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PROJECTIVE RATIONAL SMOOTHNESS OF VARIETIES OF REPRESENTATIONS FOR QUIVERS OF TYPE A

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ABSTRACT. Let $\mathbf{U}^+$ be the positive part of the quantized enveloping algebra $\mathbf{U}$ of type A_n. The change of basis between canonical, and PBW-basis of $\mathbf{U}^+$ has a geometric interpretation in terms of local intersection cohomology of some affine algebraic varieties, namely the Zariski closures of orbits of representations of a quiver of type A_n. In this paper we study the local rational smoothness of these orbit closures and, in particular, the rational smoothness of their projectivization.

1. INTRODUCTION

The present work is a sequel to [BS]. Let F be an algebraic closure of a finite field F_q with q elements, $\mathbf{d} = (d_1, d_2, \dots, d_n) \in \mathbf{N}^n$ and $G_{\mathbf{d}} = \prod_{i=1}^n GL_{d_i}(F)$. Let $\mathcal{Q}$ be a fixed quiver whose underlying graph is the Dynkin graph of type A_n. $G_{\mathbf{d}}$ acts on $E_{\mathbf{d}} = \bigoplus_{i\to j\in\mathcal{Q}} \operatorname{Hom}_F(F^{d_i}, F^{d_j})$, by conjugation. Let $\mathcal{O}$ be a $G_{\mathbf{d}}$-orbit and $\overline{\mathcal{O}}$ its Zariski closure. In [BS] the complete list of rationally smooth orbit closures was obtained. As a consequence it was shown that $\overline{\mathcal{O}}$ is rationally smooth if and only if $\overline{\mathcal{O}}$ is smooth. In this paper we will study local rational smoothness of orbit closures and establish a complete list of the projectively rationally smooth orbit closures in theorem 4.11 and theorem 4.13. Here, projectively rationally smooth means that the projectivization of the orbit closure is rationally smooth.

Rational smoothness is a topological property, which is defined using local intersection cohomology, and has been extensively studied for Schubert varieties. For a survey of some of these results, see [BL00].

Let $\mathbf{U}^+$ be the positive part of the quantized enveloping algebra $\mathbf{U}$ over $\mathbf{Q}(v)$ associated by Drinfeld and Jimbo to the root system of type A_n. Kashiwara and Lusztig have constructed independently of each other a unique canonical basis $\mathbf{B}$ of $\mathbf{U}^+$ in [Kas91] and [Lus90a]. Their construction is not restricted only to type A_n, but to a more general setting including, for example, all simply laced semisimple Lie algebras, i.e. types A, D, E. For each reduced expression $\mathbf{i}$ of the longest element w_0 of the Weyl group W of type A_n, there is also a PBW-basis $B_{\mathbf{i}}$. Some of the reduced expressions are adapted to the quiver $\mathcal{Q}$. In this case, Lusztig has shown in [Lus90a] that the entries of the transition matrix between the bases $\mathbf{B}$ and $B_{\mathbf{i}}$ have a description in terms of local intersection cohomology of orbit closures. We use this approach to study rational smoothness of orbit closures.

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This paper is organized as follows. In section 2 we fix notation and recall some results that we will need at a later stage; notably some of Lusztig's results on the geometric aspects of the canonical basis as presented in [Lus90a], and a result of Bongartz establishing the equivalence of several partial orders on the set of orbits. We study local rational smoothness of orbit closures in section 3, using a result of [BS]. We give a characterization of local rational smoothness in terms of elementary operations in theorem 3.6. In the last section we study projective rational smoothness. It turns out that the only non-smooth, projectively rationally smooth orbit closures are of type A_2 or A_3. This is proved using Bongartz's result mentioned above and the Auslander-Reiten quiver.

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2. Notation and recollections

2.1. **The quantized enveloping algebra U.** Let v be an indeterminate and $\mathbf{U}$ the quantized enveloping algebra of Drinfeld-Jimbo on the field $\mathbf{Q}(v)$ of rational functions corresponding to the simple complex Lie algebra $sl_{n+1}(\mathbf{C})$. $\mathbf{U}$ is a $\mathbf{Q}(v)$-algebra with generators E_i, F_i, K_i, $K_i^{-1}(1 \leq i \leq n)$ and relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i; \tag{r.1}$$

$$K_iE_j = \begin{cases} v^2E_jK_i, & \text{if } i = j, \\ v^{-1}E_jK_i, & \text{if } |i-j| = 1, \\ E_jK_i, & \text{if } |i-j| > 1; \end{cases} \tag{r.2}$$

$$K_iF_j = \begin{cases} v^{-2}F_jK_i, & \text{if } i = j, \\ v\ F_jK_i, & \text{if } |i-j| = 1, \\ F_jK_i, & \text{if } |i-j| > 1; \end{cases} \tag{r.3}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{(K_i - K_i^{-1})}{(v-v^{-1})} \quad \text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j; \end{cases} \tag{r.4}$$

$$\begin{cases} E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0, & \text{if } |i-j| = 1, \\ E_iE_j - E_jE_i = 0, & \text{if } |i-j| > 1; \end{cases} \tag{r.5}$$

$$\begin{cases} F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0, & \text{if } |i-j| = 1, \\ F_iF_j - F_jF_i = 0, & \text{if } |i-j| > 1. \end{cases} \tag{r.6}$$

Let $\mathbf{U}^+$ be the $\mathbf{Q}(v)$-subalgebra generated by the $E_i \quad (1 \leq i \leq n)$. Let $\overline{(\ )} : \mathbf{U} \to \mathbf{U}$ be the involution of $\mathbf{Q}$-algebras defined by

$$E_i \mapsto E_i, \quad F_i \mapsto F_i, \quad K_i \mapsto K_i^{-1} \quad \text{for all } 1 \leq i \leq n \quad \text{and} \quad v \mapsto v^{-1}.$$

Note that $\overline{\mathbf{U}^+} = \mathbf{U}^+$.

Let Q be the free abelian group with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Define an inner product $(\ ,\ )$ on Q by

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } |i-j| = 1, \\ 0, & \text{if } |i-j| > 1. \end{cases}$$

Let $R = \{\alpha \in Q \mid (\alpha, \alpha) = 2\}$. R is a root system of type A_n whose set of simple roots is $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let $R^+ = \{\alpha \in R \mid \alpha = \sum_j c_j \alpha_j$ with $c_j \in \mathbf{N}\}$ be the subset of positive roots. In our case, we have $R = \{\pm(\alpha_a + \alpha_{a+1} + \ldots + \alpha_b) \mid 1 \leq a \leq b \leq n\}$ and $R^+ = \{(\alpha_a + \alpha_{a+1} + \ldots + \alpha_b) \mid 1 \leq a \leq b \leq n\}$. The support of the root $\alpha = \sum_j c_j \alpha_j$ is by definition $\{1 \leq j \leq n \mid c_j \neq 0\}$ and we will denote it by $\mathrm{Supp}(\alpha)$. It is known that the support of a root is a connected subset of $\{1, \ldots, n\}$, i.e., $\mathrm{Supp}(\alpha) = \{a, a+1, \ldots, b\}$ with $1 \leq a \leq b \leq n$.

Each $\alpha \in R$ defines a reflection $s_\alpha : Q \to Q, \quad z \mapsto z - (z, \alpha)\ \alpha$. We will write s_i instead of s_{α_i}. Let W be the Weyl group of R. This is the subgroup of $\mathrm{Aut}(Q)$ generated by the reflections s_i, $(1 \leq i \leq n)$, and it is isomorphic to the symmetric group $\mathcal{S}_{n+1}$. Let $\ell(w)$ be the length of w with respect to the generators $\{s_1, s_2, \ldots, s_n\}$ and denote by w_0 the unique element of W of maximal length. It is known that $\ell(w_0) = \nu = n\,(n+1)/2 = \#(R^+)$.

Lusztig has defined an action of the braid group on $\mathbf{U}$ [Lus90a] and used it to define bases of PBW type (Poincaré, Birkhoff, Witt) of $\mathbf{U}^+$. We now recall these definitions.

For $i \in \{1, \ldots, n\}$, let $\tilde{T}_i : \mathbf{U} \to \mathbf{U}$ be the automorphism of $\mathbf{Q}(v)$-algebras defined by

$$\begin{array}{llll} E_i \mapsto -K_i^{-1} F_i, & F_i \mapsto -E_i K_i, & K_i \mapsto K_i^{-1}, & \\ E_j \mapsto E_j, & F_j \mapsto F_j, & K_j \mapsto K_j, & \text{if } |i-j| > 1, \\ E_j \mapsto (E_j E_i - v^{-1} E_i E_j), & F_j \mapsto (F_i F_j - v F_j F_i), & K_j \mapsto K_i K_j, & \text{if } |i-j| = 1. \end{array}$$

We have $\tilde{T}_i \tilde{T}_j \tilde{T}_i = \tilde{T}_j \tilde{T}_i \tilde{T}_j \quad$ if $|i-j| = 1$ and $\tilde{T}_i \tilde{T}_j = \tilde{T}_j \tilde{T}_i \quad$ if $|i-j| > 1$. This gives us a braid group action. Moreover, we have $\tilde{T}_i(E_j) = \tilde{T}_j^{-1}(E_i)$ if $|i-j| = 1$.

Given integers $M, N \geq 0$, we define

$$[N]! = \prod_{h=1}^{N} \frac{(v^h - v^{-h})}{(v - v^{-1})} \in \ \mathbf{Z}[v, v^{-1}], \begin{bmatrix} M+N \\ N \end{bmatrix} = \frac{[M+N]!}{[M]![N]!} \in \ \mathbf{Z}[v, v^{-1}]$$

and

$$E_i^{(N)} = \frac{E_i^N}{[N]!} \quad \text{for } 1 \leq i \leq n.$$

Let $\mathcal{I}$ be the set of sequences $\mathbf{i} = (i_1, \ldots, i_\nu)$ of elements in $\{1, \ldots, n\}$ such that $s_{i_1} \ldots s_{i_\nu}$ is a reduced expression of w_0. Each $\mathbf{i} \in \mathcal{I}$ gives rise to a total order on $R^+ = \{\alpha(\mathbf{i}, 1), \ldots, \alpha(\mathbf{i}, \nu)\}$, where $\alpha(\mathbf{i}, t) = s_{i_1} s_{i_2} \cdots s_{i_{t-1}}(\alpha_{i_t})$ for $t = 1, \ldots, \nu$. Often we will write α^t instead of $\alpha(\mathbf{i}, t)$ if there is no ambiguity on the choice of $\mathbf{i}$. We say that an element $\mathbf{c} = (c_1, \ldots, c_\nu) \in \mathbf{N}^\nu$ is of $\mathbf{i}$-homogeneity $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{N}^n$ if

$$\sum_{t=1}^{\nu} c_t\ \alpha(\mathbf{i}, t) = \sum_{k=1}^{n} d_k\ \alpha_k\ .$$

For $\mathbf{i} = (i_1, \ldots, i_\nu) \in \mathcal{I}$ and $\mathbf{c} = (c_1, \ldots, c_\nu) \in \mathbf{N}^\nu$, put

$$E_{\mathbf{i}}^{\mathbf{c}} = E_{i_1}^{(c_1)}\ \tilde{T}_{i_1}\left(E_{i_2}^{(c_2)}\right)\ \tilde{T}_{i_1}\tilde{T}_{i_2}\left(E_{i_3}^{(c_3)}\right)\ldots\ \tilde{T}_{i_1}\tilde{T}_{i_2}\cdots\tilde{T}_{i_{(\nu-1)}}\left(E_{i_\nu}^{(c_\nu)}\right).$$

Proposition 2.1. *Let* $\mathbf{i} \in \mathcal{I}$. *Then* $B_{\mathbf{i}} = \{E_{\mathbf{i}}^{\mathbf{c}} \mid \mathbf{c} \in \mathbf{N}^\nu\}$ *is a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}^+$. *We say that* $B_{\mathbf{i}}$ *is a basis of PBW type.*

Proof. [Lus90b, sect. 1.8 and 1.13]. □

We now recall Lusztig's construction of the canonical basis of $\mathbf{U}^+$.

Theorem 2.2. *Let* $\mathbf{i} \in \mathcal{I}$ *and* $\mathcal{L}_{\mathbf{i}}$ *the* $\mathbf{Z}[v^{-1}]$*-submodule of* $\mathbf{U}^+$ *generated by* $B_{\mathbf{i}}$*.*

1. $\mathcal{L}_{\mathbf{i}}$ *is independent of* $\mathbf{i}$*. We denote* $\mathcal{L}_{\mathbf{i}}$ *by* $\mathcal{L}$*.*
2. $\pi(B_{\mathbf{i}})$ *is a* $\mathbf{Z}$*-basis of* $\mathcal{L}/v^{-1}\mathcal{L}$ *independent of* $\mathbf{i}$*. Here* $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ *is the canonical projection. We denote* $\pi(B_{\mathbf{i}})$ *by* B*.*
3. *The restriction of* $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ *defines an isomorphism of* $\mathbf{Z}$*-modules* $\pi' : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/v^{-1}\mathcal{L}$ *where* $\overline{\mathcal{L}}$ *is the image of* $\mathcal{L}$ *under* $\overline{(\ )}$*. In particular,* $\mathbf{B} = \pi'^{-1}(B)$ *is a* $\mathbf{Z}$*-basis of* $\mathcal{L} \cap \overline{\mathcal{L}}$*.*
4. $\mathbf{B}$ *is a* $\mathbf{Z}[v^{-1}]$*-basis of* $\mathcal{L}$ *and a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}^+$*.* $\mathbf{B}$ *is said to be the canonical basis of* $\mathbf{U}^+$*.*
5. *Each element of* $\mathbf{B}$ *is fixed by* $\overline{(\ )} : \mathbf{U}^+ \to \mathbf{U}^+$*.*

Proof. [Lus90a]. □

For some elements $\mathbf{i} \in \mathcal{I}$, called *adapted to the quiver*, Lusztig gave a geometric description of the entries of the transition matrix between bases $\mathbf{B}$ and $B_{\mathbf{i}}$. We will describe these elements of $\mathcal{I}$ and the geometric interpretation of these entries in the following subsections.

2.2. **Quiver modules.** Recall that the set of vertices of the Dynkin graph Δ of the root system R is the set $\{1, \dots .n\}$ and that $\{i, j\}$ form an edge if and only if $|i - j| = 1$. Hence Δ is the graph:

$$1 — 2 — \cdots — (n-1) — n.$$

Let $\mathcal{Q} = (\mathcal{Q}^0, \mathcal{Q}^1)$ be a quiver whose underlying graph is the Dynkin graph Δ of R; i.e., for each edge $\{i, j\}$ of Δ we fix an orientation. (We use the notation G^0 for the set of vertices of the quiver G and G^1 for the set of arrows.) A vertex $i \in \mathcal{Q}^0$ is a sink (respectively a source) of $\mathcal{Q}$ if there is no arrow $i \to j$ (respectively $i \leftarrow j$) $\in \mathcal{Q}^1$. For $i \in \mathcal{Q}^0$, let $\mathcal{Q}_{\to}(i)$ (respectively $\mathcal{Q}_{\leftarrow}(i)$) be the maximal full subquiver of $\mathcal{Q}$ which contains i and which has only arrows $h \to k$ (respectively $h \leftarrow k$) with $h < k$.

An element $\mathbf{i} = (i_1, \dots, i_\nu) \in \mathcal{I}$ is *adapted to the quiver* $\mathcal{Q}$ if i_1 is a sink of $\mathcal{Q}_1 = \mathcal{Q}$ and i_k is a sink of the quiver $\mathcal{Q}_k = s_{i_{k-1}}(\mathcal{Q}_{k-1})$ obtained from $\mathcal{Q}_{k-1}$ by reversing the orientation of all arrows ending at i_{k-1}, where $2 \le k \le \nu$.

Proposition 2.3.
1. *There is an element* $\mathbf{i} \in \mathcal{I}$ *adapted to* $\mathcal{Q}$*.*
2. *An element* $\mathbf{i}$ *of* $\mathcal{I}$ *can be adapted to atmost one quiver.*

Proof. [BS]. □

From now on, let $\mathbf{i}$ be adapted to the quiver $\mathcal{Q}$ and write $\alpha^t = \alpha(\mathbf{i}, t)$. Let F be a fixed field. A module (or representation) $\mathbf{V} = (V_i, f_{ij})$ of $\mathcal{Q}$ is a collection of n finite dimensional F-vector spaces V_i, $(1 \le i \le n)$ and of $(n-1)$ F-linear maps $f_{ij} : V_i \to V_j$, $(i \to j \in \mathcal{Q}^1)$. A morphism from the module $\mathbf{V} = (V_i, f_{ij})$ to the module $\mathbf{V}' = (V_i', f_{ij}')$ is a collection of F-linear maps $g_i : V_i \to V_i'$, $1 \le i \le n$ such that $f_{ij}' \circ g_i = g_j \circ f_{ij}$ for each $i \to j \in \mathcal{Q}^1$. These modules and morphisms form an abelian category $\mathrm{Mod}(\mathcal{Q})$. If $\mathbf{V}$ is a module of $\mathcal{Q}$, denote by $[\mathbf{V}]$ its isomorphism class in $\mathrm{Mod}(\mathcal{Q})$.

The dimension of the module $\mathbf{V} = (V_i, f_{ij})$ is the n-tuple

$$\dim(\mathbf{V}) = (\dim_F(V_1), \dim_F(V_2), \dots, \dim_F(V_n)) \in \mathbf{N}^n.$$

A module $\mathbf{V}$ of $\mathcal{Q}$ is indecomposable if $\mathbf{V}$ cannot be written as the direct sum of proper submodules.

For $k \in \{1,\dots,n\}$, let $\mathbf{P}(k)$ be the following module of $\mathcal{Q}$: $\mathbf{P}(k)_i$ is the vector space over F with basis the set of paths $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ from k to i in $\mathcal{Q}$ and for each $i \to j \in \mathcal{Q}^1$, let $f_{ij}\colon \mathbf{P}(k)_i \to \mathbf{P}(k)_j$ be the unique F-linear map sending the basis element $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i$ to $k = k_0 \to k_1 \to k_2 \to \cdots \to k_m = i \to j$. It is easy to show that $\mathbf{P}(k) = (\mathbf{P}(k)_i, (f_{ij})_{i\to j})$ is an indecomposable projective module of $\mathcal{Q}$ and that each indecomposable projective module is isomorphic to $\mathbf{P}(k)$ for some $k \in \{1,\dots,n\}$. Similarly, for $k \in \{1,\dots,n\}$, let $\mathbf{I}(k)$ be the following module of $\mathcal{Q}$: $\mathbf{I}(k)_i$ is the vector space over F with basis the set of paths $i = k_0 \to k_1 \to k_2 \to \cdots \to k_m = k$ of i to k in $\mathcal{Q}$ and for each $i \to j \in \mathcal{Q}^1$ such that $\mathbf{I}(k)_i \neq 0$ and $\mathbf{I}(k)_j \neq 0$, let $f_{ij}\colon \mathbf{I}(k)_i \to \mathbf{I}(k)_j$ be the unique F-linear map sending the basis element $i = k_0 \to j = k_1 \to k_2 \to \cdots \to k_m = k$ to $j = k_1 \to k_2 \to \cdots \to k_m = k$. It is easy to show that $\mathbf{I}(k) = (\mathbf{I}(k)_i, (f_{ij})_{i\to j})$ is an indecomposable injective module of $\mathcal{Q}$ and that each indecomposable injective module is isomorphic to $\mathbf{I}(k)$ for some $k \in \{1,\dots,n\}$.

Theorem 2.4. *Let $\mathcal{Q}$ be a quiver and $\mathbf{i} \in \mathcal{I}$ adapted to $\mathcal{Q}$.*

1. *For all $\alpha \in R^+$, there is a unique indecomposable module (up to isomorphism), denoted $\mathbf{e}_\alpha \in \mathrm{Mod}(\mathcal{Q})$, such that $\dim(\mathbf{e}_\alpha) = (d_1,\dots,d_n)$ and $\alpha = \sum_{i=1}^n d_i\alpha_i$; any indecomposable module is isomorphic to $\mathbf{e}_\alpha$ for a unique α. This is Gabriel's theorem.*
2. *There exists a bijection $\mathbf{c} = (c_1,c_2,\dots,c_\nu) \mapsto [\mathbf{e}(\mathbf{c})]$ between $\mathbf{N}^\nu$ and the set of isomorphism classes of modules of $\mathcal{Q}$, where $\mathbf{e}(\mathbf{c}) = \bigoplus_{t=1}^{\nu} c_t\ \mathbf{e}_{\alpha^t}$. In this case, $\dim(\mathbf{e}(\mathbf{c})) = (d_1,\dots,d_n)$, where $\sum_{t=1}^{\nu} c_t\ \alpha^t = \sum_{i=1} d_i\ \alpha_i$, i.e. $\mathbf{c}$ is of $\mathbf{i}$-homogeneity $\mathbf{d}$.*
3. *If $\alpha = \alpha_a + \alpha_{a+1} + \ldots + \alpha_b$, $1 \le a \le b \le n$, then $\mathbf{e}_\alpha$ is isomorphic to the module (V_i, f_{ij}) where*

$$V_i = \begin{cases} F, & \text{if } a \le i \le b,\\ 0, & \text{otherwise;}\end{cases} \qquad \text{and } \forall i \to j \in \mathcal{Q}^1, \quad f_{ij} = \begin{cases} \mathrm{Id}_F, & \text{if } a \le i, j \le b,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. [Lus90a, sect. 4.12–4.15]. □

For each non-split exact sequence $\Upsilon\ :\ 0 \to \mathbf{U} \to \mathbf{V} \to \mathbf{W} \to 0$ of modules, we define the associated Hom-Ext sequences Hom–Ext $(\mathbf{X},\Upsilon)$ and Hom–Ext $(\Upsilon,\mathbf{X})$ by

$$\begin{aligned} \text{Hom–Ext}\,(\mathbf{X},\Upsilon) \quad &: \quad 0 \to \mathrm{Hom}\,(\mathbf{X},\mathbf{U}) \to \mathrm{Hom}\,(\mathbf{X},\mathbf{V}) \to \mathrm{Hom}\,(\mathbf{X},\mathbf{W})\\ &\quad\ \to \mathrm{Ext}\,(\mathbf{X},\mathbf{U}) \to \mathrm{Ext}\,(\mathbf{X},\mathbf{V}) \to \mathrm{Ext}\,(\mathbf{X},\mathbf{W}) \to 0\\ \text{Hom–Ext}\,(\Upsilon,\mathbf{X}) \quad &: \quad 0 \to \mathrm{Hom}\,(\mathbf{W},\mathbf{X}) \to \mathrm{Hom}\,(\mathbf{V},\mathbf{X}) \to \mathrm{Hom}\,(\mathbf{U},\mathbf{X})\\ &\quad\ \to \mathrm{Ext}\,(\mathbf{W},\mathbf{X}) \to \mathrm{Ext}\,(\mathbf{V},\mathbf{X}) \to \mathrm{Ext}\,(\mathbf{U},\mathbf{X}) \to 0. \end{aligned}$$

Note that $\mathrm{Ext}^2(\ ,\ )$ is always zero because the path algebra of $\mathcal{Q}$ is hereditary and thus these sequences are exact.

Put $[\mathbf{V},\mathbf{V}'] = \dim_F \mathrm{Hom}_{\mathcal{Q}}(\mathbf{V},\mathbf{V}')$ and $[\mathbf{V},\mathbf{V}']^1 = \dim_F \mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{V},\mathbf{V}')$. Note that $\mathrm{Hom}_{\mathcal{Q}}(\mathbf{V},\mathbf{V}')$ is the F-vector space of morphisms $g : \mathbf{V} \to \mathbf{V}'$ in $\mathrm{Mod}(\mathcal{Q})$ and $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{V},\mathbf{V}')$ is the F-vector space of extensions $0 \to \mathbf{V}' \to \mathbf{E} \to \mathbf{V} \to 0$ in $\mathrm{Mod}(\mathcal{Q})$.

For $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{N}^n$, define

$$E_{\mathbf{d}} = \bigoplus_{i \to j \in \mathcal{Q}^1} \mathrm{Hom}_F(F^{d_i}, F^{d_j}) \quad \text{and} \quad G_{\mathbf{d}} = \prod_{i=1}^{n} GL_{d_i}(F).$$

The group $G_{\mathbf{d}}$ acts on $E_{\mathbf{d}}$ by $(g \cdot f)_{i \to j} = (g_j\ f_{ij}\ g_i^{-1})_{i \to j}$. An element of $E_{\mathbf{d}}$ can be seen as a module in $\mathrm{Mod}(\mathcal{Q})$ of dimension $\mathbf{d}$. Two elements of $E_{\mathbf{d}}$ define isomorphic modules if and only if they are in the same $G_{\mathbf{d}}$-orbit. By Gabriel's theorem, there exists a bijection between the set of ν-tuples $\mathbf{c} = (c_1, \ldots, c_\nu)$ of $\mathbf{i}$-homogeneity $\mathbf{d}$ and the set of $G_{\mathbf{d}}$-orbits in $E_{\mathbf{d}}$, where $\mathbf{c} = (c_1, \ldots, c_\nu)$ corresponds to the orbit $\mathcal{O}_{\mathbf{c}}$ whose elements are isomorphic to $\mathbf{e}(\mathbf{c})$.

For the rest of this paper, let F be the algebraic closure of a finite field F_q with $q = p^e$ elements, where p is a prime number, and let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbf{N}^n$.

There is a partial order on $\mathbf{N}^\nu$ given by $\mathbf{c}' \preceq \mathbf{c}$ if $\mathbf{c}'$ and $\mathbf{c}$ have the same $\mathbf{i}$-homogeneity and the orbit $\mathcal{O}_{\mathbf{c}'}$ is contained in the Zariski closure $\overline{\mathcal{O}}_{\mathbf{c}}$ of $\mathcal{O}_{\mathbf{c}}$. Let $d(\mathbf{c}) = \dim(\mathcal{O}_{\mathbf{c}})$.

Let $\mathcal{S}$ be the set of non-split short exact sequences of modules of $\mathcal{Q}$ and let Op be the subset of $\mathcal{S}$ consisting of all sequences for which the first and the last module are indecomposable. Hence, if $\Upsilon \in Op$, then

$$\Upsilon\ :\ 0 \to \mathbf{e}_{\alpha^s} \to \mathbf{V} \to \mathbf{e}_{\alpha^t} \to 0$$

for some $s, t \in \{1, \ldots, \nu\}$ and some module $\mathbf{V}$. The elements of Op are called *elementary operations*. For $\Upsilon\ :\ 0 \to \mathbf{e}_{\alpha^s} \to \mathbf{V} \to \mathbf{e}_{\alpha^t} \to 0\ \in Op$ define $in(\Upsilon) = s$ and $out(\Upsilon) = t$ and denote by $\mathbf{op}^\Upsilon$ the vector $(op_1^\Upsilon, \ldots, op_\nu^\Upsilon) \in \mathbf{Z}^\nu$ given by

$$op_r^\Upsilon = \begin{cases} -1, & \text{if } r = s, t, \\ 1, & \text{if } \mathbf{e}_{\alpha^r} \text{ is a direct summand of } \mathbf{V}, \\ 0, & \text{otherwise.} \end{cases}$$

For all $\mathbf{c} \in \mathbf{N}^\nu$ define $Op(\mathbf{c}) = \{\Upsilon \in Op \mid \mathbf{c} + \mathbf{op}^\Upsilon \in \mathbf{N}^\nu\}$. Thus an elementary operation $\Upsilon \in Op(\mathbf{c})$ allows us to go from one orbit $\mathcal{O}_{\mathbf{c}}$ to another orbit $\mathcal{O}_{\mathbf{c}+\mathbf{op}^\Upsilon}$. As we will see in theorem 2.5 below, elementary operations do not only preserve the $\mathbf{i}$-homogeneity but they are also compatible with the partial ordering $\preceq$. Define $Op(\mathbf{c}', \mathbf{c}) = \{\Upsilon \in Op(\mathbf{c}') \mid \mathbf{c}' + \mathbf{op}^\Upsilon \preceq \mathbf{c}\}$.

The following theorem is shown in [Bon95].

Theorem 2.5. *Let $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$. Then the following four statements are equivalent:*

1. $\mathbf{c}' \preceq \mathbf{c}$.
2. *There is a sequence of elementary operations $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_k$ such that $\Upsilon_l \in Op(\mathbf{c}' + \sum_{i=1}^{l-1} \mathbf{op}^{\Upsilon_i})$ and $\mathbf{c}' + \sum_{i=1}^{k} \mathbf{op}^{\Upsilon_i} = \mathbf{c}$.*
3. $[\mathbf{e}_\alpha, \mathbf{e}(\mathbf{c}')] \geq [\mathbf{e}_\alpha, \mathbf{e}(\mathbf{c})]$ *for all indecomposable modules $\mathbf{e}_\alpha$.*
4. $[\mathbf{e}(\mathbf{c}'), \mathbf{e}_\alpha] \geq [\mathbf{e}(\mathbf{c}), \mathbf{e}_\alpha]$ *for all indecomposable modules $\mathbf{e}_\alpha$.*

2.3. **The Auslander-Reiten quiver.** In this subsection we will recall some facts about the Auslander-Reiten quiver $\Gamma = (\Gamma^0, \Gamma^1)$ of $\mathcal{Q}$. For this theory, we refer the reader to [Gab80]. We will restrict ourselves to the type A_n.

Let F be an algebraically closed field. The vertices of the Auslander-Reiten quiver are the isomorphism classes of indecomposable modules of the quiver $\mathcal{Q}$ over F and for two vertices $[\mathbf{V}], [\mathbf{W}] \in \Gamma^0$ there is an arrow $[\mathbf{V}] \to [\mathbf{W}] \in \Gamma^1$ if and only if there exists an irreducible morphism $\mathbf{V} \to \mathbf{W} \in \mathrm{Mod}_F(\mathcal{Q})$.

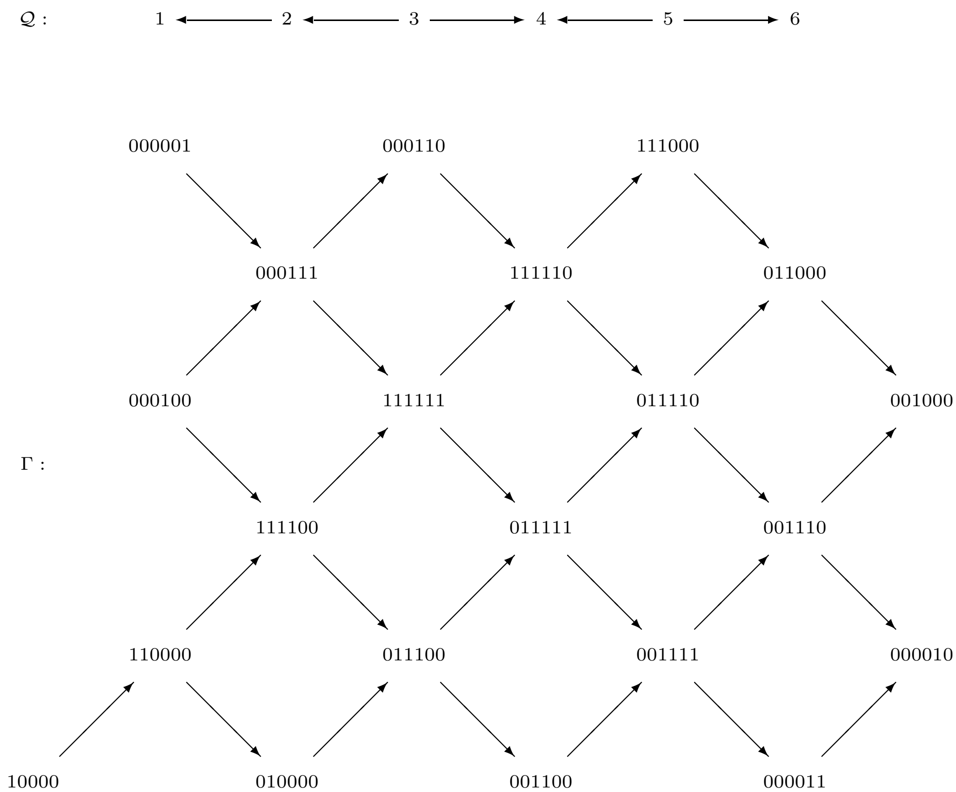

FIGURE 1. A quiver $\mathcal{Q}$ with its corresponding Auslander-Reiten quiver Γ.

By Gabriel's theorem, there is a bijection between Γ^0 and the set of positive roots $\{\alpha^1, \ldots, \alpha^\nu\}$. We do not need to determine explicitly the irreducible morphisms of Γ^1. Let $P[i] \in \{1, \ldots, \nu\}$ be such that $\mathbf{P}(i) = \mathbf{e}_{\alpha^{P[i]}}$ is the i-th indecomposable projective module.

Let $\mathbf{Z}\Delta$ be the translation quiver associated to the Dynkin graph Δ [Gab80, sect. 6.5, fig. 13]. Note that this implies a choice of indices on the vertices of Δ. Recall that $\mathbf{Z}\Delta^0 = \mathbf{Z} \times \{1, \ldots, n\}$ and $\mathbf{Z}\Delta^1 = \{(z,i) \to (z, i+1) \mid z \in \mathbf{Z},\ 1 \le i < n\} \cup \{(z,i) \to (z+1, i-1) \mid z \in \mathbf{Z},\ 1 < i \le n\}$. The translation τ is the function $\tau : \mathbf{Z}\Delta^0 \to \mathbf{Z}\Delta^0, \tau(z,i) = (z-1,i)$. There is a unique embedding Θ of Γ in $\mathbf{Z}\Delta$ such that $\Theta(\alpha^{P[1]}) = (1,1) \in \mathbf{Z}\Delta$. In particular, $\Theta(\alpha^{P[k]}) = (1 - b'_k, k)$ where b'_k is the number of arrows in the non-oriented path from 1 to k that are directed toward k. In the example shown in figure 1 we denote the root $\alpha = \sum_{i=1}^n d_i \alpha_i$ simply by the dimension $(d_1, \ldots, d_n)$ of the corresponding indecomposable module $\mathbf{e}_\alpha$.

Let $I[i] \in \{1, \ldots, \nu\}$ be such that $\mathbf{I}(i) = \mathbf{e}_{\alpha^{I[i]}}$ is the i-th indecomposable injective module. Let $R_\to[i]$ such that $\alpha^{R_\to[i]} = \sum_{h \in \mathcal{Q}^0_\to(i)} \alpha_h$ and $R_\leftarrow[i]$ such that $\alpha^{R_\leftarrow[i]} = \sum_{h \in \mathcal{Q}^0_\leftarrow(i)} \alpha_h$. The $P[i]$, $i = 1, \ldots n$, form the left boundary of Γ and the $I[i]$ the right boundary, while the $R_\to[i]$ form the bottom boundary and the $R_\leftarrow[i]$ the top boundary of Γ. Figure 2 shows an example of type A_6. For each $i = 2, \ldots, n-1$

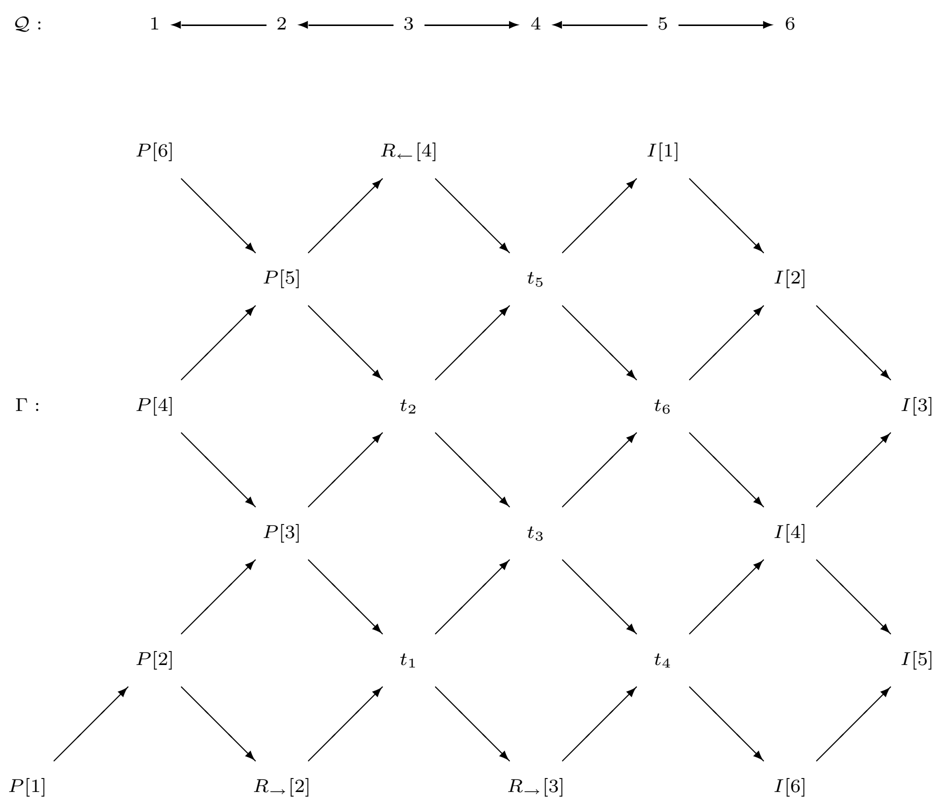

FIGURE 2. An Auslander-Reiten quiver of type A_6.
$P[1] = R_\rightarrow[1]$, $R_\rightarrow[3] = R_\rightarrow[4]$, $R_\rightarrow[5] = R_\rightarrow[6] = I[6]$,
$P[6] = R_\leftarrow[6]$, $R_\leftarrow[4] = R_\leftarrow[5]$, $R_\leftarrow[1] = R_\leftarrow[2] = R_\leftarrow[3] = I[1]$.

there is the following elementary operation $\Upsilon_i \in Op$:

$$\Upsilon_i \ : \ 0 \to \mathbf{e}_{\alpha^{P[i]}} \to \mathbf{e}_{\alpha^{R_\rightarrow[i]}} \oplus \mathbf{e}_{\alpha^{R_\leftarrow[i]}} \to \mathbf{e}_{\alpha^{I[i]}} \to 0.$$

For $i,j,k \in \mathcal{Q}^0$, let $R(i) = \{t \mid 1 \leq t \leq \nu$ and $i \in \mathrm{Supp}(\alpha^t)\}$, $R(i,j) = R(i) \cap R(j)$ and $R(i,j,k) = R(i) \cap R(j) \cap R(k)$. Often we will identify $R(i)$ (respectively $R(i,j), R(i,j,k)$) with the set of positive roots α^t with $t \in R(i)$ (respectively $R(i,j), R(i,j,k)$). Thus, in the Auslander-Reiten quiver, $R(i)$ "is" the set of vertices that are included in the rectangle having corners $P[i], R_\rightarrow[i], I[i], R_\leftarrow[i]$.

The following facts are well known.

Proposition 2.6. *Let $\alpha, \beta \in R^+$, say $\Theta(\alpha) = (x,i)$ and $\Theta(\beta) = (y,j)$ with $x, y \in \mathbf{Z}$ and $i, j \in \{1, \ldots, n\}$. Then*

1. $[\mathbf{e}_\alpha, \mathbf{e}_\beta] \in \{0,1\}$ *and* $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 1$ *iff* $x \leq y \leq x+i-1$ *and* $x+i \leq y+j \leq x+n$.
2. $[\mathbf{e}_\beta, \mathbf{e}_\alpha]^1 \in \{0,1\}$ *and* $[\mathbf{e}_\beta, \mathbf{e}_\alpha]^1 = 1$ *iff* $x+1 \leq y \leq x+i$ *and* $x+i+1 \leq y+j \leq x+n+1$.
3. *If* $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 0$ *and* $[\mathbf{e}_\beta, \mathbf{e}_\alpha]^1 = 1$, *i.e., either* $y = x+i$ *and* $x+i+1 \leq y+j < x+n+1$ *or* $y+j = x+n+1$ *and* $x+1 \leq y < x+i$, *then* $\alpha+\beta \in R^+$ *and there exists a short exact sequence* $0 \to \mathbf{e}_\alpha \to \mathbf{e}_{\alpha+\beta} \to \mathbf{e}_\beta \to 0$ *which is*

a basis of $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{e}_\beta, \mathbf{e}_\alpha)$. *Moreover,*

$$\Theta(\alpha+\beta) = \begin{cases} (x,\ y-x+j), & \text{if } y = x+i \quad \text{and} \quad x+i+1 \le y+j < x+n+1, \\ (y,\ x+i-y), & \text{if } y+j = x+n+1 \quad \text{and} \quad x+1 \le y < x+i \end{cases}$$

and if $0 \to \mathbf{e}_\alpha \to \mathbf{V} \to \mathbf{e}_\beta \to 0$ *is a non-split short exact sequence, then* $\mathbf{V}$ *is isomorphic to* $\mathbf{e}_{\alpha+\beta}$.

4. *If* $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = 1$ *and* $[\mathbf{e}_\beta, \mathbf{e}_\alpha]^1 = 1$, *i.e.,* $x+1 \le y < x+i$ *and* $x+i+1 \le y+j < x+n+1$, *then there exists a unique pair of distinct positive roots* γ, γ' *such that* $\alpha+\beta = \gamma+\gamma'$ *and there exists a short exact sequence* $0 \to \mathbf{e}_\alpha \to \mathbf{e}_\gamma \oplus \mathbf{e}_{\gamma'} \to \mathbf{e}_\beta \to 0$ *which is a basis of* $\mathrm{Ext}^1_{\mathcal{Q}}(\mathbf{e}_\beta, \mathbf{e}_\alpha)$. *Moreover,* $\Theta(\gamma) = (y,\ x+i-y)$ *and* $\Theta(\gamma') = (x,\ y-x+j)$, *and if* $0 \to \mathbf{e}_\alpha \to \mathbf{V} \to \mathbf{e}_\beta \to 0$ *is a non-split short exact sequence, then* $\mathbf{V}$ *is isomorphic to* $\mathbf{e}_\gamma \oplus \mathbf{e}_{\gamma'}$.
5. *Let* $k \in \{1, \dots, n\}$; *then* $[\mathbf{P}(k), \mathbf{e}_\alpha] = 1$ *iff* $k \in \mathrm{Supp}(\alpha)$. *Consequently, if* $\mathbf{c} \in \mathbf{N}^\nu$ *is of i-homogeneity* $\mathbf{d}$, *then* $[\mathbf{P}(k), \mathbf{e}(\mathbf{c})] = d_k$.
6. *Let* $\mathbf{c}$ *be of* $\mathbf{i}$*-homogeneity* $\mathbf{d}$. *With the notation* $c_A = \sum_{t \in A} c_t$, *for all subsets* $A \subset \{1, 2, \dots, \nu\}$, *we have*

$$[\mathbf{e}_{\alpha^{R\to[i]}}, \mathbf{e}(\mathbf{c})] = c_{R(i)\setminus R(i,j)} = d_i - c_{R(i,j)} \quad \text{if } i \to j,\ i > j$$

and

$$[\mathbf{e}_{\alpha^{R\leftarrow[i]}}, \mathbf{e}(\mathbf{c})] = c_{R(i)\setminus R(i,j)} = d_i - c_{R(i,j)} \quad \text{if } i \to j,\ i < j$$

2.4. **Local intersection cohomology of orbit closures.** The results of this subsection have been proved in [Lus90a, chapters 9–10].

Proposition 2.7. *Let* $\mathbf{c} \in \mathbf{N}^\nu$ *be of* $\mathbf{i}$*-homogeneity* $\mathbf{d}$. *Then for each* $\mathbf{c}' \preceq \mathbf{c}$, *there exists* $\omega^{\mathbf{c}}_{\mathbf{c}'} \in \mathbf{Z}[v, v^{-1}]$ *such that*

$$\overline{E^{\mathbf{c}}_{\mathbf{i}}} = \sum_{\mathbf{c}' \preceq \mathbf{c}} \omega^{\mathbf{c}}_{\mathbf{c}'}\ E^{\mathbf{c}'}_{\mathbf{i}}.$$

Moreover, $\omega^{\mathbf{c}}_{\mathbf{c}} = 1$ *and for all* $\mathbf{c}' \preceq \mathbf{c}$, $\Omega^{\mathbf{c}}_{\mathbf{c}'} \overset{def}{=} v^{d(\mathbf{c})-d(\mathbf{c}')}\omega^{\mathbf{c}}_{\mathbf{c}'}$ *is an element of* $\mathbf{Z}[v^2, v^{-2}]$.

Theorem 2.8. *Let* $\mathbf{c} \in \mathbf{N}^\nu$ *be of* $\mathbf{i}$*-homogeneity* $\mathbf{d}$ *and let* $\mathcal{E}^{\mathbf{c}} \in \mathbf{B}$ *be the unique canonical basis element such that* $\pi(\mathcal{E}^{\mathbf{c}}) = \pi(E^{\mathbf{c}}_{\mathbf{i}})$. *Then*

1. $\mathcal{E}^{\mathbf{c}} = \sum_{\mathbf{c}'} \zeta^{\mathbf{c}}_{\mathbf{c}'}\ E^{\mathbf{c}'}_{\mathbf{i}}$, *where* $\mathbf{c}'$ *runs over the set of elements of* $\mathbf{N}^\nu$ *of* $\mathbf{i}$*-homogeneity* $\mathbf{d}$, $\zeta^{\mathbf{c}}_{\mathbf{c}} = 1$ *and* $\zeta^{\mathbf{c}}_{\mathbf{c}'} \in v^{-1}\mathbf{Z}[v^{-1}]$ *for* $\mathbf{c}' \neq \mathbf{c}$.
2. *If* $\mathbf{c}' \not\preceq \mathbf{c}$, *then* $\zeta^{\mathbf{c}}_{\mathbf{c}'} = 0$.
3. *If* $\overline{(\ )}$ *is the* $\mathbf{Z}$*-linear involution of* $\mathbf{Z}[v, v^{-1}]$ *sending* v *to* v^{-1}, *then*

$$\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_{\substack{\mathbf{c}'' \\ \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}}} w^{\mathbf{c}''}_{\mathbf{c}'}\ \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}.$$

4. *If* $\mathbf{c}' \preceq \mathbf{c}$, f *is a* F_q*-rational point of the orbit* $\mathcal{O}_{\mathbf{c}'}$ *in* $E_{\mathbf{d}}$, *and* $\mathcal{H}^a_f$ *is the stalk at* f *of the* a*-th cohomology sheaf of the intersection cohomology complex of the Zariski closure* $\overline{\mathcal{O}}_{\mathbf{c}}$ *of* $\mathcal{O}_{\mathbf{c}}$ *with coefficients in* $\overline{\mathbf{Q}}_\ell$ *(extended by zero on the complement of that closure), where* ℓ *is a prime number* $\neq p$, *and with*

the F_q-structure such that the Frobenius map acts as identity on the stalks of its 0-th cohomology sheaf at the rational points of the orbit $\mathcal{O}_{\mathbf{c}}$, then

$$\mathcal{H}_f^{2a+1} = 0 \quad \textit{for all } a \quad \textit{and} \quad v^{d(\mathbf{c})-d(\mathbf{c}')}\zeta^{\mathbf{c}}_{\mathbf{c}'} = \sum_a \dim(\mathcal{H}_f^{2a})\ v^{2a}.$$

In particular, $v^{d(\mathbf{c})-d(\mathbf{c}')}\zeta^{\mathbf{c}}_{\mathbf{c}'}$ is a polynomial in v^2 with coefficients in $\mathbf{N}$.

Definition 2.9. We say that the orbit closure $\overline{\mathcal{O}}_{\mathbf{c}}$ is *rationally smooth at* $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{\mathbf{c}}$ if for all $\mathbf{c}''$ such that $\mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}$ we have $\sum_a \dim(\mathcal{H}_f^{2a})v^{2a} = 1$ for a F_q-rational point $f \in \mathcal{O}_{\mathbf{c}''}$, i.e., if $\zeta^{\mathbf{c}}_{\mathbf{c}''} = v^{d(\mathbf{c}'')-d(\mathbf{c})}$ for all $\mathbf{c}''$ such that $\mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}$.

The orbit closure $\overline{\mathcal{O}}_{\mathbf{c}}$ is *rationally smooth* if it is rationally smooth at each $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{\mathbf{c}}$, i.e., if $\zeta^{\mathbf{c}}_{\mathbf{c}'} = v^{d(\mathbf{c}')-d(\mathbf{c})}$ for all $\mathbf{c}' \preceq \mathbf{c}$.

Remark 2.10. It is known that $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth iff $\overline{\mathcal{O}_{\mathbf{c}}}$ is smooth [BS, cor. 6.9].

Given $\mathbf{c} \in \mathbf{N}^\nu$ of $\mathbf{i}$-homogeneity $\mathbf{d}$, let $\mathbf{c}^0$ be the unique element of $\mathbf{N}^\nu$ such that $\mathbf{c}^0 \preceq \mathbf{c}$ and $\dim(\mathcal{O}_{\mathbf{c}^0}) = 0$. Note that the orbit $\mathcal{O}_{\mathbf{c}^0}$ consists of one single point: the neutral element $0 \in E_{\mathbf{d}}$

Definition 2.11. The orbit closure $\overline{\mathcal{O}}_{\mathbf{c}}$ is said to be *projectively rationally smooth* if it is rationally smooth at each $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{\mathbf{c}} \setminus \{0\}$, i.e., if $\zeta^{\mathbf{c}}_{\mathbf{c}'} = v^{d(\mathbf{c}')-d(\mathbf{c})}$ for all $\mathbf{c}' \preceq \mathbf{c},\ \mathbf{c}' \neq \mathbf{c}^0$.

3. Local rational smoothness

In this section we will give some conditions for local rational smoothness of an orbit closure of type A.

Let $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^\nu$ and let $\mathcal{O}_{\mathbf{c}'}, \mathcal{O}_{\mathbf{c}}$ be the corresponding orbits such that $\mathbf{c}' \preceq \mathbf{c}$ (i.e., $\mathcal{O}_{\mathbf{c}'} \subset \overline{\mathcal{O}}_{\mathbf{c}}$). For any elementary operation $\Upsilon\ :\ 0 \to \mathbf{e}_{\alpha^s} \to V \to \mathbf{e}_{\alpha^t} \to 0 \in Op$ and $\mathbf{c} \in \mathbf{N}^\nu$, define $e(\Upsilon, \mathbf{c}) = c_s c_t$. Let us recall first two results of [BS].

Proposition 3.1. *Let $\Omega^{\mathbf{c}}_{\mathbf{c}'}$ be as in proposition 2.7. Then $\Omega^{\mathbf{c}}_{\mathbf{c}'}|_{v=1} = 0$ if $\mathbf{c}' \neq \mathbf{c}$.*

Proof. [BS, prop 5.2]. (This proposition also holds if $\mathcal{Q}$ is a quiver of type D or E.) □

Theorem 3.2. *Let $\Omega^{\mathbf{c}}_{\mathbf{c}'}$ be as in proposition 2.7. Then*

$$\left.\frac{\mathrm{d}\Omega^{\mathbf{c}}_{\mathbf{c}'}}{\mathrm{dv}}\right|_{v=1} \neq 0$$

if and only if there exists an elementary operation $\Upsilon \in Op(\mathbf{c}')$, such that $\mathbf{c} = \mathbf{c}' + \mathbf{op}^\Upsilon$. In this case Υ is unique and

$$\left.\frac{\mathrm{d}\Omega^{\mathbf{c}}_{\mathbf{c}'}}{\mathrm{dv}}\right|_{v=1} = -2\, e(\Upsilon, \mathbf{c}').$$

Proof. [BS, thm 5.4]. □

Let $\mathcal{H}_f^*$ be as in theorem 2.8 and put $a_i \overset{def}{=} \dim \mathcal{H}_f^{2i} \geq 0$. Then $a_0 = 1$ and $a_i = 0$ if $2i > d(\mathbf{c}) - d(\mathbf{c}')$. Moreover,

$$\begin{aligned}\sum_{i\geq 0} a_i\, v^{2i} &\overset{2.8.4}{=} v^{d(\mathbf{c})-d(\mathbf{c}')}\, \zeta^{\mathbf{c}}_{\mathbf{c}'} \overset{2.8.3}{=} v^{d(\mathbf{c})-d(\mathbf{c}')} \sum_{\mathbf{c}'\preceq\mathbf{c}''\preceq\mathbf{c}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}\, v^{d(\mathbf{c}')-d(\mathbf{c}'')} \\ &\overset{2.7}{=} v^{d(\mathbf{c})-d(\mathbf{c}')}\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}'}} + \sum_{\mathbf{c}'\prec\mathbf{c}''\preceq\mathbf{c}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}\, v^{d(\mathbf{c})-d(\mathbf{c}'')}.\end{aligned}$$

Note that

$$v^{d(\mathbf{c})-d(\mathbf{c}')}\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}'}} = v^{2(d(\mathbf{c})-d(\mathbf{c}'))}\, \overline{v^{d(\mathbf{c})-d(\mathbf{c}')}\, \zeta^{\mathbf{c}}_{\mathbf{c}'}} \overset{2.8.4}{=} v^{2(d(\mathbf{c})-d(\mathbf{c}'))} \sum_{i\geq 0} a_i\, v^{-2i}.$$

Whence

$$\sum_{i\geq 0} a_i\, v^{2i} - v^{2(d(\mathbf{c})-d(\mathbf{c}'))} \sum_{i\geq 0} a_i\, v^{-2i} = \sum_{\mathbf{c}'\prec\mathbf{c}''\preceq\mathbf{c}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}\, v^{d(\mathbf{c})-d(\mathbf{c}'')}.$$

Taking the derivative relative to v and evaluating at $v = 1$, we get that

$$2\sum_{i\geq 0} ia_i + 2\sum_{i\geq 0} ia_i - 2(d(\mathbf{c}) - d(\mathbf{c}')) \sum_{i\geq 0} a_i$$

is equal to

$$\sum_{\mathbf{c}'\prec\mathbf{c}''\preceq\mathbf{c}} \left(\frac{\mathrm{d}}{\mathrm{dv}} \Omega^{\mathbf{c}''}_{\mathbf{c}'}\Big|_{v=1} \overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}\Big|_{v=1} + \Omega^{\mathbf{c}''}_{\mathbf{c}'}\Big|_{v=1} \frac{\mathrm{d}}{\mathrm{dv}} \left(\overline{\zeta^{\mathbf{c}}_{\mathbf{c}''}}\, v^{d(\mathbf{c})-d(\mathbf{c}'')}\right)\Big|_{v=1}\right).$$

Now we use proposition 3.1 and theorem 3.2 to calculate this last sum and we obtain

$$\sum_{i\geq 0} a_i(d(\mathbf{c}) - d(\mathbf{c}') - 2i) = \sum_{\Upsilon\in Op(\mathbf{c}',\mathbf{c})} e(\Upsilon, \mathbf{c}')\, \overline{\zeta^{\mathbf{c}}_{\mathbf{c}'+\mathbf{op}^{\Upsilon}}}\Big|_{v=1}. \tag{3.1}$$

Remark 3.3. The left-hand side of (3.1) is a sum of natural numbers, since $a_i \geq 0$, $d(\mathbf{c}) \geq d(\mathbf{c}')$ and $a_i = 0$ for $2i > d(\mathbf{c}) - d(\mathbf{c}')$.

Remark 3.4. If $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$ then $a_0 = 1, a_i = 0$ for all $i \geq 1$ and

$$\overline{\zeta^{\mathbf{c}}_{\mathbf{c}'+\mathbf{op}^{\Upsilon}}}\Big|_{v=1} = v^{-d(\mathbf{c}'+\mathbf{op}^{\Upsilon})+d(\mathbf{c})}\Big|_{v=1} = 1 \quad \text{for all } \Upsilon \in Op(\mathbf{c}', \mathbf{c}).$$

Hence in this case (3.1) becomes

$$d(\mathbf{c}) - d(\mathbf{c}') = \sum_{\Upsilon\in Op(\mathbf{c}',\mathbf{c})} e(\Upsilon, \mathbf{c}').$$

Remark 3.5. If $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}''}$ for all $\mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}$, then again $\overline{\zeta^{\mathbf{c}}_{\mathbf{c}'+\mathbf{op}^{\Upsilon}}}\Big|_{v=1} = 1$ for all $\Upsilon \in Op(\mathbf{c}', \mathbf{c})$. In this case (3.1) implies

$$d(\mathbf{c}) - d(\mathbf{c}') \leq \sum_{\Upsilon\in Op(\mathbf{c}',\mathbf{c})} e(\Upsilon, \mathbf{c}')$$

and equality holds iff $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$.

Theorem 3.6. *$\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$ if and only if*

$$\sum_{\Upsilon\in Op(\mathbf{c}'',\mathbf{c})} e(\Upsilon, \mathbf{c}'') = d(\mathbf{c}) - d(\mathbf{c}'') \qquad \textit{for all } \mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}.$$

Proof. Suppose that the equation of the theorem is true for all $\mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}$. We will show that $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$. We proceed by induction on $d(\mathbf{c})-d(\mathbf{c}')$. If $d(\mathbf{c})-d(\mathbf{c}')=0$, then $\mathbf{c}=\mathbf{c}'$ and $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}}$.

Suppose now that $d(\mathbf{c})-d(\mathbf{c}')\geq 1$. By induction we have that $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}''}$ for all $\mathbf{c}' \prec \mathbf{c}'' \preceq \mathbf{c}$. But then remark 3.5 implies that $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$.

The other implication of the theorem is clear by remark 3.4. □

Corollary 3.7. *$\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$ if and only if for all $\mathbf{c}' \preceq \mathbf{c}'' \preceq \mathbf{c}$,*

$$\sum_{\Upsilon \in Op(\mathbf{c}'')\setminus Op(\mathbf{c}'',\mathbf{c})} e(\Upsilon,\mathbf{c}'') = \dim(E_{\mathbf{d}}) - d(\mathbf{c}) = \operatorname{codim}(\mathcal{O}_{\mathbf{c}}).$$

Proof. By the Voigt lemma (see, e.g., [Voi77] or [Gab75, sect. 1]), the codimension of the tangent space of $\mathcal{O}_{\mathbf{c}''}$ is equal to the dimension of $\operatorname{Ext}^1(\mathbf{e}(\mathbf{c}''),\mathbf{e}(\mathbf{c}''))$, hence

$$\begin{aligned}\dim(E_{\mathbf{d}}) - d(\mathbf{c}'') &= [\mathbf{e}(\mathbf{c}''),\mathbf{e}(\mathbf{c}'')]^1 = \sum_{\Upsilon\in Op(\mathbf{c}'')} e(\Upsilon,\mathbf{c}'') \\ &= \sum_{\Upsilon \in Op(\mathbf{c}'')\setminus Op(\mathbf{c}'',\mathbf{c})} e(\Upsilon,\mathbf{c}'') + \sum_{\Upsilon\in Op(\mathbf{c}'',\mathbf{c})} e(\Upsilon,\mathbf{c}'')\end{aligned}$$

and the result follows from the theorem. □

Remark 3.8. Using this result in the case $\mathbf{c}'=\mathbf{c}^0=$ the unique element in $\mathbf{N}^{\nu}$ such that $\dim(\mathcal{O}_{\mathbf{c}^0})=0$, one can prove corollary 6.9 of [BS]; that is,

$$\begin{aligned} & \overline{\mathcal{O}_{\mathbf{c}}} \text{ is rationally smooth everywhere} \\ \Leftrightarrow\ & \overline{\mathcal{O}_{\mathbf{c}}} \text{ is rationally smooth at } \overline{\mathcal{O}_{\mathbf{c}^0}} \\ \Leftrightarrow\ & \overline{\mathcal{O}_{\mathbf{c}}} \text{ is smooth everywhere.}\end{aligned}$$

In section 4 we will study projective rational smoothness of $\overline{\mathcal{O}_{\mathbf{c}}}$ using corollary 3.7 for minimal $\mathbf{c}'\neq\mathbf{c}^0$; that is, $\mathbf{c}'$ is obtained from $\mathbf{c}^0$ by performing one elementary operation.

Next we study rational smoothness of an orbit closure $\overline{\mathcal{O}_{\mathbf{c}}}$ at an orbit $\mathcal{O}_{\mathbf{c}'}$ in the case where one can go from $\mathbf{c}'$ to $\mathbf{c}$ by doing only one elementary operation. We will give a necessary condition for rational smoothness in this case in proposition 3.11.

Let $\mathbf{c}\in\mathbf{N}^{\nu}$ and fix an elementary operation $\Upsilon\ :\ 0\to\mathbf{e}_{\alpha^s}\to\mathbf{V}\to\mathbf{e}_{\alpha^t}\to 0\in Op(\mathbf{c})$. Define

$$\begin{aligned} Op_s(\mathbf{c},\mathbf{c}+\mathbf{op}^{\Upsilon}) &= \{\Upsilon'\in Op(\mathbf{c},\mathbf{c}+\mathbf{op}^{\Upsilon}) \mid \operatorname{in}(\Upsilon')=s=\operatorname{in}(\Upsilon)\},\\ Op^t(\mathbf{c},\mathbf{c}+\mathbf{op}^{\Upsilon}) &= \{\Upsilon'\in Op(\mathbf{c},\mathbf{c}+\mathbf{op}^{\Upsilon}) \mid \operatorname{out}(\Upsilon')=t=\operatorname{out}(\Upsilon)\}.\end{aligned}$$

Thus $Op_s(\mathbf{c},\mathbf{c}+\mathbf{op}^{\Upsilon})\cap Op^t(\mathbf{c},\mathbf{c}+\mathbf{o}p^{\Upsilon})=\{\Upsilon\}$.

Lemma 3.9. *Let $\Upsilon'\ :\ 0\to\mathbf{e}_\alpha\to\mathbf{M}\to\mathbf{e}_\beta\to 0$ be an elementary operation and $\gamma\in R^+$. Then*

$$[\mathbf{e}_\alpha,\mathbf{e}_\gamma]^1-[\mathbf{M},\mathbf{e}_\gamma]^1+[\mathbf{e}_\beta,\mathbf{e}_\gamma]^1=[\mathbf{e}_\alpha,\mathbf{e}_\gamma]-[\mathbf{M},\mathbf{e}_\gamma]+[\mathbf{e}_\beta,\mathbf{e}_\gamma]=[\mathbf{e}_\alpha,\mathbf{e}_\gamma][\mathbf{e}_\beta,\mathbf{e}_\gamma]^1,$$
$$[\mathbf{e}_\gamma,\mathbf{e}_\alpha]^1-[\mathbf{e}_\gamma,\mathbf{M}]^1+[\mathbf{e}_\gamma,\mathbf{e}_\beta]^1=[\mathbf{e}_\gamma,\mathbf{e}_\alpha]-[\mathbf{e}_\gamma,\mathbf{M}]+[\mathbf{e}_\gamma,\mathbf{e}_\beta]=[\mathbf{e}_\gamma,\mathbf{e}_\beta][\mathbf{e}_\gamma,\mathbf{e}_\alpha]^1.$$

Proof. The equations on the left are immediate consequences of the exactness of the sequences Hom–Ext$(\Upsilon',\mathbf{e}_\gamma)$ and Hom–Ext$(\mathbf{e}_\gamma,\Upsilon')$. By the same argument $[\mathbf{e}_\alpha,\mathbf{e}_\gamma]-[\mathbf{M},\mathbf{e}_\gamma]+[\mathbf{e}_\beta,\mathbf{e}_\gamma]=0$ if $[\mathbf{e}_\alpha,\mathbf{e}_\gamma]=0$ and $[\mathbf{e}_\gamma,\mathbf{e}_\alpha]-[\mathbf{e}_\gamma,\mathbf{M}]+[\mathbf{e}_\gamma,\mathbf{e}_\beta]=0$

if $[\mathbf{e}_\gamma, \mathbf{e}_\beta] = 0$. Now if $[\mathbf{e}_\alpha, \mathbf{e}_\gamma] = 1$ (respectively $[\mathbf{e}_\gamma, \mathbf{e}_\beta] = 1$), then as an easy consequence of proposition 2.6 we get $[\mathbf{M}, \mathbf{e}_\gamma]^1 = 0$ (respectively $[\mathbf{e}_\gamma, \mathbf{M}]^1 = 0$), and again by exactness of Hom–Ext $(\Upsilon', \mathbf{e}_\gamma)$ (respectively Hom–Ext $(\mathbf{e}_\gamma, \Upsilon')$) we obtain the lemma. □

The next lemma describes the effect of an elementary operation on the orbit dimension.

Lemma 3.10.

$$d(\mathbf{c} + \mathbf{op}^\Upsilon) - d(\mathbf{c}) = \sum_{\Upsilon' \in Op_s(\mathbf{c}, \mathbf{c}+\mathbf{op}^\Upsilon)} c_{out(\Upsilon')} + \sum_{\Upsilon' \in Op^t(\mathbf{c}, \mathbf{c}+\mathbf{op}^\Upsilon)} c_{in(\Upsilon')} - 1$$

Proof. Let us note first that for any elementary operation $\Upsilon' : 0 \to \mathbf{e}_{\alpha^{s'}} \to \mathbf{V}' \to \mathbf{e}_{\alpha^{t'}} \to 0 \in Op(\mathbf{c})$ we have $\mathbf{c} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c} + \mathbf{op}^\Upsilon$

(3.2) iff $[\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^r}]^1 \leq [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}]^1$ for all $r \in \{1, \dots, \nu\}$,

(3.3) iff $[\mathbf{e}_{\alpha^r}, \mathbf{e}_{\alpha^{t'}}][\mathbf{e}_{\alpha^r}, \mathbf{e}_{\alpha^{s'}}]^1 \leq [\mathbf{e}_{\alpha^r}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha^r}, \mathbf{e}_{\alpha^s}]^1$ for all $r \in \{1, \dots, \nu\}$.

Indeed, by theorem 2.5.4. we have $\mathbf{c} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c} + \mathbf{op}^\Upsilon$ iff $0 \leq [\mathbf{e}(\mathbf{c} + \mathbf{op}^{\Upsilon'}), \mathbf{e}_{\alpha^r}] - [\mathbf{e}(\mathbf{c} + \mathbf{op}^\Upsilon), \mathbf{e}_{\alpha^r}]$ for all $r \in \{1, \dots, \nu\}$ and this is equal to

$$\begin{aligned} &-[\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^r}] + [\mathbf{V}', \mathbf{e}_{\alpha^r}] - [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^r}] + [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^r}] - [\mathbf{V}, \mathbf{e}_{\alpha^r}] + [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}] \\ &\overset{\text{lemma } 3.9}{=} -[\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^r}]^1 + [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}]^1 \end{aligned}$$

which proves (3.2), and (3.3) can be shown by a similar argument using theorem 2.5.3.

Now

$$\begin{aligned} d(\mathbf{c} + \mathbf{op}^\Upsilon) - d(\mathbf{c}) &= \delta(\mathbf{c}) - \delta(\mathbf{c} + \mathbf{op}^\Upsilon) \\ &= [\mathbf{e}(\mathbf{c}), \mathbf{e}(\mathbf{c})]^1 - [\mathbf{e}(\mathbf{c} + \mathbf{op}^\Upsilon), \mathbf{e}(\mathbf{c} + \mathbf{op}^\Upsilon)]^1 \\ &= -\sum_{s',t'=1}^{\nu} \left(c_{s'} op_{t'}^\Upsilon + c_{t'} op_{s'}^\Upsilon + op_{s'}^\Upsilon op_{t'}^\Upsilon \right) [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^{s'}}]^1. \end{aligned}$$

Note that for any $s' \in \{1, \dots, \nu\}$

$$\begin{aligned} \sum_{t'=1}^{\nu} op_{t'}^\Upsilon [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^{s'}}]^1 &= -[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}]^1 + [\mathbf{V}, \mathbf{e}_{\alpha^{s'}}]^1 - [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 \\ &\overset{\text{lemma } 3.9}{=} -[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1. \end{aligned}$$

Similarly, for any $t' \in \{1, \dots, \nu\}$, $\sum_{s'=1}^{\nu} op_{s'}^\Upsilon [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^{s'}}]^1 = -[\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^s}]^1$. Moreover,

$$op_{s'}^\Upsilon op_{t'}^\Upsilon [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^{s'}}]^1 = \begin{cases} 1 & \text{if } s' = s \text{ and } t' = t, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $d(\mathbf{c} + \mathbf{op}^\Upsilon) - d(\mathbf{c}) = \sum_{s'} c_{s'} + \sum_{t'} c_{t'} - 1$, where the first sum is over all $s' \in \{1, \dots, \nu\}$ such that $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 = 1$ and the second sum is over all $t' \in \{1, \dots, \nu\}$ such that $[\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^s}]^1 = 1$.

The condition $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 = 1$ means that there exists an elementary operation Υ' with $in(\Upsilon') = s'$ and $\mathrm{out}(\Upsilon') = t$. If $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}] = 1$, then for any $r \in \{1, \dots, \nu\}$ such that $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}]^1 = 1$ we have $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^r}] \geq [\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^r}]$ (this is an easy consequence of proposition 2.6). Therefore, in this case $[\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}]^1 \leq [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^r}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^r}]^1$

for any $r \in \{1, \dots, \nu\}$ and by (3.2) this means that $\mathbf{c}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}+\mathbf{op}^{\Upsilon}$. Conversely, if $\mathbf{c}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}+\mathbf{op}^{\Upsilon}$, then

$$1 = [\mathbf{e}_{\alpha^{s'}}, \mathbf{e}_{\alpha^{s'}}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 \overset{(3.2)}{\leq} [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 = [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}].$$

Hence the sets $\{s' \mid [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^{s'}}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^{s'}}]^1 = 1\}$ and $\{s' \mid \exists\ \Upsilon' \in Op^t(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon})$ such that $in(\Upsilon') = s'\}$ are equal.

By a similar argument one can show that $\{t' \mid [\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha^{t'}}, \mathbf{e}_{\alpha^s}]^1 = 1\} = \{t' \mid \exists \Upsilon' \in Op_s(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon})$ such that $\mathrm{out}(\Upsilon') = t'\}$ and the lemma follows. □

Proposition 3.11. *If $\overline{\mathcal{O}_{\mathbf{c}+\mathbf{op}^{\Upsilon}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}}$, then*

$$\sum_{\Upsilon' \in Op(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon})} e(\Upsilon', \mathbf{c}+\mathbf{op}^{\Upsilon}) = 0,$$

i.e., $Op(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon}) \cap Op(\mathbf{c}+\mathbf{op}^{\Upsilon}) = \emptyset$.

Proof. Suppose $\overline{\mathcal{O}_{\mathbf{c}+\mathbf{op}^{\Upsilon}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}}$, then by theorem 3.6,

$$\sum_{\Upsilon' \in Op(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon})} e(\Upsilon', \mathbf{c}) - d(\mathbf{c}+\mathbf{op}^{\Upsilon}) + d(\mathbf{c}) = 0.$$

Applying lemma 3.10 and using $e(\Upsilon', \mathbf{c}) = c_{\mathrm{in}(\Upsilon')} c_{\mathrm{out}(\Upsilon')}$ we get

$$\begin{aligned} 0 \;=\; & \sum_{\substack{\Upsilon' \in Op_s(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon}) \\ \mathrm{out}(\Upsilon') \neq t}} c_{\mathrm{out}(\Upsilon')}(c_s - 1) + \sum_{\substack{\Upsilon' \in Op^t(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon}) \\ \mathrm{in}(\Upsilon') \neq s}} c_{\mathrm{in}(\Upsilon')}(c_t - 1) \\ & + (c_s - 1)(c_t - 1) + \sum_{\Upsilon'} e(\Upsilon', \mathbf{c}) \end{aligned}$$

(summation over all $\Upsilon' \in Op(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon}) \setminus \big(Op_s(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon}) \cup Op^t(\mathbf{c}, \mathbf{c}+\mathbf{op}^{\Upsilon})\big)$ in the last sum).

Each term in each of these sums is a natural number since $c_s, c_t \geq 1$, hence each term in each sum must be zero. Note also that $c_s - 1 = c_s + op_s^{\Upsilon}$ and $c_t - 1 = c_t + op_t^{\Upsilon}$. Thus there is no elementary operation $\Upsilon' \in Op(\mathbf{c})$ such that $\mathbf{c}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}+\mathbf{op}^{\Upsilon}$ and $\Upsilon' \in Op(\mathbf{c}+\mathbf{op}^{\Upsilon})$. □

4. Projective rational smoothness

In this section, we give the complete list of orbit closures of type A with rationally smooth projectivization (theorem 4.11 and theorem 4.13). Let $\mathcal{Q} = (\mathcal{Q}^0, \mathcal{Q}^1)$ be a quiver of type A_n and $\mathbf{i}$ adapted to $\mathcal{Q}$. Let $\mathbf{c} \in \mathbf{N}^{\nu}$ be of $\mathbf{i}$-homogeneity $\mathbf{d}$ and let $\mathbf{c}^0 \preceq \mathbf{c}$ be the unique element of $\mathbf{N}^{\nu}$ such that $\dim(\mathcal{O}_{\mathbf{c}}) = 0$ ($\mathcal{O}_{\mathbf{c}^0}$ "is" the neutral element of $E_{\mathbf{d}}$). For each arrow $i \to j \in \mathcal{Q}^1$, let $\mathbf{c}^{ij} = \mathbf{c}^0 + \mathbf{op}^{\Upsilon_{ij}}$, where Υ_{ij} : $0 \to \mathbf{e}_{\alpha_j} \to \mathbf{e}_{\alpha_i+\alpha_j} \to \mathbf{e}_{\alpha_i} \to 0$. We say that the orbit closure $\overline{\mathcal{O}_{\mathbf{c}}}$ is *projectively rationally smooth* if $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$ for all $\mathbf{c}' \preceq \mathbf{c}, \mathbf{c}' \neq \mathbf{c}^0$. Then the projectivization of $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth if and only if $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth.

Corollary 3.7 says that if $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth, then

$$\sum_{\Upsilon \in Op(\mathbf{c}^{ij}) \setminus Op(\mathbf{c}^{ij}, \mathbf{c})} e(\Upsilon, \mathbf{c}^{ij}) = \dim(E_{\mathbf{d}}) - d(\mathbf{c})$$

for all $i \to j \in \mathcal{Q}^1$ such that $\mathbf{c}^{ij} \preceq \mathbf{c}$. We will use this equation in order to characterize projectively rationally smooth orbit closures. To be more precise, first we will prove several lemmas (4.2, 4.3, 4.7) that will tell us in detail which elementary operations are in $Op(\mathbf{c}^{ij}) \setminus Op(\mathbf{c}^{ij}, \mathbf{c})$. This will allow us to calculate the left-hand side of the above equation (proposition 4.8) and finally we will get conditions on $\mathbf{c}$ by comparing this results for different arrows $i \to j \in \mathcal{Q}^1$. Roughly speaking, if the quiver $\mathcal{Q}$ is too big (i.e. $n \geq 4$), then there is no projectively rationally smooth orbit closure which is not smooth, because one gets too many conditions.

In the proofs we use the Auslander-Reiten quiver and theorem 2.5; these proofs are often very technical. Recall the notation $c_A = \sum_{t \in A} c_t$.

Lemma 4.1. *Let $\mathbf{c} \in \mathbf{N}^\nu$ be of $\mathbf{i}$-homogeneity $\mathbf{d}$. Let $h \to k \in \mathcal{Q}^1$ be any arrow and define $\mathbf{c}' \in \mathbf{N}^\nu$ by*

$$c'_t = \begin{cases} 0 & \text{if} \quad t \in R(h,k), \\ c_t + c_{R(h,k)} & \text{if} \quad \alpha^t \in \{\alpha_h, \alpha_k\}, \\ c_t & \qquad \textit{otherwise.} \end{cases}$$

In other words, if $\mathcal{O}_\mathbf{c}$ is the orbit $G_\mathbf{d} \cdot f$, $f \in E_\mathbf{d}$, then $\mathcal{O}_{\mathbf{c}'} = G_\mathbf{d} \cdot f'$ with

$$f'_{ij} = \begin{cases} f_{ij} & \text{if} \quad \{i,j\} \neq \{h,k\}, \\ 0 & \text{if} \quad \{i,j\} = \{h,k\}. \end{cases}$$

Then

$$\mathbf{c}' \preceq \mathbf{c}.$$

Proof. Define $(a_1, a_2, \ldots, a_n) \in \mathbf{N}^n$ by $a_l = 1$ if l and h are in the same component of $\mathcal{Q} \setminus \{h \to k\}$ and $a_l = 2$ if l and k are in the same component of $\mathcal{Q} \setminus \{h \to k\}$. For each $t \in F^*$, we define $\gamma(t) \in G_\mathbf{d}$ by $(\gamma(t))_l = t^{a_l} Id_l$ where $Id_l : F^{d_l} \to F^{d_l}$ is the identity.

Then

$$(\gamma(t) \cdot f)_{ij} = (\gamma(t)_j \circ f_{ij} \circ \gamma(t)_i^{-1})_{ij} = t^{a_j - a_i} f_{ij} = \begin{cases} t\, f_{ij} & \text{if } (i \to j) = (h \to k), \\ f_{ij} & \text{otherwise} \end{cases}$$

for all $t \in F^*$ and all $i \to j \in \mathcal{Q}^1$. From this action, we can conclude that $f' \in \overline{\mathcal{O}_\mathbf{c}}$, hence $\mathbf{c}' \preceq \mathbf{c}$. □

Lemma 4.2. *For any $0 \leq m \leq \min\{d_i, d_j\}$,*

$$\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c} \ \Leftrightarrow \ c_{R(i,j)} \geq m.$$

Proof. Suppose first that $c_{R(i,j)} \geq m$. Let $f \in \mathcal{O}_\mathbf{c}$ and define f' by

$$f'_{hk} = \begin{cases} f_{ij} & \text{if } (h \to k) = (i \to j), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{c}' \in \mathbf{N}^\nu$ be of the same $\mathbf{i}$-homogeneity as $\mathbf{c}$ such that $f' \in \mathcal{O}_{\mathbf{c}'}$. Then $\mathbf{c}^0 + c_{R(i,j)}\ \mathbf{op}^{\Upsilon_{ij}} = \mathbf{c}'$. But by lemma 4.1, we have $\mathbf{c}' \preceq \mathbf{c}$ whence $\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c}$.

Now let $\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c}$ and suppose, without loss of generality, that $i < j$. Then $[\mathbf{e}_{\alpha^{R \leftarrow [i]}}, \mathbf{e}(\mathbf{c})] = d_i - c_{R(i,j)}$ and $\left[\mathbf{e}_{\alpha^{R \leftarrow [i]}}, \mathbf{e}(\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}})\right] = d_i - m$, by proposition 2.6.6. But since $\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c}$, theorem 2.5 implies

$$\left[\mathbf{e}_{\alpha^{R \leftarrow [i]}}, \mathbf{e}(\mathbf{c}^0 + m\ \mathbf{op}^{\Upsilon_{ij}})\right] \geq [\mathbf{e}_{\alpha^{R \leftarrow [i]}}, \mathbf{e}(\mathbf{c})]$$

whence $c_{R(i,j)} \geq m$. □

Lemma 4.3. *Let $i \to j, h \to k \in \mathcal{Q}^1$ such that $\{i,j\} \cap \{h,k\} = \emptyset$, $\mathbf{c}^{ij} \preceq \mathbf{c}$ and $\mathbf{c}^{hk} \preceq \mathbf{c}$. Then*

$$\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{hk}} \preceq \mathbf{c}.$$

Proof. Note that $c_{R(i,j)} \geq 1$ and $c_{R(h,k)} \geq 1$, by lemma 4.2. Let $f \in \mathcal{O}_{\mathbf{c}}$ and define f' by

$$f'_{xy} = \begin{cases} f_{xy} & \text{if } (x \to y) \in \{(i \to j), (h \to k)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{c}' \in \mathbf{N}^\nu$ be of the same $\mathbf{i}$-homogeneity as $\mathbf{c}$ such that $f' \in \mathcal{O}_{\mathbf{c}'}$. Then $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{hk}} + (\mathbf{c}_{R(i,j)} - 1)\,\mathbf{op}^{\Upsilon_{ij}} + (\mathbf{c}_{R(h,k)} - 1)\,\mathbf{op}^{\Upsilon_{hk}} = \mathbf{c}'$ and so $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{hk}} \preceq \mathbf{c}'$ and by lemma 4.1, we have $\mathbf{c}' \preceq \mathbf{c}$. □

Denote by $\mathcal{Q}^0_\pm$ the subset of sources and sinks of $\mathcal{Q}^0$; i.e.,

$$\mathcal{Q}^0_\pm = \{i \in \mathcal{Q}^0 \mid \ i \text{ is a source or } i \text{ is a sink in } \mathcal{Q}\}.$$

For $t \in \{1, \ldots, \nu\}$ we say that $x \in \mathrm{Supp}(\alpha^t)$ is a source (respectively a sink) in $\mathrm{Supp}(\alpha^t)$ if there is no arrow $x \leftarrow y$ (respectively $x \to y$) with $y \in \mathrm{Supp}(\alpha^t)$. Put $\mathrm{Supp}_+(\alpha^t) = \{\text{sources in } \mathrm{Supp}(\alpha^t)\}$ and $\mathrm{Supp}_-(\alpha^t) = \{\text{sinks in } \mathrm{Supp}(\alpha^t)\}$.

Lemma 4.4. *Suppose $s, t \in \{1, \ldots, \nu\}$ and $x \in \mathcal{Q}^0$. If $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 1$, then*

$$\mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}_-(\alpha^t) \neq \emptyset \quad \text{and} \quad \mathrm{Supp}_+(\alpha^s) \cap \mathrm{Supp}(\alpha^t) \neq \emptyset.$$

In particular, if $[\mathbf{e}_{\alpha_x}, \mathbf{e}_{\alpha^t}] = 1$, then $x \in \mathrm{Supp}_-(\alpha^t)$ and if $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_x}] = 1$, then $x \in \mathrm{Supp}_+(\alpha^t)$.

Proof. Let us denote $\mathbf{e}_{\alpha^s}$ by (V_i, f_{ij}), $\mathbf{e}_{\alpha^t}$ by (V'_i, f'_{ij}) and let $g \in \mathrm{Hom}\,(\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}) \setminus \{0\}$. Let $x \in \mathcal{Q}^0$ be such that $g_x \neq 0$, hence $x \in \mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}(\alpha^t)$. Then if $x \notin \mathrm{Supp}_-(\alpha^t)$, there exists an arrow $x \to y \in \mathcal{Q}^1$ with $y \in \mathrm{Supp}(\alpha^t)$ and $g_y \circ f_{xy} = f'_{xy} \circ g_x \neq 0$. Consequently, $y \in \mathrm{Supp}(\alpha^s)$. Then if $y \notin \mathrm{Supp}_-(\alpha^t)$, we repeat the same argument until we find a $z \in \mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}_-(\alpha^t)$. The proof of $\mathrm{Supp}_+(\alpha^s) \cap \mathrm{Supp}(\alpha^t) \neq \emptyset$ is similar. □

Lemma 4.5. *Suppose $\Upsilon \ : \ 0 \to \mathbf{e}_{\alpha_j} \to \mathbf{e}_{\alpha_i + \alpha_j} \to \mathbf{e}_{\alpha_i} \to 0$ and $t \in \{1, \ldots, \nu\}$. Then*

$$\begin{aligned} [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^0)] &= \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x; \quad [\mathbf{e}(\mathbf{c}^0), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x, \\ [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij})] &= \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x - [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_i}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_j}]^1, \\ [\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] &= \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - [\mathbf{e}_{\alpha_j}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1. \end{aligned}$$

Proof. We have $c^0_s = 0$ if α^s is not a simple root and $c^0_s = d_x$ if $\alpha^s = \alpha_x$. Therefore,

$$[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^0)] = \sum_{x \in \mathcal{Q}^0} [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_x}]\, d_x = \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x$$

and

$$[\mathbf{e}(\mathbf{c}^0), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathcal{Q}^0} [\mathbf{e}_{\alpha_x}, \mathbf{e}_{\alpha^t}]\, d_x = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x$$

by the preceding lemma.

The equations for $\mathbf{e}(\mathbf{c}^{ij})$ are consequences of the equations for $\mathbf{e}(\mathbf{c}^0)$, the fact that $\mathbf{c}^{ij} = \mathbf{c}^0 + \mathbf{op}^{\Upsilon_{ij}}$ and lemma 3.9. $\square$

For $s, t \in \{1, \ldots, \nu\}$, we define $\chi_t^-(s) = |\mathrm{Supp}_-(\alpha^t) \cap \mathrm{Supp}(\alpha^s)|$, $\chi_t^+(s) = |\mathrm{Supp}_+(\alpha^t) \cap \mathrm{Supp}(\alpha^s)|$ and

$$\widetilde{\chi}_t^-(s) = \begin{cases} \chi_t^-(s) - 1 & \text{if } [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 1, \\ \chi_t^-(s) & \text{if } [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 0; \end{cases}$$

$$\widetilde{\chi}_t^+(s) = \begin{cases} \chi_t^+(s) - 1 & \text{if } [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^s}] = 1, \\ \chi_t^+(s) & \text{if } [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^s}] = 0. \end{cases}$$

Note that $\widetilde{\chi}_t^-(s) \geq 0$ and $\widetilde{\chi}_t^+(s) \geq 0$ by lemma 4.4.

Lemma 4.6. *Let $t \in \{1, \ldots, \nu\}$ and $\mathbf{c} \in \mathbf{N}^\nu$ be of $\mathbf{i}$-homogeneity $\mathbf{d}$. Then*

$$[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})] = \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^+(s) c_s$$

and

$$[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^-(s) c_s.$$

Proof. We prove only the second equation because the first is proved in a similar way. We want to calculate $[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] = \sum_{s=1}^{\nu} c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]$. The set of all $s \in \{1, \ldots, \nu\}$ such that $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \neq 0$ is a subset of the union $\bigcup_{x \in \mathrm{Supp}_-(\alpha^t)} \{s \mid x \in \mathrm{Supp}(\alpha^s)\}$ by lemma 4.4, and the function $\chi_t^-(s)$ counts the number of elements x in $\mathrm{Supp}_-(\alpha^t) \cap \mathrm{Supp}(\alpha^s)$. Hence

$$\begin{aligned} &\sum_{s=1}^{\nu} c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \\ (4.1) \quad = \quad &\left(\sum_{x \in \mathrm{Supp}_-(\alpha^t)} \sum_{s : x \in \mathrm{Supp}(\alpha^s)} c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \right) - \sum_{s=1}^{\nu} (\chi_t^-(s) - 1) c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]. \end{aligned}$$

Now $\sum_{s : x \in \mathrm{Supp}(\alpha^s)} c_s = d_x$, thus

$$\sum_{s : x \in \mathrm{Supp}(\alpha^s)} c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = d_x - \sum_{s : x \in \mathrm{Supp}(\alpha^s)} c_s (1 - [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]).$$

Moreover,

$$\sum_{x \in \mathrm{Supp}_-(\alpha^t)} \sum_{s : x \in \mathrm{Supp}(\alpha^s)} c_s (1 - [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]) = \sum_{s=1}^{\nu} \chi_t^-(s) c_s (1 - [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}])$$

and hence (4.1) becomes

$$\begin{aligned} &\left(\sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \chi_t^-(s) c_s (1 - [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]) \right) - \sum_{s=1}^{\nu} (\chi_t^-(s) - 1) c_s [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \\ &= \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^-(s) c_s. \end{aligned}$$

$\square$

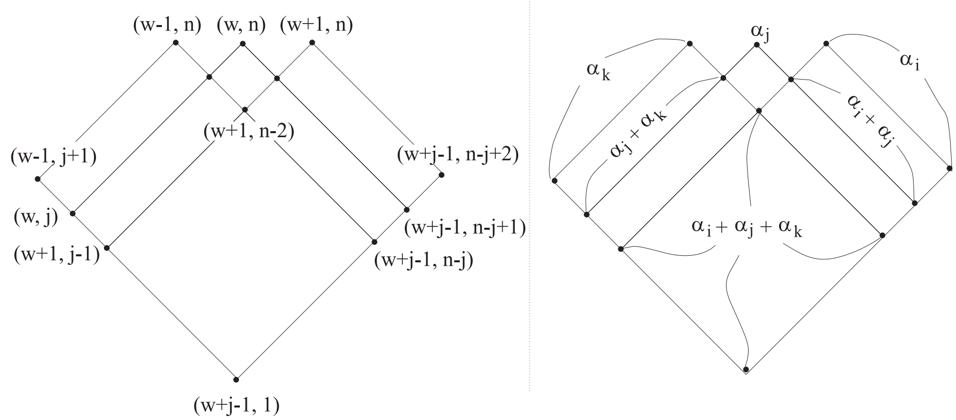

FIGURE 3. Two sketches of the Auslander-Reiten quiver in the case $i \to j \to k$.

Lemma 4.7. *Let $j \in \{2, \ldots, n-1\}$, $i = j-1$, $k = j+1$. Suppose that $\mathbf{c}^{ij} \preceq \mathbf{c}$ and $\mathbf{c}^{jk} \preceq \mathbf{c}$. Denote by Υ' the elementary operation*

$$\Upsilon' \ : \ 0 \to \mathbf{e}_{\alpha_k} \to \mathbf{e}_{\alpha_i+\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_i+\alpha_j} \to 0$$

(*respectively* $0 \to \mathbf{e}_{\alpha_i+\alpha_j} \to \mathbf{e}_{\alpha_i+\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_k} \to 0$ *depending on the orientation of the arrows* i — j — k). *Then:*

1. *If $j \notin \mathcal{Q}^0_{\pm}$, then $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c}$ and*
$$\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c} \ \Leftrightarrow \ c_{R(i,j,k)} \geq 1.$$
2. *If $j \in \mathcal{Q}^0_{\pm}$, then $\mathbf{c}^{ij} + \mathbf{o}p^{\Upsilon'} \preceq \mathbf{c}$ and*
$$\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c} \ \Leftrightarrow \ c_{\alpha_j} \leq d_j - 2.$$

Proof. Suppose, without loss of generality, that $i \to j$. We will show 1 first. Since $j \notin \mathcal{Q}^0_{\pm}$, we have $i \to j \to k$ in $\mathcal{Q}$. Denote by (w, j) the position $\Theta(\mathbf{P}(j))$ of the j-th indecomposable projectif module $\mathbf{P}(j)$ in the Auslander-Reiten quiver.

Then $\Theta(\alpha_j) = (w, n)$ and we can determine the positions of the roots $\alpha_i,\ \alpha_k,\ \alpha_i + \alpha_j,\ \alpha_j + \alpha_k$ and $\alpha_i + \alpha_j + \alpha_k$ as in the following table (there are four possible cases):

case	$\Theta(\alpha_i)$	$\Theta(\alpha_k)$	$\Theta(\alpha_i + \alpha_j)$	$\Theta(\alpha_j + \alpha_k)$	$\Theta(\alpha_i + \alpha_j + \alpha_k)$
I	(w+1, n)	(w-1, n)	(w+1, n-1)	(w, n-1)	(w+1, n-2)
II	(w+1, n)	(w-1, j+1)	(w+1, n-1)	(w, j)	(w+1, j-1)
III	(w+j-1, n-j+2)	(w-1, n)	(w+j-1, n-j+1)	(w, n-1)	(w+j-1, n-j)
IV	(w+j-1, n-j+2)	(w-1, j+1)	(w+j-1, n-j+1)	(w, j)	(w+j-1, 1)

The Auslander-Reiten quiver is illustrated in figure 3. Let us show first that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c}$. Note that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} = \mathbf{c}^{jk} + \mathbf{op}^{\Upsilon_{ij}}$. Recall that

$$\Upsilon_{ij} \ : \ 0 \to \mathbf{e}_{\alpha_j} \to \mathbf{e}_{\alpha_i+\alpha_j} \to \mathbf{e}_{\alpha_i} \to 0, \Upsilon_{jk} \ : \ 0 \to \mathbf{e}_{\alpha_k} \to \mathbf{e}_{\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_j} \to 0,$$

and that by theorem 2.5, we have

$$\mathbf{c}^{jk} + \mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c} \Leftrightarrow [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{jk} + \mathbf{op}^{\Upsilon_{ij}})] \geq [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})] \quad \forall t \in \{1, \ldots, \nu\}.$$

Moreover, $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{jk})] \geq [\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})]$ (because $\mathbf{c}^{jk} \preceq \mathbf{c}$) and

$$[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{jk}+\mathbf{op}^{\Upsilon_{ij}})] = [\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{jk})] - [\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha_i}][\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha_j}]^1$$

by lemma 3.9. Therefore, $\mathbf{c}^{jk}+\mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c}$ if and only if

$$[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{jk})] > [\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})],\ \forall t \text{ such that } [\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha_i}] = [\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha_j}]^1 = 1.$$

Fix such a t. Then on one hand, $i \in \mathrm{Supp}_+(\alpha^t)$ by lemma 4.4 and on the other hand, by proposition 2.6, we have

$$\left.\begin{array}{rl} \Theta(\alpha^t) = (w+1,\, n) & \text{in cases I and II,} \\ \Theta(\alpha^t) \in \{(w+a,\, n-a+1) \mid 1 \leq a \leq j-1\} & \text{in cases III and IV.} \end{array}\right\} \tag{4.2}$$

Therefore, $[\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha_j}] = 0$, by proposition 2.6 and thus lemma 4.5 implies

$$[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{jk})] = \sum_{x\in \mathrm{Supp}_+(\alpha^t)} d_x.$$

In order to prove that $\mathbf{c}^{jk}+\mathbf{op}^{\Upsilon_{ij}} \preceq \mathbf{c}$, it is thus enough to show that $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})] < \sum_{x\in \mathrm{Supp}_+(\alpha^t)} d_x$. Lemma 4.6 implies

$$[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})] = \sum_{x\in \mathrm{Supp}_+(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^+(s)\ c_s.$$

By (4.2), $\alpha^t = \alpha_i$ in cases I and II. Since i is a source of $\mathcal{Q}$ in cases III and IV, this implies that $i \in \mathrm{Supp}_+(\alpha^t)$ in all cases I–IV. By (4.2) and by proposition 2.6, we have $[\mathbf{e}_{\alpha^t},\mathbf{e}_{\alpha^s}] = 0$ for all $s \in R(i,j)$. But since $\mathbf{c}^{ij} \preceq \mathbf{c}$, there exists $s \in R(i,j)$ such that $c_s \neq 0$ and since $i \in \mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}_+(\alpha^t)$, we have $\widetilde{\chi}_t^+(s) \geq 1$. Therefore, $\sum_{s=1}^{\nu} \widetilde{\chi}_t^+(s)\ c_s \geq 1$ and $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})] < \sum_{x\in \mathrm{Supp}_+(\alpha^t)} d_x$, as required.

Now we prove the second part of 1, namely

$$\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c} \ \Leftrightarrow\ c_{R(i,j,k)} \geq 1.$$

Recall that $\Upsilon' : 0 \to \mathbf{e}_{\alpha_k} \to \mathbf{e}_{\alpha_i+\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_i+\alpha_j} \to 0$. Note that $t \in R(i,j,k)$ if and only if the position of α^t in the Auslander-Reiten quiver is in the rectangle having corners $(w+1,j-1),(w+1,n-2),(w+j-1,1),(w+j-1,n-j)$; more precisely,

$$t \in R(i,j,k) \Leftrightarrow \Theta(\alpha^t) \in \{(w+a,b-a) \mid 1 \leq a \leq j-1,\ j \leq b \leq n-1\}.$$

Suppose first that $\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$. Let t be such that $\Theta(\alpha^t) = (w+1,\, n-1)$. Then $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})] = d_i - c_{R(i,j,k)}$ and $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'})] = d_i - \sum_{s\in R(i,j,k)}(c_s^{ij} + op_s^{\Upsilon'})$ by proposition 2.6. But for $s \in R(i,j,k)$, we have $c_s^{ij} = 0$, $op_s^{\Upsilon'} = 1$ if $\alpha^s = \alpha_i+\alpha_j+\alpha_k$ and $op_s^{\Upsilon'} = 0$ otherwise. Thus $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'})] = d_i - 1$. But by theorem 2.5, $\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$ implies that $[\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c})] \leq [\mathbf{e}_{\alpha^t},\mathbf{e}(\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'})]$, whence $c_{R(i,j,k)} \geq 1$.

In order to prove the other implication, suppose now that $c_{R(i,j,k)} \geq 1$. By theorem 2.5, we have

$$\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c} \Leftrightarrow [\mathbf{e}(\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'}),\mathbf{e}_{\alpha^t}] \geq [\mathbf{e}(\mathbf{c}),\mathbf{e}_{\alpha^t}] \quad \forall t \in \{1,\ldots,\nu\}.$$

Moreover, $[\mathbf{e}(\mathbf{c}^{ij}),\mathbf{e}_{\alpha^t}] \geq [\mathbf{e}(\mathbf{c}),\mathbf{e}_{\alpha^t}]$ (because $\mathbf{c}^{ij} \preceq \mathbf{c}$) and

$$[\mathbf{e}(\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'}),\mathbf{e}_{\alpha^t}] = [\mathbf{e}(\mathbf{c}^{ij}),\mathbf{e}_{\alpha^t}] - [\mathbf{e}_{\alpha_k},\mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha_i+\alpha_j},\mathbf{e}_{\alpha^t}]^1$$

by lemma 3.9. Therefore, $\mathbf{c}^{ij}+\mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$ if and only if

$$[\mathbf{e}(\mathbf{c}^{ij}),\mathbf{e}_{\alpha^t}] > [\mathbf{e}(\mathbf{c}),\mathbf{e}_{\alpha^t}],\ \forall t \text{ such that } [\mathbf{e}_{\alpha_k},\mathbf{e}_{\alpha^t}] = [\mathbf{e}_{\alpha_i+\alpha_j},\mathbf{e}_{\alpha^t}]^1 = 1.$$

Fix such a t. Then, on one hand, $k \in \mathrm{Supp}_-(\alpha^t)$ by lemma 4.4 and on the other hand, by proposition 2.6, we have

(4.3)
$$\Theta(\alpha^t) \in \left\{ \begin{array}{ll} \{(w-1,\, n),\, (w,\, n-1)\} & \text{case I,} \\ \{(w-1,\, b),\, (w,\, b-1) \mid j+1 \le b \le n\} & \text{case II,} \\ \{(w-1+a,\, n-a) \mid 0 \le a \le j-1\} & \text{case III,} \\ \{(w-1+a,\, b-a) \mid 0 \le a \le j-1,\ j+1 \le b \le n\} & \text{case IV.} \end{array} \right\}$$

Therefore, $[\mathbf{e}_{\alpha_j}, \mathbf{e}_{\alpha^t}] = 0$, by proposition 2.6, and hence

$$[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x,$$

by lemma 4.5. Thus in order to prove that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$, it is enough to show $[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] < \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x$. Lemma 4.6 implies

$$[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^-(s)\ c_s. \tag{4.4}$$

By (4.3), $\alpha^t \in \{\alpha_k, \alpha_j + \alpha_k\}$ in case I. Moreover, k is a sink of $\mathcal{Q}$ in cases II and IV. In case III, the position of α^t in the Auslander-Reiten quiver is on the diagonal between α_k and $\alpha_i + \alpha_j + \alpha_k$, by (4.3). In this case, the roots on that diagonal are precisely those positive roots α such that $k \in \mathrm{Supp}(\alpha)$ and $(k+1) \notin \mathrm{Supp}(\alpha)$. (Indeed, this follows easily from proposition 2.6 and the fact that these roots are characterized by $[\mathbf{P}(k), \mathbf{e}_\alpha] = 1$ and $[\mathbf{P}(k+1), \mathbf{e}_\alpha] = 0$.) Therefore, in all cases I–IV, we have $k \in \mathrm{Supp}_-(\alpha^t)$. Let $s_0 \in R(i,j,k)$ be such that $c_{s_0} \ge 1$. Such an s_0 exists since $c_{R(i,j,k)} \ge 1$. In cases I and II we have $[\mathbf{e}_{\alpha^{s_0}}, \mathbf{e}_{\alpha^t}] = 0$, by (4.3) and proposition 2.6. But since $s_0 \in R(i,j,k)$ implies that $k \in \mathrm{Supp}(\alpha^{s_0})$, we have $\widetilde{\chi}_t^-(s_0) \ge 1$, whence

$$[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \overset{(4.4)}{\le} \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \widetilde{\chi}_t^-(s_0)\ c_{s_0} < \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x,$$

as required. Consider now cases III and IV. If $[\mathbf{e}_{\alpha^{s_0}}, \mathbf{e}_{\alpha^t}] = 0$, then the same argument as in cases I and II shows that $[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] < \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x$. Otherwise, $[\mathbf{e}_{\alpha^{s_0}}, \mathbf{e}_{\alpha^t}] = 1$. Since $s_0 \in R(i,j,k)$, proposition 2.6 combined with (4.3) implies that $t \in R(i,j,k)$; thus, in particular, $i \in \mathrm{Supp}(\alpha^t)$. Note that $i > 1$ (because if $i = 1$, then case III is case I and case IV is case II) and that i is a source in $\mathcal{Q}$. Therefore, $(i-1) \leftarrow i \in \mathcal{Q}^1$ and there exists $x_1 < i$ such that there is a path in $\mathcal{Q}$ from i to x_1 and $x_1 \in \mathrm{Supp}_-(\alpha^t)$. We will show that $x_1 \in \mathrm{Supp}(\alpha^{s_0})$ (i.e., $[\mathbf{P}(x_1), \mathbf{e}_{\alpha^{s_0}}] = 1$). Note that $\Theta(\alpha^{P[x_1]}) = (w+1, x_1)$, where $P[x_1] \in \{1, \ldots, \nu\}$ is such that $\mathbf{e}_{\alpha^{P[x_1]}} = \mathbf{P}(x_1)$. Say $\Theta(\alpha^{s_0}) = (w^0, a^0)$ and $\Theta(\alpha^t) = (w^1, a^1)$. Then since $s_0 \in R(i,j,k)$, we have

$$w+1 \le w^0 \le w+j-1 \quad \text{and} \quad w+j \le w^0 + a^0 \le w+n-1.$$

On the other hand, $[\mathbf{P}(x_1), \mathbf{e}_{\alpha^t}] = 1$ implies $w+1 \le w^1 \le w+x_1$, and $[\mathbf{e}_{\alpha^{s_0}}, \mathbf{e}_{\alpha^t}] = 1$ implies $w^0 \le w^1$. Therefore,

$$w+1 \le w^0 \le w^1 \le w+x_1 \quad \text{and} \quad w+x_1+1 \le w+j \le w^0+a^0 \le w+n-1 < w+n+1$$

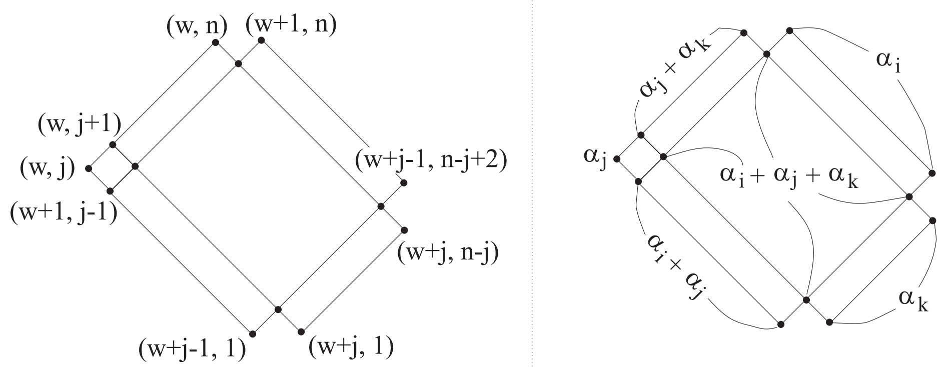

FIGURE 4. Two sketches of the Auslander-Reiten quiver in the case $i \to j \leftarrow k$.

and hence $[\mathbf{P}(x_1), \mathbf{e}_{\alpha^{s_0}}] = 1$ by proposition 2.6. Then k and x_1 are elements of $\mathrm{Supp}(\alpha^{s_0}) \cap \mathrm{Supp}_-(\alpha^t)$ and therefore $\widetilde{\chi}_t^-(s_0) \geq 1$. Consequently,

$$[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \overset{(4.4)}{\leq} \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \widetilde{\chi}_t^-(s_0)\, c_{s_0} < \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x.$$

This completes the proof of 1.

Now we show 2.

Since $j \in \mathcal{Q}^0_{\pm}$, we have $i \to j \leftarrow k$, thus $\mathbf{e}_{\alpha_j} = \mathbf{P}(j)$. Denote by (w, j) the position $\Theta(\alpha_j)$ of α_j in the Auslander-Reiten quiver. We can determine the positions of the roots $\alpha_i,\ \alpha_k,\ \alpha_i + \alpha_j,\ \alpha_j + \alpha_k$ and $\alpha_i + \alpha_j + \alpha_k$ as in the following table (there are four possible cases):

case	$\Theta(\alpha_i)$	$\Theta(\alpha_k)$	$\Theta(\alpha_i + \alpha_j)$	$\Theta(\alpha_j + \alpha_k)$	$\Theta(\alpha_i + \alpha_j + \alpha_k)$
I	(w+1, n)	(w+j, 1)	(w+1, j-1)	(w, j+1)	(w+1, j)
II	(w+1, n)	(w+j, n-j)	(w+1, j-1)	(w, n)	(w+1, n-1)
III	(w+j-1, n-j+2)	(w+j, 1)	(w+j-1, 1)	(w, j+1)	(w+j-1, 2)
IV	(w+j-1, n-j+2)	(w+j, n-j)	(w+j-1, 1)	(w, n)	(w+j-1, n-j+1)

The Auslander-Reiten quiver is illustrated in figure 4. Let us show first that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$. Recall that $\Upsilon' : 0 \to \mathbf{e}_{\alpha_i+\alpha_j} \to \mathbf{e}_{\alpha_i+\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_k} \to 0 \in Op$. By theorem 2.5, we have

$$\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c} \ \Leftrightarrow\ [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'})] \geq [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})],\ \forall t \in \{1, \ldots, \nu\}.$$

Moreover, $[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij})] \geq [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})]$ (because $\mathbf{c}^{ij} \preceq \mathbf{c}$) and

$$[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'})] = [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij})] - [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_k}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_i+\alpha_j}]^1$$

by lemma 3.9. Therefore, $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$ if and only if

$$[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij})] > [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})],\ \forall t \text{ such that } [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_k}] = [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_i+\alpha_j}]^1 = 1.$$

Fix such a t. Then, on one hand, $k \in \mathrm{Supp}_+(\alpha^t)$ by lemma 4.4 and on the other hand, by proposition 2.6,

$$\Theta(\alpha^t) \in \left\{ \begin{array}{ll} \{(w+a,\, j+1-a) \mid 2 \le a \le j\} & \text{in case I,} \\ \{(w+a,\, b-a) \mid 2 \le a \le j,\ j+1 \le b \le n\} & \text{in case II,} \\ \{(w+j,\, 1)\} & \text{in case III,} \\ \{(w+j,\, b) \mid 1 \le b \le n-j\} & \text{in case IV.} \end{array} \right\} \tag{4.5}$$

Now we apply proposition 2.6 and obtain in each case I–IV that $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_i}] = 0$ and hence

$$[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{ij})] = \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x$$

by lemma 4.5. In order to prove that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon'} \preceq \mathbf{c}$, it is thus enough to show that $[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})] < \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x$. But $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_k}][\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha_j}]^1 = 1$ (by (4.5)) and hence lemma 4.5 implies $[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{jk})] = \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x - 1$. Since $\mathbf{c}^{jk} \preceq \mathbf{c}$, we can apply theorem 2.5 and obtain $[\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c})] \le [\mathbf{e}_{\alpha^t}, \mathbf{e}(\mathbf{c}^{jk})] < \sum_{x \in \mathrm{Supp}_+(\alpha^t)} d_x$, as required.

We show now the second part of 2, namely

$$\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c} \ \Leftrightarrow\ c_{\alpha_j} \le d_j - 2.$$

Recall that $\Upsilon_{jk}\ :\ 0 \to \mathbf{e}_{\alpha_j} \to \mathbf{e}_{\alpha_j+\alpha_k} \to \mathbf{e}_{\alpha_k} \to 0$. Suppose first $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \not\preceq \mathbf{c}$, i.e., $\exists\, t$ such that

$$[\mathbf{e}_{\alpha_j}, \mathbf{e}_{\alpha^t}][\mathbf{e}_{\alpha_k}, \mathbf{e}_{\alpha^t}]^1 = 1 \text{ and } [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] = [\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] \tag{4.6}$$

(by theorem 2.5, lemma 3.9 and by the fact that $\mathbf{c}^{ij} \preceq \mathbf{c}$. Then by proposition 2.6, we have

$$\Theta(\alpha^t) \ \in\ \left\{ \begin{array}{ll} \{(w+a,\, j-a) \mid 0 \le a \le j-1\} & \text{in cases I and III,} \\ \{(w+a,\, b-a) \mid 0 \le a \le j-1,\ j \le b \le n-1\} & \text{in cases II and IV.} \end{array} \right.$$

We want to show that $c_{\alpha_j} \ge d_j - 1$, i.e., $c_{\alpha_j} = d_j - 1$ because $\mathbf{c}^{ij} \preceq \mathbf{c}$. We have to distinguish two cases each of which has several subcases.

1. $k \notin \mathrm{Supp}(\alpha^t)$, thus

$$\Theta(\alpha^t) \in \{(w+a,\, j-a) \mid 0 \le a \le j-1\}. \tag{4.7}$$

1.1 $\alpha^t = \alpha_j$. Then we have $[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] = c_{\alpha_j}$ because j is a sink in $\mathcal{Q}$ and $[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = d_j - 1$. By (4.6), this implies $c_{\alpha_j} = d_j - 1$.

1.2 $\alpha^t = \alpha_i + \alpha_j$. Then $[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 0$ and thus by lemma 4.5 $[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x$. But using (4.7) and proposition 2.6, $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 0$ and $j \in \mathrm{Supp}_-(\alpha^t) \cap \mathrm{Supp}(\alpha^s)$ for all $s \in R(j,k)$, hence $\widetilde{\chi}_t^-(s) \ge 1$ and, therefore, $[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \le \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - c_{R(j,k)}$ by lemma 4.6. But $[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}]$ by (4.6), thus $c_{R(j,k)} = 0$, which is in contradiction with the fact that $\mathbf{c}^{jk} \preceq \mathbf{c}$ by lemma 4.2.

1.3 $\mathrm{Supp}(\alpha^t) = \{a, a+1, \ldots, i, j\},\ a < i$. Then if i is not a source in $\mathcal{Q}$ or if $i = 1$ (cases I and II), we have $\Theta(\alpha_i) = (w+1, n)$ and hence $[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 0$ (by (4.7)) and, therefore, $[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x$ (by lemma 4.5); whence $c_{R(j,k)} = 0$ as in the case 1.2. Otherwise, i is a source in $\mathcal{Q}$ and there exists $x_1 < i$ such that there is a path in $\mathcal{Q}$ from i to x_1 and $x_1 \in \mathrm{Supp}_-(\alpha^t)$ (cases III and IV). Moreover, $\Theta(\alpha_i + \alpha_j) = (w+j-1, 1)$ and since $\alpha^t \ne \alpha_i + \alpha_j$, we have

$[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 1$ and $[\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] = d_j - 1 + \sum_{\substack{x \in \mathrm{Supp}_-(\alpha^t) \\ x \neq j}} d_x$ by lemma 4.5. On the other hand,

$$\begin{aligned}[\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] &\leq \sum_{s \in R(j) \cup R(x_1)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] + \sum_{\substack{x \in \mathrm{Supp}_-(\alpha^t) \\ x \neq x_1, j}} d_x \\ &= c_{\alpha_j} + \sum_{s \in R(x_1)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] + \sum_{\substack{x \in \mathrm{Supp}_-(\alpha^t) \\ x \neq x_1, j}} d_x\end{aligned}$$

because if $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 1$, $s \in R(j)$ and $s \notin R(x_1)$; then $\alpha^s = \alpha_j$ since $k \notin \mathrm{Supp}(\alpha^t)$. Therefore, (4.6) implies that $d_j + d_{x_1} - 1 \leq c_{\alpha_j} + \sum_{s \in R(x_1)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]$; whence $c_{\alpha_j} = d_j - 1$. This completes the proof in the first case.

2. $k \in \mathrm{Supp}(\alpha^t)$, thus $\Theta(\alpha^t) \in \{(w + a,\, b - a) \mid 0 \leq a \leq j - 1,\; j + 1 \leq b \leq n\}$ and hence we are in the case II or IV. Therefore, k is a source in $\mathcal{Q}$ and $k < n$. Thus there is $x_2 > k$ such that there is a path in $\mathcal{Q}$ from k to x_2 and such that x_2 is a sink in $\mathrm{Supp}(\alpha^t)$.

2.1 $i \notin \mathrm{Supp}(\alpha^t)$, thus $\Theta(\alpha^t) \in \{(w,\, b) \mid j + 1 \leq b \leq n\}$. In this case

$$\begin{aligned}\Big(\sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x\Big) - 1 &\overset{\text{lemma 4.5}}{=} [\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] \overset{(4.6)}{=} [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \\ &\overset{\text{lemma 4.6}}{=} \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^-(s) c_s \\ &\leq \sum_{s \in R(j)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] + \sum_{s \in R(x_2)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \\ &\quad - \sum_{s \in R(j, x_2)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] + \sum_{\substack{x \in \mathrm{Supp}_-(\alpha^t) \\ x \neq j, x_2}} d_x.\end{aligned}$$

Moreover, $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 0$ for all $s \in R(i)$ and hence $\sum_{s \in R(j)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] \leq d_j - c_{R(i,j)} \leq d_j - 1$. Therefore,

$$\sum_{s \in R(j)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = d_j - 1, \qquad \sum_{s \in R(x_2)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = d_{x_2}$$

and $\sum_{s \in R(j, x_2)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 0$. But

$$\sum_{s \in R(j)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] - \sum_{s \in R(j, x_2)} c_s[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = c_{\alpha_j}$$

because $i \notin \mathrm{Supp}(\alpha^t)$, whence $c_{\alpha_j} = d_j - 1$.

2.2 $i \in \mathrm{Supp}(\alpha^t)$ thus $\Theta(\alpha^t) \in \{(w + a,\, b - a) \mid 1 \leq a \leq j - 1,\, j + 1 \leq b \leq n\}$.

2.2.1 $i = 1$ or i is not a source in $\mathcal{Q}$. Then $\Theta(\alpha_i) = (w + 1,\, n)$, thus we have $[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 0$ and, therefore,

$$\begin{aligned}\sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x &\overset{\text{lemma 4.5}}{=} [\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] \overset{(4.6)}{=} [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \\ &\overset{\text{lemma 4.6}}{=} \sum_{x \in \mathrm{Supp}_-(\alpha^t)} d_x - \sum_{s=1}^{\nu} \widetilde{\chi}_t^-(s) c_s.\end{aligned}$$

Thus $\widetilde{\chi}_t^-(s)c_s = 0$ for all s. But for $s \in R(j,k)$, we have $j \in \mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}_-(\alpha^t)$ and if $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 1$ hence $j, x_2 \in \mathrm{Supp}(\alpha^s) \cap \mathrm{Supp}_-(\alpha^t)$; thus $\widetilde{\chi}_t^-(s) \geq 1$ and, therefore, $c_{R(j,k)} = 0$. This is a contradiction to the fact that $\mathbf{c}^{jk} \preceq \mathbf{c}$ by lemma 4.2.

2.2.2 i is a source in $\mathcal{Q}$ and $i \neq 1$. Thus $(i-1) \leftarrow i \rightarrow j \leftarrow k \rightarrow (k+1)$ is a subquiver of $\mathcal{Q}$ (case IV) and $\Theta(\alpha_i) = (w+j-1, n-j+2)$. If $[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 0$, then the same argument as in case 2.2.1 leads to a contradiction. Otherwise, $[\mathbf{e}_{\alpha_i}, \mathbf{e}_{\alpha^t}]^1 = 1$, $(i-1) \in \mathrm{Supp}(\alpha^t)$ and there exists $x_1 < i$ such that there is a path in $\mathcal{Q}$ from i to x_1 and such that $x_1 \in \mathrm{Supp}_-(\alpha^t)$. Then

$$\begin{aligned} & d_{x_1} + d_j - 1 + d_{x_2} + \sum_{x \in \mathrm{Supp}_-(\alpha^t) \setminus \{x_1, j, x_2\}} d_x \\ & \overset{\text{lemma 4.5}}{=} [\mathbf{e}(\mathbf{c}^{ij}), \mathbf{e}_{\alpha^t}] \overset{(4.6)}{=} [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}] \\ & = c_{\alpha_j} + \sum_{s \in R(x_1) \cup R(x_2)} c_s \, [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] + \sum_{x \in \mathrm{Supp}_-(\alpha^t) \setminus \{x_1, j, x_2\}} d_x \\ & \leq c_{\alpha_j} + d_{x_1} + d_{x_2} + \sum_{x \in \mathrm{Supp}_-(\alpha^t) \setminus \{x_1, j, x_2\}} d_x, \end{aligned}$$

whence $c_{\alpha_j} = d_j - 1$.

We have shown that $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \not\preceq \mathbf{c} \Rightarrow c_{\alpha_j} = d_j - 1$. Now, let $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c}$. Then for all $t \in \{1, \ldots, \nu\}$, $[\mathbf{e}(\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}}), \mathbf{e}_{\alpha^t}] \geq [\mathbf{e}(\mathbf{c}), \mathbf{e}_{\alpha^t}]$. For $\alpha^t = \alpha_j$, this implies $d_j - 2 \geq c_{\alpha_j}$ because j is a sink in $\mathcal{Q}$ and hence $[\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = 1$ if and only if $\alpha^s = \alpha_j$.

Thus $\mathbf{c}^{ij} + \mathbf{op}^{\Upsilon_{jk}} \preceq \mathbf{c} \Leftrightarrow c_{\alpha_j} \leq d_j - 2$ and this completes the proof of the second case and the lemma. □

Proposition 4.8. *Let $\mathbf{c} \in \mathbf{N}^\nu$ be such that $\mathbf{c}^{xy} \preceq \mathbf{c}$ for all $x \rightarrow y \in \mathcal{Q}^1$. Let $i \rightarrow j \in \mathcal{Q}^1$ and denote by h, $h \neq j$ the other vertex adjacent to i in $\mathcal{Q}$ if such a vertex exists (i.e. if $1 < i < n$) and denote by k, $k \neq i$ the other vertex adjacent to j in $\mathcal{Q}$ if such a vertex exists. Hence h—i→j—k is a subquiver of $\mathcal{Q}$. Then*

$$\sum_{\Upsilon \in Op(\mathbf{c}^{ij}) \setminus Op(\mathbf{c}^{ij}, \mathbf{c})} e(\Upsilon, \mathbf{c}^{ij}) = \begin{cases} (d_i - 1)(d_j - 1)\ \mathcal{V}(c_{R(i,j)} = 1) \\ + d_h\ \mathcal{V}(c_{R(h,i,j)} = 0 \text{ and } h \rightarrow i) \\ + d_h(d_i - 1)\ \mathcal{V}(c_{\alpha_i} = d_i - 1 \text{ and } h \leftarrow i) \\ + d_k\ \mathcal{V}(c_{R(i,j,k)} = 0 \text{ and } j \rightarrow k) \\ + d_k(d_j - 1)\ \mathcal{V}(c_{\alpha_j} = d_j - 1 \text{ and } j \leftarrow k) \end{cases}$$

where $\mathcal{V}(\mathrm{A}) = 1$ if A is true and $\mathcal{V}(\mathrm{A}) = 0$ if A is false.

Proof. Let $\Upsilon \in Op(\mathbf{c}^{ij})$, $\Upsilon : 0 \rightarrow \mathbf{e}_{\alpha^s} \rightarrow \mathbf{V} \rightarrow \mathbf{e}_{\alpha^t} \rightarrow 0$. Then $c_s^{ij} \geq 1$ and $c_t^{ij} \geq 1$ and the corresponding roots α^s, α^t are either simple or equal to $\alpha_i + \alpha_j$. Suppose $\mathbf{c}^{ij} + \mathbf{op}^\Upsilon \not\preceq \mathbf{c}$, then by lemma 4.3 we can exclude the case where $\{\alpha^s, \alpha^t\} = \{\alpha_x, \alpha_y\}$ with $\{x, y\} \cap \{i, j\} = \emptyset$. For the other cases we have

$\{\alpha^s, \alpha^t\}$	$e(\Upsilon, \mathbf{c}^{ij})$	condition equivalent to $\mathbf{c}^{ij} + \mathbf{op}^\Upsilon \not\preceq \mathbf{c}$
$\{\alpha_i, \alpha_j\}$	$(d_i - 1)(d_j - 1)$	$c_{R(i,j)} \leq 1$
$\{\alpha_h, \alpha_i\}$	$d_h(d_i - 1)$	$i \in \mathcal{Q}^0_\pm$ and $c_{\alpha_i} \geq d_i - 1$
$\{\alpha_j, \alpha_k\}$	$d_k(d_j - 1)$	$j \in \mathcal{Q}^0_\pm$ and $c_{\alpha_j} \geq d_j - 1$
$\{\alpha_i + \alpha_j, \alpha_h\}$	d_h	$i \notin \mathcal{Q}^0_\pm$ and $c_{R(h,i,j)} = 0$
$\{\alpha_i + \alpha_j, \alpha_k\}$	d_k	$j \notin \mathcal{Q}^0_\pm$ and $c_{R(i,j,k)} = 0$

where the last column is given by lemma 4.2 and lemma 4.7. Now since we have supposed that $\mathbf{c}^{ij} \preceq \mathbf{c}$, we must have that $c_{R(i,j)} \geq 1,\ c_{\alpha_i} \leq d_i - 1$ and $c_{\alpha_j} \leq d_j - 1$, and the proposition follows. □

Suppose now that $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth but not smooth, i.e., $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}'}$ for all $\mathbf{c}' \preceq \mathbf{c},\ \mathbf{c}' \neq \mathbf{c}^0$ and $\overline{\mathcal{O}_{\mathbf{c}}}$ is not rationally smooth at $\mathcal{O}_{\mathbf{c}^0}$. (Recall that orbit closures of type A_n are smooth iff they are rationally smooth [BS].) Suppose, moreover, that $\mathbf{c}^{ij} \preceq \mathbf{c}$ for all $i \to j \in \mathcal{Q}^1$.

By corollary 3.7, the codimension of $\mathcal{O}_{\mathbf{c}}$ is equal to the sum

$$\sum_{\Upsilon \in Op(\mathbf{c}^{ij}) \setminus Op(\mathbf{c}^{ij},\mathbf{c})} e(\Upsilon, \mathbf{c}^{ij})$$

and this holds for all arrows $i \to j$. We have given a formula for this sum in the preceding proposition. On the other hand,

$$\operatorname{codim}(\mathcal{O}_{\mathbf{c}}) = [\mathbf{e}(\mathbf{c}), \mathbf{e}(\mathbf{c})]^1 = \sum_{\Upsilon \in Op(\mathbf{c})} e(\Upsilon, \mathbf{c}). \tag{4.8}$$

We are going to analyse this sum now.

Lemma 4.9. *Let $\mathbf{c} \in \mathbf{N}^{\nu}$ be such that $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth but not smooth and such that $\mathbf{c}^{ij} \preceq \mathbf{c}$ for all $i \to j \in \mathcal{Q}^1$. Then $c_{R(i,j)} = 1$ for all $i \to j \in \mathcal{Q}^1$.*

Proof. Suppose the contrary (i.e. there exists $i \to j \in \mathcal{Q}^1$ such that $c_{R(i,j)} \geq 2$), then $c_{\alpha_i} < d_i - 1$ and $c_{\alpha_j} < d_j - 1$; hence, by proposition 4.8,

$$\operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \quad = \quad d_h\, \mathcal{V}(c_{R(h,i,j)} = 0 \text{ and } h \to i) + d_k\, \mathcal{V}(c_{R(i,j,k)} = 0 \text{ and } j \to k).$$

Moreover, $\operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \neq 0$ because otherwise $\overline{\mathcal{O}_{\mathbf{c}}} = E_{\mathbf{d}}$ is smooth. Suppose without loss of generality, that $i < j$. In the case $h \to i,\ j \to k$, the Auslander-Reiten quiver is illustrated in figure 5. In this situation, we have the following elementary operations:

$$0 \to R(k) \setminus R(i,j,k) \to \ldots \to R(i,j) \setminus R(i,j,k) \to 0,$$
$$0 \to R(i,j) \setminus R(h,i,j) \to \ldots \to R(h) \setminus R(h,i,j) \to 0$$

where the notation $0 \to A \to \ldots \to B \to 0$ means that for all $s \in A, t \in B$ there exists a non-split exact sequence $0 \to \mathbf{e}_s \to \mathbf{V} \to \mathbf{e}_t \to 0$. Then by (4.8),

$$\begin{aligned} \operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \quad &\geq \quad (d_k - c_{R(i,j,k)})(c_{R(i,j)} - c_{R(i,j,k)}) + (c_{R(i,j)} - c_{R(h,i,j)})(d_h - c_{R(h,i,j)}) \\ &\geq \quad 2\, d_k \mathcal{V}(c_{R(i,j,k)} = 0) + 2\, d_h \mathcal{V}(c_{R(h,i,j)} = 0) \\ &= \quad 2 \operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \end{aligned}$$

but this is impossible.

In the case $h \to i \to j \leftarrow k$ (respectively $h \leftarrow i \to j \to k$), the same argument gives $\operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \geq 2\, d_h \mathcal{V}(c_{R(h,i,j)} = 0) = 2 \operatorname{codim}(\mathcal{O}_{\mathbf{c}})$ (respectively $\operatorname{codim}(\mathcal{O}_{\mathbf{c}}) \geq 2\, d_k \mathcal{V}(c_{R(i,j,k)} = 0) = 2 \operatorname{codim}(\mathcal{O}_{\mathbf{c}})$). Thus in all cases we get a contradiction. □

We will now classify the orbit closures of type A_n that are projectively rationally smooth. The first step is the following lemma.

Lemma 4.10. *If $n \geq 4$, then there are no orbit closures $\overline{\mathcal{O}_{\mathbf{c}}}$ of type A_n that are projectively rationally smooth but not smooth and such that $\mathbf{c}^{ij} \preceq \mathbf{c}$ for all $i \to j \in \mathcal{Q}^1$.*

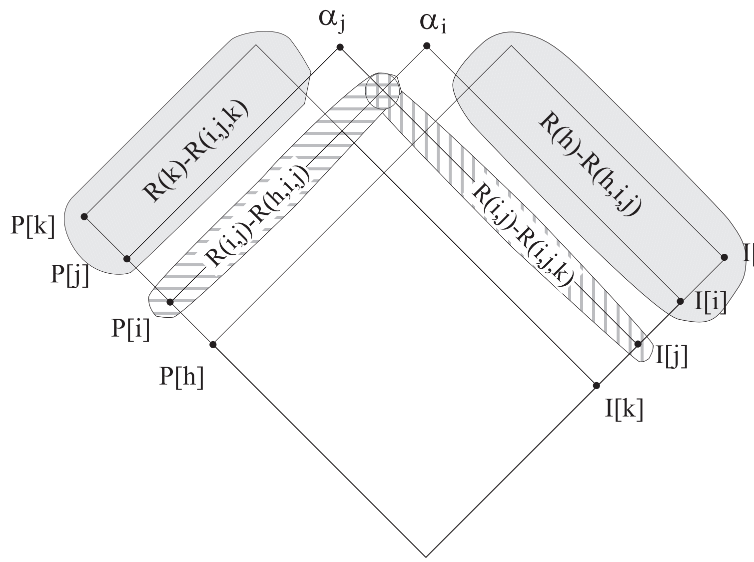

FIGURE 5. The sketch of the Auslander-Reiten quiver in the case $h \to i \to j \to k$

Proof. Suppose the contrary. By lemma 4.9, we have $c_{R(i,j)} = 1$ for all $i \to j$ and therefore there are 4 types of candidates $\mathbf{c}^1, \mathbf{c}^2, \mathbf{c}^3, \mathbf{c}^4$ classified with respect to their values on $R(1,2,3)$ and on $R(2,3,4)$. Namely, $(c^1_{R(1,2,3)}, c^1_{R(2,3,4)}) = (0,0), (c^2_{R(1,2,3)}, c^2_{R(2,3,4)}) = (1,0), (c^3_{R(1,2,3)}, c^3_{R(2,3,4)}) = (0,1), (c^4_{R(1,2,3)}, c^4_{R(2,3,4)}) = (1,1)$. We get the following table:

$\mathbf{c}$	$c_{R(1,2,3)}$	$c_{R(2,3,4)}$	c_{α_2}	c_{α_3}	$c_{\alpha_1+\alpha_2}$	$c_{\alpha_2+\alpha_3}$	$c_{\alpha_1+\alpha_2+\alpha_3}$
$\mathbf{c}^1$	0	0	d_2-2	d_3-2	1	1	0
$\mathbf{c}^2$	1	0	d_2-1	d_3-2	0	0	1
$\mathbf{c}^3$	0	1	d_2-2	d_3-1	1	0	0
$\mathbf{c}^4$	1	1	d_2-1	d_3-1	0	0	0

Note that $d_3 \geq 2$ in the cases $\mathbf{c}^1$ and $\mathbf{c}^2$ and that $d_2 \geq 2$ in the cases $\mathbf{c}^1$ and $\mathbf{c}^3$. By symmetry, we can suppose, without loss of generality, that $1 \leftarrow 2 \in \mathcal{Q}^1$. Then for the first 3 arrows there are 4 possible orientations: $\mathcal{Q}_1 = 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \ldots,\ \mathcal{Q}_2 = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \ldots,\ \mathcal{Q}_3 = 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \ldots,\ \mathcal{Q}_4 = 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \ldots$ The corresponding Auslander-Reiten quivers are illustrated in figures 6–9.

In each case, we have the elementary operations

$$0 \to R(j) \setminus R(i,j) \to \ldots \to R(i) \setminus R(i,j) \to 0$$

for all $i \to j$, hence the codimension of $\mathcal{O}_{\mathbf{c}}$ is at least $D \overset{def}{=} (d_1-1)(d_2-1) + (d_2-1)(d_3-1) + (d_3-1)(d_4-1)$. Moreover, for each case there are additional

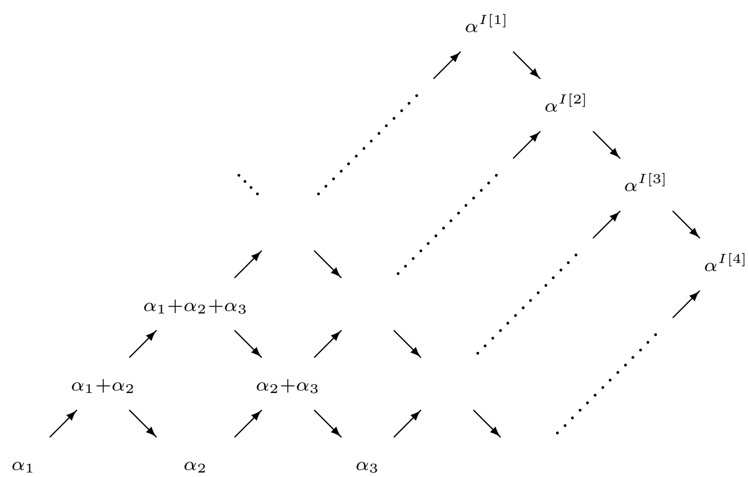

FIGURE 6. The Auslander-Reiten quiver $\Gamma_{\mathcal{Q}_1}$

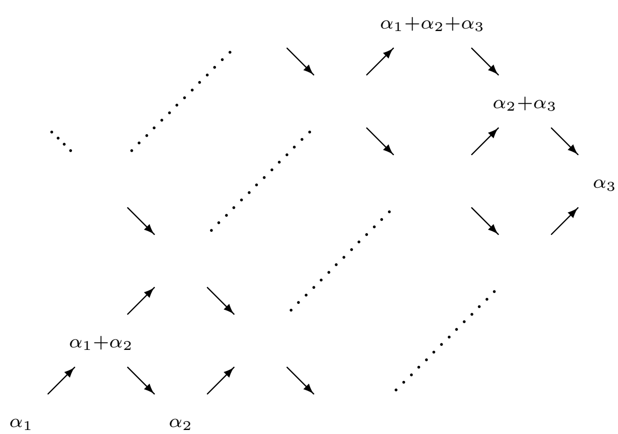

FIGURE 7. The Auslander-Reiten quiver $\Gamma_{\mathcal{Q}_2}$

elementary operations, as follows:

$$\begin{array}{cll}
0 \to \{\alpha_1+\alpha_2\} \to \ldots \to R(2,3) \setminus R(1,2,3) \to 0 & \text{if} & \mathcal{Q}_1, \mathcal{Q}_2, \\
0 \to \{\alpha_1+\alpha_2+\alpha_3\} \to \ldots \to R(3,4) \setminus R(1,2,3,4) \to 0 & \text{if} & \mathcal{Q}_1, \\
0 \to \{\alpha_2+\alpha_3\} \to \ldots \to R(3,4) \setminus R(2,3,4) \to 0 & \text{if} & \mathcal{Q}_1, \\
0 \to R(2,3,4) \to \ldots \to \{\alpha_3\} \to 0 & \text{if} & \mathcal{Q}_2, \\
0 \to R(1,2,3) \to \ldots \to \{\alpha_2\} \to 0 & \text{if} & \mathcal{Q}_3, \mathcal{Q}_4, \\
0 \to R(3,4) \setminus R(2,3,4) \to \ldots \to R(2,3) \setminus R(2,3,4) \to 0 & \text{if} & \mathcal{Q}_3, \\
0 \to \{\alpha_3\} \to \ldots \to R(2,3,4) \to 0 & \text{if} & \mathcal{Q}_4.
\end{array}$$

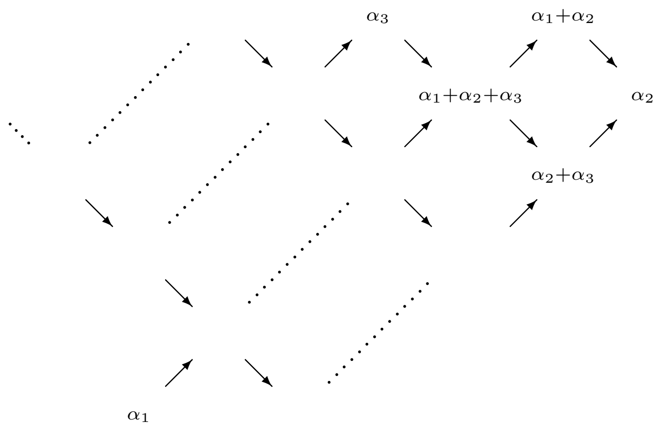

FIGURE 8. The Auslander-Reiten quiver $\Gamma_{\mathcal{Q}_3}$

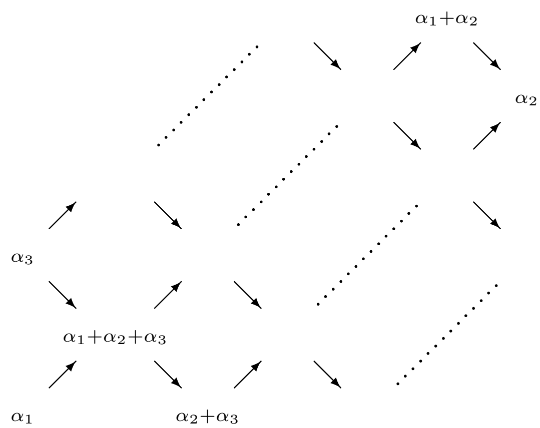

FIGURE 9. The Auslander-Reiten quiver $\Gamma_{\mathcal{Q}_4}$

Thus, the codimension of $\mathcal{O}_{\mathbf{c}}$ is at least

$$D(\mathbf{c},\mathcal{Q}) \stackrel{def}{=} \begin{cases} \left.\begin{array}{l} D + c_{\alpha_1+\alpha_2+\alpha_3}(1-c_{R(1,2,3,4)}) + c_{\alpha_1+\alpha_2}(1-c_{R(1,2,3)}) \\ \quad +c_{\alpha_2+\alpha_3}(1-c_{R(2,3,4)}) \end{array}\right\} & \text{if } \mathcal{Q}_1, \\ D + c_{\alpha_1+\alpha_2}(1-c_{R(1,2,3)}) + c_{\alpha_3}c_{R(2,3,4)} & \text{if } \mathcal{Q}_2, \\ D + c_{\alpha_2}c_{R(1,2,3)} + (1-c_{R(2,3,4)})(1-c_{R(2,3,4)}) & \text{if } \mathcal{Q}_3, \\ D + c_{\alpha_2}c_{R(1,2,3)} + c_{\alpha_3}c_{R(2,3,4)} & \text{if } \mathcal{Q}_4. \end{cases}$$

On the other hand, we can calculate the sums of proposition 4.8. Let us write $S(i,j)$ for the sum

$$\sum_{\Upsilon\in Op(\mathbf{c}^{ij})\setminus Op(\mathbf{c}^{ij},\mathbf{c})} e(\Upsilon,\mathbf{c}^{ij})$$

for all $i \to j$ and $\delta(\mathbf{c})$ for $\operatorname{codim}(\mathcal{O}_{\mathbf{c}})$. Then we have $\delta(\mathbf{c}) = S(1,2) = S(2,3) = S(3,4)$.

Suppose first that $n \geq 5$. If $4 \notin \mathcal{Q}^0_{\pm}$, then there are the following elementary operations:

$$0 \to R(3,4) \setminus R(3,4,5) \to \ldots \to R(4,5) \setminus R(3,4,5) \to 0$$

$$(\text{respectively } 0 \to R(4,5) \setminus R(3,4,5) \to \ldots \to R(3,4) \setminus R(3,4,5) \to 0),$$

and if $4 \in \mathcal{Q}^0_{\pm}$, then there are the following elementary operations:

$$0 \to R(3,4,5) \to \ldots \to \{\alpha_4\} \to 0$$

$$(\text{respectively } 0 \to \{\alpha_4\} \to \ldots \to R(3,4,5) \to 0).$$

Thus

$$\delta(\mathbf{c}) \geq \begin{cases} D(\mathbf{c},\mathcal{Q}) + (1 - c_{R(3,4,5)})^2 & \text{if } 4 \notin \mathcal{Q}^0_{\pm}, \\ D(\mathbf{c},\mathcal{Q}) + c_{\alpha_4} c_{R(3,4,5)} & \text{if } 4 \in \mathcal{Q}^0_{\pm}. \end{cases}$$

Moreover,

$$\begin{aligned} S(3,4) = A(3,4) &+ d_5 \mathcal{V}(c_{R(3,4,5)} = 0 \text{ and } 4 \notin \mathcal{Q}^0_{\pm}) \\ &+ d_5(d_4 - 1)\mathcal{V}(c_{R(3,4,5)} = 1 \text{ and } 4 \in \mathcal{Q}^0_{\pm}), \end{aligned}$$

with

$$\begin{aligned} A(3,4) = (d_3 - 1)(d_4 - 1) &+ d_2 \mathcal{V}(c_{R(2,3,4)} = 0 \text{ and } 3 \notin \mathcal{Q}^0_{\pm}) \\ &+ d_2(d_3 - 1)\mathcal{V}(c_{R(2,3,4)} = 1 \text{ and } 3 \in \mathcal{Q}^0_{\pm}). \end{aligned}$$

Thus, if $A(3,4) = 0$ and $4 \notin \mathcal{Q}^0_{\pm}$, then $c_{R(3,4,5)} = 0$ (because $S(3,4) \neq 0$), and if $A(3,4) = 0$ and $4 \in \mathcal{Q}^0_{\pm}$, then $c_{R(3,4,5)} = 1$ and $d_4 > 1$ (because $S(3,4) \neq 0$) and $c_{\alpha_4} = d_4 - 1$ (because $c_{R(3,4)} = c_{R(4,5)} = c_{R(3,4,5)} = 1$), whence $\delta(\mathbf{c}) > D(\mathbf{c},\mathcal{Q})$. Hence, if we can show that $\delta(\mathbf{c}) = D(\mathbf{c},\mathcal{Q})$, $n \geq 5$ and $A(3,4) = 0$, then we have a contradiction. We will use this result in tables 3–6 at the end of the paper.

Now let us return to the case $n \geq 4$. We can easily calculate the terms $\delta(\mathbf{c})$, $S(1,2)$, $S(2,3)$, $S(3,4)$ and we obtain a contradiction in each case. The results are given in the tables 3–6 at the end of the paper. This proves that for $n \geq 4$ there is no $\mathbf{c}$ such that $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth, non-smooth and $\mathbf{c}^{ij} \preceq \mathbf{c}$ for all $i \to j \in \mathcal{Q}^1$. □

Suppose now $n = 3$. Suppose again, without loss of generality, that $1 \leftarrow 2 \in \mathcal{Q}^1$. Then there are two possible quivers $\mathcal{Q}_1 : 1 \leftarrow 2 \leftarrow 3$ and $\mathcal{Q}_2 : 1 \leftarrow 2 \to 3$. Since $c_{R(i,j)} = 1$ for all $i \to j$ there are only two candidates $\mathbf{c}^1, \mathbf{c}^2$ classified with respect to their values on $R(1,2,3)$, namely $c^1_{R(1,2,3)} = 0$ and $c^2_{R(1,2,3)} = 1$. We can calculate easily the tables 1 and 2. Now $\delta(\mathbf{c}) = S(1,2) = S(2,3)$ and in each case we get some conditions:

For $\mathbf{c}^1$ in the case $\mathcal{Q}_1$, we must have $d_3 = (d_2 - 1)(d_3 - 1) + 1$ and $d_1 = (d_1 - 1)(d_2 - 1) + 1$ and then either $d_2 = 2$ or $d_1 = d_3 = 1$. Thus we have that $\mathbf{e}(\mathbf{c})$ is equal to

$$(d_1 - 1)\,\mathbf{e}_{\alpha_1} \oplus \mathbf{e}_{\alpha_1+\alpha_2} \oplus \mathbf{e}_{\alpha_2+\alpha_3} \oplus (d_3 - 1)\,\mathbf{e}_{\alpha_3}$$

or

$$\mathbf{e}_{\alpha_1+\alpha_2} \oplus (d_2 - 2)\,\mathbf{e}_{\alpha_2} \oplus \mathbf{e}_{\alpha_2+\alpha_3}.$$

For $\mathbf{c}^2$ in the case $\mathcal{Q}_1$ and also for $\mathbf{c}^1$ in the case $\mathcal{Q}_2$, we must have $(d_2-1)(d_3-1) = 0$ and $(d_1 - 1)(d_2 - 1) = 0$ hence $\delta(\mathbf{c}) = 0$ which is impossible because the corresponding orbit closure $\overline{\mathcal{O}_{\mathbf{c}}} = E_{\mathbf{d}}$ is smooth.

TABLE 1. Calculation results in the case Q_1, $n = 3$

$\mathbf{c}$	$\mathcal{Q}_1 : 1 \leftarrow 2 \leftarrow 3$ $\delta(\mathbf{c})$	$S(1,2)$	$S(2,3)$
$\mathbf{c}^1$	$(d_1-1)(d_2-1)$ $+(d_2-1)(d_3-1)+1$	$(d_1-1)(d_2-1)$ $+d_3$	$(d_2-1)(d_3-1)$ $+d_1$
$\mathbf{c}^2$	$(d_1-1)(d_2-1)$ $+(d_2-1)(d_3-1)$	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$

TABLE 2. Calculation results in the case Q_2, $n = 3$

$\mathbf{c}$	$\mathcal{Q}_2 : 1 \leftarrow 2 \rightarrow 3$ $\delta(\mathbf{c})$	$S(1,2)$	$S(2,3)$
$\mathbf{c}^1$	$(d_1-1)(d_2-1)$ $+(d_2-1)(d_3-1)$	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$
$\mathbf{c}^2$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$	$d_1(d_2-1)$ $+(d_2-1)(d_3-1)$

For $\mathbf{c}^2$ in the case $\mathcal{Q}_2$, we have $\delta(\mathbf{c}) = S(1,2) = S(2,3)$ hence in this case the only additional restriction is $d_2 > 1$ (since $\delta(\mathbf{c}) > 0$) and

$$\mathbf{e}(\mathbf{c}) = (d_1 - 1)\,\mathbf{e}_{\alpha_1} \oplus (d_3 - 1)\,\mathbf{e}_{\alpha_3} \oplus \mathbf{e}_{\alpha_1+\alpha_2+\alpha_3} \oplus (d_2 - 1)\,\mathbf{e}_{\alpha_2}.$$

Suppose now $n = 2$. Then $\delta(\mathbf{c}) = \mathbf{c}_{\alpha_1}\mathbf{c}_{\alpha_2}$ and $S(1,2) = (d_1 - 1)(d_2 - 1)\mathcal{V}(\mathbf{c}^{12} + \mathbf{op}^{\Upsilon_{12}} \npreceq \mathbf{c})$. Since $\delta(\mathbf{c}) = S(1,2) \neq 0$, we have $c_{\alpha_1} = d_1 - 1,\ c_{\alpha_2} = d_2 - 1$, whence $\mathbf{c} = \mathbf{c}^{12}$.

We have shown the following theorem:

Theorem 4.11. *Let $\overline{\mathcal{O}_\mathbf{c}}$ be an orbit closure of type A_n. Suppose that $\overline{\mathcal{O}_\mathbf{c}}$ is projectively rationally smooth but not smooth and that $f_{ij} \neq 0$ for each $f \in \mathcal{O}_\mathbf{c}$ and each $i \to j \in \mathcal{Q}^1$. Then $2 \leq n \leq 3$ and we have one of the following 4 cases:*

1. $\mathcal{Q} \in \{1 \to 2, 1 \leftarrow 2\}$, $d_1, d_2 > 1$ *and* $\mathbf{e}(\mathbf{c})$ *is equal to*
$$(d_1 - 1)\mathbf{e}_{\alpha_1} \oplus \mathbf{e}_{\alpha_1+\alpha_2} \oplus (d_2 - 1)\mathbf{e}_{\alpha_2}.$$
2. $\mathcal{Q} \in \{1 \to 2 \to 3, 1 \leftarrow 2 \leftarrow 3\}$ *and* $\mathbf{e}(\mathbf{c})$ *is equal to*
$$(d_1 - 1)\,\mathbf{e}_{\alpha_1} \oplus \mathbf{e}_{\alpha_1+\alpha_2} \oplus \mathbf{e}_{\alpha_2+\alpha_3} \oplus (d_3 - 1)\,\mathbf{e}_{\alpha_3}.$$
3. $\mathcal{Q} \in \{1 \to 2 \to 3, 1 \leftarrow 2 \leftarrow 3\}$ *and* $\mathbf{e}(\mathbf{c})$ *is equal to*
$$\mathbf{e}_{\alpha_1+\alpha_2} \oplus (d_2 - 2)\,\mathbf{e}_{\alpha_2} \oplus \mathbf{e}_{\alpha_2+\alpha_3}.$$
4. $\mathcal{Q} \in \{1 \to 2 \leftarrow 3, 1 \leftarrow 2 \to 3\}$, $d_2 > 1$ *and* $\mathbf{e}(\mathbf{c})$ *is equal to*
$$(d_1 - 1)\,\mathbf{e}_{\alpha_1} \oplus (d_3 - 1)\,\mathbf{e}_{\alpha_3} \oplus \mathbf{e}_{\alpha_1+\alpha_2+\alpha_3} \oplus (d_2 - 1)\,\mathbf{e}_{\alpha_2}.$$

Now we will show that these necessary conditions are also sufficient, i.e., the four types of orbit closures in the preceding theorem are projectively rationally smooth but not smooth. In case 1, $n = 2$ and by [BS, thm 6.1] $\overline{\mathcal{O}_\mathbf{c}}$ is not smooth because it is not $\{0\}$ or $E_\mathbf{d}$. Since there are no $\mathbf{c}' \in \mathbf{N}^\nu$ such that $\mathbf{c}^0 \prec \mathbf{c}' \prec \mathbf{c}$, $\overline{\mathcal{O}_\mathbf{c}}$ is projectively rationally smooth.

In all the other cases, $n = 3$ and the set $\{\mathbf{c}' \mid \mathbf{c}' \preceq \mathbf{c}\}$ is $\{\mathbf{c}^0, \mathbf{c}^{12}, \mathbf{c}^{23}, \mathbf{c}\}$ and $\mathbf{c}^0 \prec \mathbf{c}^{12} \prec \mathbf{c}$, $\mathbf{c}^0 \prec \mathbf{c}^{23} \prec \mathbf{c}$ and $\mathbf{c}^{12}, \mathbf{c}^{23}$ are not comparable. In each case theorem

6.1 in [BS] implies that $\overline{\mathcal{O}_{\mathbf{c}}}$ is not smooth and it only remains to prove that $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}^{12}}$ and at $\mathcal{O}_{\mathbf{c}^{23}}$. By symmetry it is enough to show rational smoothness at $\mathcal{O}_{\mathbf{c}^{12}}$. For this we will use theorem 3.6. In cases 2 and 3 we have

$$d(\mathbf{c}) - d(\mathbf{c}^{12}) = d_1 + 2d_2 + d_3 - 3 - (d_1 + d_2 - 1) = d_2 + d_3 - 2$$

and

$$\sum_{\Upsilon \in Op(\mathbf{c}^{12},\mathbf{c})} e(\Upsilon, \mathbf{c}^{12}) = c^{12}_{\alpha_2} c^{12}_{\alpha_3} = (d_2 - 1)d_3.$$

In both cases these two terms are equal since $d_2 = 2$ in case 2 and $d_3 = 1$ in case 3. Hence $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}^{12}}$. In case 4 we have

$$d(\mathbf{c}) - d(\mathbf{c}^{12}) = d_1 + d_2 + d_3 - 1 - (d_1 + d_2 - 1) = d_3$$

and

$$\sum_{\Upsilon \in Op(\mathbf{c}^{12},\mathbf{c})} e(\Upsilon, \mathbf{c}^{12}) = c^{12}_{\alpha_1+\alpha_2} c^{12}_{\alpha_3} = d_3.$$

Hence $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{c}^{12}}$.

Remark 4.12. Note that instead of using theorem 3.6 one can write down explicitly the equations defining $\overline{\mathcal{O}_{\mathbf{c}}}$ as an algebraic variety and then calculate the kernel of the Jacobian of $\overline{\mathcal{O}_{\mathbf{c}}}$ at a point f of $\mathcal{O}_{\mathbf{c}^{12}}$ which is equal to the tangent space $T_f\overline{\mathcal{O}_{\mathbf{c}}}$. Then one shows that the dimension of $T_f\overline{\mathcal{O}_{\mathbf{c}}}$ is equal to the dimension of $\mathcal{O}_{\mathbf{c}}$, hence $\overline{\mathcal{O}_{\mathbf{c}}}$ is smooth at $\mathcal{O}_{\mathbf{c}^{12}}$ and therefore rationally smooth at $\mathcal{O}_{\mathbf{c}^{12}}$.

Now let us consider the general case. Let $\overline{\mathcal{O}_{\mathbf{c}}}$ be an orbit closure of the quiver $\mathcal{Q}$ and $(f_{ij})_{i\to j\in\mathcal{Q}^1} \in \mathcal{O}_{\mathbf{c}}$. For any subquiver $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ we define the restriction of $\overline{\mathcal{O}_{\mathbf{c}}}$ to $\tilde{\mathcal{Q}}$ to be the orbit closure $\overline{\mathcal{O}_{\tilde{c}}}$ of the quiver $\tilde{\mathcal{Q}}$, where $\tilde{f} \in \mathcal{O}_{\tilde{c}}$ iff $\tilde{f}_{ij} = f_{ij}$ for all $i \to j \in \tilde{\mathcal{Q}}^1$.

Theorem 4.13. *Let $\overline{\mathcal{O}_{\mathbf{c}}}$ be a non-smooth orbit closure of type A_n. Then $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth if and only if there exists a subquiver $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ of type A_2 or A_3 such that*

1. *$f_{ij} = 0$ for all $f \in \overline{\mathcal{O}_{\mathbf{c}}}$ and all arrows $i \to j$ in $\mathcal{Q} \setminus \tilde{\mathcal{Q}}$, and*
2. *the restriction of $\overline{\mathcal{O}_{\mathbf{c}}}$ to $\tilde{\mathcal{Q}}$ is one of the orbit closures of theorem 4.11.*

Proof. Clearly, if 1 and 2 hold, then $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth; so we have to show the other implication.

Let $\mathbf{c}$ be of $\mathbf{i}$-homogeneity $\mathbf{d}$ and $f \in \mathcal{O}_{\mathbf{c}}$. Hence $\mathcal{O}_{\mathbf{c}} = G_{\mathbf{d}} \cdot f$. We proceed by induction on the number of arrows $i \to j \in \mathcal{Q}^1$ such that $f_{ij} = 0$. If $f_{ij} \neq 0$ for all $i \to j \in \mathcal{Q}^1$, then put $\tilde{\mathcal{Q}} = \mathcal{Q}$ and the theorem is true by theorem 4.11. Otherwise, let $i \to j \in \mathcal{Q}^1$ be such that $f_{ij} = 0$ and suppose, without loss of generality, that $i < j$. Let $\mathcal{Q}_1$ be the full subquiver of $\mathcal{Q}$ of type A_i having $\{1, \ldots, i\}$ as set of vertices and let $\mathcal{Q}_2$ be the full subquiver of $\mathcal{Q}$ of type A_{n-i} having $\{j, \ldots, n\}$ as set of vertices. Define $f^l \in E_{\mathbf{d}}$ by $f^l_{hk} = f_{hk}$ if $h \to k \in \mathcal{Q}^1_l$ and $f^l_{hk} = 0$, otherwise $(l = 1, 2)$. Let $\mathbf{c}^l \in \mathbf{N}^\nu$ be such that $c^l_t = c_t$ if $\mathrm{Supp}(\alpha^t) \subset \mathcal{Q}_l$ and $c^l_t = 0$, otherwise $(l = 1, 2)$. Note that $\mathbf{c} = \mathbf{c}^1 + \mathbf{c}^2$ and that $\mathcal{O}_{\mathbf{c}^l} \cong G_{\mathbf{d}} \cdot f^l$ $(l = 1, 2)$ ($\mathbf{c}^1$ is of $\mathbf{i}$-homogeneity $\mathbf{d}_1 = (d_1, d_2, \ldots, d_i, 0, \ldots, 0)$ and $\mathbf{c}^2$ is of $\mathbf{i}$-homogeneity $\mathbf{d}_2 = (0, \ldots, 0, d_j, d_{j+1}, \ldots, d_n)$). The orbit $\mathcal{O}_{\mathbf{c}}$ is the product $\mathcal{O}_{\mathbf{c}} = \mathcal{O}_{\mathbf{c}^1} \times \mathcal{O}_{\mathbf{c}^2}$ and the orbit closure $\overline{\mathcal{O}_{\mathbf{c}}}$ is the product of two orbit closures

$$\overline{\mathcal{O}_{\mathbf{c}}} = \overline{\mathcal{O}_{\mathbf{c}^1}} \times \overline{\mathcal{O}_{\mathbf{c}^2}}.$$

Note that $d(\mathbf{c}) \neq 0$ because $\overline{\mathcal{O}_{\mathbf{c}}}$ is not smooth and therefore at least one of $d(\mathbf{c}^1)$, $d(\mathbf{c}^2)$ is not zero. Say $d(\mathbf{c}^1) \neq 0$. We need the following lemma:

Lemma 4.14. *With the above notation we have*

1. $E_{\mathbf{i}}^{\mathbf{c}} = E_{\mathbf{i}}^{\mathbf{c}^2} E_{\mathbf{i}}^{\mathbf{c}^1}$.
2. *Each* $\mathbf{a} \preceq \mathbf{c}$ *can be written uniquely as* $\mathbf{a} = \mathbf{a}^1 + \mathbf{a}^2$ *with* $\mathbf{a}^1 \preceq \mathbf{c}^1, \mathbf{a}^2 \preceq \mathbf{c}^2$.
3. $\omega_{\mathbf{a}}^{\mathbf{c}} = \omega_{\mathbf{a}^1}^{\mathbf{c}^1}\ \omega_{\mathbf{a}^2}^{\mathbf{c}^2}$ *for each* $\mathbf{a} \preceq \mathbf{c}$.
4. $\zeta_{\mathbf{a}}^{\mathbf{c}} = \zeta_{\mathbf{a}^1}^{\mathbf{c}^1}\ \zeta_{\mathbf{a}^2}^{\mathbf{c}^2}$ *for each* $\mathbf{a} \preceq \mathbf{c}$.

Proof of the lemma. 1. For $r \in \{1, \dots, \nu\}$ let $\mathbf{b}(r) = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbf{N}^\nu$ be the vector whose only non-zero component is in the r-th position and is 1. If $c_t^1 \neq 0$ and $c_s^2 \neq 0$, then either $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^s}] = [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}] = [\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^s}]^1 = [\mathbf{e}_{\alpha^s}, \mathbf{e}_{\alpha^t}]^1 = 0$ and $E_{\mathbf{i}}^{\mathbf{b}(s)} E_{\mathbf{i}}^{\mathbf{b}(t)} = E_{\mathbf{i}}^{\mathbf{b}(t)} E_{\mathbf{i}}^{\mathbf{b}(s)}$ by [BS, prop. 3.14], or $i \in \mathrm{Supp}(\alpha^t)$, $j \in \mathrm{Supp}(\alpha^s)$, $[\mathbf{e}_{\alpha^t}, \mathbf{e}_{\alpha^s}]^1 = 1$ (because $i \to j$) and then $s < t$. Thus

$$\begin{aligned} E_{\mathbf{i}}^{\mathbf{c}^2} E_{\mathbf{i}}^{\mathbf{c}^1} &= E_{\mathbf{i}}^{c_1^2\,\mathbf{b}(1)} E_{\mathbf{i}}^{c_2^2\,\mathbf{b}(2)} \dots E_{\mathbf{i}}^{c_\nu^2\,\mathbf{b}(\nu)} E_{\mathbf{i}}^{c_1^1\,\mathbf{b}(1)} E_{\mathbf{i}}^{c_2^1\,\mathbf{b}(2)} \dots E_{\mathbf{i}}^{c_\nu^1\,\mathbf{b}(\nu)} \\ &= E_{\mathbf{i}}^{c_1\,\mathbf{b}(1)} E_{\mathbf{i}}^{c_2\,\mathbf{b}(2)} \dots E_{\mathbf{i}}^{c_\nu\,\mathbf{b}(\nu)} \\ &= E_{\mathbf{i}}^{\mathbf{c}}, \end{aligned}$$

this proves 1.

2. Let $\mathbf{a} = (a_1, a_2, \dots, a_\nu) \preceq \mathbf{c}$. For each $f \in \mathcal{O}_{\mathbf{a}}$, we have $f \in \overline{\mathcal{O}_{\mathbf{c}}}$, hence $f_{ij} = 0$. Then $a_{R(i,j)} = 0$ and $\mathbf{a} = \mathbf{a}^1 + \mathbf{a}^2$ with $\mathbf{a}^l \preceq \mathbf{c}^l$, $a_t^l = a_t$ if $\mathrm{Supp}(\alpha^t) \subset \mathcal{Q}_l$ and $a_t^l = 0$, otherwise $(l = 1, 2)$; this shows 2.

3. Using proposition 2.7 and part 1, we have

$$\sum_{\mathbf{a} \preceq \mathbf{c}} \omega_{\mathbf{a}}^{\mathbf{c}}\ E_{\mathbf{i}}^{\mathbf{a}} = \overline{E_{\mathbf{i}}^{\mathbf{c}}} = \overline{E_{\mathbf{i}}^{\mathbf{c}^2}}\ \overline{E_{\mathbf{i}}^{\mathbf{c}^1}} = \sum_{\substack{\mathbf{a}^2 \preceq \mathbf{c}^2 \\ \mathbf{a}^1 \preceq \mathbf{c}^1}} \omega_{\mathbf{a}^2}^{\mathbf{c}^2}\ \omega_{\mathbf{a}^1}^{\mathbf{c}^1}\ E_{\mathbf{i}}^{\mathbf{a}^2}\ E_{\mathbf{i}}^{\mathbf{a}^1}$$

and $E_{\mathbf{i}}^{\mathbf{a}^2}\ E_{\mathbf{i}}^{\mathbf{a}^1} = E_{\mathbf{i}}^{\mathbf{a}}$ by 1. Hence

$$\sum_{\mathbf{a} \preceq \mathbf{c}} \omega_{\mathbf{a}}^{\mathbf{c}}\ E_{\mathbf{i}}^{\mathbf{a}} = \sum_{\substack{\mathbf{a}^2 \preceq \mathbf{c}^2 \\ \mathbf{a}^1 \preceq \mathbf{c}^1}} \omega_{\mathbf{a}^2}^{\mathbf{c}^2}\ \omega_{\mathbf{a}^1}^{\mathbf{c}^1}\ E_{\mathbf{i}}^{\mathbf{a}}$$

and so $\omega_{\mathbf{a}}^{\mathbf{c}} = \omega_{\mathbf{a}^2}^{\mathbf{c}^2}\ \omega_{\mathbf{a}^1}^{\mathbf{c}^1}$, this shows 3.

4. Let $\mathbf{a} \preceq \mathbf{c}$ and $\mathbf{a} = \mathbf{a}^1 + \mathbf{a}^2$ with $\mathbf{a}^1 \preceq \mathbf{c}^1$, $\mathbf{a}^2 \preceq \mathbf{c}^2$. We proceed by induction on $|\{\mathbf{b} \in \mathbf{N}^\nu \mid \mathbf{a} \preceq \mathbf{b} \preceq \mathbf{c}\}|$. If this number is 1, then $\mathbf{a} = \mathbf{c}$ and the statement to prove is $1 = 1$. Suppose that $|\{\mathbf{b} \in \mathbf{N}^\nu \mid \mathbf{a} \preceq \mathbf{b} \preceq \mathbf{c}\}| > 1$. By theorem 2.8

$$\zeta_{\mathbf{a}^l}^{\mathbf{c}^l} = \sum_{\mathbf{b}^l : \mathbf{a}^l \preceq \mathbf{b}^l \preceq \mathbf{c}^l} \omega_{\mathbf{a}^l}^{\mathbf{b}^l}\ \overline{\zeta_{\mathbf{b}^l}^{\mathbf{c}^l}} \qquad (l = 1, 2),$$

with $\zeta_{\mathbf{b}^l}^{\mathbf{c}^l} \in v^{-1}\mathbf{Z}[v^{-1}]$. Note that for $\mathbf{b} = \mathbf{b}^1 + \mathbf{b}^2$, we have $\mathbf{a} \preceq \mathbf{b} \preceq \mathbf{c}$ and $\omega_{\mathbf{a}^1}^{\mathbf{b}^1} \omega_{\mathbf{a}^2}^{\mathbf{b}^2} = \omega_{\mathbf{a}}^{\mathbf{b}}$ by 3. Then

$$\zeta_{\mathbf{a}^1}^{\mathbf{c}^1}\ \zeta_{\mathbf{a}^2}^{\mathbf{c}^2} \quad = \sum_{\substack{\mathbf{b}^1 : \mathbf{a}^1 \preceq \mathbf{b}^1 \preceq \mathbf{c}^1 \\ \mathbf{b}^2 : \mathbf{a}^2 \preceq \mathbf{b}^2 \preceq \mathbf{c}^2}} \omega_{\mathbf{a}}^{\mathbf{b}}\ \overline{\zeta_{\mathbf{b}^1}^{\mathbf{c}^1} \zeta_{\mathbf{b}^2}^{\mathbf{c}^2}}$$

and by induction

$$\zeta_{\mathbf{a}^1}^{\mathbf{c}^1}\ \zeta_{\mathbf{a}^2}^{\mathbf{c}^2} = \overline{\zeta_{\mathbf{a}^1}^{\mathbf{c}^1} \zeta_{\mathbf{a}^2}^{\mathbf{c}^2}} + \sum_{\mathbf{b} : \mathbf{a} \prec \mathbf{b} \preceq \mathbf{c}} \omega_{\mathbf{a}}^{\mathbf{b}}\ \overline{\zeta_{\mathbf{b}}^{\mathbf{c}}}.$$

Thus $\zeta_{\mathbf{a}}^{\mathbf{c}} - \overline{\zeta_{\mathbf{a}}^{\mathbf{c}}} = \zeta_{\mathbf{a}^1}^{\mathbf{c}^1} \zeta_{\mathbf{a}^2}^{\mathbf{c}^2} - \overline{\zeta_{\mathbf{a}^1}^{\mathbf{c}^1} \zeta_{\mathbf{a}^2}^{\mathbf{c}^2}}$, whence $\zeta_{\mathbf{a}}^{\mathbf{c}} = \zeta_{\mathbf{a}^1}^{\mathbf{c}^1} \zeta_{\mathbf{a}^2}^{\mathbf{c}^2}$ and the lemma is proved. $\square$

Now if $\overline{\mathcal{O}_{\mathbf{c}}}$ is projectively rationally smooth, then $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at all $\mathcal{O}_{\mathbf{a}}$ for $\mathbf{c}^0 \neq \mathbf{a} \preceq \mathbf{c}$. Let $\mathbf{a}^1 \preceq \mathbf{c}^1$ and $\mathbf{a}^2 \preceq \mathbf{c}^2$ be such that $d(\mathbf{a}^1) \neq 0$ (such an $\mathbf{a}^1$ exists since $d(\mathbf{c}^1) \neq 0$). Then $\mathbf{a} = \mathbf{a}^1 + \mathbf{a}^2 \preceq \mathbf{c}$ and $\mathbf{a} \neq \mathbf{c}^0$. We have $\zeta_{\mathbf{a}^1}^{\mathbf{c}^1}\ \zeta_{\mathbf{a}^2}^{\mathbf{c}^2} = \zeta_{\mathbf{a}}^{\mathbf{c}}$ by the lemma, and $\zeta_{\mathbf{a}}^{\mathbf{c}} = v^{d(\mathbf{a})-d(\mathbf{c})}$ because $\overline{\mathcal{O}_{\mathbf{c}}}$ is rationally smooth at $\mathcal{O}_{\mathbf{a}}$. Moreover, by theorem 2.8.4,

$$\zeta_{\mathbf{a}^l}^{\mathbf{c}^l} = \sum_k \dim(\mathcal{H}_f^{2k})\ v^{2k}\ v^{d(\mathbf{a}^l)-d(\mathbf{c}^l)}\ (l = 1, 2).$$

Hence $\zeta_{\mathbf{a}^l}^{\mathbf{c}^l} = v^{x(l)}$ with $x(l) \geq d(\mathbf{a}^l) - d(\mathbf{c}^l)$ $(l = 1, 2)$ and $x(1) + x(2) = d(\mathbf{a}) - d(\mathbf{c})$. But

$$d(\mathbf{a}) - d(\mathbf{c}) = d(\mathbf{a}^1) + d(\mathbf{a}^2) - d(\mathbf{c}^1) - d(\mathbf{c}^2),$$

because the orbit $\mathcal{O}_{\mathbf{c}}$ is the product of the orbits $\mathcal{O}_{\mathbf{c}^1}$ and $\mathcal{O}_{\mathbf{c}^2}$ and the orbit $\mathcal{O}_{\mathbf{a}}$ is the product of the orbits $\mathcal{O}_{\mathbf{a}^1}$ and $\mathcal{O}_{\mathbf{a}^2}$. Hence $\zeta_{\mathbf{a}^l}^{\mathbf{c}^l} = v^{d(\mathbf{a}^l)-d(\mathbf{c}^l)}$ and $\overline{\mathcal{O}_{\mathbf{c}^l}}$ is rationally smooth at $\mathcal{O}_{\mathbf{a}^l}$ $(l = 1, 2)$. Since $\mathbf{a}^2 \preceq \mathbf{c}^2$ was arbitrary and $\mathbf{a}^1 \preceq \mathbf{c}^1$ was such that $d(\mathbf{a}^1) \neq 0$, we have shown that $\overline{\mathcal{O}_{\mathbf{c}^1}}$ is projectively rationally smooth and $\overline{\mathcal{O}_{\mathbf{c}^2}}$ is rationally smooth, hence smooth. Now $\overline{\mathcal{O}_{\mathbf{c}^1}}$ cannot be smooth because $\overline{\mathcal{O}_{\mathbf{c}}} = \overline{\mathcal{O}_{\mathbf{c}^1}} \times \overline{\mathcal{O}_{\mathbf{c}^2}}$ is not smooth, so by induction, we may apply the theorem to $\overline{\mathcal{O}_{\mathbf{c}^1}}$. Therefore, we only have to prove that $f_{ij} = 0$ for all $f \in \overline{\mathcal{O}_{\mathbf{c}^2}}$ and all $i \to j \in \mathcal{Q}_2^1$. Let $\mathbf{a}^1 \preceq \mathbf{c}^1$ be such that $d(\mathbf{a}^1) = 0$, then $\zeta_{\mathbf{a}^1}^{\mathbf{c}^1} \neq v^{d(\mathbf{a}^1)-d(\mathbf{c}^1)}$. Moreover, $\zeta_{\mathbf{a}^1+\mathbf{a}^2}^{\mathbf{c}} = \zeta_{\mathbf{a}^1}^{\mathbf{c}^1}\zeta_{\mathbf{a}^2}^{\mathbf{c}^2}$ implies that $\overline{\mathcal{O}_{\mathbf{c}}}$ is not rationally smooth at $\mathcal{O}_{\mathbf{a}^1+\mathbf{a}^2}$ for any $\mathbf{a}^2 \preceq \mathbf{c}^2$. Thus $\mathbf{a}^1 + \mathbf{a}^2 = \mathbf{c}^0$, hence $d(\mathbf{c}^2) = 0$, i.e. $f_{ij} = 0$ for all $f \in \overline{\mathcal{O}_{\mathbf{c}^2}}$ and all $i \to j \in \mathcal{Q}_2^1$, and this completes the proof of the theorem. □

Table 3. Calculation results in the case $\mathcal{Q}_1$

	$\mathcal{Q}_1 \; : \; 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \cdots$				
$\mathbf{c}$	$\delta(\mathbf{c}) \geq$	$S(1,2)$	$S(2,3)$	$S(3,4)$	contradiction
$\mathbf{c}^1$	$D{+}2$	$(d_1-1)(d_2-1)+d_3$			$d_2 \geq 2 \Rightarrow \delta(\mathbf{c}) > S(1,2)$
$\mathbf{c}^2$	$D{+}1$	$(d_1-1)(d_2-1)$			$\delta(\mathbf{c}) > S(1,2)$
$\mathbf{c}^3$	$D{+}1$	$(d_1-1)(d_2-1)+d_3$	$(d_2-1)(d_3-1)+d_1$	$(d_3-1)(d_4-1) + d_5 \mathcal{V}(c_{R(3,4,5)}{=}0 \text{ and } 4{\leftarrow}5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}{=}1 \text{ and } 4{\rightarrow}5)$	$d_2 \geq 2 \Rightarrow \delta(\mathbf{c}) = D{+}1 = S(1,2)$ and $(d_3-1)(d_4-1)=0$ $\Rightarrow A(3,4)=0$ and $n \geq 5$
$\mathbf{c}^4$	D	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$		$\delta(\mathbf{c}) \geq S(1,2)+S(2,3) = 2\,\delta(\mathbf{c})$

The data in the first column are the lower bounds $D(\mathbf{c}, \mathcal{Q})$ for $\delta(\mathbf{c})$.
The last column gives the contradiction to the fact that $\delta(\mathbf{c}) = S(1,2) = S(2,3) = S(3,4)$.

Table 4. Calculation results in the case $\mathcal{Q}_2$

	$\mathcal{Q}_2\ :\ 1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \cdots$				
$\mathbf{c}$	$\delta(\mathbf{c})\geq$	$S(1,2)$	$S(2,3)$	$S(3,4)$	contradiction
$\mathbf{c}^1$	$D+1$	$(d_1-1)(d_2-1)+d_3$	$(d_2-1)(d_3-1)+d_1$	$(d_3-1)(d_4-1)+d_5\mathcal{V}(c_{R(3,4,5)}=0$ and $4\rightarrow 5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}=1$ and $4\leftarrow 5)$	$d_2\geq 2 \Rightarrow \delta(\mathbf{c})=D+1=S(1,2)$ and $(d_3-1)(d_4-1)=0$ $\Rightarrow A(3,4)=0$ and $n\geq 5$
$\mathbf{c}^2$	D	$(d_1-1)(d_2-1)$			$\delta(\mathbf{c})>S(1,2)$
$\mathbf{c}^3$	$D+d_3$	$(d_1-1)(d_2-1)+d_3$	$(d_2-1)(d_3-1)$ $+d_1+d_4(d_3-1)$	$(d_3-1)(d_4-1)+d_2(d_3-1)$ $+d_5\mathcal{V}(c_{R(3,4,5)}=0$ and $4\rightarrow 5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}=1$ and $4\leftarrow 5)$	$d_2\geq 2 \Rightarrow \delta(\mathbf{c})=D+d_3=S(1,2)$ and $d_3=1\Rightarrow A(3,4)=0$ and $n\geq 5$
$\mathbf{c}^4$	$D+d_3-1$	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$ $(d_3-1)d_4$		$\delta(\mathbf{c})=S(2,3)\Rightarrow S(1,2)=0$

The data in the first column are the lower bounds $D(\mathbf{c},\mathcal{Q})$ for $\delta(\mathbf{c})$.
The last column gives the contradiction to the fact that $\delta(\mathbf{c}) = S(1,2) = S(2,3) = S(3,4)$.

TABLE 5. Calculation results in the case $\mathcal{Q}_3$

		$\mathcal{Q}_3 \ : \ 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \cdots$			
$\mathbf{c}$	$\delta(\mathbf{c})\geq$	$S(1,2)$	$S(2,3)$	$S(3,4)$	contradiction
$\mathbf{c}^1$	$D+1$	$(d_1-1)(d_2-1)$			$\delta(\mathbf{c})>S(1,2)$
$\mathbf{c}^2$	$D+d_2$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$			$\delta(\mathbf{c})>S(1,2)$
$\mathbf{c}^3$	D	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$		$\delta(\mathbf{c})\geq S(1,2)+S(2,3)=2\,\delta(\mathbf{c})$
$\mathbf{c}^4$	$D+d_2-1$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$	$(d_2-1)(d_3-1)$ $(d_2-1)d_1$	$(d_3-1)(d_4-1)+d_5\mathcal{V}(c_{R(3,4,5)}=0 \text{ and } 4\rightarrow 5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}=1 \text{ and } 4\leftarrow 5)$	$\delta(\mathbf{c})=S(1,2)\Rightarrow\delta(\mathbf{c})=D+d_2-1$ and $(d_3-1)(d_4-1)=0$ $\Rightarrow A(3,4)=0$ and $n\geq 5$

The data in the first column are the lower bounds $D(\mathbf{c},\mathcal{Q})$ for $\delta(\mathbf{c})$.
The last column gives the contradiction to the fact that $\delta(\mathbf{c}) = S(1,2) = S(2,3) = S(3,4)$.

Table 6. Calculation results in the case $\mathcal{Q}_4$

	$\mathcal{Q}_4 \ : \ 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \cdots$				
$\mathbf{c}$	$\delta(\mathbf{c})\geq$	$S(1,2)$	$S(2,3)$	$S(3,4)$	contradiction
$\mathbf{c}^1$	D	$(d_1-1)(d_2-1)$			$\delta(\mathbf{c})=S(1,2) \Rightarrow d_3=1$
$\mathbf{c}^2$	$D+d_2-1$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$	$(d_2-1)(d_3-1)$ $+(d_2-1)d_1$	$(d_3-1)(d_4-1)+d_5\mathcal{V}(c_{R(3,4,5)}=0 \text{ and } 4\leftarrow 5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}=1 \text{ and } 4\rightarrow 5)$	$\delta(\mathbf{c})=S(1,2)$ and $d_3\geq 2$ $\Rightarrow d_4=1$ and $\delta(\mathbf{c})=D+d_2-1$ $\Rightarrow A(3,4)=0$ and $n\geq 5$
$\mathbf{c}^3$	$D+d_3-1$	$(d_1-1)(d_2-1)$	$(d_2-1)(d_3-1)$ $(d_3-1)d_4$		$\delta(\mathbf{c})=S(1,2) \Rightarrow S(2,3)=0$
$\mathbf{c}^4$	$D+d_2+d_3-2$	$(d_1-1)(d_2-1)$ $+(d_2-1)d_3$	$(d_2-1)(d_3-1)$ $(d_2-1)d_1+(d_3-1)d_4$	$(d_3-1)(d_4-1)+d_2(d_3-1)$ $+d_5\mathcal{V}(c_{R(3,4,5)}=0 \text{ and } 4\leftarrow 5)$ $+(d_4-1)d_5\mathcal{V}(c_{R(3,4,5)}=1 \text{ and } 4\rightarrow 5)$	$\delta(\mathbf{c})=S(1,2) \Rightarrow d_3=1$ and $\delta(\mathbf{c})=D+d_2+d_3-2$ $\Rightarrow A(3,4)=0$ and $n\geq 5$

The data in the first column are the lower bounds $D(\mathbf{c},\mathcal{Q})$ for $\delta(\mathbf{c})$.
The last column gives the contradiction to the fact that $\delta(\mathbf{c}) = S(1,2) = S(2,3) = S(3,4)$.

References

[BL00] Sara Billey and Venkatramani Lakshmibai, *Singular loci of Schubert varieties*, Progress in Math. 182, Birkhäuser Boston, Boston, MA, 2000. MR **2001j:**14065

[Bon95] Klaus Bongartz, *Degenerations for representations of tame quivers*, Annales Scientifiques De L'école Normale Supérieure **28**, no. 5 (1995), 647–668. MR **96i:**16020

[BS] Robert Bédard and Ralf Schiffler, *Rational smoothness of varieties of representations for quivers of type A*, preprint.

[Gab75] Peter Gabriel, *Finite representation type is open*, Conference on Representations of Algebras, Carleton Math. Lecture Notes, no. 9, 1974. MR **51:**12944

[Gab80] Peter Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Representation Theory I, Lecture Notes in Mathematics 831, Springer, Berlin, 1980, pp. 1–71. MR **82i:**16030

[Kas91] M. Kashiwara, *On crystal bases of the Q-analogue of the universal enveloping algebra*, Duke Math. J. **63** (1991), 465–516. MR **93b:**17045

[Lus90a] George Lusztig, *Canonical bases arising from quantized envelopping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498. MR **90m:**17023

[Lus90b] George Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc **3** (1990), 257–296. MR **91e:**17009

[Voi77] D Voigt, *Induzierte Darstellungen in der Theorie der endlichen algebraischen Gruppen*, Lecture Notes in Math. 592, Springer-Verlag, 1977. MR **58:**5949

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