto (L, S)$ is a well-defined injective map from the set of strata of G to $G^0 \backslash \mathbf{A}$.*

By 2.2(b), g is isolated in $N_G L$. By 2.1(a) there exists a parabolic P of G^0 with Levi L such that $g \in N_G P$. We have $S \subset N_G P$. (Since S is an isolated stratum of $N_G L$, any element of S is of the form $hzgh^{-1}$ with $z \in \mathcal{Z}_L^0, h \in L$ (see 3.3(a)); clearly, $hzgh^{-1} \in N_G P$.) Thus $(L, S) \in \mathbf{A}$. Let $g' \in G$ be such that $g \asymp g'$. Let $L' = L(g')$ and let S' be the stratum of $N_G L'$ that contains g'. We show that $(L, S), (L', S')$ are in the same G^0-orbit. Replacing g by a G^0-conjugate, we may assume that $g \sim g'$. Then $T(g) = T(g')$, hence $L = L'$. Also, $g'g^{-1} \in T(g)$. Using 2.1(d) we have $T_{N_G L}(g) = T(g) = T(g') = T_{N_G L}(g')$ and $g'g^{-1} \in T_{N_G L}$, hence $g \sim g'$ (relative to $N_G L$). Hence $S = S'$. Thus the map in the lemma is well defined. We show that it is injective. Assume that $g, g' \in G$ are such that for some $h \in G^0$ we have $hLh^{-1} = L'$ where $L = L(g), L' = L(g')$ and $hgh^{-1} \asymp g'$ where $\asymp$ is relative to $N_G L'$. We must show that $g \asymp g'$ (relative to G). Replacing g by hgh^{-1}, we may assume that $L = L', g \asymp g'$ (relative to $N_G L$). Replacing g by an L-conjugate, we may assume further that $g \sim g'$ (relative to $N_G L$). We have $g'g^{-1} \in T_{N_G L}(g) = T_{N_G L}(g')$. But $T_{N_G L}(g) = T(g)$, $T_{N_G L}(g') = T(g')$, so that $g \sim g'$ (relative to G). The lemma is proved.

Proposition 3.7. *The number of strata of G is finite.*

By Lemma 3.6, it is enough to prove that $G^0 \backslash \mathbf{A}$ is finite. Since the Levi subgroups L of parabolics of G^0 fall into finitely many G^0-conjugacy classes, it is enough to show that for any such L, there are only finitely many isolated strata of $N_G L$. This follows from Lemma 2.7 and 3.3(a). The proposition is proved.

3.8. Let P be a parabolic of G^0 with Levi L. Let g be an isolated element of $N_G L \cap N_G P$. (Equivalently, g is an element of $N_G L \cap N_G P$ that is isolated in $N_G L$ (see 2.2(a)); indeed, $N_G L \cap N_G P \subset N_G L$ have the same identity component, L.) We show that

(a) $T(g) \subset (\mathcal{Z}_L \cap Z_L(g))^0$;

(b) $L \subset L(g)$;

We prove (a). Since $g_s \in N_G L \cap N_G P$, we see from 1.12(b) that $\mathcal{Z}^0_{Z_G(g_s)^0} \subset \mathcal{Z}^0_{Z_L(g_s)^0}$. Hence

$$\begin{aligned} T(g) &= (\mathcal{Z}^0_{Z_G(g_s)^0} \cap Z_G(g))^0 \subset (\mathcal{Z}^0_{Z_L(g_s)^0} \cap Z_G(g))^0 \\ &= (\mathcal{Z}^0_{Z_L(g_s)^0} \cap Z_L(g))^0 = (\mathcal{Z}_L \cap Z_L(g))^0 \end{aligned}$$

where the last equality holds since g is isolated in $N_G L$. This proves (a).

From (a) we deduce $Z_{G^0}(T(g)) \supset Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0) = L$ (the last equality comes from 1.10(a)). This proves (b).

3.9. Let P, L, g be as in 3.8. We show that the following three conditions are equivalent:

(i) $L = L(g)$;

(ii) $T(g) = (\mathcal{Z}_L \cap Z_L(g))^0$;
(iii) $Z_G(g_s)^0 \subset L$.
If (ii) holds, then $Z_{G^0}(T(g)) = Z_{G^0}((\mathcal{Z}_L \cap Z_L(g))^0) = L$ (see 1.10(a)) so that (i) holds. Assume now that (i) holds. We show that (ii) holds. By 3.8(a) it is enough to show that $(\mathcal{Z}_L \cap Z_G(g))^0 \subset T(g)$ or that $(\mathcal{Z}_{Z_{G^0}(T(g))} \cap Z_G(g))^0 \subset T(g)$ or that $(\mathcal{Z}_{Z_{G^0}(T(g))} \cap Z_G(g_s)^0 \cap Z_G(g_u))^0 \subset \mathcal{Z}_{Z_G(g_s)^0} \cap Z_G(g_u)$. It is enough to show that $\mathcal{Z}_{Z_{G^0}(T(g))} \cap Z_G(g_s)^0 \subset \mathcal{Z}_{Z_G(g_s)^0}$. This follows from $Z_G(g_s)^0 \subset Z_{G^0}(T(g))$ (a consequence of the definition of $T(g)$).

By 3.8(a), condition (ii) is equivalent to $(\mathcal{Z}_L \cap Z_G(g))^0 \subset T(g)$ and also to $(\mathcal{Z}_L \cap Z_G(g))^0 \subset \mathcal{Z}_{Z_G(g_s)^0}$. Since $(\mathcal{Z}_L \cap Z_G(g))^0 \subset Z_G(g_s)^0$ this is also equivalent to the condition $Z_G(g_s)^0 \subset Z_{G^0}((\mathcal{Z}_L \cap Z_G(g))^0)$ which by 1.10(a) is the same as (iii).

3.10. Let P, L, g be as in 3.8. For any $z \in (\mathcal{Z}_L \cap Z_L(g))^0$, the element zg is isolated in $N_G L \cap N_G P$. (This follows from 2.3(a) for $N_G L \cap N_G L$ instead of G.) We show that

(a) $\{z \in (\mathcal{Z}_L \cap Z_L(g))^0; Z_G(zg_s)^0 \subset L\}$ *is open dense in* $(\mathcal{Z}_L \cap Z_L(g))^0$.
Assume first that $g = s$ is semisimple. Since s is isolated in $N_G L$, we have $(\mathcal{Z}_L \cap Z_L(s))^0 = \mathcal{Z}^0_{Z_L(s)^0}$. Hence it is enough to show that

(b) $\{z \in \mathcal{Z}^0_{Z_L(s)^0}; Z_L(zs)^0 = Z_G(zs)^0\}$ *is open dense in* $\mathcal{Z}^0_{Z_L(s)^0}$.
By 1.12(b) we can find a maximal torus T_1 of $Z_L(s)^0$ which is also a maximal torus of $Z_G(s)^0$. Define $\mathfrak{g}_\alpha, R$ as in 1.5 in terms of s, G, T_1. We have $\mathcal{Z}^0_{Z_L(s)^0} \subset T_1$. For $z \in \mathcal{Z}^0_{Z_L(s)^0}$, $Z_G(zs)^0$ contains T_1 and is normalized by s; hence Lie $Z_G(zs)^0$ is spanned by Lie T_1 and by the $\mathfrak{g}_\alpha$ with $\alpha \in R, \alpha(zs) = 1$. Similarly, Lie $Z_L(zs)^0$ is spanned by Lie T_1 and by the $\mathfrak{g}_\alpha$ with $\alpha \in R, \alpha(zs) = 1, \mathfrak{g}_\alpha \subset \text{Lie } L$. For each $\alpha \in R$ we set $X_\alpha = \{z \in \mathcal{Z}^0_{Z_L(s)^0}; \alpha(zs) \neq 1\}$. We see that it is enough to show that

(c) $\bigcap_{\alpha \in R; \mathfrak{g}_\alpha \not\subset \text{Lie } L} X_\alpha$ *is open dense in* $\mathcal{Z}^0_{Z_L(s)^0}$.
Assume that $\mathfrak{g}_\alpha \not\subset \text{Lie } L$ but $\alpha(zs) = 1$ for all $z \in \mathcal{Z}_{Z_L(s)^0)^0}$. Then $\alpha(s) = 1$ and $\alpha|_{\mathcal{Z}^0_{Z_L(s)^0}} = 1$. Thus, $\mathfrak{g}_\alpha \subset \text{Lie } Z_{G^0}(\mathcal{Z}^0_{Z_L(s)^0}) = \text{Lie } Z_{G^0}((\mathcal{Z}_L \cap Z_L(s))^0) = \text{Lie } L$ (the last equality comes from 1.10(a)). Now $\mathfrak{g}_a \subset \text{Lie } L$ is a contradiction. Thus, for any $\alpha \in R$ such that $\mathfrak{g}_\alpha \not\subset \text{Lie } L$, the open subset X_α of $\mathcal{Z}^0_{Z_L(s)^0}$ is nonempty hence dense. It follows that (c) holds; (b) is proved.

We now consider the general case. Let u be a unipotent element in $Z_L(g_s)^0 g_u$ which is quasi-semisimple in $Z_L(g_s)$. As in 2.4, u is quasi-semisimple in $N_G L$. Hence there exists a Borel β of L and a maximal torus T of β such that β and T are normalized by u. Now U_P is normalized by g, hence by g_u, and hence by lg_u for any $l \in L$. Since $u = lg_u$ for some $l \in L$, we have $u \in N_G(U_P)$. Then u normalizes βU_P, a Borel of G^0 containing T. Thus, u is quasi-semisimple in G and $H = Z_G(u)$ is a reductive group. Similarly, $L_1 = Z_L(u)$ is a reductive group; it is a Levi of the parabolic $P_1 = Z_P(u)^0$ of H^0. By Lemma 2.5 (with G replaced by $N_G L$), g_s is isolated in $N_H(L_1)$. Also, $g_s \in N_H(L_1) \cap N_H(P_1)$. Applying the already proved part of (a) to H, L_1, P_1, g_s instead of G, L, P, g we see that

$\{z \in (\mathcal{Z}_{L_1} \cap Z_{L_1}(g_s))^0; Z_H(zg_s)^0 \subset L_1\}$ *is open dense in* $(\mathcal{Z}_{L_1} \cap Z_{L_1}(g_s))^0$.
It is then enough to show that

(d) $(\mathcal{Z}_{L_1} \cap Z_{L_1}(g_s))^0 = (\mathcal{Z}_L \cap Z_L(g))^0$

and that, for z in (d), we have $Z_H(zg_s)^0 \subset L_1$ if and only if $Z_G(zg_s)^0 \subset L$, or equivalently (see 3.9), that we have $T_H(zg_s) = (\mathcal{Z}_{L_1} \cap Z_{L_1}(zg_s))^0$ if and only if $T_G(zg) = (\mathcal{Z}_L \cap Z_L(zg))^0$. It is enough to prove (d) and that for any z in (d) we have

(e) $T_H(zg_s) = T_G(zg)$,

(f) $(\mathcal{Z}_{L_1} \cap Z_H(zg_s))^0 = (\mathcal{Z}_L \cap Z_L(zg))^0$.

As a special case of 1.7(b) we have $\mathcal{Z}_{L_1}^0 = (\mathcal{Z}_L \cap Z_L(u))^0$; hence

$$(\mathcal{Z}_{L_1} \cap Z_{L_1}(g_s))^0 = (\mathcal{Z}_L \cap Z_L(u) \cap Z_L(ug_s))^0 = (\mathcal{Z}_L \cap Z_L(g))^0,$$

so that (d) holds. (In the last equality we have used that $u \in Z_L(g_s)^0 g_u$.) Similarly, for z in (d) we have

$$(\mathcal{Z}_{L_1} \cap Z_{L_1}(zg_s))^0 = (\mathcal{Z}_L \cap Z_L(u) \cap Z_L(uzg_s))^0 = (\mathcal{Z}_L \cap Z_L(zg))^0,$$

so that (f) holds. Now (e) is shown as in the proof of Lemma 2.5 (for zg instead of g). (a) is proved.

Lemma 3.11. *Let $(L, S) \in \mathbf{A}$. Let P be a parabolic of G^0 with Levi L such that $S \subset N_G P$. Let $S^* = \{g \in S; Z_G(g_s)^0 \subset L\}$. Then S^* is an open dense subset of S.*

By 3.3(a) (for L), S is contained in a connected component δ of $N_G L \cap N_G P$. We set ${}^\delta\mathcal{Z}_L = \mathcal{Z}_L \cap Z_L(g')$ for some/any $g' \in \delta$. By 3.3(a) and 1.21(d), the action 1.21(c) of ${}^\delta\mathcal{Z}_L^0 \times L$ on S is transitive. The restriction of this action to ${}^\delta\mathcal{Z}_L^0$ is a free action (left translation) and ${}^\delta\mathcal{Z}_L^0 \backslash S$ is well defined. The conjugation action of L on S induces a transitive L-action on ${}^\delta\mathcal{Z}_L^0 \backslash S$. Hence the condition that an L-invariant subset X of S is open dense in S is equivalent to the condition that for any ${}^\delta\mathcal{Z}_L^0$-orbit $\mathcal{O}$ on S, the intersection $X \cap \mathcal{O}$ is open dense in $\mathcal{O}$. For $X = S^*$ this last condition holds by 3.10(a). The lemma is proved.

Proposition 3.12. *(a) If $(L, S) \in \mathbf{A}$, then $Y_{L,S} = \bigcup_{x \in G^0} xS^*x^{-1}$ is a stratum of G.*

(b) $(L, S) \mapsto Y_{L,S}$ is a bijection between $G^0 \backslash \mathbf{A}$ and the set of strata of G. In particular, we have a partition $G = \bigsqcup_{L,S} Y_{L,S}$ where L, S runs through a set of representatives for the G^0-orbits in $\mathbf{A}$; this is the same as the partition of G into strata.

We prove (a). By Lemma 3.11, S^* is nonempty. Hence $Y_{L,S}$ is nonempty. Now $Y_{L,S}$ is contained in a stratum of G. (It is enough to show that S^* is contained in a stratum of G. It is also enough to show that, if $g, g' \in S^*$ and $g \sim g'$ relative to $N_G L$, then $g \sim g'$ relative to G. This follows from the equalities $T_G(g) = T_{N_G L(g)}(g) = T_{N_G L}(g), T_G(g') = T_{N_G L(g')}(g') = T_{N_G L}(g')$; see 2.1(d), 3.9.) We show that $Y_{L,S}$ is a stratum of G. It is enough to show that, if $g \in S^*$ and $g' \in G, g \asymp g'$ relative to G, then $g' \in Y_{L,S}$. We may assume that $g' \sim g$ relative to G. Since $T_G(g) = T_G(g')$, we have $L(g) = Z_{G^0} T_G(g) = Z_{G^0} T_G(g') = L(g')$. Since $g \in S^*$, we have $L = L(g)$ (see 3.9). It follows that $L(g') = L$. In particular, $g' \in N_G L$. As above, we have $T_G(g) = T_{N_G L}(g), T_G(g') = T_{N_G L(g')}(g')$ and the last group equals $T_{N_G L}(g')$ since $L(g') = L$. Since $g'^{-1}g \in T_G(g) = T_G(g')$, we see that $g'^{-1}g \in T_{N_G L}(g) = T_{N_G L}(g')$. Thus $g' \sim g$ relative to $N_L G$. It follows that $g' \in S$. More precisely, since $L = L(g')$, we have $g' \in S^*$, as required. This proves (a).

From the definitions it is clear that the map in (b) is the inverse of the map $C \mapsto (L, S)$ in Lemma 3.6. The lemma is proved.

3.13. Let $(L, S) \in \mathbf{A}$. Let

$$\tilde{Y}_{L,S} = \{(g, xL) \in G \times G^0/L; x^{-1}gx \in S^*\}.$$

Define $\pi : \tilde{Y}_{L,S} \to Y_{L,S}$ by $\pi(g, xL) = g$. Now $\mathcal{W}_S = \{n \in N_{G^0}L; nSn^{-1} = S\}/L$ (a subgroup of the finite group $N_{G^0}L/L$) acts (freely) on $\tilde{Y}_{L,S}$ by $n : (g, xL) \mapsto (g, xn^{-1}L)$.

(a) *This makes* $\pi : \tilde{Y}_{L,S} \to Y_{L,S}$ *into a principal* $\mathcal{W}_S$*-bundle.*

We must show that, if $g \in G, x \in G^0, x' \in G^0$ satisfy $x^{-1}gx \in S^*, x'^{-1}gx' \in S^*$, then $x' = xn^{-1}$ with $n \in N_{G^0}L, nSn^{-1} = S$. Replacing g, x, x' by $x'^{-1}gx', x'^{-1}x, 1$, we may assume that $x' = 1$ and we must show that $xLx^{-1} = L, xSx^{-1} = S$. We have $g \in S^*, x^{-1}gx \in S^*$ and $L = L(x^{-1}gx) = x^{-1}L(g)x = x^{-1}Lx$. If $g' \in S$, then $g' = hzgh^{-1}$ for some $h \in L$, $z \in (\mathcal{Z}_L \cap Z_L(g))^0$; see 3.3(a) and 1.21(d). We have

$$x^{-1}zx \in (\mathcal{Z}_{x^{-1}Lx} \cap Z_{x^{-1}Lx}(x^{-1}gx))^0 = (\mathcal{Z}_L \cap Z_L(x^{-1}gx))^0 = (\mathcal{Z}_L \cap Z_L(g))^0$$

(since $x^{-1}gx \in gL$); hence $x^{-1}g'x = (x^{-1}hx)(x^{-1}zx)(x^{-1}gx)(x^{-1}h^{-1}x) \in S$. Thus, $x^{-1}Sx \subset S$. Since S is irreducible we have $x^{-1}Sx = S$, as required.

Since $\tilde{Y}_{L,S}$ is an irreducible variety of dimension $\dim(G^0/L) + \dim S$, it follows that

(b) $Y_{L,S}$ *is an irreducible constructible subset of* G *of dimension* $\dim(G^0/L) + \dim S$.

Lemma 3.14. *Let* $(L, S) \in \mathbf{A}$. *Let* P *be a parabolic of* G^0 *with Levi* L *such that* $S \subset N_G P$. *Let* $\bar{S}$ *be the closure of* S *in* $N_G L$. *Let* $Y'_{L,S} = \bigcup_{x \in G^0} x\bar{S}U_P x^{-1}$. *Then the closure of* $Y_{L,S}$ *in* G *is* $Y'_{L,S}$. *In particular,* $Y'_{L,S}$ *is independent of the choice of* P.

Let $X = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in \bar{S}U_P\}$. Let $\psi : X \to G$ be the first projection. Then ψ is proper since G^0/P is complete and $\bar{S}U_P$ is a closed $\mathrm{Ad}(P)$-stable subset of $N_G P$. Hence $\psi(X) = Y'_{L,S}$ is a closed subset of G. Since X is irreducible of dimension $\dim(G^0/P) + \dim S + \dim U_P = \dim(G^0/L) + \dim S$, we see that $Y'_{L,S}$ is an irreducible variety of dimension $\leq \dim(G^0/L) + \dim S$. Since $Y_{L,S}$ is an irreducible constructible subset of dimension $\dim(G^0/L) + \dim S$ of $Y'_{L,S}$ (see 3.13(b)), it follows that $Y_{L,S}$ is dense in $Y'_{L,S}$. The lemma is proved.

Proposition 3.15. *Let* $(L, S) \in \mathbf{A}$. *The closure of* $Y_{L,S}$ *in* G *is a union of strata of* G.

Let $P, \bar{S}, Y'_{L,S}$ be as in 3.14. By Lemma 3.14, it is enough to show that $Y'_{L,S}$ is a union of strata of G. Since $Y'_{L,S}$ is stable under G^0-conjugacy, it is enough to show that

$$\textit{if } g \in Y'_{L,S},\ g' \in G,\ g \sim g', \textit{ then } g' \in Y'_{L,S};$$

or the stronger statement:

(a) *if* $g \in Y'_{L,S}$ *and* $z \in T_G(g)$, *then* $zg \in Y'_{L,S}$.

Replacing (L, S, P) by a G^0-conjugate we may assume that $g \in \bar{S}U_P$. Now $g_s \in N_G P$ is semisimple, hence it normalizes some Levi of P; see 1.4(a). Hence, replacing (L, S) by a U_P-conjugate we may assume, in addition, that $g_s \in N_G L \cap N_G P$. Let f be the obvious projection of the semidirect product $(N_G L \cap N_G P)U_P$ (see 1.26) onto $N_G L \cap N_G P$ (a homomorphism of algebraic groups). Let $h = f(g)$. We have $g = hu$ where $h \in \bar{S}, u \in U_P$ and $h_s = f(g_s)$. Since $g_s \in N_G L \cap N_G P$, we have $f(g_s) = g_s$ so that $g_s = h_s$. Since the set of isolated elements of $N_G L$ is closed in

N_GL (see Lemma 2.8) and it contains S, it must also contain $\bar{S}$; thus, h is isolated in N_GL so that

(b) $T_{N_GL}(h) = (\mathcal{Z}_L \cap Z_G(h))^0$.

We show that

(c) $T_G(g) = T_G(h)$.

Using 2.1(e) and the equality $g_s = h_s$ we see that it is enough to show that $u \in Z_G(g_s)^0$. Now $g_s = h_s$ commutes with g and h hence it commutes with u. Thus $u \in U_P \cap Z_G(g_s) = U_P \cap Z_G(g_s)^0$ (see 1.11) and (c) follows. We show that

(d) $T_G(h) \subset T_{N_GL}(h)$.

That is, $(\mathcal{Z}_{Z_G(h_s)^0} \cap Z_G(h_u))^0 \subset (\mathcal{Z}_{Z_L(h_s)^0} \cap Z_G(h_u))^0$. It is enough to show that $\mathcal{Z}_{Z_G(h_s)^0} \subset \mathcal{Z}_{Z_L(h_s)^0}$. This follows from 1.12(b) since $h_s \in N_GP$. (Since $S \subset N_GP$, we have $\bar{S} \subset N_GP$; hence $h \in N_GP$.)

From (b), (c), and (d) we deduce that $T_G(g) \subset (\mathcal{Z}_L \cap Z_G(h))^0$. Hence to prove (a) it is enough to show that

$$\textit{if } g = hu, h \in \bar{S}, u \in U_P \textit{ and } z \in (\mathcal{Z}_L \cap Z_G(h))^0, \textit{ then } zg \in \bar{S}U_P.$$

It is enough to show that $z\bar{S} \subset \bar{S}$. This follows from $zS \subset S$ (see 2.3(a)). The lemma is proved.

Proposition 3.16. *For $(L, S) \in \mathbf{A}$, $Y_{L,S}$ is a locally closed (irreducible) subvariety of G. In particular, $Y_{L,S}$ is open in $Y'_{L,S}$.*

This follows from Proposition 3.15 using the following general fact.

Assume that we are given an algebraic variety V and a partition $V = \bigsqcup_{j \in J} V_j$ where V_j are irreducible constructible subsets of V (J is finite) such that for any $j \in J$, the closure of V_j is a union of subsets of the form $V_{j'}$. Then each V_j is locally closed in V.

We may assume that $J = \{1, 2, \dots, n\}$ and $j' \le j$ whenever $V_{j'}$ is contained in the closure of V_j. Then for any j, $\bigsqcup_{j';j' \le j} V_{j'}$ and $\bigsqcup_{j';j'<j} V_{j'}$ are closed in V and $V_j = \bigsqcup_{j';j' \le j} V_{j'} - \bigsqcup_{j';j'<j} V_{j'}$ is locally closed in V.

3.17. We show that, for $(L, S) \in \mathbf{A}$, $Y_{L,S}$ *is a smooth variety.* Since $\tilde{Y}_{L,S}$ is a principal $\mathcal{W}_S$-bundle over $Y_{L,S}$ (see 3.13(a)), it is enough to show that $\tilde{Y}_{L,S}$ is smooth. Consider the morphism $\tilde{Y}_{L,S} \to G^0/L, (g, xL) \mapsto xL$. This is a G^0-equivariant fibration over the homogeneous space G^0/L whose fibre at L is S^*, which is smooth (being open in the homogeneous space S).

4. Dimension estimates

4.1. The estimates in this section are generalizations of results in [L2, §1] which we follow closely.

For any parabolic P of G^0 we set $\tilde{P} = N_GP, \underline{\tilde{P}} = \tilde{P}/U_P, \underline{P} = P/U_P = \underline{\tilde{P}}^0$; let $\pi_P : \tilde{P} \to \underline{\tilde{P}}$ be the canonical map. Let $\mathcal{P}$ be a G^0-conjugacy class of parabolics of G^0. Assume that for each $P \in \mathcal{P}$ we are given a $\underline{P}$-conjugacy class $\mathbf{c}_P$ in $\underline{\tilde{P}}$ such that for any $P \in \mathcal{P}$ and any $g \in G^0$, $\mathrm{Ad}(g)$ carries $\pi_P^{-1}(\mathbf{c}_P)$ onto $\pi_{gPg^{-1}}(\mathbf{c}_{gPg^{-1}})$. Let

$$\mathbf{z} = \{(g, P_1, P_2) \in G \times \mathcal{P} \times \mathcal{P}; g \in \pi_{P_1}^{-1}(\mathcal{Z}^0_{\underline{P}_1}\mathbf{c}_{P_1}) \cap \pi_{P_2}^{-1}(\mathcal{Z}^0_{\underline{P}_2}\mathbf{c}_{P_2})\},$$

$$\mathbf{z}' = \{(g, P_1, P_2) \in G \times \mathcal{P} \times \mathcal{P}; g \in \pi_{P_1}^{-1}(\mathbf{c}_{P_1}) \cap \pi_{P_2}^{-1}(\mathbf{c}_{P_2})\}.$$

We have a partition $\mathbf{z} = \bigcup_{\mathcal{O}} \mathbf{z}_{\mathcal{O}}$ where $\mathcal{O}$ runs over the G^0-orbits in $\mathcal{P} \times \mathcal{P}$ and $\mathbf{z}_{\mathcal{O}} = \{(g, P_1, P_2) \in \mathbf{z}; (P_1, P_2) \in \mathcal{O}\}$. Similarly, we have a partition $\mathbf{z}' = \bigcup_{\mathcal{O}} \mathbf{z}'_{\mathcal{O}}$

where $\mathbf{z}'_{\mathcal{O}}=\{(g,P_1,P_2)\in\mathbf{z}';(P_1,P_2)\in\mathcal{O}\}$. We say that $\mathcal{O}$ is *good* if for some/any $(P_1,P_2)\in\mathcal{O}$, P_1,P_2 have a common Levi. We say that $\mathcal{O}$ is *bad* if it is not good. Let ν be the number of positive roots of G^0. Let $\bar\nu$ be the number of positive roots of $\underline{P}$, $\bar c=\dim\mathbf{c}_P$, $\bar r=\dim(\mathcal{Z}_{\underline{P}}\cap Z_{\underline{P}}(\gamma))^0$ for $P\in\mathcal{P},\gamma\in\mathbf{c}_P$.

Proposition 4.2. *Let* $\mathbf{c}$ *be a* G^0*-conjugacy class in* G, $c=\dim\mathbf{c}$.

(a) For any $P\in\mathcal{P},x\in\mathbf{c}_P$ *we have* $\dim(\mathbf{c}\cap\pi_P^{-1}(x))\le(c-\bar c)/2$.

(b) For any $g\in\mathbf{c}$ *we have* $\dim\{P\in\mathcal{P};g\in\pi_P^{-1}(\mathbf{c}_P)\}\le(\nu-\frac{c}{2})-(\bar\nu-\frac{\bar c}{2})$.

(c) Let $d=2\nu-2\bar\nu+\bar c+\bar r$. *Then* $\dim\mathbf{z}_{\mathcal{O}}\le d$ *if* $\mathcal{O}$ *is good and* $\dim\mathbf{z}_{\mathcal{O}}<d$ *if* $\mathcal{O}$ *is bad. Hence* $\dim\mathbf{z}\le d$.

(d) Let $d'=2\nu-2\bar\nu+\bar c$. *Then* $\dim\mathbf{z}'_{\mathcal{O}}\le d'$ *for all* $\mathcal{O}$. *Hence* $\dim\mathbf{z}'\le d$.

(We make the convention that the empty set has dimension $-\infty$.) In the case where $\mathcal{P}=\{G^0\}$, the proposition is trivial. Therefore we may assume that $\mathcal{P}\ne\{G^0\}$ and that the result is already known when G is replaced by a group of strictly smaller dimension.

We prove (c) and (d). We can map $\mathbf{z}_{\mathcal{O}}$ and $\mathbf{z}'_{\mathcal{O}}$ to $\mathcal{O}$ by $(g,P_1,P_2)\mapsto(P_1,P_2)$. We see that proving (c) and (d) for $\mathbf{z}_{\mathcal{O}},\mathbf{z}'_{\mathcal{O}}$ is the same as proving that for a fixed $(P',P'')\in\mathcal{O}$ we have

(c$'$) $\dim\{\pi_{P'}^{-1}(\mathcal{Z}^0_{\underline{P}'}\mathbf{c}_{P'})\cap\pi_{P''}^{-1}(\mathcal{Z}^0_{\underline{P}''}\mathbf{c}_{P''})\}\le d-\dim\mathcal{O}$,

(d$'$) $\dim\{\pi_{P'}^{-1}(\mathbf{c}_{P'})\cap\pi_{P''}^{-1}(\mathbf{c}_{P''})\}\le d'-\dim\mathcal{O}$,

with strict inequality in (c$'$) if $\mathcal{O}$ is bad. Choose Levi subgroups L' of P' and L'' of P'' such that L',L'' contain a common maximal torus. Then $P'\cap P''$ is a connected group with Levi $L'\cap L''$. Let $\tilde L'=N_GL'\cap\tilde P'$, $\tilde L''=N_GL''\cap\tilde P''$. By 1.26, we may identify $\tilde L'=\underline{P}'$ via $\pi_{P'}$. Similarly, we identify $\tilde L''=\underline{P}''$ via $\pi_{P''}$. Thus we regard $\mathbf{c}_{P'}\subset\tilde L',\mathbf{c}_{P''}\subset\tilde L''$. If $g\in\tilde P'\cap\tilde P''$, then, by 1.25(a),(b), we may write uniquely g in the form $zu''u=zu'v$ where $z\in\tilde L'\cap\tilde L''$, $u''\in L'\cap U_{P''},u\in U_{P'}\cap P'',u'\in L''\cap U_{P'},v\in U_{P''}\cap P'$. We see that (c$'$) and (d$'$) are equivalent to

$$\text{(c}''\text{)}\quad \begin{aligned}&\dim\{(u,v,u'',u',z)\in(U_{P'}\cap P'')\times(U_{P''}\cap P')\times(L'\cap U_{P''})\times(L''\cap U_{P'})\\&\times(\tilde L'\cap\tilde L'');u''u=u'v,zu''\in\mathcal{Z}^0_{L'}\mathbf{c}_{P'},zu'\in\mathcal{Z}^0_{L''}\mathbf{c}_{P''}\}\le d-\dim\mathcal{O},\end{aligned}$$

$$\text{(d}''\text{)}\quad \begin{aligned}&\dim\{(u,v,u'',u',z)\in(U_{P'}\cap P'')\times(U_{P''}\cap P')\times(L'\cap U_{P''})\times(L''\cap U_{P'})\\&\times(\tilde L'\cap\tilde L'');u''u=u'v,zu''\in\mathbf{c}_{P'},zu'\in\mathbf{c}_{P''}\}\le d'-\dim\mathcal{O},\end{aligned}$$

with strict inequality in (c$''$) if $\mathcal{O}$ is bad. When $(u',u'')\in(L''\cap U_{P'})\times(L'\cap U_{P''})$ is fixed, the variety

$$R=\{(u,v)\in(U_{P'}\cap P'')\times(U_{P''}\cap P');u''u=u'v\}=\{(u,v)\in U_{P'}\times U_{P''};u''u=u'v\}$$

is isomorphic to $U_{P'}\cap U_{P''}$. (Indeed, if we set $\tilde u=u'^{-1}u''uu''^{-1}\in U_{P'}$, $\tilde v=vu''^{-1}\in U_{P''}$, then R becomes $\{(\tilde u,\tilde v)\in U_{P'}\times U_{P''};\tilde u=\tilde v\}$.) Since $\dim(U_{P'}\cap U_{P''})=2\nu-2\bar\nu-\dim\mathcal{O}$, we see that (c$''$) and (d$''$) are equivalent to

$$\text{(e)}\quad \begin{aligned}&\dim\{(u'',u',z)\in(L'\cap U_{P''})\times(L''\cap U_{P'})\times(\tilde L'\cap\tilde L'');\\&\qquad zu''\in\mathcal{Z}^0_{L'}\mathbf{c}_{P'},zu'\in\mathcal{Z}^0_{L''}\mathbf{c}_{P''}\}\le\bar c+\bar r,\end{aligned}$$

$$\text{(f)}\quad \begin{aligned}&\dim\{(u'',u',z)\in(L'\cap U_{P''})\times(L''\cap U_{P'})y\times(\tilde L'\cap\tilde L'');\\&\qquad zu''\in\mathbf{c}_{P'},zu'\in\mathbf{c}_{P''}\}\le\bar c,\end{aligned}$$

with strict inequality in (e) if $\mathcal{O}$ is bad.

Let us consider the variety in (f). Let π_3 be the projection of that variety on the z-coordinate. We show that

(g) image(π_3) *is a union of finitely many* $L'\cap L''$*-conjugacy classes in the reductive group* $\tilde L'\cap\tilde L''$ *with identity component* $L'\cap L''$.

Let $H'=\tilde L'\cap\tilde P''=(\tilde L'\cap\tilde L'')(L'\cap U_{P''})$ (semidirect product) and let $f':H'\to\tilde L'\cap\tilde L''$ be the projection $zu''\mapsto z$. Let $H''=\tilde L''\cap\tilde P'=(\tilde L'\cap\tilde L'')(L''\cap U_{P'})$ (semidirect product) and let $f'':H''\to\tilde L''\cap\tilde L'$ be the projection $zu'\mapsto z$. Then image$(\pi_3)=f'(H'\cap\mathbf{c}_{P'})\cap f''(H''\cap\mathbf{c}_{P''})$. Using 1.15(a) for G or reductive groups of smaller dimension, we see that it is enough to show that

$$\text{image}(\pi_3)_s=f'(H'\cap\mathbf{c}_{P'})_s\cap f''(H''\cap\mathbf{c}_{P''})_s=f'(H'\cap(\mathbf{c}_{P'})_s)\cap f''(H''\cap(\mathbf{c}_{P''})_s)$$

is a union of finitely many (semisimple) $L'\cap L''$-conjugacy classes in $\tilde L'\cap\tilde L''$. It is enough to show that $H'\cap(\mathbf{c}_{P'})_s$ is a union of finitely many $H'^0=L'\cap P''$-conjugacy classes in H' and that $H''\cap(\mathbf{c}_{P''})_s$ is a union of finitely many $H''^0=L''\cap P'$-conjugacy classes in H''. Since $(\mathbf{c}_{P'})_s$ is a semisimple L'-conjugacy class in $\tilde L'$ and H' is a closed subgroup of $\tilde L'$, the intersection $H'\cap(\mathbf{c}_{P'})_s$ is a union of finitely many H'^0-conjugacy classes in H' (see 1.27); similarly, $H''\cap(\mathbf{c}_{P''})_s$ is a union of finitely many H''^0-conjugacy classes in H''. This proves (g).

We can write image$(\pi_3)=\chi_1\cup\chi_2\cup\dots\cup\chi_n$ where χ_i are $(L'\cap L'')$-conjugacy classes in $\tilde L'\cap\tilde L''$. The inverse image under π_3 of a point $z\in\chi_i$ is a product of two varieties of the type considered in (a) but for a smaller group (G replaced by $\tilde L'$ or $\tilde L''$), hence by the induction hypothesis it has dimension $\le(\bar c-\dim\chi_i)/2+(\bar c-\dim\chi_i)/2$. Hence $\dim\pi_3^{-1}(\chi_i)\le\bar c$. Since this holds for each $i\in[1,n]$, we see that the variety in (f) has dimension $\le\bar c$. Thus, (d) is proved (assuming the induction hypothesis).

We now consider the variety in (e). Let $\tilde p_3$ be the projection of that variety on the z-coordinate. With the earlier notation we have

$$\text{image}(\tilde p_3)=f'(H'\cap\mathcal{Z}^0_{L'}\mathbf{c}_{P'})\cap f''(H''\cap\mathcal{Z}^0_{L''}\mathbf{c}_{P''}).$$

By 1.21(d) (for $\tilde L',\tilde L''$ instead of G) we have

$$\mathcal{Z}^0_{L'}\mathbf{c}_{P'}={}^{\delta'}\mathcal{Z}^0_{L'}\mathbf{c}_{P'},\mathcal{Z}^0_{L''}\mathbf{c}_{P''}={}^{\delta''}\mathcal{Z}^0_{L''}\mathbf{c}_{P'}$$

where δ' (resp. δ'') is the connected component of $\tilde L'$ (resp. $\tilde L''$) that contains $\mathbf{c}_{P'}$ (resp. $\mathbf{c}_{P''}$). We have ${}^{\delta'}\mathcal{Z}^0_{L'}\subset\mathcal{Z}^0_{L'}\subset\mathcal{Z}^0_{L'\cap L''}\subset H'$ and $f'(\zeta h)=\zeta f'(h)$ for $\zeta\in{}^{\delta'}\mathcal{Z}^0_{L'},h\in H'$ hence $f'(H'\cap\mathcal{Z}^0_{L'}\mathbf{c}_{P'})={}^{\delta'}\mathcal{Z}^0_{L'}f'(H'\cap\mathbf{c}_{P'})$. Similarly, $f''(H''\cap\mathcal{Z}^0_{L''}\mathbf{c}_{P''})={}^{\delta''}\mathcal{Z}^0_{L''}f''(H''\cap\mathbf{c}_{P''})$. By an earlier argument we have $f'(H'\cap\mathbf{c}_{P'})=\epsilon'_1\cup\epsilon'_2\cup\dots\cup\epsilon'_r$, $f''(H''\cap\mathbf{c}_{P''})=\epsilon''_1\cup\epsilon''_2\cup\dots\cup\epsilon''_t$ where $\epsilon'_1,\epsilon'_2,\dots,\epsilon'_r,\epsilon''_1,\epsilon''_2,\dots,\epsilon''_t$ are $(L'\cap L'')$-conjugacy classes in $\tilde L'\cap\tilde L''$. Thus,

$$\text{image}(\tilde p_3)=\bigcup_{i\in[1,r],j\in[1,t]}({}^{\delta'}\mathcal{Z}^0_{L'}\epsilon'_i)\cap({}^{\delta''}\mathcal{Z}^0_{L''}\epsilon''_j).$$

Here the set corresponding to i,j is empty unless ϵ'_i,e''_j are contained in the same connected component $X=X_{ij}$ of $\tilde L'\cap\tilde L''$. In that case we have

$${}^{\delta'}\mathcal{Z}^0_{L'}\subset{}^{X}\mathcal{Z}^0_{L'\cap L''}\quad\text{and}\quad{}^{\delta''}\mathcal{Z}^0_{L''}\subset{}^{X}\mathcal{Z}^0_{L'\cap L''}.$$

(Indeed, since $L'\cap L''$ is a Levi of a parabolic of L', we have $\mathcal{Z}^0_{L'}\cap\mathcal{Z}^0_{L'\cap L''}$. Let $x\in\epsilon'_i\subset\tilde L'\cap\tilde L''$. We have $x=f'(\tilde x)$ for some $\tilde x\in H'\cap\mathcal{Z}^0_{L'}\mathbf{c}_{P'}$. If $z\in{}^{\delta'}\mathcal{Z}^0_{L'}$, then $z\in\mathcal{Z}^0_{L'\cap L''}$ and $z\tilde x=\tilde x$. Hence $f'(z)f'(\tilde x)=f'(\tilde x)f'(z)$; that is, $zx=xz$, so that

$z \in {}^X\mathcal{Z}_{L' \cap L''}$. We see that ${}^{\delta'}\mathcal{Z}^0_{L'} \subset {}^X\mathcal{Z}_{L' \cap L''}$, hence ${}^{\delta'}\mathcal{Z}^0_{L'} \subset {}^X\mathcal{Z}^0_{L' \cap L''}$. Similarly, ${}^{\delta''}\mathcal{Z}^0_{L''} \subset {}^X\mathcal{Z}^0_{L' \cap L''}$, as required.)

Using now 1.24(a) we see that ${}^{\delta'}\mathcal{Z}^0_{L'}\epsilon'_i \cap {}^{\delta''}\mathcal{Z}^0_{L''}\epsilon''_j$ is a finite union of sets of the form $({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''})\delta$ where δ is a $(L' \cap L'')$-conjugacy class in $\tilde{L}' \cap \tilde{L}''$ contained in X. We see that

$$\mathrm{image}(\tilde{p}_3) = \bigcup_{k \in [1,m]} ({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''})\delta_k$$

where $\delta_1, \delta_2, \dots, \delta_m$ are $(L' \cap L'')$-conjugacy classes in $\tilde{L}' \cap \tilde{L}''$ such that for any k, the connected component X_k of $\tilde{L}' \cap \tilde{L}''$ that contains δ_k satisfies

$${}^{\delta'}\mathcal{Z}^0_{L'} \subset {}^{X_k}\mathcal{Z}^0_{L' \cap L''} \quad \text{and} \quad {}^{\delta''}\mathcal{Z}^0_{L''} \subset {}^{X_k}\mathcal{Z}^0_{L' \cap L''}.$$

The inverse image under $\tilde{p}_3$ of a point $z \in ({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''})\delta_k$ is a product of two varieties of the type considered in (a) but for a smaller group (G replaced by $\tilde{L}'$ or $\tilde{L}''$), hence by the induction hypothesis it has dimension $\le (\bar{c} - \dim \delta_k)/2 + (\bar{c} - \dim \delta_k)/2$. Hence

$$\begin{aligned} &\dim \tilde{p}_3^{-1}(({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''})\delta_k) \le \bar{c} - \dim \delta_k + \dim({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''}\delta_k) \\ \text{(h)} \qquad &= \bar{c} - \dim \delta_k + \dim({}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''}) + \dim \delta_k \le \bar{c} + \dim({}^{\delta'}\mathcal{Z}^0_{L'}) = \bar{c} + \bar{r} \end{aligned}$$

where the equality comes from 1.23(a). Since this holds for each $k \in [1, m]$, we see that the variety in (e) has dimension $\le \bar{c} + \bar{r}$.

Moreover, if the second inequality in (h) is an equality, then $\mathcal{O}$ is good. (Indeed, in this case we have ${}^{\delta'}\mathcal{Z}^0_{L'} = {}^{\delta''}\mathcal{Z}^0_{L''}$. Taking centralizers in G^0 for both sides of the previous equality and using 1.10(a), we obtain $L' = L''$; hence $\mathcal{O}$ is good.) Thus (c) is proved (assuming the induction hypothesis).

We show that (b) is a consequence of (d). Let $\mathbf{z}'(\mathbf{c}) = \{(g, P_1, P_2) \in \mathbf{z}'; g \in \mathbf{c}\}$. If $\mathbf{z}'(\mathbf{c}) = \emptyset$, then the variety in (b) is empty and (b) follows. Hence we may assume that $\mathbf{z}'(\mathbf{c}) \ne \emptyset$. From (d) we have $\dim \mathbf{z}'(\mathbf{c}) \le d'$. We map $\mathbf{z}'(\mathbf{c})$ onto $\mathbf{c}$ by the first projection. Each fibre of this map is a product of two copies of the variety in (b). It follows that the variety in (b) has dimension equal to

$$(\dim \mathbf{z}'(\mathbf{c}) - \dim \mathbf{c})/2 \le (d' - c)/2 = \nu - \bar{\nu} + \frac{\bar{c}}{2} - \frac{c}{2}$$

and (b) is proved.

We show that (a) is a consequence of (b). Consider the variety $\{(g, P) \in \mathbf{c} \times \mathcal{P}; g \in \pi_P^{-1}(\mathbf{c}_P)\}$. Projecting it to the g-coordinate and using (b) we see that it has dimension $\le \nu - \bar{\nu} + \frac{\bar{c}}{2} + \frac{c}{2}$. If we project it to the P-coordinate, each fibre will be isomorphic to the variety $\mathbf{c} \cap \pi_P^{-1}(\mathbf{c}_P)$ (with $P \in \mathcal{P}$ fixed). Hence

$$\dim(\mathbf{c} \cap \pi_P^{-1}(\mathbf{c}_P)) \le \nu - \bar{\nu} + \frac{\bar{c}}{2} + \frac{c}{2} - \dim \mathcal{P} = \frac{\bar{c}}{2} + \frac{c}{2}.$$

Now $\mathbf{c} \cap \pi_P^{-1}(\mathbf{c}_P)$ maps onto $\mathbf{c}_P$ (via π_P) and each fibre is the variety in (a). Hence the variety in (a) has dimension $\le \frac{\bar{c}}{2} + \frac{c}{2} - \bar{c} = (c - \bar{c})/2$. The proposition is proved.

4.3. In the case where $\mathcal{O}$ is good, the variety in 4.2(e) is $\mathcal{Z}^0_{L'}\mathbf{c}_{P'} \cap \mathcal{Z}^0_{L'}\mathbf{c}_{P''}$ since $L' = L''$. This is empty if $\mathcal{Z}^0_{L'}\mathbf{c}_{P'} \ne \mathcal{Z}^0_{L'}\mathbf{c}_{P''}$ and is $\mathcal{Z}^0_{L'}\mathbf{c}_{P'}$ if $\mathcal{Z}^0_{L'}\mathbf{c}_{P'} = \mathcal{Z}^0_{L'}\mathbf{c}_{P''}$. In the last case, it follows that $\mathbf{z}_{\mathcal{O}}$ is irreducible of dimension equal to d.

4.4. The inequality in 4.2(b) can be reformulated as follows. Let us fix $P_0 \in \mathcal{P}$ and a $\underline{P}_0$-conjugacy class $\mathbf{c}_0$ in $\underline{\tilde{P}}_0$ with $\dim \mathbf{c}_0 = \bar{c}$. Let δ be the connected component of $\underline{\tilde{P}}_0$ that contains $\mathbf{c}_0$. Then for any $g \in \mathbf{c}$ we have

(a) $\dim\{xP_0 \in G^0/P_0; x^{-1}gx \in \pi_{P_0}^{-1}(\mathbf{c}_0)\} \le (\nu - \frac{c}{2}) - (\bar{\nu} - \frac{\bar{c}}{2})$.

We have the following variant of (a):

(b) $\dim\{xP_0 \in G^0/P_0; x^{-1}gx \in \pi_{P_0}^{-1}(\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0)\} \le (\nu - \frac{c}{2}) - (\bar{\nu} - \frac{\bar{c}}{2})$.

This follows from (a) by observing that, for given g, there exist finitely many $\underline{P}_0$-conjugacy classes $\mathbf{c}^1, \mathbf{c}^2, \dots, \mathbf{c}^t$ in $\underline{\tilde{P}}_0$ of dimension $\bar{c}$ such that

$$x \in G^0, x^{-1}gx \in \pi_{P_0}^{-1}(\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0) \implies x^{-1}gx \in \pi_{P_0}^{-1}(\mathbf{c}^1 \cup \dots \cup \mathbf{c}^t).$$

Since $\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0 = {}^{\delta}\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0$, it is enough to show that

(c) ${}^{\delta}\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0 \cap \pi_{P_0}(\mathbf{c} \cap \tilde{P}_0)$

is a union of finitely many $\underline{P}_0$-conjugacy classes in $\underline{\tilde{P}}_0$. (All $\underline{P}_0$-conjugacy classes contained in ${}^{\delta}\mathcal{Z}^0_{\underline{P}_0}\mathbf{c}_0$ have dimension $\bar{c}$.) Using 1.15(a) it is enough to show that the set of semisimple parts of the elements in (c), that is, ${}^{\delta}\mathcal{Z}^0_{\underline{P}_0}(\mathbf{c}_0)_s \cap \pi_{P_0}(\mathbf{c}_s \cap \tilde{P}_0)$ is a finite union of (semisimple) $\underline{P}_0$-conjugacy classes. This follows from the fact that $\mathbf{c}_s \cap \tilde{P}_0$ is a finite union of (semisimple) P_0-conjugacy classes in $\tilde{P}_0$ (see 1.27).

5. Some complexes on G

5.1. We fix a prime number l invertible in $\mathbf{k}$. We use the term "local system" instead of $\bar{\mathbf{Q}}_l$-local system. For an algebraic variety V let $\mathcal{D}(V)$ be the bounded derived category of $\bar{\mathbf{Q}}_l$-sheaves on V. For $K \in \mathcal{D}(V)$ let $\mathcal{H}^i K$ be the i-th cohomology sheaf of K.

5.2. We fix a connected component D of G. Let C be an isolated stratum of G with $C \subset D$. For any integer $n \ge 1$, invertible in $\mathbf{k}$, let $\mathcal{S}_n(C)$ be the category whose objects are the local systems on C that are equivariant for the (transitive) ${}^{D}\mathcal{Z}^0_{G^0} \times G^0$-action

(a) $(z, x) : g \to xz^n gx^{-1}$

on C.

If a local system is in $\mathcal{S}_n(C)$, then it is also in $\mathcal{S}_{n'}(C)$ for any $n' \ge 1$ invertible in $\mathbf{k}$ such that $n' \in n\mathbf{Z}$. Let $\mathcal{S}(C)$ be the category whose objects are the local systems on C that are in $\mathcal{S}_n(C)$ for some n as above.

5.3. Assume that D generates G/G^0. Then ${}^{D}\mathcal{Z}^0_{G^0}$ is a normal subgroup of G; let $G' = G/{}^{D}\mathcal{Z}^0_{G^0}$, let $\pi : G \to G'$ be the obvious map and let $D' = \pi(D)$, a connected component of G'. Let C be an isolated stratum of G and let $C' = \pi(C)$ (an isolated stratum of G'). Using $\mathcal{Z}^0_{G'^0} = \mathcal{Z}^0_{G^0}/{}^{D}\mathcal{Z}^0_{G^0}$ and the definitions we see that ${}^{D'}\mathcal{Z}^0_{G'^0} = \{1\}$. It follows that C' is single G'^0-conjugacy class in D'.

Let H be the quotient of G by the derived group of G^0. Then H is a reductive group such that H^0 is a torus. Let $f : H^0 \to H^0$ be induced by $\mathrm{Ad}(h_0)$ (with $h_0 \in D$). Then f is independent of the choice of h_0. Its image $f(H^0)$ is a closed normal subgroup of H. Let $G'' = H/f(H^0)$, a reductive group in which the torus G''^0 is central. Let $\rho : G \to G''$ be the composition of the obvious maps $G \to H \to G''$. Let $C'' = \rho(C)$, a connected component of G'' and an isolated stratum of G''. We show that

(a) *the following two conditions for a local system $\mathcal{E}$ on C are equivalent:*

(i) $\mathcal{E} \in \mathcal{S}(C)$;

(ii) $\mathcal{E} \cong \bigoplus_{i=1}^m \pi^*\mathcal{E}'_i \otimes \rho^*\mathcal{E}''_i$ *where* $\mathcal{E}'_i \in \mathcal{S}(C')$ *is irreducible,* $\mathcal{E}''_i \in \mathcal{S}(C'')$ *is of rank* 1 *and* $\pi : C \to C', \rho : C \to C''$ *are the restrictions of* π, ρ *above.*

The fact that (ii) implies (i) is immediate. We prove the converse. We may assume that $\mathcal{E}$ is irreducible. For any torus T over $\mathbf{k}$ and $n' \geq 1$ invertible in $\mathbf{k}$ let $\mu_{n'}(T) = \{t \in T; t^{n'} = 1\}$. The groups $\mu_{n'}(T)$ form an inverse system with transition maps $\mu_{n'} \to \mu_{n''}, t \mapsto t^{n'/n''}$ for n' divisible by n''; let $\mu_\infty(T)$ be the projective limit of this inverse system. Let F be any fibre of $\pi : C \to C'$ and let $g_0 \in F$. We can find $n \geq 1$ invertible in $\mathbf{k}$ such that $\mathcal{E}$ is equivariant for the transitive ${}^D\mathcal{Z}^0_{G^0} \times G^0$ action $(z, x) : g \mapsto xz^n gx^{-1}$ on C. Hence $\mathcal{E}$ corresponds to an irreducible representation of Γ/Γ^0 where $\Gamma = \{(z, x) \in {}^D\mathcal{Z}^0_{G^0} \times G^0; xz^n g_0 x^{-1} = g_0\}$. Then the local system $\mathcal{E}|_F$ is equivariant for the transitive

(b) ${}^D\mathcal{Z}^0_{G^0}$ action $z : g \mapsto z^n g$ on F.

Hence $\mathcal{E}|_F$ corresponds to a representation of $\mu_\infty({}^D\mathcal{Z}^0_{G^0})$ which factors through the finite quotient $\mu_n({}^D\mathcal{Z}^0_{G^0})$. Now $\mu_n({}^D\mathcal{Z}^0_{G^0})$ is contained in the centre of Γ by $z \mapsto (z, 1)$. Hence the image of $\mu_n({}^D\mathcal{Z}^0_{G^0}) \to \Gamma/\Gamma^0$ is contained in the centre of Γ/Γ^0. Using Schur's lemma we see that the representation of Γ/Γ^0 defining $\mathcal{E}$ restricted to $\mu_n({}^D\mathcal{Z}^0_{G^0})$ is an isotypical representation of $\mu_n({}^D\mathcal{Z}^0_{G^0})$. Hence there exists an integer $k \geq 1$ and a local system $\mathcal{L}$ of rank 1 on F, equivariant for the action (b) such that $\mathcal{E}|_F \cong \mathcal{L}^{\oplus k}$. Now $\mathcal{L}$ corresponds to a one-dimensional representation $\lambda : \mu_\infty({}^D\mathcal{Z}^0_{G^0}) \to \bar{\mathbf{Q}}_l^*$ which factors through $\mu_n({}^D\mathcal{Z}^0_{G^0})$.

The restriction of $\rho : G \to G''$ defines a finite covering of tori $\rho' : {}^D\mathcal{Z}^0_{G^0} \to G''^0$ and a finite covering $\rho_0 : F \to C''$; ρ' induces an injective homomorphism $\mu_\infty({}^D\mathcal{Z}^0_{G^0}) \to \mu_\infty(G''^0)$ and a surjective homomorphism $\mathrm{Hom}(\mu_\infty(G''^0), \bar{\mathbf{Q}}_l^*) \to \mathrm{Hom}(\mu_\infty({}^D\mathcal{Z}^0_{G^0}), \bar{\mathbf{Q}}_l^*)$ where Hom denotes continuous homomorphisms from the projective limit topology to the discrete topology. In particular, $\lambda : \mu_\infty({}^D\mathcal{Z}^0_{G^0}) \to \bar{\mathbf{Q}}_l^*$ is the restriction of some homomorphism $\tilde{\lambda} : \mu_\infty(G''^0) \to \bar{\mathbf{Q}}_l^*$ which factors through $\mu_{n'}(G''^0)$ for some n' divisible by n. To $\tilde{\lambda}$ corresponds a one-dimensional local system $\mathcal{E}''$ on C'', equivariant for the transitive G''^0-action $z : h \mapsto z^{n'}h$ on C'' and which satisfies $\rho_0^*\mathcal{E}'' = \mathcal{L}$. Clearly, $\mathcal{E}''$ is also equivariant for the ${}^D\mathcal{Z}^0_{G^0} \times G^0$ action $(z, x) : g \mapsto \rho(x)\rho(z)^{n'}g\rho(x)^{-1}$ on C''. (We have $\rho(x)\rho(z)^{n'}g\rho(x)^{-1} = \rho(z)^{n'}g$ for $g \in C''$ since G''^0 is central in G''). Moreover, $\rho : C \to C''$ is compatible with the ${}^D\mathcal{Z}^0_{G^0} \times G^0$ actions (given by $(z, x) : g \mapsto xz^{n'}gx^{-1}$ on C and as above on C''), hence $\rho^*\mathcal{E}'' \in \mathcal{S}(C)$. Let $\tilde{\mathcal{E}} = \mathcal{E} \otimes \rho^*\mathcal{E}''^*$. We have $\tilde{\mathcal{E}} \in \mathcal{S}(C)$. Moreover, $\tilde{\mathcal{E}}|_F \cong \bar{\mathbf{Q}}_l^{\oplus k}$. Since $\tilde{\mathcal{E}}$ is equivariant for the conjugation action of G^0 which permutes transtively the fibres of $C \to C'$ it follows that the restriction of $\tilde{\mathcal{E}}$ to any fibre of $C \to C'$ is isomorphic to $\bar{\mathbf{Q}}_l^{\oplus k}$. It follows that there is a local system $\mathcal{E}'$ on C' whose inverse image under $C \to C'$ is $\tilde{\mathcal{E}}$. Moreover, $\mathcal{E}'$ is automatically irreducible and G^0-equivariant (for the action $x : g \mapsto xgx^{-1}$ on C'). In this action, the subgroup ${}^D\mathcal{Z}^0_{G^0}$ of G^0 acts trivially; hence $\mathcal{E}'$ is G'^0-equivariant (for the conjugation action of G'^0). Hence $\mathcal{E}' \in \mathcal{S}(C')$. We have $\mathcal{E} \cong \pi^*\mathcal{E}' \otimes \rho^*\mathcal{E}''$ and (a) is proved.

5.4. Let $(L, S) \in \mathbf{A}$. Let P be a parabolic of G^0 with Levi L such that $S \subset N_G P$. To simplify notation, in this section we set $Y = Y_{L,S}, \tilde{Y} = \tilde{Y}_{L,S}$. As in 3.14, let $X = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in \bar{S}U_P\}$; let $\psi : X \to G$ be the first projection. Let f be the obvious projection of the semidirect product $(N_G L \cap N_G P)U_P$ (see 1.26) onto $N_G L \cap N_G P$ (a homomorphism of algebraic groups).

Lemma 5.5. $(g, xL) \mapsto (g, xP)$ *is an isomorphism* $\gamma : \tilde{Y} \xrightarrow{\sim} \psi^{-1}(Y)$.

We verify this only at the level of sets. Assume that $(g, xL), (g', x'L) \in \tilde{Y}$ have the same image under ψ. Then $g = g'$ and $x' = xp$ with $p \in P$. We have $x^{-1}gx \in S^*, x'^{-1}gx' \in S^*$, hence $p^{-1}x^{-1}gxp \in S^*$. It follows that $L(x^{-1}gx) = L = L(p^{-1}x^{-1}gxp) = p^{-1}L(x^{-1}gx)p = p^{-1}Lp$. Thus, $p^{-1}Lp = L$ so that $p \in L$ and $xL = x'L$. Thus, γ is injective.

To show that γ is surjective it is enough to show that, if $g \in S^*, x \in G^0$ satisfy $x^{-1}gx \in \bar{S}U_P$, then $u^{-1}x^{-1}gxu \in S^*$ for some $u \in U_P$, or equivalently that, if $g' \in \bar{S}U_P, x \in G^0$ satisfy $xg'x^{-1} \in S^*$, then $u^{-1}g'u \in S^*$ for some $u \in U_P$.
Now $g'_s \in N_GP$ is semisimple, hence it normalizes some Levi of P (see 1.4(a)); that is, some U_P-conjugate of L. Hence, replacing g', x by $u'^{-1}g'u', xu'$ for some $u' \in U_P$ we may assume, in addition, that $g'_s \in N_GL \cap N_GP$. We have $g' = hv$ where $h = f(g') \in \bar{S}, v \in U_P$ and $h_s = f(g'_s)$. Since $g'_s \in N_GL \cap N_GP$, we have $f(g'_s) = g'_s$ so that $g'_s = h_s$. Then $h^{-1}g' \in U_P \cap Z_G(g'_s) = U_P \cap Z_G(g'_s)^0$. Using 2.1(e), we see that $T(g') = T(h)$. By 1.22(b), we can find $h' \in S$ such that $h_s = h'_s$ and $h'^{-1}h \in Z_G(h_s)^0$. Then $T(h) = T(h')$, by 2.1(e). Thus, $T(g') = T(h')$. By 3.8(b) we have $L \subset L(h') = L(g')$. Since $xg'x^{-1} \in S^*$, we have $L(xg'x^{-1}) = L$; hence $L(g') = x^{-1}Lx$. Thus, $L \subset x^{-1}Lx$. Since $L, x^{-1}Lx$ are irreducible of the same dimension, we have $L = x^{-1}Lx$. Hence $N_GL = x^{-1}N_GLx$. Since $xg'x^{-1} \in S^* \subset N_GL$, we have $g' \in N_GL$. Thus, $g' \in N_GL \cap \bar{S}U_P$ hence $g' \in \bar{S}$. Since $g' \in x^{-1}S^*x$, we see that $x^{-1}Sx \cap \bar{S} \neq \emptyset$. Now $x^{-1}Sx$ is a stratum of N_GL since $x \in N_GL$. Since $\bar{S}$ is a union of strata of N_GL, one of which is S and the others have dimension $< \dim S$, we see that $x^{-1}Sx = S$. This, together with $x \in N_GL$, implies that $x^{-1}S^*x = S^*$. Since $g' \in x^{-1}S^*x$, we see that $g' \in S^*$. Thus, γ is surjective. The lemma is proved.

5.6. Let $\mathcal{E} \in \mathcal{S}(S)$. We define a local system $\tilde{\mathcal{E}}$ on $\tilde{Y}$ by the requirement that $b^*\mathcal{E} = a^*\tilde{\mathcal{E}}$ where $a(g,x) = (g, xL), b(g,x) = x^{-1}gx$ in the diagram

$$\tilde{Y} \xleftarrow{a} \{(g,x) \in G \times G^0; x^{-1}gx \in S^*\} \xrightarrow{b} S.$$

(We use the fact that a is a principal L-bundle and $b^*\mathcal{E}$ is L-equivariant.) For any stratum S' of $N_GL \cap N_GP$ such that $S' \subset \bar{S}$ we set $X_{S'} = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in S'U_P\}$. Then $X = \bigsqcup_{S'} X_{S'}$ (union over all $S' \subset \bar{S}$ as above; there are only finitely many S' in the union; see 3.7). By Lemma 2.8, each S' is an isolated stratum of $N_GL \cap N_GP$. Note that each $X_{S'}$ is smooth, irreducible. We define a local system $\bar{\mathcal{E}}$ on X_S by the requirement that $b'^*\mathcal{E} = a'^*\bar{\mathcal{E}}$ where $a'(g,x) = (g, xP)$, $b'(g,x) = f(x^{-1}gx)$ in the diagram

$$X_S \xleftarrow{a'} \{(g,x) \in G \times G^0; x^{-1}gx \in SU_P\} \xrightarrow{b'} S.$$

(We use the fact that a' is a principal P-bundle and $b'^*\mathcal{E}$ is P-equivariant.) It is easy to see that the restriction of $\bar{\mathcal{E}}$ to $\tilde{Y}$ (identified with an open subset of X as in Lemma 5.5) is $\tilde{\mathcal{E}}$. The intersection cohomology complexes $IC(X, \bar{\mathcal{E}})$ (on X) and $IC(\bar{S}, \mathcal{E})$ (on $\bar{S}$) are related by

(a) $a''^*IC(X, \bar{\mathcal{E}}) = b''^*IC(\bar{S}, \mathcal{E})$

where $a''(g,x) = (g, xP), b''(g,x) = f(x^{-1}gx)$ in the diagram

$$X \xleftarrow{a''} \{(g,x) \in G \times G^0; x^{-1}gx \in \bar{S}U_P\} \xrightarrow{b''} \bar{S}.$$

Here a'' is a principal P-bundle and b'' is a locally trivial fibration with smooth connected fibres. We write $\psi : X \to \bar{Y}$ for the restriction of $\psi : X \to G$. Here $\bar{Y}$ is the closure of Y in G. Recall that we have a finite covering (principal bundle)

$\pi : \tilde{Y} \to Y$ (see 3.13(a)), hence $\pi_!\tilde{\mathcal{E}}$ is a well-defined local system on Y. Thus $IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})$ is well defined (on $\bar{Y}$).

Proposition 5.7. *$\psi_!(IC(X, \bar{\mathcal{E}}))$ is canonically isomorphic to $IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})$.*

Let $K = IC(X, \bar{\mathcal{E}})$ and let $K^* = IC(X, \bar{\mathcal{E}}^*)$ where $\bar{\mathcal{E}}^*$ is defined like $\bar{\mathcal{E}}$ by replacing $\mathcal{E}$ by the dual local system $\mathcal{E}^*$. Then K^* is the Verdier dual of K with a suitable shift. Since ψ is proper, it follows that $\psi_!(K^*)$ is the Verdier dual of $\psi_!K$ with a suitable shift. We have $K|_{\tilde{Y}} = \bar{\mathcal{E}}|_{\tilde{Y}} = \tilde{\mathcal{E}}$. Using Lemma 5.5, we see that $\psi_!K|_Y = \pi_!\tilde{\mathcal{E}}$. By the definition of an intersection cohomology complex we see that it is enough to verify the following statement.

For any $i > 0$ we have $\dim \text{supp}\mathcal{H}^i(\psi_!K) < \dim Y - i$ and $\dim \text{supp}\mathcal{H}^i(\psi_!(K^*)) < \dim Y - i$.

We shall only verify this for K; the corresponding statement for K^* is entirely analogous.

If $g \in \bar{Y}$, the stalk $\mathcal{H}^i_g(\psi_!K)$ at g is equal to $H^i_c(\psi^{-1}(g), K)$. We have a partition $\psi^{-1}(g) = \bigcup_{S'}(\psi^{-1}(g) \cap X_{S'})$ where S' runs over the strata of $N_GL \cap N_GP$ contained in $\bar{S}$. If $H^i_c(\psi^{-1}(g), K) \neq 0$, then $H^i_c(\psi^{-1}(g) \cap X_{S'}, K) \neq 0$ for some S'. Hence it is enough to prove:

For any $i > 0$ and any S' as above we have $\dim\{g \in \bar{Y}; H^i_c(\psi^{-1}(g) \cap X_{S'}, K) \neq 0\} < \dim Y - i$.

Assume first that $S' \neq S$. If $H^i_c(\psi^{-1}(g) \cap X_{S'}, K) \neq 0$, then the hypercohomology spectral sequence for K on $\psi^{-1}(g) \cap X_{S'}$ shows that we can write $i = j_1 + j_2$ with $j_2 \leq 2\dim(\psi^{-1}(g) \cap X_{S'})$ and $\mathcal{H}^{j_1}(K|_{\psi^{-1}(g)\cap X_{S'}}) \neq 0$; hence $\mathcal{H}^{j_1}(K)|_{X_{S'}} \neq 0$. Using 5.6(a), we see that $\mathcal{H}^{j_1}(K)$ is a local system on $X_{S'}$ (since $\mathcal{H}^{j_1}IC(\bar{S}, \mathcal{E})$ is a local system on S', which is a $(\mathcal{Z}^0_L \times L)$-orbit on $N_GL \cap N_GP$). Thus, $X_{S'} \subset \text{supp}\mathcal{H}^{j_1}(K)$. Since $K = IC(X, \bar{\mathcal{E}})$, it follows that $j_1 < \dim X - \dim X_{S'} = \dim S - \dim S'$. Thus we have $i < 2\dim(\psi^{-1}(g) \cap X_{S'}) + \dim S - \dim S'$ and it is enough to show that

$$\dim\{g \in \bar{Y}; \dim(\psi^{-1}(g) \cap X_{S'}) > \frac{i}{2} - \frac{1}{2}(\dim S - \dim S')\} < \dim Y - i.$$

If this is violated for some $i > 0$, it would follow that the space of triples

$$\{(g, xP, x'P) \in \bar{Y} \times G^0/P \times G^0/P; x^{-1}gx \in S'U_P, x'^{-1}gx' \in S'U_P\}$$

has dimension $> \dim Y - i + i - (\dim S - \dim S') = \dim G^0/L + \dim S'$. This contradicts 4.2(c).

Next, assume that $S' = S$. If $H^i_c(\psi^{-1}(g) \cap X_S, K) \neq 0$, then $i \leq 2\dim(\psi^{-1}(g) \cap X_S)$ since $K|_{\psi^{-1}(g)\cap X_S}$ is a local system. Hence it is enough to show that for $i > 0$ we have

$$\dim\{g \in \bar{Y}; \dim(\psi^{-1}(g) \cap X_S) \geq \frac{i}{2}\} < \dim Y - i.$$

Assume that this is violated for some $i > 0$. Thus, setting $F = \{g \in \bar{Y}; \dim(\psi^{-1}(g) \cap X_S) \geq \frac{i}{2}\}$, we have $\dim F \geq \dim Y - i$. Then the space of triples

$$\{(g, xP, x'P) \in F \times G^0/P \times G^0/P; x^{-1}gx \in SU_P, x'^{-1}gx' \in SU_P\}$$

has dimension $\geq \dim G^0/L + \dim S$ (and the last inequality is strict if $\dim F > \dim Y - i$). From 4.2(c) we see that this space of triples has dimension $\leq \dim G^0/L + \dim S$; hence it has dimension equal to $\dim G^0/L + \dim S$, which forces $\dim F = \dim Y - i$. We partition our space of triples into subsets by specifying the G^0-orbit of $(xPx^{-1}, x'Px'^{-1})$ (for simultaneous conjugation). By 4.2(c), the subset

corresponding to a bad orbit has dimension $< \dim G^0/L + \dim S$. It follows that the subset corresponding to some good orbit has dimension equal to $\dim G^0/L + \dim S$. Thus there exists $n \in N_{G^0}L$ such that
(a)
$\{(g, xP, x'P) \in F \times G^0/P \times G^0/P; x^{-1}gx \in SU_P, x'^{-1}gx' \in SU_P; x^{-1}x' \in PnP\}$
has dimension equal to $\dim G^0/L + \dim S$. By 4.3, the variety
(b)
$\{(g, xP, x'P) \in G \times G^0/P \times G^0/P; x^{-1}gx \in SU_P, x'^{-1}gx' \in SU_P; x^{-1}x' \in PnP\}$
is empty if $nSn^{-1} \neq S$ and is irreducible of dimension $\dim G^0/L + \dim S$ if $nSn^{-1} = S$. It follows that we must have $nSn^{-1} = S$ and the variety (a) is dense in the variety (b). It follows that F is dense in the image I of the variety (b) under the first projection. Hence $\dim I = \dim Y - i$. If $g \in S^*$, then (g, P, nP) belongs to the variety (b). Thus, $S^* \subset I$. Since I is stable under G^0-conjugacy, we must have $Y \subset I$. Thus, $\dim I \geq \dim Y$. It follows that $\dim Y - i \geq \dim Y$, hence $i \leq 0$, contradicting $i > 0$. The proposition is proved.

6. Cuspidal local systems

6.1. In this section we fix an isolated stratum C of G. Let D be the connected component of G that contains C.

Lemma 6.2. *Let P be a parabolic of G and let $g \in C \cap N_G P$. Let $\mathbf{c}_P$ be the P/U_P-conjugacy class of the image of g in $N_G P/U_P$. Let $\delta = \dim C - \dim {}^D\mathcal{Z}^0_{G^0} - \dim \mathbf{c}_P$. Then $\dim(C \cap gU_P) \leq \delta/2$. Hence for any $\mathcal{E} \in \mathcal{S}(C)$ we have $H^i_c(C \cap gU_P, \mathcal{E}) = 0$ for $i > \delta$.*

If $y \in gU_P$, then the semisimple elements y_s, g_s normalize U_P and are in the same U_P-coset; hence, by a standard argument, are U_P-conjugate. Using the finiteness of the number of unipotent classes in $Z_G(g_s)$ (see 1.15), we deduce that gU_P is contained in the union of finitely many G^0-conjugacy classes in G. It is then enough to show that for any G^0-conjugacy class $\mathbf{c}$ in G such that $\mathbf{c} \subset C$, we have $\dim(\mathbf{c} \cap gU_P) \leq \frac{1}{2}\delta$. This follows from 4.2(a), since $\dim C = \dim \mathbf{c} + \dim {}^D\mathcal{Z}^0_{G^0}$; see 1.23(b).

6.3. Let $\mathcal{E} \in \mathcal{S}(C)$. We say that $\mathcal{E}$ is a *cuspidal local system* or that $(C, \mathcal{E})$ is a *cuspidal pair* for G if, for any P, g as in 6.2, with $P \neq G^0$, we have $H^\delta_c(C \cap gU_P, \mathcal{E}) = 0$ where δ is as in 6.2.

Lemma 6.4. *Let $\mathcal{E} \in \mathcal{S}(C)$. Assume that D generates G/G^0. Let us write $\mathcal{E} = \bigoplus_{i=1}^m \pi^*\mathcal{E}'_i \otimes \rho^*\mathcal{E}''_i$ as in 5.3(a). Then $(C, \mathcal{E})$ is a cuspidal pair for G if and only if $(C', \mathcal{E}'_i)$ is a cuspidal pair for $G' = G/{}^D\mathcal{Z}^0_{G^0}$ for $i = 1, \ldots, m$ (with C' as in 5.3).*

Since the property of being cuspidal is preserved by taking direct sums of local systems or by passage to a direct summand, we see that we may assume that $m = 1$. Next we note that, for P, g as in 6.2, we have $(\pi^*\mathcal{E}'_1 \otimes \rho^*\mathcal{E}''_1)|_{C \cap gU_P} \cong (\pi^*\mathcal{E}'_1)|_{C \cap gU_P}$ since $(\rho^*\mathcal{E}''_1)|_{C \cap gU_P} \cong \bar{\mathbf{Q}}_l$. Hence we may also assume that $\mathcal{E}''_1 = \bar{\mathbf{Q}}_l$. We set $\mathcal{E}' = \mathcal{E}'_1$. We must show that $(C, \mathcal{E})$ is a cuspidal pair for G if and only if $(C', \mathcal{E}')$ is a cuspidal pair for G'.

Assume that $(C', \mathcal{E}')$ is a cuspidal pair for G'. To show that $(C, \mathcal{E})$ is a cuspidal pair for G we must show that for any P, g as in 6.2 with $P \neq G^0$ we have $H^\delta_c(C \cap gU_P, \mathcal{E}) = 0$ where δ is as in 6.2. Let $\bar{P} = \pi(P)$, a proper parabolic of G'^0. By assumption we have $H_c^{\tilde\delta}(C'\cap\pi(g)U_{\bar P},\mathcal{E}')=0$ where $\mathbf{c}_{\bar P}$ is the $\bar P/U_{\bar P}$-conjugacy class of the image of $\pi(g)$ in $N_{G'}\bar P/U_{\bar P}$ and $\tilde\delta=\dim C'-\dim\mathbf{c}_{\bar P}$. It is clear that π restricts to an isomorphism $C\cap gU_P\xrightarrow{\sim}C'\cap\pi(g)U_{\bar P}$. Since $\dim C-\dim{}^D\mathcal{Z}^0_{G^0}=\dim C'$ and $\dim\mathbf{c}_P=\dim\mathbf{c}_{\bar P}$ ($\mathbf{c}_P$ as in 6.2), we have $\delta=\tilde\delta$. Hence $H_c^\delta(C\cap gU_P,\mathcal{E})\cong H_c^{\tilde\delta}(C'\cap\pi(g)U_{\bar P},\mathcal{E}')$ and we see that $(C,\mathcal{E})$ is a cuspidal pair for G. The reverse implication is proved in a similar way. The lemma is proved.

6.5. Lemma 6.4 shows that the study of cuspidal pairs for G can be reduced to the analogous problem in the case where ${}^D\mathcal{Z}^0_{G^0}=\{1\}$. We will show that we can further reduce to the case of a unipotent class.

Assume that ${}^D\mathcal{Z}^0_{G^0}=\{1\}$. Then C is a single G^0-conjugacy class in D. Let $\mathcal{E}\in\mathcal{S}(C)$. For any $x\in C_s$ let $\mathcal{C}^x=\{u\in Z_G(x);u\text{ unipotent, }xu\in C\}$. Then $Z_{G^0}(x)$ acts transitively (by conjugation) on $\mathcal{C}^x$. Note that any $Z_G(x)^0$-conjugacy class in $\mathcal{C}^x$ is a stratum of $Z_G(x)$. (It is enough to show that $(\mathcal{Z}^0_{Z_G(x)^0}\cap Z_G(u))^0=\{1\}$ for $u\in\mathcal{C}^x$ that is, $T_G(xu)=\{1\}$. This follows from ${}^D\mathcal{Z}^0_{G^0}=\{1\}$ and the fact that xu is isolated in G.)

Let $\mathcal{E}^x$ be the inverse image of $\mathcal{E}$ under $\mathcal{C}^x\to C,u\mapsto xu$.

Lemma 6.6. *The following three conditions are equivalent:*

(i) $(C,\mathcal{E})$ is a cuspidal pair for G;

(ii) there exists $x\in C_s$ such that for some/any $Z_G(x)^0$-conjugacy class $\mathcal{C}'$ in $\mathcal{C}^x$, the pair $(\mathcal{C}',\mathcal{E}^x|_{\mathcal{C}'})$ is cuspidal for $Z_G(x)$.

(iii) for any $x\in C_s$ and for some/any $Z_G(x)^0$-conjugacy class $\mathcal{C}'$ in $\mathcal{C}^x$, the pair $(\mathcal{C}',\mathcal{E}^x|_{\mathcal{C}'})$ is cuspidal for $Z_G(x)$.

We prove (i) assuming that (ii) holds. Assume that we are given $g\in C$ and a proper parabolic P of G^0 such that $g\in N_GP$. The P/U_P-conjugacy class of the image $\bar g$ of g in N_GP/U_P is denoted by $\mathbf{c}_P$. We must show that $H_c^\delta(C\cap gU_P,\mathcal{E})=0$ where $\delta=\dim C-\dim\mathbf{c}_P$. Since $g\in N_G(U_P)$, we have $g_s\in N_G(U_P)$; hence $(g_sU_P)_s=\mathcal{V}$ where $\mathcal{V}=\{vg_sv^{-1};v\in U_P\}\subset C_s$. Hence $y\mapsto y_s$ is a morphism $f:C\cap gU_P\to\mathcal{V}$. Since f commutes with the conjugation action of U_P on $C\cap gU_P$ and $\mathcal{V}$, we have $H_c^\delta(C\cap gU_P,\mathcal{E})\cong H_c^{\delta-2\dim\mathcal{V}}(f^{-1}(g_s),\mathcal{E})$. Now $u\mapsto g_su$ defines $\{u\in\mathcal{C}^{g_s};u\in g_uU_P\}\xrightarrow{\sim}f^{-1}(g_s)$ or equivalently, $\mathcal{C}^{g_s}\cap g_uU_Q\xrightarrow{\sim}f^{-1}(g_s)$ where $Q=P\cap Z_G(g_s)^0$ (a parabolic of $Z_G(g_s)^0$ with $U_Q=U_P\cap Z_G(g_s)=U_P\cap Z_G(g_s)^0$). It is then enough to show that $H_c^{\delta-2\dim\mathcal{V}}(\mathcal{C}^{g_s}\cap g_uU_Q,\mathcal{E}')=0$. (We have $g_u\in N_GQ\cap Z_G(g_s)$.) Since $Z_{G^0}(g_s)$ acts transitively on $\mathcal{C}^{g_s}$, we see that $\mathcal{C}^{g_s}$ is a disjoint union of finitely many $Z_G(g_s)^0$-conjugacy classes. It is enough to show that for any such conjugacy class $\mathcal{C}'$ we have $H_c^{\delta-2\dim\mathcal{V}}(\mathcal{C}'\cap g_uU_Q,\mathcal{E}')=0$. (If this holds for some $\mathcal{C}'$, then it automatically holds for all $\mathcal{C}'$, by the transitivity of the $Z_{G^0}(g_s)$-action on $\mathcal{C}^{g_s}$.) This would follow from (iii) provided we verify that

$$Q\neq Z_G(g_s)^0\quad\text{and}\quad\delta-2\dim\mathcal{V}=\delta'$$

where $\mathbf{c}_Q$ is the Q/U_Q-conjugacy class of the image of g_u in $N_{Z_G(g_s)}Q/U_Q$ and $\delta'=\dim\mathcal{C}'-\dim\mathbf{c}_Q$.

If $Q=Z_G(g_s)^0$, then $Z_G(g_s)^0\subset P$. Hence g is not isolated, a contradiction.

We now show that $\delta-2\dim\mathcal{V}=\delta'$; that is, $\dim C-\dim\mathbf{c}_P-2\dim\mathcal{V}=\dim\mathcal{C}'-\dim\mathbf{c}_Q$. Now $Q/U_Q=Z_{P/U_P}(\bar g_s)^0$ and $\mathbf{c}_Q$ may be identified with the $Z_{P/U_P}(\bar g_s)^0$-conjugacy class of $\bar g_u$ in N_GP/U_P. Consider the morphism $x\mapsto x_s,\mathbf{c}_P\to(\mathbf{c}_P)_s$; the fibre of this morphism at $\bar g_s$ is the $Z_{P/U_P}(\bar g_s)$-conjugacy class of $\bar g_u$ in N_GP/U_P which has pure dimension $\dim\mathbf{c}_Q$. We see that $\dim\mathbf{c}_P-\dim\mathbf{c}_Q=\dim(\mathbf{c}_P)_s$. We also have $\dim C = \dim \mathcal{C}' + \dim C_s$. Thus $\dim C - \dim \mathbf{c}_P - 2\dim \mathcal{V} - \dim \mathcal{C}' + \dim \mathbf{c}_Q = \dim C_s - \dim(\mathbf{c}_P)_s - 2\dim \mathcal{V}$. To prove that this is 0 it is enough to show that $(\dim G - \dim Z_G(g_s)) - (\dim L_1 - \dim Z_{L_1}(g_s)) - 2(\dim U_P - \dim Z_{U_P}(g_s)) = 0$ where L_1 is a Levi subgroup of P normalized by g_s. Since $\dim G = \dim L_1 + 2\dim U_P$ it is enough to show that $\dim Z_G(g_s) = Z_{L_1}(g_s) + 2\dim Z_{U_P}(g_s)$. This follows from the fact that $Z_G(g_s)^0$ is reductive and $Z_{L_1}(g_s)^0$ is a Levi of a parabolic of $Z_G(g_s)^0$ with unipotent radical $Z_{U_P}(g_s)$ (see 1.12(a)). This proves (i).

We prove (iii) assuming that (i) holds. Let $x \in C_s$ and let $\mathcal{C}'$ be a $Z_G(x)^0$-conjugacy class in $\mathcal{C}^x$. Assume that we are given $y \in \mathcal{C}'$ and a proper parabolic Q of $Z_G(x)^0$ such that $y \in N_{Z_G(x)}Q$. The Q/U_Q-conjugacy class of the image $\bar{y}$ of y in $N_{Z_G(x)}Q/U_Q$ is denoted by $\mathbf{c}_Q$. We must show that $H_c^{\delta'}(\mathcal{C}' \cap yU_Q, \mathcal{E}') = 0$ where $\delta' = \dim \mathcal{C}' - \dim \mathbf{c}_Q$. It is enough to show that $H_c^{\delta'}(\mathcal{C}^{g_s} \cap g_u U_Q, \mathcal{E}') = 0$. Let $g = xy = yx$. We have $g \in C$ and $x = g_s, y = g_u$. By 1.18(a), we can find a parabolic P of G^0 such that $g \in N_G P$ and $P \cap Z_G(g_s)^0 = Q$. Clearly, $P \neq G^0$. By the arguments and notation in the first part of the proof, we see that $H_c^{\delta'}(\mathcal{C}^{g_s} \cap g_u U_Q, \mathcal{E}')$ is isomorphic to $H_c^{\delta}(C \cap gU_P, \mathcal{E})$ which is 0 by assumption. This proves (iii).

Now (ii) and (iii) are equivalent since G^0 acts transitively by conjugation on C_s. The lemma is proved.

6.7. Let A be a simple perverse sheaf on G. We say that A is *admissible* if the following condition is satisfied: there exists $(L, S) \in \mathbf{A}$, a cuspidal irreducible local system $\mathcal{E} \in \mathcal{S}(S)$ and an irreducible direct summand $\tilde{\mathcal{E}}_1$ of the local system $\tilde{\mathcal{E}}$ on $\bar{Y}$ (with $\bar{Y}, \tilde{\mathcal{E}}$ defined as in 5.6 in terms of $L, S, \mathcal{E}$), such that A is isomorphic to $IC(\bar{Y}, \tilde{\mathcal{E}}_1)[\dim \bar{Y}]$ regarded as a simple perverse sheaf on G (0 on $G - \bar{Y}$).

References

[B] A. Borel, *Groupes linéaires algébriques*, Ann. Math. **64** (1956), 20-82. MR **19:**1195h

[D] J. de Siebenthal, *Sur les groupes de Lie compacts non connexes*, Comment. Math. Helv. **31** (1956), 41-89. MR **20:**926

[L1] G. Lusztig, *On the finiteness of the number of unipotent classes*, Invent. Math. **34** (1976), 201-213. MR **54:**7653

[L2] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), 205-272. MR **86d:**20050

[L3] G. Lusztig, *Character sheaves, I*, Adv. Math. **56** (1985), 193-237. MR **87b:**20055

[L4] G. Lusztig, *Introduction to character sheaves*, Proc. Symp. Pure Math. **47(1)** (1987), 165-180. MR **89e:**20079

[L5] G. Lusztig, *Classification of unipotent representations of simple p-adic groups, II*, Represent. Theory **6** (2002), 243-289.

[Sp] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel, Lecture Notes in Mathematics*, vol. 946, Springer-Verlag, New York, 1982. MR **84a:**14024

[St] R. Steinberg, *Endomorphisms of linear algebraic groups, Memoirs of Amer. Math. Soc.*, vol. 80, 1968. MR **37:**6288

Department of Mathematics, Massachusetts of Technology, Cambridge, Massachusetts 02139

E-mail address: gyuri@math.mit.edu