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RATIONAL SMOOTHNESS OF VARIETIES OF REPRESENTATIONS FOR QUIVERS OF TYPE A

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Abstract. In this paper, the authors study when the closure (in the Zariski topology) of orbits of representations of quivers of type A are rationally smooth. This is done by considering the corresponding quantized enveloping algebra $\mathbf{U}$ and studying the action of the bar involution on PBW bases. Using Ringel's Hall algebra approach to quantized enveloping algebras and also Auslander-Reiten quivers, we can describe the commutation relations between root vectors. This way we get explicit formulae for the multiplication of an element of PBW bases adapted to a quiver with a root vector and also recursive formulae to study the bar involution on PBW bases. One of the consequences of our characterization is that if the orbit closure is rationally smooth, then it is smooth.

0. Introduction

Let $\mathbf{d} = (d_1, d_2, \dots, d_n) \in \mathbf{N}^n$, F be an algebraic closure of a finite field $\mathbf{F}_q$ with q elements and $G_{\mathbf{d}} = \prod_{i=1}^n GL_{d_i}(F)$. Let $\mathcal{Q}$ be a fixed quiver whose underlying graph is the Dynkin graph of type A_n. $G_{\mathbf{d}}$ acts by conjugation on $E_{\mathbf{d}} = \bigoplus_{i \to j \in \mathcal{Q}} \mathrm{Hom}_F(F^{d_i}, F^{d_j})$.

Fix a $G_{\mathbf{d}}$-orbit $\mathcal{O}$, we will characterize which orbit closures $\overline{\mathcal{O}}$ (in the Zariski topology) are rationally smooth. Rational smoothness is a topological property of varieties defined using the local intersection cohomology groups of $\overline{\mathcal{O}}$.

Rational smoothness has been extensively studied for Schubert varieties. For a survey of some of these results, see [3]. The initial motivation was to find a criteria for rational smoothness in our situation similar to the Carrell-Peterson criteria for rational smoothness of Schubert varieties (see [6]).

If $\mathbf{U}^+$ is the positive part of the quantized enveloping algebra $\mathbf{U}$ over $\mathbf{Q}(v)$ associated by Drinfeld and Jimbo to the root system of type A_n. Kashiwara and Lusztig have constructed independently of each other a unique canonical basis $\mathbf{B}$ of $\mathbf{U}^+$ in [8] and [10]. Their construction is not restricted only to type A_n, but to a more general setting including, for example, all the simply laced semisimple Lie algebras. For each reduced expression $\mathbf{i}$ of the longest element w_0 of the Weyl group W of type A_n, there is also a PBW basis $B_{\mathbf{i}}$. Some of these reduced expressions are adapted to the quiver $\mathcal{Q}$. In this case, Lusztig has shown in [10] that the entries of the transition matrix between the bases $\mathbf{B}$ and $B_{\mathbf{i}}$ have a description using the local

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intersection cohomology groups of orbit closures. This is how we have studied the rational smoothness by using the approach of Lusztig. One important ingredient in the definition of $\mathbf{B}$ and in our approach is the action of the bar involution of $\mathbf{U}$ on the elements of $B_{\mathbf{i}}$.

This article is divided into six sections. In the first section, we fix the notation and we recall the results of Lusztig on the geometrical aspects of canonical bases of quantized enveloping algebras presented in [10]. In the second section, we describe the action of the bar involution on root vectors. In the third and fourth sections, we use Ringel's Hall algebra approach to quantized enveloping algebras and also Auslander-Reiten quivers to study commutation relations between root vectors and also the multiplication of a PBW basis element by a root vector on the left. In the fifth section, we study the components of the bar of a PBW basis element of $B_{\mathbf{i}}$ when expressed in the basis $B_{\mathbf{i}}$. In fact, we study a multiple of these components by an appropriate power of v. These components and their derivatives relative to v are evaluated at 1 in proposition 5.2 and theorem 5.4. In fact, this last theorem is the key to our study of rational smoothness. The order of the root $v = 1$ for these components seems to be related to the minimal length of finite sequences of modules $\mathbf{M} = \mathbf{M}_0, \mathbf{M}_1, \ldots, \mathbf{M}_r = \mathbf{N}$ where $\mathbf{M}_i$ is the middle term of a short exact sequence whose end terms add up to $\mathbf{M}_{i+1}$ for all i, used by Bongartz to study orbit closure in [4]. In the last section, we characterize the orbit closures $\overline{\mathcal{O}}$ which are rationally smooth. As a consequence of this characterization, if $\overline{\mathcal{O}}$ is rationally smooth, then it is smooth.

The second author with P. Caldero has recently been able to generalize theorems 5.4 and 6.8 to quivers of type A, D and E in [5]. This is possible by using the theory of dual canonical bases. The advantage of the approach of this article over the one in [5] is that it is very explicit and recursive. It could be used for computer programming. But it cannot easily be generalized to type D and E.

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1. Notation and recollections

In this section, we will fix the notation later used and recall results of Lusztig on canonical bases arising from quantized enveloping algebras. We will restrict ourselves to the type A_n.

1.1. Let v be an indeterminate and $\mathbf{U}$ the Drinfeld-Jimbo quantized enveloping algebra over $\mathbf{Q}(v)$ corresponding to the simple complex Lie algebra $sl_{n+1}(\mathbf{C})$. This is a $\mathbf{Q}(v)$-algebra with generators E_i, F_i, K_i, K_i^{-1} for $(1 \leq i \leq n)$ and relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i; \tag{r.1}$$

$$K_i E_j = \begin{cases} v^2 E_j K_i, & \text{if } i = j, \\ v^{-1} E_j K_i, & \text{if } |i-j| = 1, \\ E_j K_i, & \text{if } |i-j| > 1; \end{cases} \tag{r.2}$$

(r.3)
$$K_iF_j = \begin{cases} v^{-2}F_jK_i, & \text{if } i=j,\\ v\,F_jK_i, & \text{if } |i-j|=1,\\ F_jK_i, & \text{if } |i-j|>1; \end{cases}$$

(r.4)
$$E_iF_j - F_jE_i = \delta_{ij}\frac{(K_i - K_i^{-1})}{(v-v^{-1})} \qquad \text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i=j,\\ 0, & \text{if } i\neq j; \end{cases}$$

(r.5)
$$\begin{cases} E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0, & \text{if } |i-j|=1,\\ E_iE_j - E_jE_i = 0, & \text{if } |i-j|>1; \end{cases}$$

(r.6)
$$\begin{cases} F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0, & \text{if } |i-j|=1,\\ F_iF_j - F_jF_i = 0, & \text{if } |i-j|>1. \end{cases}$$

$\mathbf{U}^+$ will denote the $\mathbf{Q}(v)$-subalgebra of $\mathbf{U}$ generated by the elements E_i ($1 \leq i \leq n$). Let $(^-) : \mathbf{U} \to \mathbf{U}$ be the $\mathbf{Q}$-algebra involution defined by

$$E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_i \mapsto K_i^{-1} \qquad \text{for all } 1 \leq i \leq n \text{ and } v \mapsto v^{-1}.$$

This maps $\mathbf{U}^+$ into itself.

1.2. Let Q be the free abelian group with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Define an inner product $(\ ,\ )$ on Q by

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{if } i=j,\\ -1, & \text{if } |i-j|=1,\\ 0, & \text{if } |i-j|>1. \end{cases}$$

Let $R = \{\alpha \in Q \mid (\alpha,\alpha) = 2\}$. This is a root system of type A_n whose set of simple roots is $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $R^+ = \{\alpha \in R \mid \alpha = \sum_j c_j\alpha_j \text{ with } c_j \in \mathbf{N}\}$ be the subset of positive roots. In our case, we have that $R = \{\pm(\alpha_a + \alpha_{a+1} + \ldots + \alpha_b) \mid 1 \leq a \leq b \leq n\}$ and $R^+ = \{(\alpha_a + \alpha_{a+1} + \ldots + \alpha_b) \mid 1 \leq a \leq b \leq n\}$. The support of the root $\alpha = \sum_j c_j\alpha_j$ is defined to be $\{1 \leq j \leq n \mid c_j \neq 0\}$ and we will denote it by $\mathrm{Supp}(\alpha)$. It is known that the support of a root is a connected subset of $[1,n]$; i.e., $\mathrm{Supp}(\alpha) = [a,b] = \{a, a+1, \ldots, b\}$ for some $1 \leq a \leq b \leq n$.

Any $\alpha \in R$ defines a reflection $s_\alpha : Q \to Q$, $z \mapsto z - (z,\alpha)\,\alpha$. We will write s_i instead of s_{α_i}. W will denote the Weyl group of R. This is the subgroup of $\mathrm{Aut}(Q)$ generated by the reflections s_i for $1 \leq i \leq n$ and it is isomorphic to the symmetric group on $(n+1)$ elements. Let $\ell(w)$ be the length of w with respect to the generators $\{s_1, s_2, \ldots, s_n\}$. Let w_0 be the unique element of W of maximal length. It is known that if $\nu = \#(R^+)$, then $\ell(w_0) = \nu = n\,(n+1)/2$.

1.3. Lusztig has defined an action of the braid group on $\mathbf{U}$ in [9] and used it to give bases of type PBW of $\mathbf{U}^+$. We now recall these definitions.

For $i \in [1,n]$, let $\tilde{T}_i : \mathbf{U} \to \mathbf{U}$ be the $\mathbf{Q}(v)$-algebra automorphism defined by

$$\begin{array}{llll} E_i \mapsto -K_i^{-1}F_i, & F_i \mapsto -E_iK_i, & K_i \mapsto K_i^{-1}, & \\ E_j \mapsto E_j, & F_j \mapsto F_j, & K_j \mapsto K_j, & \text{if } |i-j|>1,\\ E_j \mapsto (E_jE_i - v^{-1}E_iE_j), & F_j \mapsto (F_iF_j - vF_jF_i), & K_j \mapsto K_iK_j, & \text{if } |i-j|=1. \end{array}$$

We have $\tilde{T}_i\tilde{T}_j\tilde{T}_i = \tilde{T}_j\tilde{T}_i\tilde{T}_j$ if $|i-j|=1$ and $\tilde{T}_i\tilde{T}_j = \tilde{T}_j\tilde{T}_i$ if $|i-j|>1$. This gives us a braid group action. Note that it is easy to check that $\tilde{T}_i(E_j) = \tilde{T}_j^{-1}(E_i)$ whenever $|i-j|=1$.

1.4. Given integers $M, N \geq 0$, we define

$$[N]! = \prod_{h=1}^{N} \frac{(v^h - v^{-h})}{(v - v^{-1})} \in \mathbf{Z}[v, v^{-1}], \quad \begin{bmatrix} M+N \\ N \end{bmatrix} = \frac{[M+N]!}{[M]![N]!} \in \mathbf{Z}[v, v^{-1}]$$

and $E_i^{(N)} = E_i^N/[N]!$ for $1 \leq i \leq n$. U (resp. U^+) will denote the $\mathbf{Z}[v, v^{-1}]$-subalgebra of $\mathbf{U}$ (resp. $\mathbf{U}^+$) generated by the elements $E_i^{(N)}$, $F_i^{(N)}$, K_i and K_i^{-1} (resp. $E_i^{(N)}$) for $1 \leq i \leq n$, $N \geq 0$.

1.5. We will denote the set of sequences $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ in $[1, n]$ such that $s_{i_1}s_{i_2}\ldots s_{i_\nu}$ is a reduced expression of w_0 by $\mathcal{I}$.

To $\mathbf{i} \in \mathcal{I}$, we can associate the sequence $\alpha(\mathbf{i}, 1)$, $\alpha(\mathbf{i}, 2)$, $\ldots, \alpha(\mathbf{i}, \nu)$ defined by $\alpha(\mathbf{i}, k) = s_{i_1}s_{i_2}\ldots s_{i_{k-1}}(\alpha_{i_k})$ for $k = 1, 2, \ldots, \nu$. It is well known that this sequence contains each positive root once and exactly once.

An element $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ is said to have $\mathbf{i}$-homogeneity $\mathbf{d} \in \mathbf{N}^n$ if and only if

$$\sum_{k=1}^{\nu} c_k \, \alpha(\mathbf{i}, k) = \sum_{i=1}^{n} d_i \, \alpha_i \qquad \text{where} \qquad \mathbf{d} = (d_1, d_2, \ldots, d_n).$$

1.6. For $\mathbf{i} = (i_1, i_2, \ldots, i_\nu) \in \mathcal{I}$ and $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$, we set

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)}\tilde{T}_{i_1}\left(E_{i_2}^{(c_2)}\right)\tilde{T}_{i_1}\tilde{T}_{i_2}\left(E_{i_3}^{(c_3)}\right)\ldots\tilde{T}_{i_1}\tilde{T}_{i_2}\ldots\tilde{T}_{i_{(\nu-1)}}\left(E_{i_\nu}^{(c_\nu)}\right).$$

Proposition 1.7. *Let* $\mathbf{i} \in \mathcal{I}$. *Then* $B_{\mathbf{i}} = \{E_{\mathbf{i}}^{\mathbf{c}} \mid \mathbf{c} \in \mathbf{N}^\nu\}$ *is a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}^+$. $B_{\mathbf{i}}$ *is said to be a basis of PBW type.*

Proof. This is a consequence of the proof of proposition 1.13 in [9]. □

We will now recall Lusztig's construction of the canonical basis of $\mathbf{U}^+$.

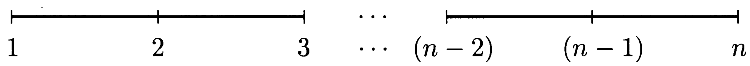
Theorem 1.8 (Lusztig). *Let* $\mathbf{i} \in \mathcal{I}$ *and* $\mathcal{L}_{\mathbf{i}}$ *be the* $\mathbf{Z}[v^{-1}]$*-submodule of* $\mathbf{U}^+$ *generated by* $B_{\mathbf{i}}$. *Then*

(a) $\mathcal{L}_{\mathbf{i}}$ *is independent of* $\mathbf{i}$. *We denote* $\mathcal{L}_{\mathbf{i}}$ *by* $\mathcal{L}$.
(b) $\pi(B_{\mathbf{i}})$ *is a* $\mathbf{Z}$*-basis of* $\mathcal{L}/v^{-1}\mathcal{L}$ *independent of* $\mathbf{i}$ *where* $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ *is the canonical projection. We denote* $\pi(B_{\mathbf{i}})$ *by* B.
(c) *The restriction of* $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ *defines an isomorphism of* $\mathbf{Z}$*-modules* $\pi' : \mathcal{L} \cap \bar{\mathcal{L}} \to \mathcal{L}/v^{-1}\mathcal{L}$ *where* $\bar{\mathcal{L}}$ *is the image of* $\mathcal{L}$ *under* $(^-)$. *In particular,* $\mathbf{B} = \pi^{-1}(B)$ *is a* $\mathbf{Z}$*-basis of* $\mathcal{L} \cap \bar{\mathcal{L}}$.
(d) $\mathbf{B}$ *is a* $\mathbf{Z}[v^{-1}]$*-basis of* $\mathcal{L}$ *and a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}^+$. $\mathbf{B}$ *is said to be the canonical basis of* $\mathbf{U}^+$.
(e) *Each element of* $\mathbf{B}$ *is fixed by* $(^-) : \mathbf{U}^+ \to \mathbf{U}^+$.

Proof. (a) and (b) are proved in proposition 2.3 in [10]. (c), (d) and (e) are proved in theorem 3.2 also in [10]. □

For some elements $\mathbf{i} \in \mathcal{I}$, Lusztig gave a geometric description of the entries of the transition matrix between bases $\mathbf{B}$ and $B_{\mathbf{i}}$. In the following paragraphs, we will describe these elements of $\mathcal{I}$ and the geometric interpretation of these entries.

1.9**.** Recall that the Dynkin graph Δ of the root system R is the graph whose set of vertices is $\{1, 2, \ldots, n\}$ and $\{i, j\}$ form an edge if and only if $|i - j| = 1$. Δ is thus the graph:



In the following, $\mathcal{Q}$ will denote a quiver whose underlying graph is the Dynkin graph Δ of R, i.e. each edge $\{i, j\}$ in Δ is given an orientation. A vertex i is a sink (resp. source) of $\mathcal{Q}$ if there is no arrow $i \to j$ (resp. $i \leftarrow j$) in $\mathcal{Q}$.

An element $\mathbf{i} = (i_1, i_2, \ldots, i_\nu) \in \mathcal{I}$ is said to be adapted to the quiver $\mathcal{Q}$ if i_1 a sink of $\mathcal{Q}_1 = \mathcal{Q}$, i_2 a sink of the quiver $\mathcal{Q}_2 = s_{i_1}(\mathcal{Q})$ obtained from $\mathcal{Q}$ by reversing the orientation of all the arrows ending with i_1, i_3 a sink of the quiver $\mathcal{Q}_3 = s_{i_2}(\mathcal{Q}_2)$ obtained from $\mathcal{Q}_2$ by reversing the orientation of all the arrows ending with i_2 and so on; i.e., i_k is a sink of the quiver $\mathcal{Q}_k = s_{i_{k-1}}(\mathcal{Q}_{k-1})$ obtained from $\mathcal{Q}_{k-1}$ by reversing the orientation of all the arrows ending with i_{k-1}, where $2 \leq k \leq \nu$.

Proposition 1.10. (a) *There is an element* $\mathbf{i}$ *of* $\mathcal{I}$ *adapted to the quiver* $\mathcal{Q}$.

(b) *An element* $\mathbf{i}$ *of* $\mathcal{I}$ *can be adapted to atmost one quiver.*

(c) *If* $\mathbf{i} = (i_1, i_2, \ldots, i_\nu)$ *is an element of* $\mathcal{I}$ *adapted to the quiver* $\mathcal{Q}$ *and* $j \in [1, n]$ *is defined by* $w_0(\alpha_{i_1}) = -\alpha_j$, *then* $\mathbf{i}' = (i_2, i_3, \ldots, i_\nu, j)$ *is an element of* $\mathcal{I}$ *and is adapted to the quiver* $s_{i_1}(\mathcal{Q})$.

Proof. (a) is proposition 4.12 (b) of [10]. (b) and (c) are proved in 4.14 of [10]. □

1.11**.** Let F be a fixed field. A module (or representation) $\mathbf{V} = (V_i, f_{ij})$ of $\mathcal{Q}$ is a collection of finite dimensional F-vector spaces V_i, $i \in [1, n]$ and of F-linear maps $f_{ij} : V_i \to V_j$ defined for all arrows $i \to j$ in $\mathcal{Q}$. A morphism from the module $\mathbf{V} = (V_i, f_{ij})$ to the module $\mathbf{V}' = (V'_i, f'_{ij})$ is a collection of F-linear maps $g_i : V_i \to V'_i$ for $i \in [1, n]$ such that $f'_{ij}\ g_i = g_j\ f_{ij}$ for all arrows $i \to j$ in $\mathcal{Q}$. These modules and morphisms form an abelian category $\mathrm{Mod}(\mathcal{Q})$. If $\mathbf{V}$ is a module of $\mathcal{Q}$, then $[\mathbf{V}]$ will denote the isomorphism class of $\mathbf{V}$ in $\mathrm{Mod}(\mathcal{Q})$.

The dimension of the module $\mathbf{V} = (V_i, f_{ij})$ is the n-tuple

$$\dim(\mathbf{V}) = (\dim_F(V_1), \dim_F(V_2), \ldots, \dim_F(V_n)) \in \mathbf{N}^n.$$

If i is a source (resp. a sink) of $\mathcal{Q}$, then $\mathrm{Mod}^-(\mathcal{Q}, i)$ (resp. $\mathrm{Mod}^+(\mathcal{Q}, i)$) denote the full subcategory of $\mathrm{Mod}(\mathcal{Q})$ whose objects are the modules $\mathbf{V} = (V_h, f_{hh'})$ such that $\bigoplus_j f_{ij} : V_i \to \bigoplus_j V_j$ is injective (resp. $\bigoplus_j f_{ji} : \bigoplus_j V_j \to V_i$ is surjective) and we define the reflection functor $\Phi_i^- : \mathrm{Mod}(\mathcal{Q}) \to \mathrm{Mod}^+(s_i(\mathcal{Q}), i)$ (resp. $\Phi_i^+ : \mathrm{Mod}(\mathcal{Q}) \to \mathrm{Mod}^-(s_i(\mathcal{Q}), i)$) by associating to an object $\mathbf{V} = (V_h, f_{hh'})$ the object $\mathbf{V}' = (V'_h, f'_{hh'})$ of $\mathrm{Mod}^+(s_i(\mathcal{Q}), i)$ (resp. $\mathrm{Mod}^-(s_i(\mathcal{Q}), i)$) defined by

$$V'_h = \begin{cases} V_h, & \text{if } h \neq i, \\ \mathrm{coker}(\bigoplus_j f_{ij} : V_i \to \bigoplus_j V_j), & \text{if } h = i. \end{cases}$$

$$\left(\text{resp. } V'_h = \begin{cases} V_h, & \text{if } h \neq i, \\ \ker(\bigoplus_j f_{ji} : \bigoplus_j V_j \to V_i), & \text{if } h = i \end{cases}\right).$$

The maps $f'_{hh'}$ are the obvious one. Φ_i^- and Φ_i^+ are defined in the obvious way on morphisms.

It is known that the restriction of Φ_i^+ to the subcategory $\mathrm{Mod}^+(\mathcal{Q},i)$ defines an equivalence of

$$\mathrm{Mod}^+(\mathcal{Q},i) \cong \mathrm{Mod}^-(s_i(\mathcal{Q}),i)$$

whose inverse is given by the restriction of Φ_i^- to $\mathrm{Mod}^-(s_i(\mathcal{Q}),i)$. Moreover, if a module $\mathbf{V}$ in $\mathrm{Mod}^+(\mathcal{Q},i)$ corresponds under Φ_i^+ to $\mathbf{V}'$ in $\mathrm{Mod}^-(s_i(\mathcal{Q}),i)$, then $\dim(\mathbf{V}) = (d_1, d_2, \ldots, d_n)$ and $\dim(\mathbf{V}') = (d'_1, d'_2, \ldots, d'_n)$ are related by

$$d'_1\,\alpha_1 + d'_2\,\alpha_2 + \cdots + d'_n\,\alpha_n = s_i(d_1\,\alpha_1 + d_2\,\alpha_2 + \ldots + d_n\,\alpha_n).$$

A module of $\mathcal{Q}$ is said to be indecomposable if it cannot be written as the direct sum of proper submodules.

For $k \in \{1, 2, \ldots, n\}$, denote by $\mathbf{P}(k)$ the following module of $\mathcal{Q}$: $\mathbf{P}(k)_i$ is the vector space over F with basis the set of paths $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ from k to i in $\mathcal{Q}$ and for any arrow $i \to j$ in $\mathcal{Q}$, let $f_{ij} \colon \mathbf{P}(k)_i \to \mathbf{P}(k)_j$ be defined by sending the basis element $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ to $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i \to j$. It is easy to prove that $\mathbf{P}(k) = (\mathbf{P}(k)_i, (f_{ij})_{i\to j}))$ is an indecomposable projective module of $\mathcal{Q}$ and that all indecomposable projective modules are isomorphic to some $\mathbf{P}(k)$ for $k \in \{1, 2, \ldots, n\}$.

Theorem 1.12. *Let $\mathcal{Q}$ be a quiver and $\mathbf{i} \in \mathcal{I}$ adapted to $\mathcal{Q}$.*

(a) *For any $\alpha \in R^+$, there is a unique indecomposable module (up to isomorphism) denoted $\mathbf{e}_\alpha \in \mathrm{Mod}(\mathcal{Q})$ such that $\dim(\mathbf{e}_\alpha) = (d_1, d_2, \ldots, d_n)$ and $\alpha = \sum_{i=1}^n d_i\alpha_i$; any indecomposable module is isomorphic to $\mathbf{e}_\alpha$ for a unique α. If $\alpha = \alpha(\mathbf{i}, k) = s_{i_1}s_{i_2}\ldots s_{i_{k-1}}(\alpha_{i_k})$, then $\mathbf{e}_\alpha = \Phi^-_{i_1}\Phi^-_{i_2}\ldots\Phi^-_{i_{k-1}}(e_{i_k})$, where e_{i_k} is the simple module (V_i, f_{ij}) in $\mathrm{Mod}(\mathcal{Q}_k)$ defined by $V_{i_k} = F$, $V_j = 0$ for $j \neq i_k$ and $f_{ij} = 0$ for all arrows $i \to j$ in $\mathcal{Q}_k$. This is Gabriel's theorem. In particular, the classification of indecomposable modules is independent of the ground field.*

(b) *There is a bijection $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \mapsto \mathbf{V_c}$ between $\mathbf{N}^\nu$ and the set of isomorphism classes of modules of $\mathcal{Q}$, where $\mathbf{V_c}$ is the direct sum of c_k copies of $\mathbf{e}_{\alpha(\mathbf{i},k)}$ for $k = 1, 2, \ldots, \nu$. In this case, $\dim(\mathbf{V_c}) = (d_1, d_2, \ldots, d_n)$, where $\sum_{k=1}^{\nu} c_k\ \alpha(\mathbf{i}, k) = \sum_{i=1} d_i\ \alpha_i$, i.e. $\mathbf{c}$ has $\mathbf{i}$-homogeneity $\mathbf{d}$.*

(c) *If $\alpha = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$ for $1 \le a \le b \le n$, then $\mathbf{e}_\alpha$ is isomorphic to the module (V_i, f_{ij}) where*

$$V_i = \begin{cases} F, & \text{if } a \le i \le b, \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{ij} = \begin{cases} \mathrm{Id}_F, & \text{if } a \le i, j \le b, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{whenever } i \to j \text{ in } \mathcal{Q}.$$

Proof. (a) is proved for example in proposition 4.12 in [10]. (b) is proved for example in 4.15 in [10]. (c) is easily obtained. □

1.13**.** We will abbreviate $\dim_F \mathrm{Hom}_{\mathcal{Q}}(\mathbf{V}, \mathbf{V}')$ by $\langle \mathbf{V}, \mathbf{V}' \rangle_{\mathcal{Q}}$ and $\dim_F \mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{V}, \mathbf{V}')$ by $\langle \mathbf{V}, \mathbf{V}' \rangle^1_{\mathcal{Q}}$. Note that here $\mathrm{Hom}_{\mathcal{Q}}(\mathbf{V}, \mathbf{V}')$ is the F-vector space of morphisms $g : \mathbf{V} \to \mathbf{V}'$ in $\mathrm{Mod}(\mathcal{Q})$ and $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{V}, \mathbf{V}')$ is the F-vector space of extensions $0 \to \mathbf{V}' \to \mathbf{E} \to \mathbf{V} \to 0$ in $\mathrm{Mod}(\mathcal{Q})$

Lemma 1.14. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_\nu) \in \mathcal{I}$ *adapted to the quiver* $\mathcal{Q}$ *and* $\mathbf{i}' = (i_2, i_3, \ldots, i_\nu, j) \in \mathcal{I}$ *adapted to the quiver* $s_{i_1}(\mathcal{Q}) = \mathcal{Q}'$ *where* j *is defined by* $w_0(\alpha_{i_1}) = -\alpha_j$. *Let* $\mathbf{e}_\beta$ *(resp.* $\mathbf{e}'_\beta$*) be an indecomposable module of* $\mathcal{Q}$ *(resp.* $\mathcal{Q}'$*) of dimension* $\beta \in R^+$.

(a) *If* $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} \neq 0$ *for* $1 \leq h,\ k \leq \nu$, *then* $h \leq k$.

(b) *For* $1 < h < k \leq \nu$, *then*

$$\langle \mathbf{e}_{\alpha(\mathbf{i},\,h)}, \mathbf{e}_{\alpha(\mathbf{i},\,k)} \rangle_{\mathcal{Q}} \neq 0 \qquad \text{if and only if} \qquad \langle \mathbf{e}_{\alpha(\mathbf{i}',\,h-1)}, \mathbf{e}_{\alpha(\mathbf{i}',\,k-1)} \rangle_{\mathcal{Q}'} \neq 0.$$

(c) *For* $1 < h < k \leq \nu$, *then*

$$\langle \mathbf{e}_{\alpha(\mathbf{i},\,k)}, \mathbf{e}_{\alpha(\mathbf{i},\,h)} \rangle^1_{\mathcal{Q}} \neq 0 \qquad \text{if and only if} \qquad \langle \mathbf{e}_{\alpha(\mathbf{i}',\,k-1)}, \mathbf{e}_{\alpha(\mathbf{i}',\,h-1)} \rangle^1_{\mathcal{Q}'} \neq 0.$$

Proof. (a) is proved in proposition 4.12 of [10].

(b) Since $h, k > 1$, then $\alpha(\mathbf{i}, h)$ and $\alpha(\mathbf{i}, k)$ are both distinct from α_{i_1} and consequently the indecomposable modules $\mathbf{e}_{\alpha(\mathbf{i},h)}$ and $\mathbf{e}_{\alpha(\mathbf{i},k)}$ are both in $\mathrm{Mod}^+(\mathcal{Q}, i_1)$. Since $h, k \leq \nu$, then $\alpha(\mathbf{i}', h-1)$ and $\alpha(\mathbf{i}', k-1)$ are both distinct from α_{i_1} and consequently the indecomposable modules $\mathbf{e}'_{\alpha(\mathbf{i}',h-1)}$ and $\mathbf{e}'_{\alpha(\mathbf{i}',k-1)}$ are both in $\mathrm{Mod}^-(\mathcal{Q}', i_1)$. In [10], it is proved that $\Phi^+_{i_1}(\mathbf{e}_{\alpha(\mathbf{i},h)})$ is isomorphic to $\mathbf{e}'_{\alpha(\mathbf{i}',h-1)}$ and $\Phi^+_{i_1}(\mathbf{e}_{\alpha(\mathbf{i},k)})$ is isomorphic to $\mathbf{e}'_{\alpha(\mathbf{i}',k-1)}$ when $h, k > 1$. Since $\Phi^+_{i_1}$ is an equivalence of $\mathrm{Mod}^+(\mathcal{Q}, i_1) \cong \mathrm{Mod}^-(s_{i_1}(\mathcal{Q}), i_1)$, we get that $\mathrm{Hom}_{\mathcal{Q}}(\mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)}) \neq 0$ if and only if $\mathrm{Hom}_{\mathcal{Q}'}(\mathbf{e}'_{\alpha(\mathbf{i}',h-1)}, \mathbf{e}'_{\alpha(\mathbf{i}',k-1)}) \neq 0$. This proves (b).

(c) First note that if the modules $\mathbf{V}$ and $\mathbf{V}'$ of $\mathcal{Q}$ are in $\mathrm{Mod}^+(\mathcal{Q}, i_1)$ and $0 \to \mathbf{V}' \to \mathbf{E} \to \mathbf{V} \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathcal{Q})$, then $\mathbf{E}$ is also in $\mathrm{Mod}^+(\mathcal{Q}, i_1)$. This can be proved either from the definition of $\mathrm{Mod}^+(\mathcal{Q}, i_1)$ or by noting that a module $\mathbf{M}$ belongs to $\mathrm{Mod}^+(\mathcal{Q}, i_1)$ if and only if $\mathrm{Hom}_{\mathcal{Q}}(\mathbf{M}, \mathbf{e}_{\alpha_{i_1}}) = 0$ and by applying the functor $\mathrm{Hom}_{\mathcal{Q}}(\,\cdot\,, \mathbf{e}_{\alpha_{i_1}})$ to the exact sequence $0 \to \mathbf{V}' \to \mathbf{E} \to \mathbf{V} \to 0$ to show that $\mathrm{Hom}_{\mathcal{Q}}(\mathbf{E}, \mathbf{e}_{\alpha_{i_1}}) = 0$.

Similarly if the modules $\mathbf{V}$, $\mathbf{V}'$ of $\mathcal{Q}'$ are in $\mathrm{Mod}^-(\mathcal{Q}', i_1)$ and $0 \to \mathbf{V}' \to \mathbf{E}' \to \mathbf{V} \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathcal{Q}')$, then $\mathbf{E}'$ is also in $\mathrm{Mod}^-(\mathcal{Q}', i_1)$. This is proved by noting that a module $\mathbf{M}$ belongs to $\mathrm{Mod}^-(\mathcal{Q}', i_1)$ if and only if $\mathrm{Hom}_{\mathcal{Q}'}(\mathbf{e}_{\alpha_{i_1}}, \mathbf{M}) = 0$ and by applying the functor $\mathrm{Hom}_{\mathcal{Q}'}(\mathbf{e}_{\alpha_{i_1}}, \,\cdot\,)$ to the exact sequence $0 \to \mathbf{V}' \to \mathbf{E}' \to \mathbf{V} \to 0$ to show that $\mathrm{Hom}_{\mathcal{Q}'}(\mathbf{e}_{\alpha_{i_1}}, \mathbf{E}') = 0$.

For $1 < h < k \leq \nu$, then, as we have noted in the proof of (b), the indecomposable modules $\mathbf{e}_{\alpha(\mathbf{i},h)}$ and $\mathbf{e}_{\alpha(\mathbf{i},k)}$ are in $\mathrm{Mod}^+(\mathcal{Q}, i_1)$ and consequently, if $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{E} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathcal{Q})$, then $\mathbf{E}$ is in $\mathrm{Mod}^+(\mathcal{Q}, i_1)$. By applying the reflection functor $\Phi^+_{i_1}$ to such an exact sequence and because $\Phi^+_{i_1}(\mathbf{e}_{\alpha(\mathbf{i},h)})$ and $\Phi^+_{i_1}(\mathbf{e}_{\alpha(\mathbf{i},k)})$ are respectively isomorphic to the modules $\mathbf{e}'_{\alpha(\mathbf{i}',h-1)}$ and $\mathbf{e}'_{\alpha(\mathbf{i}',k-1)}$ of $\mathcal{Q}'$, we get a short exact sequence $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \Phi^+_{i_1}(\mathbf{E}) \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ in $\mathrm{Mod}(\mathcal{Q}')$. This follows easily from the snake lemma. Moreover, it is easy to check that if the sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{E} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ splits, then the sequence $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \Phi^+_{i_1}(\mathbf{E}) \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ also splits.

Similarly when $1 < h < k \leq \nu$, then the indecomposable modules $\mathbf{e}'_{\alpha(\mathbf{i}',h-1)}$ and $\mathbf{e}'_{\alpha(\mathbf{i}',k-1)}$ are in $\mathrm{Mod}^-(\mathcal{Q}', i_1)$ and consequently, if $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \mathbf{E}' \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ is a short exact sequence in $\mathrm{Mod}(\mathcal{Q}')$, then $\mathbf{E}'$ is in $\mathrm{Mod}^-(\mathcal{Q}', i_1)$. By applying the reflection functor $\Phi^-_{i_1}$ to such an exact sequence and because $\Phi^-_{i_1}(\mathbf{e}'_{\alpha(\mathbf{i}',h-1)})$ and $\Phi^-_{i_1}(\mathbf{e}'_{\alpha(\mathbf{i}',k-1)})$ are respectively isomorphic to the modules $\mathbf{e}_{\alpha(\mathbf{i},h)}$

and $\mathbf{e}_{\alpha(\mathbf{i},k)}$ of $\mathcal{Q}$, we get a short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \Phi^-_{i_1}(\mathbf{E}') \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ in $\mathrm{Mod}(\mathcal{Q})$. This also follows from the snake lemma. Moreover, it is easy to check that if the sequence $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \mathbf{E}' \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ splits, then the sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \Phi^-_{i_1}(\mathbf{E}') \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ also splits.

If $\langle \mathbf{e}_{\alpha(\mathbf{i},\,k)}, \mathbf{e}_{\alpha(\mathbf{i},\,h)}\rangle^1_{\mathcal{Q}} \neq 0$, then there exists a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{E} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ of modules of $\mathcal{Q}$. By applying the reflection functor $\Phi^+_{i_1}$, we get a nonsplit short exact sequence $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \Phi^+_{i_1}(\mathbf{E}) \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$. It is nonsplit, for otherwise, by applying $\Phi^-_{i_1}$, we would get that $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{E} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ is split, contradicting our hypothesis. So $\langle \mathbf{e}'_{\alpha(\mathbf{i}',\,k-1)}, \mathbf{e}'_{\alpha(\mathbf{i}',\,h-1)}\rangle^1_{\mathcal{Q}'} \neq 0$. Similarly if $\langle \mathbf{e}'_{\alpha(\mathbf{i}',\,k-1)}, \mathbf{e}'_{\alpha(\mathbf{i}',\,h-1)}\rangle^1_{\mathcal{Q}'} \neq 0$, then there exists a nonsplit short exact sequence $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \mathbf{E}' \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ of modules of $\mathcal{Q}'$. By applying the reflection functor $\Phi^-_{i_1}$, we get a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \Phi^-_{i_1}(\mathbf{E}') \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$. It is nonsplit, for otherwise by applying $\Phi^+_{i_1}$, we would get that $0 \to \mathbf{e}'_{\alpha(\mathbf{i}',h-1)} \to \mathbf{E}' \to \mathbf{e}'_{\alpha(\mathbf{i}',k-1)} \to 0$ is split, contradicting our hypothesis. So $\langle \mathbf{e}_{\alpha(\mathbf{i},\,k)}, \mathbf{e}_{\alpha(\mathbf{i},\,h)}\rangle^1_{\mathcal{Q}} \neq 0$. This concludes our proof. □

In the following, $\mathbf{i}$ will be an element of $\mathcal{I}$ adapted to the quiver $\mathcal{Q}$.

1.15**.** Let $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbf{N}^n$. Let

$$E_{\mathbf{d}} = \bigoplus_{i \to j} \mathrm{Hom}_F(F^{d_i}, F^{d_j})$$

sum over all arrows $i \to j$ in $\mathcal{Q}$ and let $G_{\mathbf{d}} = \prod_{i=1}^n GL_{d_i}(F)$. The group $G_{\mathbf{d}}$ acts naturally on $E_{\mathbf{d}}$ by

$$(g_i)_i : (f_{ij})_{i \to j} \mapsto (g_j\ f_{ij}\ g_i^{-1})_{i \to j}.$$

An element of $E_{\mathbf{d}}$ can be seen as a module in $\mathrm{Mod}(\mathcal{Q})$ of dimension $\mathbf{d}$. Two elements of $E_{\mathbf{d}}$ define isomorphic modules if and only if they are in the same $G_{\mathbf{d}}$-orbit. Using theorem 1.12 (b), we get that there exists a bijection between the set of ν-tuples $\mathbf{c} = (c_1, c_2, \ldots, c_\nu)$ of $\mathbf{i}$-homogeneity $\mathbf{d}$ and the set of $G_{\mathbf{d}}$-orbits on $E_{\mathbf{d}}$ where $\mathbf{c} = (c_1, c_2, \ldots, c_\nu)$ corresponds to the orbit $\mathcal{O}_{\mathbf{c}}$ whose elements are isomorphic to $\mathbf{V}_{\mathbf{c}}$.

For the rest of this section, F will denote an algebraic closure of a finite field $\mathbf{F}_q$ with $q = p^e$ elements, p being a prime number, and $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbf{N}^n$.

1.16**.** For $\mathbf{c}', \mathbf{c} \in \mathbf{N}^\nu$, we will write $\mathbf{c}' \preceq \mathbf{c}$ if $\mathbf{c}', \mathbf{c}$ have the same $\mathbf{i}$-homogeneity and the orbit $\mathcal{O}_{\mathbf{c}'}$ is contained in the Zariski closure $\overline{\mathcal{O}_{\mathbf{c}}}$ of $\mathcal{O}_{\mathbf{c}}$. This gives us a partial order on $\mathbf{N}^\nu$. We will write the dimension $\dim(\mathcal{O}_{\mathbf{c}})$ of the orbit $\mathcal{O}_{\mathbf{c}}$ by $d(\mathbf{c})$.

Proposition 1.17. *Let $\mathbf{c}$ be an element of $\mathbf{N}^\nu$ of $\mathbf{i}$-homogeneity $\mathbf{d}$.*

(a) $\overline{E^{\mathbf{c}}_{\mathbf{i}}} = \sum_{\mathbf{c}'} \omega^{\mathbf{c}}_{\mathbf{c}'}\ E^{\mathbf{c}'}_{\mathbf{i}}$ *where $\mathbf{c}'$ runs over the set of elements of $\mathbf{N}^\nu$ of $\mathbf{i}$-homogeneity $\mathbf{d}$, $\omega^{\mathbf{c}}_{\mathbf{c}'} \in \mathbf{Z}[v, v^{-1}]$ and $\omega^{\mathbf{c}}_{\mathbf{c}} = 1$. Moreover, $\omega^{\mathbf{c}}_{\mathbf{c}'} \neq 0$ implies that $\mathbf{c}' \preceq \mathbf{c}$.*

(b) *If we set $\Omega^{\mathbf{c}}_{\mathbf{c}'} = v^{d(\mathbf{c}) - d(\mathbf{c}')}\omega^{\mathbf{c}}_{\mathbf{c}'}$ for $\mathbf{c}' \preceq \mathbf{c}$, then $\Omega^{\mathbf{c}}_{\mathbf{c}'} \in \mathbf{Z}[v^2, v^{-2}]$.*

Proof. (a) is proved in theorem 9.13 of [10]. (b) is proposition 7.14 of [10]. □

Theorem 1.18. *Let $\mathbf{c}$ be an element in $\mathbf{N}^\nu$ of $\mathbf{i}$-homogeneity $\mathbf{d}$ and let $\mathcal{E}^{\mathbf{c}}$ be the unique element of $\mathbf{B}$ such that $\pi(\mathcal{E}^{\mathbf{c}}) = \pi(E^{\mathbf{c}}_{\mathbf{i}})$.*

(a) $\mathcal{E}^{\mathbf{c}} = \sum_{\mathbf{c}'} \zeta^{\mathbf{c}}_{\mathbf{c}'} E^{\mathbf{c}'}_{\mathbf{i}}$ *where* $\mathbf{c}'$ *runs over the set of elements of* $\mathbf{N}^\nu$ *of* $\mathbf{i}$*-homogeneity* $\mathbf{d}$, $\zeta^{\mathbf{c}}_{\mathbf{c}} = 1$ *and* $\zeta^{\mathbf{c}}_{\mathbf{c}'} \in v^{-1}\mathbf{Z}[v^{-1}]$ *for* $\mathbf{c}' \neq \mathbf{c}$.
(b) *If* $\mathbf{c}' \not\preceq \mathbf{c}$, *then* $\zeta^{\mathbf{c}}_{\mathbf{c}'} = 0$.
(c) *If* $(^-)$ *is the* $\mathbf{Z}$*-linear involution of* $\mathbf{Z}[v, v^{-1}]$ *sending* v *to* v^{-1}, *then*

$$\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \omega^{\mathbf{c}''}_{\mathbf{c}'} \, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}.$$

(d) *If* $\mathbf{c}' \preceq \mathbf{c}$, f *is an* $\mathbf{F}_q$*-rational point of the orbit* $\mathcal{O}_{\mathbf{c}'}$ *in* $E_{\mathbf{d}}$ *and* $\mathcal{H}^a_f(\overline{\mathcal{O}_{\mathbf{c}}})$ *is the stalk at* f *of the* a^{th} *cohomology sheaf of the intersection complex of the Zariski closure* $\overline{\mathcal{O}}_{\mathbf{c}}$ *of* $\mathcal{O}_{\mathbf{c}}$ *with coefficients in* $\bar{\mathbf{Q}}_\ell$ *extended by zero on the complement of that closure, where* ℓ *is a prime number* $\neq p$, *and with the* $\mathbf{F}_q$*-structure such that the Frobenius map acts as identity on the stalks of its* 0th *cohomology sheaf at the rational points of the orbit* $\mathcal{O}_{\mathbf{c}}$, *then* $\mathcal{H}^{2a+1}_f(\overline{\mathcal{O}_{\mathbf{c}}}) = 0$ *for all* a *and*

$$v^{d(\mathbf{c})-d(\mathbf{c}')}\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_a \dim(\mathcal{H}^{2a}_f(\overline{\mathcal{O}_{\mathbf{c}}})) \; v^{2a}.$$

In particular, $v^{d(\mathbf{c})-d(\mathbf{c}')}\zeta^{\mathbf{c}}_{\mathbf{c}'}$ *is a polynomial in* v^2 *with coefficients in* $\mathbf{N}$.

Proof. (a) is 9.11 of [10]. (b) is theorem 9.13 of [10]. (c) is 9.12 of [10]. (d) is corollary 10.7 of [10]. □

2. Bar involution of root vectors

2.1. In this section, we will express the action of the bar involution on an element $\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})$ of $\mathbf{U}^+$ with respect to the PBW basis $B_{\mathbf{i}}$. Here $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$ is adapted to a quiver.

An element of $\mathbf{U}^+$ of the form $\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})$ is said to be a root vector relative to $\mathbf{i}$.

Proposition 2.2. *Let* $\mathcal{Q}$ *be a quiver of* Δ *and* $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$ *adapted to* $\mathcal{Q}$. *For* $1 \leq a \leq b \leq n$, *denote by* $\Delta[a,b]$*: the full subgraph of* Δ *whose set of vertices is* $[a,b]$ *and by* $\mathcal{Q}[a,b]$*: the subquiver of* $\mathcal{Q}$ *whose graph is* $\Delta[a,b]$. *If* $\alpha(\mathbf{i}, k) = s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$, *then*

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) = \sum_{\mathcal{Q}'} (-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}')} E_{\mathcal{Q}'},$$

where the sum is over all quivers $\mathcal{Q}'$ *whose graph is* $\Delta[a,b]$, $\delta(\mathcal{Q}[a,b],\mathcal{Q}')$ *is the number of edges of* $\Delta[a,b]$ *with opposite orientation in* $\mathcal{Q}[a,b]$ *and* $\mathcal{Q}'$ *and* $E_{\mathcal{Q}'} = E_a$ *if* $a = b$ *and*

$$E_{\mathcal{Q}'} = \begin{cases} E_b E_{\mathcal{Q}'[a,b-1]}, & \text{if } (b-1) \leftarrow b \text{ in } \mathcal{Q}', \\ E_{\mathcal{Q}'[a,b-1]} E_b, & \text{if } (b-1) \rightarrow b \text{ in } \mathcal{Q}' \end{cases} \qquad \text{if } a < b.$$

Proof. We prove this by induction on k. If $k = 1$, then $\alpha(\mathbf{i}, 1) = \alpha_{i_1}$ and the result is obvious. Assume now that the result is true whenever $k < m$ for all sequences $\mathbf{j} = (j_1, j_2, \dots, j_\nu) \in \mathcal{I}$ adapted to a quiver. We consider now the case where $k = m$.

Let $j \in [1, n]$ be defined by $w_0(\alpha_{i_1}) = -\alpha_j$. Then we know by proposition 1.10 (c) that $\mathbf{j} = (i_2, i_3, \dots, i_\nu, j) \in \mathcal{I}$ and is adapted to the quiver $s_{i_1}(\mathcal{Q}) = \mathcal{Q}_1$. We get that

$$\alpha(\mathbf{j},m-1)=s_{i_2}s_{i_3}\dots s_{i_{m-1}}(\alpha_{i_m})=s_{i_1}(\alpha(\mathbf{i},m))=\begin{cases}\sum_{i=a}^{b}\alpha_i, & \text{if } a<i_1<b,\\ \sum_{i=a+1}^{b}\alpha_i, & \text{if } i_1=a,\\ \sum_{i=a}^{b-1}\alpha_i, & \text{if } i_1=b,\\ \sum_{i=a-1}^{b}\alpha_i, & \text{if } i_1=(a-1),\\ \sum_{i=a}^{b+1}\alpha_i, & \text{if } i_1=(b+1),\\ \sum_{i=a}^{b}\alpha_i, & \text{otherwise.}\end{cases}$$

By induction hypothesis, we have

$$\tilde{T}_{i_2}\tilde{T}_{i_3}\dots\tilde{T}_{i_{m-1}}(E_{i_m})=\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a',b'],\mathcal{Q}_1')}E_{\mathcal{Q}_1'}$$

where $a'\le b'$ are defined by $\alpha(\mathbf{j},m-1)=\sum_{a'}^{b'}\alpha_i$ and the sum is over all quivers $\mathcal{Q}_1'$ of the graph $\Delta[a',b']$. We want to compute

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\tilde{T}_{i_3}\dots\tilde{T}_{i_{m-1}}(E_{i_m})=\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a',b'],\mathcal{Q}_1')}\ \tilde{T}_{i_1}\left(E_{\mathcal{Q}_1'}\right).$$

Note that i_1 is a sink of $\mathcal{Q}$ and a source of $\mathcal{Q}_1=s_{i_1}(\mathcal{Q})$. There are six cases to consider:

(1) $a<i_1<b$; (2) $i_1=a$; (3) $i_1=b$;

(4) $i_1=(a-1)$; (5) $i_1=(b+1)$; (6) $i_1<(a-1)$ or $i_1>(b+1)$.

In what follows, i_1 will be denoted by c.

We first consider the case (1): $a<i_1<b$. In this case, $a'=a$ and $b'=b$. If $\mathcal{P}_{<c}$ is a quiver whose graph is $\Delta[a,c-1]$, $\mathcal{P}_{>c}$ is a quiver whose graph is $\Delta[c+1,b]$ and we consider all the quivers $\mathcal{Q}_1'$ whose graph is $\Delta[a,b]$ such that $\mathcal{Q}_1'[a,c-1]=\mathcal{P}_{<c}$ and $\mathcal{Q}_1'[c+1,b]=\mathcal{P}_{>c}$, then there are four such quivers corresponding to the possible orientation of the edges $\{c-1,c\}$ and $\{c,c+1\}$. Using the definition of $E_{\mathcal{Q}_1'}$, we can easily prove that there exists two disjoint subsets $\{j_1,j_2,\dots,j_x\}$, $\{j_1',j_2',\dots,j_y'\}$ with $j_1>j_2>\dots>j_x$, $j_1'>j_2'>\dots>j_y'$, $\{j_1,j_2,\dots,j_x\}\cup\{j_1',j_2',\dots,j_y'\}=[a,b]\setminus\{c-1,c,c+1\}$ such that for all four quivers $\mathcal{Q}_1'$ whose graph is $\Delta[a,b]$ with $\mathcal{Q}_1'[a,c-1]=\mathcal{P}_{<c}$ and $\mathcal{Q}_1'[c+1,b]=\mathcal{P}_{>c}$, we have

$$E_{\mathcal{Q}_1'}=\begin{cases}E_{j_1}E_{j_2}\dots E_{j_x}E_{c-1}E_{c+1}E_cE_{j_y'}\dots E_{j_2'}E_{j_1'}, & \text{if } (c-1)\to c\leftarrow(c+1) \text{ in } \mathcal{Q}_1',\\ E_{j_1}E_{j_2}\dots E_{j_x}E_{c+1}E_cE_{c-1}E_{j_y'}\dots E_{j_2'}E_{j_1'}, & \text{if } (c-1)\leftarrow c\leftarrow(c+1) \text{ in } \mathcal{Q}_1',\\ E_{j_1}E_{j_2}\dots E_{j_x}E_{c-1}E_cE_{c+1}E_{j_y'}\dots E_{j_2'}E_{j_1'}, & \text{if } (c-1)\to c\to(c+1) \text{ in } \mathcal{Q}_1',\\ E_{j_1}E_{j_2}\dots E_{j_x}E_cE_{c-1}E_{c+1}E_{j_y'}\dots E_{j_2'}E_{j_1'}, & \text{if } (c-1)\leftarrow c\to(c+1) \text{ in } \mathcal{Q}_1'\end{cases}$$

and

$$\delta(\mathcal{Q}_1[a,b],\mathcal{Q}_1')=\sigma+\begin{cases}2, & \text{if } (c-1)\to c\leftarrow(c+1) \text{ in } \mathcal{Q}_1',\\ 1, & \text{if } (c-1)\leftarrow c\leftarrow(c+1) \text{ in } \mathcal{Q}_1',\\ 1, & \text{if } (c-1)\to c\to(c+1) \text{ in } \mathcal{Q}_1',\\ 0 & \text{if } (c-1)\leftarrow c\to(c+1) \text{ in } \mathcal{Q}_1'\end{cases}$$

where $\sigma=\delta(\mathcal{Q}_1[a,c-1],\mathcal{Q}_1'[a,c-1])+\delta(\mathcal{Q}_1[c+1,b],\mathcal{Q}_1'[c+1,b])$.

Consequently,

$$\text{(1)}\qquad \sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a,b],\mathcal{Q}_1')}T_{i_1}(E_{\mathcal{Q}_1'})=(-v)^{-\sigma}E_{j_1}E_{j_2}\dots E_{j_x}\ T_c(X)\ E_{j_y'}\dots E_{j_1'}$$

where the sum is over all four quivers $\mathcal{Q}_1'$ whose graph is $\Delta[a,b]$ such that $\mathcal{Q}_1'[a,c-1]=\mathcal{P}_{<c}$ and $\mathcal{Q}_1'[c+1,b]=\mathcal{P}_{>c}$ and

$$X=(-v)^{-2}E_{c-1}E_{c+1}E_c+(-v)^{-1}E_{c+1}E_cE_{c-1}+(-v)^{-1}E_{c-1}E_cE_{c+1}+E_cE_{c-1}E_{c+1}.$$

But

$$X=\tilde{T}_{c-1}(E_c)E_{c+1}-v^{-1}E_{c+1}\tilde{T}_{c-1}(E_c)=\tilde{T}_c^{-1}(E_{c-1})E_{c+1}-v^{-1}E_{c+1}\tilde{T}_c^{-1}(E_{c-1})$$

and

$$\begin{aligned}T_c(X)&=E_{c-1}\tilde{T}_c(E_{c+1})-v^{-1}\tilde{T}_c(E_{c+1})E_{c-1}\\&=E_{c-1}E_{c+1}E_c-v^{-1}E_{c-1}E_cE_{c+1}-v^{-1}E_{c+1}E_cE_{c-1}+v^{-2}E_cE_{c-1}E_{c+1},\end{aligned}$$

because $T_c^{-1}(E_{c-1})=T_{c-1}(E_c)$.

Thus the right-hand side of equation (1) is equal to

$$\begin{aligned}&(-v)^{-\sigma}E_{j_1}E_{j_2}\dots E_{j_x}E_{c-1}E_{c+1}E_cE_{j'_y}\dots E_{j'_2}E_{j'_1}\\+\quad&(-v)^{-(\sigma+1)}E_{j_1}E_{j_2}\dots E_{j_x}E_{c-1}E_cE_{c+1}E_{j'_y}\cdots E_{j'_2}E_{j'_1}\\+\quad&(-v)^{-(\sigma+1)}E_{j_1}E_{j_2}\dots E_{j_x}E_{c+1}E_cE_{c-1}E_{j'_y}\dots E_{j'_2}E_{j'_1}\\+\quad&(-v)^{-(\sigma+1)}E_{j_1}E_{j_2}\dots E_{j_x}E_cE_{c+1}E_{c-1}E_{j'_y}\dots E_{j'_2}E_{j'_1}\end{aligned}$$

and this is equal to

$$\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}_1')}E_{\mathcal{Q}_1'}$$

where the sum is over all four quivers $\mathcal{Q}_1'$ whose graph is $\Delta[a,b]$ such that $\mathcal{Q}_1'[a,c-1]=\mathcal{P}_{<c}$ and $\mathcal{Q}_1'[c+1,b]=\mathcal{P}_{>c}$. Note that we have used the fact that c is a sink of $\mathcal{Q}$.

So

$$\begin{aligned}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m})&=\sum_{\mathcal{P}_{>c},\mathcal{P}_{<c}}\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a',b'],\mathcal{Q}_1')}\tilde{T}_{i_1}(E_{\mathcal{Q}_1'})\\&=\sum_{\mathcal{P}_{>c},\mathcal{P}_{<c}}\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}_1')}(E_{\mathcal{Q}_1'})\\&=\sum_{\mathcal{Q}'}(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}')}(E_{\mathcal{Q}'})\end{aligned}$$

where $(\mathcal{P}_{>c},\mathcal{P}_{<c})$ runs in the sum $\sum_{\mathcal{P}_{>c},\mathcal{P}_{<c}}$ over all pairs of quivers with $\mathcal{P}_{>c}$, a quiver of the graph $\Delta[c+1,b]$ and $\mathcal{P}_{<c}$, a quiver of the graph $\Delta[a,c-1]$, $\mathcal{Q}_1'$ runs in the sum $\sum_{\mathcal{Q}_1'}$ over all four quivers whose graph is $\Delta[a,b]$ such that $\mathcal{Q}_1'[a,c-1]=\mathcal{P}_{<c}$ and $\mathcal{Q}_1'[c+1,b]=\mathcal{P}_{>c}$ and $\mathcal{Q}'$ runs in the sum $\sum_{\mathcal{Q}'}$ over all quivers whose graph is $\Delta[a,b]$. This ends the proof in case (1).

Consider now the case (2): $i_1=a$. In this case, $a'=a+1$ and $b'=b$. So

$$\tilde{T}_{i_2}\tilde{T}_{i_3}\dots\tilde{T}_{i_{m-1}}(E_{i_m})=\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a+1,b],\mathcal{Q}_1')}E_{\mathcal{Q}_1'}$$

and

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m})=\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a+1,b],\mathcal{Q}_1')}\tilde{T}_{i_1}(E_{\mathcal{Q}_1'})$$

where the sum is over all quivers $\mathcal{Q}_1'$ of $\Delta[a+1,b]$.

Using the definition of $E_{\mathcal{Q}'_1}$, we get that

$$E_{\mathcal{Q}'_1} = \begin{cases} E_{\mathcal{Q}'_1[a+2,b]} E_{a+1}, & \text{if } (a+1) \leftarrow (a+2) \text{ in } \mathcal{Q}'_1, \\ E_{a+1} E_{\mathcal{Q}'_1[a+2,b]}, & \text{if } (a+1) \rightarrow (a+2) \text{ in } \mathcal{Q}'_1 \end{cases}$$

and

$$\tilde{T}_a(E_{\mathcal{Q}'_1}) = \begin{cases} E_{\mathcal{Q}'_1[a+2,b]} \ (E_{a+1}E_a - v^{-1}E_aE_{a+1}), & \text{if } (a+1) \leftarrow (a+2) \text{ in } \mathcal{Q}'_1, \\ (E_{a+1}E_a - v^{-1}E_aE_{a+1}) \ E_{\mathcal{Q}'_1[a+2,b]}, & \text{if } (a+1) \rightarrow (a+2) \text{ in } \mathcal{Q}'_1. \end{cases}$$

Note that if $\mathcal{Q}'$ is a quiver of the graph $\Delta[a,b]$ with $\mathcal{Q}'[a+1,b] = \mathcal{Q}'_1$, then

$$E_{\mathcal{Q}'} = \begin{cases} E_{\mathcal{Q}'[a+2,b]} \ E_{a+1} \ E_a, & \text{if } a \leftarrow (a+1) \leftarrow (a+2) \text{ in } \mathcal{Q}', \\ E_{\mathcal{Q}'[a+2,b]} \ E_a \ E_{a+1}, & \text{if } a \rightarrow (a+1) \leftarrow (a+2) \text{ in } \mathcal{Q}', \\ E_{a+1} \ E_a \ E_{\mathcal{Q}'[a+2,b]}, & \text{if } a \leftarrow (a+1) \rightarrow (a+2) \text{ in } \mathcal{Q}', \\ E_a \ E_{a+1} \ E_{\mathcal{Q}'[a+2,b]}, & \text{if } a \rightarrow (a+1) \rightarrow (a+2) \text{ in } \mathcal{Q}'. \end{cases}$$

Using the definition of the function δ, it is now easy to check that

$$\tilde{T}_{i_1}\tilde{T}_{i_2} \dots \tilde{T}_{i_{m-1}}(E_{i_m}) = \sum_{\mathcal{Q}'} (-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}')} E_{\mathcal{Q}'}.$$

This ends the proof in case (2).

The case (3): $i_1 = b$ is proved in the same manner as in case (2).

We now consider the case (4): $i_1 = (a-1)$. In this case $a' = (a-1)$ and $b' = b$. So

$$\tilde{T}_{i_2}\tilde{T}_{i_3} \dots \tilde{T}_{i_{m-1}}(E_{i_m}) = \sum_{\mathcal{Q}'_1} (-v)^{-\delta(\mathcal{Q}_1[a-1,b],\mathcal{Q}'_1)} E_{\mathcal{Q}'_1}$$

where the sum is over all quivers $\mathcal{Q}'_1$ of $\Delta[a-1,b]$. If $\mathcal{P}$ is a quiver whose graph is $\Delta[a,b]$ and we consider all the quivers $\mathcal{Q}'_1$ of $\Delta[a-1,b]$ such that $\mathcal{Q}'_1[a,b] = \mathcal{P}$, then there are two such quivers corresponding to the possible orientation of the edge $\{a-1,a\}$ and, for these, using the definition of $E_{\mathcal{Q}'_1}$, we get that

$$E_{\mathcal{Q}'_1} = \begin{cases} E_{\mathcal{Q}'_1[a+1,b]} E_a E_{a-1}, & \text{if } (a-1) \leftarrow a \leftarrow (a+1) \text{ in } \mathcal{Q}'_1, \\ E_{\mathcal{Q}'_1[a+1,b]} E_{a-1} E_a, & \text{if } (a-1) \rightarrow a \leftarrow (a+1) \text{ in } \mathcal{Q}'_1, \\ E_a E_{a-1} E_{\mathcal{Q}'_1[a+1,b]}, & \text{if } (a-1) \leftarrow a \rightarrow (a+1) \text{ in } \mathcal{Q}'_1, \\ E_{a-1} E_a E_{\mathcal{Q}'_1[a+1,b]}, & \text{if } (a-1) \rightarrow a \rightarrow (a+1) \text{ in } \mathcal{Q}'_1. \end{cases}$$

Thus

$$\begin{aligned} &\sum_{\mathcal{Q}'_1} (-v)^{-\delta(\mathcal{Q}_1[a-1,b],\mathcal{Q}'_1)} E_{\mathcal{Q}'_1} \\ &\qquad = \begin{cases} (-v)^{-\sigma} E_{\mathcal{Q}'_1[a+1,b]} \ (-v^{-1}E_aE_{a-1} + E_{a-1}E_a) & \text{if } a \leftarrow (a+1) \text{ in } \mathcal{P}, \\ (-v)^{-\sigma} (-v^{-1}E_aE_{a-1} + E_{a-1}E_a) \ E_{\mathcal{Q}'_1[a+1,b]} & \text{if } a \rightarrow (a+1) \text{ in } \mathcal{P} \end{cases} \\ &\qquad = \begin{cases} (-v)^{-\sigma} E_{\mathcal{Q}'_1[a+1,b]} \ \tilde{T}_a(E_{a-1}) & \text{if } a \leftarrow (a+1) \text{ in } \mathcal{P}, \\ (-v)^{-\sigma} \tilde{T}_a(E_{a-1}) \ E_{\mathcal{Q}'_1[a+1,b]} & \text{if } a \rightarrow (a+1) \text{ in } \mathcal{P}. \end{cases} \end{aligned}$$

where the sum is over the two quivers $\mathcal{Q}_1'$ of $\Delta[a-1,b]$ such that $\mathcal{Q}_1'[a,b]=\mathcal{P}$ and $\sigma=\delta(\mathcal{Q}_1[a,b],\mathcal{Q}_1'[a,b])$. Note that $\tilde{T}_a(E_{a-1})=\tilde{T}_{a-1}^{-1}(E_a)$. So we get that

$$\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a-1,b],\mathcal{Q}_1')}\tilde{T}_{i_1}(E_{\mathcal{Q}_1'})$$
$$=\begin{cases}(-v)^{-\sigma}E_{\mathcal{Q}_1'[a+1,b]}\ E_a & \text{if } a\leftarrow(a+1) \text{ in } \mathcal{P}\\ (-v)^{-\sigma}E_a\ E_{\mathcal{Q}_1'[a+1,b]} & \text{if } a\rightarrow(a+1) \text{ in } \mathcal{P}\end{cases}=(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{P})}E_{\mathcal{P}}$$

where the sum is over the two quivers $\mathcal{Q}_1'$ of $\Delta[a-1,b]$ such that $\mathcal{Q}_1'[a,b]=\mathcal{P}$ and σ is as above. We get that

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m})=\sum_{\mathcal{P}}\sum_{\mathcal{Q}_1'}(-v)^{-\delta(\mathcal{Q}_1[a-1,b],\mathcal{Q}_1')}T_{i_1}(E_{\mathcal{Q}_1'})$$
$$=\sum_{\mathcal{P}}(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{P})}E_{\mathcal{P}}$$

where $\mathcal{P}$ runs in the sum $\sum_{\mathcal{P}}$ over all quivers of the graph $\Delta[a,b]$ and $\mathcal{Q}_1'$ runs in the sum $\sum_{\mathcal{Q}_1'}$ over all quivers $\mathcal{Q}_1'$ of $\Delta[a-1,b]$ such that $\mathcal{Q}_1'[a,b]=\mathcal{P}$. This ends the proof in case (4).

The case (5): $i_1=(b+1)$ is proved in the same manner as in case (4). The case (6) is obvious. □

Proposition 2.3 (With the same notation as in 2.2)**.** *Let $\mathcal{Q}$ be a quiver of Δ and $\mathbf{i}=(i_1,i_2,\dots,i_\nu)\in\mathcal{I}$ adapted to $\mathcal{Q}$. For $1\le a\le b\le n$, denote by $\Delta[a,b]$ the full subgraph of Δ whose set of vertices is $[a,b]$ and by $\mathcal{Q}[a,b]$ the subquiver of $\mathcal{Q}$ whose graph is $\Delta[a,b]$. If $\alpha(\mathbf{i},k)=s_{i_1}s_{i_2}\dots s_{i_{k-1}}(\alpha_{i_k})=\alpha_a+\alpha_{a+1}+\dots+\alpha_b$, then*

$$\overline{\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})}=\sum_J(v^{-1}-v)^{|J|}E_{\mathbf{i}}^{\mathbf{c}(J)},$$

where the sum is over all subsets J of the set of edges of $\Delta[a,b]$, $\mathbf{c}(J)\in\mathbf{N}^\nu$ is defined by $\mathbf{c}(J)=(c_1(J),c_2(J),\dots,c_\nu(J))$ with $c_m(J)\in\{0,1\}$ and $c_m(J)=1$ if and only if $\operatorname{Supp}(\alpha(\mathbf{i},m))$ *is one of the connected components of $\Delta[a,b]$ minus the edges in J.*

Proof. $E_{\mathbf{i}}^{\mathbf{c}(J)}$ is a product of root vectors relative to $\mathbf{i}$. Using proposition 2.2, we can express each of these factors as a sum of monomials in the E_i and consequently, $E_{\mathbf{i}}^{\mathbf{c}(J)}$ is a sum of monomials $E_{\mathcal{Q}'}$ where $\mathcal{Q}'$ is a quiver of $\Delta[a,b]$. Write $\mathbf{c}(J)=(c_1,c_2,\dots,c_\nu)$ and let $a\le i<j\le b$ be such that $\{i,j\}$ is an edge in J. Then there exist $1\le k\ne k'\le\nu$ such that $c_k=c_{k'}=1$, $i\in\operatorname{Supp}(\alpha(\mathbf{i},k))$ and $j\in\operatorname{Supp}(\alpha(\mathbf{i},k'))$. In this case, $(\alpha(\mathbf{i},k),\alpha(\mathbf{i},k'))=-1$ and $\alpha(\mathbf{i},k)+\alpha(\mathbf{i},k')\in R^+$. If $i\to j$ in $\mathcal{Q}[a,b]$, then we easily get that there exists a short exact sequence $0\to\mathbf{e}_{\alpha(\mathbf{i},k')}\to\mathbf{e}_{\alpha(\mathbf{i},k)+\alpha(\mathbf{i},k')}\to\mathbf{e}_{\alpha(\mathbf{i},k)}\to 0$ of modules in $\operatorname{Mod}(\mathcal{Q})$. By applying lemma 1.14 (a), we get that $k'<k$. Consquently, when we express $E_{\mathbf{i}}^{\mathbf{c}(J)}$ as a sum of monomials $E_{\mathcal{Q}'}$ using proposition 2.2, we see that E_j will appear on the left of E_i in $E_{\mathcal{Q}'}$. If $i\leftarrow j$ in $\mathcal{Q}[a,b]$, then we easily get that there exists a short exact sequence $0\to\mathbf{e}_{\alpha(\mathbf{i},k)}\to\mathbf{e}_{\alpha(\mathbf{i},k)+\alpha(\mathbf{i},k')}\to\mathbf{e}_{\alpha(\mathbf{i},k')}\to 0$ of modules in $\operatorname{Mod}(\mathcal{Q})$. By applying lemma 1.14 (a), we get that $k<k'$. Consequently, when we express $E_{\mathbf{i}}^{\mathbf{c}(J)}$ as a sum of monomials $E_{\mathcal{Q}'}$ using proposition 2.2 we see that E_j will appear on the right of E_i in $E_{\mathcal{Q}'}$. Consequently, the monomial $E_{\mathcal{Q}'}$ appears in this sum for $E_{\mathbf{i}}^{\mathbf{c}(J)}$ with a nonzero coefficient if and only if the edges in J have opposite

orientation in $\mathcal{Q}[a,b]$ and $\mathcal{Q}'$ and, in this case, it is easy to see that this coefficient is $(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}')\,+\,|J|}$ by applying proposition 2.2. Thus

$$(2)\qquad \sum_J (v^{-1}-v)^{|J|} E_{\mathbf{i}}^{\mathbf{c}(J)} = \sum_J (v^{-1}-v)^{|J|} \sum_{\mathcal{Q}'} (-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}')\,+\,|J|} E_{\mathcal{Q}'}$$

where J runs in the sum $\sum_J$ over all subsets of the set of edges of $\Delta[a,b]$ and $\mathcal{Q}'$ runs in the sum $\sum_{\mathcal{Q}'}$ over all quivers of $\Delta[a,b]$ for which the edges in J have opposite orientation in $\mathcal{Q}[a,b]$ and in $\mathcal{Q}'$.

Clearly the right-hand side of equation (2) is equal to

$$\sum_{\mathcal{Q}''}\left[\sum_K (v^{-1}-v)^{|K|}(-v)^{|K|}\right](-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}'')} E_{\mathcal{Q}''}$$

where $\mathcal{Q}''$ runs in the sum $\sum_{\mathcal{Q}''}$ over all quivers of the graph $\Delta[a,b]$ and K runs in the sum $\sum_K$ over all the subsets of the set of edges of $\Delta[a,b]$ that have an opposite orientation in $\mathcal{Q}[a,b]$ and in $\mathcal{Q}''$.

Note that

$$\sum_K (v^{-1}-v)^{|K|}(-v)^{|K|} = \sum_K (v^2-1)^{|K|} = \sum_{i=0}^{m}\binom{m}{i}(v^2-1)^i = v^{2m} = v^{2\,\delta(\mathcal{Q}[a,b],\mathcal{Q}'')}$$

where K runs in the sum $\sum_K$ over all the subsets of the set of edges of $\Delta[a,b]$ that have an opposite orientation in $\mathcal{Q}[a,b]$ and in $\mathcal{Q}''$ and m is the number of such edges.

Consequently, equation (2) becomes

$$\begin{aligned}\sum_J (v^{-1}-v)^{|J|} E_{\mathbf{i}}^{\mathbf{c}(J)} &= \sum_{\mathcal{Q}''} v^{2\,\delta(\mathcal{Q}[a,b],\mathcal{Q}'')}(-v)^{-\delta(\mathcal{Q}[a,b],\mathcal{Q}'')} E_{\mathcal{Q}''}\\ &= \sum_{\mathcal{Q}''}(-v)^{\delta(\mathcal{Q}[a,b],\mathcal{Q}'')} E_{\mathcal{Q}''} = \overline{\tilde{T}_{i_1}\tilde{T}_{i_2}\ldots\tilde{T}_{i_{k-1}}(E_{i_k})}\end{aligned}$$

where J runs in the sum $\sum_J$ over all subsets of the set of edges of $\Delta[a,b]$ and $\mathcal{Q}''$ runs in the sum $\sum_{\mathcal{Q}''}$ over all quivers of the graph $\Delta[a,b]$. The last equation follows by applying the bar involution to the formula in proposition 2.2. □

3. Commutation relations between root vectors

3.1. We will describe in this section the commutation relations between root vectors relative to $\mathbf{i}\in\mathcal{I}$ adapted to a quiver $\mathcal{Q}$. First we will recall Ringel's Hall algebra approach to quantized enveloping algebras. We will follow the presentation given by Lusztig in [10] for this. In the case A_n, we can use this approach to give the commutation relations between root vectors.

3.2. Let $\mathbf{F}_q$ be a finite field with q elements. $\mathcal{Q}$ will denote a quiver whose underlying graph is the Dynkin graph of type A_n and $\mathbf{i}=(i_1,i_2,\ldots,i_\nu)\in\mathcal{I}$ is adapted to $\mathcal{Q}$. Let $\mathbf{z}=(z_1,z_2,\ldots,z_n)\in\mathbf{Z}^n$ be such that $z_i-z_j=1$ whenever $i\to j$ is an arrow in $\mathcal{Q}$. There is such a $\mathbf{z}$.

From now on, $\mathbf{b}(k)$ will denote the vector $(0,\ldots,0,1,0,\ldots,0)\in\mathbf{N}^\nu$ whose only nonzero component is in the kth column and is 1, where $1\le k\le\nu$.

3.3. $R\mathcal{Q}$ is defined to be the $\mathbf{C}$-vector space with basis $([\mathbf{V}])$ indexed by the isomorphism classes $[\mathbf{V}]$ of modules $\mathbf{V}$ of $\mathcal{Q}$ over $\mathbf{F}_q$ with the $\mathbf{C}$-algebra structure given by

$$[\mathbf{V}'] \cdot [\mathbf{V}''] = \sum_{[\mathbf{V}]} g_{\mathbf{V},\mathbf{V}',\mathbf{V}''}\ [\mathbf{V}]$$

where the sum is over all isomorphism classes $[\mathbf{V}]$ of modules $\mathbf{V}$ of $\mathcal{Q}$, and $g_{\mathbf{V},\mathbf{V}',\mathbf{V}''}$ is the number of submodules of $\mathbf{V}$ that are isomorphic to $\mathbf{V}''$ and are such that the corresponding quotient module is isomorphic to $\mathbf{V}'$. Note that the sum is finite because $\dim([\mathbf{V}]) = \dim([\mathbf{V}']) + \dim([\mathbf{V}''])$ whenever $g_{\mathbf{V},\mathbf{V}',\mathbf{V}''} \neq 0$.

3.4. We can view $\mathbf{C}$ as a $\mathbf{Z}[v, v^{-1}]$-algebra with v acting as multiplication by a fixed square root $q^{1/2}$ of q. Let U_q, U_q^+ be the $\mathbf{C}$-algebras obtained respectively from U, U^+ by extension of scalar.

U_q^0 will denote the subalgebra of U_q generated by K_i, K_i^{-1} for $1 \leq i \leq n$. Let $\widetilde{R\mathcal{Q}} = U_q^0 \otimes_{\mathbf{C}} R\mathcal{Q}$. $\widetilde{R\mathcal{Q}}$ is an associative algebra containing U_q^0 and $R\mathcal{Q}$ as subalgebras with the commutation rule

$$K_i\ [\mathbf{V}]\ K_i^{-1} = q^{\sum_j d_j(\alpha_i,\alpha_j)/2}[\mathbf{V}]$$

where $\mathbf{V}$ is a module of $\mathcal{Q}$ of dimension $\dim(\mathbf{V}) = (d_1, d_2, \ldots, d_n)$.

Let $U_q^{\geq 0} = U_q^0 \otimes_{\mathbf{C}} U_q^+$. $U_q^{\geq 0}$ can be identified with the subalgebra of U_q generated by U_q^0 and U_q^+.

For any $N \in \mathbf{N}$ and $\alpha \in R^+$, let

$$[[N]]! = \prod_{s=1}^{N} \frac{(q^s - 1)}{(q-1)} \qquad \text{and} \qquad [\mathbf{e}_\alpha]^{((N))} = \frac{1}{[[N]]!}[\mathbf{e}_\alpha]^N.$$

Proposition 3.5. (a) (Ringel) *There is a unique algebra isomorphism* $U_q^{\geq 0} \to \widetilde{R\mathcal{Q}}$ *such that* $K_i \mapsto K_i$, $K_i^{-1} \mapsto K_i^{-1}$, $E_i \mapsto K_i^{z_i}[\mathbf{e}_i]$, *where* $\mathbf{e}_i$ *is the simple module of* $\mathcal{Q}$ *whose dimension* $\dim(\mathbf{e}_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$ *has only one nonzero component in the* ith *column and is equal to 1. In particular, there is an imbedding of algebras* $\Xi : R\mathcal{Q} \hookrightarrow U_q$ *such that* $\Xi([\mathbf{e}_j]) = K_j^{-z_j} E_j$ *for all* $1 \leq j \leq n$.

(b) *For* $\mathbf{c} \in \mathbf{N}^\nu$, *we have in* $R\mathcal{Q}$ *that*

$$[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}]^{((c_1))}[\mathbf{e}_{\alpha(\mathbf{i},2)}]^{((c_2))} \ldots [\mathbf{e}_{\alpha(\mathbf{i},\nu)}]^{((c_\nu))}.$$

(c) *Let* $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ *be of* $\mathbf{i}$-*homogeneity* $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbf{N}^n$. *Let* $\mathbf{c}_k \in \mathbf{N}^\nu$ *be such that its* kth *coordinate is* c_k *and its other coordinates are zero. Let* $\mathbf{d}^k = (d_1^k, d_2^k, \ldots, d_n^k)$ *be the* $\mathbf{i}$-*homogeneity of* $\mathbf{c}_k$. *Let* $r_{\mathbf{d}} = \sum_i d_i\,(d_i - 1)\,(2z_i - 1)/2 - \sum_{i\to j} d_i\, d_j\, z_j$, $\delta(\mathbf{c}) = -\sum_{h<k,i} d_i^h\, d_i^k + \sum_{h<k,i\to j} d_j^h\, d_i^k$, $f_{\mathbf{c}} = r_{\mathbf{d}} - \delta(\mathbf{c})$ *and* $K(\mathbf{d}) = K_1^{-z_1 d_1} K_2^{-z_2 d_2} \ldots K_n^{-z_n d_n}$. *Then* $\Xi([\mathbf{V_c}]) = q^{f_{\mathbf{c}}/2}\, K(\mathbf{d})\, E_{\mathbf{i}}^{\mathbf{c}}$.

Proof. For a proof of (a), see proposition 5.7 in [10]. For (b), see the proof of lemma 5.4 in [10]. Finally for (c), see 7.13 in [10]. □

3.6. In the following, we will describe the product $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ in $R\mathcal{Q}$ for $k > 1$. This will be presented in the next four lemmas. First we can make some general observations.

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbf{N}^n$ such that $\sum_{i=1}^n d_i\, \alpha_i = \alpha(\mathbf{i}, k) + \alpha(\mathbf{i}, 1)$. Let $1 \leq a \leq b \leq n$ such that $\mathrm{Supp}(\alpha(\mathbf{i}, k)) = [a, b]$. We also have $\alpha(\mathbf{i}, 1) = \alpha_{i_1}$. Note that if the isomorphism class $[\mathbf{V_c}]$ with $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$ appears in the product $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ with nonzero coefficient, then $\mathbf{c}$ has $\mathbf{i}$-homogeneity $\mathbf{d}$. This follows easily from the definition of the product in $R\mathcal{Q}$. Note also that, if $c_j \neq 0$ and i_1 is not in the support $\mathrm{Supp}(\alpha(\mathbf{i}, j))$ of $\alpha(\mathbf{i}, j)$, then $j = k$. For otherwise, if $\mathbf{M}$ is a submodule of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$, then we get that $\mathbf{V_c}/\mathbf{M}$ is not isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ because it has an indecomposable direct summand isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},j)}$.

Lemma 3.7 (With the notation of 3.6)**.** *If $i_1 < (a-1)$ or $i_1 > (b+1)$, then*

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}].$$

Proof. By our observations in 3.6, the fact that the support of a root is a connected subset of $\{1, 2, \dots, n\}$ and the fact that $i_1 < (a-1)$ or $i_1 > (b+1)$, we get that if the isomorphism class $[\mathbf{V_c}]$ appears with nonzero coefficient in the product $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$, then $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$ must be

$$c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise.} \end{cases}$$

We must now count how many submodules $\mathbf{M}$ of $\mathbf{V_c}$ are isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and are such that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$. Here it is obvious that there is only one submodule $\mathbf{M}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and, for this unique submodule, $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$. Thus the coefficient of $[\mathbf{V_c}]$ in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is 1. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}]$. This proves the lemma. □

Lemma 3.8 (With the notation of 3.6)**.** *If $i_1 = a$ or $i_1 = b$, then*

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = q\ [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}].$$

Proof. We will prove the lemma in the case $i_1 = a$. The other case $i_1 = b$ can be treated similarly. By our observations in 3.6, the fact that the support of a root is connected and the $\mathbf{i}$-homogeneity $\mathbf{d} = (d_1, d_2, \dots, d_n)$ is given by

$$d_i = \begin{cases} 2, & \text{if } i = a, \\ 1, & \text{if } a < i \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Then, if the isomorphism class $[\mathbf{V_c}]$ appears with nonzero coefficient in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$, we get that $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$ must be

$$c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise.} \end{cases}$$

Let us write $(\mathbf{V_c}) = (V_i, f_{ij})$ for the rest of this proof. Then V_{i_1} has dimension 2 over $\mathbf{F}_q$. We know that $(\mathbf{e}_{\alpha(\mathbf{i},1)})_i = 0$ for all $i = 1, 2, \dots, n$ except when $i = i_1$ in which case it is $\mathbf{F}_q$. We see easily that each 1-dimensional vector subspace of V_{i_1} gives rise to a submodule of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and all submodules of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ are of this form. So there are $(q+1)$ submodules $\mathbf{M}$ of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$. For each of these submodules $\mathbf{M}$, we get that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ except for one of them: the submodule corresponding to the image of $f_{(i_1+1)\ i_1} : V_{i_1+1} \to V_{i_1}$. Note that in our situation, $f_{(i_1+1)\ i_1}$ has rank 1 and its image is a 1-dimensional subspace. Thus the coefficient of $[\mathbf{V_c}]$ in

$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is q. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}]$. This proves the lemma. □

Lemma 3.9 (With the notation of 3.6)**.** *If* $i_1 = (a-1)$ *or* $i_1 = (b+1)$*, then*

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}] \ + \ [\mathbf{e}_{\alpha(\mathbf{i},m)}]$$

where $\alpha(\mathbf{i},m) = \alpha(\mathbf{i},1) + \alpha(\mathbf{i},k)$*. (There is such an* $m \neq 1,k$*, because the sum* $\alpha(\mathbf{i},1) + \alpha(\mathbf{i},k)$ *is a positive root in this case.)*

Proof. We will prove the lemma in the case $i_1 = (a-1)$. The other case $i_1 = (b+1)$ can be treated similarly. By our observations in 3.6, the fact that the support of a root is connected and the $\mathbf{i}$-homogeneity $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is given by

$$d_i = \begin{cases} 1, & \text{if } (a-1) \leq i \leq b, \\ 0, & \text{otherwise;} \end{cases}$$

then, if the isomorphism class $[\mathbf{V_c}]$ appears with nonzero coefficient in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$, we get that there are only two possibilities for $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$:

$$\text{either} \quad c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise} \end{cases} \qquad \text{or} \qquad c_j = \begin{cases} 1, & \text{if } j = m, \\ 0, & \text{otherwise.} \end{cases}$$

Here m is defined above.

Let us write $(\mathbf{V_c}) = (V_i, f_{ij})$ for the rest of this proof.

In the first possibility, i.e.,

$$c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise;} \end{cases}$$

then we have $V_{i_1} = \mathbf{F}_q$ and $f_{(i_1+1)\ i_1} : V_{i_1+1} \to V_{i_1}$ is 0. Thus there is a unique submodule $\mathbf{M}$ of $\mathbf{V_c}$ such that $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and for this unique submodule, we have that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$. In other words, the coefficient of $[\mathbf{V_c}]$ in the product $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is 1. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}]$.

In the second possibility, i.e.,

$$c_j = \begin{cases} 1, & \text{if } j = m, \\ 0, & \text{otherwise;} \end{cases}$$

then we have $V_{i_1} = \mathbf{F}_q$ and $f_{(i_1+1)\ i_1} : V_{i_1+1} \to V_{i_1}$ is an isomorphism. Thus there is a unique submodule $\mathbf{M}$ of $\mathbf{V_c}$ such that $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and for this unique submodule, we have that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$. In other words, the coefficient of $[\mathbf{V_c}]$ in the product $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is 1. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},m)}]$. This completes the proof. □

Lemma 3.10 (With the notation of 3.6)**.** *If* $a < i_1 < b$*, then*

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = q\ [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}] \ + \ (q-1)\ [\mathbf{e}_{\alpha(\mathbf{i},m)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},m')}]$$

where $\{\alpha(\mathbf{i},m), \alpha(\mathbf{i},m')\} = \{\alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1},\ \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b\}$ *with* $m < m'$*. (There are such integers* m*,* $m' \neq 1$*,* k*, with* $1 < m < m' < \nu$ *because both sums* $\alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1}$*,* $\alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b$ *are two distinct positive roots in this case.)*

Proof. By our observations in 3.6, the fact that the support of a root is connected and the $\mathbf{i}$-homogeneity $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ is given by

$$d_i = \begin{cases} 2, & \text{if } i = i_1, \\ 1, & \text{if } a \leq i \leq b \text{ and } i \neq i_1, \\ 0, & \text{otherwise;} \end{cases}$$

then, if the isomorphism class $[V_{\mathbf{c}}]$ appears with nonzero coefficient in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$, we get that there are only two possibilities for $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$:

$$\text{either} \quad c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad c_j = \begin{cases} 1, & \text{if } j = m \text{ or } m', \\ 0, & \text{otherwise.} \end{cases}$$

Here m, m' are defined above.

Let us write $(\mathbf{V_c}) = (V_i, f_{ij})$ for the rest of this proof.

In the first possibility, i.e.,

$$c_j = \begin{cases} 1, & \text{if } j = 1 \text{ or } k, \\ 0, & \text{otherwise;} \end{cases}$$

then we have that V_{i_1} has dimension 2 over $\mathbf{F}_q$, that both $f_{(i_1+1)i_1} : V_{i_1+1} \to V_{i_1}$ and $f_{(i_1-1)i_1} : V_{i_1-1} \to V_{i_1}$ are of rank 1 and have the same image in V_{i_1}. We see easily that each 1-dimensional vector subspace of V_{i_1} gives rise to a submodule of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and all submodules of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ are of this form. So there are $(q+1)$ submodules $\mathbf{M}$ of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$. For each of these submodules $\mathbf{M}$, we get that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ except for one of them: the submodule corresponding to the image of $f_{(i_1+1)i_1} : V_{i_1+1} \to V_{i_1}$, which is also the image of $f_{(i_1-1)i_1} : V_{i_1-1} \to V_{i_1}$. Thus the coefficient of $[\mathbf{V_c}]$ in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is q. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}]$.

In the second possibility, i.e.,

$$c_j = \begin{cases} 1, & \text{if } j = m \text{ or } m', \\ 0, & \text{otherwise;} \end{cases}$$

then we have that V_{i_1} has dimension 2 over $\mathbf{F}_q$, that both $f_{(i_1+1)i_1} : V_{i_1+1} \to V_{i_1}$ and $f_{(i_1-1)i_1} : V_{i_1-1} \to V_{i_1}$ are of rank 1, but the intersection of their images is the zero subspace 0. Easily we see that each 1-dimensional vector subspace of V_{i_1} gives rise to a submodule of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ and all submodules of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$ are of this form. So there are $(q+1)$ submodules $\mathbf{M}$ of $\mathbf{V_c}$ isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},1)}$. For each of these submodules $\mathbf{M}$, we get that $\mathbf{V_c}/\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ except for two of them: the submodules corresponding respectively to the image of $f_{(i_1+1)i_1} : V_{i_1+1} \to V_{i_1}$ and the image of $f_{(i_1-1)i_1} : V_{i_1-1} \to V_{i_1}$. Thus the coefficient of $[\mathbf{V_c}]$ in $[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}]$ is $(q-1)$. By proposition 3.5 (b), we get that $[\mathbf{V_c}] = [\mathbf{e}_{\alpha(\mathbf{i},m)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},m')}]$. □

To state the commutation relations between the root vectors relative to $\mathbf{i}$, we will also need to use the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ of $\mathcal{Q}$. We will now recall some notation and results on $\Gamma_{\mathcal{Q}}$. For this theory, we refer the reader either to section 6.5 in [7] or to section 2.2 in [11] or to the book [2]. We will restrict ourselves to the type A_n. F will denote an algebraically closed field.

3.11. The vertices of the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ are the isomorphism classes of indecomposable modules of the quiver $\mathcal{Q}$ over F and two isomorphism classes $[\mathbf{V}]$ and $[\mathbf{W}]$ of indecomposable modules of $\mathcal{Q}$ are linked by an arrow $[\mathbf{V}] \to [\mathbf{W}]$ in $\Gamma_{\mathcal{Q}}$ if and only if there exists an irreducible morphism $\mathbf{V} \to \mathbf{W}$ in $\mathrm{Mod}(\mathcal{Q})$.

As we saw above, the set of isomorphism classes of indecomposable modules of $\mathcal{Q}$ is in bijection with R^+ and we will represent below each vertex $[\mathbf{e}_\alpha]$ of $\Gamma_{\mathcal{Q}}$ by simply writing the corresponding positive root $\alpha = \dim(\mathbf{e}_\alpha)$. We won't need to explicitly determine the irreducible morphisms between two vertices that are linked together in $\Gamma_{\mathcal{Q}}$; we will just draw the arrow in $\Gamma_{\mathcal{Q}}$ corresponding to the fact that there are irreducible morphisms.

The Auslander-Reiten quiver can be computed in a very combinatorial way using the dimension type of the indecomposable projective modules and the additivity property of the dimension types on the Auslander-Reiten sequences.

Let $\mathbf{N}\mathcal{Q}$ be the following quiver: its set of vertices is $\mathbf{N} \times \{1, 2, \ldots, n\}$ and, whenever there is an arrow $i \leftarrow j$ in $\mathcal{Q}$, we draw one arrow $(z, i) \to (z, j)$ and one arrow $(z, j) \to (z + 1, i)$ for each $z \in \mathbf{N}$. Define $A(\mathcal{Q})$ as the full subquiver of $\mathbf{N}\mathcal{Q}$ of all vertices (z, i) such that $1 \leq z \leq (n + 1 + a_i - b_i)/2$ where, for each $i \in \{1, 2, \ldots, n\}$, a_i (respectively b_i) is the number of arrows in the unoriented path from i to $(n + 1 - i)$ that are directed towards i (respectively $(n + 1 - i)$).

There is a unique isomorphism $\Psi\colon \Gamma_{\mathcal{Q}} \to A(\mathcal{Q})$ of quivers such that $\Psi([\mathbf{P}(k)]) = (1, k)$ for each $k \in \{1, 2, \ldots, n\}$. From the dimension types of the indecomposable projective modules, we can then easily compute $\Gamma_{\mathcal{Q}}$ using this isomorphism Ψ and the additivity property of the dimension on the Auslander-Reiten sequences.

Let $\mathbf{Z}\Delta$ denote the translation quiver associated to the Dynkin graph Δ as presented in figure 13 of section 6.5 of [7]. Note that this implies a choice of indices for the vertices of Δ. Recall that the set of vertices of $\mathbf{Z}\Delta$ is $\mathbf{Z} \times \{1, 2, \ldots, n\}$ and the arrows are $(z, i) \to (z, i + 1)$ for $z \in \mathbf{Z}$, $1 \leq i < n$ and $(z, i) \to (z + 1, i - 1)$ for $z \in \mathbf{Z}$ and $1 < i \leq n$. The translation τ is the function on the set of vertices of $\mathbf{Z}\Delta$ defined by $\tau(z, i) = (z - 1, i)$. There is a unique embedding Θ of $\Gamma_{\mathcal{Q}}$ (or $A(\mathcal{Q})$ under the isomorphism Ψ) into $\mathbf{Z}\Delta$ such that $[\mathbf{P}(1)] = \Psi^{-1}(1, 1)$ is mapped to the vertex $(1, 1)$ of $\mathbf{Z}\Delta$. In particular, $\Theta([\mathbf{P}(k)]) = (1 - b'_k,\ k)$ where b'_k is the number of arrows in the unoriented path from 1 to k that are directed toward k.

$\Gamma_{\mathcal{Q}}$ comes equipped with a translation $\tau = D \circ Tr$ where Tr is the transpose (see chapter IV of [2] for the definition) and D is a duality (see chapter II of [2] for the definition). We will just list some of the properties of τ that are verified in our situation.

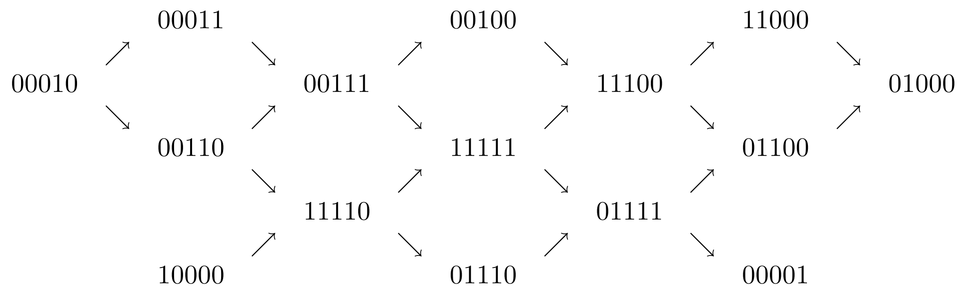
(a) If $\mathbf{P}$ is a projective module, then $\tau(\mathbf{P}) = 0$.
(b) If $\mathbf{V}$ and $\mathbf{W}$ are modules of $\mathcal{Q}$ without projective summands, then $\mathbf{V}$ and $\mathbf{W}$ are isomorphic if and only if $\tau(\mathbf{V})$ and $\tau(\mathbf{W})$ are isomorphic.
(c) $\tau(\bigoplus_{i=1}^m \mathbf{V}(i))$ is isomorphic to $\bigoplus_{i=1}^m \tau(\mathbf{V}(i))$ where $\mathbf{V}(1), \mathbf{V}(2), \ldots, \mathbf{V}(m)$ are modules of $\mathcal{Q}$.
(d) τ induces a bijection $[\mathbf{V}] \mapsto [\tau(\mathbf{V})]$ (also denoted by τ) from the set of indecomposable nonprojective modules of $\mathcal{Q}$ into the set of indecomposable noninjective modules of $\mathcal{Q}$ with $Tr \circ D$ as inverse.
(e) If $\mathbf{V}$ is an indecomposable nonprojective module of $\mathcal{Q}$ and $\Theta([\mathbf{V}]) = (k, i)$ for some $i \in \{1, 2, \ldots, n\}$ and $k \in \mathbf{Z}$, where Θ is the unique embedding of $\Gamma_{\mathcal{Q}}$ into $\mathbf{Z}\Delta$ defined above, then $\Theta(\tau[\mathbf{V}]) = (k - 1, i)$. In other words, the τ defined by $D \circ Tr$ on $\Gamma_{\mathcal{Q}}$ is the restriction of the τ defined above on $\mathbf{Z}\Delta$.

(f) If $\mathbf{V}$ and $\mathbf{W}$ are two modules for $\mathcal{Q}$, then

$$\dim(\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{V},\mathbf{W})) = \dim(\mathrm{Hom}_{\mathcal{Q}}(\mathbf{W},\tau(\mathbf{V}))).$$

In the example below, we write the root $\alpha = \sum_{i=1}^{n} d_i\alpha_i$ by simply displaying the values $(d_1, d_2, \ldots, d_n)$ in the same pattern as the Dynkin graph Δ and we have identified $\alpha \in R^+$ with the vertex $[\mathbf{e}_\alpha]$ of $\Gamma_{\mathcal{Q}}$.

Example 3.12. For the quiver $\mathcal{Q}$: $1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5$ with underlying graph of type A_5, the Auslander-Reiten quiver is



3.13. Let α, $\beta \in R^+$. Write $\Theta([\mathbf{e}_\alpha]) = (y, j)$ and $\Theta([\mathbf{e}_\beta]) = (x, i)$ with $x, y \in \mathbf{Z}$ and $i, j \in [1, n]$. The following facts are well known:

(a) $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathcal{Q}} \in \{0, 1\}$ and $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathcal{Q}} = 1$ if and only if $x - (n-i) \le y \le x$ and $x + 1 \le y + j \le x + i$.

(b) $\langle \mathbf{e}_\beta, \mathbf{e}_\alpha \rangle^1_{\mathcal{Q}} \in \{0, 1\}$ and $\langle \mathbf{e}_\beta, \mathbf{e}_\alpha \rangle^1_{\mathcal{Q}} = 1$ if and only if $x-1-(n-i) \le y \le x-1$ and $x \le y + j \le x - 1 + i$ excluding the case where $y = x - 1 - (n-i)$ and $y + j = x$ simultanously.

(c) If $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathcal{Q}} = 0$ and $\langle \mathbf{e}_\beta, \mathbf{e}_\alpha \rangle^1_{\mathcal{Q}} = 1$, i.e. either ($y = x - 1 + (n-i)$ and $x < y + j \le x - 1 + i$) or ($x - 1 - (n-i) < y \le x - 1$ and $y + j = x$), then $\alpha + \beta \in R^+$ and there exists a short exact sequence of the form $0 \to \mathbf{e}_\alpha \to \mathbf{e}_{\alpha+\beta} \to \mathbf{e}_\beta \to 0$ that is a basis of $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{e}_\beta, \mathbf{e}_\alpha)$. Also $\Theta([\mathbf{e}_{\alpha+\beta}])$ is equal to

$$\begin{cases} (x,\ i+j-n-1), & \text{if } y = x-1-(n-i) \text{ and } x < y+j \le x-1+i, \\ (x-j,\ i+j), & \text{if } x-1-(n-i) < y \le (x-1) \text{ and } y+j = x. \end{cases}$$

Moreover, if $0 \to \mathbf{e}_\alpha \to \mathbf{V} \to \mathbf{e}_\beta \to 0$ is a nonsplit short exact sequence, then $\mathbf{V}$ is isomorphic to $\mathbf{e}_{\alpha+\beta}$.

(d) If $\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle_{\mathcal{Q}} = 1$ and $\langle \mathbf{e}_\beta, \mathbf{e}_\alpha \rangle^1_{\mathcal{Q}} = 1$, i.e. $x - (n-i) \le y \le x - 1$ and $x + 1 \le y + j \le x - 1 + i$, then there is a unique pair of distinct positive roots γ, γ' such that $\alpha + \beta = \gamma + \gamma'$ and such that there exists a short exact sequence of the form $0 \to \mathbf{e}_\alpha \to \mathbf{e}_\gamma \oplus \mathbf{e}_{\gamma'} \to \mathbf{e}_\beta \to 0$ that is a basis of $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{e}_\beta, \mathbf{e}_\alpha)$. Also $\Theta([\mathbf{e}_\gamma]) = (x,\ y + j - x)$ and $\Theta([\mathbf{e}_{\gamma'}]) = (y,\ x + i - y)$. Moreover, if $0 \to \mathbf{e}_\alpha \to \mathbf{V} \to \mathbf{e}_\beta \to 0$ is a nonsplit short exact sequence, then $\mathbf{V}$ is isomorphic to $\mathbf{e}_\gamma \oplus \mathbf{e}_{\gamma'}$.

(e) For $k \in [1, n]$, then $\langle \mathbf{P}(k), \mathbf{e}_\alpha \rangle_{\mathcal{Q}} = 1$ if and only if $k \in \mathrm{Supp}(\alpha)$.

Proposition 3.14. *If $1 \le h < k \le \nu$ are such that $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$ and $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 0$, then*

$$E_{\mathbf{i}}^{\mathbf{b}(k)}\, E_{\mathbf{i}}^{\mathbf{b}(h)} = E_{\mathbf{i}}^{\mathbf{b}(h)}\, E_{\mathbf{i}}^{\mathbf{b}(k)}.$$

Proof. We will first consider the case where $h = 1$. So $\alpha(\mathbf{i}, h) = \alpha_{i_1}$. Let a, b be defined by $1 \leq a \leq b \leq n$ and $\alpha(\mathbf{i}, k) = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$. Since $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$, we must have $i_1 \leq (a-1)$ or $i_1 \geq (b+1)$. If $i_1 = (a-1)$, then it is easy to construct a nonsplit short exact sequence

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{e}_\beta \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

where $\beta = \alpha_{a-1} + \alpha_a + \cdots + \alpha_b$. But this is in contradiction with our hypothesis $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 0$. Thus $i_1 \neq (a-1)$. In the same way, we can prove that $i_1 \neq (b+1)$. The hypothesis of lemma 3.7 is satisfied and we have in $R\mathcal{Q}$ that

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}].$$

Applying the imbedding of algebras $\Xi : R\mathcal{Q} \hookrightarrow U_q$ to this equation, we get by proposition 3.5 (c) that

$$\begin{aligned} &q^{f_{\mathbf{b}(k)}/2} K_a^{-z_a} \ldots K_b^{-z_b} \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})\ q^{f_{\mathbf{b}(1)}/2} K_{i_1}^{-z_{i_1}} E_{i_1} \\ &\qquad = q^{f_{\mathbf{b}(1)}/2} K_{i_1}^{-z_{i_1}} E_{i_1}\ q^{f_{\mathbf{b}(k)}/2} K_a^{-z_a} \ldots K_b^{-z_b} \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k}). \end{aligned}$$

Consequently, $\tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1} = E_{i_1}\ \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})$ is verified in U_q for all q. From this, we get that $\tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1} = E_{i_1}\ \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})$ in $\mathbf{U}$.

We now consider the case where $h > 1$. Let $\mathbf{j} = (i_h, i_{h+1}, \ldots, i_\nu, j_1, j_2, \ldots, j_{h-1}) \in \mathcal{I}$ adapted to the quiver $\mathcal{Q}' = s_{i_{h-1}} \ldots s_{i_2} s_{i_1}(\mathcal{Q})$ where j_1, j_2, $\ldots$, j_{h-1} are defined by $w_0(\alpha_{i_1}) = -\alpha_{j_1}$, $w_0(\alpha_{i_2}) = -\alpha_{j_2}, \ldots, w_0(\alpha_{i_{h-1}}) = -\alpha_{j_{h-1}}$. In what follows, $\mathbf{e}'_\beta$ will denote an indecomposable module of $\mathcal{Q}'$ of dimension $\beta \in R^+$. By using repeatedly lemma 1.14, we get that $\langle \mathbf{e}'_{\alpha(\mathbf{j},1)}, \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)} \rangle_{\mathcal{Q}'} = 0$ and $\langle \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)}, \mathbf{e}'_{\alpha(\mathbf{j},1)} \rangle^1_{\mathcal{Q}'} = 0$. By applying the first part of our proof to $\mathbf{j}$ and $\mathcal{Q}'$, we get that

$$\tilde{T}_{i_h} \tilde{T}_{i_{h+1}} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_h} = E_{i_h}\ \tilde{T}_{i_h} \tilde{T}_{i_{h+1}} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})$$

in $\mathbf{U}$. Applying the automorphism $\tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{h-1}}$, we get

$$\begin{aligned} &\tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k})\ \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{h-1}}(E_{i_h}) \\ &\qquad = \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{h-1}}(E_{i_h})\ \tilde{T}_{i_1} \tilde{T}_{i_2} \ldots \tilde{T}_{i_{k-1}}(E_{i_k}) \end{aligned}$$

in $\mathbf{U}$. □

Proposition 3.15. *If* $1 \leq h < k \leq \nu$ *are such that* $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 1$ *and* $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 0$*, then*

$$E_{\mathbf{i}}^{\mathbf{b}(k)}\, E_{\mathbf{i}}^{\mathbf{b}(h)} = v\, E_{\mathbf{i}}^{\mathbf{b}(h)}\, E_{\mathbf{i}}^{\mathbf{b}(k)}.$$

Proof. We will first consider the case where $h = 1$. So $\alpha(\mathbf{i}, h) = \alpha_{i_1}$. Let a, b be defined by $1 \leq a \leq b \leq n$ and $\alpha(\mathbf{i}, k) = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$. Since $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 1$, we must have $a \leq i_1 \leq b$. If $a < i_1 < b$, then it is easy to construct a nonsplit short exact sequence

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{e}_\beta \oplus \mathbf{e}_{\beta'} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

with $\beta = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1}$ and $\beta' = \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b$. But this is in contradiction with our hypothesis $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 0$. If $i_1 = a$ or b, then it is easy to check that all short exact sequences

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{V} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

must split. The argument is similar to the one used in lemma 3.8. Thus $i_1 \in \{a, b\}$.

We will now assume that $i_1 = a$. The proof in the case $i_1 = b$ is analogous and is left to the reader. The hypothesis of lemma 3.8 is satisfied and we have in $R\mathcal{Q}$ that

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = q\,[\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}].$$

Applying the imbedding of algebras $\Xi : R\mathcal{Q} \hookrightarrow U_q$ to this equation, we get by proposition 3.5 (c) that

$$\begin{aligned} &q^{f_{\mathbf{b}(k)}/2} K_a^{-z_a} \cdots K_b^{-z_b} \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})\; q^{f_{\mathbf{b}(1)}/2} K_a^{-z_a} E_a \\ &\qquad = q\, q^{f_{\mathbf{b}(1)}/2} K_a^{-z_a} E_a\; q^{f_{\mathbf{b}(k)}/2}\; K_a^{-z_a} \dots K_b^{-z_b} \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k}). \end{aligned}$$

By the defining relations of U_q, we get that

$$\begin{aligned} &q^{(f_{\mathbf{b}(1)} + f_{\mathbf{b}(k)} + z_a)/2} K_a^{-z_a} \dots K_b^{-z_b} K_a^{-z_a} \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k}) E_a \\ &\qquad = q^{(f_{\mathbf{b}(1)} + f_{\mathbf{b}(k)} + 2z_a - z_{a+1} + 2)/2} K_a^{-z_a} K_a^{-z_a} \dots K_b^{-z_b} E_a\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k}). \end{aligned}$$

Since $a \leftarrow (a+1)$ in $\mathcal{Q}$, $i_1 = a$ being a sink, we have $z_{a+1} = z_a + 1$. Consequently, $\tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})\; E_{i_1} = q^{1/2} E_{i_1}\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})$ is verified in U_q for all q. From this, we get that $\tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})\; E_{i_1} = v\; E_{i_1}\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})$ in $\mathbf{U}$.

We now consider the case where $h > 1$. Let $\mathbf{j} = (i_h, i_{h+1}, \dots, i_\nu, j_1, j_2, \dots, j_{h-1}) \in \mathcal{I}$ adapted to the quiver $\mathcal{Q}' = s_{i_{h-1}} \dots s_{i_2} s_{i_1}(\mathcal{Q})$ where $j_1,\ j_2, \dots, j_{h-1}$ are defined by $w_0(\alpha_{i_1}) = -\alpha_{j_1},\ w_0(\alpha_{i_2}) = -\alpha_{j_2}, \dots, w_0(\alpha_{i_{h-1}}) = -\alpha_{j_{h-1}}$. In what follows, $\mathbf{e}'_\beta$ will denote an indecomposable module of $\mathcal{Q}'$ of dimension $\beta \in R^+$. By using repeatedly lemma 1.14, we get that $\langle \mathbf{e}'_{\alpha(\mathbf{j},1)},\, \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)} \rangle_{\mathcal{Q}'} = 1$ and $\langle \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)},\, \mathbf{e}'_{\alpha(\mathbf{j},1)} \rangle^1_{\mathcal{Q}'} = 0$. Note that we have also used 3.13 (a). By applying the first part of our proof but to $\mathbf{j}$ and $\mathcal{Q}'$, we get that

$$\tilde{T}_{i_h} \tilde{T}_{i_{h+1}} \dots \tilde{T}_{i_{k-1}}(E_{i_k})\; E_{i_h} = v\; E_{i_h}\; \tilde{T}_{i_h} \tilde{T}_{i_{h+1}} \dots \tilde{T}_{i_{k-1}}(E_{i_k})$$

in $\mathbf{U}$. Applying the automorphism $\tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{h-1}}$, we get

$$\begin{aligned} &\tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k})\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{h-1}}(E_{i_h}) \\ &\qquad\qquad = v\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{h-1}}(E_{i_h})\; \tilde{T}_{i_1} \tilde{T}_{i_2} \dots \tilde{T}_{i_{k-1}}(E_{i_k}) \end{aligned}$$

in $\mathbf{U}$. □

Proposition 3.16. *If $1 \le h < k \le \nu$ are such that $\langle \mathbf{e}_{\alpha(\mathbf{i},h)},\, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$ and $\langle \mathbf{e}_{\alpha(\mathbf{i},k)},\, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 1$, then*

$$E_{\mathbf{i}}^{\mathbf{b}(k)}\, E_{\mathbf{i}}^{\mathbf{b}(h)} = v^{-1}\, E_{\mathbf{i}}^{\mathbf{b}(h)}\, E_{\mathbf{i}}^{\mathbf{b}(k)} + E_{\mathbf{i}}^{\mathbf{b}(m)},$$

where m is defined by $\alpha(\mathbf{i}, m) = \alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k)$. Note that by 3.13 (c)*, $\alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k) \in R^+$, $h < m < k$ and the position $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}])$ in $\Gamma_{\mathcal{Q}}$ is known.*

Proof. We will first consider the case where $h = 1$. So $\alpha(\mathbf{i}, h) = \alpha_{i_1}$. Let a, b be defined by $1 \le a \le b \le n$ and $\alpha(\mathbf{i}, k) = \alpha_a + \alpha_{a+1} + \dots + \alpha_b$. Since $\langle \mathbf{e}_{\alpha(\mathbf{i},h)},\, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$, we must have $i_1 \le (a-1)$ or $i_1 \ge (b+1)$. If $i_1 < (a-1)$ or $i_1 > (b+1)$, then it is easy to check using dimension arguments that all short exact sequences

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{V} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

must split. Since $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)}\rangle^1_{\mathcal{Q}} = 1$, then we must exclude the cases where $i_1 < (a-1)$ or $i_1 > (b+1)$. If $i_1 = (a-1)$ or $(b+1)$, then it is easy to construct a nonsplit short exact sequence

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{e}_{\alpha(\mathbf{i},1)+\alpha(\mathbf{i},k)} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0.$$

Thus the only cases to consider are when $i_1 \in \{(a-1),(b+1)\}$. We will now assume that $i_1 = (a-1)$. The proof in the case $i_1 = (b+1)$ is analogous and is left to the reader. The hypothesis of lemma 3.9 is satisfied and we have in $R\mathcal{Q}$ that

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}]$$

where $\alpha(\mathbf{i},m) = \alpha(\mathbf{i},1) + \alpha(\mathbf{i},k) = \alpha_{a-1} + \alpha_a + \cdots + \alpha_b$. Applying the imbedding of algebras $\Xi : R\mathcal{Q} \hookrightarrow U_q$ to this equation, we get by proposition 3.5 (c) that

$$\begin{aligned} &q^{f_{\mathbf{b}(k)}/2}\ K_a^{-z_a} \dots K_b^{-z_b} \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ q^{f_{\mathbf{b}(1)}/2} K_{(a-1)}^{-z_{(a-1)}} E_{(a-1)} \\ &\qquad = \quad q^{f_{\mathbf{b}(1)}/2} K_{(a-1)}^{-z_{(a-1)}} E_{(a-1)}\ q^{f_{\mathbf{b}(k)}/2}\ K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) \\ &\qquad\quad + \quad q^{f_{\mathbf{b}(m)}/2}\ K_{(a-1)}^{-z_{(a-1)}} K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m}). \end{aligned}$$

By the defining relations of U_q, we get that

$$\begin{aligned} &q^{(f_{\mathbf{b}(1)}+f_{\mathbf{b}(k)}-z_{(a-1)})/2}\ K_{(a-1)}^{-z_{(a-1)}} K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{(a-1)} \\ &\qquad = \quad q^{(f_{\mathbf{b}(1)}+f_{\mathbf{b}(k)}-z_a)/2} K_{(a-1)}^{-z_{(a-1)}} K_a^{-z_a}\dots K_b^{-z_b} E_{(a-1)}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) \\ &\qquad\quad + \quad q^{f_{\mathbf{b}(m)}/2}\ K_{(a-1)}^{-z_{(a-1)}} K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m}). \end{aligned}$$

Since $(a-1) \leftarrow a$ in $\mathcal{Q}$, $i_1 = (a-1)$ being a sink, we have $z_a = z_{(a-1)} + 1$. Consequently,

$$\begin{aligned} &\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{(a-1)} \\ &\qquad = q^{-(1/2)} E_{(a-1)}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) + q^{N/2}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m}) \end{aligned}$$

where $N = f_{\mathbf{b}(m)} - f_{\mathbf{b}(1)} - f_{\mathbf{b}(k)} + z_{(a-1)}$. But by proposition 3.5 (c) and the fact that $(a-1)$ is a sink of $\mathcal{Q}$, we get

$$f_{\mathbf{b}(1)} = 0, \quad f_{\mathbf{b}(k)} = -\sum_{\substack{i\to j \\ a\le i,j\le b}} z_j, \quad f_{\mathbf{b}(m)} = -\sum_{\substack{i\to j \\ (a-1)\le i,j\le b}} z_j = f_{\mathbf{b}(k)} - z_{(a-1)},$$

and finally $N = 0$.

Thus

$$\begin{aligned} &\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{(a-1)} \\ &\qquad = q^{-(1/2)} E_{(a-1)}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) + \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m}) \end{aligned}$$

is verified in U_q for all q. From this, we can conclude that

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\, E_{(a-1)} = v^{-1} E_{(a-1)}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) + \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m-1}}(E_{i_m})$$

in $\mathbf{U}$.

We now consider the case where $h > 1$. Let $\mathbf{j} = (i_h, \dots, i_\nu, j_1, j_2, \dots, j_{h-1}) \in \mathcal{I}$ adapted to the quiver $\mathcal{Q}' = s_{i_{h-1}}\dots s_{i_2}s_{i_1}(\mathcal{Q})$ where $j_1, j_2, \dots, j_{h-1}$ are defined by $w_0(\alpha_{i_1}) = -\alpha_{j_1}$, $w_0(\alpha_{i_2}) = -\alpha_{j_2}, \dots, w_0(\alpha_{i_{h-1}}) = -\alpha_{j_{h-1}}$. In what follows, $\mathbf{e}'_\beta$ will denote an indecomposable module of $\mathcal{Q}'$ of dimension $\beta \in R^+$. By using repeatedly lemma 1.14, we get that $\langle \mathbf{e}'_{\alpha(\mathbf{j},1)}, \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)}\rangle_{\mathcal{Q}'} = 0$ and $\langle \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)}, \mathbf{e}'_{\alpha(\mathbf{j},1)}\rangle_{\mathcal{Q}'} = 1$.

Note that we have also used 3.13 (b). By applying the first part of our proof to $\mathbf{j}$ and $\mathcal{Q}'$, we get that

$$\tilde{T}_{i_h}\tilde{T}_{i_{h+1}}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_h} = v^{-1}E_{i_h}\tilde{T}_{i_h}\tilde{T}_{i_{h+1}}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) + \tilde{T}_{i_h}\tilde{T}_{i_{h+1}}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}})$$

where $\alpha(\mathbf{j}, m'-h+1) = \alpha(\mathbf{j},1) + \alpha(\mathbf{j}, k-h+1)$. Applying the automorphism $\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}$, we get

$$\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}(E_{i_h}) = v^{-1}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}(E_{i_h})\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k}) + \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}}).$$

To conclude the proof, we also have to check that $m' = m$. This follows because

$$\begin{aligned}\alpha(\mathbf{i}, m') &= s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j}, m'-h+1)) \\ &= s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j},1)) + s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j}, k-h+1)) \\ &= \alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k)\end{aligned}$$

and $\alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k) = \alpha(\mathbf{i}, m)$ from the definition of m. So $m = m'$. □

Proposition 3.17. *If $1 \le h < k \le \nu$ are such that $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)}\rangle_{\mathcal{Q}} = 1$ and $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)}\rangle^1_{\mathcal{Q}} = 1$, then*

$$E_{\mathbf{i}}^{\mathbf{b}(k)}\,E_{\mathbf{i}}^{\mathbf{b}(h)} = E_{\mathbf{i}}^{\mathbf{b}(h)}\,E_{\mathbf{i}}^{\mathbf{b}(k)} + (v-v^{-1})\,E_{\mathbf{i}}^{\mathbf{b}(m)}E_{\mathbf{i}}^{\mathbf{b}(m')},$$

where m, m' are defined by $m < m'$ and $\alpha(\mathbf{i}, m)$, $\alpha(\mathbf{i}, m')$ is the unique pair of distinct positive roots given in 3.13 (d) *relative to $\mathbf{e}_\alpha = \mathbf{e}_{\alpha(\mathbf{i},h)}$ and $\mathbf{e}_\beta = \mathbf{e}_{\alpha(\mathbf{i},k)}$. Note that $h < m < m' < k$ and the positions $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}])$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}])$ in $\Gamma_{\mathcal{Q}}$ are known.*

Proof. We will first consider the case where $h = 1$. So $\alpha(\mathbf{i}, h) = \alpha_{i_1}$. Let a, b be defined by $1 \le a \le b \le n$ and $\alpha(\mathbf{i}, k) = \alpha_a + \alpha_{a+1} + \cdots + \alpha_b$. Since $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)}\rangle_{\mathcal{Q}} = 1$, then we must have $a \le i_1 \le b$. We saw in the proof of proposition 3.15 that, if $i_1 = a$ or b, then all short exact sequences $0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{V} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ must split, while if $a < i_1 < b$, then there exists a nonsplit short exact sequence

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},1)} \to \mathbf{e}_\gamma \oplus \mathbf{e}_{\gamma'} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

where $\gamma = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1}$ and $\gamma' = \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b$. From this and because $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)}\rangle^1_{\mathcal{Q}} = 1$, then $a < i_1 < b$. The hypothesis of lemma 3.10 is satisfied and we have in $R\mathcal{Q}$ that

$$[\mathbf{e}_{\alpha(\mathbf{i},k)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},1)}] = q\ [\mathbf{e}_{\alpha(\mathbf{i},1)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},k)}]\ +\ (q-1)\ [\mathbf{e}_{\alpha(\mathbf{i},m)}] \cdot [\mathbf{e}_{\alpha(\mathbf{i},m')}]$$

where $m < m'$ and $\{\alpha(\mathbf{i}, m), \alpha(\mathbf{i}, m')\} = \{\alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1},\ \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b\}$ is the unique pair of distinct positive roots given in 3.13 (d). We will now assume that $\alpha(\mathbf{i}, m) = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1}$ and $\alpha(\mathbf{i}, m') = \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b$. The proof in the case where $\alpha(\mathbf{i}, m) = \alpha_{i_1} + \cdots + \alpha_{b-1} + \alpha_b$ and $\alpha(\mathbf{i}, m') = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{i_1}$ is analogous and left to the reader. Applying the imbedding of algebras $\Xi : R\mathcal{Q} \hookrightarrow U_q$

to this equation, we get by proposition 3.5 (c) that

$$\begin{aligned}
&q^{f_{\mathbf{b}(k)}/2}\ K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ q^{f_{\mathbf{b}(1)}/2}K_{i_1}^{-z_{i_1}}E_{i_1}\\
&= q\ q^{f_{\mathbf{b}(1)}/2}K_{i_1}^{-z_{i_1}}E_{i_1}\ q^{f_{\mathbf{b}(k)}/2}\ K_a^{-z_a}\dots K_b^{-z_b}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\\
&\quad+(q-1)\,q^{f_{\mathbf{b}(m)}/2}\ K_a^{-z_a}\dots K_{i_1}^{-z_{i_1}}\tilde{T}_{i_1}\dots\tilde{T}_{i_{m-1}}(E_{i_m})q^{f_{\mathbf{b}(m')}/2}\ K_{i_1}^{-z_{i_1}}\dots K_{(b-1)}^{-z_{(b-1)}}\\
&\qquad\times K_b^{-z_b}\tilde{T}_{i_1}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}}).
\end{aligned}$$

By the defining relations of U_q, we get that

$$\begin{aligned}
&q^{(f_{\mathbf{b}(1)}+f_{\mathbf{b}(k)})/2}\ K_a^{-z_a}\dots K_b^{-z_b}K_{i_1}^{-z_{i_1}}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1}\\
&= q^{(f_{\mathbf{b}(1)}+f_{\mathbf{b}(k)}+2-z_{(i_1-1)}+2z_{i_1}-z_{(i_1+1)})/2}K_{i_1}^{-z_{i_1}}\ K_a^{-z_a}\dots K_b^{-z_b}E_{i_1}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\\
&\quad+(q-1)\ q^{(f_{\mathbf{b}(m)}+f_{\mathbf{b}(m')}+z_{i_1}-z_{(i_1+1)})/2}\ K_a^{-z_a}\dots K_{i_1}^{-z_{i_1}}K_{i_1}^{-z_{i_1}}\dots K_b^{-z_b}\\
&\qquad\times\tilde{T}_{i_1}\dots\tilde{T}_{i_{m-1}}(E_{i_m})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}}).
\end{aligned}$$

Since i_1 is a sink in $\mathcal{Q}$, we have $z_{i_1+1}=z_{i_1}+1$, $z_{i_1-1}=z_{i_1}+1$ and consequently $-z_{i_1-1}+2z_{i_1}-z_{i_1+1}=-2$. From this we get that

$$\begin{aligned}
&\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1}\\
&\quad= E_{i_1}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})+(q-1)\ q^{N/2}\tilde{T}_{i_1}\dots\tilde{T}_{i_{m-1}}(E_{i_m})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}})
\end{aligned}$$

where $N=(f_{\mathbf{b}(m)}+f_{\mathbf{b}(m')}-1-f_{\mathbf{b}(1)}-f_{\mathbf{b}(k)})$. By proposition 3.5 (c),

$$f_{\mathbf{b}(1)}=0,\quad f_{\mathbf{b}(k)}=-\sum_{\substack{i\to j\\ a\le i,j\le b}}z_j,\quad f_{\mathbf{b}(m)}=-\sum_{\substack{i\to j\\ a\le i,j\le i_1}}z_j,\quad f_{\mathbf{b}(m')}=-\sum_{\substack{i\to j\\ i_1\le i,j\le b}}z_j$$

and finally $N=-1$. Thus

$$\begin{aligned}
&\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1}\\
&= E_{i_1}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})+(q^{(1/2)}-q^{-(1/2)})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m-1}}(E_{i_m})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}})
\end{aligned}$$

in U_q for all q. From this, we can conclude that

$$\begin{aligned}
&\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_1}\\
&\qquad= E_{i_1}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})+(v-v^{-1})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m-1}}(E_{i_m})\tilde{T}_{i_1}\dots\tilde{T}_{i_{m'-1}}(E_{i_{m'}})
\end{aligned}$$

in $\mathbf{U}$.

We now consider the case where $h>1$. Let $\mathbf{j}=(i_h,\dots,i_\nu,j_1,j_2,\dots,j_{h-1})\in\mathcal{I}$ adapted to the quiver $\mathcal{Q}'=s_{i_{h-1}}\dots s_{i_2}s_{i_1}(\mathcal{Q})$ where $j_1,j_2,\dots,j_{h-1}$ are defined by $w_0(\alpha_{i_1})=-\alpha_{j_1}$, $w_0(\alpha_{i_2})=-\alpha_{j_2},\dots,w_0(\alpha_{i_{h-1}})=-\alpha_{j_{h-1}}$. In what follows, $\mathbf{e}'_\beta$ will denote an indecomposable module of $\mathcal{Q}'$ of dimension $\beta\in R^+$. By using repeatedly lemma 1.14, we get that $\langle\mathbf{e}'_{\alpha(\mathbf{j},1)},\mathbf{e}'_{\alpha(\mathbf{j},k-h+1)}\rangle_{\mathcal{Q}'}=1$ and $\langle\mathbf{e}'_{\alpha(\mathbf{j},k-h+1)},\mathbf{e}'_{\alpha(\mathbf{j},1)}\rangle^1_{\mathcal{Q}'}=1$. Note that we have also used 3.13. By applying the first part of our proof to $\mathbf{j}$ and $\mathcal{Q}'$, we get that

$$\begin{aligned}
&\tilde{T}_{i_h}\tilde{T}_{i_{h+1}}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ E_{i_h}\\
&= E_{i_h}\tilde{T}_{i_h}\tilde{T}_{i_{h+1}}\dots\tilde{T}_{i_{k-1}}(E_{i_k})+(v-v^{-1})\tilde{T}_{i_h}\dots\tilde{T}_{i_{m''-1}}(E_{i_{m''}})\tilde{T}_{i_h}\dots\tilde{T}_{i_{m'''-1}}(E_{i_{m'''}})
\end{aligned}$$

where m'', m''' are defined by $m''<m'''$ and $\alpha(\mathbf{j},m''-h+1)$, $\alpha(\mathbf{j},m'''-h+1)$ is the unique pair of distinct positive roots such that $\alpha(\mathbf{j},1)+\alpha(\mathbf{j},k-h+1)=$

$\alpha(\mathbf{j}, m''-h+1)+\alpha(\mathbf{j}, m'''-h+1)$ and such that there exists a nonsplit short exact sequence of the form

$$0 \to \mathbf{e}'_{\alpha(\mathbf{j},1)} \to \mathbf{e}'_{\alpha(\mathbf{j},m''-h+1)} \oplus \mathbf{e}'_{\alpha(\mathbf{j},m'''-h+1)} \to \mathbf{e}'_{\alpha(\mathbf{j},k-h+1)} \to 0$$

of representations of $\mathcal{Q}'$. Applying the automorphism $\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}$, we get

$$\begin{aligned}\tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})\ \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}(E_{i_h})& \\ = \quad \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{h-1}}(E_{i_h})\ \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{k-1}}(E_{i_k})& \\ +(v-v^{-1})\ \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m''-1}}(E_{i_{m''}})\ \tilde{T}_{i_1}\tilde{T}_{i_2}\dots\tilde{T}_{i_{m'''-1}}(E_{i_{m'''}}).&\end{aligned}$$

We must now prove that $m = m''$ and $m' = m'''$. First note, that since

$$\begin{aligned}s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j},1)) + s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j},k-h+1))& \\ = s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j},m''-h+1)) + s_{i_1}s_{i_2}\dots s_{i_{h-1}}(\alpha(\mathbf{j},m'''-h+1)),&\end{aligned}$$

we have $\alpha(\mathbf{i},h)+\alpha(\mathbf{i},k) = \alpha(\mathbf{i},m'')+\alpha(\mathbf{i},m''')$. Moreover, by applying the functor $\Phi^-_{i_1}\Phi^-_{i_2}\dots\Phi^-_{i_{h-1}}$ to the above nonsplit short exact sequence and using an argument similar to the one used in lemma 1.14, we get a nonsplit short exact sequence

$$0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{e}_{\alpha(\mathbf{i},m'')} \oplus \mathbf{e}_{\alpha(\mathbf{i},m''')} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$$

of representations of $\mathcal{Q}$. Thus by 3.13 (d), we can conclude that $m'' = m$ and $m''' = m'$. □

3.18**.** Combining the four previous propositions with our observations in 3.13, it is then easy to describe the commutation relations between root vectors. We have illustrated some of these relations in the example below.

Example 3.19. Let W be the Weyl group of type A_5. Let us consider the element $\mathbf{i} = (4,1,3,5,2,4,1,3,5,2,4,1,3,5,4)$ of $\mathcal{I}$ adapted to the quiver $\mathcal{Q}$: $1 \leftarrow 2 \to 3 \to 4 \leftarrow 5$. We have already computed the Auslander-Reiten quiver of $\mathcal{Q}$ in example 3.12.

By applying proposition 3.16, we get that

$$\tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4\tilde{T}_1\tilde{T}_3(E_5)\ E_4 = v^{-1}E_4\ \tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4\tilde{T}_1\tilde{T}_3(E_5) + \tilde{T}_4\tilde{T}_1(E_3).$$

In fact, this corresponds to the case $h = 1$, $k = 9$ in proposition 3.16. We have $\alpha(\mathbf{i},1) = \alpha_4$, $\alpha(\mathbf{i},9) = \alpha_3$. We have $\langle \mathbf{e}_{\alpha(\mathbf{i},1)}, \mathbf{e}_{\alpha(\mathbf{i},9)}\rangle_{\mathcal{Q}} = 0$, $\langle \mathbf{e}_{\alpha(\mathbf{i},9)}, \mathbf{e}_{\alpha(\mathbf{i},1)}\rangle^1_{\mathcal{Q}} = 1$ and $m = 3$ by 3.13. Note that in this special case, we could directly verify the commutation relations because $\tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4\tilde{T}_1\tilde{T}_3(E_5) = E_3$ and $\tilde{T}_4\tilde{T}_1(E_3) = \tilde{T}_4(E_3) = E_3E_4 - v^{-1}E_4E_3$.

By applying proposition 3.17, we get that

$$\begin{aligned}\tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5(E_2)\ \tilde{T}_4\tilde{T}_1(E_3)& \\ = \tilde{T}_4\tilde{T}_1(E_3)\ \tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5(E_2)& \\ +(v-v^{-1})\ \tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2(E_4)\ \tilde{T}_4\tilde{T}_1\tilde{T}_3\tilde{T}_5\tilde{T}_2\tilde{T}_4(E_1).&\end{aligned}$$

In fact, this corresponds to the case $h = 3$, $k = 10$ in proposition 3.17. We have $\alpha(\mathbf{i},3) = \alpha_3+\alpha_4$, $\alpha(\mathbf{i},10) = \alpha_2+\alpha_3+\alpha_4+\alpha_5$. By 3.13, we have $\langle \mathbf{e}_{\alpha(\mathbf{i},1)}, \mathbf{e}_{\alpha(\mathbf{i},10)}\rangle_{\mathcal{Q}} = 1$, $\langle \mathbf{e}_{\alpha(\mathbf{i},10)}, \mathbf{e}_{\alpha(\mathbf{i},1)}\rangle^1_{\mathcal{Q}} = 1$, $m = 6$ and $m' = 7$. Note that this commutation relation could also be checked directly.

4. Multiplication by a root vector

4.1. In this section, we will describe the result of the multiplication of a root vector with an element of a PBW basis element . This formula is given in theorem 4.16. We will need some preliminary lemmas for this. First we recall some results on almost split sequences (also called Auslander-Reiten sequences) and Grothendieck groups of artin algebras. We will state these results not in full generality as they appeared in [1] and [2] but rather as they are needed for our situation.

From now on, we fix a quiver $\mathcal{Q}$ whose graph is of type A_n and $\mathbf{i} = (i_1, i_2, \dots, i_\nu) \in \mathcal{I}$ adapted to $\mathcal{Q}$.

4.2. Let $\mathbf{V}$, $\mathbf{V}'$ and $\mathbf{V}''$ be three modules of the quiver $\mathcal{Q}$. A morphism $f : \mathbf{V} \to \mathbf{V}''$ (respectively $g : \mathbf{V}' \to \mathbf{V}$) is said to be right (respectively left) almost split if

(a) it is not a split epimorphism (respectively monomorphism);
(b) any morphism $\mathbf{M} \to \mathbf{V}''$ (respectively $\mathbf{V}' \to \mathbf{M}'$) which is not a split epimorphism (respectively monomorphism) factors through f (respectively g).

An exact sequence $0 \to \mathbf{V}' \xrightarrow{g} \mathbf{V} \xrightarrow{f} \mathbf{V}'' \to 0$ is said to be an almost split sequence if g is left almost split and f is right almost split.

4.3. Let $\mathbf{K}(\mathcal{Q}, 0)$ be the free abelian group with basis the isomorphism classes $[\mathbf{M}]$ of modules $\mathbf{M}$ of $\mathcal{Q}$ modulo the subgroup generated by the elements of the form $[\mathbf{V}]+[\mathbf{W}]-[\mathbf{V}\oplus\mathbf{W}]$. It is well known that the set of isomorphism classes $[\mathbf{M}]$ of indecomposable modules of $\mathcal{Q}$ is a basis of $\mathbf{K}(\mathcal{Q}, 0)$. Because of theorem 1.12, this means that $\{[\mathbf{e}_\alpha] \mid \alpha \in R^+\}$ is a basis of $\mathbf{K}(\mathcal{Q}, 0)$ and $[\mathbf{V}_\mathbf{c}]$ is equal to $\sum_{k=1}^{\nu} c_k\ [\mathbf{e}_{\alpha(\mathbf{i},k)}]$ in $\mathbf{K}(\mathcal{Q}, 0)$ for $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$.

$\mathbf{K}_+(\mathcal{Q}, 0)$ will be the subset of $\mathbf{K}(\mathcal{Q}, 0)$ consisting of the elements $[\mathbf{V}_\mathbf{c}]$ for $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$. $\langle\ \ ,\ \ \rangle_\mathcal{Q} : \mathbf{K}(\mathcal{Q}, 0) \times \mathbf{K}(\mathcal{Q}, 0) \to \mathbf{Z}$ (respectively $\langle\ \ ,\ \ \rangle^1_\mathcal{Q}$: $\mathbf{K}(\mathcal{Q}, 0) \times \mathbf{K}(\mathcal{Q}, 0) \to \mathbf{Z}$) will be the unique bilinear form such that, whenever $\mathbf{V}$ and $\mathbf{W}$ are modules of $\mathcal{Q}$, we have $\langle[\mathbf{V}], [\mathbf{W}]\rangle_\mathcal{Q} = \langle\mathbf{V}, \mathbf{W}\rangle_\mathcal{Q}$ (respectively $\langle[\mathbf{V}], [\mathbf{W}]\rangle^1_\mathcal{Q} = \langle\mathbf{V}, \mathbf{W}\rangle^1_\mathcal{Q}$). Define also $[\mathbf{V}] \circ_\mathcal{Q} [\mathbf{W}] = \langle[\mathbf{V}], [\mathbf{W}]\rangle_\mathcal{Q} - \langle[\mathbf{W}], [\mathbf{V}]\rangle^1_\mathcal{Q}$ for $[\mathbf{V}]$, $[\mathbf{W}] \in \mathbf{K}(\mathcal{Q}, 0)$.

Let $\mathbf{V}''$ be an indecomposable module of $\mathcal{Q}$. If $\mathbf{V}''$ is nonprojective, then there is up to isomorphism a unique almost split sequence $0 \to \mathbf{V}' \xrightarrow{g} \mathbf{V} \xrightarrow{f} \mathbf{V}'' \to 0$. We then associate to $\mathbf{V}''$, the element $\mathbf{r}_{\mathbf{V}''} = [\mathbf{V}'] + [\mathbf{V}''] - [\mathbf{V}]$ in $\mathbf{K}(\mathcal{Q}, 0)$. If $\mathbf{V}''$ is projective, we define $\mathbf{r}_{\mathbf{V}''} = [\mathbf{V}''] - [\underline{r}\mathbf{V}''] \in \mathbf{K}(\mathcal{Q}, 0)$ where $\underline{r}\mathbf{V}''$ is the unique maximal proper submodule of $\mathbf{V}''$.

From now on, we will denote the element $\mathbf{r}_{\mathbf{e}_\alpha}$ of $\mathbf{K}(\mathcal{Q}, 0)$ by $\mathbf{r}_\alpha$. Here $\alpha \in R^+$. It is well known that

$$\langle[\mathbf{e}_\alpha], \mathbf{r}_\beta\rangle_\mathcal{Q} = \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{if } \alpha \neq \beta \end{cases}$$

for α, $\beta \in R^+$. For a proof, see, for example, [1].

4.4. Let α be a positive root and denote its position $\Theta([\mathbf{e}_\alpha])$ in the Auslander-Reiten quiver $\Gamma_\mathcal{Q}$ by $(x, i) \in \mathbf{Z} \times \{1, 2, \dots, n\}$. A partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ is said to be α-admissible if and only if

(a) $\lambda_1 \leq i$ and $\ell \leq (n - i + 1)$;
(b) $(x - a + 1, i + a - b)$ is the position $\Theta([\mathbf{e}_\beta])$ for some indecomposable nonprojective module $\mathbf{e}_\beta$ of $\mathcal{Q}$ for $1 \leq a \leq \ell$ and $1 \leq b \leq \lambda_a$.

$\Lambda(\alpha)$ will denote the set of α-admissible partitions. Note that ℓ could be 0 and, in that case, the partition λ can only be the empty partition $\emptyset$ and it is always α-admissible for each positive root α. We will call ℓ: the height of λ and we will denote it by $\mathrm{ht}(\lambda)$.

4.5. Let α be a positive root and $(x,i)=\Theta([\mathbf{e}_\alpha])$ be its position in $\Gamma_{\mathcal{Q}}$. To $\lambda\in\Lambda(\alpha)$, we associate the element $\mathbf{t}(\lambda,\alpha)\in\mathbf{K}(\mathcal{Q},0)$ defined by

$$\mathbf{t}(\lambda,\alpha)=\begin{cases}0, & \text{if } \lambda=\emptyset,\\ \sum\limits_{\substack{1\le a\le\ell\\ 1\le b\le\lambda_a}}\mathbf{r}_{\Theta^{-1}(x-a+1,i+a-b)}, & \text{if } \lambda:\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_\ell>0 \text{ with } \ell>0.\end{cases}$$

It is easy to see that if λ and μ are two α-admissible partitions and $\mathbf{t}(\lambda,\alpha)=\mathbf{t}(\mu,\alpha)$, then $\lambda=\mu$.

4.6. Let α be a positive root and $\mathbf{c}\in\mathbf{N}^\nu$. $\Lambda(\alpha,\mathbf{c})$ will denote the subset of $\Lambda(\alpha)$ consisting of α-admissible partitions λ such that $[\mathbf{V}_\mathbf{c}]+[\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha)\in\mathbf{K}_+(\mathcal{Q},0)$. By 4.3 above, this condition is equivalent to $\langle[\mathbf{V}_\mathbf{c}]+[\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha),\mathbf{r}_\beta\rangle_{\mathcal{Q}}\in\mathbf{N}$ for all $\beta\in R^+$. Note that $\Lambda(\alpha,\mathbf{c})$ is not empty because the empty partition is always an element of $\Lambda(\alpha,\mathbf{c})$.

If $\lambda\in\Lambda(\alpha,\mathbf{c})$, then $J(\lambda,\alpha)$ will denote the set

$$\{1\le j\le\nu \mid \langle[\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha),\mathbf{r}_{\alpha(\mathbf{i},j)}\rangle_{\mathcal{Q}}=1\}.$$

This set $J(\lambda,\alpha)$ is never empty. If λ is the empty partition $\emptyset$, then $J(\emptyset,\alpha)=\{k\}$, where k is the unique integer such that $\alpha(\mathbf{i},k)=\alpha$. If λ is not the empty partition, then it is an easy consequence of the next lemma that $J(\lambda,\alpha)$ is not empty.

Lemma 4.7. (a) *Let $\alpha,\beta\in R^+$ and $\Theta([\mathbf{e}_\alpha])=(x,i)$ the position of $[\mathbf{e}_\alpha]$ in $\Gamma_{\mathcal{Q}}$. Let $\lambda:\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_\ell>0$ be a nonempty α-admissible partition of height $\mathrm{ht}(\lambda)=\ell$. Then $\langle\mathbf{t}(\lambda,\alpha),\mathbf{r}_\beta\rangle_{\mathcal{Q}}$ is equal to 1, if $\Theta([\mathbf{e}_\beta])\in\{(x,i)\}\cup\{(x-j,i+j-\lambda_j)\mid 1\le j\le\ell \text{ and } \lambda_j\ne\lambda_{j+1}\}$, is equal to -1, if $\Theta([\mathbf{e}_\beta])\in\{(x-j+1,i+j-\lambda_j-1)\mid 1<j\le\ell \text{ and } \lambda_j\ne\lambda_{j-1}\}\cup\{(x,i-\lambda_1),(x-\ell,i+\ell)\}$, and is equal to 0, otherwise.*

(b) *Let $\mathbf{c}\in\mathbf{N}^\nu$, $\alpha\in R^+$ and $1\le h\le\nu$. Set $\Lambda_\sigma(\alpha,\mathbf{c},h)=\{\lambda\in\Lambda(\alpha,\mathbf{c})\mid\langle\mathbf{t}(\lambda,\alpha),\mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}}=\sigma\}$ for $\sigma\in\{-1,0,1\}$. Then $\Lambda(\alpha,\mathbf{c})$ is the disjoint union $\Lambda_{-1}(\alpha,\mathbf{c},h)\cup\Lambda_0(\alpha,\mathbf{c},h)\cup\Lambda_1(\alpha,\mathbf{c},h)$.*

Proof. (a) Let $\Theta([\mathbf{e}_\beta])=(y,j)$ be the position of the vertex $[\mathbf{e}_\beta]$ in $\Gamma_{\mathcal{Q}}$. Define $\Upsilon(\alpha,\lambda,\beta)=\{(x-a+1,i+a-b)\mid 1\le a\le\ell, 1\le b\le\lambda_a\}\cap\{(y,j),(y+1,j),(y,j+1),(y+1,j-1)\}$. Write

$$\mathbf{t}(\lambda,\alpha)=\sum_{\gamma\in R^+}m_\gamma[\mathbf{e}_\gamma] \quad \text{where } m_\gamma\in\mathbf{Z}.$$

Using the definition of $\mathbf{r}_\gamma$ for $\gamma\in R^+$ and the fact that $\lambda\in\Lambda(\alpha)$, we can now compute m_β by analyzing $\Upsilon(\alpha,\lambda,\beta)$.

If $\Upsilon(\alpha,\lambda,\beta)=\emptyset$, then $m_\beta=0$. If $\Upsilon(\alpha,\lambda,\beta)=\{(y,j)\}$ or $\{(y+1,j)\}$, then $m_\beta=1$, while if $\Upsilon(\alpha,\lambda,\beta)=\{(y+1,j-1)\}$ or $\{(y,j+1)\}$, then $m_\beta=-1$.

If $|\Upsilon(\alpha,\lambda,\beta)|=2$, then $\Upsilon(\alpha,\lambda,\beta)$ can only be one of the following four cases: $\{(y,j),(y+1,j-1)\}$, $\{(y,j),(y,j+1)\}$, $\{(y+1,j),(y+1,j-1)\}$, $\{(y+1,j),(y,j+1)\}$. The other two cases $\{(y,j),(y+1,j)\}$ and $\{(y,j+1),(y+1,j-1)\}$ have to be excluded because λ is a partition in $\Lambda(\alpha)$. For these four allowable cases, we get that $m_\beta = 0$.

If $|\Upsilon(\alpha,\lambda,\beta)| = 3$, then $\Upsilon(\alpha,\lambda,\beta)$ can only be one case: $\{(y,j+1),(y+1,j),(y+1,j-1)\}$. The other three cases have to be excluded, because λ is a partition in $\Lambda(\alpha)$. If $\Upsilon(\alpha,\lambda,\beta) = \{(y,j+1),(y+1,j),(y+1,j-1)\}$, then $m_\beta = -1$.

Finally, if $\Upsilon(\alpha,\lambda,\beta) = \{(y,j),(y+1,j),(y,j+1),(y+1,j-1)\}$, then $m_\beta = 0$.

Using these computations of m_β and 4.3, (a) is proved.

(b) From (a), $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} \in \{-1,0,1\}$ when λ is a nonempty α-admissible partition. If $\lambda = \emptyset$ is the empty partition, then $\mathbf{t}(\lambda,\alpha) = 0$ and $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 0$. From this, we get that $\Lambda(\alpha,\mathbf{c})$ is the disjoint union $\Lambda_{-1}(\alpha,\mathbf{c},h) \cup \Lambda_0(\alpha,\mathbf{c},h) \cup \Lambda_1(\alpha,\mathbf{c},h)$. □

Lemma 4.8. *Let α, β be two distinct positive roots and λ an α-admissible partition.*

(a) *If $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = 1$, then $\langle [\mathbf{e}_\alpha], [\mathbf{e}_\beta]\rangle^1_{\mathcal{Q}} = 1$.*
(b) *If $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = -1$, then $\langle [\mathbf{e}_\beta], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 1$.*

Proof. Let $\Theta([\mathbf{e}_\alpha]) = (x,i)$ be the position of $[\mathbf{e}_\alpha]$ in $\Gamma_{\mathcal{Q}}$.

(a) If $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = 1$, then λ is a nonempty α-admissible partition $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ of height $\mathrm{ht}(\lambda) = \ell$. Since $\beta \neq \alpha$ and by lemma 4.7 (a), we get that the position $\Theta([\mathbf{e}_\beta])$ of $[\mathbf{e}_\beta]$ in $\Gamma_{\mathcal{Q}}$ is in the set $\{(x-j, i+j-\lambda_j) \mid 1 \leq j \leq \ell \text{ and } \lambda_j \neq \lambda_{j+1}\}$. If $\Theta([\mathbf{e}_\beta]) = (x-j, i+j-\lambda_j)$ for $1 \leq j \leq \ell$ with $\lambda_j \neq \lambda_{j+1}$, then $x-1-(n-i) \leq x-j \leq x-1$ because $1 \leq j \leq \ell \leq (n-i+1)$, λ being an α-admissible partition, and $x \leq x+i-\lambda_j \leq x-1+i$ because $0 < \lambda_j \leq \lambda_1 \leq i$, λ being an α-admissible partition. By 3.13 (b), we can conclude that $\langle [\mathbf{e}_\alpha], [\mathbf{e}_\beta]\rangle^1_{\mathcal{Q}} = 1$.

(b) If $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = -1$, then λ is a nonempty α-admissible partition $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell > 0$ of height $\mathrm{ht}(\lambda) = \ell$. By lemma 4.7 (a), we get that the position $\Theta([\mathbf{e}_\beta])$ of $[\mathbf{e}_\beta]$ in $\Gamma_{\mathcal{Q}}$ is in the set $\{(x-j+1, i+j-\lambda_j-1) \mid 1 < j \leq \ell \text{ and } \lambda_j \neq \lambda_{j-1}\} \cup \{(x, i-\lambda_1), (x-\ell, i+\ell)\}$. If $\Theta([\mathbf{e}_\beta]) = (x-j+1, i+j-\lambda_j-1)$ for $1 < j \leq \ell$ with $\lambda_j \neq \lambda_{j-1}$, then $x-(n-i) \leq x-j+1 \leq x$ because $1 < j \leq \ell \leq (n-i+1)$, λ being an α-admissible partition, and $x+1 \leq x+i-\lambda_j \leq x+i$, because $\lambda_j < \lambda_{j-1} \leq \lambda_1 \leq i$, λ being an α-admissible partition. Because of 3.13 (a), we can conclude that $\langle [\mathbf{e}_\beta], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 1$.

If $\Theta([\mathbf{e}_\beta]) = (x, i-\lambda_1)$, then $\lambda_1 < i$ and using 3.13 (a), we can conclude that $\langle [\mathbf{e}_\beta], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 1$. If $\Theta([\mathbf{e}_\beta]) = (x-\ell, i+\ell)$, then $\ell \leq (n-i)$ and again using 3.13 (a), we can conclude that $\langle [\mathbf{e}_\beta], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 1$. □

4.9. If $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ and $\mathbf{c} \neq (0,0,\ldots,0)$, then $\rho(\mathbf{c})$ is defined as the smallest integer j such that $1 \leq j \leq \nu$ and $c_j \neq 0$. We also define $\hat{\mathbf{c}} \in \mathbf{N}^\nu$ by $[\mathbf{V}_{\hat{\mathbf{c}}}] = [\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, where $\rho(\mathbf{c}) = h$. Obviously, either $\hat{\mathbf{c}} = (0,0,\ldots,0)$ or $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$ and $\rho(\hat{\mathbf{c}}) > \rho(\mathbf{c})$.

Lemma 4.10. *Let $\alpha = \alpha(\mathbf{i},k) \in R^+$ and $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ with $\mathbf{c} \neq (0,0,\ldots,0)$ and $\rho(\mathbf{c}) = h < k$.*

(a) *If $\lambda \in \Lambda(\alpha,\mathbf{c})$, then $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} \in \{0,1\}$. So $\Lambda_{-1}(\alpha,\mathbf{c},h) = \emptyset$.*
(b) *$\Lambda(\alpha,\hat{\mathbf{c}}) \subseteq \Lambda(\alpha,\mathbf{c})$ and $\Lambda_0(\alpha,\mathbf{c},h) = \Lambda(\alpha,\hat{\mathbf{c}})$.*
(c) *If the set $\Lambda_1(\alpha,\mathbf{c},h)$ is not empty, then $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} \neq 0$.*
(d) *$[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha)$ is equal to 0 if $\lambda \in \Lambda_0(\alpha,\mathbf{c},h)$ and to -1 if $\lambda \in \Lambda_1(\alpha,\mathbf{c},h)$.*
(e) *$h \notin J(\lambda,\alpha)$ for $\lambda \in \Lambda(\alpha,\mathbf{c})$.*

(f) *If* $\lambda \in \Lambda(\alpha, \mathbf{c})$ *and* $\mathbf{c}' \in \mathbf{N}^\nu$ *is such that* $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$, *then* $\mathbf{c}' \neq (0, 0, \ldots, 0)$ *and* $\rho(\mathbf{c}') \geq h$.

Proof. (a) Let $\Theta([\mathbf{e}_\alpha]) = (x, i)$ be the position of $[\mathbf{e}_\alpha]$ in $\Gamma_\mathcal{Q}$. Let $\lambda \in \Lambda(\alpha, \mathbf{c})$ be such that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = -1$. Obviously $\lambda \neq \emptyset$. Write $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ with $\mathrm{ht}(\lambda) = \ell > 0$. By lemma 4.7, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$ is either $(x - j + 1, i + j - \lambda_j - 1)$ for some $1 < j \leq \ell$ and $\lambda_j \neq \lambda_{j-1}$ or is $(x, i - \lambda_1)$ or is $(x - \ell, i + \ell)$. We will first show that there exist $a < h$ such that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = 1$ in all these cases.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x - j + 1, i + j - \lambda_j - 1)$ for some $1 < j \leq \ell$ and $\lambda_j \neq \lambda_{j-1}$, then $(x - j + 1, i + j - 1 - \lambda_{j-1})$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the corresponding indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - j + 1, i + j - \lambda_{j-1} - 1)$, then $a < h$ and by lemma 4.7, $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = 1$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x, i - \lambda_1)$ and assume that p is the largest integer such that $p \leq \ell$ and $\lambda_p = \lambda_1$, then $(x - p, i - \lambda_p + p)$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the corresponding indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - p, i - \lambda_p + p)$, then $a < h$ and by lemma 4.7, $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = 1$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x - \ell, i + \ell)$, then $(x - \ell, i + \ell - \lambda_\ell)$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the corresponding indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - \ell, i + \ell - \lambda_\ell)$, then $a < h$ and by lemma 4.7, $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = 1$.

As we have seen, there exists $a < h$ such that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = 1$. Since $a < h = \rho(\mathbf{c}) < k$, we get that $\langle [\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_\mathcal{Q} = -1$. This contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$. This proves (a).

(b) If $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$, then $[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \in \mathbf{K}_+(\mathcal{Q}, 0)$. Since $[\mathbf{V}_\mathbf{c}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$ with $c_h > 0$, then $[\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \in \mathbf{K}_+(\mathcal{Q}, 0)$ and $\lambda \in \Lambda(\alpha, \mathbf{c})$.

If $\hat{\mathbf{c}} = (0, 0, \ldots, 0)$, then $\Lambda(\alpha, \hat{\mathbf{c}}) = \{\text{empty partition } \emptyset\}$. If $\Theta([\mathbf{e}_\alpha]) = (x, i)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (y, j)$ and, because $\mathbf{c} = (0, 0, \ldots, 0, c_h, 0, \ldots, 0)$, then there is at most one nonempty partition in $\Lambda(\alpha, \mathbf{c})$. This will happen when $x - n + i \leq y < x$ and $x + 1 \leq y + j < x + i$ and in this case this partition λ is of height $\mathrm{ht}(\lambda) = (x - y)$ and is equal to $\lambda : (x - y + i - j) \geq (x - y + i - j) \geq \cdots \geq (x - y + i - j)$. But for this partition λ, we have that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = 1$. Consequently, $\Lambda_0(\alpha, \mathbf{c}, h) = \{\text{empty partition } \emptyset\}$ and we have the equality $\Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda_0(\alpha, \mathbf{c}, h)$. Assume now that $\hat{\mathbf{c}} \neq (0, 0, \ldots, 0)$.

If $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$, then we have $h < \rho(\hat{\mathbf{c}})$ and $\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} \geq 0$. But $\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = -\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} \geq 0$ and $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} \in \{0, 1\}$ by (a). Thus $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = 0$. In other words, $\Lambda(\alpha, \hat{\mathbf{c}}) \subseteq \Lambda_0(\alpha, \mathbf{c}, h)$.

If $\lambda \in \Lambda_0(\alpha, \mathbf{c}, h)$, then $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle = 0$. Assume that $\lambda \notin \Lambda(\alpha, \hat{\mathbf{c}})$, then $[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \notin \mathbf{K}_+(\mathcal{Q}, 0)$. Note that $\lambda \in \Lambda(\alpha, \mathbf{c})$ implies that $[\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \in \mathbf{K}_+(\mathcal{Q}, 0)$. Since $[\mathbf{V}_\mathbf{c}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, then $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = 1$. This contradicts the fact that $\lambda \in \Lambda_0(\alpha, \mathbf{c}, h)$. We have proved that $\Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda_0(\alpha, \mathbf{c}, h)$.

(c) If $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h)$, then $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_\mathcal{Q} = 1$ and λ is different from the empty partition $\emptyset$. By lemma 4.8 (a), we get that $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_\mathcal{Q} \neq 0$.

(d) Write $(x, i) = \Theta([\mathbf{e}_\alpha])$ for the position of $[\mathbf{e}_\alpha]$ in $\Gamma_\mathcal{Q}$. Note that we have $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{t}(\lambda, \alpha) \rangle_\mathcal{Q} \in \{0, 1\}$ by 4.3 and 4.5. If $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{t}(\lambda, \alpha) \rangle_\mathcal{Q} = 1$, then λ is a nonempty partition. Write $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ with $\mathrm{ht}(\lambda) = \ell$. From the definition of $\mathbf{t}(\lambda, \alpha)$, we get that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x - a + 1, i + a - b)$ with $1 \leq a \leq \ell$

and $1 \le b \le \lambda_a$. Let a' be the largest integer such that $a \le a' \le \ell$ with $\lambda_{a'} = \lambda_a$. Then $(x - a', i + a' - \lambda_{a'})$ is the position $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h')}])$ of an indecomposable module of $\mathcal{Q}$. Here we have used the fact that λ is an α-admissible partition. Since $a' \ge a$, then $h' < h$ and $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = 1$ by lemma 4.7 (a). But from this and $h' < h = \rho(\mathbf{c}) < k$, we get that $\langle [\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$. So $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{t}(\lambda,\alpha)\rangle_{\mathcal{Q}} = 0$ for all $\lambda \in \Lambda(\alpha, \mathbf{c})$.

$\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} \in \{0,1\}$ because $\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \tau(\mathbf{t}(\lambda,\alpha))\rangle_{\mathcal{Q}}$ and if λ is the nonempty partition $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$ with $\ell > 0$, then

$$\tau(\mathbf{t}(\lambda,\alpha)) = \sum_{\substack{1 \le a \le \ell \\ 1 \le b \le \lambda_a}} \mathbf{r}_{\Theta^{-1}(x-a, i+a-b)}.$$

Now we want to show that $\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = 1$ if and only if $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h)$.

If $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h)$, in other words $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$, then λ is a nonempty partition $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$ with $\ell > 0$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x - j, i + j - \lambda_j)$ for some $1 \le j \le \ell$ and $\lambda_j \ne \lambda_{j+1}$ by lemma 4.7. Consequently, $\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \tau(\mathbf{t}(\lambda,\alpha))\rangle_{\mathcal{Q}} = 1$ from the expression of $\tau(\mathbf{t}(\lambda,\alpha))$ above.

If $\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \tau(\mathbf{t}(\lambda,\alpha))\rangle_{\mathcal{Q}} = 1$, then λ is a nonempty partition $\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_\ell > 0$ with $\ell > 0$. Since $\langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \tau(\mathbf{t}(\lambda,\alpha))\rangle_{\mathcal{Q}} = 1$ and

$$\tau(\mathbf{t}(\lambda,\alpha)) = \sum_{\substack{1 \le a \le \ell \\ 1 \le b \le \lambda_a}} \mathbf{r}_{\Theta^{-1}(x-a, i+a-b)},$$

we get that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x-a, i+a-b)$ where $1 \le a \le \ell$ and $1 \le b \le \lambda_a$. Let a' be the largest integer such that $a \le a' \le \ell$ and $\lambda_{a'} = \lambda_a$. Then $(x-(a'-1), i+a'-\lambda_{a'})$ is the position $\Theta([\mathbf{e}_\beta])$ for some indecomposable nonprojective module of $\mathcal{Q}$ and $(x - a', i + a' - \lambda_{a'})$ is the position $\Theta(\tau([\mathbf{e}_\beta])) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},h')}])$ of an indecomposable module of $\mathcal{Q}$. If $b < \lambda_a$ or $a < a'$, then $h' < h$ and by lemma 4.7, we get that $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = 1$. But from this, we get that $\langle [\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$. So $a = a'$, $b = \lambda_a$ and $\lambda_a > \lambda_{a+1}$. But by lemma 4.7, we get that $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$ and $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h)$.

Finally,

$$\begin{aligned}[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha) &= \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{t}(\lambda,\alpha)\rangle_{\mathcal{Q}} - \langle \mathbf{t}(\lambda,\alpha), [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} \\ &= \begin{cases} 0, & \text{if } \lambda \in \Lambda_0(\alpha, \mathbf{c}, h), \\ -1, & \text{if } \lambda \in \Lambda_1(\alpha, \mathbf{c}, h). \end{cases}\end{aligned}$$

This proves (d).

(e) Write $\Theta([\mathbf{e}_\alpha]) = (x, i)$ for the position of $[\mathbf{e}_\alpha]$ in $\Gamma_{\mathcal{Q}}$. If $h \in J(\lambda, \alpha)$, then $\langle [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$. Since $h < k$, then $\langle [\mathbf{e}_\alpha], \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 0$ and we get that $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = -1$. So $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$ belongs to the set $\{(x-j+1, i+j-\lambda_j-1) \mid 1 < j \le \ell \text{ and } \lambda_j \ne \lambda_{j-1}\} \cup \{(x, i-\lambda_1), (x-\ell, i+\ell)\}$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x-j+1, i+j-\lambda_j-1)$ for some $1 < j \le \ell$ with $\lambda_j \ne \lambda_{j-1}$, then $(x - j + 1, i + j - 1 - \lambda_{j-1})$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - j + 1, i + j - 1 - \lambda_{j-1})$, then $a < h$ and, by lemma 4.7, $\langle \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},a)}\rangle_{\mathcal{Q}} = 1$. Then $\langle [\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha), \mathbf{r}_{\alpha(\mathbf{i},a)}\rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x, i - \lambda_1)$ and assume that p is the largest integer such that $p \leq \ell$ and $\lambda_p = \lambda_1$, then $(x - p, i - \lambda_p + p)$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - p, i - \lambda_p + p)$, then $a < h$ and, by lemma 4.7, $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$. Then $\langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}]) = (x - \ell, i + \ell)$, then $(x - \ell, i + \ell - \lambda_\ell)$ is the position of an indecomposable module of $\mathcal{Q}$, because λ is α-admissible. If we take the indecomposable module $[\mathbf{e}_{\alpha(\mathbf{i},a)}]$ such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x - \ell, i + \ell - \lambda_\ell)$, then $a < h$ and, by lemma 4.7, $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$. Then $\langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$.

In all cases, we get a contradiction. So $h \notin J(\lambda, \alpha)$.

(f) If $\mathbf{c}' = (0, 0, \ldots, 0)$, then $\mathbf{t}(\lambda, \alpha) = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha]$. In this case, λ cannot be the empty partition, because $[\mathbf{V}_{\mathbf{c}}] \in \mathbf{K}_+(\mathcal{Q}, 0)$. We have that

$$\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_\beta \rangle_{\mathcal{Q}} = \langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha], \mathbf{r}_\beta \rangle_{\mathcal{Q}} \geq 0 \quad \text{for all } \beta \in R^+,$$

because $[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] \in \mathbf{K}_+(\mathcal{Q}, 0)$. But by lemma 4.7 (a), there is at least a positive root $\gamma \in R^+$ such that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_\gamma \rangle_{\mathcal{Q}} = -1$. In this way, we get a contradiction and consequently $\mathbf{c}' \neq (0, 0, \ldots, 0)$.

Assume now that $\rho(\mathbf{c}') < h$, then there exists h', $1 \leq h' < h$ such that

$$\langle [\mathbf{V}_{\mathbf{c}'}], \mathbf{r}_{\alpha(\mathbf{i},h')} \rangle_{\mathcal{Q}} = \langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h')} \rangle_{\mathcal{Q}} > 0.$$

From this, we get that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h')} \rangle_{\mathcal{Q}} < 0$. By lemma 4.7 (a), $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h')} \rangle_{\mathcal{Q}} = -1$. We can proceed as in the proof of (a) to show that there exists $a < h'$ such that $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$. We leave this proof to the reader. Thus $\langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda \in \Lambda(\alpha, \mathbf{c})$. So $\rho(\mathbf{c}') \geq h$. This concludes the proof of the lemma. □

Lemma 4.11. *Let $\alpha = \alpha(\mathbf{i}, k) \in R^+$ and $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ with $\mathbf{c} \neq (0, 0, \ldots, 0)$ and $\rho(\mathbf{c}) = h < k$. If $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 0$, then $\Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda(\alpha, \mathbf{c})$.*

Proof. Because of lemma 4.10 (c), we get that $\Lambda_1(\alpha, \mathbf{c}, h)$ is empty. By lemmas 4.7 (b) and 4.10 (a) and (b), we have that $\Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda_0(\alpha, \mathbf{c}, h) = \Lambda(\alpha, \mathbf{c})$. □

4.12**.** Let $\alpha = \alpha(\mathbf{i}, k) \in R^+$ and $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$ with $\mathbf{c} \neq (0, 0, \ldots, 0)$ and $\rho(\mathbf{c}) = h < k$. Assume that $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},k)}] \rangle_{\mathcal{Q}} = 0$ and $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 1$. In this case, $\alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k)$ is a positive root $\alpha(\mathbf{i}, m)$ with $h < m < k$ as we have noted in 3.13 (c). Let $(x, i) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$ (resp. $(y, j) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$) be the position of $[\mathbf{e}_{\alpha(\mathbf{i},k)}]$ (resp. $[\mathbf{e}_{\alpha(\mathbf{i},h)}]$) in $\Gamma_{\mathcal{Q}}$. As noted in 3.13 (c), we have either $(y = x - 1 - (n - i)$ and $x < y + j \leq x - 1 + i)$ or $(x - 1 - (n - i) < y \leq x - 1$ and $y + j = x)$. For each of these cases, we can compute the position $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}])$ of $[\mathbf{e}_{\alpha(\mathbf{i},m)}]$ in $\Gamma_{\mathcal{Q}}$: $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}])$ is equal to $(x, i + j - n - 1)$ if $(y = x - 1 - (n - i)$ and $x < y + j \leq x - 1 + i)$ and it is equal to $(x - j, i + j)$ if $(x - 1 - (n - i) < y \leq x - 1$ and $y + j = x)$.

Lemma (With the above hypothesis and notation)**.** (a) *If $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, $y = x - 1 - (n - i)$ and $x < y + j \leq x - 1 + i$, then the height* $\mathrm{ht}(\lambda')$ *of λ' is strictly smaller than $(n + 1 - i)$. Moreover, if $\overline{\lambda'}$ is the partition with*

$(n-i+1)$ parts defined by

$$\overline{\lambda'} :(n-j+1+\lambda'_1) \geq (n-j+1+\lambda'_2) \geq \cdots \geq (n-j+1+\lambda'_{\ell'}) > (n-j+1) \geq \cdots \geq (n-j+1)$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'}$ where $\mathrm{ht}(\lambda') = \ell' > 0$, and by

$$\overline{\lambda'} : (n-j+1) \geq (n-j+1) \geq \cdots \geq (n-j+1),$$

if λ' is the empty partition $\emptyset$, then $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$.

(b) *If $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, $x-1-(n-i) < y \leq x-1$ and $y+j = x$, then the largest part of λ' is strictly smaller than i. Moreover, if $\overline{\lambda'}$ is the partition with $(j + \mathrm{ht}(\lambda'))$ parts defined by*

$$\overline{\lambda'} : i \geq i \geq \cdots \geq i > \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ where $\mathrm{ht}(\lambda') = \ell' > 0$, and by

$$\overline{\lambda'} : i \geq i \geq \cdots \geq i,$$

if λ' is the empty partition $\emptyset$, then $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$.

(c) *$\Lambda_0(\alpha, \mathbf{c}, h) = \Lambda(\alpha, \hat{\mathbf{c}})$ and $\Lambda_1(\alpha, \mathbf{c}, h) = \{\overline{\lambda'} \mid \lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})\}$.*

(d) *If $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, then*

$$\mathbf{t}(\overline{\lambda'}, \alpha) = \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}].$$

(e) *If $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, then $h \notin J(\lambda', \alpha(\mathbf{i}, m))$ and $J(\overline{\lambda'}, \alpha) = J(\lambda', \alpha(\mathbf{i}, m))$.*

Proof. (a) Assume $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ with $\mathrm{ht}(\lambda') = \ell' \geq (n+1-i)$. Then $(x-\ell', i+j-(n+1)+\ell'-\lambda'_{\ell'})$ is the position $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h')}])$ of an indecomposable module of $\mathcal{Q}$. Here we use the fact that λ' is an $\alpha(\mathbf{i}, m)$-admissible partition. Since $\ell' \geq (n+1-i)$, then $h' < h$. From lemma 4.7 (a), we have $\langle \mathbf{t}(\lambda', \alpha(\mathbf{i}, m), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = 1$. Since $h' < h < m$, we get that $\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$. So $\ell' < (n+1-i)$. From the definition of $\overline{\lambda'}$ and of $\mathbf{t}(\overline{\lambda'}, \alpha)$ and $\mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$, we get

$$\begin{aligned}\mathbf{t}(\overline{\lambda'}, \alpha) &= \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + \sum_{\substack{1 \leq a \leq (n+1-i) \\ 1 \leq b \leq (n+1-j)}} \mathbf{r}_{\Theta^{-1}(x-a+1, i+a-b)} \\ &= \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}].\end{aligned}$$

Thus

$$\begin{aligned}&[\mathbf{V}_\mathbf{c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\overline{\lambda'}, \alpha) \\ &\quad = \big([\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_\alpha]\big) - \big(\mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}]\big) \\ &\quad = \big([\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))\big) + (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}] \in \mathbf{K}_+(\mathcal{Q}, 0)\end{aligned}$$

because $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ and $\rho(\mathbf{c}) = h \Rightarrow c_h \geq 1$. This implies that $\overline{\lambda'} \in \Lambda(\alpha, \mathbf{c})$.

$\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$ because

$$\langle \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = \langle \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$$

by lemma 4.7 applied to $\mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ when λ' is nonempty. Note that if λ' is the empty partition, the result is easily obtained. This proves (a).

(b) Assume $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ with $\mathrm{ht}(\lambda') = \ell' > 0$ and $\lambda'_1 \geq i$. Let a be the largest integer such that $1 \leq a \leq \ell'$ and $\lambda'_a = \lambda'_1$. Then $(x-j-a, i+j+a-\lambda'_a)$ is the position $\Theta([\mathbf{e}_{\alpha(\mathbf{i},h')}])$ of an indecomposable module of $\mathcal{Q}$. Here we use the fact that λ' is an $\alpha(\mathbf{i},m)$-admissible partition. Since $\lambda'_1 \geq i$, then $h' < h$. From lemma 4.7 (a), we have $\langle \mathbf{t}(\lambda', \alpha(\mathbf{i},m)), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = 1$. Since $h' < h < m$, we get that $\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i},m)), \mathbf{r}_{\alpha(\mathbf{i},h')}\rangle_{\mathcal{Q}} = -1$ and this contradicts the fact that $\lambda' \in \Lambda(\alpha(\mathbf{i},m), \hat{\mathbf{c}})$. So $\lambda'_1 < i$. From the definitions of $\overline{\lambda'}$, $\mathbf{t}(\overline{\lambda'}, \alpha)$ and $\mathbf{t}(\lambda', \alpha(\mathbf{i},m))$, we get

$$\begin{aligned}\mathbf{t}(\overline{\lambda'}, \alpha) &= \mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + \sum_{\substack{1\leq a\leq j\\ 1\leq b\leq i}} \mathbf{r}_{\Theta^{-1}(x-a+1, i+a-b)} \\ &= \mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}].\end{aligned}$$

Thus

$$\begin{aligned}&[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\overline{\lambda'}, \alpha) \\ &\quad = \big([\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_\alpha]\big) - \big(\mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}]\big) \\ &\quad = \big([\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i},m))\big) + (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}] \in \mathbf{K}_+(\mathcal{Q}, 0)\end{aligned}$$

because $\lambda' \in \Lambda(\alpha(\mathbf{i},m), \hat{\mathbf{c}})$ and $\rho(\mathbf{c}) = h \;\Rightarrow\; c_h \geq 1$. This implies that $\overline{\lambda'} \in \Lambda(\alpha, \mathbf{c})$. $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$ because

$$\langle \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = \langle \mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$$

by lemma 4.7 applied to $\mathbf{t}(\lambda', \alpha(\mathbf{i},m))$ when λ' is nonempty. Note that if λ' is the empty partition, the result is easily obtained. This proves (b).

(c) We have $\Lambda_0(\alpha, \mathbf{c}, h) = \Lambda(\alpha, \hat{\mathbf{c}})$ by lemma 4.10 (b). We have also seen in (a) and (b) that

$$\{\overline{\lambda'} \mid \lambda' \in \Lambda(\alpha(\mathbf{i},m), \hat{\mathbf{c}})\} \subseteq \Lambda_1(\alpha, \mathbf{c}, h).$$

Let $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h)$, then λ cannot be the empty partition. Write $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ with $\mathrm{ht}(\lambda) = \ell > 0$. Since $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$, we get that the position $(y, j) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$ is $(x - a, i + a - \lambda_a)$ for some $1 \leq a \leq \ell$ and $\lambda_a \neq \lambda_{a+1}$ by lemma 4.7 (a). Note that this position cannot be $(x, i) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$. There are two situations to consider corresponding to (a) and (b) above.

If $y = x - 1 - (n - i)$ and $x < y + j \leq (x - 1 + i)$, then $y = x - a = x - 1 - (n - i)$ and $a = n - i + 1$. In this case, $\ell \leq (n - i + 1)$ because $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}]) = (x, i)$ and λ must be α-admissible. So $a = \ell = n - i + 1$. Also $j = i + a - \lambda_a$ implies that $\lambda_a = i + (n - i + 1) - j = (n - j + 1)$. Now consider the partition $\lambda' : \lambda_1 - (n - j + 1) \geq \lambda_2 - (n - j + 1) \geq \cdots \geq \lambda_\ell - (n - j + 1) = 0$. Either λ' is the empty partition $\emptyset$ or $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ with $\mathrm{ht}(\lambda') = \ell' > 0$. If $\lambda' = \emptyset$, then $\lambda' \in \Lambda(\alpha(\mathbf{i},m), \hat{\mathbf{c}})$ and $\lambda = \overline{\lambda'}$. Assume now λ' is nonempty. Write $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ with $\ell' = \mathrm{ht}(\lambda') > 0$. Since $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (x, i + j - n - 1)$, $\Theta([\mathbf{e}_\alpha]) = (x, i)$, λ is an α-admissible partition and λ' is obtained by subtracting $(n - j + 1)$ to each nonzero part of λ, we get that λ' is an $\alpha(\mathbf{i},m)$-admissible partition. To prove that $\lambda' \in \Lambda(\alpha(\mathbf{i},m), \hat{\mathbf{c}})$, we can first observe that

$$\begin{aligned}\mathbf{t}(\lambda, \alpha) &= \mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + \sum_{\substack{1\leq a'\leq (n+1-i)\\ 1\leq b'\leq (n+1-j)}} \mathbf{r}_{\Theta^{-1}(x-a'+1, i+a'-b')} \\ &= \mathbf{t}(\lambda', \alpha(\mathbf{i},m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}].\end{aligned}$$

So

$$\begin{aligned}
&[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) \\
&\qquad = \big([\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}]\big) - \big(\mathbf{t}(\lambda, \alpha) - [\mathbf{e}_\alpha] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - [\mathbf{e}_{\alpha(\mathbf{i},h)}]\big) \\
&\qquad = \big([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)\big) - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}]
\end{aligned}$$

and

$$\begin{aligned}
&\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \\
&\qquad\qquad = \langle (([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)) - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_\beta \rangle_{\mathcal{Q}} \\
&\qquad\qquad = \begin{cases} \langle (([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \geq 0, & \text{if } \beta \neq \alpha(\mathbf{i}, h), \\ (c_h - 1) - (c_h - 1) = 0, & \text{if } \beta = \alpha(\mathbf{i}, h) \end{cases}
\end{aligned}$$

because $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h) \Rightarrow \langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 1$. Thus $[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) \in \mathbf{K}_+(\mathcal{Q}, 0)$ and $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$. Clearly, $\overline{\lambda'} = \lambda$. So $\Lambda_1(\alpha, \mathbf{c}, h) = \{\overline{\lambda'} \mid \lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})\}$.

If $x - 1 - (n - i) < y \leq (x - 1)$ and $y + j = x$, then $y = x - a = y + j - a$ and $1 \leq a = j \leq \ell$. Also $j = i + a - \lambda_a = i + j - \lambda_j$ implies that $\lambda_j = i$. Since $\lambda_1 \leq i$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j = i$, we have $\lambda_1 = \lambda_2 = \cdots = \lambda_j = i$ and $\lambda_j > \lambda_{j+1}$. Consider now the partition λ' defined as either the empty partition $\emptyset$ if $j = \ell$ or as $\lambda' : \lambda_{j+1} \geq \lambda_{j+2} \geq \cdots \geq \lambda_\ell > 0$ if $j < \ell$. If $\lambda' = \emptyset$, then $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ and $\lambda = \overline{\lambda'}$. Assume now that $j < \ell$ and $\lambda' : \lambda_{j+1} \geq \lambda_{j+2} \geq \cdots \geq \lambda_\ell > 0$. Since $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (x - j, i + j)$, $\Theta([\mathbf{e}_\alpha]) = (x, i)$, λ is an α-admissible partition and the definition of λ', we get that λ' is an $\alpha(\mathbf{i}, m)$-admissible partition. To prove that $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, we can first observe that

$$\begin{aligned}
\mathbf{t}(\lambda, \alpha) &= \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + \sum_{\substack{1 \leq a' \leq j \\ 1 \leq b' \leq i}} \mathbf{r}_{\Theta^{-1}(x - a' + 1, i + a' - b')} \\
&= \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}].
\end{aligned}$$

So

$$\begin{aligned}
&[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) \\
&\qquad = \big([\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}]\big) - \big(\mathbf{t}(\lambda, \alpha) - [\mathbf{e}_\alpha] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - [\mathbf{e}_{\alpha(\mathbf{i},h)}]\big) \\
&\qquad = \big([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)\big) - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}]
\end{aligned}$$

and

$$\begin{aligned}
&\langle [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \\
&\qquad\qquad = \langle (([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)) - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_\beta \rangle_{\mathcal{Q}} \\
&\qquad\qquad = \begin{cases} \langle (([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \geq 0, & \text{if } \beta \neq \alpha(\mathbf{i}, h), \\ (c_h - 1) - (c_h - 1) = 0, & \text{if } \beta = \alpha(\mathbf{i}, h) \end{cases}
\end{aligned}$$

because $\lambda \in \Lambda_1(\alpha, \mathbf{c}, h) \Rightarrow \langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 1$. Thus $[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) \in \mathbf{K}_+(\mathcal{Q}, 0)$ and $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$. Clearly $\overline{\lambda'} = \lambda$. So $\Lambda_1(\alpha, \mathbf{c}, h) = \{\overline{\lambda'} \mid \lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})\}$ and (c) is proved.

(d) follows from our proof of (a) and (b).

(e) Assume that $h \in J(\lambda', \alpha(\mathbf{i}, m))$, then

$$\begin{aligned} 1 &= \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} \\ &= \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 1 - \langle \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} \end{aligned}$$

implies that $\overline{\lambda'} \in \Lambda_0(\alpha, \mathbf{c}, h)$. But this is impossible because we saw in (a) and (b) that $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$. So $h \notin J(\lambda', \alpha(\mathbf{i}, m))$.

If $a \in J(\lambda', \alpha(\mathbf{i}, m))$, then $a \neq h$ and

$$\begin{aligned} 1 &= \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} \\ &= \langle [\mathbf{e}_{\alpha}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}] - \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}}. \end{aligned}$$

So $a \in J(\overline{\lambda'}, \alpha)$ and $J(\lambda', \alpha(\mathbf{i}, m)) \subseteq J(\overline{\lambda'}, \alpha)$.

If $a \in J(\overline{\lambda'}, \alpha)$, then $a \neq h$ by lemma 4.10 (e) and

$$\begin{aligned} 1 &= \langle [\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{\lambda'}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) - [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} \\ &= \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}}. \end{aligned}$$

So $a \in J(\lambda', \alpha(\mathbf{i}, m))$ and $J(\overline{\lambda'}, \alpha) \subseteq J(\lambda', \alpha(\mathbf{i}, m))$. This concludes the proof of (e). □

4.13. Let $\alpha = \alpha(\mathbf{i}, k) \in R^+$ and $\mathbf{c} = (c_1, c_2, \dots, c_\nu) \in \mathbf{N}^\nu$ with $\mathbf{c} \neq (0, 0, \dots, 0)$ and $\rho(\mathbf{c}) = h < k$. Assume that $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},k)}] \rangle_{\mathcal{Q}} = 1$ and $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 1$. As noted in 3.13 (d), there exists a unique pair of distinct positive roots $\alpha(\mathbf{i}, m)$, $\alpha(\mathbf{i}, m')$ such that $\alpha(\mathbf{i}, h) + \alpha(\mathbf{i}, k) = \alpha(\mathbf{i}, m) + \alpha(\mathbf{i}, m')$, $h < m < m' < k$ and there exists a nonsplit short exact sequence of the form $0 \to \mathbf{e}_{\alpha(\mathbf{i},h)} \to \mathbf{e}_{\alpha(\mathbf{i},m)} \oplus \mathbf{e}_{\alpha(\mathbf{i},m')} \to \mathbf{e}_{\alpha(\mathbf{i},k)} \to 0$ that is a basis of $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)})$. Let $(x, i) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$ (resp. $(y, j) = \Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$) be the position of $[\mathbf{e}_{\alpha(\mathbf{i},k)}]$ (resp. $[\mathbf{e}_{\alpha(\mathbf{i},h)}]$) in the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$. As noted in 3.13 (d), we have $(x - n + i) \leq y \leq (x - 1)$ and $(x + 1) \leq (y + j) \leq (x - 1 + i)$. We have either $\big(\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (x, y + j - x)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (y, x + i - y)\big)$ or $\big(\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (y, x + i - y)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (x, y + j - x)\big)$.

Lemma (With the above hypothesis and notation)**.** (a) *If* $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (x, y + j - x)$ *and* $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (y, x + i - y)$, *then the height* $\mathrm{ht}(\lambda')$ *of* λ' *is strictly smaller than* $(x - y)$ *and the largest part* λ''_1 *of* λ'' *is strictly smaller than* $(x - y + i - j)$. *Moreover, let* $\overline{(\lambda', \lambda'')}$ *be the partition with* $(x - y) + \mathrm{ht}(\lambda'')$ *parts defined by*

$$\begin{aligned} (x - y + i - j + \lambda'_1) &\geq (x - y + i - j + \lambda'_2) \geq \cdots \geq (x - y + i - j + \lambda'_{\ell'}) \\ &> (x - y + i - j) \geq \cdots \geq (x - y + i - j) > \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_{\ell''}, \end{aligned}$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \dots \lambda'_{\ell'} > 0$ *where* $\mathrm{ht}(\lambda') = \ell' > 0$ *and* $\lambda'' : \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_{\ell''} > 0$ *where* $\mathrm{ht}(\lambda'') = \ell'' > 0$; *by*

$$(x - y + i - j) \geq (x - y + i - j) \geq \cdots \geq (x - y + i - j) > \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_{\ell''},$$

if λ' *is the empty partition* $\emptyset$ *and* $\lambda'' : \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_{\ell''} > 0$ *where* $\mathrm{ht}(\lambda'') = \ell'' > 0$; *by*

$$\begin{aligned} (x - y + i - j + \lambda'_1) &\geq (x - y + i - j + \lambda'_2) \geq \cdots \geq (x - y + i - j + \lambda'_{\ell'}) \\ &> (x - y + i - j) \geq \cdots \geq (x - y + i - j), \end{aligned}$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \dots \lambda'_{\ell'} > 0$ *where* $\mathrm{ht}(\lambda') = \ell' > 0$ *and* λ'' *is the empty partition* $\emptyset$*; and by*

$$(x-y+i-j) \geq (x-y+i-j) \geq \cdots \geq (x-y+i-j),$$

if both λ' *and* λ'' *are the empty partition* $\emptyset$*; then* $\overline{(\lambda', \lambda'')} \in \Lambda_1(\alpha, \mathbf{c}, h)$.

(b) *If* $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (y, x+i-y)$ *and* $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (x, y+j-x)$*, then the largest part* λ'_1 *of* λ' *is strictly smaller than* $(x-y+i-j)$ *and the height* $\mathrm{ht}(\lambda'')$ *of* λ'' *is strictly smaller than* $(x-y)$. *Moreover, let* $\overline{(\lambda', \lambda'')}$ *be the partition with* $(x-y)+\mathrm{ht}(\lambda')$ *parts defined by*

$$\begin{aligned}(x-y+i-j+\lambda''_1) \geq (x-y+i-j+\lambda''_2) \geq \cdots \geq (x-y+i-j+\lambda''_{\ell''})\\ > x-y+i-j \geq \cdots \geq x-y+i-j > \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0,\end{aligned}$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \dots \lambda'_{\ell'} > 0$ *where* $\mathrm{ht}(\lambda') = \ell' > 0$ *and* $\lambda'' : \lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_{\ell''} > 0$ *where* $\mathrm{ht}(\lambda'') = \ell'' > 0$*; by*

$$\begin{aligned}&(x-y+i-j+\lambda''_1) \geq (x-y+i-j+\lambda''_2) \geq \dots\\ &\geq (x-y+i-j+\lambda''_{\ell''}) > (x-y+i-j) \geq \cdots \geq (x-y+i-j) > 0,\end{aligned}$$

if λ' *is the empty partition* $\emptyset$ *and* $\lambda'' : \lambda''_1 \geq \lambda''_2 \geq \dots \lambda''_{\ell''} > 0$ *where* $\mathrm{ht}(\lambda'') = \ell'' > 0$*; by*

$$\begin{aligned}(x-y+i-j) \geq (x-y+i-j) \geq \cdots \geq (x-y+i-j) > \lambda'_1\\ \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0,\end{aligned}$$

if $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{\ell'} > 0$ *where* $\mathrm{ht}(\lambda') = \ell' > 0$ *and* λ'' *is the empty partition* $\emptyset$ *and by*

$$(x-y+i-j) \geq (x-y+i-j) \geq \cdots \geq (x-y+i-j) > 0,$$

if both λ' *and* λ'' *are the empty partition* $\emptyset$*, then* $\overline{(\lambda', \lambda'')} \in \Lambda_1(\alpha, \mathbf{c}, h)$.

(c) $\Lambda_0(\alpha, \mathbf{c}, h) = \Lambda(\alpha, \hat{\mathbf{c}})$ *and* $\Lambda_1(\alpha, \mathbf{c}, h) = \{\overline{(\lambda', \lambda'')} \mid \lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}}), \lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})\}$.

(d) *If* $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ *and* $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$*, then* $\mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)$ *is equal to* $\mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')) + [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] - [\mathbf{e}_{\alpha(\mathbf{i},m')}]$. *Also,*

$$[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))) = 1,$$

$$[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) = 1,$$

$$([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) \ \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))) = 0 \quad \textit{and}$$

$$([\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)) \ \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = -1.$$

(e) *If* $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ *and* $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$*, then*

$$h \notin J(\lambda', \alpha(\mathbf{i}, m)) \cup J(\lambda'', \alpha(\mathbf{i}, m')),$$

$$J(\lambda', \alpha(\mathbf{i}, m)) \cap J(\lambda'', \alpha(\mathbf{i}, m')) = \emptyset \quad \textit{and}$$

$$J(\overline{(\lambda', \lambda'')}, \alpha) = J(\lambda', \alpha(\mathbf{i}, m)) \cup J(\lambda'', \alpha(\mathbf{i}, m')).$$

(f) *If* $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$ *and* $\mathbf{c}' \in \mathbf{N}^\nu$ *is such that* $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))$*, then*

$$\Lambda(\alpha(\mathbf{i}, m), \mathbf{c}') = \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}}).$$

Proof. (a) The argument to prove that $\mathrm{ht}(\lambda')$ is strictly smaller than $(x-y)$ is similar to the one used in lemma 4.12 (a). The argument to prove that λ''_1 is strictly smaller than $(x-y+i-j)$ is similar to the one used in lemma 4.12 (b). We leave this part of the proof to the reader.

We have

$$\begin{aligned}
&\mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)\\
&\quad= \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) + \sum_{\substack{1\le a\le (x-y)\\ 1\le b\le (x-y+i-j)}} \mathbf{r}_{\Theta^{-1}(x-a+1,i+a-b)}\\
&\quad= \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] - [\mathbf{e}_{\alpha(\mathbf{i},m')}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}].
\end{aligned}$$

Consequently,

$$\begin{aligned}
&[\mathbf{V_c}] + [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)\\
&= [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) - [\mathbf{e}_\alpha]\\
&\quad + [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - [\mathbf{e}_{\alpha(\mathbf{i},h)}]\\
&= [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) + (c_h-1)[\mathbf{e}_{\alpha(\mathbf{i},h)}].
\end{aligned}$$

Note that if $\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)), \mathbf{r}_\beta\rangle_{\mathcal{Q}} \neq 0$, then

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = 0 \quad \text{for } \beta\in R^+;$$

similarly, if $\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')), \mathbf{r}_\beta\rangle_{\mathcal{Q}} \neq 0$, then

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)), \mathbf{r}_\beta\rangle_{\mathcal{Q}} = 0 \quad \text{for } \beta\in R^+.$$

This follows because λ is a $\alpha(\mathbf{i},m)$-admissible partition, λ' is a $\alpha(\mathbf{i},m')$-admissible partition and the respective positions of $[\mathbf{e}_{\alpha(\mathbf{i},m)}]$ and $[\mathbf{e}_{\alpha(\mathbf{i},m')}]$ in the Auslander-Reiten quiver.

From this observation, the formula above for $\mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)$ and the fact that $\lambda'\in\Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$ and $\lambda''\in\Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$ we get that $\langle[\mathbf{V_c}]+[\mathbf{e}_\alpha]-\mathbf{t}(\overline{(\lambda',\lambda'')},\alpha),\mathbf{r}_\beta\rangle_{\mathcal{Q}}\geq 0$ for all $\beta\in R^+$ and $\overline{(\lambda',\lambda'')}\in\Lambda(\alpha,\mathbf{c})$.

$\overline{(\lambda',\lambda'')}\in\Lambda_1(\alpha,\mathbf{c},h)$ because $\langle\mathbf{t}(\overline{(\lambda',\lambda'')},\alpha),\mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}}$ is equal to

$$\langle \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) + [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_\alpha] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] - [\mathbf{e}_{\alpha(\mathbf{i},m')}], \mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}} = 1$$

by lemma 4.7 (a) applied to $\mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ when λ' is a nonempty partition and to $\mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ when λ'' is a nonempty partition. Note that if λ' and/or λ'' is the empty partition, the result is easily obtained.

(b) The proof is similar to the proof of (a) and it is left to the reader.

(c) $\Lambda_0(\alpha,\mathbf{c},h)=\Lambda(\alpha,\hat{\mathbf{c}})$ by lemma 4.10 (b). We also have by (a) and (b) that

$$\{\overline{(\lambda',\lambda'')} \mid \lambda'\in\Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}}), \lambda''\in\Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})\} \subseteq \Lambda_1(\alpha,\mathbf{c},h).$$

Let $\lambda\in\Lambda_1(\alpha,\mathbf{c},h)$, then λ cannot be the empty partition. Write $\lambda:\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_\ell>0$ with $\mathrm{ht}(\lambda)=\ell>0$. Since $\langle\mathbf{t}(\lambda,\alpha),\mathbf{r}_{\alpha(\mathbf{i},h)}\rangle_{\mathcal{Q}}=1$, we get that the position $(y,j)=\Theta([\mathbf{e}_{\alpha(\mathbf{i},h)}])$ is $(x-a,i+a-\lambda_a)$ for some $1\leq a\leq\ell$ and $\lambda_a\neq\lambda_{a+1}$ by lemma 4.7 (a). Note that this position cannot be $(x,i)=\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$. There are two situations to consider corresponding to (a) and (b) above. In both cases, $y=x-a$ and $j=i+a-\lambda_a$ with $\lambda_a\neq\lambda_{a+1}$. Thus $\lambda_a=x-y+i-j$.

If $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (x, y+j-x)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (y, x+i-y)$, then we can consider the following partitions λ' and λ'' defined by

$$\lambda' : (\lambda_1-\lambda_a) \geq (\lambda_2-\lambda_a) \geq \cdots \geq (\lambda_{a-1}-\lambda_a) \quad \text{and} \quad \lambda'' : \lambda_{a+1} \geq \lambda_{a+2} \geq \cdots \geq \lambda_\ell;$$

while if $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m)}]) = (y, x+i-y)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m')}]) = (x, y+j-x)$, then we can consider the following partitions λ' and λ'' defined by

$$\lambda' : \lambda_{a+1} \geq \lambda_{a+2} \geq \cdots \geq \lambda_\ell \quad \text{and} \quad \lambda'' : (\lambda_1-\lambda_a) \geq (\lambda_2-\lambda_a) \geq \cdots \geq (\lambda_{a-1}-\lambda_a).$$

It is easy to check in both cases that λ' is an $\alpha(\mathbf{i},m)$-admissible partition, $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$, λ'' is an $\alpha(\mathbf{i},m')$-admissible partition, $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$ and $\lambda = \overline{(\lambda',\lambda'')}$. The proof of these statements is very similar to the argument used in the proof of lemma 4.12 (c).

(d) The formula for $\mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)$ has been proved in (a) and (b). We have $\langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],[\mathbf{e}_{\alpha(\mathbf{i},m')}]\rangle_{\mathcal{Q}} = 1$ and $\langle[\mathbf{e}_{\alpha(\mathbf{i},m')}],[\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = 0$ because of the respective positions of $[\mathbf{e}_{\alpha(\mathbf{i},h)}]$ and $[\mathbf{e}_{\alpha(\mathbf{i},m')}]$ in the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ and using 3.13 (a) and (b).

When we write $\mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ and $\tau(\mathbf{t}(\lambda'',\alpha(\mathbf{i},m')))$ in the basis $\{\mathbf{r}_\beta \mid \beta \in R^+\}$, then the coefficient of $\mathbf{r}_{\alpha(\mathbf{i},h)}$ in each case is 0, because of the position of $[\mathbf{e}_{\alpha(\mathbf{i},m')}]$ in $\Gamma_{\mathcal{Q}}$ and the fact that $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$. Consequently, $\langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],\mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))\rangle_{\mathcal{Q}}$ $= 0$ and $\langle\mathbf{t}(\lambda'',\alpha(\mathbf{i},m')),[\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = \langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],\tau(\mathbf{t}(\lambda'',\alpha(\mathbf{i},m')))\rangle_{\mathcal{Q}} = 0$. From all of this, we can conclude that $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))) = 1$.

The proof that $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))) = 1$ is similar to the one above.

If we express $[\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ in the basis $\{[\mathbf{e}_\beta] \mid \beta \in R^+\}$, we can tell by lemma 4.7 (a) for which $\beta \in R^+$ the coefficient of $[\mathbf{e}_\beta]$ is nonzero and also we know the position of these $[\mathbf{e}_\beta]$ in $\Gamma_{\mathcal{Q}}$. For each of these β, we get using 3.13 (a) and (b), the fact that $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$ and $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$ and the definition of $\mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ that $\langle[\mathbf{e}_\beta],[\mathbf{e}_{\alpha(\mathbf{i},m')}]\rangle_{\mathcal{Q}} = 0$, $\langle[\mathbf{e}_{\alpha(\mathbf{i},m')}],[\mathbf{e}_\beta]\rangle^1_{\mathcal{Q}} = 0$, $\langle[\mathbf{e}_\beta],\mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))\rangle_{\mathcal{Q}} = 0$ and

$$\langle\mathbf{t}(\lambda'',\alpha(\mathbf{i},m')),[\mathbf{e}_\beta]\rangle^1_{\mathcal{Q}} = \langle[\mathbf{e}_\beta],\tau(\mathbf{t}(\lambda'',\alpha(\mathbf{i},m')))\rangle_{\mathcal{Q}} = 0.$$

Consequently,

$$([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))) \ \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))) = 0.$$

We have

$$[\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda',\lambda'')},\alpha) = [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) - [\mathbf{e}_{\alpha(\mathbf{i},h)}]$$

and

$$\begin{aligned}&([\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)) \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}]\\ &= ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) - [\mathbf{e}_{\alpha(\mathbf{i},h)}]) \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}].\end{aligned}$$

If we express $[\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ in the basis $\{[\mathbf{e}_\beta] \mid \beta \in R^+\}$, we can tell by lemma 4.7 (a) for which $\beta \in R^+$ the coefficient of $[\mathbf{e}_\beta]$ is nonzero and the position of $[\mathbf{e}_\beta]$ in the Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$. For each of these β, we get using 3.13 (a) and (b) and the fact that $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$, that $\langle[\mathbf{e}_\beta],[\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle_{\mathcal{Q}} = 0$ and $\langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],[\mathbf{e}_\beta]\rangle^1_{\mathcal{Q}} = 0$. Consequently, $([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))) \ \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = 0$. In the same way, we get that $([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))) \ \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = 0$. We also have $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle_{\mathcal{Q}} - \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 1$. Putting all of this together, we get that

$$([\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)) \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = -[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = -1.$$

(e) Assume that $h \in J(\lambda', \alpha(\mathbf{i}, m))$, then $\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 1$ and, by what we have noted above, this implies that

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 0.$$

Thus

$$\begin{aligned} 1 &= \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} \\ &= \langle [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha) - [\mathbf{e}_{\alpha(\mathbf{i},m')}] + \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')) + [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} \end{aligned}$$

and we get $\langle [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 0$. So $\overline{(\lambda', \lambda'')} \in \Lambda_0(\alpha, \mathbf{c}, h)$. But this contradicts the fact that $\overline{(\lambda', \lambda'')} \in \Lambda_1(\alpha, \mathbf{c}, h)$. So $h \notin J(\lambda', \alpha(\mathbf{i}, m))$. The same way we can prove that $h \notin J(\lambda'', \alpha(\mathbf{i}, m'))$. Thus $h \notin J(\lambda', \alpha(\mathbf{i}, m)) \cup J(\lambda'', \alpha(\mathbf{i}, m'))$.

$J(\lambda', \alpha(\mathbf{i}, m)) \cap J(\lambda'', \alpha(\mathbf{i}, m')) = \emptyset$ because we noted before that if $\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \neq 0$, then $\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_\beta \rangle_{\mathcal{Q}} = 0$ and if $\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_\beta \rangle_{\mathcal{Q}} \neq 0$, then $\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_\beta \rangle_{\mathcal{Q}} = 0$.

If $a \in J(\lambda', \alpha(\mathbf{i}, m))$, then $a \neq h$ by lemma 4.10 (e) and

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1 \Rightarrow \langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 0.$$

Thus $\langle [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}}$ is equal to

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')) - [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$$

and $a \in J(\overline{(\lambda', \lambda'')}, \alpha)$. Similarly, if $a \in J(\lambda'', \alpha(\mathbf{i}, m'))$, then $a \in J(\overline{(\lambda', \lambda'')}, \alpha)$. So

$$J(\lambda', \alpha(\mathbf{i}, m)) \cup J(\lambda'', \alpha(\mathbf{i}, m')) \subseteq J(\overline{(\lambda', \lambda'')}, \alpha).$$

If $a \in J(\overline{(\lambda', \lambda'')}, \alpha)$, then $a \neq h$ by lemma 4.10 (e) and

$$\begin{aligned} 1 &= \langle [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} \\ &= \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')) - [\mathbf{e}_{\alpha(\mathbf{i},h)}], \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}}. \end{aligned}$$

Thus

$$\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} + \langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1.$$

Both of the summands on the left-hand side of the previous equation cannot be zero. But as we saw above if one of them is different from 0, then the other must be 0. Consequently, we must have either $\langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$ or $\langle [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')), \mathbf{r}_{\alpha(\mathbf{i},a)} \rangle_{\mathcal{Q}} = 1$. From all of this, we get that

$$J(\lambda', \alpha(\mathbf{i}, m)) \cup J(\lambda'', \alpha(\mathbf{i}, m')) = J(\overline{(\lambda', \lambda'')}, \alpha).$$

□

4.14. With the notation of 4.12 and 4.13, it is easy to see that the map $\lambda' \mapsto \overline{\lambda'}$ from $\Lambda(\alpha(\mathbf{i}, \hat{\mathbf{c}}))$ to $\Lambda_1(\alpha, \mathbf{c}, h)$ and the map $(\lambda', \lambda'') \mapsto \overline{(\lambda', \lambda'')}$ from $\Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}}) \times \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$ to $\Lambda_1(\alpha, \mathbf{c}, h)$ are both injective and consequently they are both bijective.

4.15. Let $\alpha = \alpha(\mathbf{i}, k) \in R^+$ and $\mathbf{c} \in \mathbf{N}^\nu$. Define $g_{k,\mathbf{c},\mathbf{c}'} \in \mathbf{Z}[v, v^{-1}]$ for $\mathbf{c}' \in \mathbf{N}^\nu$ by

$$E_{\mathbf{i}}^{\mathbf{b}(k)} \ E_{\mathbf{i}}^{\mathbf{c}} = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} g_{k,\mathbf{c},\mathbf{c}'} \ E_{\mathbf{i}}^{\mathbf{c}'}.$$

Recall that $\mathbf{b}(k)$ has been defined in 3.2. Note also that this sum is a finite sum.

Theorem 4.16. *Let $\alpha = \alpha(\mathbf{i}, k) \in R^+$ and $\mathbf{c} \in \mathbf{N}^\nu$. If $\mathbf{c}' \in \mathbf{N}^\nu$, then $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for an α-admissible partition $\lambda \in \Lambda(\alpha, \mathbf{c})$. Moreover, in this case, λ is uniquely determined by $\mathbf{c}'$ and reciprocally $\mathbf{c}'$ is uniquely determined by λ. Also,*

$$g_{k,\mathbf{c},\mathbf{c}'} = v^{[\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} \ (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c'_j}).$$

Proof. If $\mathbf{c} = (0, 0, \ldots, 0)$, then clearly $\Lambda(\alpha, \mathbf{c})$ has only one element: the empty partition, $E_{\mathbf{i}}^{\mathbf{c}} = 1$ and

$$g_{k,\mathbf{c},\mathbf{c}'} = \begin{cases} 1, & \text{if } \mathbf{c}' = \mathbf{b}(k), \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbf{c}' = \mathbf{b}(k)$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\emptyset, \alpha)$ because $[\mathbf{V}_{\mathbf{c}}] = 0$ in $\mathbf{K}(\mathcal{Q}, 0)$. Moreover, for $\mathbf{c}' = \mathbf{b}(k)$, we have that $[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\emptyset, \alpha)) = [\mathbf{e}_\alpha] \circ_{\mathcal{Q}} [\mathbf{e}_\alpha] = 1$, $J(\emptyset, \alpha) = \{k\}$ and

$$v^{[\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} \ (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c'_j}) = v \ (v - v^{-1})^{-1} \ (1 - v^{-2}) = 1.$$

The theorem is verified for $\mathbf{c} = (0, 0, \ldots, 0)$.

We will now assume that $\mathbf{c} \neq (0, 0, \ldots, 0)$. Let $\rho(\mathbf{c}) = \min\{1 \leq j \leq \nu \mid c_j \neq 0\}$. We will prove the theorem by decreasing induction on $\rho(\mathbf{c})$ starting with $\rho(\mathbf{c}) = \nu$. First we can note that for some cases, the theorem is easily verified. If $k \leq \rho(\mathbf{c})$, then there is only one $\mathbf{c}' \in \mathbf{N}^\nu$ for which $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$; it is given by $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha]$. The kth component c'_k of $\mathbf{c}'$ is 1 if $k < \rho(\mathbf{c})$ and is $(c_k + 1)$ if $k = \rho(\mathbf{c})$. For this unique $\mathbf{c}'$, we have that

$$g_{k,\mathbf{c},\mathbf{c}'} = \begin{cases} 1, & \text{if } k < \rho(\mathbf{c}), \\ (v^{(c_k+1)} - v^{-(c_k+1)})/(v - v^{-1}), & \text{if } k = \rho(\mathbf{c}). \end{cases}$$

$\Lambda(\alpha, \mathbf{c})$ has only one element: the empty partition $\lambda = \emptyset$. For otherwise if λ is a nonempty partition belonging to $\Lambda(\alpha, \mathbf{c})$, then, by lemma 4.7, there exists h such that $1 \leq h < k$ and $\langle \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 1$ and consequently

$$\langle [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha), \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = -1$$

because $h < k \leq \rho(\mathbf{c})$ implies that $\langle [\mathbf{V}_{\mathbf{c}}], \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 0$ and $\langle [\mathbf{e}_\alpha], \mathbf{r}_{\alpha(\mathbf{i},h)} \rangle_{\mathcal{Q}} = 0$. This contradicts the fact that $[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \in \mathbf{K}_+(\mathcal{Q}, 0)$. For the unique $\mathbf{c}'$ above such that $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$, we have that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ if and only if λ is the empty partition $\emptyset$. Moreover, we also have that

$$[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\emptyset, \alpha)) = ([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha]) \circ_{\mathcal{Q}} [\mathbf{e}_\alpha] = \begin{cases} 1, & \text{if } k < \rho(\mathbf{c}), \\ (c_k + 1), & \text{if } k = \rho(\mathbf{c}), \end{cases}$$

and $J(\emptyset, \alpha) = \{k\}$. If $k < \rho(\mathbf{c})$, then

$$v^{[\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} \ (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c'_j}) = v \ (v - v^{-1})^{-1} \ (1 - v^{-2}) = 1;$$

while, if $k = \rho(\mathbf{c})$, then

$$v^{[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c'_j})$$
$$= v^{c_k+1} (v - v^{-1})^{-1} (1 - v^{-2(c_k+1)}) = (v^{(c_k+1)} - v^{-(c_k+1)})/(v - v^{-1}).$$

Thus the theorem is verified if $k \leq \rho(\mathbf{c})$.

If $\rho(\mathbf{c}) = \nu$, then the result is true by what we have proved above and because $k \leq \nu = \rho(\mathbf{c})$. Assume now that the proposition is true whenever $\rho(\mathbf{c}) > h$ and consider $\mathbf{c} \in \mathbf{N}^\nu$ such that $\rho(\mathbf{c}) = h$. By what we have proved above, we can assume that $k > h$. We will have to consider four cases corresponding to the possibilities for $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}}$ and $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}}$. For $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$, then $\mathbf{c}(j)$ will denote the element of $\mathbf{N}^\nu$ whose jth coordinate is c_j and its other coordinates are zero.

If $\langle \mathbf{e}_{\alpha(\mathbf{i},h)}, \mathbf{e}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$ and $\langle \mathbf{e}_{\alpha(\mathbf{i},k)}, \mathbf{e}_{\alpha(\mathbf{i},h)} \rangle^1_{\mathcal{Q}} = 0$, then $E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} = E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)}$ by using repeatedly proposition 3.14. For $\hat{\mathbf{c}} \in \mathbf{N}^\nu$ defined by $[\mathbf{V}_{\hat{\mathbf{c}}}] = [\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, we have either $\hat{\mathbf{c}} = (0, 0, \ldots, 0)$ or $\hat{\mathbf{c}} \neq (0, 0, \ldots, 0)$ and $\rho(\hat{\mathbf{c}}) > \rho(\mathbf{c})$. In either case, we can assume by induction that the theorem is verified for $\hat{\mathbf{c}}$. So

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} \, E_{\mathbf{i}}^{\mathbf{c}''}$$

where $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for an α-admissible partition $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$. Moreover, in this case λ is unique and

$$g_{k,\hat{\mathbf{c}},\mathbf{c}''} = v^{[\mathbf{V}_{\mathbf{c}''}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c''_j}).$$

In particular, $\rho(\mathbf{c}'') > h$ whenever $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$. In fact, if $\hat{\mathbf{c}} = (0, 0, \ldots, 0)$, then $\mathbf{c}'' = \mathbf{b}(k)$ and obviously $\rho(\mathbf{c}'') > h$. If $\hat{\mathbf{c}} \neq (0, 0, \ldots, 0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < k$, then we get that $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$, because $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}'') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq k$, then $\mathbf{c}'' = \hat{\mathbf{c}} + \mathbf{b}(k)$ and $\rho(\mathbf{c}'') \geq k > h$.

We have

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}} = E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = E_{\mathbf{i}}^{\mathbf{c}(h)} \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''}$$
$$= \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''+\mathbf{c}(h)} = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} g_{k,\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}.$$

So $g_{k,\mathbf{c},\mathbf{c}'} = g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$ and $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] = [\mathbf{V}_{\mathbf{c}'}] - c_h \, [\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some α-admissible partition $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$. Since $[\mathbf{V}_{\mathbf{c}}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$ and $\Lambda(\alpha, \mathbf{c}) = \Lambda(\alpha, \hat{\mathbf{c}})$ by lemma 4.11, we get that $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some α-admissible partition $\lambda \in \Lambda(\alpha, \mathbf{c})$. If $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ and λ is the unique element of $\Lambda(\alpha, \mathbf{c})$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$, then

$$g_{k,\mathbf{c},\mathbf{c}'} = g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$$
$$= v^{[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} \left(1 - v^{-2(c'_j - \mathbf{c}(h)_j)}\right).$$

But

$$[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) = ([\mathbf{V}_{\mathbf{c}'}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]) \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))$$
$$= [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)),$$

because $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_\alpha] \rangle_{\mathcal{Q}} = 0$, $\langle [\mathbf{e}_\alpha], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 0$ by hypothesis and $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha) = 0$ by lemma 4.10 (b) and (d). By lemma 4.10 (e), $h \notin J(\lambda,\alpha)$ and

$$\prod_{j \in J(\lambda,\alpha)} (1 - v^{-2(c'_j - \mathbf{c}(h)_j)}) = \prod_{j \in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right).$$

So

$$g_{k,\mathbf{c},\mathbf{c}'} = v^{[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right)$$

and the theorem is verified in this first case.

If $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},k)}] \rangle_{\mathcal{Q}} = 1$ and $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}] \rangle^1_{\mathcal{Q}} = 0$, then $E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} = v^{c_h} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)}$ by using repeatedly proposition 3.15. For $\hat{\mathbf{c}} \in \mathbf{N}^\nu$ defined by $[\mathbf{V}_{\hat{\mathbf{c}}}] = [\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, we have either $\hat{\mathbf{c}} = (0,0,\ldots,0)$ or $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$ and $\rho(\hat{\mathbf{c}}) > \rho(\mathbf{c})$. In either case, we can assume by induction that the theorem is verified for $\hat{\mathbf{c}}$. So

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} \, E_{\mathbf{i}}^{\mathbf{c}''}$$

where $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for an α-admissible partition $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$. Moreover, in this case λ is unique and

$$g_{k,\hat{\mathbf{c}},\mathbf{c}''} = v^{[\mathbf{V}_{\mathbf{c}''}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c''_j}).$$

In particular, $\rho(\mathbf{c}'') > h$ whenever $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$. In fact, if $\hat{\mathbf{c}} = (0,0,\ldots,0)$, then $\mathbf{c}'' = \mathbf{b}(k)$ and obviously $\rho(\mathbf{c}'') > h$. If $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < k$, then we get that $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$, because $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}'') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq k$, then $\mathbf{c}'' = \hat{\mathbf{c}} + \mathbf{b}(k)$ and $\rho(\mathbf{c}'') \geq k > h$.

We have

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}} = E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = v^{c_h} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = v^{c_h} E_{\mathbf{i}}^{\mathbf{c}(h)} \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''}$$
$$= \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} v^{c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''+\mathbf{c}(h)} = \sum_{\mathbf{c}' \in \mathbf{N}^\nu} g_{k,\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}.$$

So $g_{k,\mathbf{c},\mathbf{c}'} = v^{c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$ and $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] = [\mathbf{V}_{\mathbf{c}'}] - c_h \, [\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some α-admissible partition $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$. Since $[\mathbf{V}_{\mathbf{c}}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$ and $\Lambda(\alpha,\mathbf{c}) = \Lambda(\alpha,\hat{\mathbf{c}})$ by lemma 4.11, we get that $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some α-admissible partition $\lambda \in \Lambda(\alpha,\mathbf{c})$. If $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ and λ is the unique element of $\Lambda(\alpha,\mathbf{c})$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$, then

$$g_{k,\mathbf{c},\mathbf{c}'} = v^{c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$$
$$= v^{c_h} v^{[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} \, (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} \left(1 - v^{-2(c'_j - \mathbf{c}(h)_j)}\right).$$

But

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) &= ([\mathbf{V}_{\mathbf{c}'}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]) \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) \\ &= \Big([\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))\Big) - c_h,\end{aligned}$$

because $\langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],[\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 1$, $\langle[\mathbf{e}_\alpha],[\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = 0$ by hypothesis and $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha) = 0$ by lemma 4.10 (b) and (d). By lemma 4.10 (e), $h \notin J(\lambda,\alpha)$ and

$$\prod_{j\in J(\lambda,\alpha)} (1 - v^{-2(c'_j - \mathbf{c}(h)_j)}) = \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right).$$

So

$$\begin{aligned}g_{k,\mathbf{c},\mathbf{c}'} &= v^{c_h} v^{\big([\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))\big)-c_h} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right) \\ &= v^{[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right)\end{aligned}$$

and the theorem is verified in the second case.

If $\langle[\mathbf{e}_{\alpha(\mathbf{i},h)}],[\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} = 0$ and $\langle[\mathbf{e}_\alpha],[\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle^1_{\mathcal{Q}} = 1$, then

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} = v^{-c_h} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} + E_{\mathbf{i}}^{\mathbf{c}(h)-\mathbf{b}(h)} E_{\mathbf{i}}^{\mathbf{b}(m)}$$

by using repeatedly propositions 3.15 and 3.16 where m is defined by $\alpha(\mathbf{i},m) = \alpha(\mathbf{i},h) + \alpha(\mathbf{i},k)$. For $\hat{\mathbf{c}} \in \mathbf{N}^\nu$ defined by $[\mathbf{V}_{\hat{\mathbf{c}}}] = [\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, we have either $\hat{\mathbf{c}} = (0,0,\ldots,0)$ or $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$ and $\rho(\hat{\mathbf{c}}) > \rho(\mathbf{c})$. In either case, we can assume by induction that the theorem is verified for $\hat{\mathbf{c}}$. So

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}''\in\mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''}$$

where $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for an α-admissible partition $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$. Moreover, in this case λ is unique and

$$g_{k,\hat{\mathbf{c}},\mathbf{c}''} = v^{[\mathbf{V}_{\mathbf{c}''}] \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c''_j}\right).$$

In particular, $\rho(\mathbf{c}'') > h$ whenever $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$. In fact, if $\hat{\mathbf{c}} = (0,0,\ldots,0)$, then $\mathbf{c}'' = \mathbf{b}(k)$ and obviously $\rho(\mathbf{c}'') > h$. If $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < k$, then we get that $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$, because $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}'') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq k$, then $\mathbf{c}'' = \hat{\mathbf{c}} + \mathbf{b}(k)$ and $\rho(\mathbf{c}'') \geq k > h$.

We also have

$$E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}'''\in\mathbf{N}^\nu} g_{m,\hat{\mathbf{c}},\mathbf{c}'''} E_{\mathbf{i}}^{\mathbf{c}'''}$$

where $g_{m,\hat{\mathbf{c}},\mathbf{c}'''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ for an $\alpha(\mathbf{i},m)$-admissible partition $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$. Moreover, in this case λ' is unique and

$$g_{m,\hat{\mathbf{c}},\mathbf{c}'''} = v^{[\mathbf{V}_{\mathbf{c}'''}] \circ_{\mathcal{Q}}([\mathbf{e}_{\alpha(\mathbf{i},m)}]-\mathbf{t}(\lambda',\alpha(\mathbf{i},m)))} (v-v^{-1})^{-1} \prod_{j\in J(\lambda',\alpha(\mathbf{i},m))} \left(1 - v^{-2c'''_j}\right).$$

In particular, $\rho(\mathbf{c}''') > h$ whenever $g_{m,\hat{\mathbf{c}},\mathbf{c}'''} \neq 0$. In fact, if $\hat{\mathbf{c}} = (0,0,\ldots,0)$, then $\mathbf{c}''' = \mathbf{b}(m)$ and obviously $\rho(\mathbf{c}''') = m > h$. If $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < m$, then we get that $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ for some

$\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$, because $g_{m,\hat{\mathbf{c}},\mathbf{c}'''} \neq 0$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}''') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq m$, then $\mathbf{c}''' = \hat{\mathbf{c}} + \mathbf{b}(m)$ and $\rho(\mathbf{c}''') \geq m > h$.

We have

$$\begin{aligned} E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}} &= E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \left(v^{-c_h} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} + E_{\mathbf{i}}^{\mathbf{c}(h)-\mathbf{b}(h)} E_{\mathbf{i}}^{\mathbf{b}(m)} \right) E_{\mathbf{i}}^{\hat{\mathbf{c}}} \\ &= \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} v^{-c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''+\mathbf{c}(h)} + \sum_{\mathbf{c}''' \in \mathbf{N}^\nu} g_{m,\hat{\mathbf{c}},\mathbf{c}'''} E_{\mathbf{i}}^{\mathbf{c}'''+\mathbf{c}(h)-\mathbf{b}(h)} \\ &= \sum_{\mathbf{c}' \in \mathbf{N}^\nu} g_{k,\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}. \end{aligned}$$

Note that if $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and $g_{m,\hat{\mathbf{c}},\mathbf{c}'''} \neq 0$, then $\mathbf{c}'' + \mathbf{c}(h) \neq \mathbf{c}''' + \mathbf{c}(h) - \mathbf{b}(h)$ in the above sums. For otherwise, we would have $\mathbf{c}'' = \mathbf{c}''' - \mathbf{b}(h)$. But $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda_0(\alpha, \mathbf{c}, h)$ and $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ for some $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ because $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and $g_{m,\hat{\mathbf{c}},\mathbf{c}'''} \neq 0$. Consequently,

$$[\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) + [\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$$

and

$$\mathbf{t}(\lambda, \alpha) = \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) + [\mathbf{e}_\alpha] + [\mathbf{e}_{\alpha(\mathbf{i},h)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] = \mathbf{t}(\overline{\lambda'}, \alpha)$$

by lemma 4.12 (d) where $\overline{\lambda'}$ is defined in lemma 4.12 (a) and (b) and $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$. This implies that $\lambda = \overline{\lambda'}$. But this is impossible because $\lambda \in \Lambda_0(\alpha, \mathbf{c}, h)$, while $\overline{\lambda'} \in \Lambda_1(\alpha, \mathbf{c}, h)$.

Thus $g_{k,\mathbf{c},\mathbf{c}'}$ is equal to $v^{-c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$, if $[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ with $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$ and is equal to $g_{m,\hat{\mathbf{c}},\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}$, if $[\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ with $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$.

First note that $[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] = [\mathbf{V}_{\mathbf{c}'}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}}) = \Lambda_0(\alpha, \mathbf{c}, h)$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda_0(\alpha, \mathbf{c}, h)$ because $[\mathbf{V}_{\mathbf{c}}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$. Secondly observe that $[\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] = [\mathbf{V}_{\mathbf{c}'}] - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ for some $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] - [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\overline{\lambda'}, \alpha(\mathbf{i}, m))$ for some $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ because of lemma 4.12 (d). Finally, because $\Lambda_1(\alpha, \mathbf{c}, h) = \{\overline{\lambda'} \mid \lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})\}$ and $\Lambda(\alpha, \mathbf{c})$ is the disjoint union $\Lambda_0(\alpha, \mathbf{c}, h) \cup \Lambda_1(\alpha, \mathbf{c}, h)$, we get that $g_{k,\mathbf{c},\mathbf{c}'} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda(\alpha, \mathbf{c})$. Moreover,

$$g_{k,\mathbf{c},\mathbf{c}'} = \begin{cases} v^{-c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}, & \text{if } [\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \text{ with } \lambda \in \Lambda_0(\alpha, \mathbf{c}, h), \\ g_{m,\hat{\mathbf{c}},\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}, & \text{if } [\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha) \text{ with } \lambda \in \Lambda_1(\alpha, \mathbf{c}, h). \end{cases}$$

We can now compute $g_{k,\mathbf{c},\mathbf{c}'}$.

If $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for some $\lambda \in \Lambda_0(\alpha, \mathbf{c}, h)$, then we see that this is equivalent to $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$, $[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ and

$$\begin{aligned} g_{k,\mathbf{c},\mathbf{c}'} &= v^{-c_h} g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)} \\ &= v^{-c_h} v^{[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda, \alpha)} \left(1 - v^{-2(c'_j - \mathbf{c}(h)_j)} \right). \end{aligned}$$

But

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) \\ &= [\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) \\ &= \big([\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))\big) + c_h,\end{aligned}$$

because $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} [\mathbf{e}_\alpha] = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} - \langle [\mathbf{e}_\alpha], [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle_{\mathcal{Q}}^1 = -1$ by hypothesis and $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha) = 0$ by lemma 4.10 (d). Also $h \notin J(\lambda,\alpha)$ by lemma 4.10 (e). So

$$\prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2(c'_j - \mathbf{c}(h)_j)}\right) = \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right).$$

Consequently,

$$\begin{aligned}g_{k,\mathbf{c},\mathbf{c}'} &= v^{-c_h} v^{([\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha)))+c_h} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right) \\ &= v^{[\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right).\end{aligned}$$

If $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some $\lambda \in \Lambda_1(\alpha,\mathbf{c},h)$, then $\lambda = \overline{\lambda'}$ for some $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$, $[\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ and

$$\begin{aligned}g_{k,\mathbf{c},\mathbf{c}'} &= g_{m,\hat{\mathbf{c}},\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)} \\ &= v^{[\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] \ \circ_{\mathcal{Q}}([\mathbf{e}_{\alpha(\mathbf{i},m)}]-\mathbf{t}(\lambda',\alpha(\mathbf{i},m)))} (v-v^{-1})^{-1} \\ &\quad \times \prod_{j\in J(\lambda',\alpha(\mathbf{i},m))} \left(1 - v^{-2(c'_j+\mathbf{b}(h)_j-\mathbf{c}(h)_j)}\right).\end{aligned}$$

But

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))) \\ &= [\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\overline{\lambda'},\alpha) + [\mathbf{e}_{\alpha(\mathbf{i},h)}]) \\ &= [\mathbf{V}_{\mathbf{c}'+\mathbf{b}(h)-\mathbf{c}(h)}] \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\overline{\lambda'},\alpha)) \\ &= \big([\mathbf{V}_{\mathbf{c}'}] + (c_h-1)[\mathbf{e}_{\alpha(\mathbf{i},h)}]\big) \ \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)) \\ &= [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))\end{aligned}$$

by lemma 4.12 (d) and because $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} [\mathbf{e}_\alpha] = \langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_\alpha]\rangle_{\mathcal{Q}} - \langle [\mathbf{e}_\alpha], [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle_{\mathcal{Q}}^1 = -1$ by hypothesis and, by lemma 4.10 (d), $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \ \circ_{\mathcal{Q}} \mathbf{t}(\lambda,\alpha) = -1$ when $\lambda \in \Lambda_1(\alpha,\mathbf{c},h)$. Note that $h \notin J(\lambda',\alpha(\mathbf{i},m))$ and $J(\lambda,\alpha) = J(\overline{\lambda'},\alpha) = J(\lambda',\alpha(\mathbf{i},m))$ by lemma 4.12 (e). Consequently,

$$\prod_{j\in J(\lambda',\alpha(\mathbf{i},m))} \left(1 - v^{-2(c'_j+\mathbf{b}(h)_j-\mathbf{c}(h)_j)}\right) = \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right)$$

and

$$g_{k,\mathbf{c},\mathbf{c}'} = v^{[\mathbf{V}_{\mathbf{c}'}] \ \circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))} (v-v^{-1})^{-1} \prod_{j\in J(\lambda,\alpha)} \left(1 - v^{-2c'_j}\right).$$

The theorem is verified in the third case.

If $\langle [\mathbf{e}_{\alpha(\mathbf{i},h)}], [\mathbf{e}_{\alpha(\mathbf{i},k)}]\rangle_{\mathcal{Q}} = 1$ and $\langle [\mathbf{e}_{\alpha(\mathbf{i},k)}], [\mathbf{e}_{\alpha(\mathbf{i},h)}]\rangle_{\mathcal{Q}}^1 = 1$, then

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} = E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} + v^{(c_h-1)}(v-v^{-1}) E_{\mathbf{i}}^{\mathbf{c}(h)-\mathbf{b}(h)} E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\mathbf{b}(m')}$$

where m, m' are defined in proposition 3.17. Here $h < m < m' < k$. This formula is obtained by using repeatedly propositions 3.15 and 3.17.

For $\hat{\mathbf{c}} \in \mathbf{N}^\nu$ defined by $[\mathbf{V}_{\hat{\mathbf{c}}}] = [\mathbf{V}_{\mathbf{c}}] - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]$, we have either $\hat{\mathbf{c}} = (0,0,\ldots,0)$ or $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$ and $\rho(\hat{\mathbf{c}}) > \rho(\mathbf{c})$. In either case, we can assume by induction that the theorem is verified for $\hat{\mathbf{c}}$. So

$$E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{\mathbf{c}''}$$

where $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for an α-admissible partition $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$. Moreover, in this case, λ is unique and

$$g_{k,\hat{\mathbf{c}},\mathbf{c}''} = v^{[\mathbf{V}_{\mathbf{c}''}] \circ_{\mathcal{Q}} ([\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda,\alpha)} (1 - v^{-2c''_j}).$$

In particular, $\rho(\mathbf{c}'') > h$ whenever $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$. In fact, if $\hat{\mathbf{c}} = (0,0,\ldots,0)$, then $\mathbf{c}'' = \mathbf{b}(k)$ and obviously $\rho(\mathbf{c}'') > h$. If $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < k$, then we get that $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}})$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}'') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq k$, then $\mathbf{c}'' = \hat{\mathbf{c}} + \mathbf{b}(k)$ and $\rho(\mathbf{c}'') \geq k > h$.

We also have

$$E_{\mathbf{i}}^{\mathbf{b}(m')} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}'''' \in \mathbf{N}^\nu} g_{m',\hat{\mathbf{c}},\mathbf{c}''''} E_{\mathbf{i}}^{\mathbf{c}''''}$$

where $g_{m',\hat{\mathbf{c}},\mathbf{c}''''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ for an $\alpha(\mathbf{i},m')$-admissible partition $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$. Moreover, in this case λ'' is unique and

$$g_{m',\hat{\mathbf{c}},\mathbf{c}''''} = v^{[\mathbf{V}_{\mathbf{c}''''}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda'',\alpha(\mathbf{i},m'))} (1 - v^{-2c''''_j}).$$

If $g_{m',\hat{\mathbf{c}},\mathbf{c}''''} \neq 0$, then $\mathbf{c}'''' \neq (0,0,\ldots,0)$ and $\rho(\mathbf{c}'''') > h$. In fact, if $\hat{\mathbf{c}} = (0,0,\ldots,0)$, then $\mathbf{c}'''' = \mathbf{b}(m')$ and obviously $\rho(\mathbf{c}'''') = m' > h$. If $\hat{\mathbf{c}} \neq (0,0,\ldots,0)$, then $\rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) < m'$, then $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ for some $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$ and by lemma 4.10 (f), we get that $\rho(\mathbf{c}'''') \geq \rho(\hat{\mathbf{c}}) > h$. If $\rho(\hat{\mathbf{c}}) \geq m'$, then $\mathbf{c}'''' = \hat{\mathbf{c}} + \mathbf{b}(m')$ and $\rho(\mathbf{c}'''') \geq m' > h$.

By induction hypothesis, we can write when $g_{m',\hat{\mathbf{c}},\mathbf{c}''''} \neq 0$ that

$$E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\mathbf{c}''''} = \sum_{\mathbf{c}''' \in \mathbf{N}^\nu} g_{m,\mathbf{c}'''',\mathbf{c}'''} E_{\mathbf{i}}^{\mathbf{c}'''}$$

where $g_{m,\mathbf{c}'''',\mathbf{c}'''} \neq 0$ if and only if $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\mathbf{c}''''}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ for a $\alpha(\mathbf{i},m)$-admissible partition $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\mathbf{c}'''')$.

Moreover, in this case λ' is unique and

$$g_{m,\mathbf{c}'''',\mathbf{c}'''} = v^{[\mathbf{V}_{\mathbf{c}'''}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda',\alpha(\mathbf{i},m))} (1 - v^{-2c'''_j}).$$

If $g_{m,\mathbf{c}'''',\mathbf{c}'''} \neq 0$ in the above sum, then $\rho(\mathbf{c}''') > h$. In fact, we saw above that $\mathbf{c}'''' \neq (0,0,\ldots,0)$ and $\rho(\mathbf{c}'''') > h$. If $\rho(\mathbf{c}'''') < m$, then by lemma 4.10 (f) and because $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\mathbf{c}''''}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))$ for some $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\mathbf{c}'''')$, we get that $\rho(\mathbf{c}''') \geq \rho(\mathbf{c}'''') > h$. If $\rho(\mathbf{c}'''') \geq m$, then $\mathbf{c}''' = \mathbf{c}'''' + \mathbf{b}(m)$ and $\rho(\mathbf{c}''') \geq m > h$.

Consequently,

$$E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\mathbf{b}(m')} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}''',\mathbf{c}''''} g_{m',\hat{\mathbf{c}},\mathbf{c}''''}\ g_{m,\mathbf{c}'''',\mathbf{c}'''}\ E_{\mathbf{i}}^{\mathbf{c}'''}$$

where in the sum $(\mathbf{c}''', \mathbf{c}'''')$ runs over the set of pairs of elements of $\mathbf{N}^\nu$ such that $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))$ and $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\mathbf{c}''''}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ with $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$ and $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \mathbf{c}'''')$.

For a fixed $\mathbf{c}''' \in \mathbf{N}^\nu$, there exists at most one $\mathbf{c}'''' \in \mathbf{N}^\nu$ such that $g_{m',\hat{\mathbf{c}},\mathbf{c}''''} \neq 0$ and $g_{m,\mathbf{c}'''',\mathbf{c}'''} \neq 0$. Assume $\mathbf{c}^\heartsuit$, $\mathbf{c}^\clubsuit \in \mathbf{N}^\nu$ are such that $g_{m',\hat{\mathbf{c}},\mathbf{c}^\heartsuit} \neq 0$, $g_{m,\mathbf{c}^\heartsuit,\mathbf{c}'''} \neq 0$, $g_{m',\hat{\mathbf{c}},\mathbf{c}^\clubsuit} \neq 0$ and $g_{m,\mathbf{c}^\clubsuit,\mathbf{c}'''} \neq 0$, then

$$\begin{aligned}
[\mathbf{V}_{\mathbf{c}^\heartsuit}] &= [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda''_\heartsuit, \alpha(\mathbf{i}, m')) \quad \text{for some } \lambda''_\heartsuit \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}});\\
[\mathbf{V}_{\mathbf{c}^\clubsuit}] &= [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda''_\clubsuit, \alpha(\mathbf{i}, m')) \quad \text{for some } \lambda''_\clubsuit \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}});\\
[\mathbf{V}_{\mathbf{c}'''}] &= [\mathbf{V}_{\mathbf{c}^\heartsuit}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda'_\heartsuit, \alpha(\mathbf{i}, m)) \quad \text{for some } \lambda'_\heartsuit \in \Lambda(\alpha(\mathbf{i}, m), \mathbf{c}^\heartsuit);\\
[\mathbf{V}_{\mathbf{c}'''}] &= [\mathbf{V}_{\mathbf{c}^\clubsuit}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda'_\clubsuit, \alpha(\mathbf{i}, m)) \quad \text{for some } \lambda'_\clubsuit \in \Lambda(\alpha(\mathbf{i}, m), \mathbf{c}^\clubsuit).
\end{aligned}$$

Note that $\Lambda(\alpha(\mathbf{i}, m), \mathbf{c}^\heartsuit) = \Lambda(\alpha(\mathbf{i}, m), \mathbf{c}^\clubsuit) = \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ by lemma 4.13 (f). Consequently, using lemma 4.13 (d) we get that

$$\begin{aligned}
[\mathbf{V}_{\mathbf{c}'''}] &= [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'_\heartsuit, \alpha(\mathbf{i}, m)) - \mathbf{t}(\lambda''_\heartsuit, \alpha(\mathbf{i}, m'))\\
&= [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\overline{(\lambda'_\heartsuit, \lambda''_\heartsuit)}, \alpha(\mathbf{i}, k)).
\end{aligned}$$

Similarly,

$$[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\overline{(\lambda'_\clubsuit, \lambda''_\clubsuit)}, \alpha(\mathbf{i}, k)).$$

Thus $\mathbf{t}(\overline{(\lambda'_\heartsuit, \lambda''_\heartsuit)}, \alpha(\mathbf{i}, k)) = \mathbf{t}(\overline{(\lambda'_\clubsuit, \lambda''_\clubsuit)}, \alpha(\mathbf{i}, k))$ and $\overline{(\lambda'_\heartsuit, \lambda''_\heartsuit)} = \overline{(\lambda'_\clubsuit, \lambda''_\clubsuit)}$. From 4.14, we get $\lambda'_\heartsuit = \lambda'_\clubsuit$ and $\lambda''_\heartsuit = \lambda''_\clubsuit$. This implies that $\mathbf{c}^\heartsuit = \mathbf{c}^\clubsuit$.

So

$$E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\mathbf{b}(m')} E_{\mathbf{i}}^{\hat{\mathbf{c}}} = \sum_{\mathbf{c}''' \in \mathbf{N}^\nu} g_{m',\hat{\mathbf{c}},\mathbf{c}''''}\ g_{m,\mathbf{c}'''',\mathbf{c}'''} E_{\mathbf{i}}^{\mathbf{c}'''}$$

where $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))$ and $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\mathbf{c}''''}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ for $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$ and $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$. We can now compute $E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}}$.

$$\begin{aligned}
E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}} &= E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\hat{\mathbf{c}}}\\
&= \left(E_{\mathbf{i}}^{\mathbf{c}(h)} E_{\mathbf{i}}^{\mathbf{b}(k)} + v^{(c_h-1)}(v - v^{-1}) E_{\mathbf{i}}^{\mathbf{c}(h)-\mathbf{b}(h)} E_{\mathbf{i}}^{\mathbf{b}(m)} E_{\mathbf{i}}^{\mathbf{b}(m')} \right) E_{\mathbf{i}}^{\hat{\mathbf{c}}}\\
&= \sum_{\mathbf{c}'' \in \mathbf{N}^\nu} g_{k,\hat{\mathbf{c}},\mathbf{c}''} E_{\mathbf{i}}^{(\mathbf{c}''+\mathbf{c}(h))}\\
&\qquad + \sum_{\mathbf{c}''' \in \mathbf{N}^\nu} v^{(c_h-1)}(v - v^{-1})\ g_{m',\hat{\mathbf{c}},\mathbf{c}''''}\ g_{m,\mathbf{c}'''',\mathbf{c}'''} E_{\mathbf{i}}^{\mathbf{c}'''+\mathbf{c}(h)-\mathbf{b}(h)}\\
&= \sum_{\mathbf{c}'} g_{k,\mathbf{c},\mathbf{c}'} E_{\mathbf{i}}^{\mathbf{c}'}
\end{aligned}$$

where $\mathbf{c}''''$ in the sum $\sum_{\mathbf{c}'''}$ is given by $[\mathbf{V}_{\mathbf{c}'''}] = [\mathbf{V}_{\mathbf{c}''''}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))$ and $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))$ with $\lambda' \in \Lambda(\alpha(\mathbf{i}, m), \hat{\mathbf{c}})$ and $\lambda'' \in \Lambda(\alpha(\mathbf{i}, m'), \hat{\mathbf{c}})$. Note also that in the sum $\sum_{\mathbf{c}''}$, we have $[\mathbf{V}_{\mathbf{c}''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda, \alpha)$ for $\lambda \in \Lambda(\alpha, \hat{\mathbf{c}})$.

In particular, $[\mathbf{V}_{\mathbf{c}''+\mathbf{c}(h)}] = [\mathbf{V}_{\mathbf{c}''}] + c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for some $\lambda \in \Lambda(\alpha,\hat{\mathbf{c}}) = \Lambda_0(\alpha,\mathbf{c},h)$ and

$$\begin{aligned}
&[\mathbf{V}_{\mathbf{c}'''+\mathbf{c}(h)-\mathbf{b}(h)}] \\
&\qquad = [\mathbf{V}_{\mathbf{c}}] - [\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m')) \\
&\qquad = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\overline{(\lambda',\lambda'')},\alpha)
\end{aligned}$$

for some unique $\overline{(\lambda',\lambda'')} \in \Lambda_1(\alpha,\mathbf{c},h)$ with $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$ and $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$ by lemma 4.13 (d).

From this we can conclude that if $g_{k,\hat{\mathbf{c}},\mathbf{c}''} \neq 0$ and $g_{m',\hat{\mathbf{c}},\mathbf{c}''''}g_{m,\mathbf{c}'''',\mathbf{c}'''} \neq 0$ in the above sums, then $\mathbf{c}'' + \mathbf{c}(h) \neq \mathbf{c}''' + \mathbf{c}(h) - \mathbf{b}(h)$ and $g_{k,\mathbf{c},\mathbf{c}'}$ is equal to $g_{k,\hat{\mathbf{c}},\mathbf{c}'-\mathbf{c}(h)}$ if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for $\lambda \in \Lambda_0(\alpha,\mathbf{c},h)$ and is equal to $v^{c_h-1}(v-v^{-1})\ g_{m',\hat{\mathbf{c}},\mathbf{c}''''}\ g_{m,\mathbf{c}'''',\mathbf{c}'-\mathbf{c}(h)+\mathbf{b}(h)}$ if $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for $\lambda \in \Lambda_1(\alpha,\mathbf{c},h)$ where in the second case, the one for which $\lambda \in \Lambda_1(\alpha,\mathbf{c},h)$, then $[\mathbf{V}_{\mathbf{c}''''}] = [\mathbf{V}_{\hat{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))$ with $\lambda = \overline{(\lambda',\lambda'')}$ such that $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$ and $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$. We can now compute $g_{k,\mathbf{c},\mathbf{c}'}$.

If $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for $\lambda \in \Lambda_0(\alpha,\mathbf{c},h)$, then

$$\begin{aligned}
g_{k,\mathbf{c},\mathbf{c}'} &= v^{[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))}(v-v^{-1})^{-1}\prod_{j\in J(\lambda,\alpha)}(1-v^{-2(c'_j-\mathbf{c}(h)_j)}) \\
&= v^{[\mathbf{V}_{\mathbf{c}'}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))}(v-v^{-1})^{-1}\prod_{j\in J(\lambda,\alpha)}(1-v^{-2c'_j})
\end{aligned}$$

because

$$\begin{aligned}
&[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha)) \\
&\qquad\qquad = [\mathbf{V}_{\mathbf{c}'}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha)) - c_h[\mathbf{e}_{\alpha(\mathbf{i},h)}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha)) \\
&\qquad\qquad = [\mathbf{V}_{\mathbf{c}'}]\circ_{\mathcal{Q}}([\mathbf{e}_\alpha]-\mathbf{t}(\lambda,\alpha))
\end{aligned}$$

from the fact that $[\mathbf{e}_{\alpha(\mathbf{i},h)}]\circ_{\mathcal{Q}}[\mathbf{e}_\alpha] = 0$ and $[\mathbf{e}_{\alpha(\mathbf{i},h)}]\circ_{\mathcal{Q}}\mathbf{t}(\lambda,\alpha) = 0$, λ being an element of $\Lambda_0(\alpha,\mathbf{c},h)$, and because $h \notin J(\lambda,\alpha)$.

If $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_\alpha] - \mathbf{t}(\lambda,\alpha)$ for $\lambda = \overline{(\lambda',\lambda'')} \in \Lambda_1(\alpha,\mathbf{c},h)$ for $\lambda' \in \Lambda(\alpha(\mathbf{i},m),\hat{\mathbf{c}})$ and $\lambda'' \in \Lambda(\alpha(\mathbf{i},m'),\hat{\mathbf{c}})$, then

$$\begin{aligned}
g_{k,\mathbf{c},\mathbf{c}'} &= v^{c_h-1+\epsilon}(v-v^{-1})^{-1}\prod_{j\in J(\lambda'',\alpha(\mathbf{i},m'))}\left(1-v^{-2c''''_j}\right) \\
&\qquad \times \prod_{j\in J(\lambda',\alpha(\mathbf{i},m))}\left(1-v^{-2(c'_j-\mathbf{c}(h)_j+\mathbf{b}(h)_j)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\epsilon = &\left([\mathbf{V}_{\mathbf{c}''''}]\circ_{\mathcal{Q}}\left([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'',\alpha(\mathbf{i},m'))\right)\right) \\
&+ \left([\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)+\mathbf{b}(h)}]\circ_{\mathcal{Q}}\left([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda',\alpha(\mathbf{i},m))\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
[\mathbf{V}_{\mathbf{c}''''}] &= [\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)+\mathbf{b}(h)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + \mathbf{t}(\lambda',\alpha(\mathbf{i},m)) \\
&= [\mathbf{V}_{\mathbf{c}'}] - (c_h-1)[\mathbf{e}_{\alpha(\mathbf{i},h)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + \mathbf{t}(\lambda',\alpha(\mathbf{i},m)).
\end{aligned}$$

We get that $[\mathbf{V}_{\mathbf{c}''''}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')))$ is equal to

$$([\mathbf{V}_{\mathbf{c}'}] - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}] - [\mathbf{e}_{\alpha(\mathbf{i},m)}] + \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m')));$$

and, finally, that

$$[\mathbf{V}_{\mathbf{c}''''}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))) = [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} \big([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))\big) - (c_h - 1),$$

because $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} \big([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))\big) = 1$ and $([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) \circ_{\mathcal{Q}} \big([\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))\big) = 0$ by lemma 4.13 (d).

Also,

$$\begin{aligned}
[\mathbf{V}_{\mathbf{c}'-\mathbf{c}(h)+\mathbf{b}(h)}] &\circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) \\
&= ([\mathbf{V}_{\mathbf{c}'}] - (c_h - 1)[\mathbf{e}_{\alpha(\mathbf{i},h)}]) \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) \\
&= \big([\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)))\big) - (c_h - 1),
\end{aligned}$$

because $[\mathbf{e}_{\alpha(\mathbf{i},h)}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m))) = 1$ by lemma 4.13 (d).

So

$$\begin{aligned}
\epsilon &= [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} \big([\mathbf{e}_{\alpha(\mathbf{i},m)}] + [\mathbf{e}_{\alpha(\mathbf{i},m')}] - \mathbf{t}(\lambda', \alpha(\mathbf{i}, m)) - \mathbf{t}(\lambda'', \alpha(\mathbf{i}, m'))\big) - 2(c_h - 1) \\
&= [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} \Big([\mathbf{e}_{\alpha(\mathbf{i},h)}] + [\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)\Big) - 2(c_h - 1) \\
&= [\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} \Big([\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)\Big) - (c_h - 1)
\end{aligned}$$

because $[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = ([\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)) \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = (c_h - 1)$. This follows from the fact that $\rho(\mathbf{c}) = h$ and from $([\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha)) \circ_{\mathcal{Q}} [\mathbf{e}_{\alpha(\mathbf{i},h)}] = -1$ by lemma 4.13 (d).

Note also that $c_j'''' = c_j'$ if $j \in J(\lambda'', \alpha(\mathbf{i}, m'))$ and $h \notin J(\lambda', \alpha(\mathbf{i}, m))$.

From lemma 4.13 and what we got above, we can write

$$\begin{aligned}
g_{k,\mathbf{c},\mathbf{c}'} &= v^{[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha}] - \mathbf{t}(\overline{(\lambda', \lambda'')}, \alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\overline{(\lambda', \lambda'')}, \alpha)} (1 - v^{-2c_j'}) \\
&= v^{[\mathbf{V}_{\mathbf{c}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha}] - \mathbf{t}(\lambda, \alpha))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda, \alpha)} (1 - v^{-2c_j'}).
\end{aligned}$$

This concludes the proof of the theorem. □

5. Values of $\Omega^{\mathbf{c}}_{\mathbf{c}'}$ and its derivative at $v = 1$

In this section, we will evaluate $\Omega^{\mathbf{c}}_{\mathbf{c}'}(1)$ and $(d\Omega^{\mathbf{c}}_{\mathbf{c}'}/dv)(1)$ where $\mathbf{c}$, $\mathbf{c}'$ have the same $\mathbf{i}$-homogeneity and $\mathbf{c}' \preceq \mathbf{c}$. This will be used in the next section when we characterize which orbit closures $\overline{\mathcal{O}_{\mathbf{c}}}$ are rationally smooth.

5.1. We will now fix the notation for the rest of this section. Let $\mathbf{c} \in \mathbf{N}^{\nu}$, $\mathbf{c} \neq 0$ with $\rho(\mathbf{c}) = k$. Assume that $\alpha(\mathbf{i}, k) = \alpha_r + \alpha_{r+1} + \cdots + \alpha_s$ with $1 \leq r \leq s \leq n$. Let $\Delta[r, s]$ denote the full subgraph of Δ whose set of vertices is $\{r, r+1, \ldots, s\}$. We have

$$\begin{bmatrix} c_k \\ 1 \end{bmatrix} E^{\mathbf{c}}_{\mathbf{i}} = E^{\mathbf{b}(k)}_{\mathbf{i}} E^{\mathbf{c}^-}_{\mathbf{i}} \quad \text{where } \mathbf{c}^- = \mathbf{c} - \mathbf{b}(k).$$

Recall that $\mathbf{b}(k)$ is the unique element of $\mathbf{N}^\nu$ whose only nonzero component is in the kth column and is equal to 1. By taking the bar involution on both sides of this last equation, we get

$$\begin{bmatrix} c_k \\ 1 \end{bmatrix} \overline{E_{\mathbf{i}}^{\mathbf{c}}} = \overline{E_{\mathbf{i}}^{\mathbf{b}(k)} E_{\mathbf{i}}^{\mathbf{c}^-}}$$

$$\Rightarrow \quad \begin{bmatrix} c_k \\ 1 \end{bmatrix} \sum_{\mathbf{c}'} \omega_{\mathbf{c}'}^{\mathbf{c}} E_{\mathbf{i}}^{\mathbf{c}'} = \left(\sum_H (-1)^{|H|} (v - v^{-1})^{|H|} E_{\mathbf{i}}^{\mathbf{c}(H)} \right) \left(\sum_{\mathbf{c}^{(0)}} \omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-} E_{\mathbf{i}}^{\mathbf{c}^{(0)}} \right)$$

where $\mathbf{c}'$ runs over all the elements of $\mathbf{N}^\nu$ which have the same $\mathbf{i}$-homogeneity as $\mathbf{c}$ and such that $\mathbf{c}' \preceq \mathbf{c}$, H runs over all the subsets of the set of edges of $\Delta[r,s]$ and $\mathbf{c}^{(0)}$ runs over all the elements of $\mathbf{N}^\nu$ which have the same $\mathbf{i}$-homogeneity as $\mathbf{c}^-$ and such that $\mathbf{c}^{(0)} \preceq \mathbf{c}^-$. Recall that $\mathbf{c}(H)$ has been defined in proposition 2.3.

Using repeatedly theorem 4.16, we can express the right-hand side of the last equation in the PBW basis $B_{\mathbf{i}}$ and in this way we get an expression for $\omega_{\mathbf{c}'}^{\mathbf{c}}$ in term of $\omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}$.

If H is a subset of the set of edges of $\Delta[r,s]$, then $1 \le m_{|H|}(H) < m_{(|H|-1)}(H) < \cdots < m_1(H) < m_0(H) \le \nu$ will denote the positions of the nonzero components of $\mathbf{c}(H)$. With this notation, we get

$$(3) \quad \frac{(v^{c_k} - v^{-c_k})}{(v - v^{-1})} \omega_{\mathbf{c}'}^{\mathbf{c}} = \sum_{\mathbf{c}^{(0)}, H, \underline{\lambda}(H)} (-1)^{|H|} (v - v^{-1})^{|H|} \left[\prod_{h=0}^{|H|} g_{m_h(H), \mathbf{c}^{(h)}, \mathbf{c}^{(h+1)}} \right] \omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}$$

where $\mathbf{c}^{(0)}$ runs over all the elements of $\mathbf{N}^\nu$ which have the same $\mathbf{i}$-homogeneity as $\mathbf{c}^-$ and such that $\mathbf{c}^{(0)} \preceq \mathbf{c}^-$; H runs over all the subsets of the set of edges of $\Delta[r,s]$; $\underline{\lambda}(H) = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(|H|)})$ with $\lambda^{(h)}$ is an $\alpha(\mathbf{i}, m_h(H))$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i}, m_h(H)), \mathbf{c}^{(h)})$ and $[\mathbf{V}_{\mathbf{c}^{(h+1)}}] = [\mathbf{V}_{\mathbf{c}^{(h)}}] + [\mathbf{e}_{\alpha(\mathbf{i}, m_h(H))}] - \mathbf{t}(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))$ for $h = 0, 1, 2, \ldots, |H|$ and finally $\mathbf{c}' = \mathbf{c}^{(|H|+1)}$.

Recall that by theorem 4.16 if $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_\nu)$, $\tilde{\mathbf{c}}' = (\tilde{c}_1', \tilde{c}_2', \ldots, \tilde{c}_\nu') \in \mathbf{N}^\nu$ are such that $[\mathbf{V}_{\tilde{\mathbf{c}}'}] = [\mathbf{V}_{\tilde{\mathbf{c}}}] + [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda, \alpha(\mathbf{i}, m))$ for some $\lambda \in \Lambda(\alpha(\mathbf{i}, m), \tilde{\mathbf{c}})$. Then

$$g_{m, \tilde{\mathbf{c}}, \tilde{\mathbf{c}}'} = v^{[\mathbf{V}_{\tilde{\mathbf{c}}'}] \circ_{\mathcal{Q}} ([\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda, \alpha(\mathbf{i},m)))} (v - v^{-1})^{-1} \prod_{j \in J(\lambda, \alpha(\mathbf{i},m))} \left(1 - v^{-2\tilde{c}_j'}\right)$$

with $J(\lambda, \alpha(\mathbf{i}, m)) = \{1 \le j \le \nu \mid \langle [\mathbf{e}_{\alpha(\mathbf{i},m)}] - \mathbf{t}(\lambda, \alpha(\mathbf{i}, m)), \mathbf{r}_{\alpha(\mathbf{i},j)} \rangle_{\mathcal{Q}} = 1\}$.

Also recall that $\Omega_{\tilde{\mathbf{c}}'}^{\tilde{\mathbf{c}}} = v^{d(\tilde{\mathbf{c}}) - d(\tilde{\mathbf{c}}')} \omega_{\tilde{\mathbf{c}}'}^{\tilde{\mathbf{c}}}$ for $\tilde{\mathbf{c}}', \tilde{\mathbf{c}} \in \mathbf{N}^\nu$. Substituting this in equation (3), we get that

$$(4) \quad \frac{(v^{2c_k} - 1)}{(v^2 - 1)} \Omega_{\mathbf{c}'}^{\mathbf{c}} =$$

$$\sum_{\mathbf{c}^{(0)}, H, \underline{\lambda}(H)} (-1)^{|H|} v^{\epsilon(\mathbf{c}, \mathbf{c}^{(0)}, H, \underline{\lambda}(H))} \left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} (1 - v^{-2c_j^{(h+1)}})}{(1 - v^{-2})} \right] \Omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}$$

with

$$\epsilon(\mathbf{c}, \mathbf{c}^{(0)}, H, \underline{\lambda}(H)) = (c_k - 2) + d(\mathbf{c}) - d(\mathbf{c}') - d(\mathbf{c}^-) + d(\mathbf{c}^{(0)}) + \sum_{h=0}^{|H|} [\mathbf{V}_{\mathbf{c}^{(h+1)}}] \circ_{\mathcal{Q}} \left([\mathbf{e}_{\alpha(\mathbf{i}, m_h(H))}] - \mathbf{t}(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H))) \right)$$

and $\mathbf{c}^{(0)}, H, \underline{\lambda}(H)$ are as above.

Proposition 5.2. *If $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ have the same $\mathbf{i}$-homogeneity with $\mathbf{c}' \preceq \mathbf{c}$, then*

$$\Omega^{\mathbf{c}}_{\mathbf{c}'}(1) = \begin{cases} 1, & \text{if } \mathbf{c}' = \mathbf{c}, \\ 0, & \text{if } \mathbf{c}' \neq \mathbf{c}. \end{cases}$$

Proof. We proceed by induction on $|\mathbf{c}| = \sum_{j=1}^{\nu} c_j$ for $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$. If $|\mathbf{c}| = 0$, then $\mathbf{c} = (0, 0, \ldots, 0)$ and, if $\mathbf{c}' \in \mathbf{N}^\nu$ is such that $\mathbf{c}' \preceq \mathbf{c} = 0$, then $\mathbf{c}' = \mathbf{c} = 0$. In this case, we have $\Omega^{\mathbf{c}}_{\mathbf{c}} = 1$ and the proposition is verified. From now on, we will use the same notation as in 5.1.

Assume the proposition is true whenever $|\mathbf{c}| < p$. Consider now $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ such that $|\mathbf{c}| = p$ and $\mathbf{c}' \preceq \mathbf{c}$. If $\mathbf{c}' = \mathbf{c}$, then $\Omega^{\mathbf{c}}_{\mathbf{c}'} = 1$ and the proposition is true. Assume now that $\mathbf{c}' \neq \mathbf{c}$. From equation (4), we get that $c_k\ \Omega^{\mathbf{c}}_{\mathbf{c}'}(1)$ is equal to

$$(*) \quad \sum_{\mathbf{c}^{(0)}, H, \underline{\lambda}(H)} (-1)^{|H|} \left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} \left(1 - v^{-2c_j^{(h+1)}}\right)}{(1 - v^{-2})} \right] \Bigg|_{v=1} \Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}(1)$$

with $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ as in 5.1. Since $|\mathbf{c}^-| = |\mathbf{c}| - 1 < p$, then

$$\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}(1) = \begin{cases} 1, & \text{if } \mathbf{c}^{(0)} = \mathbf{c}^-, \\ 0, & \text{if } \mathbf{c}^{(0)} \neq \mathbf{c}^- \end{cases}$$

by induction hypothesis. So we can restrict ourselves to the case where $\mathbf{c}^{(0)} = \mathbf{c}^-$ in the sum $(*)$ above.

Recall that $J(\lambda, \alpha)$ is never empty when $\alpha \in R^+$ and $\lambda \in \Lambda(\alpha, \tilde{\mathbf{c}})$ for $\tilde{\mathbf{c}} \in \mathbf{N}^\nu$. So

$$\left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} \left(1 - v^{-2c_j^{(h+1)}}\right)}{(1 - v^{-2})} \right] \Bigg|_{v=1} = 0$$

unless $H = \emptyset$, $\underline{\lambda}(H) = (\lambda^{(0)})$ and $|J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H)))| = 1$. Note that in the latter case $m_0(H) = k$ and $\alpha(\mathbf{i}, m_0(H)) = \alpha(\mathbf{i}, k)$. Since $\lambda^{(0)} \in \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^{(0)}) = \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^-)$ and $\rho(\mathbf{c}^-) \geq \rho(\mathbf{c}) = k$, then $\lambda^{(0)}$ must be the empty partition, $\mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = 0$ and $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}}]$. But this is impossible because $\mathbf{c}' \neq \mathbf{c}$. Consequently, the sum $(*)$ above is 0 when $\mathbf{c}' \neq \mathbf{c}$ and $\Omega^{\mathbf{c}}_{\mathbf{c}'}(1) = 0$. □

5.3. It is possible to give another proof of the previous proposition valid not only for type A, but also for type D and E. Since $\Omega^{\mathbf{c}}_{\mathbf{c}} = 1$, we only have to show that $\Omega^{\mathbf{c}}_{\mathbf{c}'}(1) = 0$ when $\mathbf{c}' \prec \mathbf{c}$. Let $Z^{\mathbf{c}}_{\mathbf{c}'} = v^{d(\mathbf{c}) - d(\mathbf{c}')} \zeta^{\mathbf{c}}_{\mathbf{c}'}$. We saw in theorem 1.18 that

$$\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \omega^{\mathbf{c}''}_{\mathbf{c}'}\ \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}} \quad \text{and} \quad Z^{\mathbf{c}}_{\mathbf{c}'} = \sum_j \dim(\mathcal{H}^{2j}_f(\overline{\mathcal{O}}_{\mathbf{c}}))\ v^{2j}$$

where f is an $\mathbf{F}_q$-rational point of the orbit $\mathcal{O}_{\mathbf{c}'}$.

For $\mathbf{c}' \prec \mathbf{c}$, we get that

$$Z^{\mathbf{c}}_{\mathbf{c}'} = \Omega^{\mathbf{c}}_{\mathbf{c}'} + \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \prec \mathbf{c}}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\ v^{2(d(\mathbf{c}) - d(\mathbf{c}''))} \overline{Z^{\mathbf{c}}_{\mathbf{c}''}} + v^{2(d(\mathbf{c}) - d(\mathbf{c}'))} \overline{Z^{\mathbf{c}}_{\mathbf{c}'}}.$$

By induction, we can assume that $\Omega_{\mathbf{c}'}^{\mathbf{c}''}(1) = 0$ whenever $\mathbf{c}' \neq \mathbf{c}''$. Consequently, when we evaluate at $v = 1$, we get

$$\sum_j \dim(\mathcal{H}_f^{2j}(\overline{\mathcal{O}}_{\mathbf{c}})) = \Omega_{\mathbf{c}'}^{\mathbf{c}}(1) + \sum_j \dim(\mathcal{H}_f^{2j}(\overline{\mathcal{O}}_{\mathbf{c}}))$$

and $\Omega_{\mathbf{c}'}^{\mathbf{c}}(1) = 0$.

Theorem 5.4. *If $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ have the same $\mathbf{i}$-homogeneity and are such that $\mathbf{c}' \preceq \mathbf{c}$, then $(d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1) \neq 0$ if and only if there exists a pair of positive roots $\alpha(\mathbf{i}, a)$, $\alpha(\mathbf{i}, b)$ where $1 \leq a < b \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$ in $\mathbf{K}(\mathcal{Q}, 0)$. In this case, the pair $\{\alpha(\mathbf{i}, a), \alpha(\mathbf{i}, b)\}$ and the isomorphism type of $\mathbf{M}$ are unique, $\langle [\mathbf{e}_{\alpha(\mathbf{i},b)}], [\mathbf{e}_{\alpha(\mathbf{i},a)}] \rangle_{\mathcal{Q}}^1 \neq 0$, $c'_a > 0$, $c'_b > 0$ and, finally,*

$$\left. \frac{d\Omega_{\mathbf{c}'}^{\mathbf{c}}}{dv} \right|_{v=1} = -2c'_a c'_b.$$

Proof. We will use the same notation as in 5.1. We proceed by induction on $|\mathbf{c}| = \sum_{j=1}^{\nu} c_j$ for $\mathbf{c} = (c_1, c_2, \ldots, c_\nu) \in \mathbf{N}^\nu$. If $|\mathbf{c}| = 0$, then $\mathbf{c} = (0, 0, \ldots, 0)$ and, if $\mathbf{c}' \in \mathbf{N}^\nu$ is such that $\mathbf{c}' \preceq \mathbf{c}$, then $\mathbf{c}' = \mathbf{c} = (0, 0, \ldots, 0)$. In this case, $\Omega_{\mathbf{c}}^{\mathbf{c}} = 1$ and $(d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1) = 0$. Obviously in this case for all nonsplit short exact sequences $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ where $\alpha(\mathbf{i}, a)$, $\alpha(\mathbf{i}, b)$ is a pair of positive roots with $1 \leq a < b \leq \nu$, then $[\mathbf{V}_{\mathbf{c}'}] \neq [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$.

Assume now that $|\mathbf{c}| \neq 0$ and that the proposition is true whenever $|\mathbf{c}| < p$. Consider now $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ such that $|\mathbf{c}| = p$ and $\mathbf{c}' \preceq \mathbf{c}$. If $\mathbf{c}' = \mathbf{c}$, then $\Omega_{\mathbf{c}'}^{\mathbf{c}} = 1$, $(d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1) = 0$ and it is obvious that the proposition is true for this pair $\mathbf{c}, \mathbf{c}'$.

Assume now that $\mathbf{c}' \neq \mathbf{c}$. Taking the derivative on both sides of equation (4) and evaluating at $v = 1$, we get using proposition 5.2 that $c_k\, (d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1)$ is equal to

$$(\dagger) \qquad \sum_{\mathbf{c}^{(0)}, H, \underline{\lambda}(H)} (-1)^{|H|} \big(S_1(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) + S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) + S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \big)$$

where the triple $\mathbf{c}^{(0)}, H, \underline{\lambda}(H)$ is as in 5.1, $S_1(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is equal to

$$\epsilon(\mathbf{c}, \mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \left. \left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} \left(1 - v^{-2c_j^{(h+1)}}\right)}{(1 - v^{-2})} \right] \right|_{v=1} \Omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}(1),$$

$S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is equal to

$$\frac{d}{dv} \left. \left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} \left(1 - v^{-2c_j^{(h+1)}}\right)}{(1 - v^{-2})} \right] \right|_{v=1} \Omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}(1)$$

and $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is equal to

$$\left. \left[\frac{\prod_{h=0}^{|H|} \prod_{j \in J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))} \left(1 - v^{-2c_j^{(h+1)}}\right)}{(1 - v^{-2})} \right] \right|_{v=1} \frac{d\Omega_{\mathbf{c}^{(0)}}^{\mathbf{c}^-}}{dv}(1).$$

Assume now that $(d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1) \neq 0$, we want to prove that there exists a pair of positive roots $\alpha(\mathbf{i}, a), \alpha(\mathbf{i}, b)$ with $1 \leq a < b \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$ in $\mathbf{K}(\mathcal{Q}, 0)$. Since $(d\Omega_{\mathbf{c}'}^{\mathbf{c}}/dv)(1) \neq 0$, one of the summands in (†) must be different from 0. In other words, there exists a triple $\mathbf{c}^{(0)}$, $H, \underline{\lambda}(H) = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(|H|)})$ as in 5.1 such that either $S_1(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$ or $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$ or $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$. We will now analyse each of these cases.

If $S_1(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$, then we have $\mathbf{c}^{(0)} = \mathbf{c}^-$ because $\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}(1) \neq 0$. Also $H = \emptyset$, $\underline{\lambda}(H) = (\lambda^{(0)})$, $m_0(H) = k$, $|J(\lambda^{(0)}, \alpha(\mathbf{i}, k))| = 1$ for

$$\left[\frac{\prod_{h=0}^{|H|}\prod_{j\in J(\lambda^{(h)},\alpha(\mathbf{i},m_h(H)))}\left(1-v^{-2c_j^{(h+1)}}\right)}{(1-v^{-2})}\right]\Bigg|_{v=1} \neq 0.$$

Note that $\lambda^{(0)}$ is an $\alpha(\mathbf{i}, k)$-admissible partition such that $\lambda^{(0)} \in \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^{(0)}) = \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^-)$. Since $\rho(\mathbf{c}^-) \geq \rho(\mathbf{c}) = k$, then $\lambda^{(0)}$ is the empty partition. So we must have $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}}]$. But this is impossible because $\mathbf{c}' \neq \mathbf{c}$ and consequently, all the summands $S_1(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in the summation (†) are equal to 0.

If $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$, then we have $\mathbf{c}^{(0)} = \mathbf{c}^-$ because $\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}(1) \neq 0$. Since $|J(\lambda^{(h)}, \alpha(\mathbf{i}, m_h(H)))| \geq 1$ for all $h \in H$, we get that $|H| \leq 1$ if we want

$$\frac{d}{dv}\left[\frac{\prod_{h=0}^{|H|}\prod_{j\in J(\lambda^{(h)},\alpha(\mathbf{i},m_h(H)))}\left(1-v^{-2c_j^{(h+1)}}\right)}{(1-v^{-2})}\right]\Bigg|_{v=1} \neq 0.$$

If $|H| = 0$, then $m_0(H) = k$, $\underline{\lambda}(H) = (\lambda^{(0)})$ where $\lambda^{(0)}$ is an $\alpha(\mathbf{i}, k)$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i}, m_0(H)), \mathbf{c}^{(0)}) = \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^-)$. Since $\rho(\mathbf{c}^-) \geq \rho(\mathbf{c}) = k$, then $\lambda^{(0)}$ must be the empty partition. Consequently, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}}]$. This is impossible because $\mathbf{c}' \neq \mathbf{c}$.

If $|H| = 1$, then $1 \leq m_1(H) < m_0(H) \leq \nu$, $\underline{\lambda}(H) = (\lambda^{(0)}, \lambda^{(1)})$ is such that $\lambda^{(0)}$ is an $\alpha(\mathbf{i}, m_0(H))$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i}, m_0(H)), \mathbf{c}^{(0)}) = \Lambda(\alpha(\mathbf{i}, m_0(H)), \mathbf{c}^-)$ and $\lambda^{(1)}$ is a $\alpha(\mathbf{i}, m_1(H))$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i}, m_1(H)), \mathbf{c}^{(1)})$. We must have $\alpha(\mathbf{i}, m_0(H)) + \alpha(\mathbf{i}, m_1(H)) = \alpha(\mathbf{i}, k)$ and consequently $m_1(H) < k < m_0(H)$ because $\mathbf{i}$ is adapted to a quiver. This is a property of the total order associated to a reduced expression $\mathbf{i}$ of the longest element w_0 of W adapted to a quiver $\mathcal{Q}$. We can look at the positions $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}])$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}])$ in the Auslander-Reiten quiver.

Write $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}]) = (x, i)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}]) = (y, j)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}]) = (y', j')$. We have that $\langle[\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}], [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}]\rangle^1_{\mathcal{Q}} = 1$ and $\langle[\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}], [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}]\rangle_{\mathcal{Q}} = 0$ because $\alpha(\mathbf{i}, k) = \alpha(\mathbf{i}, m_0(H)) + \alpha(\mathbf{i}, m_1(H))$. Because of this, there are two possibilities to consider for these positions. In the first possibility, $y + j = x + i$, $y' = x$ and $y' + j' = y$, while in the second possibility $y = x$, $y' = y + j - n - 1$ and $y' + j' = x + i$. For both of these possibilities, we must consider the set of $\alpha(\mathbf{i}, m_0(H))$-admissible partitions $\lambda^{(0)}$ belonging to $\Lambda(\alpha(\mathbf{i}, m_0(H)), \mathbf{c}^-)$ with $|J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H)))| = 1$. Using lemma 4.7, we can determine these $\lambda^{(0)}$.

In the first possibility: $y + j = x + i$, $y' = x$ and $y' + j' = y$, we must have either $\lambda^{(0)} \neq \emptyset$ and $\lambda^{(0)} : j \geq j \geq j \geq \cdots \geq j$ with height $\mathrm{ht}(\lambda^{(0)}) = \ell < (y - x) = (i - j)$ or we must have $\lambda^{(0)} = \emptyset$.

If $\lambda^{(0)} \neq \emptyset$ and $\lambda^{(0)} : j \geq j \geq j \geq \cdots \geq j$ with $\mathrm{ht}(\lambda^{(0)}) = \ell < (y - x) = (i - j)$, then $[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\Theta^{-1}(y-\ell,j+\ell)}] -$

$[\mathbf{e}_{\Theta^{-1}(y-\ell,\ell)}]$ and $\lambda^{(1)}$ can only be $\emptyset$ because $\rho(\mathbf{c}^{(1)}) \geq k$. Thus

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'}] &= [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] \\ &= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\Theta^{-1}(y-\ell,j+\ell)}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\Theta^{-1}(y-\ell,\ell)}] \\ &= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\Theta^{-1}(y-\ell,j+\ell)}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\Theta^{-1}(y-\ell,\ell)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}].\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ such that $a = m_1(H)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}]) = (y-\ell, j+\ell)$ and the module $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\Theta^{-1}(y-\ell,\ell)}$, we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \subset \mathrm{Supp}(\alpha(\mathbf{i},k))$ in this case.

If $\lambda^{(0)} = \emptyset$, then $[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}]$ and $\lambda^{(1)}$ can only be $\emptyset$ because $\rho(\mathbf{c}^{(1)}) \geq k$. Thus

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'}] &= [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] \\ &= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}].\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ such that $a = m_1(H)$ and $b = m_0(H)$ and the module $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)}$, we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 0$ in this case.

In the second possibility: $y = x$, $y' = (y+j-n-1)$ and $y'+j' = x+i$, we must have either $\lambda^{(0)} \neq \emptyset$ and $\lambda^{(0)} : p \geq p \geq p \geq \cdots \geq p$ with $p < (j-i)$ and height $\mathrm{ht}(\lambda^{(0)}) = (n-j+1)$ or we must have $\lambda^{(0)} = \emptyset$.

If $\lambda^{(0)} \neq \emptyset$ and $\lambda^{(0)} : p \geq p \geq p \geq \cdots \geq p$ with $p < (j-i)$ and height $\mathrm{ht}(\lambda^{(0)}) = (n-j+1)$, then

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}^{(1)}}] &= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) \\ &= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\Theta^{-1}(x,j-p)}] - [\mathbf{e}_{\Theta^{-1}(y+j-n-1,x-y+n+1-p)}]\end{aligned}$$

and $\lambda^{(1)}$ can only be $\emptyset$ because $\rho(\mathbf{c}^{(1)}) \geq k$. Thus

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'}] &= [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] \\ &= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\Theta^{-1}(x,j-p)}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\Theta^{-1}(y+j-n-1,x-y+n+1-p)}] \\ &= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\Theta^{-1}(x,j-p)}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\Theta^{-1}(y+j-n-1,x-y+n+1-p)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}].\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ such that $a = m_1(H)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}]) = (x, j-p)$ and the module $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\Theta^{-1}(y+j-n-1,x-y+n+1-p)}$, we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \subset \mathrm{Supp}(\alpha(\mathbf{i},k))$ in this case.

If $\lambda^{(0)} = \emptyset$, then $[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}]$ and $\lambda^{(1)}$ can only be $\emptyset$ because $\rho(\mathbf{c}^{(1)}) \geq k$. Thus

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'}] &= [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] \\ &= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},m_0(H))}] + [\mathbf{e}_{\alpha(\mathbf{i},m_1(H))}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}].\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ such that $a = m_1(H)$ and $b = m_0(H)$ and the module $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)}$, we see that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 0$ in this case.

Finally, we must analyse what is happening when $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H)) \neq 0$. Since $(d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}/dv)(1) \neq 0$ and $|\mathbf{c}^-| = p-1 < |\mathbf{c}|$, we get by induction hypothesis that there exists a pair of positive roots $\alpha(\mathbf{i},a')$, $\alpha(\mathbf{i},b')$ with $1 \leq a' < b' \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a')} \to \mathbf{M}' \to \mathbf{e}_{\alpha(\mathbf{i},b')} \to 0$ of modules of $\mathcal{Q}$

such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{M}'] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$. Also $H = \emptyset$, $\underline{\lambda}(H) = (\lambda^{(0)})$, $m_0(H) = k$, $|J(\lambda^{(0)}, \alpha(\mathbf{i}, k))| = 1$ for

$$\left[\frac{\prod_{h=0}^{|H|}\prod_{j\in J(\lambda^{(h)},\alpha(\mathbf{i},m_h(H)))}\left(1 - v^{-2c_j^{(h+1)}}\right)}{(1-v^{-2})}\right]\Bigg|_{v=1} \neq 0.$$

Recall also that $\lambda^{(0)}$ is an $\alpha(\mathbf{i}, k)$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^{(0)})$. There are three cases to study: $a' \geq k$; $a' < k$ and $\lambda^{(0)} \neq \emptyset$ and finally $a' < k$ and $\lambda^{(0)} = \emptyset$.

If $a' \geq k$, then $\rho(\mathbf{c}^{(0)}) \geq k$ because $b' > a'$. This implies that $\lambda^{(0)} = \emptyset$ and $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{M}'] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$. Taking the positive roots $\alpha(\mathbf{i}, a)$, $\alpha(\mathbf{i}, b)$ such that $a = a'$, $b = b'$ and $\mathbf{M} = \mathbf{M}'$, we see that our statement is verified in this case. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}], [\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}}$ could be 0 or 1.

If $a' < k$ and $\lambda^{(0)} \neq \emptyset$, we must consider the positions of $[\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and $[\mathbf{e}_{\alpha(\mathbf{i},a')}]$. Write $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}]) = (x, i)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a')}]) = (y, j)$. Since $\lambda^{(0)} \in \Lambda(\alpha(\mathbf{i}, k), \mathbf{c}^{(0)})$, $|J(\lambda^{(0)}, \alpha(\mathbf{i}, k))| = 1$, $\rho(\mathbf{c}) = k$ and there is a unique component in the first $(k-1)$ columns of $\mathbf{c}^{(0)}$ different from 0 (it is equal to 1 and is in the (a')th column), then we must have either $\lambda^{(0)} : p \geq p \geq \cdots \geq p$ with height $\mathrm{ht}(\lambda^{(0)}) = (n - i + 1)$, $y = x + i - n - 1$ and $y + j = x + i - p$ or we must have $\lambda^{(0)} : i \geq i \geq i \geq \cdots \geq i$ with height $\mathrm{ht}(\lambda^{(0)}) = \ell$, $y = x - \ell$ and $y + j = x$. The fact that there is a unique component in the first $(k-1)$ columns of $\mathbf{c}^{(0)}$ different from 0 follows from the fact that when we write $\mathbf{M}'$ as a direct sum of indecomposable modules $\mathbf{e}_{\alpha(\mathbf{i},b'')}$, where the b'' are integers between 1 and ν, then $b'' \geq k$ for these b'' because $\rho(\mathbf{c}^-) \geq k$ and $\mathbf{c}^{(0)} \in \mathbf{N}^{\nu}$. From this and because there exists a nonzero homomorphism from $\mathbf{M}'$ to $\mathbf{e}_{\alpha(\mathbf{i},b')}$, we get that $b' > k$. If $\lambda^{(0)} : p \geq p \geq \cdots \geq p$ with height $\mathrm{ht}(\lambda^{(0)}) = (n - i + 1)$, $y = x + i - n - 1$ and $y + j = x + i - p$, then $p = n + 1 - j$, $\langle[\mathbf{e}_{\alpha(\mathbf{i},a')}], [\mathbf{e}_{\alpha(\mathbf{i},b')}]\rangle_{\mathcal{Q}} = 0$, $\langle[\mathbf{e}_{\alpha(\mathbf{i},b')}], [\mathbf{e}_{\alpha(\mathbf{i},a')}]\rangle^1_{\mathcal{Q}} = 1$ and $\mathbf{M}'$ is an indecomposable module $\mathbf{e}_{\alpha(\mathbf{i},a'')}$ where $\alpha(\mathbf{i}, a'') = \alpha(\mathbf{i}, a') + \alpha(\mathbf{i}, b')$. In fact, if $\langle[\mathbf{e}_{\alpha(\mathbf{i},a')}], [\mathbf{e}_{\alpha(\mathbf{i},b')}]\rangle_{\mathcal{Q}} = 1$, then we get a contradiction with the fact that $\mathbf{V}_{\mathbf{c}^{(0)}} \in \mathbf{K}_+(\mathcal{Q}, 0)$. If we write the position of $[\mathbf{e}_{\alpha(\mathbf{i},a'')}]$ by $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a'')}]) = (y', j')$, then $y' > x$ and $y' + j' = y + j$. Consequently, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b')}]) = (y', x+i-y')$. Let $\alpha(\mathbf{i}, a)$ be the positive root such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x, y - x + j)$. Then $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$ and

$$\begin{aligned}
[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] &= [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) \\
&= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a')}] \\
&= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}].
\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i}, a)$, $\alpha(\mathbf{i}, b)$ with a as above and $b = b'$ and $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},a'')}$ we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}], [\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i}, a)) \supset \mathrm{Supp}(\alpha(\mathbf{i}, k))$ in this situation.

If $\lambda^{(0)} : i \geq i \geq i \geq \cdots \geq i$ with height $\mathrm{ht}(\lambda^{(0)}) = \ell$, $y = x - \ell$ and $y + j = x$, then $\ell = j$, $\langle[\mathbf{e}_{\alpha(\mathbf{i},a')}], [\mathbf{e}_{\alpha(\mathbf{i},b')}]\rangle_{\mathcal{Q}} = 0$, $\langle[\mathbf{e}_{\alpha(\mathbf{i},b')}], [\mathbf{e}_{\alpha(\mathbf{i},a')}]\rangle^1_{\mathcal{Q}} = 1$ and $\mathbf{M}'$ is an indecomposable module $\mathbf{e}_{\alpha(\mathbf{i},a'')}$ where $\alpha(\mathbf{i}, a'') = \alpha(\mathbf{i}, a') + \alpha(\mathbf{i}, b')$. In fact, if $\langle[\mathbf{e}_{\alpha(\mathbf{i},a')}], [\mathbf{e}_{\alpha(\mathbf{i},b')}]\rangle_{\mathcal{Q}} = 1$, then we get a contradiction with the fact that $\mathbf{V}_{\mathbf{c}^{(0)}} \in \mathbf{K}_+(\mathcal{Q}, 0)$. If we write the position of $[\mathbf{e}_{\alpha(\mathbf{i},a'')}]$ by $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a'')}]) = (y', j')$, then $y' + j' > x + i$ and $y' = y$. Consequently, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b')}]) = (x, y' + j' - x)$. Let $\alpha(\mathbf{i}, a)$ be the positive root such that $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (y, x + i - y)$. Then $[\mathbf{V}_{\mathbf{c}^{(0)}}] =$

$[\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$ and

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] &= [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i},k))\\ &= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a')}]\\ &= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a'')}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}].\end{aligned}$$

Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ with a as above and $b = b'$ and $\mathbf{M} = \mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},a'')}$ we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}], [\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}} = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$ here.

Finally, if $a' < k$ and $\lambda^{(0)} = \emptyset$, then $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{M}'] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$ and $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{M}'] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]$. Taking the positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ with $a = a'$ and $b = b'$ and $\mathbf{M} = \mathbf{M}'$, we see using 3.13 that our statement is verified in this situation. Note that $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}], [\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}}$ could be 0 or 1.

We can conclude that whenever $(d\Omega^{\mathbf{c}}_{\mathbf{c}'}/dv)(1) \neq 0$, then there exists a pair of positive roots $\alpha(\mathbf{i},a), \alpha(\mathbf{i},b)$ with $1 \leq a < b \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$ in $\mathbf{K}(\mathcal{Q}, 0)$.

If there exists a pair of positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ where $1 \leq a < b \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, then this pair $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ is unique. In fact, this follows because $a = \min\{1 \leq j \leq \nu \mid \langle[\mathbf{V}_{\mathbf{c}'}] - [\mathbf{V}_{\mathbf{c}}], \mathbf{r}_{\alpha(\mathbf{i},j)}\rangle_{\mathcal{Q}} \neq 0\}$ and $b = \max\{1 \leq j \leq \nu \mid \langle[\mathbf{V}_{\mathbf{c}'}] - [\mathbf{V}_{\mathbf{c}}], \mathbf{r}_{\alpha(\mathbf{i},j)}\rangle_{\mathcal{Q}} \neq 0\}$. By 3.13, the isomorphism type of $\mathbf{M}$ is unique. Obviously $\langle \mathbf{e}_{\alpha(\mathbf{i},b)}, \mathbf{e}_{\alpha(\mathbf{i},a)}\rangle^1_{\mathcal{Q}} \neq 0$. We also have $c'_a, c'_b > 0$ because

$$c'_a = \langle[\mathbf{V}_{\mathbf{c}'}], \mathbf{r}_{\alpha(\mathbf{i},a)}\rangle_{\mathcal{Q}} = \langle[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}], \mathbf{r}_{\alpha(\mathbf{i},a)}\rangle_{\mathcal{Q}} = (c_a + 1)$$

and

$$c'_b = \langle[\mathbf{V}_{\mathbf{c}'}], \mathbf{r}_{\alpha(\mathbf{i},b)}\rangle_{\mathcal{Q}} = \langle[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}], \mathbf{r}_{\alpha(\mathbf{i},b)}\rangle_{\mathcal{Q}} = (c_b + 1)$$

using 3.13.

If there exists a pair of positive roots $\alpha(\mathbf{i},a)$, $\alpha(\mathbf{i},b)$ where $1 \leq a < b \leq \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, then we want to prove that $(d\Omega^{\mathbf{c}}_{\mathbf{c}'}/dv)(1) \neq 0$. In fact, we will show that

$$\left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = -2c'_a c'_b.$$

From this we can conclude that the derivative of $\Omega^{\mathbf{c}}_{\mathbf{c}'}$ relative to v is nonzero because $c'_a, c'_b > 0$.

There are eight cases to study:

(1) $a \geq k$,
(2) $a < k$ and $\langle[\mathbf{M}], \mathbf{r}_{\alpha(\mathbf{i},k)}\rangle_{\mathcal{Q}} = 0$,
(3) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ and $c_k = 1$,
(4) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ and $c_k > 1$,
(5) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \subset \mathrm{Supp}(\alpha(\mathbf{i},k))$,
(6) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k > 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \subset \mathrm{Supp}(\alpha(\mathbf{i},k))$,

(7) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$ and, finally,

(8) $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k > 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$.

Before we begin to study all of these cases, we will make some observations. If the coefficient of $[\mathbf{e}_{\alpha(\mathbf{i},j)}]$ when the element $[\mathbf{M}]$ of $\mathbf{K}(\mathcal{Q},0)$ is written in the basis $\{[\mathbf{e}_\beta] \mid \beta \in R^+\}$ is different from 0, then $j \geq k$ because $\rho(\mathbf{c}) = k$ and $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] \in \mathbf{K}_+(\mathcal{Q},0)$. By a similar argument, we get that in the case where $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$, then $k' > k$. Since there exists a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$, we get, using 3.13 and the above observation, that $b > k$. If $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \subset \mathrm{Supp}(\alpha(\mathbf{i},k))$, then there exists a unique integer b' such that $\alpha(\mathbf{i},a) + \alpha(\mathbf{i},b') = \alpha(\mathbf{i},k)$ and in this case we also have $b' > k$. While if $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$, then there exists a unique integer a' such that $\alpha(\mathbf{i},a') + \alpha(\mathbf{i},k) = \alpha(\mathbf{i},a)$ and in this case we also have $a' < a$ and $\alpha(\mathbf{i},a') + \alpha(\mathbf{i},b) = \alpha(\mathbf{i},k')$. Both of these results are easy consequences of the fact that $\alpha(\mathbf{i},a) + \alpha(\mathbf{i},b) = \alpha(\mathbf{i},k) + \alpha(\mathbf{i},k')$ and the description of the positive roots in type A_n. As for the inequalities $b' > k$ and $a' < a$, they follow because $a < k$, $\alpha(\mathbf{i},a) + \alpha(\mathbf{i},b') = \alpha(\mathbf{i},k)$, $\alpha(\mathbf{i},a') + \alpha(\mathbf{i},k) = \alpha(\mathbf{i},a)$ and the fact that the reduced expression $\mathbf{i}$ is adapted to a quiver. We will now study the eight cases above. For this, we use our previous analysis of when a summand of (†) is nonzero.

In case 1: $a \geq k$, then it is only for a summand $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get a nonzero term. This happens when $\mathbf{c}^{(0)}$ is such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $H = \emptyset$ and $\underline{\lambda}(H) = (\lambda^{(0)})$ with $\lambda^{(0)} = \emptyset$. In this case, $m_0(H) = k$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = \{k\}$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i},k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)^0 \left.\left[\frac{(1 - v^{-2c_k^{(1)}})}{(1 - v^{-2})}\right]\right|_{v=1} \left.\frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\right|_{v=1}.$$

We have that

$$c_k^{(1)} = \begin{cases} c_k, & \text{if } a > k, \\ (c_k + 1) = c'_a, & \text{if } a = k \end{cases}$$

and

$$\left.\frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\right|_{v=1} = -2c_a^{(0)} c_b^{(0)} = \begin{cases} -2c'_a c'_b, & \text{if } a > k, \\ -2c_k c'_b, & \text{if } a = k \end{cases}$$

by induction hypothesis. Consequently, $c_k\ (d\Omega^{\mathbf{c}}_{\mathbf{c}'}/dv)(1) = -2c_k c'_a c'_b$ and the result follows because $c_k > 0$.

In case 2: $a < k$ and $\langle [\mathbf{M}], \mathbf{r}_{\alpha(\mathbf{i},k)} \rangle_{\mathcal{Q}} = 0$, then it is only for a summand $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get a nonzero term. This happens when $\mathbf{c}^{(0)}$ is such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $H = \emptyset$ and $\underline{\lambda}(H) = (\lambda^{(0)})$ with $\lambda^{(0)} = \emptyset$. In this case, $m_0(H) = k$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = \{k\}$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i},k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)^0 \left.\left[\frac{(1 - v^{-2c_k^{(1)}})}{(1 - v^{-2})}\right]\right|_{v=1} \left.\frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\right|_{v=1}.$$

Since $\langle[\mathbf{M}], \mathbf{r}_{\alpha(\mathbf{i},k)}\rangle_{\mathcal{Q}} = 0$, this means that, when the element $[\mathbf{M}]$ of $\mathbf{K}(\mathcal{Q}, 0)$ is written in the basis $\{[\mathbf{e}_\alpha] \mid \alpha \in R^+\}$, the coefficient of $[\mathbf{e}_{\alpha(\mathbf{i},k)}]$ is 0. As noted above, $b > k$. Because of these observations and $\rho(\mathbf{c}^-) \geq k$, we get $c_k^{(1)} = c_k$, $c_a^{(0)} = 1 = c'_a$ and $c_b^{(0)} = c'_b$. By induction hypothesis, we get

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = c_k^{(1)}(-2c_a^{(0)}c_b^{(0)}) = -2c_k c'_a c'_b$$

and the result follows because $c_k > 0$.

In case 3: $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ and $c_k = 1$, then it is only for a summand $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get a nonzero term. This happens when $\mathbf{c}^{(0)} = \mathbf{c}^-$, H is the unique subset of the set of edges of $\Delta[r,s]$ such that $|H| = 1$ and $\mathbf{c}(H)$ is the unique element of $\mathbf{N}^\nu$ whose only nonzero entries are in the ath and bth columns and are equal to 1 and $\underline{\lambda}(H) = (\lambda^{(0)}, \lambda^{(1)})$ with both $\lambda^{(0)}$ and $\lambda^{(1)}$ being the empty partition. In this case, $m_0(H) = b$, $m_1(H) = a$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = \{b\}$, $J(\lambda^{(1)}, \alpha(\mathbf{i}, m_1(H))) = \{a\}$, $[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, b)) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - \mathbf{t}(\lambda^{(1)}, \alpha(\mathbf{i}, a)) = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}]$ and

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)^1 \frac{d}{dv}\left.\left[\frac{(1 - v^{-2c_b^{(1)}})(1 - v^{-2c_a^{(2)}})}{(1 - v^{-2})}\right]\right|_{v=1}.$$

As remarked above, $b > k$. So $\rho(\mathbf{c}^{(1)}) \geq k$. We have $c_b^{(1)} = c'_b$ and $c_a^{(2)} = 1 = c'_a$ because $a < k$. Thus

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)\frac{d}{dv}\left.\left(1 - v^{-2c'_b}\right)\right|_{v=1} = -2c'_b = -2c'_a c'_b.$$

The result follows because $c_k = 1$.

In case 4: $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$ and $c_k > 1$, it is both for summands $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ and $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get nonzero terms. We get a nonzero summand $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ when $\mathbf{c}^{(0)} = \mathbf{c}^-$, H is the unique subset of the set of edges of $\Delta[r,s]$ such that $|H| = 1$ and $\mathbf{c}(H)$ is the unique element of $\mathbf{N}^\nu$ whose only nonzero entries are in the ath and bth columns and are equal to 1 and $\underline{\lambda}(H) = (\lambda^{(0)}, \lambda^{(1)})$ with both $\lambda^{(0)}$ and $\lambda^{(1)}$ being the empty partition $\emptyset$. For this triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$, we get $m_0(H) = b$, $m_1(H) = a$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = \{b\}$, $J(\lambda^{(1)}, \alpha(\mathbf{i}, m_1(H))) = \{a\}$, $[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, b)) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - \mathbf{t}(\lambda^{(1)}, \alpha(\mathbf{i}, a)) = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}]$ and the summand $(-1)^{|H|}S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ of (†) corresponding to this triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ is

$$(-1)^1 \frac{d}{dv}\left.\left[\frac{(1 - v^{-2c_b^{(1)}})(1 - v^{-2c_a^{(2)}})}{(1 - v^{-2})}\right]\right|_{v=1} = -2c'_b = -2c'_a c'_b.$$

This last equality follows because $b > k$ as noted previously, $\rho(\mathbf{c}^{(1)}) \geq k$, $c_b^{(1)} = c'_b$, $c_a^{(2)} = 1 = c'_a$.

We get a nonzero summand $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ when $\mathbf{c}^{(0)}$ is such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $H = \emptyset$ and $\underline{\lambda}(H) = (\lambda^{(0)})$ with $\lambda^{(0)} = \emptyset$. In this case, $m_0(H) = k$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, m_0(H))) = \{k\}$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and the summand $(-1)^{|H|}S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ of (†)

corresponding to this triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ is

$$(-1)^0 \left[\frac{(1-v^{-2c_k^{(1)}})}{(1-v^{-2})}\right]\Bigg|_{v=1} \frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\Bigg|_{v=1} = c_k^{(1)}(-2c_a^{(0)}c_b^{(0)}) = -2(c_k-1)c'_a c'_b$$

by induction hypothesis. This last equality follows because $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}$, $b>k$ as remarked previously, $c_k^{(1)}=(c_k-1)$, $c_a^{(0)}=1=c'_a$ and $c_b^{(0)}=c'_b$. Consequently,

$$c_k\, \frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\Bigg|_{v=1} = -2c'_a c'_b - 2(c_k-1)\, c'_a c'_b = -2c_k\, c'_a c'_b$$

and the result follows because $c_k>0$.

In case 5: $a<k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)}\oplus\mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k'\neq k$, $c_k=1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a))\subset\mathrm{Supp}(\alpha(\mathbf{i},k))$, it is only for a summand $S_2(\mathbf{c}^{(0)},H,\underline{\lambda}(H))$ in (†) that we can get a nonzero term. This happens for the following unique triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$. Set $\mathbf{c}^{(0)}=\mathbf{c}^-$. H is the unique subset of the set of edges of $\Delta[r,s]$ such that $|H|=1$ and $\mathbf{c}(H)$ is the unique element of $\mathbf{N}^\nu$ whose only nonzero entries are in the ath and (b')th columns and are equal to 1, where b' is the unique integer such that $\alpha(\mathbf{i},a)+\alpha(\mathbf{i},b')=\alpha(\mathbf{i},k)$. Here $b'>k$. The existence of b' follows from the fact that $\mathrm{Supp}(\alpha(\mathbf{i},a))\subset\mathrm{Supp}(\alpha(\mathbf{i},k))$ as we have noted previously. So $m_0(H)=b'$ and $m_1(H)=a$. To define $\underline{\lambda}(H)=(\lambda^{(0)},\lambda^{(1)})$, we need to consider the positions $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b')}])$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}])$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}])$ in the Auslander-Reiten quiver. Write $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])=(x,i)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b')}])=(y,j)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}])=(y',j')$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}])=(y'',j'')$. Here we have $\langle[\mathbf{e}_{\alpha(\mathbf{i},b')}],[\mathbf{e}_{\alpha(\mathbf{i},a)}]\rangle^1_{\mathcal{Q}}=1$ and $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b')}]\rangle_{\mathcal{Q}}=0$ by our construction of b'. We have $\langle[\mathbf{e}_{\alpha(\mathbf{i},a)}],[\mathbf{e}_{\alpha(\mathbf{i},b)}]\rangle_{\mathcal{Q}}=1$ and $\langle[\mathbf{e}_{\alpha(\mathbf{i},b)}],[\mathbf{e}_{\alpha(\mathbf{i},a)}]\rangle^1_{\mathcal{Q}}=1$ because of 3.13 and the fact that $\mathbf{M}$ is $\mathbf{e}_{\alpha(\mathbf{i},k)}\oplus\mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k'\neq k$. Because of this and 3.13, there are two possibilities for these positions. In the first, we have $y'=x<y''<y$, $(x+i)=(y+j)=(y''+j'')$ and $(y'+j')=y$; while in the second, we have $(y'+j')=(x+i)<(y''+j'')<(y+j)$, $x=y''=y$ and $y'=(y+j-n-1)$. For both of these possibilities, we will define $\underline{\lambda}(H)=(\lambda^{(0)},\lambda^{(1)})$ where $\lambda^{(0)}$ is a $\alpha(\mathbf{i},b')$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i},b'),\mathbf{c}^-)$ and $\lambda^{(1)}$ is a $\alpha(\mathbf{i},a)$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i},a),\mathbf{c}^{(1)})$.

In the first possibility $y'=x<y''<y$, $(x+i)=(y+j)=(y''+j'')$ and $(y'+j')=y$, then we set $\lambda^{(0)}$ to be the partition $\lambda^{(0)}: j\geq j\geq\cdots\geq j$ with height $\mathrm{ht}(\lambda^{(0)})=(y-y'')$ and $\lambda^{(1)}=\emptyset$, then $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k')}])=(y'',y-y'')$ and

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}^{(1)}}] &= [\mathbf{V}_{\mathbf{c}^-}]+[\mathbf{e}_{\alpha(\mathbf{i},b')}]-\mathbf{t}(\lambda^{(0)},\alpha(\mathbf{i},b'))=[\mathbf{V}_{\mathbf{c}^-}]+[\mathbf{e}_{\alpha(\mathbf{i},b)}]-[\mathbf{e}_{\alpha(\mathbf{i},k')}]\\ &=[\mathbf{V}_{\mathbf{c}}]+[\mathbf{e}_{\alpha(\mathbf{i},b)}]-[\mathbf{e}_{\alpha(\mathbf{i},k)}]-[\mathbf{e}_{\alpha(\mathbf{i},k')}]\in\mathbf{K}_+(\mathcal{Q},0)\end{aligned}$$

implies that $\lambda^{(0)}$ as defined is in $\Lambda(\alpha(\mathbf{i},b'),\mathbf{c}^-)$. Note that $[\mathbf{V}_{\mathbf{c}^{(1)}}]\in\mathbf{K}_+(\mathcal{Q},0)$ because $a<k$, $\rho(\mathbf{c})=k$ and $[\mathbf{V}_{\mathbf{c}}]+[\mathbf{e}_{\alpha(\mathbf{i},a)}]-[\mathbf{e}_{\alpha(\mathbf{i},k)}]-[\mathbf{e}_{\alpha(\mathbf{i},k')}]+[\mathbf{e}_{\alpha(\mathbf{i},b)}]\in\mathbf{K}_+(\mathcal{Q},0)$ with $a<k<k'$. Obviously, $\lambda^{(1)}\in\Lambda(\alpha(\mathbf{i},a),\mathbf{c}^{(1)})$ and we have

$$\begin{aligned}[\mathbf{V}_{\mathbf{c}^{(2)}}] &= [\mathbf{V}_{\mathbf{c}^{(1)}}]+[\mathbf{e}_{\alpha(\mathbf{i},a)}]-\mathbf{t}(\lambda^{(1)},\alpha(\mathbf{i},a))=[\mathbf{V}_{\mathbf{c}^{(1)}}]+[\mathbf{e}_{\alpha(\mathbf{i},a)}]\\ &=[\mathbf{V}_{\mathbf{c}}]+[\mathbf{e}_{\alpha(\mathbf{i},a)}]-[\mathbf{e}_{\alpha(\mathbf{i},k)}]-[\mathbf{e}_{\alpha(\mathbf{i},k')}]+[\mathbf{e}_{\alpha(\mathbf{i},b)}]=[\mathbf{V}_{\mathbf{c}'}].\end{aligned}$$

In the second possibility $(y'+j')=(x+i)<(y''+j'')<(y+j)$, $x=y=y''$ and $y'=(y+j-n-1)$, then we set $\lambda^{(0)}$ to be the partition $\lambda^{(0)}:(j-j'')\geq(j-j'')\geq\cdots\geq(j-j'')$ with height $\mathrm{ht}(\lambda^{(0)})=(n-j+1)$ and $\lambda^{(1)}=\emptyset$, then

$\Theta([\mathbf{e}_{\alpha(\mathbf{i},k')}]) = (y', y'' + j'' - y')$ and

$$[\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, b')) = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}]$$
$$= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] \in \mathbf{K}_+(\mathcal{Q}, 0)$$

implies that $\lambda^{(0)}$ as defined is in $\Lambda(\alpha(\mathbf{i}, b'), \mathbf{c}^-)$. Note that $[\mathbf{V}_{\mathbf{c}^{(1)}}] \in \mathbf{K}_+(\mathcal{Q}, 0)$ because $a < k$, $\rho(\mathbf{c}) = k$ and $[\mathbf{V}_{\mathbf{c}}]+[\mathbf{e}_{\alpha(\mathbf{i},a)}]-[\mathbf{e}_{\alpha(\mathbf{i},k)}]-[\mathbf{e}_{\alpha(\mathbf{i},k')}]+[\mathbf{e}_{\alpha(\mathbf{i},b)}] \in \mathbf{K}_+(\mathcal{Q}, 0)$ with $a < k < k'$. Obviously, $\lambda^{(1)} \in \Lambda(\alpha(\mathbf{i}, a), \mathbf{c}^{(1)})$ and we have

$$[\mathbf{V}_{\mathbf{c}^{(2)}}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - \mathbf{t}(\lambda^{(1)}, \alpha(\mathbf{i}, a)) = [\mathbf{V}_{\mathbf{c}^{(1)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}]$$
$$= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] = [\mathbf{V}_{\mathbf{c}'}].$$

For this triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ in both possibilities, we get that $J(\lambda^{(0)}, \alpha(\mathbf{i}, b')) = \{b\}$, $J(\lambda^{(1)}, \alpha(\mathbf{i}, a)) = \{a\}$ and

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)^1 \frac{d}{dv}\left.\left[\frac{(1 - v^{-2c_b^{(1)}})(1 - v^{-2c_a^{(2)}})}{(1 - v^{-2})}\right]\right|_{v=1} \Omega^{\mathbf{c}^-}_{\mathbf{c}^-}(1)$$
$$= (-1)\left.\frac{d}{dv}\left(1 - v^{-2c_b'}\right)\right|_{v=1} = -2c_a'c_b'.$$

This last equality follows because $\rho(\mathbf{c}^{(1)}) \geq k$, $c_b^{(1)} = c_b'$ and $c_a^{(2)} = 1 = c_a'$. The result follows because $c_k = 1$.

In case 6: $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k > 1$ and $\mathrm{Supp}(\alpha(\mathbf{i}, a)) \subset \mathrm{Supp}(\alpha(\mathbf{i}, k))$, it is both for summands $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ and $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get nonzero terms. We get a nonzero summand $S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ for the unique triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ constructed as in case 5 and for this triple, the nonzero summand $(-1)^{|H|}S_2(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is equal to

$$(-1)^1 \frac{d}{dv}\left.\left[\frac{(1 - v^{-2c_b^{(1)}})(1 - v^{-2c_a^{(2)}})}{(1 - v^{-2})}\right]\right|_{v=1} \Omega^{\mathbf{c}^-}_{\mathbf{c}^-}(1) = -2c_a'c_b'$$

as in case 5. We get a nonzero summand $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ when $\mathbf{c}^{(0)}$ is such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$, $H = \emptyset$ and $\lambda^{(0)}$ is the empty partition $\emptyset$. Note $\mathbf{c}^{(0)} \in \mathbf{N}^\nu$ because $c_k > 1$ and $[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] = [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] \in \mathbf{K}_+(\mathcal{Q}, 0)$ implies that $c^-_{k'} > 0$. For this triple, we get $m_0(H) = k$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = \{k\}$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}]+[\mathbf{e}_{\alpha(\mathbf{i},k)}]-\mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}]+[\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and the corresponding nonzero summand $(-1)^{|H|}S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is

$$\left.\left[\frac{(1 - v^{-2c_k^{(1)}})}{(1 - v^{-2})}\right]\right|_{v=1} \left.\frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\right|_{v=1} = c_k^{(1)}(-2c_a^{(0)}c_b^{(0)}) = -2(c_k - 1)c_a'c_b'$$

by induction hypothesis. Here $c_k^{(1)} = (c_k - 1)$, $c_a^{(0)} = 1 = c_a'$ and $c_b^{(0)} = c_b'$. Consequently,

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = -2c_a'c_b' - 2(c_k - 1)c_a'c_b' = -2c_kc_a'c_b'$$

and the result follows because $c_k > 0$.

In case 7: $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k = 1$ and $\mathrm{Supp}(\alpha(\mathbf{i}, a)) \supset \mathrm{Supp}(\alpha(\mathbf{i}, k))$, it is for a summand $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get a nonzero term. This happens for the following unique triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$. As we have noticed previously, if $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$, then there exists a unique integer a' such that $\alpha(\mathbf{i},a') + \alpha(\mathbf{i},k) = \alpha(\mathbf{i},a)$ and, moreover, here $a' < a$ and $\alpha(\mathbf{i},a') + \alpha(\mathbf{i},b) = \alpha(\mathbf{i},k')$. Because of this last equation, there exists a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a')} \to \mathbf{e}_{\alpha(\mathbf{i},k')} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of $\mathcal{Q}$. Take $\mathbf{c}^{(0)}$ such that $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$. Here $\mathbf{c}^{(0)} \in \mathbf{N}^\nu$ because

$$\begin{aligned}[]
[\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}]& \\
= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b')}] &\in \mathbf{K}_+(\mathcal{Q},0)
\end{aligned}$$

implies that $c^-_{k'} > 0$. Take H to be the empty set. Then $m_0(H) = k$. As for $\underline{\lambda}(H) = (\lambda^{(0)})$ where $\lambda^{(0)}$ is an $\alpha(\mathbf{i},k)$-admissible partition belonging to $\Lambda(\alpha(\mathbf{i},k), \mathbf{c}^{(0)})$, we need to consider the positions $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}])$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a')}])$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k')}])$. Write $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k)}]) = (x,i)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a')}]) = (y,j)$ and $\Theta([\mathbf{e}_{\alpha(\mathbf{i},k')}]) = (y',j')$. Since $\langle [\mathbf{e}_{\alpha(\mathbf{i},a')}], [\mathbf{e}_{\alpha(\mathbf{i},b)}] \rangle_{\mathcal{Q}} = 0$, $\langle [\mathbf{e}_{\alpha(\mathbf{i},b)}], [\mathbf{e}_{\alpha(\mathbf{i},a')}] \rangle^1_{\mathcal{Q}} = 1$, $\langle [\mathbf{e}_{\alpha(\mathbf{i},a)}], [\mathbf{e}_{\alpha(\mathbf{i},b)}] \rangle_{\mathcal{Q}} = 1$ and $\langle [\mathbf{e}_{\alpha(\mathbf{i},b)}], [\mathbf{e}_{\alpha(\mathbf{i},a)}] \rangle^1_{\mathcal{Q}} = 1$, we see using 3.13 that there are two possibilities for these positions, either we have $(y+j) = (y'+j') < (x+i)$, $y < x < y'$ and $y = (x+i-n-1)$ or we have $y = y' < x$, $(y+j) < (x+i) < (y'+j')$ and $(y+j) = x$. For each of these possibilities, we will define $\lambda^{(0)}$. In the first possibility: $(y+j) = (y'+j') < (x+i)$, $y < x < y'$ and $y = (x+i-n-1)$, then $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (x, y'+j'-x)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}]) = (y', x+i-y')$ and set $\lambda^{(0)}$ to be the partition $\lambda^{(0)} : (x+i-y-j) \geq (x+i-y-j) \geq \cdots \geq (x+i-y-j)$ of height $\mathrm{ht}(\lambda^{(0)}) = (n-i+1)$. In the second possibility: $y = y' < x$, $(y+j) < (x+i) < (y'+j')$ and $(y+j) = x$, then $\Theta([\mathbf{e}_{\alpha(\mathbf{i},a)}]) = (y, x+i-y)$, $\Theta([\mathbf{e}_{\alpha(\mathbf{i},b)}]) = (x, y'+j'-x)$ and set $\lambda^{(0)}$ to be the partition $\lambda^{(0)} : i \geq i \geq \cdots \geq i$ of height $\mathrm{ht}(\lambda^{(0)}) = (x-y)$. For both possibilities, we get that $\lambda^{(0)} \in \Lambda(\alpha(\mathbf{i},k), \mathbf{c}^{(0)})$ and

$$\begin{aligned}[]
[\mathbf{V}_{\mathbf{c}^{(1)}}] &= [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i},k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a')}] \\
&= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a')}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},a')}] \\
&= [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] \\
&= [\mathbf{V}_{\mathbf{c}}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] = [\mathbf{V}_{\mathbf{c}'}].
\end{aligned}$$

We get $J(\lambda^{(0)}, \alpha(\mathbf{i},k)) = \{a\}$ and

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = (-1)^0 \left.\left[\frac{(1-v^{-2c^{(1)}_a})}{(1-v^{-2})}\right]\right|_{v=1} \left.\frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\right|_{v=1} = c^{(1)}_a \left(-2c^{(0)}_{a'} c^{(0)}_b\right)$$

by induction hypothesis. Since $a' < a < k'$, we get $c^{(0)}_a = 0$ and $c^{(1)}_a = 1$. We have also $c^{(0)}_{a'} = 1 = c'_a$ and $c^{(0)}_b = c'_b$. Thus

$$c_k \left.\frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\right|_{v=1} = -2c'_b = -2c'_a c'_b$$

and the result follows because $c_k = 1$.

In case 8: $a < k$, $\mathbf{M}$ is isomorphic to $\mathbf{e}_{\alpha(\mathbf{i},k)} \oplus \mathbf{e}_{\alpha(\mathbf{i},k')}$ with $k' \neq k$, $c_k > 1$ and $\mathrm{Supp}(\alpha(\mathbf{i},a)) \supset \mathrm{Supp}(\alpha(\mathbf{i},k))$, it is for summands $S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ in (†) that we can get nonzero terms. There are two triples $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ corresponding to nonzero summands. The first triple is defined as in case 7 and proceeding as in case 7, we get that the corresponding nonzero summand $(-1)^{|H|}S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is

$$(-1)^0 \left[\frac{(1-v^{-2c_a^{(1)}})}{(1-v^{-2})}\right]\Bigg|_{v=1} \frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\Bigg|_{v=1} = -2c'_a c'_b.$$

The other triple $\mathbf{c}^{(0)}$, H, $\underline{\lambda}(H)$ for which the corresponding summand is nonzero is defined as follows. $\mathbf{c}^{(0)}$ is defined by $[\mathbf{V}_{\mathbf{c}^{(0)}}] = [\mathbf{V}_{\mathbf{c}^-}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$. Note that $\mathbf{c}^{(0)} \in \mathbf{N}^\nu$ because $c_k > 1$ and also $[\mathbf{V}_\mathbf{c}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{e}_{\alpha(\mathbf{i},k)}] - [\mathbf{e}_{\alpha(\mathbf{i},k')}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}] \in \mathbf{K}_+(\mathcal{Q}, 0)$ implies that $c^-_{k'} > 0$. Set $H = \emptyset$ and $\lambda^{(0)}$ is the empty partition $\emptyset$. For this second triple $m_0(H) = k$, $J(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = \{k\}$, $[\mathbf{V}_{\mathbf{c}'}] = [\mathbf{V}_{\mathbf{c}^{(1)}}] = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}] - \mathbf{t}(\lambda^{(0)}, \alpha(\mathbf{i}, k)) = [\mathbf{V}_{\mathbf{c}^{(0)}}] + [\mathbf{e}_{\alpha(\mathbf{i},k)}]$ and the corresponding nonzero summand $(-1)^{|H|}S_3(\mathbf{c}^{(0)}, H, \underline{\lambda}(H))$ is

$$(-1)^0 \left[\frac{(1-v^{-2c_k^{(1)}})}{(1-v^{-2})}\right]\Bigg|_{v=1} \frac{d\Omega^{\mathbf{c}^-}_{\mathbf{c}^{(0)}}}{dv}\Bigg|_{v=1} = c_k^{(1)}\left(-2c_a^{(0)}c_b^{(0)}\right) = -2(c_k-1)c'_a c'_b$$

by induction hypothesis and because $c_k^{(1)} = (c_k - 1)$, $c_a^{(0)} = 1 = c'_a$ and $c_b^{(0)} = c'_b$. Consequently, we get

$$c_k \frac{d\Omega^{\mathbf{c}}_{\mathbf{c}'}}{dv}\Bigg|_{v=1} = -2c'_a c'_b - 2(c_k-1)c'_a c'_b = -2c_k c'_a c'_b$$

and the result follows because $c_k > 0$. This concludes the proof of the theorem. □

6. Rational smoothness

In this section, we will characterize which orbit closures $\overline{\mathcal{O}_\mathbf{c}}$ are rationally smooth. As a consequence, we will show that if $\overline{\mathcal{O}_\mathbf{c}}$ is rationally smooth, then $\overline{\mathcal{O}_\mathbf{c}}$ is smooth.

6.1. The orbit closure $\overline{\mathcal{O}_\mathbf{c}}$ is said to be rationally smooth if and only if $\zeta^\mathbf{c}_{\mathbf{c}'} = v^{d(\mathbf{c}')-d(\mathbf{c})}$ for all $\mathbf{c}' \in \mathbf{N}^\nu$ with the same $\mathbf{i}$-homogeneity as $\mathbf{c}$ and such that $\mathbf{c}' \preceq \mathbf{c}$. By theorem 1.18, this is the same as asking that

$$Z^\mathbf{c}_{\mathbf{c}'} = v^{d(\mathbf{c})-d(\mathbf{c}')}\zeta^\mathbf{c}_{\mathbf{c}'} = \sum_j \dim \mathcal{H}^{2j}_f(\overline{\mathcal{O}_\mathbf{c}})\ v^{2j} = 1$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with the same $\mathbf{i}$-homogeneity as $\mathbf{c}$ and such that $\mathbf{c}' \preceq \mathbf{c}$. Here f is an $\mathbf{F}_q$-rational point of the orbit $\mathcal{O}_{\mathbf{c}'}$ and $\mathcal{H}^{2j}_f(\overline{\mathcal{O}_\mathbf{c}})$ is the stalk at f of the $(2j)$th cohomology sheaf of the intersection complex of the Zariski closure $\overline{\mathcal{O}_\mathbf{c}}$ of $\mathcal{O}_\mathbf{c}$.

6.2. Fix $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbf{N}^n$ and $\mathbf{c} \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$. For the rest of this section, u will denote v^2. Recall that $Z^\mathbf{c}_{\mathbf{c}'}$ and $\Omega^\mathbf{c}_{\mathbf{c}'}$ are both polynomials in $v^2 = u$. Here $\mathbf{c}' \in \mathbf{N}^\nu$ has $\mathbf{i}$-homogeneity $\mathbf{d}$ and $\mathbf{c}' \preceq \mathbf{c}$.

Proposition 6.3. *The orbit closure $\overline{\mathcal{O}_\mathbf{c}}$ is rationally smooth if and only if*

$$\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\ u^{d(\mathbf{c})-d(\mathbf{c}'')} = 1$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$. In the above sum, $\mathbf{c}'' \in \mathbf{N}^\nu$ and has $\mathbf{i}$-homogeneity $\mathbf{d}$.

Proof. As we saw in theorem 1.18, we have

$$\zeta_{\mathbf{c}'}^{\mathbf{c}} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \omega_{\mathbf{c}'}^{\mathbf{c}''} \, \overline{\zeta_{\mathbf{c}''}^{\mathbf{c}}}$$

for all $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{c}' \preceq \mathbf{c}$. Recall also that $\Omega_{\mathbf{c}'}^{\mathbf{c}''} = v^{d(\mathbf{c}'')-d(\mathbf{c}')}\omega_{\mathbf{c}'}^{\mathbf{c}''}$.

If $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth, then $\zeta_{\mathbf{c}'}^{\mathbf{c}} = v^{d(\mathbf{c}')-d(\mathbf{c})}$ and $\overline{\zeta_{\mathbf{c}''}^{\mathbf{c}}} = v^{d(\mathbf{c})-d(\mathbf{c}'')}$ in the above equation. Thus

$$v^{d(\mathbf{c}')-d(\mathbf{c})} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} v^{d(\mathbf{c}')-d(\mathbf{c}'')} \, \Omega_{\mathbf{c}'}^{\mathbf{c}''} \, v^{d(\mathbf{c})-d(\mathbf{c}'')} \text{ and } \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \Omega_{\mathbf{c}'}^{\mathbf{c}''} \, v^{2(d(\mathbf{c})-d(\mathbf{c}''))} = 1.$$

Conversely, if

$$\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \Omega_{\mathbf{c}'}^{\mathbf{c}''} \, u^{d(\mathbf{c})-d(\mathbf{c}'')} = 1$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$, then we get easily that

$$v^{d(\mathbf{c}')-d(\mathbf{c})} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \omega_{\mathbf{c}'}^{\mathbf{c}''} \, \overline{v^{d(\mathbf{c}'')-d(\mathbf{c})}}$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$. By 7.10 in [10], we can conclude that $\zeta_{\mathbf{c}'}^{\mathbf{c}} = v^{d(\mathbf{c}')-d(\mathbf{c})}$ for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$. So $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth. □

Corollary 6.4. *If the orbit closure $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth, then*

$$-\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}}} \left.\frac{d\Omega_{\mathbf{c}'}^{\mathbf{c}''}}{du}\right|_{u=1} = d(\mathbf{c}) - d(\mathbf{c}')$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$.

Proof. Since $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth, we saw in proposition 6.3 that

$$\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} \Omega_{\mathbf{c}'}^{\mathbf{c}''} \, u^{d(\mathbf{c})-d(\mathbf{c}'')} = u^{d(\mathbf{c})-d(\mathbf{c}')} + \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}}} \Omega_{\mathbf{c}'}^{\mathbf{c}''} \, u^{d(\mathbf{c})-d(\mathbf{c}'')} = 1$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$. By taking the derivative relative to u evaluated at $u = 1$, we get

$$(d(\mathbf{c}) - d(\mathbf{c}')) + \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}}} \left.\frac{d\Omega_{\mathbf{c}'}^{\mathbf{c}''}}{du}\right|_{u=1} + \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}}} \Omega_{\mathbf{c}'}^{\mathbf{c}''}(1) \, (d(\mathbf{c}) - d(\mathbf{c}'')) = 0.$$

By proposition 5.2, $\Omega_{\mathbf{c}'}^{\mathbf{c}''}(1) = 0$ when $\mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}$. Consequently,

$$-\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}}} \left.\frac{d\Omega_{\mathbf{c}'}^{\mathbf{c}''}}{du}\right|_{u=1} = d(\mathbf{c}) - d(\mathbf{c}')$$

for all $\mathbf{c}' \in \mathbf{N}^\nu$ with $\mathbf{i}$-homogeneity $\mathbf{d}$ such that $\mathbf{c}' \preceq \mathbf{c}$. □

6.5**.** For a subset J of the set Δ^1 of edges of the Dynkin graph Δ, we define

$$E_{\mathbf{d}}(J) = \{(f_{ij})_{i\to j} \in E_{\mathbf{d}} \mid f_{ij} = 0 \quad \text{if } \{i,j\} \in \Delta^1 \setminus J\}.$$

The following properties are easily proved and left to the reader. Let J, J' be two subsets of the set Δ^1 of edges of Δ.

(a) $E_{\mathbf{d}}(J)$ is a linear subspace of $E_{\mathbf{d}}$ of dimension $\dim(E_{\mathbf{d}}(J)) = \sum_{\{i,j\}\in J} d_i\, d_j$. In particular, $E_{\mathbf{d}}(J)$ is a smooth variety.

(b) $E_{\mathbf{d}}(J)$ is a $G_{\mathbf{d}}$-stable closed subset of $E_{\mathbf{d}}$ and it is a finite union of $G_{\mathbf{d}}$-orbits. As a consequence and because the field F is algebraically closed, we get that there is a unique open dense $G_{\mathbf{d}}$-orbit in $E_{\mathbf{d}}(J)$. We will denote this orbit by $\mathcal{O}(J)$.

(c) $E_{\mathbf{d}}(J) \cap E_{\mathbf{d}}(J') = E_{\mathbf{d}}(J \cap J')$ and $E_{\mathbf{d}}(J) + E_{\mathbf{d}}(J') = E_{\mathbf{d}}(J \cup J')$.

(d) $E_{\mathbf{d}}(J) \subseteq E_{\mathbf{d}}(J')$ if $J \subseteq J'$.

(e) $E_{\mathbf{d}}(\emptyset) = \{0\}$ and $E_{\mathbf{d}}(\Delta^1) = E_{\mathbf{d}}$.

6.6**.** Because of 6.5 (c) and (e), we see that for each $G_{\mathbf{d}}$-orbit $\mathcal{O}_{\mathbf{c}}$ in $E_{\mathbf{d}}$, there is a unique smallest subset $J(\mathbf{c})$ of Δ^1 for which $\mathcal{O}_{\mathbf{c}} \subseteq E_{\mathbf{d}}(J(\mathbf{c}))$. In fact

$$J(\mathbf{c}) = \bigcap_{\mathcal{O}_{\mathbf{c}} \subseteq E_{\mathbf{d}}(J)} J.$$

Since $E_{\mathbf{d}}(J)$ is closed, then $\mathcal{O}_{\mathbf{c}} \subseteq E_{\mathbf{d}}(J)$ if and only if $\overline{\mathcal{O}_{\mathbf{c}}} \subseteq E_{\mathbf{d}}(J)$. As a consequence, if $\mathbf{c}'' \preceq \mathbf{c}$, then we get easily that $J(\mathbf{c}'') \subseteq J(\mathbf{c})$. Note also that we don't necessarily have $\mathcal{O}_{\mathbf{c}} = \mathcal{O}(J(\mathbf{c}))$. In fact, we will prove in theorem 6.8 that we get this equality precisely when $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth.

6.7**.** Let $\mathbf{c}^{\min} = (c_1^{\min}, c_2^{\min}, \ldots, c_\nu^{\min}) \in \mathbf{N}^\nu$ be defined by

$$c_k^{\min} = \begin{cases} d_i, & \text{if } \alpha(\mathbf{i}, k) = \alpha_i \text{ for some } i,\ 1 \le i \le n \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{O}_{\mathbf{c}^{\min}}$ is the minimal $G_{\mathbf{d}}$-orbit of $E_{\mathbf{d}}$ and it consists of only the null element of $E_{\mathbf{d}}$.

Theorem 6.8. (a) *We have the equality*

$$-\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}}} \left.\frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du}\right|_{u=1} = \dim(E_{\mathbf{d}}(J(\mathbf{c}))).$$

(b) *We have the inequality*

$$-\sum_{\substack{\mathbf{c}'' \\ \mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}}} \left.\frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du}\right|_{u=1} \ge d(\mathbf{c})$$

with the equality if and only if $\overline{\mathcal{O}_{\mathbf{c}}}$ *is rationally smooth. Moreover, in this latter case,* $\overline{\mathcal{O}_{\mathbf{c}}} = E_{\mathbf{d}}(J(\mathbf{c}))$.

Proof. (a) By the chain rule and theorem 5.4, we get for $\mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}$ that

$$\left.\frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du}\right|_{u=1} \neq 0$$

if and only if there exists a pair of positive roots $\alpha(\mathbf{i}, a)$, $\alpha(\mathbf{i}, b)$ where $1 \le a < b \le \nu$ and a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha(\mathbf{i},a)} \to \mathbf{M} \to \mathbf{e}_{\alpha(\mathbf{i},b)} \to 0$ of modules of

$\mathcal{Q}$ such that $[\mathbf{V}_{\mathbf{c}^{\min}}] = [\mathbf{V}_{\mathbf{c}''}] + [\mathbf{e}_{\alpha(\mathbf{i},a)}] - [\mathbf{M}] + [\mathbf{e}_{\alpha(\mathbf{i},b)}]$. Here $[\mathbf{V}_{\mathbf{c}^{\min}}] = \sum_{i=1}^{n} d_i\ [\mathbf{e}_{\alpha_i}]$. Thus because $[\mathbf{V}_{\mathbf{c}^{\min}}] - [\mathbf{e}_{\alpha(\mathbf{i},a)}] + [\mathbf{M}] - [\mathbf{e}_{\alpha(\mathbf{i},b)}] = [\mathbf{V}_{\mathbf{c}''}] \in \mathbf{K}_+(\mathcal{Q}, 0)$, we must have that both $\alpha(\mathbf{i}, a)$ and $\alpha(\mathbf{i}, b)$ are simple roots. So $\alpha(\mathbf{i}, a) = \alpha_j$ and $\alpha(\mathbf{i}, b) = \alpha_i$ for some $1 \leq i, j \leq n$, $i \neq j$. From the chain rule, theorem 5.4 and the expression for $\mathbf{c}^{\min}$, we can also add that in this case

$$\left.\frac{d\Omega^{\mathbf{c}''}_{\mathbf{c}^{\min}}}{du}\right|_{u=1} = -d_i\ d_j.$$

Since there exists also a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha_j} \to \mathbf{M} \to \mathbf{e}_{\alpha_i} \to 0$ of modules of $\mathcal{Q}$, we have that $\{i, j\}$ must be an edge of the Dynkin graph Δ and that $\mathbf{M}$ must be isomorphic to $\mathbf{e}_{\alpha_i+\alpha_j}$. Consequently, in this case $\mathbf{c}'' = (c''_1, c''_2, \ldots, c''_\nu)$ is given by

$$c''_k = \begin{cases} (d_i - 1), & \text{if } \alpha(\mathbf{i}, k) = \alpha_i, \\ (d_j - 1), & \text{if } \alpha(\mathbf{i}, k) = \alpha_j, \\ 1, & \text{if } \alpha(\mathbf{i}, k) = \alpha_i + \alpha_j, \\ d_{i'}, & \text{if } \alpha(\mathbf{i}, k) = \alpha_{i'} \text{ with } i' \neq i, j, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbf{c}'' \preceq \mathbf{c}$, the edge $\{i, j\} \in J(\mathbf{c})$. This follows from the fact that $\{i, j\} \in J(\mathbf{c}'')$ and because $J(\mathbf{c}'') \subseteq J(\mathbf{c})$ from 6.6.

Now let $\{i, j\}$ be an edge in $J(\mathbf{c})$ with $i \to j$ in $\mathcal{Q}$. Consider the element $\mathbf{c}'' = (c''_1, c''_2, \ldots, c''_\nu) \in \mathbf{Z}^\nu$ defined by

$$c''_k = \begin{cases} (d_i - 1), & \text{if } \alpha(\mathbf{i}, k) = \alpha_i, \\ (d_j - 1), & \text{if } \alpha(\mathbf{i}, k) = \alpha_j, \\ 1, & \text{if } \alpha(\mathbf{i}, k) = \alpha_i + \alpha_j, \\ d_{i'}, & \text{if } \alpha(\mathbf{i}, k) = \alpha_{i'} \text{ with } i' \neq i, j, \\ 0, & \text{otherwise;} \end{cases}$$

we want to prove that $\mathbf{c}'' \in \mathbf{N}^\nu$ and it has $\mathbf{i}$-homogeneity $\mathbf{d}$, that $\mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}$ and, finally, that

$$\left.\frac{d\Omega^{\mathbf{c}''}_{\mathbf{c}^{\min}}}{du}\right|_{u=1} = -d_i\ d_j.$$

Since $\{i, j\} \in J(\mathbf{c})$, there exists an element $f = (f_{i'j'})_{i' \to j'} \in \mathcal{O}_{\mathbf{c}}$ such that $f_{ij} \neq 0$; in fact, $f_{ij} \neq 0$ for all $f = (f_{i'j'})_{i' \to j'} \in \mathcal{O}_{\mathbf{c}}$ and $\mathrm{rk}(f_{ij})$ is constant on the orbit $\mathcal{O}_{\mathbf{c}}$. This implies that $d_i, d_j > 0$ and we can conclude that $\mathbf{c}'' \in \mathbf{N}^\nu$. Obviously, $\mathbf{c}''$ has $\mathbf{i}$-homogeneity $\mathbf{d}$ and $\mathbf{c}^{\min} \prec \mathbf{c}''$. From our definition of $\mathbf{c}''$, the fact that $\{i, j\}$ is an edge of Δ and that there exists a nonsplit short exact sequence $0 \to \mathbf{e}_{\alpha_j} \to \mathbf{M} \to \mathbf{e}_{\alpha_i} \to 0$ of modules of $\mathcal{Q}$ because $i \to j$, we get that $[\mathbf{V}_{\mathbf{c}^{\min}}] = [\mathbf{V}_{\mathbf{c}''}] + [\mathbf{e}_{\alpha_i}] - [\mathbf{e}_{\alpha_i+\alpha_j}] + [\mathbf{e}_{\alpha_j}]$ and from theorem 5.4

$$\left.\frac{d\Omega^{\mathbf{c}''}_{\mathbf{c}^{\min}}}{du}\right|_{u=1} = -d_i\ d_j \neq 0.$$

We still have to prove that $\mathbf{c}'' \preceq \mathbf{c}$.

Since Δ is a tree, there exists $(a_1, a_2, \ldots, a_n) \in \mathbf{Z}^n$ such that if $\{i', j'\}$ is an edge different from $\{i, j\}$ and $i' \to j'$, then $a_{i'} + 1 = a_{j'}$ and if $\{i', j'\} = \{i, j\}$, then $a_i = a_j$. In fact, there is an infinity of such solutions and we could choose an n-tuple $(a_1, a_2, \ldots, a_n) \in \mathbf{N}^n$. Fix such an n-tuple. For each $t \in F^*$, we define the element $\gamma(t) \in G_{\mathbf{d}}$ by the requirement that, for $1 \leq i' \leq n$, its i'-component is $(\gamma(t))_{i'} = t^{a_{i'}} I_{d_{i'}}$ where $I_{d_{i'}} : F^{d_{i'}} \to F^{d_{i'}}$ is the identity. Let $f = (f_{i'j'})_{i' \to j'} \in \mathcal{O}_{\mathbf{c}}$. Then, for all $t \in F^*$, we get easily from our choice of $(a_1, a_2, \ldots, a_n)$ that $\gamma(t) \cdot f = (f'_{i'j'})_{i' \to j'}$ where

$$f'_{i'j'} = t^{a_{j'} - a_{i'}} f_{i'j'} = \begin{cases} t f_{i'j'}, & \text{if } \{i', j'\} \neq \{i, j\}, \\ f_{ij}, & \text{if } \{i', j'\} = \{i, j\} \end{cases} \quad \text{for all arrows } i' \to j' \in \mathcal{Q}.$$

From this action, we can conclude that $\tilde{f} = (\tilde{f}_{i'j'})_{i' \to j'} \in E_{\mathbf{d}}$ defined by

$$\tilde{f}_{i'j'} = \begin{cases} 0, & \text{if } \{i', j'\} \neq \{i, j\}, \\ f_{ij}, & \text{if } \{i', j'\} = \{i, j\} \end{cases}$$

belongs to the orbit closure $\overline{\mathcal{O}_{\mathbf{c}}}$. If we denote by $\mathrm{rk}(f_{ij})$ the rank of f_{ij}, then it is easy to see that $\tilde{f} \in \mathcal{O}_{\tilde{\mathbf{c}}}$ where $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_\nu) \in \mathbf{N}^\nu$ is given by

$$\tilde{c}_k = \begin{cases} d_i - \mathrm{rk}(f_{ij}), & \text{if } \alpha(\mathbf{i}, k) = \alpha_i, \\ d_j - \mathrm{rk}(f_{ij}), & \text{if } \alpha(\mathbf{i}, k) = \alpha_j, \\ \mathrm{rk}(f_{ij}), & \text{if } \alpha(\mathbf{i}, k) = \alpha_i + \alpha_j, \\ d_{i'}, & \text{if } \alpha(\mathbf{i}, k) = \alpha_{i'} \text{ with } i' \neq i, j, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\tilde{\mathbf{c}} \preceq \mathbf{c}$. Note that $\mathrm{rk}(f_{ij}) \geq 1$ because $\{i, j\} \in J(\mathbf{c})$. We can easily compare $\tilde{\mathbf{c}}$ and $\mathbf{c}''$ because then we are reduced to the A_2 case and the rank of the $(i \to j)$-component gives the order relation. Since $\mathrm{rk}(f_{ij}) \geq 1$, we get that $\mathbf{c}'' \preceq \tilde{\mathbf{c}}$. So $\mathbf{c}'' \preceq \mathbf{c}$.

From all of what precedes, we get that

$$- \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}}} \left. \frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du} \right|_{u=1} = \sum_{\{i,j\} \in J(\mathbf{c})} d_i \; d_j = \dim(E_{\mathbf{d}}(J(\mathbf{c}))).$$

This concludes the proof of (a).

(b) Since $\mathcal{O}_{\mathbf{c}} \subseteq E_{\mathbf{d}}(J(\mathbf{c}))$, we can conclude that

$$d(\mathbf{c}) \leq \dim(E_{\mathbf{d}}(J(\mathbf{c}))) = - \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}}} \left. \frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du} \right|_{u=1}.$$

This shows the inequality.

If

$$- \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}^{\min} \prec \mathbf{c}'' \preceq \mathbf{c}}} \left. \frac{d\Omega_{\mathbf{c}^{\min}}^{\mathbf{c}''}}{du} \right|_{u=1} = d(\mathbf{c}),$$

then $\dim(\mathcal{O}_{\mathbf{c}}) = \dim(E_{\mathbf{d}}(J(\mathbf{c})))$. Consequently, $\mathcal{O}_{\mathbf{c}}$ is the unique open dense orbit in $E_{\mathbf{d}}(J(\mathbf{c}))$ and $\overline{\mathcal{O}_{\mathbf{c}}} = E_{\mathbf{d}}(J(\mathbf{c}))$ is smooth; in particular, it is rationally smooth.

Conversely, if $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth, then we get by using corollary 6.4 for $\mathbf{c}' = \mathbf{c}^{\min}$ that

$$-\sum_{\substack{\mathbf{c}''\\ \mathbf{c}^{\min}\prec\mathbf{c}''\preceq\mathbf{c}}} \left.\frac{d\Omega^{\mathbf{c}''}_{\mathbf{c}^{\min}}}{du}\right|_{u=1} = d(\mathbf{c})$$

because $d(\mathbf{c}^{\min}) = 0$. □

6.9**.** From our proof of 6.8, we get

Corollary. *If $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth, then $\overline{\mathcal{O}_{\mathbf{c}}}$ is smooth.*

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