

## DEGENERATE PRINCIPAL SERIES FOR EVEN-ORTHOGONAL GROUPS

DUBRAVKA BAN AND CHRIS JANTZEN

ABSTRACT. Let  $F$  be a  $p$ -adic field of characteristic 0 and  $G = O(2n, F)$  (resp.  $SO(2n, F)$ ). A maximal parabolic subgroup of  $G$  has the form  $P = MU$ , with Levi factor  $M \cong GL(k, F) \times O(2(n-k), F)$  (resp.  $M \cong GL(k, F) \times SO(2(n-k), F)$ ). We consider a one-dimensional representation of  $M$  of the form  $\chi \circ \det_k \otimes \text{triv}_{(n-k)}$ , with  $\chi$  a one-dimensional representation of  $F^\times$ ; this may be extended trivially to get a representation of  $P$ . We consider representations of the form  $\text{Ind}_P^G(\chi \circ \det_k \otimes \text{triv}_{(n-k)}) \otimes 1$ . (Our results also work when  $G = O(2n, F)$  and the inducing representation is  $(\chi \circ \det_k \otimes \det_{(n-k)}) \otimes 1$ , using  $\det_{(n-k)}$  to denote the nontrivial character of  $O(2(n-k), F)$ .) More generally, we allow Zelevinsky segment representations for the inducing representations.

In this paper, we study the reducibility of such representations. We determine the reducibility points, give Langlands data and Jacquet modules for each of the irreducible composition factors, and describe how they are arranged into composition series. For  $O(2n, F)$ , we use Jacquet module methods to obtain our results; the results for  $SO(2n, F)$  are obtained via an analysis of restrictions to  $SO(2n, F)$ .

### 1. INTRODUCTION

Let  $F$  be a  $p$ -adic field with  $\text{char} F = 0$ .

The basic purpose of this paper is to study degenerate principal series for  $O(2n, F)$  and  $SO(2n, F)$  (though we work in a more general setting, what might be called generalized degenerate principal series). This paper completes the analysis of reducibility for degenerate principal series for classical  $p$ -adic groups; the corresponding results for  $SL(n, F)$  (cf. [B-Z] and [Tad1]),  $Sp(2n, F)$  and  $SO(2n+1, F)$  (cf. [Jan3]) are already known. We remark that while a fair amount of work on degenerate principal series for  $Sp(2n, F)$ ,  $SO(2n+1, F)$  had been done prior to [Jan3] (cf. [Gus], [K-R], [Jan1], [Jan2], [Tad4]), relatively little has been done for  $SO(2n, F)$  or  $O(2n, F)$  (though [Jan2] contains some results for  $SO(2n, F)$ ). There are also results on degenerate principal series available for other  $p$ -adic groups (e.g., see [Mu], [K-S], [Ch]). For real and complex groups, there is significantly more available on degenerate principal series; we refer the reader to [H-L] for a discussion of these cases.

In [Jan3] (generalized) degenerate principal series for  $SO(2n+1, F)$ ,  $Sp(2n, F)$  are analyzed. Structural similarities between the two families of groups allow them to be treated together. A careful study of Jacquet modules—made possible by the results from [Tad3]—allowed us to determine the number of irreducible subquotients

---

Received by the editors May 7, 2002 and, in revised form, September 22, 2003.  
2000 *Mathematics Subject Classification*. Primary 22E50.

and give their Langlands data. This, in turn, made it possible to reconstruct the composition series.

These structural similarities are shared by  $O(2n, F)$  (but not  $SO(2n, F)$ ). Thus, we first focus on (generalized) degenerate principal series for  $O(2n, F)$ . Now, there were two obstacles to including  $O(2n, F)$  in [Jan3]: (1) the Jacquet module results in [Tad3] did not apply to  $O(2n, F)$ , and (2) the Langlands classification required the underlying algebraic group to be connected. Since then, the Jacquet module structures of [Tad3] have been extended to  $O(2n, F)$  (cf. [Ban1]); the Langlands classification was extended (to quasi-split groups with abelian component group) in [B-J1]. (We also need multiplicity one for the Langlands classification; for  $O(2n, F)$ , this follows from [B-J2] or the argument for Lemma 3.4 in [Jan4].) Thus, generalized degenerate principal series for  $O(2n, F)$  may be handled in exactly the same manner as in [Jan3].

We do our analysis for  $SO(2n, F)$  by using [G-K] to study restrictions of representations from  $O(2n, F)$  to  $SO(2n, F)$ . (This approach to the study of representations of non-connected groups has been used by [Gol2], [G-H], e.g., though they use information for the connected component to obtain information for the non-connected group.) The key tools for doing our analysis are Proposition 4.5 in [B-J1] and the results of section 2 [G-K]. If  $\pi$  is an irreducible representation of  $O(2n, F)$ , these allow us to obtain the Langlands data for the component(s) of  $\pi|_{SO(2n, F)}$  from the Langlands data for  $\pi$ .

Let us now discuss the contents in greater detail. In the next section, we introduce notation and review some results which will be needed in the rest of the paper.

We begin by discussing the generalized degenerate principal series for  $O(2n, F)$  considered in this paper. As in [B-Z], we let  $\nu = |det|$  for general linear groups and use  $\times$  to denote parabolic induction (cf. section 2 for a more detailed explanation). If  $\rho_0$  is an irreducible, unitary, supercuspidal representation of  $GL(r_0, F)$ , then

$$\nu^{\frac{-k+1}{2}} \rho_0 \times \nu^{\frac{-k+1}{2}+1} \rho_0 \times \cdots \times \nu^{\frac{k-1}{2}} \rho_0$$

has a unique irreducible subrepresentation which we denote by  $\zeta(\rho_0, k)$ . We note that if  $\rho_0 = 1_{F^\times}$ , we have  $\zeta(\rho_0, k) = \text{triv}_{GL(k, F)}$ . As in [Tad2], we use  $\rtimes$  to denote parabolic induction for classical groups (cf. section 2 for a more detailed explanation). Let  $\rho$  be an irreducible, unitary, supercuspidal representation of  $GL(r, F)$  and  $\sigma$  an irreducible, supercuspidal representation of  $O(2m, F)$  (or  $Sp(2m, F)$ ,  $SO(2m+1, F)$ ,  $SO(2m, F)$ ). (We note that such a  $\sigma$  is necessarily unitarizable.) We say  $(\rho, \sigma)$  satisfies (C0) if (1)  $\rho \rtimes \sigma$  is reducible and (2)  $\nu^x \rho \rtimes \sigma$  is irreducible for all  $x \in \mathbb{R} \setminus \{0\}$ . If  $(\rho, \sigma)$  satisfies (C0), then

$$\nu^{-\ell+1} \rho \times \nu^{-\ell+2} \rho \times \cdots \times \nu^{-1} \rho \times \rho \rtimes \sigma$$

has two irreducible subrepresentations which we denote  $\zeta_1(\rho, \ell; \sigma)$  and  $\zeta_2(\rho, \ell; \sigma)$ . We note that if  $\rho = 1_{F^\times}$  and  $\sigma = 1_{O(0, F)}$  (with  $O(0, F)$  the trivial group), then  $\zeta_1(\rho, \ell; \sigma) = \text{triv}_{O(2\ell, F)}$  and  $\zeta_2(\rho, \ell; \sigma) = \text{det}_{O(2\ell, F)}$ .

In the third section, we discuss the generalized degenerate principal series  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_i(\rho, \ell; \sigma)$ ,  $\alpha \in \mathbb{R}$ , for  $O(2n, F)$ . The case  $\rho_0 \cong \rho$  is covered by Proposition 3.2 (for  $k = 1, \ell \geq 1$ ), Proposition 3.3 (for  $\ell = 0, k \geq 2$ ), and Theorem 3.4 (for  $k \geq 2, \ell \geq 1$ ). The case  $\rho_0 \not\cong \rho$  is covered by Theorem 3.5 and Remark 3.6. In particular, we determine for which values of  $\alpha$  reducibility occurs. When there is reducibility, we identify the irreducible subquotients (by giving their Langlands

data), describe their composition series structure, and give information on their Jacquet modules.

At this point, several remarks are in order. First, the arguments needed to obtain the results in section 3 are essentially the same as those used in [Jan3]. For this reason, we are rather brief in our discussion; we are content to summarize the results, with suitable references to [Jan3], and thereby avoid repeating long arguments which contain nothing new. Second, the roles of  $\zeta_1(\rho, \ell; \sigma)$  and  $\zeta_2(\rho, \ell; \sigma)$  are interchangeable; either can serve as the  $\zeta_1(\rho, \ell; \sigma)$  in the results. Thus, section 3 may also be used to deal with degenerate principal series where the one-dimensional representation of the orthogonal group is *det* rather than *triv*. Finally, the results of section 3 apply equally well to  $Sp(2n, F)$  and  $SO(2n + 1, F)$ . However, for these groups, the results do not say anything about degenerate principal series. The difference lies in the conditions on  $(\rho, \sigma)$ : For degenerate principal series we want  $\rho = 1_{F^\times}$  and  $\sigma$  trivial (for the rank 0 classical group, i.e., the trivial group). For  $O(2n, F)$ , this means  $(\rho, \sigma)$  satisfies (C0); for  $Sp(2n, F)$ ,  $SO(2n + 1, F)$ , this requires that  $(\rho, \sigma)$  satisfies (C1/2), (C1), respectively (cf. [Jan3] for a more detailed discussion regarding  $Sp(2n, F)$ ,  $SO(2n + 1, F)$ ).

Our analysis of generalized degenerate principal series for  $SO(2n, F)$  is done by combining the results for  $O(2n, F)$  (section 3) with results on the restriction of representations from  $O(2n, F)$  to  $SO(2n, F)$  (section 4). We note that our results on restriction from  $O(2n, F)$  to  $SO(2n, F)$  are built in part on the results of section 2 [G-K] and Proposition 4.5 [B-J1]. In particular, in combination these may be used to tell, from the Langlands data of an irreducible representation  $\pi$  of  $O(2n, F)$ , whether its restriction to  $SO(2n, F)$  is reducible and determine the Langlands data of the components of the restriction.

Included in section 4 is a discussion of cuspidal reducibility. We recall what this means for  $Sp(2n, F)$ ,  $SO(2n + 1, F)$ . Suppose  $\rho$  is an irreducible, unitary, supercuspidal representation of  $GL(r, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $Sp(2m, F)$  or  $SO(2m + 1, F)$ . If  $\rho \not\cong \tilde{\rho}$  ( $\tilde{\rho}$  the contragredient of  $\rho$ ), then  $\nu^x \rho \rtimes \sigma$  is irreducible for all  $x \in \mathbb{R}$ ; if  $\rho \cong \tilde{\rho}$ , there exists a unique  $\alpha(\rho, \sigma) \geq 0$  such that  $\nu^\alpha \rho \rtimes \sigma$  is reducible (cf. [Sil2], [Sil3]). Characterizations of the cuspidal reducibility  $\alpha(\rho, \sigma)$  (based on certain conjectures) are given in [Mœ], [Zh]. (Also noteworthy are the results of [Sh1], [Sh2] in the generic case and the examples from [M-R] and [Re].) Using our study of restriction/induction between  $SO(2n, F)$  and  $O(2n, F)$ , the results of [Mœ], [Zh] may be extended to  $O(2n, F)$  (cf. Corollary 4.4), noting that [Mœ], [Zh] also deal with cuspidal reducibility for  $SO(2n, F)$ . This is obtained from Proposition 4.3, which relates the cuspidal reducibility for  $(\rho, \sigma)$  to that of  $(\rho, \sigma_0)$ , where  $\sigma_0$  is a component of the restriction of  $\sigma$  to  $SO(2m, F)$ .

In section 5, we deal with generalized degenerate principal series for  $SO(2n, F)$ . Here, we have two situations to consider. Suppose  $(\rho, \sigma)$  ( $\sigma$  an irreducible, supercuspidal representation of  $O(2m, F)$ ) satisfies (C0). Let  $\sigma_0$  be a component of the restriction of  $\sigma$  to  $SO(2m, F)$ . Then, either (1)  $(\rho, \sigma_0)$  satisfies (C0), or (2)  $\nu^x \rho \rtimes \sigma_0$  is irreducible for all  $x \in \mathbb{R}$  (cf. Proposition 4.3 for a precise characterization). In the first case (i.e.,  $(\rho, \sigma_0)$  (C0)), the generalized degenerate principal series  $\nu^\alpha \zeta(\rho_0) \rtimes \zeta_1(\rho, \ell; \sigma_0)$  (for  $SO(2n, F)$ ) behave like the generalized degenerate principal series  $\nu^\alpha \zeta(\rho_0) \rtimes \zeta_1(\rho, \ell; \sigma)$  (for  $O(2n, F)$ ). In fact, the results are sufficiently similar such that we include their statements in section 3, although the proofs are in section 5. In the second case (i.e.,  $\nu^x \rho \rtimes \sigma_0$  irreducible for all  $x \in \mathbb{R}$ ), the results

are not as similar. In this case, the results on generalized degenerate principal series for  $\rho_0 \cong \rho$  are given in Proposition 5.1 (for  $k = 1, \ell \geq 1$ ), Proposition 5.2 (for  $\ell = 0, k \geq 2$ ), and Theorem 5.3 (for  $k \geq 2, \ell \geq 1$ ). The results for  $\rho_0 \not\cong \rho$  are covered by Theorem 5.5 and Remark 5.6. Jacquet module information is included in Propositions 5.1 and 5.2. For Theorems 5.3 and 5.5, a brief discussion of how to calculate Jacquet modules is given in Remark 5.4. We note that this is the case which includes the actual degenerate principal series for  $SO(2n, F)$ .

We close this introduction with a sort of user’s guide, to help easily find the appropriate results on degenerate principal series. For  $O(2n, F)$ , the results on degenerate principal series of the form  $\text{Ind}(|\det_{GL(k,F)}|^x \otimes \text{triv}_{O(2\ell,F)})$  ( $x \in \mathbb{R}$ ) may be found by taking  $\rho = 1_{F^\times}$  and  $\sigma = 1_{O(0,F)}$  ( $O(0, F)$  is the trivial group) in Proposition 3.2 (for  $k = 1$ ), Proposition 3.3 (for  $\ell = 0$ ), or Theorem 3.4 (when  $k \geq 2$  and  $\ell \geq 1$ ). The results for degenerate principal series of the form  $\text{Ind}(|\det_{GL(k,F)}|^x \cdot (\chi \circ \det_{GL(k,F)}) \otimes \text{triv}_{O(2\ell,F)})$  ( $\chi$  a unitary character of  $F^\times$ ) may be found by taking  $\rho_0 = \chi, \rho = 1_{F^\times}, \sigma = 1_{O(0,F)}$  in Theorem 3.5 (if  $\chi$  is of order two) or Remark 3.6 (if order  $\chi > 2$ ). To deal with degenerate principal series of the form  $\text{Ind}(|\det_{GL(k,F)}|^x \cdot (\chi \circ \det_{GL(k,F)}) \otimes \det_{O(2\ell,F)})$ , use the same results, but with the roles of  $\zeta_1(\rho, \ell; \sigma)$  and  $\zeta_2(\rho, \ell; \sigma)$  reversed.

For  $SO(2n, F)$ , the results on degenerate principal series of the form  $\text{Ind}(|\det_{GL(k,F)}|^x \otimes \text{triv}_{SO(2\ell,F)})$  ( $x \in \mathbb{R}$ ) may be found by taking  $\rho = 1_{F^\times}$  and  $\sigma = 1_{SO(0,F)}$  ( $SO(0, F)$  is the trivial group) in Proposition 5.1 (for  $k = 1$ ), Proposition 5.2 (for  $\ell = 0$ ), or Theorem 5.3 (when  $k \geq 2$  and  $\ell \geq 1$ ). The results for degenerate principal series of the form  $\text{Ind}(|\det_{GL(k,F)}|^x \cdot (\chi \circ \det_{GL(k,F)}) \otimes \text{triv}_{SO(2\ell,F)})$  ( $\chi$  a unitary character of  $F^\times$ ) may be found by taking  $\rho_0 = \chi, \rho = 1_{F^\times}, \sigma = 1_{SO(0,F)}$  in Theorem 5.5 (if  $\chi$  is of order two) or Remark 5.6 (if order  $\chi > 2$ ).

**Acknowledgment.** We would like to close by thanking the referees for valuable comments and corrections.

## 2. PRELIMINARIES

In this section, we introduce notation and state the Langlands classification for  $SO(2n, F)$  and  $O(2n, F)$ . An explanation how such a form of the Langlands classification follows from the general results in [B-J1] can be found in the Appendix.

Let  $F$  be a  $p$ -adic field with  $\text{char} F = 0$ . We make use of results from [Gol1], [Gol2] in this paper, hence we need this assumption.

In most of this paper, we work with the components (irreducible subquotients) of a representation rather than with the actual composition series. Suppose that  $\pi_1, \pi_2$  are smooth finite length representations. We write  $\pi_1 = \pi_2$  if  $\pi_1$  and  $\pi_2$  have the same components with the same multiplicities. We write  $\pi_1 \cong \pi_2$  if  $\pi_1$  and  $\pi_2$  are actually equivalent.

The special orthogonal group  $SO(2n, F)$ ,  $n \geq 1$ , is the group

$$SO(2n, F) = \{X \in SL(2n, F) \mid {}^\tau X X = I_{2n}\}.$$

Here  ${}^\tau X$  denotes the matrix of  $X$  transposed with respect to the second diagonal. For  $n = 1$ , we get

$$SO(2, F) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \mid \lambda \in F^\times \right\} \cong F^\times.$$

$SO(0, F)$  is defined to be the trivial group. The orthogonal group  $O(2n, F)$ ,  $n \geq 1$ , is the group

$$O(2n, F) = \{X \in GL(2n, F) \mid {}^\tau XX = I_{2n}\}.$$

We have

$$O(2n, F) = SO(2n, F) \rtimes \{1, s\},$$

where

$$s = \begin{bmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{bmatrix} \in O(2n, F)$$

and it acts on  $SO(2n, F)$  by conjugation. We take  $O(0, F)$  to be the trivial group.

In the group  $SO(2n, F)$ , fix the minimal parabolic subgroup  $P_\theta$  consisting of all upper triangular matrices in  $SO(2n, F)$  and the maximal split torus  $A_\theta$  consisting of all diagonal matrices in  $SO(2n, F)$ .

Let  $M^0$  be the standard Levi subgroup of  $G^0 = SO(2n, F)$ . We denote by  $i_{G^0, M^0}$  the functor of parabolic induction and by  $r_{M^0, G^0}$  the Jacquet functor (cf. [B-Z]). Let  $G = O(2n, F)$ . We use the notation  $i_{G, G^0}$  and  $r_{G^0, G}$  for induction and restriction of representations.

Suppose that  $\rho_1, \dots, \rho_k$  are representations of  $GL(n_1, F), \dots, GL(n_k, F)$  and  $\tau_0$  a representation of  $SO(2m, F)$ . Let  $G^0 = SO(2n, F)$ , where  $n = n_1 + \dots + n_k + m$ . Let

$$M^0 = \{diag(g_1, \dots, g_k, h, {}^\tau g_k^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), h \in SO(2m, F)\}.$$

Then  $M^0$  is a standard Levi subgroup of  $G^0$  (cf. Appendix or [Ban2]). Following [B-Z], [Tad2], set

$$\rho_1 \times \dots \times \rho_k \rtimes \tau_0 = i_{G^0, M^0}(\rho_1 \otimes \dots \otimes \rho_k \otimes \tau_0).$$

If  $m = 0$  and  $n_k > 1$ , then  $sM^0s$  is another standard Levi subgroup of  $G^0$ . Let  $\mathbf{1}_0$  denote the trivial representation of  $SO(0, F)$ . We write  $\rho_1 \otimes \dots \otimes \rho_{k-1} \otimes s(\rho_k \otimes \mathbf{1}_0)$  for the corresponding representation of  $sM^0s$ . According to [Ban2],

$$\begin{aligned} s(\rho_1 \times \dots \times \rho_k \rtimes \mathbf{1}_0) &= \rho_1 \times \dots \times \rho_{k-1} \rtimes s(\rho_k \rtimes \mathbf{1}_0) \\ &= i_{G^0, sM^0s}(\rho_1 \otimes \dots \otimes \rho_{k-1} \otimes s(\rho_k \otimes \mathbf{1}_0)). \end{aligned}$$

As in [B-Z], we set  $\nu = |det|$  for general linear groups. Let  $\rho$  be an irreducible representation of  $GL(n, F)$ . We say that  $\rho$  is essentially square-integrable (resp., essentially tempered) if there exists  $e(\rho) \in \mathbb{R}$  such that  $\nu^{-e(\rho)}\rho$  is square-integrable (resp., tempered).

Now, we discuss the Langlands classification for  $SO(2n, F)$ (cf. Appendix). Let  $\rho_i$ ,  $i = 1, \dots, k$ , be irreducible essentially square-integrable representations of  $GL(n_i, F)$  and  $\tau_0$  an irreducible tempered representation of  $SO(2m, F)$ . Suppose that  $m \geq 1$  and  $e(\rho_1) \leq \dots \leq e(\rho_k) < 0$ . Then the representation  $\rho_1 \times \dots \times \rho_k \rtimes \tau_0$  has a unique irreducible subrepresentation which we denote by  $L(\rho_1, \dots, \rho_k; \tau_0)$ . (We note that this formulation—using weak inequalities with essentially square-integrable representations in lieu of strict inequalities with essentially tempered representations—follows easily from the fact that an irreducible tempered representation of  $GL(n, F)$  has the form  $\delta_1 \times \dots \times \delta_j$  with  $\delta_i$  irreducible and square-integrable.) If  $m = 0$ , then

$$sL(\rho_1, \dots, \rho_k; \mathbf{1}_0) \not\cong L(\rho_1, \dots, \rho_k; \mathbf{1}_0).$$

We have  $sL(\rho_1, \dots, \rho_k; 1_0) = L(s(\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0))$ . Both  $\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0$  and  $s(\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0)$  appear in (2) and (3) of Proposition 6.3, i.e., constitute Langlands data. Further, any Langlands datum in (2) or (3) of Proposition 6.3 may be written as either  $\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0$  or  $s(\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0)$  with  $\rho_1, \dots, \rho_k$  as above. To simplify notation, etc., we write  $sL(\rho_1, \dots, \rho_k; 1_0)$  rather than  $L(s(\rho_1 \otimes \dots \otimes \rho_k \otimes 1_0))$  in these cases.

At times, it will be convenient not to have to worry about listing  $\rho_1, \dots, \rho_k$  in increasing order. So, if  $\rho_1, \dots, \rho_k$  satisfy  $e(\rho_i) < 0$ , then there is some permutation  $\rho_{\sigma_1}, \dots, \rho_{\sigma_k}$  which satisfies  $e(\rho_{\sigma_1}) \leq \dots \leq e(\rho_{\sigma_k}) < 0$ . Then, by  $L(\rho_1, \dots, \rho_k; \tau_0)$  we mean  $L(\rho_{\sigma_1}, \dots, \rho_{\sigma_k}; \tau_0)$ .

Suppose that  $\rho_1, \dots, \rho_k$  are representations of  $GL(n_1, F), \dots, GL(n_k, F)$  and  $\tau$  a representation of  $O(2m, F)$ . Let  $G = O(2n, F)$ , where  $n = n_1 + \dots + n_k + m$ . Let

$$M = \{diag(g_1, \dots, g_k, h, {}^\tau g_k^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), h \in O(2m, F)\}.$$

Then  $M$  is a standard Levi subgroup of  $G$  (cf. Appendix or [B-J1]). Set

$$\rho_1 \times \dots \times \rho_k \rtimes \tau = i_{M,G}(\rho_1 \otimes \dots \otimes \rho_k \otimes \tau).$$

In the case  $m = 0$ , we denote the trivial representation of  $O(0, F)$  by 1.

Now, we give the Langlands classification for  $O(2n, F)$  (cf. Appendix). Let  $\rho_i, i = 1, \dots, k$ , be an irreducible essentially square-integrable representation of  $GL(n_i, F)$  and  $\tau$  an irreducible tempered representation of  $O(2m, F)$ . Suppose that  $e(\rho_1) \leq \dots \leq e(\rho_k) < 0$ . Then the representation  $\rho_1 \times \dots \times \rho_k \rtimes \tau$  has a unique irreducible subrepresentation which we denote by  $L(\rho_1, \dots, \rho_k; \tau)$ .

Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(n, F)$  and  $k \in \mathbb{Z}, k > 0$ . Then the representation

$$\nu^{-\frac{k+1}{2}} \rho \times \nu^{-\frac{k+1}{2}+1} \rho \times \dots \times \nu^{\frac{k-1}{2}} \rho$$

has a unique irreducible subrepresentation which we denote by  $\zeta(\rho, k)$  and a unique irreducible quotient which we denote by  $\delta(\rho, k)$  (cf. [Zel]).

Suppose that  $\sigma$  is an irreducible supercuspidal representation of  $SO(2m, F)$  (respectively,  $O(2m, F)$ ) satisfying

$$(C0) \ \rho \rtimes \sigma \text{ is reducible and } \nu^\alpha \rho \rtimes \sigma \text{ is irreducible for all } \alpha \in \mathbb{R} \text{ with } \alpha \neq 0.$$

Then  $\rho \rtimes \sigma$  is the direct sum of two irreducible tempered representations. We write

$$\rho \rtimes \sigma = T_1(\rho; \sigma) \oplus T_2(\rho; \sigma).$$

Let  $i = 1, 2$  and  $\ell > 1$ . By Jacquet module considerations, the representation

$$\nu^{-\ell+1} \rho \times \nu^{-\ell+2} \rho \times \dots \times \nu^{-1} \rho \rtimes T_i(\rho; \sigma)$$

has a unique irreducible subrepresentation which we denote by  $\zeta_i(\rho, \ell; \sigma)$  and a unique irreducible quotient which we denote by  $\delta([\nu\rho, \nu^{\ell-1}\rho]; T_i(\rho; \sigma))$  (it is square-integrable for  $\ell > 0$ ). For convenience, we also use the segment notation of Zelevinsky: let

$$[\nu^\beta \rho, \nu^{\beta+m} \rho] = \nu^\beta \rho, \nu^{\beta+1} \rho, \dots, \nu^{\beta+m} \rho.$$

Then, e.g., we have  $\zeta_i(\rho, \ell; \sigma) = L([\nu^{-\ell+1} \rho, \nu^{-1} \rho]; T_i(\rho; \sigma))$ .

Let  $\rho$  be an irreducible supercuspidal representation of  $GL(m, F)$ ,  $m$  odd. Suppose that  $\rho \cong \tilde{\rho}$ . Then  $\rho \rtimes 1_0$  is an irreducible tempered representation of  $SO(2m, F)$ . By Jacquet module considerations, the representation

$$\nu^{-\ell+1} \rho \times \nu^{-\ell+2} \rho \times \dots \times \nu^{-1} \rho \rtimes \rho \rtimes 1_0$$

has a unique irreducible subrepresentation which we denote by  $\zeta(\rho, \ell; 1_0)$  and a unique irreducible quotient which we denote by  $\delta([\nu\rho, \nu^{\ell-1}\rho]; \rho \rtimes 1_0)$  (it is square-integrable for  $\ell > 0$ ). Similarly, if  $\sigma_0$  is an irreducible admissible representation of  $SO(2n, F)$  such that  $s\sigma_0 \not\cong \sigma_0$ , then  $\rho \rtimes \sigma_0$  is an irreducible tempered representation of  $SO(2(m+n), F)$ . By Jacquet module considerations, the representation

$$\nu^{-\ell+1}\rho \times \nu^{-\ell+2}\rho \times \cdots \nu^{-1}\rho \times \rho \rtimes \sigma_0$$

has a unique irreducible subrepresentation which we denote by  $\zeta(\rho, \ell; \sigma_0)$  and a unique irreducible quotient which we denote by  $\delta([\nu\rho, \nu^{\ell-1}\rho]; \rho \rtimes \sigma_0)$  (it is square-integrable for  $\ell > 0$ ).

We introduce an additional piece of notation for Jacquet modules. Let  $G = SO(2n, F)$  (respectively,  $G = O(2n, F)$ ) and  $\pi$  a representation of  $G$ . We write

$$s_{(m)}\pi = r_{M,G}(\pi),$$

where  $M$  is the standard Levi subgroup of  $G$  isomorphic to  $GL(m, F) \times SO(2(n-m), F)$  (respectively,  $GL(m, F) \times O(2(n-m), F)$ ).

### 3. DEGENERATE PRINCIPAL SERIES FOR $O(2n, F)$

In this section, we determine the reducibility for (generalized) degenerate principal series for  $O(2n, F)$ . Suppose  $(\rho, \sigma)$  satisfy (C0) (for  $O(2(m+r), F)$ ). We analyze the reducibility of  $\zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  below. If  $\rho_0 \cong \rho$ , the results are given in Proposition 3.2 (for  $k = 1$ ), Proposition 3.3 (for  $\ell = 0$ ), and Theorem 3.4 (for  $k \geq 2, \ell \geq 1$ ). If  $\rho_0 \not\cong \rho$ , the results are given in Theorem 3.5. We note that the results consist of determining the reducibility points, the Langlands data of the irreducible subquotients which appear, the composition series structure, and certain information on Jacquet modules. As the arguments required are essentially the same as in [Jan3], we omit the (rather lengthy) details.

The results also apply to  $SO(2n+1, F)$ ,  $Sp(2n, F)$ , and  $SO(2n, F)$  when  $(\rho, \sigma)$  satisfies (C0) (where  $\sigma$  is an irreducible supercuspidal representation of the corresponding group). For  $SO(2n+1, F)$ ,  $Sp(2n, F)$ , the proofs are the same as for  $O(2n, F)$ ; for  $SO(2n, F)$ , the proofs are given in section 5. To allow this sort of generality in the results, we use  $G(n, F)$  to denote any of these groups.

We start with a preliminary result. While there are strong reasons to believe that  $\zeta_i(\rho, \ell; \sigma)$  is unitary, to the best of our knowledge this remains unknown. The following lemma serves as a substitute for the unitarity of  $\zeta_i(\rho, \ell; \sigma)$ . (A similar result should have been included in [Jan3].)

**Lemma 3.1.** *Suppose  $\rho, \rho_0$  are irreducible unitary supercuspidal representations of  $GL(m, F), GL(m_0, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $G(r, F)$ . Further, assume that  $(\rho, \sigma)$  satisfies (C0). Then, if  $\pi = \zeta(\rho_0, k) \rtimes \zeta_i(\rho, \ell; \sigma)$  is reducible, it decomposes as the direct sum of two irreducible, inequivalent representations.*

*Proof.* Let  $\hat{\cdot}$  denote the duality of [Aub] (also, cf. [S-S]); for  $O(2n, F)$  see [Jan6]). Then,  $\hat{\pi} = \delta(\rho_0, k) \rtimes \delta_i(\rho, \ell; \sigma)$ . By [Gol1], [Gol2], e.g., we know that if this reduces, it is the direct sum of two inequivalent, irreducible subrepresentations. By Théorème 1.7 and Corollaire 3.9 [Aub],  $\pi$  must have two inequivalent, irreducible subquotients. It remains to show that  $\pi$  decomposes as a direct sum.

Suppose  $\pi$  is reducible. Let  $\pi_1$  and  $\pi_2$  denote the irreducible subquotients. We claim that to show that  $\pi = \pi_1 \oplus \pi_2$ , it is enough to show that  $s_{(km_0)}(\pi_i)$  contains

$\zeta(\rho_0, k) \otimes \zeta_i(\rho, \ell; \sigma)$ . One can see from [Tad3] (also, cf. pp. 74 and 75, [Jan2]) that the two copies of  $\zeta(\rho_0, k) \otimes \zeta_i(\rho, \ell; \sigma)$  in  $s_{(km_0)}(\pi)$  are the only irreducible subquotients of  $s_{(km_0)}(\pi)$  having unitary central character (all the rest have central character with the exponent having negative real part). Thus, if  $s_{(km_0)}(\pi_i)$  has  $\zeta(\rho_0, k) \otimes \zeta_i(\rho, \ell; \sigma)$  as an irreducible subquotient, it follows from central character considerations and Frobenius reciprocity that  $\pi_i$  is a subrepresentation of  $\zeta(\rho_0, k) \rtimes \zeta_i(\rho, \ell; \sigma)$ . The claim follows. Finally, to see that  $s_{(km_0)}(\pi_i)$  contains  $\zeta(\rho_0, k) \otimes \zeta_i(\rho, \ell; \sigma)$ , we again look to the dual side, where we have  $\hat{\pi} = \hat{\pi}_1 \oplus \hat{\pi}_2$ . By Frobenius reciprocity,  $s_{(km_0)}(\hat{\pi}_i)$  contains  $\delta(\rho_0, k) \otimes \delta_i(\rho, \ell; \sigma)$ . By Théorème 1.7, [Aub], we can conclude  $s_{(km_0)}(\pi_i)$  contains  $\zeta(\rho_0, k) \otimes \zeta_i(\rho, \ell; \sigma)$ , as needed.  $\square$

**Proposition 3.2.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $G(r, F)$  with  $(\rho, \sigma)$  satisfying (C0). Let  $\pi = \nu^\alpha \rho \rtimes \zeta_1(\rho, \ell; \sigma)$  with  $\alpha \in \mathbb{R}$ ,  $\ell \geq 1$ . Then,  $\pi$  is reducible if and only if  $\alpha \in \{\pm 1, \pm \ell\}$ . Suppose  $\pi$  is reducible. By contragredience, we may assume that  $\alpha \leq 0$ .*

$$(1) \quad \alpha = -1, \ell = 1$$

$$\pi = \pi_1 + \pi_2 + \pi_3 \text{ with}$$

$$\pi_1 = L(\nu^{-1}\rho; T_1(\rho; \sigma)), \quad \pi_2 = \delta(\nu\rho; T_1(\rho; \sigma)), \quad \pi_3 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

*In this case,  $\pi_1$  is the unique irreducible subrepresentation,  $\pi_2$  is the unique irreducible quotient, and  $\pi_3$  is a subquotient. We have*

$$s_{(m)}\pi_1 = \nu^{-1}\rho \otimes T_1(\rho; \sigma),$$

$$s_{(m)}\pi_2 = \nu\rho \otimes T_1(\rho; \sigma),$$

$$s_{(m)}\pi_3 = \rho \otimes L(\nu^{-1}\rho; \sigma).$$

$$(2) \quad \alpha = -1, \ell > 1$$

$$\pi = \pi_1 + \pi_2 \text{ with}$$

$$\pi_1 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-1}\rho; T_1(\rho; \sigma)), \quad \pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))).$$

*In this case,  $\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient. We have*

$$(a) \quad \ell = 2$$

$$s_{(m)}\pi_1 = 2\nu^{-1}\rho \otimes L(\nu^{-1}\rho; T_1(\rho; \sigma)) + \nu^{-1}\rho \otimes L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma),$$

$$s_{(m)}\pi_2 = \nu^{-1}\rho \otimes \delta(\nu\rho; T_1(\rho; \sigma)) + \nu\rho \otimes L(\nu^{-1}\rho; T_1(\rho; \sigma)).$$

$$(b) \quad \ell > 2.$$

$$s_{(m)}\pi_1 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho, \nu^{-1}\rho], \nu^{-1}\rho; T_1(\rho; \sigma)) \\ + \nu^{-1}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)),$$

$$s_{(m)}\pi_2 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))) \\ + \nu\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)).$$

$$(3) \quad \alpha = -\ell, \ell > 1.$$

$$\pi = \pi_1 + \pi_2 \text{ with}$$

$$\pi_1 = L([\nu^{-\ell}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)), \quad \pi_2 = L(\nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)).$$



In this case,  $\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient. We have

$$\begin{aligned} s_{(m)}\pi_1 &= \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)) \\ s_{(m)}\pi_2 &= \nu^{-\ell+1}\rho \otimes L(\nu^{-\ell}\rho, [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)) \\ &\quad + \nu^\ell\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)). \end{aligned}$$

*Proof.* For  $O(2n, F)$  (and  $SO(2n+1, F)$ ,  $Sp(2n, F)$ ), this is proved as in Proposition 3.1, [Jan3] (also, cf. [Tad4]).  $\square$

**Proposition 3.3.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $G(r, F)$  with  $(\rho, \sigma)$  satisfying (C0). Let  $\pi = \nu^\alpha\zeta(\rho, k) \rtimes \sigma$  with  $\alpha \in \mathbb{R}$ ,  $k \geq 2$ . Then  $\pi$  is reducible if and only if  $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \dots, \frac{k-1}{2}\}$ . Suppose  $\pi$  is reducible. By contragredience, we may assume that  $\alpha \leq 0$ . Write  $\alpha = \frac{-k+1}{2} + j$  with  $0 \leq j \leq \frac{k-1}{2}$ .*

(1)  $j = \frac{k-1}{2}$   
 $\pi = \pi_1 + \pi_2$  with

$$\pi_i = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$ . In this case,  $\pi = \pi_1 \oplus \pi_2$ . We have

$$\begin{aligned} s_{(m)}\pi_i &= 2\nu^{\frac{-k+1}{2}}\rho \otimes L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+3}{2}}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma)) \\ &\quad + \nu^{\frac{-k+1}{2}}\rho \otimes L(\nu^{-\frac{k}{2}+1}\delta(\rho, 2), \nu^{-\frac{k}{2}+2}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma) \end{aligned}$$

for  $i = 1, 2$ .

(2)  $0 \leq j < \frac{k-1}{2}$   
 $\pi = \pi_1 + \pi_2 + \pi_3$  with

$$\pi_i = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$  and

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

In this case,  $\pi_3$  is the unique irreducible quotient and  $\pi_1 \oplus \pi_2$  is a subrepresentation.

(a)  $j = 0 = \frac{k-2}{2}$  ( $k = 2$ ),

$$s_{(m)}\pi_i = \nu^{-1}\rho \otimes T_i(\rho; \sigma)$$

for  $i = 1, 2$ .

$$s_{(m)}\pi_3 = \rho \otimes L(\nu^{-1}\rho; \sigma).$$

(b)  $j = 0$ ,  $k > 2$ ,

$$s_{(m)}\pi_i = \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$ .

$$\begin{aligned} s_{(m)}\pi_3 &= \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma) \\ &\quad + \rho \otimes L([\nu^{-k+1}\rho, \nu^{-1}\rho]; \sigma). \end{aligned}$$

(c)  $j = \frac{k-2}{2}, k \geq 4$  ( $k$  even),

$$s_{(m)}\pi_i = \nu^{-\frac{k}{2}}\rho \otimes L([\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma)) \\ + \nu^{-\frac{k}{2}+1}\rho \otimes L([\nu^{-\frac{k}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+2}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$ .

$$s_{(m)}\pi_3 = \nu^{-\frac{k}{2}+1}\rho \otimes L(\nu^{-\frac{k}{2}}\rho, \nu^{-\frac{k+3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

(d)  $0 < j < \frac{k-2}{2}$ ,

$$s_{(m)}\pi_i = \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma)) \\ + \nu^{-j}\rho \otimes L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$ .

$$s_{(m)}\pi_3 = \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma) \\ + \nu^{-j}\rho \otimes L([\nu^{-k+j+1}\rho, \nu^{-j-1}\rho], \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

*Proof.* For  $O(2n, F)$ , this is essentially the same as Proposition 3.11, [Jan3]. (For  $SO(2n+1, F)$ ,  $Sp(2n, F)$ , this is Proposition 3.11, [Jan3].)  $\square$

**Theorem 3.4.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $G(r, F)$  with  $(\rho, \sigma)$  satisfying (C0). Let  $\pi = \nu^\alpha \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$ . Suppose  $k \geq 2$  and  $\ell \geq 1$  (the cases  $k = 1$  and  $\ell = 0$  are covered by Propositions 3.2 and 3.3 above). Then,  $\pi$  is reducible if and only if*

$$\alpha \in \{ \pm(\ell + \frac{k-1}{2}), \pm(\ell + \frac{k-1}{2} - 1), \dots, \pm(\ell + \frac{-k+1}{2}) \}$$

$$\cup \{ \{ \frac{-k-1}{2}, \frac{-k-1}{2} + 1, \dots, \frac{k+1}{2} \} \setminus \{0 \text{ if } k = 2\ell - 1\} \}.$$

(We note that these sets need not be disjoint.) Let  $S_1$  denote the first set and  $S_2$  the second. Suppose  $\pi$  is reducible. By contragredience, we may restrict our attention to the case  $\alpha \leq 0$ .

(1)  $\alpha \notin S_2$ .

In this case, we have  $\pi = \pi_1 + \pi_2$ , where

$$\pi_1 = L([\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{\alpha+\frac{k-1}{2}}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)),$$

$$\pi_2 = L([\nu^{\alpha-\frac{k-1}{2}}\rho, \nu^{-\ell-1}\rho],$$

$$\nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{\alpha+\frac{k}{2}}\delta(\rho, 2), [\nu^{\alpha+\frac{k-1}{2}+2}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)).$$

$\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(2)  $\alpha = \frac{-k-1}{2}$ .

One component of  $\pi$  is the following:

$$\pi_1 = L([\nu^{-k}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; T_1(\rho; \sigma)).$$

The other components are described below.

(a)  $\ell = 1$  (so  $k > \ell - 1$ ).

In this case, there are two additional components:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))),$$

and

$$\pi_3 = L([\nu^{-k}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

$\pi_1$  is the unique irreducible subrepresentation,  $\pi_2$  is the unique irreducible quotient, and  $\pi_3$  is a subquotient.

(b)  $k > \ell - 1 > 0$ .

In this case, there are three additional components:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))),$$

$$\pi_3 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{3}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma))),$$

$$\pi_4 = L([\nu^{-k}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

$\pi_1$  is the unique irreducible subrepresentation,  $\pi_3$  is the unique irreducible quotient, and  $\pi_2 \oplus \pi_4$  is a subquotient.

(c)  $\ell - 1 = k$ .

In this case, there is one additional component:

$$\pi_2 = L([\nu^{-k}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))).$$

$\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(d)  $\ell - 1 > k$ .

In this case, there is one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; T_1(\rho; \sigma))).$$

$\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(3)  $\alpha \in S_2$ .

Write  $\alpha = \frac{-k+1}{2} + j$ , with  $0 \leq j \leq \frac{k-1}{2}$ . One component of  $\pi$  is  $\pi_1$ , where  $\pi_1$  is defined as follows:

$$\pi_1 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes T_1(\rho; \sigma)).$$

The remaining components are described below, on a case by case basis.

(a)  $k - j - 1 > j > \ell - 1$ .

We have two additional components:

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); T_2(\rho; \sigma)),$$

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); T_2(\rho; \sigma)).$$

$\pi_3$  is the unique irreducible quotient and  $\pi_1 \oplus \pi_2$  is a subrepresentation.

(b)  $k - j - 1 = j > \ell - 1$ .

We have one additional component:

$$\pi_2 = L([\nu^{\frac{-k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); T_2(\rho; \sigma)).$$

In this case,  $\pi = \pi_1 \oplus \pi_2$ .

(c)  $k - j - 1 > j = \ell - 1$ .

We have one additional component:

$$\pi_2 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); T_2(\rho; \sigma)).$$

$\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(d)  $k - j - 1 > \ell - 1 > j$ .

We have three additional components:

$$\begin{aligned}\pi_2 &= L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \\ &\quad \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma))), \\ \pi_3 &= L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-j-\frac{5}{2}}\delta(\rho, 2), \\ &\quad \nu^{-j-1}\delta(\rho, 3), \nu^{-j}\delta(\rho, 3), \dots, \nu^{-1}\delta(\rho, 3); \delta(\nu\rho; T_1(\rho; \sigma))), \\ \pi_4 &= L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \\ &\quad \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); T_2(\rho; \sigma)).\end{aligned}$$

$\pi_1$  is the unique irreducible subrepresentation,  $\pi_3$  is the unique irreducible quotient, and  $\pi_2 \oplus \pi_4$  is a subquotient.

(e)  $k - j - 1 = \ell - 1 > j$ .

We have one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-k+\ell-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-2}\rho], \nu^{-k+\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-k+\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma))).$$

$\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(f)  $\ell - 1 > k - j - 1 > j$ .

(i) If  $j = 0$ , the representation  $\pi_2$  below is the only other component.

In this case,  $\pi_1$  is the unique irreducible subrepresentation and  $\pi_2$  is the unique irreducible quotient.

(ii) If  $j > 0$ , there are two additional components:

$$\begin{aligned}\pi_2 &= L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \\ &\quad \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma))), \\ \pi_3 &= L([\nu^{-\ell+1}\rho, \nu^{-k+j-1}\rho], [\nu^{-j}\rho, \nu^{-2}\rho], \\ &\quad \nu^{-k+j+\frac{1}{2}}\delta(\rho, 2), \nu^{-k+j+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma))).\end{aligned}$$

In this case,  $\pi_2$  is the unique irreducible quotient and  $\pi_1 \oplus \pi_3$  is a subrepresentation.

(g)  $\ell - 1 > k - j - 1 = j$ .

We have one additional component:

$$\pi_2 = L([\nu^{-\ell+1}\rho, \nu^{-\frac{k-3}{2}}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-2}\rho], \nu^{-\frac{k}{2}}\delta(\rho, 2), \nu^{-\frac{k}{2}+1}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; T_1(\rho; \sigma)))$$

In this case,  $\pi = \pi_1 \oplus \pi_2$ .

We note that the case  $k - j - 1 = j = \ell - 1$  is a point of irreducibility.

*Proof.* For  $O(2n, F)$  (and  $SO(2n+1, F)$ ,  $Sp(2n, F)$ ), the proof that the reducibility points are as stated is essentially the same argument used in Theorem 4.1, [Jan3]. We will not repeat the arguments here, just restrict ourselves to a comment on the most difficult case: the irreducibility of  $\zeta(\rho, 2\ell - 1) \rtimes \zeta_1(\rho, \ell; \sigma)$ . The analogous case in [Jan3] is covered by Lemma 4.3, [Jan3]. A more efficient argument is given in section 6, [Jan5]. For  $\zeta(\rho, 2\ell - 1) \rtimes \zeta_1(\rho, \ell; \sigma)$ , the [Jan5] argument only works for  $\ell > 2$ . Thus, the most efficient way to deal with this case seems to be to use [Gol1], [Gol2] for  $\ell = 1$ , a Jacquet module argument similar (but simpler) than that of Lemma 4.3, [Jan3] for  $\ell = 2$ , then the [Jan5] argument for  $\ell > 2$ .

The proof that the  $\pi$  has the irreducible subquotients indicated is similar to the proof of Theorem 6.1, [Jan3]. It is an inductive argument, with the induction on

$k + \ell$  (the parabolic rank of the supercuspidal support). As it is a rather lengthy argument, and there are no new ideas involved, we do not go through the details. Instead, we just restrict ourselves to a few remarks.

As in [Jan3], the induction focuses on  $s_{(m)}\pi$ . For cases (2) and (3) (noting that case (2) is essentially  $j = -1$ ), we observe that  $\pi = \nu^{-\frac{k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  has

$$\begin{aligned} s_{(m)}\pi &= \nu^{-k+j+1}\rho \otimes \nu^{-\frac{k}{2}+j+1}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma) \\ &\quad + \nu^{-j}\rho \otimes \nu^{-\frac{k}{2}+j}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma) \\ &\quad + \nu^{-\ell+1}\rho \otimes \nu^{-\frac{k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma). \end{aligned}$$

We let

$$\begin{aligned} \tau' &= \nu^{-k+j+1}\rho \otimes \nu^{-\frac{k}{2}+j+1}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma), \\ \tau'' &= \nu^{-j}\rho \otimes \nu^{-\frac{k}{2}+j}\zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma), \\ \tau''' &= \nu^{-\ell+1}\rho \otimes \nu^{-\frac{k+1}{2}+j}\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma). \end{aligned}$$

For  $\tau'$ , we have  $k' = k-1, j' = j, \ell' = \ell$  (in the obvious notation) so that  $k' - j' - 1 = k - j - 2, j' = j, \ell' - 1 = \ell - 1$ . Similarly, for  $\tau''$  we have  $k'' - j'' - 1 = k - j - 1, j'' = j - 1, \ell'' - 1 = \ell - 1$  and for  $\tau'''$ ,  $k''' - j''' - 1 = k - j - 1, j''' = j, \ell''' - 1 = \ell - 2$ . Further, by inductive hypothesis, we know that  $\tau', \tau'', \tau'''$  decompose according to the theorem.

The proof of the theorem is broken into subcases based on how  $\tau', \tau'', \tau'''$  decompose (with respect to the theorem). The particular case of the theorem governing the decomposition of  $\tau'$  is given in the second column in the table below, and is easily determined from  $k' - j' - 1, j', \ell' - 1$ . One note: if  $j = \frac{k+1}{2}$ , in order to avoid having  $\alpha' > 0$ , we replace  $\tau' = \nu^{-\frac{k+1}{2}}\rho \otimes \nu^{\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma)$  with  $\nu^{-\frac{k+1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \zeta(\rho, \ell; \sigma) = \tau''$  (so then  $k' - j' - 1 = k'' - j'' - 1, j' = j'', \ell' - 1 = \ell'' - 1$ ). The third and fourth columns have the corresponding information for  $\tau''$  and  $\tau'''$ , respectively. The final column indicates which components of  $\tau', \tau'', \tau'''$  are contained in  $s_{(m)}\pi_i$  for each component  $\pi_i$  of  $\pi$ . Note that this is part of the induction; we assume the table gives the Jacquet modules for lower values of  $k + \ell$  and verifies it for the  $k + \ell$  under consideration.

We note that the notation in the tables is the obvious notation; e.g., if  $\tau'$  decomposes according to case 3a, then  $\tau'_2$  is the second component in part 3a of the statement of the theorem.

The proof that the composition series have the indicated structure is similar to the proof of Theorem 7.1, [Jan3]. Again, since there are no new ideas involved, we omit the details.

**Table 1.** Let  $\pi = \nu^\alpha\zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  be as in the statement of Theorem 3.4. The components of  $\pi$  are denoted by  $\pi_1, \pi_2, \dots$ ; the corresponding Jacquet modules are  $s_{(m)}\pi_1, s_{(m)}\pi_2, \dots$ . An explicit description of  $\pi_1, \pi_2, \dots$  is given in Theorem 3.4. The representations  $\tau', \tau'', \tau'''$  are the induced representations arising in  $s_{(m)}\pi$  ( $s_{(m)}\pi = \tau' + \tau'' + \tau'''$ ), possibly reducible. They are described in the proof of Theorem 3.4 and decompose as indicated in the table; the final column indicates which components of  $\tau', \tau'', \tau'''$  are contained in  $s_{(m)}\pi_i$  for each component  $\pi_i$  of  $\pi$ .

Case (for $\pi$ )	for $\tau'$	for $\tau''$	for $\tau'''$	components
1. ( $\alpha$ ) $k = 2, \alpha = -(\ell + \frac{1}{2})$	3.1	irr	irr	$s_{(m)}\pi_1 = \tau'_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau'' + \tau'''$
( $\beta$ ) $k = 2, \alpha = -(\ell - \frac{1}{2})$	irr	3.1	1	$s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau''_2 + \tau'''_2$
( $\gamma$ ) $k > 2, \alpha = -(\ell + \frac{k-1}{2})$	1	irr	irr	$s_{(m)}\pi_1 = \tau'_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau'' + \tau'''$
( $\delta$ ) $k > 2, \alpha = -(\ell + \frac{-k+1}{2})$	irr	1	1	$s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau''_2 + \tau'''_2$
( $\epsilon$ ) $k > 2, -(\ell + \frac{k-1}{2}) < \alpha, \alpha < -(\ell + \frac{-k+1}{2})$	1	1	1	$s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'''_2$
2a.				
( $\alpha$ ) $\ell = 1, k = 2$	3.1	irr	irr	$s_{(m)}\pi_1 = \tau'_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''$ $s_{(m)}\pi_3 = \tau'_3 + \tau'''$
( $\beta$ ) $\ell = 1, k > 2$	2a	irr	irr	$s_{(m)}\pi_1 = \tau'_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''$ $s_{(m)}\pi_3 = \tau'_3 + \tau'''$
2b.				
( $\alpha$ ) $\ell = 2, k = 2$	3.1	3.1	2a	$s_{(m)}\pi_1 = \tau'_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''_1$ $s_{(m)}\pi_3 = \tau'_2 + \tau''_2$ $s_{(m)}\pi_4 = \tau'''_3$
( $\beta$ ) $\ell = 2, k > 2$	2b	1	2a	$s_{(m)}\pi_1 = \tau'_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''_1$ $s_{(m)}\pi_3 = \tau'_3 + \tau'_2 + \tau'''_2$ $s_{(m)}\pi_4 = \tau'_4 + \tau'''_3$
( $\gamma$ ) $\ell > 2, k = \ell$	2c	1	2b	$s_{(m)}\pi_1 = \tau'_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''_1 + \tau'''_2$ $s_{(m)}\pi_3 = \tau'_2 + \tau'''_3$ $s_{(m)}\pi_4 = \tau'''_4$
( $\delta$ ) $\ell > 2, k > \ell$	2b	1	2b	$s_{(m)}\pi_1 = \tau'_1 + \tau'''_1$ $s_{(m)}\pi_2 = \tau'_2 + \tau''_1 + \tau'''_2$ $s_{(m)}\pi_3 = \tau'_3 + \tau'_2 + \tau'''_3$ $s_{(m)}\pi_4 = \tau'_4 + \tau'''_4$
2c.				
( $\alpha$ ) $k = 2, \ell = 3$	3.1	irr	2b	$s_{(m)}\pi_1 = \tau'_1 + \tau'''_1 + \tau'''_4$ $s_{(m)}\pi_2 = \tau'_2 + \tau'' + \tau'''_2 + \tau'''_3$
( $\beta$ ) $k > 2, \ell = k + 1$	2d	irr	2b	$s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_4$ $s_{(m)}\pi_2 = \tau'_2 + \tau'' + \tau'''_2 \tau'''_3$

<p>2d.</p> <p>(<math>\alpha</math>) <math>k = 2, \ell = 4</math></p> <p>(<math>\beta</math>) <math>k = 2, \ell &gt; 4</math></p> <p>(<math>\gamma</math>) <math>k &gt; 2, \ell = k + 2</math></p> <p>(<math>\delta</math>) <math>k &gt; 2, \ell &gt; k + 2</math></p>	<p>3.1 irr 2c</p> <p>3.1 irr 2d</p> <p>2d irr 2c</p> <p>2d irr 2d</p>	<p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_1'' + \tau_2'''</math></p>
<p>3a.</p> <p>(<math>\alpha</math>) <math>\ell = 1, j = 1, k = 4</math></p> <p>(<math>\beta</math>) <math>\ell = 1, j = 1, k &gt; 4</math></p> <p>(<math>\gamma</math>) <math>\ell = 1, j &gt; 1, k = 2j + 2</math></p> <p>(<math>\delta</math>) <math>\ell = 1, j &gt; 1, k &gt; 2j + 2</math></p> <p>(<math>\epsilon</math>) <math>\ell &gt; 1, j = \ell, k = 2j + 2</math></p> <p>(<math>\zeta</math>) <math>\ell &gt; 1, j = \ell, k &gt; 2j + 2</math></p> <p>(<math>\eta</math>) <math>\ell &gt; 1, j &gt; \ell, k = 2j + 2</math></p> <p>(<math>\theta</math>) <math>\ell &gt; 1, j &gt; \ell, k &gt; 2j + 2</math></p>	<p>3b 3c 3.2</p> <p>3a 3c 3.2</p> <p>3b 3a 3.2</p> <p>3a 3a 3.2</p> <p>3b 3c 3a</p> <p>3a 3c 3a</p> <p>3b 3a 3a</p> <p>3a 3a 3a</p>	<p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_2'' + \tau_3'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_3' + \tau_2'' + \tau_3'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_3 = \tau_3' + \tau_3'' + \tau_3'''</math></p>
<p>3b.</p> <p>(<math>\alpha</math>) <math>\ell = 1, j = 1, k = 3</math></p> <p>(<math>\beta</math>) <math>\ell = 1, j &gt; 1, k = 2j + 1</math></p> <p>(<math>\gamma</math>) <math>\ell &gt; 1, j = \ell, k = 2j + 1</math></p> <p>(<math>\delta</math>) <math>\ell &gt; 1, j &gt; \ell, k = 2j + 1</math></p>	<p>3c 3c 3.2</p> <p>3a 3a 3.2</p> <p>3c 3c 3b</p> <p>3a 3a 3b</p>	<p>(n.b. <math>\tau' = \tau''</math>)</p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_3' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_3'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_2' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2'' + \tau_2'''</math></p> <p><math>s^{(m)}\pi_1 = \tau_1' + \tau_3' + \tau_1'' + \tau_1'''</math></p> <p><math>s^{(m)}\pi_2 = \tau_2' + \tau_2'' + \tau_3'' + \tau_2'''</math></p>

<p>3c.</p> <p>(<math>\alpha</math>) <math>j = 0, \ell = 1, k = 2</math></p> <p>(<math>\beta</math>) <math>j = 0, \ell = 1, k &gt; 2</math></p> <p>(<math>\gamma</math>) <math>j &gt; 0, \ell = j + 1, k = 2j + 2</math></p> <p>(<math>\delta</math>) <math>j &gt; 0, \ell = j + 1, k &gt; 2j + 2</math></p>	<p>irr 3.1 3.2</p> <p>3c 2a 3.2</p> <p>irr 3d 3a</p> <p>3c 3d 3a</p>	<p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau''_2 + \tau''_3 + \tau'''_2 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'_3</math></p> <p style="padding-left: 40px;"><math>+ \tau'''_2 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'_2 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_3 + \tau''_4 + \tau'_2 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'_2 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau'_3 + \tau'_4</math></p> <p style="padding-left: 40px;"><math>+ \tau'''_2 + \tau'''_3</math></p>
<p>3d.</p> <p>(<math>\alpha</math>) <math>j = 0, \ell = 2, k = 3</math></p> <p>(<math>\beta</math>) <math>j = 0, \ell = 2, k &gt; 3</math></p> <p>(<math>\gamma</math>) <math>j = 0, \ell &gt; 2, k = \ell + 1</math></p> <p>(<math>\delta</math>) <math>j = 0, \ell &gt; 2, k &gt; \ell + 1</math></p> <p>(<math>\epsilon</math>) <math>j &gt; 0, \ell = j + 2, k = 2j + 3</math></p> <p>(<math>\zeta</math>) <math>j &gt; 0, \ell = j + 2, k &gt; 2j + 3</math></p> <p>(<math>\eta</math>) <math>j &gt; 0, \ell &gt; j + 2,</math> <math>k = \ell + j + 1</math></p> <p>(<math>\theta</math>) <math>j &gt; 0, \ell &gt; j + 2,</math> <math>k &gt; \ell + j + 1</math></p>	<p>3e 2b 3c</p> <p>3d 2b 3c</p> <p>3e 2b 3d</p> <p>3d 2b 3d</p> <p>3e 3d 3c</p> <p>3d 3d 3c</p> <p>3e 3d 3d</p> <p>3d 3d 3d</p>	<p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2</math></p> <p><math>s_{(m)}\pi_3 = \tau''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau''_4 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2</math></p> <p><math>s_{(m)}\pi_3 = \tau'_3 + \tau''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau'_4 + \tau''_4 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_3 = \tau''_3 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau''_4 + \tau'''_4</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2</math></p> <p><math>s_{(m)}\pi_3 = \tau''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau''_4 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_3 = \tau'_3 + \tau''_3 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau'_4 + \tau''_4 + \tau'''_4</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'_3</math></p> <p><math>s_{(m)}\pi_3 = \tau'_3 + \tau''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau'_4 + \tau''_4 + \tau'''_2</math></p> <p><math>s_{(m)}\pi_1 = \tau'_1 + \tau''_1 + \tau'''_1</math></p> <p><math>s_{(m)}\pi_2 = \tau'_2 + \tau''_2 + \tau'_3</math></p> <p><math>s_{(m)}\pi_3 = \tau'_3 + \tau''_3 + \tau'''_3</math></p> <p><math>s_{(m)}\pi_4 = \tau'_4 + \tau''_4 + \tau'''_4</math></p>



3e.	( $\alpha$ ) $j = 0, \ell = 2, k = 2$	irr	3.1	3c	$s_{(m)}\pi_1 = \tau^j + \tau_1'' + \tau_1''' + \tau_2'''$ $s_{(m)}\pi_2 = \tau_2''$
	( $\beta$ ) $j = 0, \ell > 2, k = \ell$	3f(i)	2c	3d	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1''' + \tau_4'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2''' + \tau_3'''$
	( $\gamma$ ) $j > 0, \ell = j + 2,$ $k = \ell + j$	3g	3e	3c	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1''' + \tau_2'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2''$
	( $\delta$ ) $j > 0, \ell > j + 2,$ $k = \ell + j$	3f(ii)	3e	3d	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1''' + \tau_4'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2''' + \tau_3'''$
3f(i).	( $\alpha$ ) $j = 0, k = 2, \ell = 3$	irr	3.1	3e	$s_{(m)}\pi_1 = \tau^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$
	( $\beta$ ) $j = 0, k = 2, \ell > 3$	irr	3.1	3f(i)	$s_{(m)}\pi_1 = \tau^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$
	( $\gamma$ ) $j = 0, k > 2, \ell = k + 1$	3f(i)	2d	3e	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$
	( $\delta$ ) $j = 0, k > 2, \ell > k + 1$	3f(i)	2d	3f(i)	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$
3f(ii).	( $\alpha$ ) $j = 1, k = 4, \ell = 4$	3g	3f(i)	3e	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3^j$
	( $\beta$ ) $j = 1, k = 4, \ell > 4$	3g	3f(i)	3f(ii)	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2^j + \tau_3'''$
	( $\gamma$ ) $j = 1, k > 4, \ell = k$	3f(ii)	3f(i)	3e	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3^j$
	( $\delta$ ) $j = 1, k > 4, \ell > k$	3f(ii)	3f(i)	3f(ii)	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2^j + \tau_3'''$
	( $\epsilon$ ) $j > 1, k = 2j + 2,$ $\ell = j + 3$	3g	3f(ii)	3e	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2^j + \tau_3'''$
	( $\zeta$ ) $j > 1, k = 2j + 2,$ $\ell > j + 3$	3g	3f(ii)	3f(ii)	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_2^j + \tau_3'' + \tau_3'''$
	( $\eta$ ) $j > 1, k > 2j + 2,$ $\ell = k - j + 1$	3f(ii)	3f(ii)	3e	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3^j + \tau_3''$
	( $\theta$ ) $j > 1, k > 2j + 2,$ $\ell > k - j + 1$	3f(ii)	3f(ii)	3(ii)	$s_{(m)}\pi_1 = \tau_1^j + \tau_1'' + \tau_1'''$ $s_{(m)}\pi_2 = \tau_2^j + \tau_2'' + \tau_2'''$ $s_{(m)}\pi_3 = \tau_3^j + \tau_3'' + \tau_3'''$

3g.				(n.b. $\tau' = \tau''$ )
( $\alpha$ ) $j = 1, k = 3, \ell = 3$	3f(i)	3f(i)	irr	$s_{(m)}\pi_1 = \tau'_1 + \tau'_2 + \tau''_1 + \tau'''$ $s_{(m)}\pi_2 = \tau''_2$
( $\beta$ ) $j = 1, k = 3, \ell > 3$	3f(i)	3f(i)	3g	$s_{(m)}\pi_1 = \tau'_1 + \tau'_2 + \tau''_1 + \tau'''$ $s_{(m)}\pi_2 = \tau''_2 + \tau'''$
( $\gamma$ ) $j > 1, k = 2j + 1,$ $\ell = j + 2$	3f(ii)	3f(ii)	irr	$s_{(m)}\pi_1 = \tau'_1 + \tau'_2 + \tau''_1 + \tau'''$ $s_{(m)}\pi_2 = \tau'_3 + \tau''_2 + \tau'''$
( $\delta$ ) $j > 1, k = 2j + 1,$ $\ell > j + 2$	3f(ii)	3f(ii)	3g	$s_{(m)}\pi_1 = \tau'_1 + \tau'_2 + \tau''_1 + \tau'''$ $s_{(m)}\pi_2 = \tau'_3 + \tau''_2 + \tau'''$

□

Before proceeding to the next result, we pause to make a couple of observations about the preceding theorem.

For case 3, we write  $\alpha = \frac{-k+1}{2} + j, 0 \leq j \leq \frac{k-1}{2}$ . As noted in the proof, we may also write case 2 this way, using  $j = -1$ . Now,  $s_{(m)}\pi$  is the sum of three terms (see proof) which are tensor products whose first factors are  $\nu^{-(k-j-1)}\rho, \nu^{-j}\rho, \nu^{-(\ell-1)}\rho$ , resp. The relations among the exponents  $k-j-1, j, \ell-1$  govern the decomposition of 2 and 3 into cases. The reader may observe the similarity between neighboring cases; e.g., one observes that taking  $\ell-1 = j$  in case 3(d), suitably interpreted, gives the irreducible subquotients for 3(c). To make this comparison, Zelevinsky segments (resp., sequences of generalized Steinbergs) of length  $-1$  should be treated as missing; segments (resp., sequences of generalized Steinbergs) of length  $< -1$  should have the entire representation treated as missing.

We also make an observation regarding the generalized Steinbergs which appear in the Langlands data. First, we note that any term appearing in the minimal Jacquet module  $s_{min}\pi$  (i.e., the Jacquet module with respect to the smallest standard parabolic subgroup having nonzero Jacquet module) has the form

$$\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \dots \otimes \nu^{x_{k+\ell}}\rho \otimes \sigma$$

with  $\nu^{x_1}\rho \otimes \nu^{x_2}\rho \otimes \dots \otimes \nu^{x_{k+\ell}}\rho$  a shuffle (i.e., a permutation preserving the relative orders; cf. section 4 [K-R]) of  $\nu^{-k+j+1}\rho \otimes \nu^{-k+j+2}\rho \otimes \dots \otimes \nu^x\rho, \nu^{-j}\rho \otimes \nu^{-j+1}\rho \otimes \dots \otimes \nu^{-x-1}\rho$ , and  $\nu^{-\ell+1}\rho \otimes \nu^{-\ell+2}\rho \otimes \dots \otimes \rho$ . Therefore, one can have at most three consecutive  $x_i, x_{i+1}, x_{i+2}$  which are decreasing. This is why we do not get  $\delta(\rho, n)$  for  $n > 3$ . Similar but subtler considerations may be used to constrain the possible tempered representations of orthogonal groups which appear.

We now turn to what might be considered a generalized version of ramified degenerate principal series. Here, we need a bit of additional notation. Suppose  $\rho \not\cong \rho_0$  are representations of  $GL(m, F)$  and  $GL(m_0, F)$ , with both  $(\rho, \sigma)$  and  $(\rho_0, \sigma)$  satisfying (C0). Again, let  $\rho \rtimes \sigma = T_1(\rho; \sigma) + T_2(\rho; \sigma)$  and  $\rho_0 \rtimes \sigma = T_1(\rho_0; \sigma) + T_2(\rho_0; \sigma)$ . By [Gol1], [Gol2],  $\rho \rtimes T_i(\rho_0; \sigma)$  and  $\rho_0 \rtimes T_j(\rho; \sigma)$  have a common component. Denote this common component by  $T_{i,j}(\rho_0, \rho; \sigma)$ .

**Theorem 3.5.** *Suppose that  $\rho, \rho_0$  are irreducible unitary supercuspidal representations of  $GL(m, F), GL(m_0, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $G(r, F)$  such that both  $(\rho, \sigma)$  and  $(\rho_0, \sigma)$  satisfy (C0). Let  $\pi = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  with  $\alpha \in \mathbb{R}, k \geq 1$ . Then  $\pi$  is reducible if and only if  $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \dots, \frac{k-1}{2}\}$ . Suppose  $\pi$  is reducible. By contragredience, we may assume that  $\alpha \leq 0$ . Write  $\alpha = \frac{-k+1}{2} + j$  with  $0 \leq j \leq \frac{k-1}{2}$ .*

(1)  $j = \frac{k-1}{2}$   
 $\pi = \pi_1 + \pi_2$  with  
 $\pi_i = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k+1}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{-\frac{k+1}{2}}\rho_0, \nu^{-1}\rho_0]; T_{i,1}(\rho_0, \rho; \sigma))$

for  $i = 1, 2$ . In this case,  $\pi = \pi_1 \oplus \pi_2$ .

(2)  $0 \leq j < \frac{k-1}{2}$   
 $\pi = \pi_1 + \pi_2 + \pi_3$  with  
 $\pi_i = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_{i,1}(\rho_0, \rho; \sigma))$

for  $i = 1, 2$  and

$$\pi_3 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-2}\rho_0], [\nu^{-j-\frac{1}{2}}\delta(\rho_0, 2), \nu^{-j+\frac{1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2)]; T_1(\rho; \sigma)).$$

In this case,  $\pi_3$  is the unique irreducible quotient and  $\pi_1 \oplus \pi_2$  is a subrepresentation.

*Proof.* For  $O(2n, F)$  (and  $SO(2n + 1, F)$ ,  $Sp(2n, F)$ ), the arguments from section 5, [Jan3] may be used to identify the irreducible subquotients and their Jacquet modules. Alternatively, and more directly, they may be obtained from the results of [Jan6]. The arguments from section 7, [Jan3] may be used to determine the composition series. □

*Remark 3.6.* Let  $\rho$  and  $\sigma$  be as in the preceding theorem. Suppose  $\rho_0$  is an irreducible unitary supercuspidal representation of  $GL(m_0, F)$  with  $\rho_0 \not\cong \tilde{\rho}_0$ . Then,  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  is irreducible for all  $\alpha \in \mathbb{R}$ . (This also follows from [Jan6] or an argument like that in [Jan3].)

#### 4. RESTRICTIONS OF REPRESENTATIONS

Our analysis of generalized degenerate principal series for  $SO(2n, F)$  is based on our results for  $O(2n, F)$  and the connection between induced representations for  $SO(2n, F)$  and those for  $O(2n, F)$ . To establish the connection between generalized degenerate principal series for  $G^0 = SO(2n, F)$  and  $G = O(2n, F)$ , we study the restriction from  $G$  to  $G^0$  in general (Lemmas 4.1 and 4.2), for Langlands data (Lemma 4.6) and for representations  $T_i(\rho; \sigma)$  and  $\zeta_i(\rho, \ell; \sigma)$ . Proposition 4.3 and Corollary 4.4 discuss cuspidal reducibility. In Lemma 4.5, we describe the connection between composition series for representations of  $G$  and  $G^0$ .

Recall that for  $n \geq 1$ ,

$$O(2n, F) = SO(2n, F) \rtimes \{1, s\},$$

where  $s$  is defined in section 2. We denote by  $\hat{s}$  the nontrivial character of  $O(2n, F)$  defined by

$$\begin{aligned} \hat{s}(g) &= 1, \\ \hat{s}(gs) &= -1, \end{aligned}$$

for every  $g \in SO(2n, F)$ .

**Lemma 4.1.** *Let  $G = O(2n, F)$ ,  $G^0 = SO(2n, F)$ , with  $n > 0$ .*

- (1) For any admissible representation  $\pi_0$  of  $G^0$  and any admissible representation  $\pi$  of  $G$  ( $\pi_0, \pi$  not necessarily irreducible), we have

$$\begin{aligned} r_{G^0, G} \circ i_{G, G^0}(\pi_0) &\cong \pi_0 \oplus s\pi_0, & i_{G, G^0}(s\pi_0) &\cong i_{G, G^0}(\pi_0), \\ i_{G, G^0} \circ r_{G^0, G}(\pi) &\cong \pi \oplus \hat{s}\pi, & r_{G^0, G}(\hat{s}\pi) &\cong r_{G^0, G}(\pi). \end{aligned}$$

- (2) Let  $\sigma$  be an irreducible admissible representation of  $G$ . Suppose that  $\sigma_0$  is an irreducible subquotient of  $r_{G^0, G}(\sigma)$ . Then,

$$\sigma_0 \cong s\sigma_0 \text{ if and only if } \sigma \not\cong \hat{s}\sigma.$$

- (a) If  $\sigma_0 \cong s\sigma_0$ , then

$$\begin{aligned} i_{G, G^0}(\sigma_0) &\cong \sigma \oplus \hat{s}\sigma, \\ r_{G^0, G}(\sigma) &\cong \sigma_0. \end{aligned}$$

- (b) If  $\sigma_0 \not\cong s\sigma_0$ , then

$$\begin{aligned} i_{G, G^0}(\sigma_0) &\cong \sigma, \\ r_{G^0, G}(\sigma) &\cong \sigma_0 \oplus s\sigma_0. \end{aligned}$$

*Proof.* 1. The first statement follows from [B-Z], Theorem 5.2, noting that since both  $G^0$  and  $sG^0$  are open in  $G$ , either can start the filtration, hence both  $\pi_0$  and  $s\pi_0$  appear as subrepresentations.

An isomorphism  $i_{G, G^0} \circ r_{G^0, G}(\pi) \cong \pi \oplus \hat{s}\pi$  can be constructed in the following way: Let  $V$  denote the space of  $\pi$ . For  $v \in V$ , define  $\varphi_v : G \rightarrow V$  and  $\psi_v : G \rightarrow V$  by

$$\varphi_v(x) = \pi(x)v, \quad \psi_v(x) = \hat{s}\pi(x)v.$$

Then  $\varphi : V \rightarrow i_{G, G^0}(V)$  given by  $\varphi(v) = \varphi_v$  and  $\psi : V \rightarrow i_{G, G^0}(V)$  defined by  $\psi(v) = \psi_v$  are intertwining operators and  $\varphi \oplus \psi$  is an isomorphism between  $\pi \oplus \hat{s}\pi$  and  $i_{G, G^0} \circ r_{G^0, G}(\pi)$ .

The proof that  $i_{G, G^0}(s\pi_0) \cong i_{G, G^0}(\pi_0)$  is direct, using the intertwining operator  $\varphi$  between  $i_{G, G^0}(\pi_0)$  and  $i_{G, G^0}(s\pi_0)$  given by

$$\varphi(f)(g) = f(sg),$$

where  $f \in i_{G, G^0}(\pi_0)$ ,  $g \in G$ .

The last statement follows from [B-Z], Proposition 1.9.

2. This follows from the results in section 2, [G-K] (cf. Lemma 2.1, [B-J1]).  $\square$

**Lemma 4.2.** Let  $\sigma$  be an admissible representation of  $O(2m, F)$ ,  $m > 0$ , and  $\sigma_0$  an admissible representation of  $SO(2m, F)$ . Let  $\rho$  be an admissible representation of  $GL(n, F)$ . Set  $G = O(2(m+n), F)$ ,  $G^0 = SO(2(m+n), F)$ .

- (1)

$$\hat{s}(\rho \rtimes \sigma) \cong \rho \rtimes \hat{s}\sigma, \quad s(\rho \rtimes \sigma_0) \cong \rho \rtimes s\sigma_0.$$

- (2) Suppose that  $\sigma$  is irreducible and  $\sigma_0$  is an irreducible subquotient of  $r_{SO(2m, F), O(2m, F)}(\sigma)$ .

- (a) If  $\sigma_0 \cong s\sigma_0$ , then

$$\begin{aligned} i_{G, G^0}(\rho \rtimes \sigma_0) &= \rho \rtimes \sigma + \hat{s}(\rho \rtimes \sigma), \\ r_{G^0, G}(\rho \rtimes \sigma) &= \rho \rtimes \sigma_0. \end{aligned}$$

(b) If  $\sigma_0 \not\cong s\sigma_0$ , then

$$\begin{aligned} i_{G,G^0}(\rho \rtimes \sigma_0) &= \rho \rtimes \sigma, \\ r_{G^0,G}(\rho \rtimes \sigma) &= \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0. \end{aligned}$$

*Proof.* 1. The first statement follows from [B-Z], Proposition 1.9, the second from [Ban2], Corollary 4.1.

2. Let  $M^0$  (respectively,  $M$ ) be the standard Levi subgroup of  $G^0$  (respectively,  $G$ ) isomorphic to  $GL(n, F) \times SO(2m, F)$  (respectively,  $GL(n, F) \times O(2m, F)$ ).

(a) Suppose  $\sigma_0 \cong s\sigma_0$ . Then  $\rho \rtimes \sigma_0 \cong s(\rho \rtimes \sigma_0)$ . According to Lemma 4.1,  $i_{O(2m,F),SO(2m,F)}(\sigma_0) = \sigma + \hat{s}\sigma$ . We have

$$\begin{aligned} \rho \rtimes \sigma + \hat{s}(\rho \rtimes \sigma) &= i_{G,M}(\rho \otimes \sigma) + i_{G,M}(\rho \otimes \hat{s}\sigma) \\ &= i_{G,M}(\rho \otimes (\sigma + \hat{s}\sigma)) = i_{G,M^0}(\rho \otimes \sigma_0) = i_{G,G^0}(\rho \rtimes \sigma_0). \end{aligned}$$

By Lemma 4.2,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma_0 + s(\rho \rtimes \sigma_0) = 2\rho \rtimes \sigma_0.$$

On the other hand,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = r_{G^0,G}(\rho \rtimes \sigma) + r_{G^0,G}(\hat{s}(\rho \rtimes \sigma)) = 2r_{G^0,G}(\rho \rtimes \sigma).$$

It follows that  $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$ .

(b) Suppose that  $\sigma_0 \not\cong s\sigma_0$ . According to Lemma 4.1,  $i_{O(2m,F),SO(2m,F)}(\sigma_0) = \sigma$ . It follows that

$$\rho \rtimes \sigma = i_{G,M}(\rho \otimes \sigma) = i_{G,M^0}(\rho \otimes \sigma_0) = i_{G,G^0}(\rho \rtimes \sigma_0).$$

Further,

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0$$

and

$$r_{G^0,G} \circ i_{G,G^0}(\rho \rtimes \sigma_0) = r_{G^0,G}(\rho \rtimes \sigma).$$

It follows that  $r_{G^0,G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0 + \rho \rtimes s\sigma_0$ .  $\square$

The following proposition relates the reducibility of  $\nu^x \rho \rtimes \sigma$  and that of  $\nu^x \rho \rtimes \sigma_0$  (for  $x \in \mathbb{R}$ ). This proposition tells us the conditions under which  $(\rho, \sigma)$  satisfy (C0) implies  $(\rho, \sigma_0)$  satisfy (C0) (noting the relevance of the latter to generalized degenerate principal series for  $SO(2n, F)$  discussed in section 3). In addition, as a corollary, we deduce that cuspidal reducibility for  $O(2n, F)$  has a characterization like that for  $SO(2n+1, F)$ ,  $Sp(2n, F)$ , and  $SO(2n, F)$ .

**Proposition 4.3.** *Suppose  $\rho$  is an irreducible unitary supercuspidal representation of  $GL(m, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $O(2r, F)$ . Suppose  $\sigma_0$  is an irreducible subquotient of  $r_{SO(2r,F),O(2r,F)}(\sigma)$ .*

(1)  $r > 0$  and  $s \cdot \sigma_0 \cong \sigma_0$ .

*For all  $x \in \mathbb{R}$ , we have  $\nu^x \rho \rtimes \sigma$  is reducible if and only if  $\nu^x \rho \rtimes \sigma_0$  is reducible.*

(2)  $r = 0$  or  $\sigma_0 \not\cong s \cdot \sigma_0$ .

*Here, there are two possibilities:*

(a)  $m$  odd with  $\rho \cong \tilde{\rho}$ .

*In this case,  $\nu^x \rho \rtimes \sigma_0$  is irreducible for all  $x \in \mathbb{R}$ . However,  $\nu^x \rho \rtimes \sigma$  is irreducible for all  $x \in \mathbb{R} \setminus \{0\}$  and reducible for  $x = 0$ .*

(b)  $m$  even or  $\rho \not\cong \tilde{\rho}$ .

For all  $x \in \mathbb{R}$ , we have  $\nu^x \rho \rtimes \sigma$  is reducible if and only if  $\nu^x \rho \rtimes \sigma_0$  is reducible.

*Proof.* Let  $M$  be the standard Levi subgroup of  $G$  isomorphic to  $GL(m, F) \times O(2r, F)$ . For (1), we show that  $\nu^x \rho \rtimes \sigma$  is irreducible if and only if  $\nu^x \rho \rtimes \sigma_0$  is irreducible. For (2), we show two things: (i)  $\nu^x \rho \rtimes \sigma$  irreducible implies  $\nu^x \rho \rtimes \sigma_0$  irreducible, and (ii)  $\nu^x \rho \rtimes \sigma_0$  irreducible implies  $\nu^x \rho \rtimes \sigma$  irreducible unless  $m$  is odd,  $x = 0$ , and  $\rho \cong \tilde{\rho}$ , in which case  $\nu^x \rho \rtimes \sigma$  reduces.

We start with (1), so we may assume  $r > 0$  and  $\sigma_0 \cong s\sigma_0$ . By Lemma 4.2,

$$r_{G^0, G}(\nu^x \rho \rtimes \sigma) = \nu^x \rho \rtimes \sigma_0.$$

Therefore, if  $\nu^x \rho \rtimes \sigma_0$  is irreducible, then  $\nu^x \rho \rtimes \sigma$  is irreducible. On the other hand, suppose that  $\nu^x \rho \rtimes \sigma$  is irreducible. We have

$$r_{M, G}(\nu^x \rho \rtimes \sigma) = \nu^x \rho \otimes \sigma + \nu^{-x} \tilde{\rho} \otimes \sigma.$$

Since  $\sigma \not\cong \hat{s}\sigma$ , Lemma 4.2(1) implies

$$r_{M, G}(\hat{s}(\nu^x \rho \rtimes \sigma)) = \nu^x \rho \otimes \hat{s}\sigma + \nu^{-x} \tilde{\rho} \otimes \hat{s}\sigma.$$

It follows that  $\nu^x \rho \rtimes \sigma \not\cong \hat{s}(\nu^x \rho \rtimes \sigma)$ . By Lemma 4.1,  $\nu^x \rho \rtimes \sigma_0 = r_{G^0, G}(\nu^x \rho \rtimes \sigma)$  is irreducible.

We now address (2) for  $r > 0$ , so we may assume  $r > 0$  and  $\sigma_0 \not\cong s\sigma_0$ . By Lemma 4.2,

$$i_{G, G^0}(\nu^x \rho \rtimes \sigma_0) = \nu^x \rho \rtimes \sigma.$$

Therefore, if  $\nu^x \rho \rtimes \sigma$  is irreducible, then  $\nu^x \rho \rtimes \sigma_0$  is irreducible. On the other hand, suppose that  $\nu^x \rho \rtimes \sigma_0$  is irreducible. We have

$$r_{M^0, G^0}(\nu^x \rho \rtimes \sigma_0) = \nu^x \rho \otimes \sigma_0 + \nu^{-x} \tilde{\rho} \otimes s^m \sigma_0.$$

By Lemma 4.2(1),

$$r_{M^0, G^0}(s(\nu^x \rho \rtimes \sigma_0)) = \nu^x \rho \otimes s\sigma_0 + \nu^{-x} \tilde{\rho} \otimes s^{m+1} \sigma_0.$$

Thus,  $\nu^x \rho \rtimes \sigma_0 \not\cong s(\nu^x \rho \rtimes \sigma_0)$  unless  $m$  is odd,  $x = 0$ , and  $\rho \cong \tilde{\rho}$ . By Lemma 4.1,  $\nu^x \rho \rtimes \sigma = i_{G, G^0}(\nu^x \rho \rtimes \sigma_0)$  is irreducible. If  $m$  is odd,  $x = 0$ , and  $\rho \cong \tilde{\rho}$ , then Theorem 6.11 of [Gol1] tells us that  $\rho \rtimes \sigma_0$  is irreducible and Theorem 3.3 of [Gol2] implies that  $\rho \rtimes \sigma$  has two components.

Finally, we address (2) for  $r = 0$ . Then  $\sigma_0 = 1_0$ ,  $\sigma = 1$ , both trivial representations of the trivial group.

First, since  $M = M^0$ , we have

$$i_{G, M}(\nu^x \rho \otimes 1) \cong i_{G, G^0} \circ i_{G^0, M^0}(\nu^x \rho \otimes 1_0).$$

Therefore, if  $i_{G, M}(\nu^x \rho \otimes 1)$  is irreducible, then so is  $i_{G^0, M^0}(\nu^x \rho \otimes 1_0)$ .

On the other hand, suppose  $i_{G^0, M^0}(\nu^x \rho \otimes 1_0)$  is irreducible. Suppose  $x \neq 0$ ; without loss of generality,  $x < 0$ . Then,  $i_{G^0, M^0}(\nu^x \rho \otimes 1_0) = L(\nu^x \rho \otimes 1_0)$ . Since

$$s \cdot L(\nu^x \rho) = L(s \cdot (\nu^x \rho \otimes 1_0)) \not\cong L(\nu^x \rho \otimes 1_0),$$

we see that  $i_{G, G^0} \circ i_{G^0, M^0}(\nu^x \rho \otimes 1_0)$  is irreducible, as needed. When  $x = 0$ , Theorems 3.1 and 3.3 of [Gol2] tell us that  $i_{G^0, M^0}(\rho \otimes 1_0)$  and  $i_{G, M}(\rho \otimes 1)$  have the same number of components (implying the irreducibility of  $i_{G, M}(\rho \otimes 1)$ ) unless  $\rho$  is a self-contragredient representation of  $GL_m(F)$  with  $m$  odd, in which  $i_{G, M}(\rho \otimes 1)$  has twice as many components as  $i_{G^0, M^0}(\rho \otimes 1_0)$  (implying the reducibility of  $i_{G, M}(\rho \otimes 1)$ ), as needed.  $\square$

It is worth noting that the results may be interpreted as follows: If  $\nu^x \rho \rtimes \sigma$  is irreducible, then  $\nu^x \rho \rtimes \sigma_0$  is irreducible. On the other hand, if  $\nu^x \rho \rtimes \sigma_0$  is irreducible, then  $\nu^x \rho \rtimes \sigma$  is irreducible unless  $\nu^x \rho \otimes \sigma$  is unitary (so  $x = 0$ ) and the pair  $(P, \rho \otimes \sigma)$  is ramified (in the sense of Harish-Chandra, cf. [Sil2]) but  $(P^0, \rho \otimes \sigma_0)$  is unramified.

**Corollary 4.4.** *Assume the conjectures needed for [Mœ] or [Zh]. With notation as above, we have the following:*

- (1) *If  $\rho \not\cong \tilde{\rho}$ , then  $\nu^x \rho \rtimes \sigma$  is irreducible for all  $x \in \mathbb{R}$ .*
- (2) *If  $\rho \cong \tilde{\rho}$ , then there is a unique  $\alpha \geq 0$  with  $\alpha \in \frac{1}{2}\mathbb{Z}$  such that  $\nu^x \rho \rtimes \sigma$  is irreducible and  $\nu^x \rho \rtimes \sigma$  is irreducible for all  $x \in \mathbb{R} \setminus \{\pm\alpha\}$ . (The specific value of  $\alpha$  may be determined by the preceding theorem and the results of [Mœ], [Zh].)*

In section 3, we claimed that when  $(\rho, \sigma)$  and  $(\rho, \sigma_0)$  both satisfy (C0), the composition series for the generalized degenerate principal series  $\nu^\alpha \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  and those of  $\nu^\alpha \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma_0)$  have the same form. Lemmas 4.5 and 4.6 are used (in section 5) to show that this is indeed the case.

**Lemma 4.5.** *Let  $\sigma$  be an irreducible admissible representation of  $O(2m, F)$ ,  $m > 0$ , and  $\sigma_0$  an irreducible subquotient of  $r_{SO(2m, F), O(2m, F)}(\sigma)$ . Let  $\rho$  be an admissible representation of  $GL(n, F)$ . Set  $G = O(2(m+n), F)$  and  $G^0 = SO(2(m+n), F)$ .*

- (1) *Suppose that  $\sigma_0 \cong s\sigma_0$  and that  $\pi_0 \cong s\pi_0$  for every irreducible subquotient  $\pi_0$  of  $\rho \rtimes \sigma_0$ . Then  $\rho \rtimes \sigma$  and  $\rho \rtimes \sigma_0$  have the same number of irreducible subquotients. Assume, in addition, that for any two irreducible subquotients  $\pi$  and  $\pi'$  of  $\rho \rtimes \sigma$ ,  $\hat{s}\pi \not\cong \pi'$ . Then*

$$0 \subset \pi_1 \subset \cdots \subset \pi_k = \rho \rtimes \sigma$$

*is a composition series for  $\rho \rtimes \sigma$  if and only if*

$$0 \subset r_{G^0, G}(\pi_1) \subset \cdots \subset r_{G^0, G}(\pi_k) = \rho \rtimes \sigma_0$$

*is a composition series for  $\rho \rtimes \sigma_0$ .*

- (2) *Suppose that  $\sigma \cong \hat{s}\sigma$  and that  $\pi \cong \hat{s}\pi$  for every irreducible subquotient  $\pi$  of  $\rho \rtimes \sigma$ . Then  $\rho \rtimes \sigma$  and  $\rho \rtimes \sigma_0$  have the same number of irreducible subquotients. Assume, in addition, that for any two irreducible subquotients  $\pi_0$  and  $\pi'_0$  of  $\rho \rtimes \sigma_0$ ,  $s\pi_0 \not\cong \pi'_0$ . Then*

$$0 \subset \pi_1^0 \subset \cdots \subset \pi_k^0 = \rho \rtimes \sigma_0$$

*is a composition series for  $\rho \rtimes \sigma_0$  if and only if*

$$0 \subset i_{G, G^0}(\pi_1) \subset \cdots \subset i_{G, G^0}(\pi_k) = \rho \rtimes \sigma$$

*is a composition series for  $\rho \rtimes \sigma$ .*

*Proof.* 1. According to Lemma 4.2,  $r_{G^0, G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$ . Let

$$\rho \rtimes \sigma = \pi_1 + \cdots + \pi_k,$$

where  $\pi_1, \dots, \pi_k$  are irreducible. For  $i = 1, \dots, k$ , let  $\pi_i^0$  be an irreducible subquotient of  $r_{G^0, G}(\pi_i)$ . Then  $\pi_i^0$  is an irreducible component of  $r_{G^0, G}(\rho \rtimes \sigma) = \rho \rtimes \sigma_0$ . Since  $\pi_i^0 \cong s\pi_i^0$ , we have (Lemma 4.1)

$$r_{G^0, G}(\pi_i) = \pi_i^0.$$

It follows that

$$\rho \rtimes \sigma_0 = r_{G^0, G}(\rho \rtimes \sigma) = r_{G^0, G}(\pi_1 + \cdots + \pi_k) = \pi_1^0 + \cdots + \pi_k^0.$$

We deal with composition series inductively. In order to do this, we have to work in slightly greater generality. To this end, suppose that  $\pi$  is an admissible representation of  $G$  such that  $\hat{s}\pi \cong \pi$  and  $\pi' \cong \hat{s}\pi'$  for every irreducible subquotient  $\pi'$  of  $\pi$ . Further, assume the following: (1) if  $\pi', \pi''$  are irreducible subquotients of  $\pi$ , then  $\pi' \not\cong \hat{s}\pi''$ , and (2) if  $\pi_i$  is an irreducible subquotient of  $\pi$ , then  $\pi_i^0$  is an irreducible subquotient of  $\pi^0 = r_{G^0, G}(\pi)$ . We note that these assumptions hold when  $\pi = \rho \rtimes \sigma$ . Now, let  $\pi_1$  be an irreducible subquotient of  $\pi$ . We prove that  $\pi_1$  is a subrepresentation of  $\pi$  if and only if  $\pi_1^0$  is a subrepresentation of  $\pi_0$ . The statement then follows by the induction on the number of irreducible subquotients.

Suppose that  $\pi_1$  is a subrepresentation of  $\pi$ . Then we have the exact sequence

$$0 \longrightarrow \pi_1 \longrightarrow \pi \longrightarrow \pi/\pi_1 \longrightarrow 0.$$

The functor  $r_{G^0, G}$  is exact ([B-Z], Proposition 1.9), so we have the exact sequence

$$0 \longrightarrow r_{G^0, G}(\pi_1) \longrightarrow r_{G^0, G}(\pi) \longrightarrow r_{G^0, G}(\pi/\pi_1) \longrightarrow 0;$$

i.e.,

$$0 \longrightarrow \pi_1^0 \longrightarrow \pi_0 \longrightarrow \pi_0/\pi_1^0 \longrightarrow 0,$$

so  $\pi_1^0$  is a subrepresentation of  $\pi_0$ .

Conversely, assume  $\pi_1^0 \hookrightarrow \pi_0$ . Then the exact sequence

$$0 \longrightarrow \pi_1^0 \longrightarrow \pi_0 \longrightarrow \pi/\pi_1^0 \longrightarrow 0$$

implies

$$0 \longrightarrow i_{G, G^0}(\pi_1^0) \longrightarrow i_{G, G^0}(\pi_0) \longrightarrow i_{G, G^0}(\pi_0/\pi_1^0) \longrightarrow 0,$$

so, by Lemma 4.2, we have

$$0 \longrightarrow \pi_1^0 \oplus \hat{s}\pi_1^0 \longrightarrow \pi \oplus \hat{s}\pi \longrightarrow i_{G, G^0}(\pi_0/\pi_1^0) \longrightarrow 0.$$

Therefore,  $\pi_1$  is a subrepresentation of  $\pi \oplus \hat{s}\pi$ . By the assumption,  $\pi_1$  is not a component of  $\hat{s}\pi$ . We conclude that  $\pi_1$  is a subrepresentation of  $\pi$ .

The proof of (2) is similar to that of (1). □

**Lemma 4.6.** *Let  $\rho_i, i = 1, \dots, k$  be an irreducible essentially square-integrable representation of  $GL(n_i, F)$  and  $\sigma$  an irreducible tempered representation of  $O(2m, F)$ ,  $m \geq 0$ . If  $m > 0$ , let  $\sigma_0$  be an irreducible subquotient of  $r_{SO(2m, F), O(2m, F)}(\sigma)$ . If  $m = 0$ , let  $\sigma_0 = 1_0$ . Set  $n = n_1 + \dots + n_k + m$ ,  $G = O(2n, F)$ ,  $G^0 = SO(2n, F)$ . Suppose that  $e(\rho_1) \leq \dots \leq e(\rho_k) < 0$ . Then*

$$\tau = \rho_1 \otimes \dots \otimes \rho_k \otimes \sigma \quad \text{and} \quad \tau_0 = \rho_1 \otimes \dots \otimes \rho_k \otimes \sigma_0$$

are Langlands data for  $G$  and  $G^0$  and

$$\begin{aligned} sL(\tau_0) &= L(s\tau_0), \\ \hat{s}L(\tau) &= L(\hat{s}\tau). \end{aligned}$$

Moreover,  $L(\tau)$  is a component of  $i_{G, G^0}(L(\tau_0))$ .

*Proof.* Let  $M^0$  (respectively,  $M$ ) denote the standard Levi subgroup corresponding to  $\tau_0$  (respectively,  $\tau$ ).

The first equality follows from [B-J1], Proposition 4.5. According to Lemma 4.2,

$$i_{G, M}(\hat{s}\tau) \cong \hat{s}i_{G, M}(\tau).$$

Now,  $L(\hat{s}\tau)$  is the unique irreducible subrepresentation of  $i_{G, M}(\hat{s}\tau)$  and  $\hat{s}L(\tau)$  is the unique irreducible subrepresentation of  $\hat{s}i_{G, M}(\tau)$ . It follows that  $\hat{s}L(\tau) \cong L(\hat{s}\tau)$ .



To prove that  $L(\tau)$  is a component of  $i_{G,G^0}(L(\tau_0))$ , suppose that  $\sigma_0 \cong s\sigma_0$ . Then  $\tau_0 \cong s\tau_0$ ,  $\tau \not\cong \hat{s}\tau$ ,  $i_{M,M^0}(\tau_0) \cong \tau \oplus \hat{s}\tau$ . We have

$$i_{G,G^0} \circ i_{G^0,M^0}(\tau_0) \cong i_{G,M}(\tau \oplus \hat{s}\tau) \cong i_{G,M}(\tau) \oplus i_{G,M}(\hat{s}\tau).$$

This representation has two irreducible subrepresentations,  $L(\tau)$  and  $L(\hat{s}\tau)$ .  $L(\tau_0)$  is the unique irreducible subrepresentation of  $i_{G^0,M^0}(\tau_0)$ . Since  $sL(\tau_0) \cong L(s\tau_0) \cong L(\tau_0)$ ,  $i_{G,G^0}(L(\tau_0))$  is the direct sum of two irreducible representations. They are subrepresentations of  $i_{G,G^0} \circ i_{G^0,M^0}(\tau_0)$ . It follows that

$$i_{G,G^0}(L(\tau_0)) \cong L(\tau) \oplus \hat{s}L(\tau) \cong L(\tau) \oplus L(\hat{s}\tau).$$

In the case  $\sigma_0 \not\cong s\sigma_0$ , the proof is similar. □

**Lemma 4.7.** *Let  $\rho$  be an irreducible supercuspidal unitary representation of  $GL(n, F)$  and  $\sigma$  an irreducible supercuspidal representation of  $O(2m, F)$ ,  $m \geq 0$ . If  $m > 0$ , let  $\sigma_0$  be an irreducible subquotient of  $r_{SO(2m,F),O(2m,F)}(\sigma)$ . If  $m = 0$ , let  $\sigma_0 = 1_0$ . Set  $G = O(2(m+n), F)$ ,  $G^0 = SO(2(m+n), F)$ .*

(1) *Suppose that  $\rho \rtimes \sigma_0$  is reducible. Let*

$$\rho \rtimes \sigma_0 \cong T_1(\rho; \sigma_0) \oplus T_2(\rho; \sigma_0)$$

*be the decomposition of  $\rho \rtimes \sigma_0$  into the direct sum of two inequivalent irreducible subrepresentations. Then there exists a decomposition*

$$\rho \rtimes \sigma \cong T_1(\rho; \sigma) \oplus T_2(\rho; \sigma)$$

*into the direct sum of two inequivalent irreducible subrepresentations such that, for  $i = 1, 2$ ,  $T_i(\rho; \sigma_0)$  (resp.,  $\zeta_i(\rho, \ell; \sigma_0)$ ) is an irreducible subquotient of  $r_{G^0,G}(T_i(\rho; \sigma))$  (resp.,  $r_{SO(2m+2\ell n,F),O(2m+2\ell n,F)}(\zeta_i(\rho, \ell; \sigma))$ ).*

(a) *If  $m > 0$  and  $\sigma_0 \cong s\sigma_0$ , then*

$$\begin{aligned} s(T_i(\rho; \sigma_0)) &\cong T_i(\rho; \sigma_0), \\ s(\zeta_i(\rho, \ell; \sigma_0)) &\cong \zeta_i(\rho, \ell; \sigma_0). \end{aligned}$$

(b) *If  $m = 0$  or  $m > 0$ ,  $\sigma_0 \not\cong s\sigma_0$ , then  $n$  is even and*

$$\begin{aligned} s(T_i(\rho; \sigma_0)) &\not\cong T_i(\rho; \sigma_0), \\ s(\zeta_i(\rho, \ell; \sigma_0)) &\not\cong \zeta_i(\rho, \ell; \sigma_0). \end{aligned}$$

(2) *Suppose that  $n$  is odd and  $\rho \cong \tilde{\rho}$ . If  $m > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ .*

(a)  *$\rho \rtimes \sigma_0$  is an irreducible tempered representation and*

$$\rho \rtimes \sigma_0 \cong s(\rho \rtimes \sigma_0).$$

(b)  *$\rho \rtimes \sigma$  is reducible and it is the direct sum of two inequivalent tempered representations*

$$\begin{aligned} \rho \rtimes \sigma &\cong T_1(\rho; \sigma) \oplus T_2(\rho; \sigma), \\ r_{G^0,G}(T_1(\rho; \sigma)) &= r_{G^0,G}(T_2(\rho; \sigma)) = \rho \rtimes \sigma_0, \\ r_{G^0,G}(\zeta_1(\rho, \ell; \sigma)) &= r_{G^0,G}(\zeta_2(\rho, \ell; \sigma)) = \zeta(\rho, \ell; \sigma_0). \end{aligned}$$

5. DEGENERATE PRINCIPAL SERIES FOR  $SO(2n, F)$

In this section, we deal with generalized degenerate principal series for  $SO(2n, F)$ . Let  $\rho, \rho_0$  be irreducible unitary supercuspidal representations of  $GL(m, F)$ ,  $GL(m_0, F)$ . Let  $\sigma_0$  (resp.,  $\sigma$ ) be an irreducible supercuspidal representation of  $SO(2r, F)$  (resp.  $O(2r, F)$ ) such that  $\sigma_0$  is a component of  $r_{SO(2r, F), O(2r, F)}(\sigma)$ . (We are allowing the possibility that  $r = 0$  here.) Further, suppose that  $(\rho, \sigma)$  satisfies (C0) (which implies  $\rho \cong \tilde{\rho}$ ). There are two possibilities (cf. Proposition 4.3): (1)  $(\rho, \sigma_0)$  also satisfies (C0), and (2)  $\nu^x \rho \rtimes \sigma_0$  is irreducible for all  $x \in \mathbb{R}$  (which can happen only if  $m$  is odd and either  $r = 0$  or  $\sigma_0 \not\cong s\sigma_0$ ). In the first case, the results for  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma_0)$  are given in section 3. The proofs in section 3 are for  $O(2n, F)$ ; the proofs for  $SO(2n, F)$  are handled in this section. In the second case, the results for  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma_0)$  are different than those in section 3; both the statements and proofs are given in this section. In both cases, the proofs are built from the results on  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma)$  and our study of  $r_{G^0, G}, i_{G, G^0}$  from section 4.

We start by dealing with the first case, where  $(\rho, \sigma_0)$  also satisfies (C0). We prove Theorem 3.4. Set

$$\begin{aligned} \pi_0 &= \nu^\alpha \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma_0), \\ \pi &= \nu^\alpha \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell; \sigma). \end{aligned}$$

First, consider the case  $r \neq 0$ .

The irreducible subquotients in Theorem 3.4 are described as Langlands subrepresentations. Let

$$\{L(\tau) \mid \tau \in T\}$$

be the set of all irreducible subquotients appearing in the statement of Theorem 3.4. For  $\tau \in T$ , denote by  $\tau_0$  the representation obtained by replacing  $\sigma$  by  $\sigma_0$  (noting that since  $(\rho, \sigma_0)$  also satisfies (C0), we have  $T_i(\rho, \sigma_0)$  and  $\delta(\nu\rho; T_i(\rho, \sigma_0))$  defined). We have two possibilities:  $\sigma_0 \cong s\sigma_0$  or  $\sigma_0 \not\cong s\sigma_0$ .

1. Suppose that  $\sigma_0 \cong s\sigma_0$ . Then  $\zeta_1(\rho, \ell; \sigma_0) \cong s\zeta_1(\rho, \ell; \sigma_0)$  (Lemma 4.7) and  $\pi_0 \cong s\pi_0$  (Lemma 4.2). By inspecting all  $\tau \in T$ , we conclude that  $s\tau_0 \cong \tau_0$  and

$$sL(\tau_0) \cong L(s\tau_0) \cong L(\tau_0).$$

Also, for  $\tau, \tau' \in T$  we have

$$L(\tau) \not\cong \hat{s}L(\tau').$$

Therefore, we are in the situation described by Lemma 4.5, (1). According to Lemma 4.6,

$$L(\tau_0) \cong r_{G^0, G}(L(\tau)).$$

We conclude that  $\pi$  and  $\pi_0$  have the same number of irreducible components and the same composition series structure.

Note that

$$r_{G^0, G}(\pi) = \pi_0$$

and

$$r_{M^0, G^0}(\pi_0) = r_{M^0, G^0} \circ r_{G^0, G}(\pi) = r_{M^0, M} \circ r_{M, G}(\pi).$$

Therefore, Jacquet modules of  $\pi_0$  can be found by taking restrictions of Jacquet modules of  $\pi$ . Let  $\tau', \tau'', \tau'''$  be as in the proof of Theorem 3.4; i.e.,

$$\tau' = \nu^{-k+j+1} \rho \otimes \nu^{\frac{-k}{2}+j+1} \zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau'' = \nu^{-j} \rho \otimes \nu^{\frac{-k}{2}+j} \zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau''' = \nu^{-\ell+1} \rho \otimes \nu^{\frac{-k+1}{2}+j} \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma).$$

Let

$$\tau'_0 = \nu^{-k+j+1} \rho \otimes \nu^{\frac{-k}{2}+j+1} \zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma_0),$$

$$\tau''_0 = \nu^{-j} \rho \otimes \nu^{\frac{-k}{2}+j} \zeta(\rho, k-1) \rtimes \zeta_1(\rho, \ell; \sigma_0),$$

$$\tau'''_0 = \nu^{-\ell+1} \rho \otimes \nu^{\frac{-k+1}{2}+j} \zeta(\rho, k) \rtimes \zeta_1(\rho, \ell-1; \sigma_0).$$

Then, from  $\sigma_0 \cong s\sigma_0$  and Lemmas 4.2 and 4.7, we can conclude

$$\begin{aligned} \tau'_0 &= r_{M^0, M}(\tau'), \\ \tau''_0 &= r_{M^0, M}(\tau''), \\ \tau'''_0 &= r_{M^0, M}(\tau''') \end{aligned}$$

and

$$s_{(m)}(\pi_0) = \tau'_0 + \tau''_0 + \tau'''_0.$$

2. Suppose that  $\sigma \cong \hat{s}\sigma$ , i.e.  $\sigma_0 \not\cong s\sigma_0$ . Condition (C0) implies that this is possible only for  $m$  even. Then  $\zeta_1(\rho, \ell; \sigma) \cong \hat{s}\zeta_1(\rho, \ell; \sigma)$ ,  $\pi \cong \hat{s}\pi$  and  $\tau \cong \hat{s}\tau$ , for all  $\tau \in T$ . Theorem 3.4 for  $SO(2n, F)$  follows from Lemma 4.5, (2) and Theorem 3.4 for  $O(2n, F)$ . According to Lemma 4.2,

$$r_{G^0, G}(\pi) = \pi_0 + s\pi_0.$$

We have

$$r_{M^0, G}(\pi) = r_{M^0, G^0}(\pi_0) + sr_{M^0, G^0}(\pi_0).$$

To find Jacquet modules of  $\pi_0$ , we select half of the components of restrictions of Jacquet modules of  $\pi$ . More precisely, we take the components containing  $\sigma_0$  (not  $s\sigma_0$ ).

Now, consider the case  $r = 0$ . Note that to have  $(\rho, 1_0)$  satisfy (C0), we must have  $m$  even. By Lemma 4.6,  $s\zeta_1(\rho, \ell; 1_0) \not\cong \zeta_1(\rho, \ell; 1_0)$ . We apply the same reasoning as in the case  $\sigma_0 \cong s\sigma_0$ .

Propositions 3.2, 3.3 and Theorem 3.5 can be proved in a similar way. However, for Proposition 3.3, the case  $r = 0$ ,  $m$  even has to be considered separately. We now give the proof of Proposition 3.3 for  $r = 0$ ,  $m$  even. Let

$$\begin{aligned} \pi_0 &= \nu^\alpha \zeta(\rho, k) \rtimes 1_0, \\ \pi &= \nu^\alpha \zeta(\rho, k) \rtimes 1. \end{aligned}$$

Then

$$\pi = i_{G, G^0}(\pi_0).$$

If  $\pi$  is irreducible, then  $\pi_0$  is irreducible.

Suppose that  $\pi$  is reducible. Recall that (Lemma 4.7)

$$T_i(\rho; 1_0) \not\cong sT_i(\rho; 1_0),$$

so

$$T_i(\rho; 1) \cong \hat{s}T_i(\rho; 1)$$

and therefore

$$T_1(\rho; 1) \not\cong \hat{s}T_2(\rho; 1).$$

According to Lemma 4.1 or Lemma 4.7,

$$T_i(\rho; 1) = i_{O(2m,F),SO(2m,F)}(T_i(\rho; 1_0)).$$

By inspecting all the cases in Proposition 3.3, we see that there is one-to-one correspondence between components of  $\pi_0$  and irreducible components of  $\pi$ .

Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$ ,  $m$  odd. Suppose  $\rho \cong \tilde{\rho}$ . Let  $\sigma_0$  be an irreducible supercuspidal representation of  $SO(2n, F)$ ,  $n \geq 0$ . If  $n > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ .

Then,  $\rho \rtimes \sigma_0$  is irreducible, but  $\rho \rtimes \sigma$  is reducible (cf. Proposition 4.3).

**Proposition 5.1.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$ ,  $m$  odd. Suppose  $\rho \cong \tilde{\rho}$ . Let  $\sigma_0$  be an irreducible supercuspidal representation of  $SO(2n, F)$ ,  $n \geq 0$ . If  $n > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ . Let  $\pi_0 = \nu^\alpha \rho \rtimes \zeta(\rho, \ell; \sigma_0)$  with  $\alpha \in \mathbb{R}$ ,  $\ell \geq 1$ . Then,  $\pi_0$  is reducible if and only if  $\alpha \in \{\pm 1, \pm \ell\}$ . Suppose  $\pi_0$  is reducible. By contragredience, we may assume  $\alpha \leq 0$ .*

(1)  $\alpha = -1, \ell = 1$   
 $\pi_0 = \pi_1^0 + \pi_2^0 + \pi_3^0 + \pi_4^0$  with

$$\pi_1^0 = L(\nu^{-1}\rho; \rho \rtimes \sigma_0), \quad \pi_2^0 = \delta(\nu\rho; \rho \rtimes \sigma_0), \quad \pi_3^0 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0), \quad \pi_4^0 = s\pi_3^0.$$

*In this case,  $\pi_1^0$  is the unique irreducible subrepresentation,  $\pi_2^0$  is the unique irreducible quotient, and  $\pi_3^0 \oplus \pi_4^0$  is a subquotient. We have*

$$s_{(m)}\pi_1^0 = \nu^{-1}\rho \otimes \rho \rtimes \sigma_0,$$

$$s_{(m)}\pi_2^0 = \nu\rho \otimes \rho \rtimes \sigma_0,$$

$$s_{(m)}\pi_3^0 = \rho \otimes L(\nu^{-1}\rho; \sigma_0).$$

(2)  $\alpha = -1, \ell > 1$   
 $\pi_0 = \pi_1^0 + \pi_2^0$  with

$$\pi_1^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-1}\rho; \rho \rtimes \sigma_0), \quad \pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

*In this case,  $\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient. We have*

(a)  $\ell = 2$ ,

$$s_{(m)}\pi_1^0 = 2\nu^{-1}\rho \otimes L(\nu^{-1}\rho; \rho \rtimes \sigma_0) + \nu^{-1}\rho \otimes L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0) \\ + \nu^{-1}\rho \otimes s \cdot L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0)$$

$$s_{(m)}\pi_2^0 = \nu^{-1}\rho \otimes \delta(\nu\rho; \rho \rtimes \sigma_0) + \nu\rho \otimes L(\nu^{-1}\rho; \rho \rtimes \sigma_0).$$

(b)  $\ell > 2$ ,

$$s_{(m)}\pi_1^0 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho, \nu^{-1}\rho], \nu^{-1}\rho; \rho \rtimes \sigma_0) + \nu^{-1}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

$$s_{(m)}\pi_2^0 = \nu^{-\ell+1}\rho \otimes L([\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)) + \nu\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

$$(3) \quad \alpha = -\ell, \ell > 1$$

$$\pi_0 = \pi_1^0 + \pi_2^0 \text{ with}$$

$$\pi_1^0 = L([\nu^{-\ell}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \quad \pi_2^0 = L(\nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

In this case,  $\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient. We have

$$s_{(m)}\pi_1^0 = \nu^{-\ell}\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0),$$

$$s_{(m)}\pi_2^0 = \nu^{-\ell+1}\rho \otimes L(\nu^{-\ell}\rho, [\nu^{-\ell+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

$$+ \nu^\ell\rho \otimes L([\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0).$$

*Proof.* First, we verify that  $\pi_0$  is reducible if and only if  $\alpha \in \{\pm 1, \pm \ell\}$ . Let

$$\pi = \nu^\alpha \rho \rtimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau = \nu^\alpha \rho \otimes \zeta_1(\rho, \ell; \sigma),$$

$$\tau_0 = \nu^\alpha \rho \otimes \zeta(\rho, \ell; \sigma_0).$$

We show that  $\pi$  is irreducible if and only if  $\pi_0$  is irreducible. The result then follows from Proposition 3.2. Notice that  $\tau_0 \cong s\tau_0$  and therefore  $\pi_0 \cong s\pi_0$ . By Lemmas 4.1 and 4.2,

$$i_{M, M^0}(\tau_0) = \tau + \hat{s}\tau,$$

$$i_{G, G^0}(\pi_0) = i_{G, M}(\tau + \hat{s}\tau) = \pi + \hat{s}\pi,$$

$$r_{G^0, G}(\pi) = \pi_0.$$

Therefore, if  $\pi_0$  is irreducible, then  $\pi$  is irreducible. Conversely, suppose that  $\pi$  is irreducible. We can prove by Jacquet module considerations that  $\hat{s}\pi \not\cong \pi$ . According to Lemma 4.1,  $\pi_0 = r_{G^0, G}(\pi)$  is irreducible. Thus, we have the reducibility points claimed.

Now, suppose that  $\pi$  is reducible. We verify (1).

First, Proposition 3.2 tells us that  $\pi$  decomposes as  $\pi = \pi_1 + \pi_2 + \pi_3$  where

$$\pi_1 = L(\nu^{-1}\rho; T_1(\rho; \sigma)), \quad \pi_2 = \delta(\nu\rho; T_1(\rho; \sigma)), \quad \pi_3 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

Then

$$\pi_0 = r_{G^0, G}(\pi) = r_{G^0, G}(\pi_1 + \pi_2 + \pi_3) = \pi_1^0 + \pi_2^0 + \pi_3^0 + s\pi_3^0,$$

where

$$\pi_1^0 = L(\nu^{-1}\rho; \rho \rtimes \sigma_0), \quad \pi_2^0 = \delta(\nu\rho; \rho \rtimes \sigma_0), \quad \pi_3^0 = L(\nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0).$$

Note that  $\pi_1^0 \cong s\pi_1^0$ ,  $\pi_2^0 \cong s\pi_2^0$ ,  $\pi_3^0 \not\cong s\pi_3^0$ .

Let  $N^0$  be the standard Levi subgroup of  $G^0 = SO(4m + 2n, F)$  isomorphic to  $GL(m, F) \times SO(2m + 2n, F)$ . Then

$$s_{(m)}\pi_1^0 = r_{N^0, G^0}(\pi_1^0) = r_{N^0, G^0} \circ r_{G^0, G}(\pi_1) = r_{N^0, N} \circ r_{N, G}(\pi_1) = r_{N^0, N}(s_{(m)}\pi_1).$$

Similarly,  $s_{(m)}\pi_2^0 = r_{N^0, N}(s_{(m)}\pi_2)$ . For  $\pi_3^0$ , we have

$$s_{(m)}\pi_3^0 + s \cdot s_{(m)}\pi_3^0 = r_{N^0, G^0}(\pi_3^0 + s\pi_3^0)$$

$$= r_{N^0, N} \circ r_{N, G}(\pi_3)$$

$$= r_{N^0, N}(s_{(m)}\pi_3)$$

$$= \rho \otimes L(\nu^{-1}\rho; \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0).$$

Let  $M^0$  be the standard Levi subgroup of  $G^0 = SO(4m + 2n, F)$  isomorphic to  $GL(m, F) \times GL(m, F) \times SO(2m + 2n, F)$ . From the Langlands data, we see that

$$\begin{aligned} r_{N^0, G^0}(\pi_3) &\geq \nu^{-\frac{1}{2}}\delta(\rho, 2) \otimes \sigma_0 \\ &\quad \downarrow \\ r_{M^0, G^0}(\pi_3) &\geq \rho \otimes \nu^{-1}\rho \otimes \sigma_0. \end{aligned}$$

It follows that

$$s_{(m)}\pi_3^0 = \rho \otimes L(\nu^{-1}\rho; \sigma_0).$$

Proposition 3.2 and the exactness of the functor  $r_{G^0, G}$  tell us that  $\pi_1^0$  is a subrepresentation,  $\pi_2^0$  a quotient, and  $\pi_3^0 \oplus \pi_4^0$  a subquotient of  $\pi_0$ . To see that  $\pi_1^0$  is the unique irreducible subrepresentation, suppose that  $\pi_i^0$  is a subrepresentation of  $\pi_0$ . Applying  $i_{G, G^0}$  to the exact sequence

$$0 \longrightarrow \pi_i^0 \longrightarrow \pi_0 \longrightarrow \pi_0/\pi_i^0 \longrightarrow 0,$$

we obtain

$$0 \longrightarrow i_{G, G^0}(\pi_i^0) \longrightarrow \pi \oplus \hat{s}\pi \longrightarrow i_{G, G^0}(\pi_0/\pi_i^0) \longrightarrow 0.$$

It follows that  $i_{G, G^0}(\pi_i^0)$  is a subrepresentation of  $\pi \oplus \hat{s}\pi$ . Since  $\pi_i \hookrightarrow i_{G, G^0}(\pi_i^0)$ , we have  $\pi_i \hookrightarrow \pi \oplus \hat{s}\pi$ . Therefore,  $\pi_i \cong \pi_1$  or  $\pi_i \cong \hat{s}\pi_1$ . By checking all irreducible subquotients of  $\pi$ , we obtain  $\pi_i \cong \pi_1$ . It follows that  $\pi_1^0$  is the unique subrepresentation of  $\pi_0$ . The proof that  $\pi_2^0$  is the unique quotient of  $\pi_0$  is similar.

(2) and (3) are done analogously. In these cases, the composition series for  $\pi_0$  follow from Lemma 4.5. □

**Proposition 5.2.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$ ,  $m$  odd. Suppose  $\rho \cong \tilde{\rho}$ . Let  $\sigma_0$  be an irreducible supercuspidal representation of  $SO(2n, F)$ ,  $n \geq 0$ . If  $n > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ . Let  $\pi_0 = \nu^\alpha\zeta(\rho, k) \rtimes \sigma_0$  with  $\alpha \in \mathbb{R}$ ,  $k \geq 2$ . Then  $\pi_0$  is reducible if and only if  $\alpha \in \{-\frac{k+1}{2}, -\frac{k+3}{2}, \dots, \frac{k-1}{2}\} \setminus \{0\}$ . Suppose  $\pi_0$  is reducible. By contragredience, we may assume that  $\alpha \leq 0$ . Write  $\alpha = -\frac{k+1}{2} + j$  with  $0 \leq j \leq \frac{k-3}{2}$ . Then  $\pi_0 = \pi_1^0 + \pi_2^0$  with*

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

and

$$\pi_2^0 = s^{j+1}L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0).$$

In this case,  $\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  the unique irreducible quotient.

(1)  $j = 0 = \frac{k-2}{2}$  ( $k = 2$ ),

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \rho \otimes sL(\nu^{-1}\rho; \sigma_0). \end{aligned}$$

(2)  $j = 0, k > 2$ ,

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-k+1}\rho \otimes L([\nu^{-k+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \nu^{-k+1}\rho \otimes sL([\nu^{-k+2}\rho, \nu^{-2}\rho], \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0) \\ &\quad + \rho \otimes sL([\nu^{-k+1}\rho, \nu^{-1}\rho]; \sigma_0). \end{aligned}$$

(3)  $j = \frac{k-2}{2}$ ,  $k \geq 4$  ( $k$  even),

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-\frac{k}{2}}\rho \otimes L([\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &\quad + \nu^{-\frac{k}{2}+1}\rho \otimes sL([\nu^{-\frac{k}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k}{2}+2}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \nu^{-\frac{k}{2}+1}\rho \otimes s^{j+1}L(\nu^{-\frac{k}{2}}\rho, \nu^{-\frac{k+3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0). \end{aligned}$$

(4)  $0 < j < \frac{k-2}{2}$ ,

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-k+j+1}\rho \otimes L([\nu^{-k+j+2}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &\quad + \nu^{-j}\rho \otimes sL([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j+1}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \nu^{-k+j+1}\rho \otimes s^{j+1}L([\nu^{-k+j+2}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0) \\ &\quad + \nu^{-j}\rho \otimes s^{j+1}L([\nu^{-k+j+1}\rho, \nu^{-j-1}\rho], \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0). \end{aligned}$$

*Proof.* Let  $\pi = \nu^\alpha \zeta(\rho, k) \rtimes \sigma$ . Then  $\pi = i_{G, G^0}(\pi_0)$ . We consider the cases from Proposition 3.3.

(i) If  $j = \frac{k-1}{2}$  (i.e.,  $\alpha = 0$ ), then  $\pi = \pi_1 + \pi_2$  with

$$\pi_i = L([\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho]; T_i(\rho; \tau))$$

for  $i = 1, 2$ . We have

$$\begin{aligned} \pi_0 + s\pi_0 &= r_{G^0, G} \circ i_{G, G^0}(\pi_0) = r_{G^0, G}(\pi) \\ &= r_{G^0, G}(\pi_1 + \pi_2) = 2L([\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0). \end{aligned}$$

It follows that

$$\pi_0 = L([\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

is irreducible.

(ii) Let  $0 \leq j < \frac{k-1}{2}$ . By Proposition 3.3,  $\pi = \pi_1 + \pi_2 + \pi_3$  with

$$\pi_i = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; T_i(\rho; \sigma))$$

for  $i = 1, 2$  and

$$\pi_3 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma).$$

We have

$$\begin{aligned} \pi_0 + s\pi_0 &= r_{G^0, G} \circ i_{G, G^0}(\pi_0) = r_{G^0, G}(\pi) = r_{G^0, G}(\pi_1 + \pi_2 + \pi_3) \\ &= 2L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0) \\ &\quad + L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0) \\ &\quad + sL([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0). \end{aligned}$$

It follows that  $\pi_0 = \pi_1^0 + \pi_2^0$  with

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho]; \rho \rtimes \sigma_0)$$

and

$$\pi_2^0 = s^\epsilon L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0),$$

where  $\epsilon = 0$  or  $1$ . Consider the standard Levi subgroup of  $G^0$ ,

$$Q^0 \cong \underbrace{GL(m, F) \times \cdots \times GL(m, F)}_k \times SO(2n, F).$$

Let

$$\begin{aligned} \psi &= \nu^{-k+j+1}\rho \otimes \nu^{-j-2}\rho \otimes (\nu^{-j}\rho \otimes \nu^{-j-1}\rho) \\ &\quad \otimes (\nu^{-j+1}\rho \otimes \nu^{-j}\rho) \otimes \cdots \otimes (\nu^{-1}\rho \otimes \nu^{-2}\rho) \otimes (\rho \otimes \nu^{-1}\rho) \otimes \sigma_0. \end{aligned}$$

By Frobenius reciprocity,

$$r_{Q^0, G^0}(L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0)) \geq \psi$$

and

$$\begin{aligned} r_{s(Q^0), G^0}(sL([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \\ \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \sigma_0)) \geq s\psi. \end{aligned}$$

We apply Corollary 5.3 and Lemma 6.2 of [Ban3] to compute  $r_{Q^0, G^0}(\pi_0)$ . We observe that, for  $j$  odd, the multiplicity of  $\psi$  in  $r_{Q^0, G^0}(\pi_0)$  is 1 and the multiplicity of  $s\psi$  in  $r_{s(Q^0), G^0}(\pi_0)$  is 0. It follows that  $\epsilon = 0$ . Similarly, for  $j$  even, the multiplicity of  $\psi$  in  $r_{Q^0, G^0}(\pi_0)$  is 0 and the multiplicity of  $s\psi$  in  $r_{s(Q^0), G^0}(\pi_0)$  is 1. It follows that  $\epsilon = 1$ .

Now,

$$s_{(m)}\pi_0 + s \cdot s_{(m)}\pi_0 = r_{N^0, G^0}(\pi_0 + s\pi_0) = r_{N^0, G}(\pi) = r_{N^0, N}(s_{(m)}\pi_1 + s_{(m)}\pi_2 + s_{(p)}\pi_3).$$

Consider the case (a) from the Proposition 3.3. Then

$$s_{(m)}\pi_0 + s \cdot s_{(m)}\pi_0 = 2\nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes L(\nu^{-1}\rho; \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0).$$

It follows that

$$s_{(m)}\pi_0 = \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes L(\nu^{-1}\rho; \sigma_0)$$

and

$$\begin{aligned} s_{(m)}\pi_1^0 &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0), \\ s_{(m)}\pi_2^0 &= \rho \otimes s^\epsilon L(\nu^{-1}\rho; \sigma_0), \end{aligned}$$

where  $\epsilon = 0$  or  $1$ .

To determine  $\epsilon$ , observe that in general,

$$s_{(m)}(\pi) = \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma + \nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho} \otimes \nu^{\alpha-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma.$$

Since  $m$  is odd, an odd number of sign changes are required to produce  $\nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho}$ . Therefore,

$$s_{(m)}(\pi_0) = \nu^{\alpha+\frac{-k+1}{2}}\rho \otimes \nu^{\alpha+\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma_0 + \nu^{-\alpha+\frac{-k+1}{2}}\tilde{\rho} \otimes s(\nu^{\alpha-\frac{1}{2}}\zeta(\rho, k-1) \rtimes \sigma_0).$$

When  $j = 0 = \frac{k-2}{2}$  ( $k = 2$ ), we get  $\alpha = -\frac{1}{2}$  and

$$\begin{aligned} s_{(m)}(\pi_0) &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes s(\nu^{-1}\rho \rtimes \sigma_0) \\ &= \nu^{-1}\rho \otimes (\rho \rtimes \sigma_0) + \rho \otimes sL(\nu^{-1}\rho; \sigma_0). \end{aligned}$$

Thus, we see that  $\epsilon = 1$ .

For the remaining cases, the proofs are similar.  $\square$

**Theorem 5.3.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(m, F)$ ,  $m$  odd. Suppose  $\rho \cong \tilde{\rho}$ . Let  $\sigma_0$  be an irreducible supercuspidal representation of  $SO(2n, F)$ ,  $n \geq 0$ . If  $n > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ . Let  $\pi_0 =$*



$\nu^\alpha \zeta(\rho, k) \rtimes \zeta(\rho, \ell; \sigma_0)$ . Suppose  $k \geq 2$  and  $\ell \geq 1$  (the cases  $k = 1$  and  $\ell = 0$  are covered by Propositions 5.1 and 5.2 above). Then,  $\pi_0$  is reducible if and only if

$$\alpha \in \left\{ \pm(\ell + \frac{k-1}{2}), \pm(\ell + \frac{k-1}{2} - 1), \dots, \pm(\ell + \frac{-k+1}{2}) \right\}$$

$$\cup \left\{ \frac{-k-1}{2}, \frac{-k-1}{2} + 1, \dots, \frac{k+1}{2} \right\} \setminus \{0 \text{ if } k = 2\ell - 1\}.$$

(We note that these sets need not be disjoint.) Let  $S_1$  denote the first set, and  $S_2$  the second. Suppose  $\pi_0$  is reducible. By contragredience, we may restrict our attention to the case  $\alpha \leq 0$ .

(1)  $\alpha \notin S_2$ .

In this case, we have  $\pi_0 = \pi_1^0 + \pi_2^0$ , where

$$\pi_1^0 = L([\nu^{\alpha - \frac{k-1}{2}} \rho, \nu^{\alpha + \frac{k-1}{2}} \rho], [\nu^{-\ell+1} \rho, \nu^{-1} \rho]; \rho \rtimes \sigma_0),$$

$$\pi_2^0 = L([\nu^{\alpha - \frac{k-1}{2}} \rho, \nu^{-\ell-1} \rho], \nu^{-\ell + \frac{1}{2}} \delta(\rho, 2), \nu^{-\ell + \frac{3}{2}} \delta(\rho, 2), \dots, \nu^{\alpha + \frac{k}{2}} \delta(\rho, 2), [\nu^{\alpha + \frac{k-1}{2} + 2} \rho, \nu^{-1} \rho]; \rho \rtimes \sigma_0).$$

$\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(2)  $\alpha = \frac{-k-1}{2}$ .

One component of  $\pi_0$  is the following:

$$\pi_1^0 = L([\nu^{-k} \rho, \nu^{-1} \rho], [\nu^{-\ell+1} \rho, \nu^{-1} \rho]; \rho \rtimes \sigma_0).$$

The other components are described below.

(a)  $\ell = 1$  (so  $k > \ell - 1$ ).

In this case, there are three additional components:

$$\pi_2^0 = L([\nu^{-k} \rho, \nu^{-2} \rho]; \delta(\nu \rho; \rho \rtimes \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-k} \rho, \nu^{-2} \rho], \nu^{-\frac{1}{2}} \delta(\rho, 2); \sigma_0)$$

and

$$\pi_4^0 = s\pi_3^0.$$

$\pi_1^0$  is the unique irreducible subrepresentation,  $\pi_2^0$  is the unique irreducible quotient, and  $\pi_3^0 \oplus \pi_4^0$  is a subquotient.

(b)  $k > \ell - 1 > 0$ .

In this case, there are four additional components:

$$\pi_2^0 = L([\nu^{-k} \rho, \nu^{-2} \rho], [\nu^{-\ell+1} \rho, \nu^{-1} \rho]; \delta(\nu \rho; \rho \rtimes \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-k} \rho, \nu^{-\ell-1} \rho], \nu^{-\ell + \frac{1}{2}} \delta(\rho, 2), \nu^{-\ell + \frac{3}{2}} \delta(\rho, 2), \dots, \nu^{-\frac{3}{2}} \delta(\rho, 2); \delta(\nu \rho; \rho \rtimes \sigma_0))$$

$$\pi_4^0 = L([\nu^{-k} \rho, \nu^{-\ell-1} \rho], \nu^{-\ell + \frac{1}{2}} \delta(\rho, 2), \nu^{-\ell + \frac{3}{2}} \delta(\rho, 2), \dots, \nu^{-\frac{1}{2}} \delta(\rho, 2); \sigma_0)$$

$$\pi_5^0 = s\pi_4^0.$$

$\pi_1^0$  is the unique irreducible subrepresentation,  $\pi_3^0$  is the unique irreducible quotient, and  $\pi_2^0 \oplus \pi_4^0 \oplus \pi_5^0$  is a subquotient.

(c)  $\ell - 1 = k$ .

In this case, there is one additional component:

$$\pi_2^0 = L([\nu^{-k} \rho, \nu^{-2} \rho], [\nu^{-k} \rho, \nu^{-1} \rho]; \delta(\nu \rho; \rho \rtimes \sigma_0)).$$

$\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(d)  $\ell - 1 > k$ .

In this case, there is one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-2}\rho], [\nu^{-k}\rho, \nu^{-1}\rho]; \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

$\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(3)  $\alpha \in S_2$ .

Write  $\alpha = \frac{-k+1}{2} + j$ , with  $0 \leq j \leq \frac{k-1}{2}$ . One component of  $\pi_0$  is  $\pi_1^0$ , where  $\pi_1^0$  is defined as follows:

$$\pi_1^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho]; \rho \times \rho \rtimes \sigma_0).$$

The remaining components are described below, on a case by case basis.

(a)  $k - j - 1 > j > \ell - 1$ .

We have two additional components:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-1}\rho], [\nu^{-j}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0),$$

$$\pi_3^0 = L([\nu^{-k+j+1}\rho, \nu^{-j-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

$\pi_3^0$  is the unique irreducible quotient and  $\pi_1^0 \oplus \pi_2^0$  is a subrepresentation.

(b)  $k - j - 1 = j > \ell - 1$ .

We have one additional component:

$$\pi_2^0 = L([\nu^{-\frac{k+1}{2}}\rho, \nu^{-1}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

In this case,  $\pi_0 = \pi_1^0 \oplus \pi_2^0$ .

(c)  $k - j - 1 > j = \ell - 1$ .

We have one additional component:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-\ell+1}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

$\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(d)  $k - j - 1 > \ell - 1 > j$ .

We have three additional components:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \rtimes \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-j-\frac{5}{2}}\delta(\rho, 2), \nu^{-j-1}\delta(\rho, 3), \nu^{-j}\delta(\rho, 3), \dots, \nu^{-1}\delta(\rho, 3); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

$$\pi_4^0 = L([\nu^{-k+j+1}\rho, \nu^{-\ell-1}\rho], [\nu^{-j}\rho, \nu^{-1}\rho], \nu^{-\ell+\frac{1}{2}}\delta(\rho, 2), \nu^{-\ell+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \rho \rtimes \sigma_0).$$

$\pi_1^0$  is the unique irreducible subrepresentation,  $\pi_3^0$  is the unique irreducible quotient, and  $\pi_2^0 \oplus \pi_4^0$  is a subquotient.

(e)  $k - j - 1 = \ell - 1 > j$ .

We have one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-k+\ell-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-2}\rho], \nu^{-k+\ell-\frac{1}{2}}\delta(\rho, 2), \nu^{-k+\ell+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \rtimes \sigma_0)).$$

$\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(f)  $\ell - 1 > k - j - 1 > j$ .

(i) If  $j = 0$ , the representation  $\pi_2^0$  below is the only other component. In this case,  $\pi_1^0$  is the unique irreducible subrepresentation and  $\pi_2^0$  is the unique irreducible quotient.

(ii) If  $j > 0$ , there are two additional components:

$$\pi_2^0 = L([\nu^{-k+j+1}\rho, \nu^{-2}\rho], [\nu^{-\ell+1}\rho, \nu^{-j-2}\rho], \nu^{-j-\frac{1}{2}}\delta(\rho, 2), \nu^{-j+\frac{1}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \times \sigma_0)),$$

$$\pi_3^0 = L([\nu^{-\ell+1}\rho, \nu^{-k+j-1}\rho], [\nu^{-j}\rho, \nu^{-2}\rho], \nu^{-k+j+\frac{1}{2}}\delta(\rho, 2), \nu^{-k+j+\frac{3}{2}}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \times \sigma_0)).$$

In this case,  $\pi_2^0$  is the unique irreducible quotient and  $\pi_1^0 \oplus \pi_3^0$  is a subrepresentation.

(g)  $\ell - 1 > k - j - 1 = j$ .

We have one additional component:

$$\pi_2^0 = L([\nu^{-\ell+1}\rho, \nu^{-\frac{k-3}{2}}\rho], [\nu^{-\frac{k+1}{2}}\rho, \nu^{-2}\rho], \nu^{-\frac{k}{2}}\delta(\rho, 2), \nu^{-\frac{k}{2}+1}\delta(\rho, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho, 2); \delta(\nu\rho; \rho \times \sigma_0)).$$

In this case,  $\pi_0 = \pi_1^0 \oplus \pi_2^0$ .

We note that the case  $k - j - 1 = j = \ell - 1$  is a point of irreducibility.

*Proof.* Let

$$\pi = \nu^\alpha \zeta(\rho, k) \times \zeta_1(\rho, \ell; \sigma).$$

As in the proof of Proposition 5.1, we obtain

$$\pi_0 = r_{G^0, G}(\pi).$$

The irreducible subquotients and composition series structure of  $\pi$  are described in Theorem 3.4. We argue as above to obtain the corresponding results for  $\pi_0$ .  $\square$

*Remark 5.4.* Let  $\pi_0, \pi$  be as in the proof of Theorem 5.3. The proof of Theorem 3.4 describes Jacquet modules of components of  $\pi$ . We may apply restriction to obtain Jacquet modules of irreducible components of  $\pi_0$ , as follows: Let  $\pi_i$  be a component of  $\pi$ . If  $r_{G^0, G}(\pi_i) = \pi_i^0$  is irreducible, then

$$s_{(m)}(\pi_i^0) = r_{G^0, G}(s_{(m)}(\pi_i)).$$

If  $r_{G^0, G}(\pi_i)$  is reducible, then  $r_{G^0, G}(\pi_i) = \pi_i^0 + s \cdot \pi_i^0$  and

$$(1) \quad r_{G^0, G}(s_{(m)}(\pi_i)) = s_{(m)}(\pi_i^0) + s \cdot s_{(m)}(\pi_i^0).$$

The components of  $s_{(m)}(\pi_i^0)$  can be computed from (1) up to  $s$ . We can determine whether  $s$  appears in a component of  $s_{(m)}(\pi_i^0)$  by observing that if  $r_{G^0, G}(\pi_i)$  is reducible, then  $\pi_i^0 \not\cong s\pi_i^0$ . Such components appear only in cases 2(a) and (b) (in particular,  $\pi_3^0, \pi_4^0$  for 2(a) and  $\pi_4^0, \pi_5^0$  for 2(b)). In either case, exactly one of  $\pi_i^0, s\pi_i^0$  appears as a component of  $\nu^{-\frac{k+\ell-1}{2}}\zeta(\rho, k + \ell) \times \sigma_0$ , hence has Jacquet module described in Proposition 5.2.

Suppose  $\rho \not\cong \rho_0$  are representations of  $GL(m, F)$  and  $GL(m_0, F)$ . We continue to assume  $\rho \cong \tilde{\rho}$  with  $m$  odd and either  $\sigma_0 \not\cong s\sigma_0$  or  $\sigma_0 = 1_0$ . In order to build on Theorem 3.5, we want  $(\rho_0, \sigma)$  to satisfy (C0). We can characterize this in terms of

$\rho_0$  and  $\sigma_0$  by requiring that either  $(\rho_0, \sigma_0)$  satisfies (C0) or  $\rho_0 \cong \tilde{\rho}_0$  with  $m_0$  odd (by Proposition 4.3). Then, by [Gol1],  $\rho_0 \times \rho \rtimes \sigma_0$  is the direct sum of two irreducible subrepresentations. Write

$$\rho_0 \times \rho \rtimes \sigma_0 = T_1(\rho_0, \rho; \sigma_0) + T_2(\rho_0, \rho; \sigma_0).$$

**Theorem 5.5.** *Suppose  $\rho \not\cong \rho_0$  are irreducible unitary supercuspidal representations of  $GL(m, F)$  and  $GL(m_0, F)$ . Assume that  $\rho \cong \tilde{\rho}$  and  $m$  odd. Let  $\sigma_0$  be an irreducible supercuspidal representation of  $SO(2n, F)$ ,  $n \geq 0$ . If  $n > 0$ , suppose that  $\sigma_0 \not\cong s\sigma_0$ . We also assume that either  $(\rho_0, \sigma_0)$  satisfies (C0) or  $\rho_0 \cong \tilde{\rho}_0$  with  $m_0$  odd. Let  $\pi_0 = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta(\rho, \ell; \sigma_0)$  with  $\alpha \in \mathbb{R}$ ,  $k \geq 1$ . Then  $\pi_0$  is reducible if and only if  $\alpha \in \{\frac{-k+1}{2}, \frac{-k+3}{2}, \dots, \frac{k-1}{2}\}$ . Suppose  $\pi_0$  is reducible. By contragredience, we may assume that  $\alpha \leq 0$ . Write  $\alpha = \frac{-k+1}{2} + j$  with  $0 \leq j \leq \frac{k-1}{2}$ .*

- (1)  $j = \frac{k-1}{2}$   
 $\pi_0 = \pi_1^0 + \pi_2^0$  with  
 $\pi_i^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0], [\nu^{\frac{-k+1}{2}}\rho_0, \nu^{-1}\rho_0]; T_i(\rho_0, \rho; \sigma_0))$   
for  $i = 1, 2$ . In this case,  $\pi_0 = \pi_1^0 \oplus \pi_2^0$ .
- (2)  $0 \leq j < \frac{k-1}{2}$   
 $\pi_0 = \pi_1^0 + \pi_2^0 + \pi_3^0$  with  
 $\pi_i^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-1}\rho_0], [\nu^{-j}\rho_0, \nu^{-1}\rho_0]; T_i(\rho_0, \rho; \sigma_0))$   
for  $i = 1, 2$  and  
 $\pi_3^0 = L([\nu^{-\ell+1}\rho, \nu^{-1}\rho], [\nu^{-k+j+1}\rho_0, \nu^{-j-2}\rho_0], \nu^{-j-\frac{1}{2}}\delta(\rho_0, 2),$   
 $\nu^{-j+\frac{1}{2}}\delta(\rho_0, 2), \dots, \nu^{-\frac{1}{2}}\delta(\rho_0, 2); \rho \rtimes \sigma_0).$

In this case,  $\pi_3^0$  is the unique irreducible quotient and  $\pi_1^0 \oplus \pi_2^0$  is a subrepresentation.

*Proof.* Let  $\pi = \nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$ . The irreducible subquotients and composition series structure of  $\pi$  are described in Theorem 3.4. We argue as above to obtain the corresponding results for  $\pi_0$ . The Jacquet modules may be determined as in (the first part of) Remark 5.4. □

*Remark 5.6.* Let  $\rho$  and  $\sigma$  be as in the preceding corollary. Suppose  $\rho_0$  is an irreducible unitary supercuspidal representation of  $GL(m_0, F)$  with  $\rho_0 \not\cong \tilde{\rho}_0$ . Then,  $\nu^\alpha \zeta(\rho_0, k) \rtimes \zeta_1(\rho, \ell; \sigma)$  is irreducible for all  $\alpha \in \mathbb{R}$ .

## 6. APPENDIX

Let  $G$  be the group of  $F$ -points of a quasi-split reductive algebraic group defined over  $F$ . Let  $G^0$  denote the connected component of the identity in  $G$ . For convenience, suppose that

$$G = G^0 \rtimes C,$$

with  $C$  a finite abelian group (that  $G$  is the semidirect product of  $G^0$  and  $C$  is not required, but it is easier to formulate in that case). We now describe how to get from the general formulation of the Langlands classification ([B-W], [Sil1], [B-J1]) to the explicit description for even-orthogonal groups given in section 2 (patterned after [Tad2], [Jan2]).

We recall that we call an irreducible representation of  $G$  tempered if its restriction to  $G^0$  is tempered (cf. Definition 2.5, [B-J1]).

Let  $\Pi$  be the set of simple roots for  $G^0$ . For  $\Phi \subset \Pi$ , we let  $P_\Phi = M_\Phi U_\Phi$  be the standard parabolic subgroup of  $G^0$  determined by  $\Phi$ . Fix an order on  $\Pi$ . Then, there is a lexicographic order on subsets of  $\Pi$ . We define

$$X_C = \{\Phi \subset \Pi \mid \Phi \text{ is maximal among } \{c \cdot \Phi\}_{c \in C}\}.$$

Let  $C(\Phi) = \{c \in C \mid c \cdot \Phi = \Phi\}$ . We call  $P = MU_\Phi$ , where  $M_\Phi \leq M \leq M_\Phi \rtimes C(\Phi)$  and  $\Phi \in X_C$ , a standard parabolic subgroup of  $G$ .

Set  $P^0 = P_\Phi$ . Let  $A$  be the split component of  $M_\Phi$ ,  $\mathfrak{a}$  the real Lie algebra of  $A$ , and  $\mathfrak{a}^*$  its dual. Let  $\Pi(P^0, A) \subset \mathfrak{a}^*$  denote the set of simple roots corresponding to the pair  $(P^0, A)$ . We set

$$\begin{aligned} \mathfrak{a}_-^* &= \{x \in \mathfrak{a}^* \mid \langle x, \alpha \rangle < 0, \forall \alpha \in \Pi(P^0, A)\}, \\ \mathfrak{a}_-^*(C) &= \{x \in \mathfrak{a}_-^* \mid x \succeq c \cdot x, \forall c \in C(\Phi)\}, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is a  $C(\Phi)$ -invariant inner product on  $\mathfrak{a}^* \times \mathfrak{a}^*$  and  $\succeq$  is the lexicographic order inherited from the order on  $\Pi$  (cf. section 3, [B-J1] for details).

**Definition 6.1.** A set of Langlands data for  $G$  is a triple  $(P, x, \tau)$  with the following properties:

- (1)  $P = MU$  is a standard parabolic subgroup of  $G$ .
- (2)  $x \in \mathfrak{a}_-^*(C)$ .
- (3)  $M = M_\Phi \rtimes C(\Phi, x)$ , where  $C(\Phi, x) = \{c \in C(\Phi) \mid c \cdot x = x\}$ .
- (4)  $\tau \in Irr(M)$  is tempered.

**Theorem 6.2** (The Langlands classification). *There is a bijective correspondence*

$$Lang(G) \longleftrightarrow Irr(G),$$

where  $Lang(G)$  denotes the set of all triples of Langlands data. Furthermore, if  $(P, x, \tau) \leftrightarrow \pi$  under this correspondence, then  $\pi$  is the unique irreducible subrepresentation of  $i_{G,M}(exp\ x \otimes \tau)$ .

We now consider the group  $SO(2n, F)$ . The maximal split torus  $A_\emptyset$  of  $SO(2n, F)$  is

$$A_\emptyset = \{diag(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1}) \mid a_i \in F^\times\} \cong (F^\times)^n.$$

Let  $a$  denote the usual isomorphism of  $(F^\times)^n$  into  $A_\emptyset$ , defined by

$$a(a_1, \dots, a_n) = diag(a_1, \dots, a_n, a_n^{-1}, \dots, a_1^{-1}).$$

The group  $X(A_\emptyset)_F$  of  $F$ -rational characters of  $A_\emptyset$  has a basis  $\{e_1^0, \dots, e_n^0\}$ , where  $e_i^0$  is defined by

$$e_i^0(a(a_1, \dots, a_n)) = a_i.$$

Thus,  $\mathfrak{a}_0^* = \{x_1 e_1^0 + \dots + x_n e_n^0 \mid x_i \in \mathbb{R}\}$ . The roots of  $SO(2n, F)$  form a root system of type  $D_n$ . The set of simple roots is  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ , where  $\alpha_i = e_i^0 - e_{i-1}^0$ , for  $1 \leq i \leq n - 1$ , and  $\alpha_n = e_{n-1}^0 + e_n^0$ . For our order on  $\Pi$ , we take  $\alpha_i > \alpha_j$  if  $i < j$ .

Let  $\Phi \subset \Pi$ . We now describe the standard parabolic subgroup  $P_\Phi = M_\Phi U_\Phi$ . Write  $\Phi$  in the form  $\Phi = \Pi \setminus \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ , where  $i_1 < i_2 < \dots < i_k$ . We have two cases.

A) If  $\alpha_{n-1} \in \Phi$  or  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \notin \Phi$ , then

$$M_\Phi = \{ \text{diag}(g_1, \dots, g_k, h, {}^\tau g_k^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F), h \in SO(2(n-m), F) \},$$

where  $n_1 = i_1, n_1 + n_2 = i_2, \dots, n_1 + \dots + n_k = i_k = m$ . We have

$$M_\Phi \cong GL(n_1, F) \times GL(n_2, F) \times \dots \times GL(n_k, F) \times SO(2(n-m), F).$$

B) If  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \in \Phi$ , then

$$M_\Phi = sM_{\Phi'}s,$$

where  $\Phi' = s(\Phi)$ ,

$$M_{\Phi'} = \{ \text{diag}(g_1, \dots, g_k, {}^\tau g_k^{-1}, \dots, {}^\tau g_1^{-1}) \mid g_i \in GL(n_i, F) \}.$$

Now, we will describe the set  $\mathfrak{a}_-^*$  which appears in the definition of Langlands data. We have four cases.

A) Suppose that  $\alpha_{n-1} \in \Phi$  or  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \notin \Phi$ . Then

$$A = \{ a(\underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k}, \underbrace{1, \dots, 1}_{n-m}) \mid a_i \in F^\times \}.$$

The basis for  $X(A)_F$  is  $\{e_1, \dots, e_k\}$ , where

$$e_j : a(\underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k}, \underbrace{1, \dots, 1}_{n-m}) \mapsto a_j.$$

We have  $\mathfrak{a}^* = \{x_1 e_1 + \dots + x_k e_k \mid x_j \in \mathbb{R}\}$ . The set  $\Pi(P, A) = \{\beta_1, \dots, \beta_k\} \subset \mathfrak{a}^*$  corresponds to  $\Pi \setminus \Phi = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \mathfrak{a}_0^*$ . For  $1 \leq j \leq k-1$ ,  $\beta_j = e_{j-1} - e_j$ .

A.1) If  $\alpha_{n-1} \in \Phi$ ,  $\alpha_n \in \Phi$ , then  $\beta_k = e_k$  ( $\alpha_{i_k} = \alpha_n = e_m^0 - e_{m+1}^0$ ). Take  $x = x_1 e_1 + \dots + x_k e_k \in \mathfrak{a}^*$ . Then

$$x \in \mathfrak{a}_-^* \Leftrightarrow \begin{cases} x_1 - x_2 < 0, \\ \vdots \\ x_{k-1} - x_k < 0, \\ x_k < 0. \end{cases}$$

This implies  $x_1 < \dots < x_k < 0$ , so

$$\mathfrak{a}_-^* = \{x_1 e_1 + \dots + x_k e_k \mid x_1 < \dots < x_k < 0\}.$$

A.2) If  $\alpha_{n-1} \in \Phi$ ,  $\alpha_n \notin \Phi$ , then  $\beta_k = 2e_k$  ( $\alpha_{i_k} = \alpha_n = e_{n-1}^0 + e_n^0$ ), and

$$\mathfrak{a}_-^* = \{x_1 e_1 + \dots + x_k e_k \mid x_1 < \dots < x_k < 0\}.$$

A.3) If  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \notin \Phi$ , then  $n_k = 1$ ,  $\beta_k = e_{k-1} + e_k$  and

$$\mathfrak{a}_-^* = \{x_1 e_1 + \dots + x_k e_k \mid x_1 < \dots < x_{k-1} < -|x_k|\}.$$

B) Suppose that  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \in \Phi$ . Then

$$A = \{ a(\underbrace{a_1, \dots, a_1}_{n_1}, \dots, \underbrace{a_k, \dots, a_k}_{n_k-1}, a_k^{-1}) \mid a_i \in F^\times \},$$

and we have  $\beta_k = 2e_k$  ( $\alpha_{i_k} = \alpha_n = e_{n-1}^0 - e_n^0$ ),

$$\mathfrak{a}_-^* = \{x_1 e_1 + \dots + x_k e_k \mid x_1 < \dots < x_k < 0\}.$$

For  $x = x_1 e_1 + \cdots + x_k e_k \in \mathfrak{a}^*$ , we have

$$\exp x = \nu^{x_1} \otimes \cdots \otimes \nu^{x_k} \otimes 1.$$

This means that the value of  $\exp x$  on  $m = \text{diag}(g_1, \dots, g_k, h, {}^\tau g_k^{-1}, \dots, {}^\tau g_1^{-1})$  is

$$\exp x(m) = |\det g_1|^{x_1} \cdots |\det g_k|^{x_k}.$$

**Proposition 6.3** (The Langlands classification for  $SO(2n, F)$ ). (i) Let  $\rho_i$ ,  $i = 1, \dots, k$ , be an irreducible essentially tempered representation of  $GL(n_i, F)$  and  $\tau_0$  an irreducible tempered representation of  $SO(2m, F)$ .

- (1) Suppose that  $m \geq 1$  and  $e(\rho_1) < \cdots < e(\rho_k) < 0$ . Then the representation  $\rho_1 \times \cdots \times \rho_k \rtimes \tau_0$  has a unique irreducible subrepresentation  $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau_0)$ .
- (2) Suppose that  $m = 0$ ,  $n_k > 1$  and  $e(\rho_1) < \cdots < e(\rho_k) < 0$ . Then the representation  $\rho_1 \times \cdots \times \rho_k \rtimes 1_0$  (resp.  $\rho_1 \times \cdots \times \rho_{k-1} \times s(\rho_k \rtimes 1_0)$ ) has a unique irreducible subrepresentation  $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0)$  (resp.  $L(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0))$ ). Further,  $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0) \not\cong L(\rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes s(\rho_k \otimes 1_0))$ .
- (3) Suppose that  $m = 0$ ,  $n_k = 1$  and  $e(\rho_1) < \cdots < e(\rho_{k-1}) < -|e(\rho_k)| < 0$ . Then the representation  $\rho_1 \times \cdots \times \rho_k \rtimes 1_0$  has a unique irreducible subrepresentation  $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes 1_0)$ .

(ii) Let  $\sigma_0$  be an irreducible admissible representation of  $SO(2n, F)$ . Then there exists a unique datum as in (i) such that  $\sigma_0 \cong L(\cdot)$ .

One remark is in order. We note that  $SO(2, F) \cong F^\times$ . Thus, a unitary character of  $F^\times$  may be viewed as a tempered representation of  $SO(2, F)$ . This allows us to have the  $< 0$  in part 3; if  $e(\rho_k) = 0$ , it is covered by part 1.

Now, we follow Theorem 6.2 to obtain the Langlands classification for  $O(2n, F)$ . First, we have to determine the set  $X_C$ . Since  $\alpha_{n-1} < \alpha_n$  in the order on  $\Pi$ , we easily see that

$$\Phi \in X_C \Leftrightarrow (\alpha_{n-1} \in \Phi \text{ or } \alpha_{n-1} \notin \Phi, \alpha_n \notin \Phi).$$

- A.1) If  $\alpha_{n-1} \in \Phi$ ,  $\alpha_n \in \Phi$ , then  $C(\Phi) = \{1, s\}$ . For every  $x \in \mathfrak{a}_-^*$ , we have  $s \cdot x = x$ , so  $\mathfrak{a}_-^*(C) = \mathfrak{a}_-^*$  and  $C(\Phi, x) = \{1, s\}$ .
- A.2) If  $\alpha_{n-1} \in \Phi$ ,  $\alpha_n \notin \Phi$ , then  $C(\Phi) = \{1\}$ . It follows that  $\mathfrak{a}_-^*(C) = \mathfrak{a}_-^*$  and  $C(\Phi, x) = \{1\}$ , for every  $x \in \mathfrak{a}_-^*(C)$ .
- A.3) If  $\alpha_{n-1} \notin \Phi$ ,  $\alpha_n \notin \Phi$ , then  $C(\Phi) = \{1, s\}$ . Take  $x \in \mathfrak{a}_-^*$ . Then  $x = x_1 e_1 + \cdots + x_k e_k$ , where  $x_1 < \cdots < x_{k-1} < -|x_k|$ . The action of  $s$  on  $x$  is given by

$$s \cdot x = x_1 e_1 + \cdots + x_{k-1} e_{k-1} - x_k e_k.$$

The condition  $x \succeq s \cdot x$  implies  $\langle x, \alpha_{k-1} \rangle \geq \langle x, \alpha_k \rangle$ . This gives  $x_{k-1} - x_k \geq x_{k-1} + x_k$ , so  $x_k \leq 0$ . It follows that

$$\mathfrak{a}_-^*(C) = \{x_1 e_1 + \cdots + x_k e_k \mid x_1 < \cdots < x_k \leq 0\}.$$

If  $x_k < 0$ , then  $C(\Phi, x) = \{1\}$ . If  $x_k = 0$ , then  $C(\Phi, x) = \{1, s\}$ .

**Proposition 6.4** (The Langlands classification for  $O(2n, F)$ ). (i) Let  $\rho_i$ ,  $i = 1, \dots, k$ , be an irreducible essentially tempered representation of  $GL(n_i, F)$  and  $\tau$  an irreducible tempered representation of  $O(2m, F)$ . Suppose that  $e(\rho_1) < \cdots < e(\rho_k) < 0$ . Then the representation  $\rho_1 \times \cdots \times \rho_k \rtimes \tau$  has a unique irreducible subrepresentation  $L(\rho_1 \otimes \cdots \otimes \rho_k \otimes \tau)$ .

(ii) Let  $\sigma$  be an irreducible admissible representation of  $O(2n, F)$ . Then there exists a unique datum as in (i) such that  $\sigma \cong L(\cdot)$ .

*Remark 6.5.* (1) In section 2, a minor variation on Propositions 6.3 and 6.4 is used: Instead of having  $\rho_1, \dots, \rho_k$  essentially tempered with  $\epsilon(\rho_1) < \dots < \epsilon(\rho_k) < 0$  (resp.,  $\epsilon(\rho_1) < \dots < \epsilon(\rho_{k-1}) < -|\epsilon(\rho_k)| < 0$  for Proposition 6.3 (3), we take  $\rho_1, \dots, \rho_k$  essentially square-integrable with  $\epsilon(\rho_1) \leq \dots \leq \epsilon(\rho_k) < 0$  (resp.,  $\epsilon(\rho_1) \leq \dots \leq \epsilon(\rho_{k-1}) \leq -|\epsilon(\rho_k)| < 0$ ). This is justified by the following: If  $\delta_1, \dots, \delta_m$  are irreducible, square-integrable representations of  $GL(n_1, F), \dots, GL(n_m, F)$ , then  $\delta_1 \times \dots \times \delta_m$  is an irreducible, tempered representation. Further, any irreducible, tempered representation of  $GL(n, F)$  can be written this way.

(2) We also need multiplicity one in the Langlands classification, i.e., that the Langlands subrepresentation appears with multiplicity one in the corresponding induced representation. We refer the reader to [B-W] in the connected case and [B-J2] in the non-connected case. (We remark that for  $O(2n, F)$ , multiplicity one may also be shown directly using the arguments of Lemma 3.4, [Jan4].)

## REFERENCES

- [Aub] A.-M. Aubert, Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif  $p$ -adique, *Trans. Amer. Math. Soc.*, **347**(1995), 2179-2189, and Erratum à "Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif  $p$ -adique", *Trans. Amer. Math. Soc.*, **348**(1996), 4687-4690. MR **95i**:22025
- [Ban1] D. Ban, Parabolic induction and Jacquet modules of representations of  $O(2n, F)$ , *Glasnik. Mat.*, **34**(54)(1999), 147-185. MR **2001m**:22033
- [Ban2] D. Ban, Self-duality in the case of  $SO(2n, F)$ , *Glasnik. Mat.*, **34**(54)(1999), 187-196. MR **2002a**:22022
- [Ban3] D. Ban, Jacquet modules of parabolically induced representations and Weyl groups, *Can. J. Math.*, **53**(2001), 675-695. MR **2002j**:22020
- [B-J1] D. Ban and C. Jantzen, The Langlands classification for non-connected  $p$ -adic groups, *Isr. J. Math.*, **126**(2001), 239-261. MR **2002i**:22018
- [B-J2] D. Ban and C. Jantzen, The Langlands classification for non-connected  $p$ -adic groups II: Multiplicity one, *Proc. Amer. Math. Soc.*, **131**(2003), 3297-3304.
- [B-Z] I. Bernstein and A. Zelevinsky, Induced representations of reductive  $p$ -adic groups I, *Ann. Sci. Éc. Norm. Sup.*, **10** (1977), 441-472. MR 58:28310
- [B-W] A. Borel and N. Wallach, *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*, Princeton University Press, Princeton, 1980. MR **83c**:22018
- [Ch] S. Choi, Degenerate principal series for exceptional  $p$ -adic groups Ph.D. thesis, University of Michigan, 2002.
- [G-K] S. Gelbart and A. Knapp,  $L$ -indistinguishability and  $R$  groups for the special linear group, *Adv. in Math.*, **43**(1982), 101-121. MR **83j**:22009
- [Gol1] D. Goldberg, Reducibility of induced representations for  $Sp(2n)$  and  $SO(n)$ , *Amer. J. Math.*, **116**(1994), 1101-1151. MR **95g**:22016
- [Gol2] D. Goldberg, Reducibility for non-connected  $p$ -adic groups with  $G^\circ$  of prime index, *Can. J. Math.*, **47**(1995), 344-363. MR **96d**:22003
- [G-H] D. Goldberg and R. Herb, Some results on the admissible representations of non-connected reductive  $p$ -adic groups, *Ann. Sci. Éc. Norm. Sup.*, **30**(1997), 97-146. MR **98b**:22033
- [Gus] R. Gustafson, The degenerate principal series for  $Sp(2n)$ , *Mem. Amer. Math. Soc.*, **248**(1981), 1-81. MR **83e**:22021
- [H-L] R. Howe and S. Lee, Degenerate principal series representations of  $GL_n(\mathbb{C})$  and  $GL_n(\mathbb{R})$ , *J. Funct. Anal.*, **166**(1999), 244-309. MR **2000g**:22023



- [Jan1] C. Jantzen, Degenerate principal series for symplectic groups, *Mem. Amer. Math. Soc.*, **488**(1993), 1-111. MR **93g**:22018
- [Jan2] C. Jantzen, Degenerate principal series for orthogonal groups, *J. Reine Angew. Math.*, **441**(1993), 61-98. MR **94**:22022
- [Jan3] C. Jantzen, Degenerate principal series for symplectic and odd-orthogonal groups, *Mem. Amer. Math. Soc.*, **590**(1996), 1-100. MR **97d**:22020
- [Jan4] C. Jantzen, On supports of induced representations for symplectic and odd-orthogonal groups, *Amer. J. Math.*, **119**(1997), 1213-1262. MR **99b**:22028
- [Jan5] C. Jantzen, Some remarks on degenerate principal series, *Pac. J. Math.*, **186**(1998), 67-87. MR **99j**:22018
- [Jan6] C. Jantzen, On duality and supports of induced representations for even-orthogonal groups, preprint.
- [K-R] S. Kudla and S. Rallis, Ramified degenerate principal series representations for  $Sp(n)$ , *Isr. J. Math.*, **78**(1992), 209-256. MR **94a**:22035
- [K-S] S. Kudla and J. Sweet, Degenerate principal series representations for  $U(n, n)$ , *Isr. J. Math.*, **98**(1997), 253-306. MR **98h**:22021
- [Mœ] C. Mœglin, Normalisation des opérateurs d'entrelacement et réductibilité des induites des cuspidales; le cas des groupes classiques  $p$ -adiques, *Ann. of Math.*, **151**(2000), 817-847. MR **2002b**:22032
- [Mu] G. Muić, The unitary dual of  $p$ -adic  $G_2$ , *Duke Math. J.*, **90**(1997), 465-493. MR **98k**:22073
- [M-R] F. Murnaghan and J. Repka, Reducibility of some induced representations of split classical  $p$ -adic groups, *Comp. Math.*, **114**(1998), 263-313. MR **99m**:22021
- [Re] M. Reeder, Hecke algebras and harmonic analysis on  $p$ -adic groups, *Amer. J. Math.*, **119**(1997), 225-248. MR **99c**:22025
- [S-S] P. Schneider and U. Stuhler, Representation theory and sheaves on the Bruhat-Tits building, *Publ. Math. IHES*, **85**(1997), 97-191. MR **98m**:22023
- [Sh1] F. Shahidi, A proof of Langlands conjecture on Plancherel measures; complementary series for  $p$ -adic groups, *Ann. of Math.*, **132**(1990), 273-330. MR **91m**:11095
- [Sh2] F. Shahidi, Twisted endoscopy and reducibility of induced representations for  $p$ -adic groups, *Duke Math. J.*, **66**(1992), 1-41. MR **93d**:22034
- [Sil1] A. Silberger, The Langlands quotient theorem for  $p$ -adic groups, *Math. Ann.*, **236**(1978), 95-104. MR **58**:22413
- [Sil2] A. Silberger, *Introduction to Harmonic Analysis on Reductive  $p$ -adic Groups*, Princeton University Press, Princeton, 1979. MR **81m**:22025
- [Sil3] A. Silberger, Special representations of reductive  $p$ -adic groups are not integrable, *Ann. of Math.*, **111**(1980), 571-587. MR **82k**:22015
- [Tad1] M. Tadić, Notes on representations of non-archimedean  $SL(n)$ , *Pac. J. Math.*, **152**(1992), 375-396. MR **92k**:22029
- [Tad2] M. Tadić, Representations of  $p$ -adic symplectic groups, *Comp. Math.*, **90**(1994), 123-181. MR **95a**:22025
- [Tad3] M. Tadić, Structure arising from induction and Jacquet modules of representations of classical  $p$ -adic groups, *J. Algebra*, **177**(1995), 1-33. MR **97b**:22023
- [Tad4] M. Tadić, On reducibility of parabolic induction, *Isr. J. Math.*, **107**(1998), 29-91. MR **2001d**:22012
- [Zel] A. Zelevinsky, Induced representations of reductive  $p$ -adic groups II, On irreducible representations of  $GL(n)$ , *Ann. Sci. Éc. Norm. Sup.*, **13** (1980), 165-210. MR **83g**:22012
- [Zh] Y. Zhang, L-packets and reducibilities, *J. Reine Angew. Math.*, **510**(1999), 83-102. MR **2000e**:22009

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901  
*E-mail address:* `dban@math.siu.edu`

DEPARTMENT OF MATHEMATICS, EAST CAROLINA UNIVERSITY, GREENVILLE, NORTH CAROLINA 27858  
*E-mail address:* `jantzenc@mail.ecu.edu`