

LEFT CELLS IN TYPE B_n WITH UNEQUAL PARAMETERS

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ABSTRACT. Kazhdan and Lusztig have shown that the partition of the symmetric group \mathfrak{S}_n into left cells is given by the Robinson–Schensted correspondence. The aim of this paper is to provide a similar description of the left cells in type B_n for a special class of choices of unequal parameters. This is based on a generalization of the Robinson–Schensted correspondence in type B_n . We provide an explicit description of the left cell representations and show that they are irreducible and constructible.

1. INTRODUCTION

The Robinson–Schensted correspondence is a classical combinatorial instrument which gives rise to a partition of the symmetric group \mathfrak{S}_n into pieces which are indexed by the various standard tableaux of size n (with a filling by the numbers $1, \dots, n$). Kazhdan and Lusztig have given a completely different description of that partition by using the construction of a new basis (the “Kazhdan–Lusztig basis”) of the Iwahori–Hecke algebra of \mathfrak{S}_n . In this context, the pieces in the partition are called *left cells*. Now the definition of left cells makes sense for any (finite or infinite) Coxeter group, using the Kazhdan–Lusztig basis of the one-parameter or even multi-parameter Iwahori–Hecke algebra. One of the important aspects of this construction is that each left cell gives rise to a representation of the Iwahori–Hecke algebra where the underlying vector space has a natural basis indexed by the elements in that left cell.

Now, in the case of finite Coxeter groups and one-parameter Iwahori–Hecke algebras, the decomposition of the left cell representations into irreducible representations is completely known. For the symmetric group \mathfrak{S}_n , Kazhdan and Lusztig [11] showed that each left cell representation actually is irreducible. In the remaining types, the left cell representations are no longer irreducible and Lusztig [15] showed how they decompose into irreducibles.

This paper is concerned with the multi-parameter case. Note that, as far as finite Coxeter groups are concerned, we only have to deal with the dihedral groups and Coxeter groups of type F_4 and B_n . For type B_n and a special choice of the parameters (which allows a geometric interpretation), Lusztig [13] showed that, again, all left cell representations are irreducible. For the dihedral groups and type F_4 , results in the multi-parameter case have been obtained by Geck and Pfeiffer [8], Geck [4] and Lusztig [17].

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In this paper, we consider Coxeter groups of type B_n with diagram and parameters of the corresponding Hecke algebra \mathcal{H}_n given as follows:

$$B_n \quad \begin{array}{c} q^c \quad q \quad q \quad \dots \quad q \\ \circ \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \end{array} \quad \text{where } c \geq 1.$$

It is easy to see that the left cells are independent of the particular value of c , as long as c is sufficiently large. We are precisely interested in this “asymptotic” case, where c is sufficiently large. We shall actually prove a result which implies that all the left cell representations in that case are irreducible. This gives a new construction of “integral” forms of the irreducible representations; the first such construction is due to Dipper, James and Murphy [2]. Our result involves a generalization of the classical Robinson–Schensted correspondence. This is the subject of Section 3; the generalization does not only work for the Coxeter groups of type B_n but for all complex reflection groups of type $G(e, 1, n)$ (see [10]).

Analogously to the case of \mathfrak{S}_n , the generalized Robinson–Schensted correspondence gives rise to a partition of the Coxeter group W_n of type B_n into pieces which are indexed by pairs of standard bitableaux of total size n (with a filling by the numbers $1, \dots, n$).

Our aim is to show that the left cells in type B_n with the above choice of the parameters are given by the generalized Robinson–Schensted correspondence. As a consequence, we obtain that the left cell representations of \mathcal{H}_n are irreducible and we retrieve the classical parametrisation of irreducible \mathcal{H}_n -modules.

The paper is organized as follows. After introducing the general set-up in §2, we define in §3 the generalized Robinson–Schensted correspondence and give its first properties (Knuth correspondence, compatibility with parabolic subgroups...). In §4, we define a decomposition of elements of W_n which sounds like *Clifford theory for elements*. This decomposition gives a new description of Robinson–Schensted cells in terms of cells for symmetric groups (Proposition 4.8). One of the main tools developed in this section is Proposition 4.4.

In §5, we recall the basic notions and results concerning the construction of the Kazhdan–Lusztig basis and left cells in the multi-parameter case for general Coxeter groups. For this purpose, it is convenient to work in the general setting described by Lusztig (see [13] and [17]). In §6, we come back to W_n : we will replace the parameter q^c by a new variable Q and work with the Iwahori–Hecke algebra \mathcal{H}_n of type B_n with two independent parameters q and Q . The main results of this section are the following. First, Kazhdan–Lusztig polynomials are polynomials only in q (Theorem 6.3 (a)). Second, we obtain a kind of grading for left cells (Theorems 6.3 (b) and 6.6). The last section is devoted to the proof of the main results of this paper, namely the explicit description of Kazhdan–Lusztig left cells (Theorem 7.7) and the fact that left cells representations are irreducible and explicitly determined (Proposition 7.11).

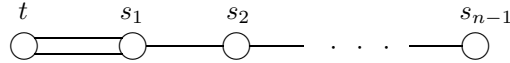
2. THE SET-UP

We introduce in this section all the notation we need concerning the Coxeter group of type B_n . This group has a presentation with a set of generators $S_n =$

$\{t, s_1, \dots, s_{n-1}\}$ and defining relations

$$\begin{cases} t^2 = 1, & s_i^2 = 1 \text{ for } i \geq 1, \\ ts_1ts_1 = s_1ts_1t, & ts_i = s_it \text{ for } i \geq 2, \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } i \geq 1, & s_is_j = s_js_i \text{ if } |i - j| \geq 2. \end{cases}$$

We visualize this presentation by the diagram



A group with a presentation as above can be naturally realized as the finite reflection group of type $G(2, 1, n)$, that is, as the subgroup of $GL_n(\mathbb{C})$ consisting of all matrices whose nonzero coefficients are 1 or -1 and where there is precisely one nonzero coefficient in each row and each column. For our purposes, it will be more convenient to work with a different realisation, using permutations.

2.1. B_n as a permutation group. Let $n \geq 1$ and consider the set

$$I_n = I_n^+ \cup I_n^-,$$

where

$$I_n^+ = \{1, 2, \dots, n\} \quad \text{and} \quad I_n^- = -I_n^+ = \{-1, -2, \dots, -n\}.$$

We denote by $\mathfrak{S}(I_n)$ the group of permutations of the set I_n and we set

$$W_n := \{\pi \in \mathfrak{S}(I_n) \mid \forall i \in I_n, \pi(-i) = -\pi(i)\}.$$

In other words, if $w_n \in \mathfrak{S}(I_n)$ is defined by $I_n \rightarrow I_n, i \mapsto -i$, then W_n is the centralizer of w_n in $\mathfrak{S}(I_n)$. We define the following transpositions in $\mathfrak{S}(I_n)$:

$$t_i := (i, -i) \quad \text{for } 1 \leq i \leq n.$$

Then $w_n = t_1t_2 \cdots t_n$. The order formulas for centralizers in symmetric groups (see [18, Chap. I]) show that $|W_n| = 2^n \cdot n!$. It is easily checked that t_1, t_2, \dots, t_n generate a subgroup $N_n \subseteq W_n$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Furthermore, the elements

$$\begin{cases} s_1 & := (1, 2) \cdot (-1, -2), \\ s_2 & := (2, 3) \cdot (-2, -3), \\ & \vdots \\ s_{n-1} & := ((n-1), n) \cdot (-(n-1), -n), \end{cases}$$

generate a subgroup $\mathfrak{S}_n \subseteq W_n$ which is isomorphic to the symmetric group of degree n . We set $\Sigma_n = \{s_1, s_2, \dots, s_{n-1}\}$. Since $N_n \cap \mathfrak{S}_n = \{1\}$, we have $|\mathfrak{S}_n N_n| = |\mathfrak{S}_n| \cdot |N_n| = 2^n \cdot n!$ and so $W_n = N_n \mathfrak{S}_n$. We now set

$$S_n = \{t, s_1, s_2, \dots, s_{n-1}\} \quad \text{where } t = t_1.$$

The previous discussion shows that

$$W_n = \langle S_n \rangle.$$

In terms of these generators, the above transpositions t_i are given recursively by $t_{i+1} = s_it_is_i$ for $1 \leq i \leq n-1$. Finally, since the generators t, s_1, \dots, s_{n-1} satisfy the relations specified by the above diagram and since $|W_n| = 2^n \cdot n!$, we conclude that these relations form a set of defining relations for W_n .

Remark 2.2. The fact that each element $\pi \in W_n$ commutes with $w_n = t_1 t_2 \cdots t_n$ implies that π is uniquely determined by the images of $1, \dots, n$. Indeed, if we know $\pi(i)$, then we also know $\pi(-i) = -\pi(i)$. Thus, if we set

$$\pi(i) = \varepsilon_i \cdot p_i \quad \text{where } 1 \leq i \leq n \text{ and } \varepsilon_i \in \{+1, -1\},$$

then the sequence of numbers p_1, \dots, p_n is a permutation of $1, \dots, n$. Thus, we shall represent π by the array

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \varepsilon_1 \cdot p_1 & \varepsilon_2 \cdot p_2 & \cdots & \varepsilon_n \cdot p_n \end{pmatrix} \in W_n. \quad \square$$

2.3. Length function, Bruhat ordering. Using this choice of generators, we can define the length function $\ell : W_n \rightarrow \mathbb{N}$. It is easily checked that w_n is the longest element of W_n : we have $\ell(w_n) = n^2$. If $w \in W_n$, we denote by $\ell_t(w)$ the number of occurrences of t in a reduced decomposition of w . This does not depend on the choice of the reduced decomposition. We set $\ell_s(w) = \ell(w) - \ell_t(w)$. It is easily checked that, if w and w' are elements of W_n such that $\ell(w w') = \ell(w) + \ell(w')$, then $\ell_t(w w') = \ell_t(w) + \ell_t(w')$ and $\ell_s(w w') = \ell_s(w) + \ell_s(w')$. For instance, we have, for every $i \in I_n$,

$$\ell_t(t_i) = 1, \quad \ell_s(t_i) = 2(i-1) \quad \text{and} \quad \ell(t_i) = 2i-1.$$

So $\ell_t(w_n) = n$ and $\ell_s(w_n) = n^2 - n$. This implies that, for every $w \in W_n$, we have $\ell_t(w) \leq n$.

We denote by \leq the Bruhat order on W_n defined by the set of generators S_n . We write $x < y$ to say that $x \leq y$ and $x \neq y$. If $w \in W_n$, we define its left descent set $\mathcal{L}(w)$ and its right descent set $\mathcal{R}(w)$ as follows:

$$\mathcal{L}(w) = \{s \in S_n \mid sw < w\}$$

and

$$\mathcal{R}(w) = \{s \in S_n \mid ws < w\}.$$

3. ON THE ROBINSON–SCHENSTED CORRESPONDENCE

In this section we describe a generalization of the classical Robinson–Schensted correspondence (which is concerned with the symmetric group \mathfrak{S}_n) to the Coxeter group of type B_n . For more details on the classical correspondence see Knuth [12, 5.1.4] or Fulton [3, Part I].

3.1. A generalized Robinson–Schensted correspondence. Let us first introduce some more notation. If λ is a partition, and if T is a standard tableau of shape λ , we set $|T| = |\lambda|$ (the number $|\lambda|$ is called the *size* of T). A *bipartition* (of n) is a pair (λ, μ) of partitions (such that $|\lambda| + |\mu| = n$). A *bitableau* is a pair of tableaux. If (T_1, T_2) is a bitableau such that T_1 is of shape λ and T_2 is of shape μ , we say that (λ, μ) is the *shape* of (T_1, T_2) and that $|\lambda| + |\mu|$ is the *size* of (T_1, T_2) . The bitableau is said to be *n-standard* if T_1 and T_2 are standard tableaux, if $|T_1| + |T_2| = n$ and if the filling of T_1 and T_2 is the set I_n^+ .

In order to generalize the Robinson–Schensted correspondence to W_n , we work with the realisation of W_n as a subgroup of $\mathfrak{S}(I_n)$ and use Remark 2.2 to represent the elements of W_n . Thus, let $\pi \in W_n$. Then we define a pair of *n-standard*

bitableaux:

$$(A_n(\pi), B_n(\pi)) \quad \text{where} \quad \begin{cases} A_n(\pi) = (A_n^+(\pi), A_n^-(\pi)), \\ B_n(\pi) = (B_n^+(\pi), B_n^-(\pi)) \end{cases}$$

and $A_n(\pi), B_n(\pi)$ have the same shape. This is done as follows.

Apply the Knuth insertion procedure as follows: insert successively the numbers p_i into two initially empty tableaux $A_n^+(\pi), A_n^-(\pi)$; more precisely, insert p_i into $A_n^{\varepsilon_i}(\pi)$. Note that at each step this yields a new box, located on the a_i th row and b_i th column, say, of $A_n^{\varepsilon_i}(\pi)$. Now add a box containing i to $B_n^{\varepsilon_i}(\pi)$ on its (a_i, b_i) position (“keep the record”).

Example 3.2. Let us consider an element $\pi \in W_7$ represented, as in Remark 2.2, by the array

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -4 & 3 & 6 & -1 & 7 & -2 & 5 \end{pmatrix}.$$

Then $(A(\pi), B(\pi))$ is equal to

$$A(\pi) = \begin{array}{|c|c|c|} \hline 3 & 5 & 7 \\ \hline 6 & & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & \\ \hline \end{array} \quad B(\pi) = \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 7 & & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 6 \\ \hline 4 & \\ \hline \end{array}$$

□

Theorem 3.3 (see also Okada [19, Th. 3.3]). *With the above notation, the following hold:*

- (a) *The map $\pi \mapsto (A_n(\pi), B_n(\pi))$ is a bijection from W_n onto the set of all pairs of n -standard bitableaux of the same shape.*
- (b) *For any $\pi \in W_n$, we have $A_n(\pi^{-1}) = B_n(\pi)$ and $B_n(\pi^{-1}) = A_n(\pi)$.*
- (c) *The number of generalized Robinson–Schensted cells equals the number of involutions in W_n .*

Proof. The proof uses essentially the same argument as for the symmetric group [12, 5.1.4, Theorems A and B].

(a) It is clear from the construction that, for any $\pi \in W_n$, $A_n(\pi)$ and $B_n(\pi)$ always have the same shape; furthermore: (i) since they are obtained through the bumping procedure, $A_n^+(\pi)$ and $A_n^-(\pi)$ are standard tableaux, (ii) $B_n^+(\pi)$ and $B_n^-(\pi)$ are standard tableaux also, since we always add elements on their periphery in increasing order.

Conversely, given a pair (A, B) of n -standard bitableaux of the same shape (with $A = (A^+, A^-)$ and $B = (B^+, B^-)$), we can find the corresponding array (and so the element $\pi \in W_n$) as follows: for $i = n, \dots, 2, 1$, let (ν, a, b) be defined by the fact that the number i appears in the B^ν tableau, on its a th row and b th column. We set then $\varepsilon_i = \nu$. Now let p_i be the element x that is removed when applying the deleting algorithm (inverse of the insertion algorithm) to the (ν, a, b) box of A . The two constructions we have described are inverses of each other.

(b) The argument is identical to the one in [12, 5.1.4, Theorem B].

(c) is an easy consequence of (a) and (b). □

Definition 3.4. Let \mathbb{T} be an n -standard bitableau. Then the set

$$\mathbb{T}(W_n) := \{w \in W_n \mid B_n(w) = \mathbb{T}\}$$

is nonempty and will be called a *generalized Robinson–Schensted cell* (or generalized RS-cell for short) of W_n . We have a partition

$$W_n = \coprod_{\mathbb{T}} \mathbb{T}(W_n),$$

where \mathbb{T} runs over the set of all n -standard bitableaux. An example is given in Table 1.

Remark 3.5. If $\pi \in W_n$ then, by construction, $\ell_t(\pi) = |A_n^-(\pi)| = |B_n^-(\pi)|$. Therefore, the function ℓ_t is constant on generalized Robinson–Schensted cells. Moreover, if $\pi \in \mathfrak{S}_n$, then $A_n^-(\pi) = B_n^-(\pi) = \emptyset$ and $(A_n^+(\pi), B_n^+(\pi))$ is the pair of standard tableaux associated to π via the usual Robinson–Schensted correspondence. \square

Example 3.6. We have

$$A_n(1) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & n \\ \hline \end{array} \quad \emptyset \qquad B_n(1) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & n \\ \hline \end{array} \quad \emptyset$$

On the other hand,

$$A_n(w_n) = \emptyset \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & n \\ \hline \end{array} \qquad B_n(w_n) = \emptyset \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & \dots & n \\ \hline \end{array}$$

Moreover, $\{1\}$ and $\{w_n\}$ are generalized Robinson–Schensted left cells. \square

3.7. A generalization of a theorem of Knuth. Knuth has given a purely group theoretical description of the partition of \mathfrak{S}_n into Robinson–Schensted cells. We wish to generalize that statement to the Coxeter groups of type B_n . We begin with some general definitions. First, we set $S'_n = S_n \cup \{t_1, t_2, \dots, t_n\}$. Then the extended left descent set of $w \in W_n$ is defined by

$$\mathcal{L}'(w) := \{u \in S'_n \mid \ell(uw) < \ell(w)\} = \{u \in S'_n \mid uw < w\}.$$

Let $x, y \in W_n$ and $s \in \Sigma_n = S_n - \{t\}$; then we define

$$x \xrightarrow{s}_L y \quad \stackrel{\text{def}}{\iff} \quad y = sx, \ell(y) > \ell(x) \text{ and } \mathcal{L}'(x) \not\subseteq \mathcal{L}'(y),$$

and we write $x \xleftarrow{s}_L y$ if $x \xrightarrow{s}_L y$ or $y \xrightarrow{s}_L x$. Finally, we write $x \longleftrightarrow_L y$ if there exists a sequence $x = x_1, x_2, \dots, x_k = y$ and $\rho_i \in \Sigma_n$ such that $x_i \xleftarrow{\rho_i}_L x_{i+1}$ for all i .

Proposition 3.8. *Let $x, y \in W_n$. Then*

$$B_n(x) = B_n(y) \quad \text{if and only if} \quad x \longleftrightarrow_L y.$$

Proof. We first define “admissible transformations” in W_n . Let $x \in W_n$ be represented, as in Remark 2.2, by the array

$$x = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n \\ \varepsilon_1 \cdot p_1 & \varepsilon_2 \cdot p_2 & \dots & \varepsilon_i \cdot p_i & \varepsilon_{i+1} \cdot p_{i+1} & \dots & \varepsilon_n \cdot p_n \end{pmatrix}$$

with $p_j \in I_n^+$ and $\varepsilon_j \in \{+1, -1\}$; interchanging $\varepsilon_i p_i$ and $\varepsilon_{i+1} p_{i+1}$ is an admissible transformation if we are in one of the following situations:

- (a) $2 \leq i \leq n-1$, $\varepsilon_{i-1} = \varepsilon_i = \varepsilon_{i+1}$ and p_{i-1} lies between p_i and p_{i+1}
- (b) $1 \leq i \leq n-2$, $\varepsilon_i = \varepsilon_{i+1} = \varepsilon_{i+2}$ and p_{i+2} lies between p_i and p_{i+1}
- (c) $\varepsilon_i = -\varepsilon_{i+1}$.

TABLE 1. Generalized Robinson–Schensted cells in type B_3

Generalized RS-cell	B -bitableaux
$\{1\}$	$\boxed{1} \boxed{2} \boxed{3}$ —
$\{s_2, s_1s_2\}$	$\begin{array}{ c c } \hline \boxed{1} & \boxed{2} \\ \hline \boxed{3} & \\ \hline \end{array}$ —
$\{s_1, s_2s_1\}$	$\begin{array}{ c c } \hline \boxed{1} & \boxed{3} \\ \hline \boxed{2} & \\ \hline \end{array}$ —
$\{s_1s_2s_1\}$	$\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{2} \\ \hline \boxed{3} \\ \hline \end{array}$ —
$\{t, s_1t, s_2s_1t\}$	$\boxed{2} \boxed{3}$ $\boxed{1}$
$\{ts_1, s_1ts_1, s_2s_1ts_1\}$	$\boxed{1} \boxed{3}$ $\boxed{2}$
$\{ts_1s_2, s_1ts_1s_2, s_2s_1ts_1s_2\}$	$\boxed{1} \boxed{2}$ $\boxed{3}$
$\{ts_2, s_1ts_2, s_1s_2s_1t\}$	$\begin{array}{ c } \hline \boxed{2} \\ \hline \boxed{3} \\ \hline \end{array}$ $\boxed{1}$
$\{ts_2s_1, s_1ts_2s_1, s_1s_2s_1ts_1\}$	$\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{3} \\ \hline \end{array}$ $\boxed{2}$
$\{ts_1s_2s_1, s_1ts_1s_2s_1, s_1s_2s_1ts_1s_2\}$	$\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{2} \\ \hline \end{array}$ $\boxed{3}$
$\{ts_1ts_1s_2s_1, ts_1s_2s_1ts_1s_2, s_1ts_1s_2s_1ts_1s_2\}$	$\boxed{1}$ $\boxed{2} \boxed{3}$
$\{ts_1ts_1s_2, ts_2s_1ts_1s_2, s_1ts_2s_1ts_1s_2\}$	$\boxed{2}$ $\boxed{1} \boxed{3}$
$\{ts_1ts_1, ts_2s_1ts_1, s_1ts_2s_1ts_1\}$	$\boxed{3}$ $\boxed{1} \boxed{2}$
$\{ts_1ts_2s_1, ts_1s_2s_1ts_1, s_1ts_1s_2s_1ts_1\}$	$\boxed{1}$ $\begin{array}{ c } \hline \boxed{2} \\ \hline \boxed{3} \\ \hline \end{array}$
$\{ts_1ts_2, ts_1s_2s_1t, s_1ts_1s_2s_1t\}$	$\boxed{2}$ $\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{3} \\ \hline \end{array}$
$\{ts_1t, ts_2s_1t, s_1ts_2s_1t\}$	$\boxed{3}$ $\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{2} \\ \hline \end{array}$
$\{ts_1ts_2s_1t\}$	— $\begin{array}{ c } \hline \boxed{1} \\ \hline \boxed{2} \\ \hline \boxed{3} \\ \hline \end{array}$
$\{ts_1ts_1s_2s_1t, ts_1ts_2s_1ts_1s_2\}$	— $\begin{array}{ c c } \hline \boxed{1} & \boxed{3} \\ \hline \boxed{2} & \\ \hline \end{array}$
$\{ts_1ts_2s_1ts_1, ts_1ts_1s_2s_1ts_1\}$	— $\begin{array}{ c c } \hline \boxed{1} & \boxed{2} \\ \hline \boxed{3} & \\ \hline \end{array}$
$\{ts_1ts_1s_2s_1ts_1s_2\}$	— $\boxed{1} \boxed{2} \boxed{3}$

Now note that (c)-type transformations do not change the relative ordering of the numbers $x(j)$ belonging to I_n^+ or to I_n^- respectively. Then, by applying [12, 5.1.4, Ex. 4] we get that two elements $x, y \in W_n$ have the same A -bitableaux if and only if each of them can be obtained from the other through a finite number of admissible transformations.

As for the group theoretical description of these admissible transformations, we obtain the following as an easy consequence of [20, Ex. 9.10]:

- $xs_i < x \iff \begin{matrix} x(i), x(i+1) \in I_n^+ \text{ and } x(i+1) < x(i) \\ \text{or } x(i), x(i+1) \in I_n^- \text{ and } x(i+1) < x(i) \\ \text{or } x(i) \in I_n^+, x(i+1) \in I_n^- \end{matrix}$
- $\ell(xt_i) < \ell(x) \iff x(i) \in I_n^-$.

The proof of the proposition is now complete. □

4. A THIRD CONSTRUCTION OF GENERALIZED ROBINSON-SCHENSTED CELLS

4.1. Preliminaries. The parabolic subgroup \mathfrak{S}_n of W_n generated by $\Sigma_n = \{s_1, \dots, s_{n-1}\}$ is isomorphic to the symmetric group of degree n . We denote by X_n the set of elements $w \in W_n$ which are of minimal length in $w\mathfrak{S}_n$ (they are usually called *distinguished left coset representatives* of \mathfrak{S}_n in W_n). If $0 \leq l \leq n$, we set

$$X_n^{(l)} = \{w \in X_n \mid \ell_t(w) = l\}.$$

Let us give a description of $X_n^{(l)}$. We define $r_1 = t$ and, for $1 \leq i \leq n - 1$, we set $r_{i+1} = s_i r_i = s_i \dots s_2 s_1 t$. Then

$$X_n^{(l)} = \{r_{i_1} r_{i_2} \dots r_{i_l} \mid 1 \leq i_1 < i_2 < \dots < i_l \leq n\}.$$

We have for instance $X_n^{(0)} = \{1\}$. Note also that

$$\ell(r_{i_1} r_{i_2} \dots r_{i_l}) = i_1 + i_2 + \dots + i_l$$

if $1 \leq i_1 < i_2 < \dots < i_l \leq n$. Therefore, in the subset $X_n^{(l)}$, there is a unique element of minimal length which will be denoted by $a_l = r_1 r_2 \dots r_l$ (note that $a_0 = 1$). It is also clear that a_l has minimal length among all the elements of t -length equal to l (in W_n).

Let us make some further notation. We denote by:

- \mathfrak{S}_l the parabolic subgroup of W_n generated by Σ_l and by σ_l its longest element.
- W_l the parabolic subgroup of W_n generated by $S_l = \{t\} \cup \Sigma_l$ and by w_l its longest element.
- $\mathfrak{S}_{l,n-l}$ the parabolic subgroup of W_n generated by $\Sigma_{l,n-l} = S_n \setminus \{t, s_l\}$ and by $\sigma_{l,n-l}$ its longest element.
- $W_{l,n-l}$ the parabolic subgroup of W_n generated by $S_{l,n-l} = S_n \setminus \{s_l\}$ and by $w_{l,n-l}$ its longest element.
- $Y_{l,n-l}$ the set of distinguished left coset representatives of $\mathfrak{S}_{l,n-l}$ in \mathfrak{S}_n .

One can immediately check that

$$a_l = w_{l,n-l} \sigma_{l,n-l} = \sigma_{l,n-l} w_{l,n-l} = w_l \sigma_l = \sigma_l w_l.$$

In particular, $a_l^2 = 1$ and conjugacy by a_l stabilizes $S_l, \Sigma_l, S_{l,n-l}$ and $\Sigma_{l,n-l}$. In particular, a_l normalizes $W_l, \mathfrak{S}_l, W_{l,n-l}$ and $\mathfrak{S}_{l,n-l}$. Note also that

$$|X_n^{(l)}| = \binom{n}{l} = \frac{n!}{l!(n-l)!}.$$

One can notice that $|X_n^{(l)}| = |Y_{l,n-l}|$ and that $|X_n^{(l)} \cdot \mathfrak{S}_{l,n-l}| = n! = |\mathfrak{S}_n|$. This is not a coincidence, as will be shown in this subsection (see Proposition 4.4). We need the following elementary lemma (the proof is left to the reader).

Lemma 4.2. *If $1 \leq i \leq n$, then*

$$r_i^{-1}(j) = \begin{cases} j+1 & \text{if } 1 \leq j \leq i-1, \\ -1 & \text{if } j = i, \\ j & \text{if } i+1 \leq j \leq n. \end{cases}$$

Therefore, if $1 \leq j < i-1$, then $r_i^{-1}s_jr_i = s_{j+1}$.

Corollary 4.3. *Let $a \in X_n^{(l)}$ with $1 \leq l \leq n-1$ and let $s \in \Sigma_n$. Then $a^{-1}sa \neq s_l$.*

Proof. Let us write $a = r_{i_1} \dots r_{i_l}$ with $1 \leq i_1 < \dots < i_l \leq n$. We want to prove by induction on $l \geq 1$ that $as_l a^{-1} \notin \Sigma_n$.

Let us assume first that $l = 1$. We write $i = i_1$. Then $(r_i s_l r_i^{-1})(i) = (r_i s_l)(-1) = r_i(-2) = -r_i(2) < 0$. So $r_i s_l r_i^{-1} \notin \Sigma_n$.

Let us assume now that $l \geq 2$ and that the result holds for $l' < l$. Two cases may occur:

First case. Assume that $i_l \geq l+1$. Therefore, by Lemma 4.2, we have $r_{i_l} s_l r_{i_l}^{-1} = s_{l-1}$ and the result follows by induction.

Second case. Assume that $i_l \leq l$. In this case, we have $i_l = l$ and $a = a_l$. But,

$$a_l s_l a_l^{-1}(l+1) = a_l s_l(l+1) = a_l(l) = -1.$$

So $a_l s_l a_l^{-1} \notin \Sigma_n$. □

Proposition 4.4. *Let l be a nonnegative integer such that $0 \leq l \leq n$. Then $X_n^{(l)} = Y_{l,n-l} a_l$ and $X_n^{(l)} \cdot \mathfrak{S}_{l,n-l} = \mathfrak{S}_n a_l$.*

Proof. Since $|Y_{l,n-l}| = |X_n^{(l)}|$, it is sufficient to prove that $wa_l \in X_n^{(l)}$ for every $w \in Y_{l,n-l}$ to prove the first equality. So let $w \in Y_{l,n-l}$ and let $s \in \{s_1, s_2, \dots, s_{n-1}\}$. We need to prove that $\ell(wa_l s) = \ell(wa_l) + 1$. If $a_l s \in (X_n^{(l)})^{-1}$, then $\ell(a_l s) = \ell(a_l) + 1$ (because a_l is the element of $X_n^{(l)}$ of minimal length) and

$$\ell(wa_l s) = \ell(w) + \ell(a_l s) = \ell(w) + \ell(a_l) + 1 = \ell(wa_l) + 1.$$

If $a_l s \notin (X_n^{(l)})^{-1}$, then, by Deodhar's Lemma [8, Lemma 2.1.2], we have $a_l s = s' a_l$ for some $s' \in \{s_1, s_2, \dots, s_{n-1}\}$. But, by Corollary 4.3, we have $s' \neq s_l$. So $\ell(ws') = \ell(w) + 1$. Therefore,

$$\ell(wa_l s) = \ell(ws' a_l) = \ell(ws') + \ell(a_l) = \ell(w) + \ell(a_l) + 1 = \ell(wa_l) + 1.$$

Let us prove now the second equality. Since $|X_n^{(l)} \cdot \mathfrak{S}_{l,n-l}| = |\mathfrak{S}_n|$, we only need to show that $wa_l \in X_n^{(l)} \cdot \mathfrak{S}_{l,n-l}$ for every $w \in \mathfrak{S}_n$. So let $w \in \mathfrak{S}_n$. We write $w = yw'$ with $y \in Y_{l,n-l}$ and $w' \in \mathfrak{S}_{l,n-l}$. Then $wa_l = ya_l a_l^{-1} w' a_l$. But, by (a), we have that $ya_l \in X_n^{(l)}$. Moreover, a_l normalizes $\mathfrak{S}_{l,n-l}$, so $a_l^{-1} w' a_l \in \mathfrak{S}_{l,n-l}$. Therefore, $wa_l \in X_n^{(l)} \cdot \mathfrak{S}_{l,n-l}$. □

Remark 4.5. (a) Let $\pi_1, \pi_2 \in \mathfrak{S}_n$. A simple computation together with Proposition 3.8 shows that we have

$$B(\pi_1) = B(\pi_2) \iff B(\sigma_n \pi_1) = B(\sigma_n \pi_2).$$

(b) Let $\sigma = \pi\rho \in \mathfrak{S}_{l,n-l}$, with $\pi \in \mathfrak{S}_l$ and $\rho \in \mathfrak{S}_{[l+1,n]}$ (where $\mathfrak{S}_{[l+1,n]}$ is the parabolic subgroup of \mathfrak{S}_n generated by s_{l+1}, \dots, s_{n-1}). Let $(A(\pi), B(\pi))$ and $(A(\rho), B(\rho))$ be the pairs of standard tableaux associated by the (classical) Robinson-Schensted correspondence to π and ρ respectively. Then the pair of tableaux associated to $\sigma \in \mathfrak{S}_n$ is given by $(A(\pi) \cdot A(\rho), B(\pi) \cdot B(\rho))$. (For the definition of the product of tableaux see [3, 1.2].)

4.6. A decomposition of elements of W_n . If $w \in W_n$ and if $l = \ell_t(w)$, then, by Proposition 4.4, there exist uniquely determined elements $a_w, b_w \in Y_{l,n-l}$, and $\sigma_w \in \mathfrak{S}_{l,n-l}$ such that

$$w = a_w a_l \sigma_w b_w^{-1}.$$

Moreover, we have

$$\ell(w) = \ell(a_w) + \ell(a_l) + \ell(\sigma_w) + \ell(b_w).$$

If w and w' are two elements of W_n , we write $w \rightsquigarrow_L w'$ if

$$\ell_t(w) = \ell_t(w'), \quad \sigma_w \longleftrightarrow_L \sigma_{w'} \quad \text{and} \quad b_w = b_{w'}.$$

It is obvious that \rightsquigarrow_L is an equivalence relation. We write $\sigma_l \sigma_w = \sigma'_w \sigma''_w$ with $\sigma'_w \in \mathfrak{S}_l$ and $\sigma''_w \in \mathfrak{S}_{[l+1,n]}$. Then

$$w = a_w w_l \sigma'_w \sigma''_w b_w^{-1}.$$

Moreover, w_l, σ'_w and σ''_w commute with each other. By Remark 4.5 we have $w \rightsquigarrow_L w'$ if and only if

$$\ell_t(w) = \ell_t(w'), \quad \sigma'_w \longleftrightarrow_L \sigma'_{w'}, \quad \sigma''_w \longleftrightarrow_L \sigma''_{w'} \quad \text{and} \quad b_w = b_{w'}.$$

4.7. Bitableaux and decomposition. Note that by the (usual) Robinson-Schensted correspondence, the element $\sigma'_w \sigma''_w$ is associated to a pair of n -standard bitableaux of shape (λ, μ) with $|\lambda| = l$ and $|\mu| = n - l$; also note that (by Proposition 3.8) the elements a_w and b_w do not affect the shape of the bitableaux of $w_l \sigma'_w \sigma''_w$. This implies that the shape of the n -standard bitableaux associated to w by the generalized Robinson-Schensted correspondence is exactly (μ, λ) . As for the filling of the n -standard bitableaux associated to w : $A_n^+(w)$ is obtained by the action of a_w on the A -tableau of σ'' , $A_n^-(w)$ is obtained by the action of a_w on the A -tableau of σ' , while $B_n^+(w)$ is obtained by the action of b_w on the B -tableau of σ'' and finally $B_n^-(w)$ is obtained by the action of b_w on the B -tableau of σ' .

The above remark has as a direct consequence the next proposition.

Proposition 4.8. *Let w and w' be two elements of W_n . Then the following are equivalent:*

- (1) $w \longleftrightarrow_L w'$;
- (2) $w \rightsquigarrow_L w'$.

Proof. Follows immediately from Proposition 3.8 and Remark 4.7. □

Remark 4.9. Note that Proposition 4.8 implies that every equivalence class for \rightsquigarrow_L contains a unique involution.

We end this subsection with a result about the above decomposition of elements of W_n and the Bruhat order.

Proposition 4.10. *Let x and y be two elements of W_n such that $\ell_t(x) = \ell_t(y) = l$ and $x \leq y$. Then:*

- (a) $a_x \leq a_y$ and $b_x \leq b_y$.
- (b) If $y \in X_n^{(l)} \cdot \mathfrak{S}_{l, n-l}$, then $x \in X_n^{(l)} \cdot \mathfrak{S}_{l, n-l}$.
- (c) If $b_x = b_y = b$, then $xb \leq yb$ and the map $[xb, yb] \rightarrow [x, y]$, $z \mapsto zb^{-1}$ is an isomorphism between the two intervals.

Proof. (a) By [17, Lemma 9.10 (f)], we have $a_x a_l \leq a_y a_l$. But, a_l is minimal in $\mathfrak{S}_n a_l$. Therefore, again by [17, Lemma 9.10 (f)], we get that $a_x \leq a_y$.

On the other hand, we have $x^{-1} \leq y^{-1}$. So, by the previous result, we have $a_{x^{-1}} \leq a_{y^{-1}}$. But Proposition 4.4 together with a simple computation show that $a_{x^{-1}} = b_x$. Hence $b_x \leq b_y$.

(b) By Proposition 4.4, there exists $w \in \mathfrak{S}_n$ such that $y = wa_l$. So there exists a reduced expression $x = w'a'$ where $w' \leq w$ and $a' \leq a_l$. But $\ell_t(x) = l$ so $\ell_t(a') = l$. Therefore, $a' = a_l$. Moreover, $w' \in \mathfrak{S}_n$. So $x = w'a_l \in X_n^{(l)} \cdot \mathfrak{S}_{l, n-l}$ by Proposition 4.4.

(c) Let us write $x = \alpha\beta$ (reduced expression) where $\alpha \leq yb$ and $\beta \leq b^{-1}$. But, by (b), we have $\alpha \in X_n^{(l)} \cdot \mathfrak{S}_{l, n-l}$. So $\beta = b^{-1}$ and $xb = \alpha \leq yb$.

Now, let us prove that the map $f : [xb, yb] \rightarrow [x, y]$, $z \mapsto zb^{-1}$ is an increasing bijection. First note that the map is well defined, injective and increasing. We need to show that it is surjective. Let $z \in [x, y]$. By (a), we have $b = b_x \leq b_z \leq b_y = b$. So $b_z = b$. In particular, the previous result shows that $xb \leq zb \leq yb$. Therefore, $zb \in [xb, yb]$ and $f(zb) = z$. Let $u, v \in [x, y]$ such that $u \leq v$; then there exist $u', v' \in [xb, yb]$ such that $u = u'b^{-1}$ and $v = v'b^{-1}$. Now $u' \leq v'$ for the same reasons for which $xb \leq yb$. \square

5. KAZHDAN–LUSZTIG POLYNOMIALS IN THE UNEQUAL PARAMETER CASE

In this section, we recall the basic constructions from Lusztig [13]. Let (W, S) be a Coxeter system and $\ell : W \rightarrow \mathbb{N}_0$ the corresponding length function. Let $\varphi : W \rightarrow \Gamma$ be a map into an abelian group Γ such that $\varphi(w_1 w_2) = \varphi(w_1) \varphi(w_2)$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Note that this condition implies that φ is determined by its images on S and that $\varphi(s) = \varphi(s')$ whenever $s, s' \in S$ are conjugate in W . We set

$$\varphi(w) = v_w \quad \text{for any } w \in W.$$

Let \mathcal{H} be the generic Iwahori-Hecke algebra associated with (W, S) over the ring $A = \mathbb{Z}[\Gamma]$; then \mathcal{H} has a basis $\{T_w \mid w \in W\}$ such that the multiplication is given by

$$T_s^2 = T_1 + (v_s - v_s^{-1})T_s \quad \text{for } s \in S$$

and $T_{w_1} T_{w_2} = T_{w_1 w_2}$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Each T_s ($s \in S$) is invertible in \mathcal{H} ; we have

$$T_s^{-1} = T_s + (v_s^{-1} - v_s)T_1 \quad \text{for all } s \in S.$$

(note that this is the basis usually denoted by $\{\tilde{T}_w \mid w \in W\}$; see [13]). The construction of a Kazhdan–Lusztig basis of \mathcal{H} depends on one further ingredient, namely, the choice of a total ordering on Γ which is compatible with the group

structure. Thus, we assume that we have fixed a multiplicatively closed subset $\Gamma_+ \subset \Gamma$ such that we have a disjoint union $\Gamma = \Gamma_+ \cup \{1\} \cup \Gamma_-$, where $\Gamma_- = \{g^{-1} \mid g \in \Gamma_+\}$. Note that this means, in particular, that $1_\Gamma \notin \Gamma_+$. We assume that

$$v_s \in \Gamma_+ \quad \text{for all } s \in S.$$

Example 5.1. Let Γ be the infinite cyclic group generated by some indeterminate v over \mathbb{C} , and $\Gamma_+ = \{v^m \mid m \geq 1\}$. Let $\{c_s \mid s \in S\}$ be a collection of positive integers such that $c_s = c_t$ whenever $s, t \in S$ are conjugate in W . Then the above requirements are satisfied for the unique map $\varphi: W \rightarrow \Gamma$ such that $\varphi(s) = v^{c_s}$ for all $s \in S$. □

Remark 5.2. One should keep in mind that the choice of $\varphi: W \rightarrow \Gamma$ in Example 5.1 is the most important one as far as applications to representations of reductive algebraic groups are concerned; see [14] and [16]. However, more general choices of $\varphi: W \rightarrow \Gamma$ have applications to the representation theory of Iwahori–Hecke algebras, via the construction of left cell representations. For example, the determination of the left cells for a two-parameter algebra of type F_4 in [8, Chap. 11] has lead to a construction of the irreducible representations in terms of W -graphs in this case. These W -graphs in turn yield a complete set of irreducible representations for any semisimple specialization of that algebra.

5.3. The Kazhdan–Lusztig basis. Let $a \mapsto \bar{a}$ be the involution of $\mathbb{Z}[\Gamma]$ which takes g to g^{-1} for any $g \in \Gamma$. We extend it to a map $\mathcal{H} \rightarrow \mathcal{H}$, $h \mapsto \bar{h}$, by the formula

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_w^{-1} \quad (a_w \in \mathbb{Z}[\Gamma]).$$

Then $h \mapsto \bar{h}$ is in fact a ring involution. In this set-up, let $\{C_w \mid w \in W\}$ be the basis of \mathcal{H} constructed in [13, Prop. 2] (formerly denoted by $\{C'_w \mid w \in W\}$). We have

$$(a) \quad C_w = \sum_{\substack{y \in W \\ y \leq w}} P_{y,w}^* T_y \quad \text{with } P_{y,w}^* \in A,$$

where $P_{w,w}^* = 1$ and $P_{y,w}^* \in \mathbb{Z}[\Gamma_-]$ if $y < w$. Here, $\mathbb{Z}[\Gamma_-]$ denotes the set of integral linear combinations of elements in Γ_- .

We have $\overline{P_{y,w}^*} - P_{y,w}^* = \sum_{y < x \leq w} R_{y,x} P_{x,w}^*$, where the coefficients $R_{x,w} \in \mathbb{Z}[\Gamma]$ are

defined by $\overline{T_w} = \sum_{y \leq w} R_{y,w} T_y$. Note also that

$$(b) \quad P_{y^{-1},w^{-1}}^* = P_{y,w}^* \quad \text{for all } y, w \in W \text{ such that } y \leq w.$$

This immediately follows from the discussion in [13, §6].

5.4. The M -polynomials. Let $w \in W$ and $s \in S$ be such that $sw > w$. As in [13, §3], for each $y \in W$ such that $sy < y < w < sw$ we define an element $M_{y,w}^s \in \mathbb{Z}[\Gamma]$ by the inductive condition

$$(M1) \quad M_{y,w}^s + \sum_{\substack{y < z < w \\ sz < z}} P_{y,z}^* M_{z,w}^s - v_s P_{y,w}^* \in \mathbb{Z}[\Gamma_-]$$

and the symmetry condition

$$(M2) \quad \overline{M_{y,w}^s} = M_{y,w}^s.$$

With the above definition we have the following multiplication formulas; see [13, Prop. 4]. Let $w \in W$ and $s \in S$. Then we have

$$C_s C_w = \begin{cases} C_{sw} + \sum_{\substack{z < w \\ sz < z}} M_{z,w}^s C_z & \text{if } sw > w, \\ (v_s + v_s^{-1}) C_w & \text{if } sw < w. \end{cases}$$

The proof for the above multiplication rules in [13] actually provides a recursion formula for the computation of $P_{y,w}^*$. First recall that $P_{w,w}^* = 1$ for all $w \in W$ and $P_{y,w}^* = 0$ unless $y \leq w$. Now let y and w be two elements of W such that $y < w$ and let $s \in S$ such that $sw < w$. Then:

$$(1) \quad P_{y,w}^* = v_s P_{y,sw}^* + P_{sy,sw}^* - \sum_{\substack{y \leq z < sw \\ sz < z}} P_{y,z}^* M_{z,sw}^s \quad \text{if } sy < y,$$

$$(2) \quad P_{y,w}^* = v_s^{-1} P_{sy,w}^* \quad \text{if } sy > y.$$

We conclude by an obvious lemma concerning the degree of the M -polynomials.

Lemma 5.5. *Let $s \in S$ and $y, w \in W$ be such that $sy < y < w < sw$. Then $v_s^{-1} M_{y,w}^s \in \mathbb{Z}[\Gamma_-]$.*

Proof. If $sy < y < w < sw$, we set $\tilde{M}_{y,w}^s = v_s^{-1} M_{y,w}^s$. Then condition (M1) implies that

$$\tilde{M}_{y,w}^s + \sum_{\substack{y < z < w \\ sz < z}} P_{y,z}^* \tilde{M}_{z,w}^s - P_{y,w}^* \in \mathbb{Z}[\Gamma_-].$$

Since $P_{y,w}^* \in \mathbb{Z}[\Gamma_-]$, the result follows immediately by induction on $\ell(w) - \ell(y)$. \square

5.6. The longest element. Assume that W is finite and let $w_0 \in W$ be the unique element of maximal length. Then, for $y \leq w$ in W , we have the following relation:

$$(a) \quad \sum_{\substack{z \in W \\ y \leq z \leq w}} (-1)^{\ell(w) - \ell(z)} P_{y,z}^* P_{zw_0, zw_0}^* = \begin{cases} 1 & \text{if } y = w, \\ 0 & \text{if } y < w. \end{cases}$$

Furthermore, if $y, w \in W$ and $s \in S$ are such that $sy < y < w < sw$, then

$$(b) \quad M_{ww_0, yw_0}^s = -(-1)^{\ell(w) - \ell(y)} M_{y,w}^s.$$

In the equal parameter case, these relations are already contained in Kazhdan and Lusztig [11]; for the general case, see [4, §2]. Passing to the polynomials $P_{y,w} = v_y^{-1} v_w P_{y,w}^*$, we obtain

$$(c) \quad \sum_{\substack{z \in W \\ y \leq z \leq w}} (-1)^{\ell(w) - \ell(z)} P_{y,z} P_{zw_0, zw_0} = \begin{cases} 1 & \text{if } y = w, \\ 0 & \text{if } y < w. \end{cases}$$

The last relation will be helpful in the computation of certain Kazhdan–Lusztig polynomials in type B_n .

5.7. Left cells. We recall the definition of left cells from [17, §8.1]. Let \leq_L be the preorder relation on W generated by the relation:

$$(L) \quad \begin{cases} y \leq_{L,s} w \text{ if there exists some } s \in S \text{ such that } C_y \text{ appears with} \\ \text{nonzero coefficient in } C_s C_w \text{ (expressed in the } C_w\text{-basis)}. \end{cases}$$

TABLE 2. Left cells in type B_3 with unequal parameters

Parameters v, v^2, v^2
$\{1\}, \{t\}, \{s_1ts_1s_2s_1ts_1s_2\}, \{ts_1ts_1s_2s_1ts_1s_2\}, \{s_1ts_1, s_2s_1ts_1\},$ $\{ts_1ts_1, ts_2s_1ts_1\}, \{s_1ts_1s_2, s_2s_1ts_1s_2\}, \{ts_1ts_1s_2, ts_2s_1ts_1s_2\},$ $\{s_1, s_2s_1, ts_1\}, \{s_1t, s_2s_1t, ts_1t\}, \{s_2, s_1s_2, ts_1s_2\}, \{ts_2, s_1ts_2, ts_1ts_2\},$ $\{ts_2s_1, s_1ts_2s_1, ts_1ts_2s_1\}, \{ts_2s_1t, s_1ts_2s_1t, ts_1ts_2s_1t\},$ $\{s_1s_2s_1, ts_1s_2s_1, s_1ts_1s_2s_1\}, \{s_1s_2s_1t, ts_1s_2s_1t, s_1ts_1s_2s_1t\},$ $\{s_1s_2s_1ts_1, ts_1s_2s_1ts_1, s_1ts_1s_2s_1ts_1\},$ $\{s_1s_2s_1ts_1s_2, ts_1ts_1s_2s_1, ts_1s_2s_1ts_1s_2\},$ $\{s_1ts_2s_1ts_1, ts_1ts_2s_1ts_1, ts_1ts_1s_2s_1ts_1\},$ $\{s_1ts_2s_1ts_1s_2, ts_1ts_1s_2s_1t, ts_1ts_2s_1ts_1s_2\}$
Parameters v^3, v^2, v^2
$\{1\}, \{ts_1t\}, \{s_1s_2s_1ts_1s_2\}, \{ts_1ts_1s_2s_1ts_1s_2\}, \{s_1, s_2s_1\}, \{s_2, s_1s_2\},$ $\{t, s_1t, s_2s_1t\}, \{ts_2, s_1ts_2, ts_1ts_2\}, \{ts_1, s_1ts_1, s_2s_1ts_1\},$ $\{ts_2s_1, s_1ts_2s_1, ts_1ts_2s_1\}, \{ts_1s_2, s_1ts_1s_2, s_2s_1ts_1s_2\},$ $\{ts_2s_1t, s_1ts_2s_1t, ts_1ts_2s_1t\}, \{ts_1ts_2s_1ts_1, ts_1ts_1s_2s_1ts_1\},$ $\{s_1s_2s_1, ts_1s_2s_1, s_1ts_1s_2s_1\}, \{ts_1ts_1s_2s_1t, ts_1ts_2s_1ts_1s_2\},$ $\{ts_1ts_1, ts_2s_1ts_1, s_1ts_2s_1ts_1\}, \{s_1s_2s_1t, ts_1s_2s_1t, s_1ts_1s_2s_1t\},$ $\{ts_1ts_1s_2, ts_2s_1ts_1s_2, s_1ts_2s_1ts_1s_2\},$ $\{s_1s_2s_1ts_1, ts_1s_2s_1ts_1, s_1ts_1s_2s_1ts_1\},$ $\{ts_1ts_1s_2s_1, ts_1s_2s_1ts_1s_2, s_1ts_1s_2s_1ts_1s_2\}$

The equivalence relation associated with \leq_L will be denoted by \sim_L and the corresponding equivalence classes are called the *left cells* of W . Lusztig [17, §8.3] has associated to each left cell \mathfrak{C} a representation of \mathcal{H} .

Remark 5.8. When (W, S) is the Coxeter group of type A_{n-1} , Kazhdan and Lusztig [11, §5] proved that two elements $y, w \in W$ are in the same left cell if and only if they are in the same Robinson-Schensted cell (i.e. $B(y) = B(w)$).

Example 5.9. Assume that $(W, S) = (W_n, S_n)$ as defined in §2. Let v be an indeterminate and $\Gamma := \{v^j \mid j \in \mathbb{Z}\}$. Let $c, d \geq 1$. As in Example 5.1, let $\varphi: W \rightarrow \Gamma$ be defined by

$$\varphi(t) = v^c \text{ and } \varphi(s_i) = v^d \text{ for } 1 \leq i \leq n-1.$$

Let $\Gamma_+ = \{v^j \mid j > 0\}$. Assume that $(c, d) \in \{(1, 2), (3, 2)\}$. Then Lusztig [13, Theorem 11] has shown that the corresponding left cell representations are all irreducible. As an example, we give the distribution of the elements of W_3 into left cells in Table 2. (The computation was done using the CHEVIE system [6].) We only point out here that the generalized Robinson-Schensted cells in Table 1 appear to be completely unrelated to the left cells in Table 2. Our main result, which will be proven in Section 7 will show that the Robinson-Schensted cells are related to the case where $d = 1$ and c is large enough; we shall call it *the asymptotic case*.

5.10. Left cells and parabolic subgroups. If J is a subset of S , we denote by W_J the parabolic subgroup of W generated by J and by X_J the set of distinguished left coset representatives of W_J in W .

The next result, due to Geck [5], will be used in the sequel.

Theorem 5.11 (Geck [5, Th. 1 and §4]). *With the above notation, we have:*

- (a) *Let \mathfrak{C} be a left cell of W_J . Then $X_J \cdot \mathfrak{C}$ is a union of left cells of W .*
- (b) *Let $z, u \in X_J$ and $x, y \in W_J$. Then we have*

$$zx \leq_L uy \implies x \leq_L y \quad \text{in } W_J$$

and

$$zx \sim_L uy \implies x \sim_L y \quad \text{in } W_J.$$

6. TYPE B_n IN THE ASYMPTOTIC CASE

Now consider the group W_n , with generators $S_n = \{t, s_1, \dots, s_{n-1}\}$ as in Section 2. Let $A = \mathbb{Z}[V, V^{-1}, v, v^{-1}]$ where V and v are independent indeterminates and $\Gamma = \{V^i v^j \mid i, j \in \mathbb{Z}\}$ (which is an abelian group under multiplication). We define $\varphi: W_n \rightarrow \Gamma$ by

$$v_t = V \quad \text{and} \quad v_{s_i} = v \quad \text{for } 1 \leq i \leq n-1.$$

In particular, we have $\varphi(w) = v_w = V^{\ell_t(w)} v^{\ell_s(w)}$ for $w \in W_n$. Let \mathcal{H}_n be the corresponding generic Iwahori–Hecke algebra over A , with quadratic relations

$$\begin{aligned} T_t^2 &= T_1 + (V - V^{-1})T_t, \\ T_{s_i}^2 &= T_1 + (v - v^{-1})T_{s_i} \quad \text{for } 1 \leq i \leq n-1. \end{aligned}$$

Hypothesis: *We fix a lexicographic ordering of Γ where*

$$(H) \quad \Gamma_+ = \{V^i v^j \mid i > 0, \text{ any } j\} \cup \{v^j \mid j > 0\}.$$

We have a corresponding Kazhdan–Lusztig basis $\{C_w\}$ in \mathcal{H}_n , corresponding Kazhdan–Lusztig polynomials $P_{y,w}$, and corresponding left cells in W_n . Note that all these depend on the choice of Γ_+ . We begin with the following remark.

Remark 6.1. Let $\mathcal{H}_n(c, d)$ denote the Hecke algebra defined in Example 5.9. Assume in this remark that c/d is sufficiently large. Then the Kazhdan–Lusztig basis $\{C_w\}$ of \mathcal{H}_n specializes (via $V \mapsto v^c$ and $v \mapsto v^d$) to the Kazhdan–Lusztig basis of $\mathcal{H}_n(c, d)$. Similarly, the M -polynomials are obtained by the same specialization. (This follows essentially from the fact that we only have a finite list of Kazhdan–Lusztig polynomials and M -polynomials for our given group W_n . All these polynomials are two-variable Laurent polynomials in v and V . Hence it is clear that if we specialize V and v as above, then the specializations of all those polynomials will remain in $\mathbb{Z}[\Gamma_-]$.)

In particular, the left cells of W_n defined by the ordering (H) are the left cells defined by Example 5.9.

It can be proved that the best bound for “sufficiently large” is $c/d > n - 1$ (a proof of this fact will appear in a forthcoming paper).

6.2. Some properties of the polynomials $M_{y,w}^s$. We will now establish some basic properties of the polynomials $P_{y,w}$ and of the M -polynomials. We set $Q = V^2$ and $q = v^2$. It is known ([13, Prop. 2] and [8, Exercise 11.1]) that $P_{y,w} \in \mathbb{Z}[Q, q]$

and that its constant term is 1. The next theorem gives some consequences of the very particular choice of the ordering (H).

Theorem 6.3. *With the above choice of $\Gamma_+ \subset \Gamma$, the following hold:*

- (a) *For all $y, w \in W_n$ with $y \leq w$, we have $P_{y,w} \in \mathbb{Z}[q]$.*
- (b) *Let $y, w \in W_n$ and $1 \leq i \leq n - 1$ be such that $s_i y < y < w < s_i w$. Then we have $M_{y,w}^{s_i} = 0$ unless $\ell_t(y) = \ell_t(w)$. Furthermore, if $\ell_t(y) = \ell_t(w)$, then $P_{y,w} \in \mathbb{Z}[q]$ has degree at most $(\ell(w) - \ell(y) - 1)/2$ and $M_{y,w}^{s_i} \in \mathbb{Z}$ is the coefficient of $q^{(\ell(w) - \ell(y) - 1)/2}$.*

Proof. Let us make a preliminary remark. By combining the symmetry condition (M2) in the definition 5.4 of M -polynomials with Lemma 5.5 and with our choice for Γ_+ we obtain that $M_{y,w}^{s_i} \in \mathbb{Z}$, for all $y, w \in W_n$ and $1 \leq i \leq n - 1$ such that $s_i y < y < w < s_i w$.

(a) We will proceed by induction. For this purpose, it will be convenient to consider all groups W_n at the same time. Note that we have standard embeddings $W_0 \subset W_1 \subset W_2 \subset W_3 \subset \dots$, where $W_0 = \{1\}$. Furthermore, if $y, w \in W_n$ lie in W_m for some $m < n$, then $P_{y,w}$ computed with respect to W_m is the same as $P_{y,w}$ computed with respect to W_n ; a similar result also holds for the M -polynomials. (This immediately follows, for example, from the recursion formulas for the Kazhdan-Lusztig polynomials in [8, 11.1].) Now let $W := \bigcup_{n \geq 0} W_n$. For a pair (y, w) of elements in W such that $y \leq w$, we set

$$\lambda(y, w) = (n(w), \ell(w) - \ell(y), \ell_t(w), \ell_s(w))$$

where $n(w) := \min\{n \geq 0 \mid w \in W_n\}$. Let \prec be the usual lexicographical ordering of these quadruples. We write $\lambda(y', w') \prec \lambda(y, w)$ if $\lambda(y', w') \leq \lambda(y, w)$ and $\lambda(y', w') \neq \lambda(y, w)$.

Now let (y, w) be a pair of elements in W such that $y \leq w$. If $\lambda(y, w) = (0, 0, 0, 0)$, then $y = w = 1$ and there is nothing to prove. Now assume that $\lambda(y, w) \neq (0, 0, 0, 0)$ and that the assertions hold for all pairs (y', w') of elements in W such that $\lambda(y', w') \prec \lambda(y, w)$. Let $n = n(w)$. First we show that $P_{y,w} \in \mathbb{Z}[q]$. We distinguish several cases.

Case 1. Suppose there exists some $1 \leq i \leq n - 1$ such that $s_i w < w$. Then we can apply the recursion formulas and see that $P_{y,w} \in \mathbb{Z}[q]$ by induction. Note that each term in that formula involves a Kazhdan–Lusztig polynomial or an M -polynomial associated with a pair (y', w') such that $n(w') \leq n(w)$, $\ell(w') - \ell(y') \leq \ell(w) - \ell(y)$, $\ell_t(w') = \ell_t(w)$ and $\ell_s(w') \leq \ell_s(w)$, where at least one inequality is strict. Thus, we have $\lambda(y', w') \prec \lambda(y, w)$.

Case 2. Suppose there exists some $1 \leq i \leq n - 1$ such that $ws_i < w$. Then we use the fact that $P_{y,w} = P_{y',w'}$ where $y' = y^{-1}$ and $w' = w^{-1}$ and that $\lambda(y', w') = \lambda(y, w)$; see (5.3)(c). We have $s_i w' < w'$, so we can apply the argument in Case 1 to conclude that $P_{y,w} \in \mathbb{Z}[q]$.

Case 3. Suppose that $s_i w > w$ and $ws_i > w$ for all $1 \leq i \leq n - 1$. Then the only remaining possibility is that $tw < w$ and $wt < w$. This means that w is a “bi-grassmannian” in the sense of Geck and Kim [7]. So we have $w = a_l$ with $0 \leq l \leq n$ (see [7, Prop. 4.2]). In particular, since we assumed that $n = n(w)$, we must have $l = n$. If $ty > y$, we have $P_{y,w} = P_{ty,w}$ by the recursion formulas for $P_{y,w}$. Hence we can apply induction since

$$\lambda(ty, w) = (n(w), \ell(w) - \ell(y) - 1, \ell_t(w), \ell_s(w)) \prec \lambda(y, w).$$

So we have $P_{y,w} \in \mathbb{Z}[q]$ in this case. Let us now assume that $ty < y$. Using the relation in 5.6(c), we can express $P_{y,w}$ as follows:

$$P_{y,w} = -(-1)^{\ell(w)-\ell(y)}P_{ww_n,yw_n} - \sum_{\substack{z \in W \\ y < z < w}} (-1)^{\ell(w)-\ell(z)}P_{y,z}P_{ww_n,zw_n};$$

here, w_n is again the longest element in W_n . Let (y', w') be a pair of elements in W_n occurring in the above sum; that is, we have $(y', w') = (y, z)$ or $(y', w') = (ww_n, zw_n)$ where $y < z < w$. Then we have $n(w') \leq n(w)$ and $\ell(w') - \ell(y') < \ell(w) - \ell(y)$ and so $\lambda(y', w') \prec \lambda(y, w)$. Thus, by induction, all terms in the above sum lie in $\mathbb{Z}[q]$. So it remains to check if we can also apply induction to the term P_{ww_n,yw_n} . This is indeed so. For, we certainly have $n(yw_n) \leq n(w)$ and $\ell(yw_n) - \ell(ww_n) = \ell(w) - \ell(y)$. Next note that $\ell_t(w) = n$. On the other hand, we have $\ell_t(yw_n) = n - \ell_t(y) < n$ since $ty < y$ by assumption. Thus, we have $\lambda(ww_n, yw_n) \prec \lambda(y, w)$ and so we can apply induction to P_{ww_n,yw_n} .

(b) Let us prove the remaining assertions in (b). We have already seen that $P_{y,w} \in \mathbb{Z}[q]$. Thus, if $\ell_t(y) < \ell_t(w)$, then $vP_{y,w}^*$ is a negative power of V times a Laurent polynomial in v and so $vP_{y,w}^* \in \mathbb{Z}[\Gamma_-]$ by the definition of Γ_+ . Hence we have $M_{y,w}^{s_i} = 0$ in this case. Now assume that $\ell_t(y) = \ell_t(w)$. Since $P_{y,w} \in \mathbb{Z}[q]$, we have $P_{y,w}^* \in \mathbb{Z}[q^{-1}]$ if $\ell(w) - \ell(y)$ is even. This shows that $M_{y,w}^{s_i} = 0$ whenever $\ell(w) - \ell(y)$ is even.

Finally, the fact that $P_{y,w}^* = v^{\ell(y)-\ell(w)}P_{y,w}$ is a polynomial in v^{-1} with constant term zero also implies that $\deg_q(P_{y,w}^*) \leq (\ell(w) - \ell(y) - 1)/2$ and that $M_{y,w}^{s_i}$ is the coefficient of $v^{\ell(w)-\ell(y)-1}$. Thus, (b) is proved. \square

6.4. On the polynomials $M_{y,w}^t$. Our aim here is to prove that, if $y, w \in W_n$ are such that $ty < y < w < tw$ and $\ell_t(y) < \ell_y(w)$, then $M_{y,w}^t = 0$. Before proving this, we need the following lemma.

Lemma 6.5. *Assume that $n \geq 2$ and let $w \in W_n$ satisfying the following four properties:*

- (1) $ws_i > w$ for every $i \in \{1, 2, \dots, n - 1\}$.
- (2) $s_iw > w$ for every $i \in \{2, \dots, n - 1\}$.
- (3) $tw > w$.
- (4) $w \notin W_{n-1}$.

Then $w = r_2r_3 \dots r_n$.

Proof. We set $l = \ell_t(w)$. By (1), we have $w \in X_n^{(l)}$. So there exist $1 \leq i_1 < i_2 < \dots < i_l \leq n$ such that $w = r_{i_1}r_{i_2} \dots r_{i_l}$. By (3), we have that $i_1 \geq 2$. By (2) we have that $i_j - i_{j-1} = 1$ for every $j \in \{1, 2, \dots, l\}$. By (4), we have that $i_l = n$. So $l = n - 1$ and $w = r_2r_3 \dots r_n$. \square

We are now ready to prove the following theorem.

Theorem 6.6. *Let y and w be two elements of W_n such that $ty < y < w < tw$. If $M_{y,w}^t \neq 0$, then $\ell_t(y) = \ell_t(w)$.*

Proof. Assume that the theorem does not hold. Let n be minimal such that there exists y and w in W_n such that $ty < y < w < tw$, $M_{y,w}^t \neq 0$ and $\ell_t(y) < \ell_t(w)$. It is obvious that $n \geq 2$. We choose such a pair (y, w) in such a way that $\ell(w)$ is minimal.

Assume first that there exists $i \in \{1, 2, \dots, n - 1\}$ such that $ws_i < w$. We set $w' = ws_i$. By Theorem 6.3 (b), we have

$$C_w = C_{w'}C_{s_i} + \sum_{x < w' \text{ and } \ell_t(x) = \ell_t(w)} \alpha_x C_x$$

for some $\alpha_x \in \mathbb{Z}[\Gamma]$. Therefore,

$$C_t C_w = C_t C_{w'} C_{s_i} + \sum_{x < w' \text{ and } \ell_t(x) = \ell_t(w)} \alpha_x C_t C_x.$$

But, by minimality of w , $C_t C_{w'}$ (respectively $C_t C_x$) has a nonzero coordinate on C_z only if $\ell_t(z) = \ell_t(w') = \ell_t(w)$ (respectively $\ell_t(z) = \ell_t(x) = \ell_t(w)$) or if $z = tw'$ (or $z = tx$). So, by Theorem 6.3 (b), $C_t C_{w'} C_{s_i}$ has a nonzero coordinate on C_z only if $\ell_t(z) = \ell_t(w') = \ell_t(w)$. Therefore, $M_{y,w}^t \neq 0$ implies that $\ell_t(y) \geq \ell_t(w)$, which is contrary to our hypothesis. So we have:

(1) $ws_i > w$ for every $i \in \{1, 2, \dots, n - 1\}$.

By a similar argument, and using the fact that C_t and C_{s_i} commute if $i \geq 2$, we have:

(2) $s_i w > w$ for every $i \in \{2, \dots, n - 1\}$.

By hypothesis, we have

(3) $tw > w$.

By the minimality of n , we have

(4) $w \notin W_{n-1}$.

Therefore, by (1), (2), (3) and (4) and by Lemma 6.5, we get that $w = r_2 r_3 \dots r_n = s_1 s_2 \dots s_{n-1} r_1 r_2 \dots r_{n-1}$. Now, let us write $y = zy'$ with $y' \in W_{n-1} = W_{S_{n-1}}$ and $z \in X_{S_{n-1}}$. Note that $s_1 s_2 \dots s_{n-1} \in X_{S_{n-1}}$. So, by Theorem 5.11, we have $y' \leq_L r_1 \dots r_{n-1}$. Therefore, by the minimality of n , we get that $\ell_t(y) \geq \ell_t(y') \geq \ell_t(w') = \ell_t(w)$. So $\ell_t(y) = \ell_t(w)$. □

As a consequence of Theorem 6.3 (b) and of Theorem 6.6 we obtain the following statement.

Corollary 6.7. *If y and w are such that $y \leq_L w$, then $\ell_t(y) \geq \ell_t(w)$. If, moreover, $y \sim_L w$, then $\ell_t(y) = \ell_t(w)$.*

7. MAIN RESULT AND CONSEQUENCES

We prove in this section the main result of this paper; namely, the fact that left cells coincide with Robinson-Schensted generalized left cells for our choice of parameters (see Theorem 7.7). Note that Corollary 6.7 and Remark 3.5 are first evidences.

7.1. Some preliminaries. The next results relate some different Kazhdan-Lusztig polynomials by using the decomposition of elements of W_n defined in §4: $w \in W_n$ is written as $w = a_w a_l \sigma_w b_w^{-1}$ (reduced expression), where $l = \ell_t(w)$, $a_w, b_w \in Y_{l,n-l}$, $\sigma_w \in \mathfrak{S}_{l,n-l}$.

Proposition 7.2. *Let x and y be two elements of W_n such that $x \leq y$, $\ell_t(x) = \ell_t(y)$ and $b_x = b_y = b$. Then:*

- (a) $R_{x,y} = R_{xb,yb}$.
- (b) $P_{x,y}^* = P_{xb,yb}^*$.

- (c) $P_{x,y} = P_{xb,yb}$.
- (d) If $s \in S_n$ is such that $sx < x < y < sy$, then $M_{x,y}^s = M_{xb,yb}^s$.

Proof. Note that equalities (a), (b), (c) and (d) for $s \in \Sigma_n$ have a meaning because of Proposition 4.10. It is also easy to check that $tz < z \iff tzb < zb$.

Now let us write $b = s_{i_1} s_{i_2} \dots s_{i_r}$ where $i_j \in \{1, 2, \dots, n-1\}$ and $r = \ell(b)$. Then

$$R_{a_x a_1 \sigma_x, a_y a_1 \sigma_y} = R_{a_x a_1 \sigma_x s_{i_1}, a_y a_1 \sigma_y s_{i_1}} = R_{a_x a_1 \sigma_x s_{i_1} s_{i_2}, a_y a_1 \sigma_y s_{i_1} s_{i_2}} = \dots = R_{x,y}$$

by Lusztig, [17, Lemma 4.4]. This proves (a).

(b) and (d) follow immediately from (a), from Proposition 4.10 (b) and from the fact that $P_{x,y}^*$ and $M_{x,y}^s$ are defined inductively using the polynomials $R_{l,?}$. Moreover, (b) clearly implies (c). \square

Lemma 7.3. *Let x and y be two elements of W_n such that $\ell_t(x) = \ell_t(y)$ and $x \leq_L y$. Then $b_x \leq b_y$. In particular, if $x \sim_L y$, then $b_x = b_y$.*

Proof. By Corollary 6.7, we may assume that there exists some $s \in S_n$ such that C_x appears with a nonzero coefficient in $C_s C_y$. Two cases may occur. If $x \leq y$, then $b_x \leq b_y$ by Proposition 4.10 (a). Otherwise, we have $x = sy > y$. Since $\ell_t(x) = \ell_t(y)$, this implies that $s \neq t$. Therefore, $b_x = b_y$ by Proposition 4.4. \square

Corollary 7.4. *Let x and y be two elements of W_n such that $\ell_t(x) = \ell_t(y)$ and $b_x = b_y = b$. Then the following are equivalent:*

- (1) $x \leq_L y$.
- (2) $xb \leq_L yb$.

Proof. This follows immediately from Propositions 4.10 and 7.2. \square

Corollary 7.5. *Let x and y be two elements of W_n such that $\ell_t(x) = \ell_t(y)$ and $b_x = b_y = b$. Then the following are equivalent:*

- (1) $x \sim_L y$.
- (2) $xb \sim_L yb$.

7.6. Left cells in type B_n . We are now ready to prove the following result:

Theorem 7.7. *Let x and y be two elements of W_n . Then the following are equivalent:*

- (1) $B_n(x) = B_n(y)$, that is, x and y lie in the same generalized RS-cell.
- (2) $x \rightsquigarrow_L y$.
- (3) $x \sim_L y$, that is, x and y lie in the same left cell.

Proof. The equivalence between (1) and (2) has been proved in Theorem 4.8.

First, we prove that (3) implies (2). So we assume that $x \sim_L y$. Then, by Corollary 6.7, we have $\ell_t(x) = \ell_t(y) = l$. Moreover, by Lemma 7.3, we have $b_x = b_y$. Let $b = b_x = b_y$. Then it follows, by Corollary 7.5, that $xb \sim_L yb$. But $xb, yb \in X_n^{(l)} \cdot \mathfrak{S}_{l, n-l}$, and by Theorem 5.11(b) we get that $\sigma_x \sim_L \sigma_y$. Now by [11, §5] and [1], this implies that $\sigma_x \longleftrightarrow_L \sigma_y$. Thus $x \rightsquigarrow_L y$ as desired.

We prove that (2) implies (3) by using a counting argument. We have just seen that each left cell of W_n is contained in a generalized Robinson-Schensted cell, that is

$$\#\{\text{left cells}\} \geq \#\{\text{generalized RS-cells}\}.$$

On the other hand, since any irreducible representation of W_n is realized over \mathbb{Q} , it is then well known that the number of involutions of W_n equals the number of

irreducible direct summands of the left regular representation of W_n . Thus, since the representations carried by the left cells give a direct sum decomposition of the left regular representation, we have

$$\#\{\text{left cells}\} \leq \#\text{involutions in } W_n = \#\{\text{generalized RS-cells}\}.$$

□

7.8. Characters afforded by left cells representations. Let K be the fraction field of A . We are now interested in the representation theory of the K -algebra $\mathcal{H}_n^K = K \otimes_A \mathcal{H}_n$. This algebra is split semisimple. Let \mathfrak{L} be a left cell in W_n and let $x_0 \in \mathfrak{L}$. We denote by $\mathcal{I}_{\leq_L \mathfrak{L}}$ (resp. $\mathcal{I}_{<_L \mathfrak{L}}$) the left ideal of \mathcal{H}_n having $(C_x)_{x \leq_L x_0}$ (resp. $(C_x)_{x <_L x_0}$) as A -basis. We set $\mathcal{V}_{\mathfrak{L}} = \mathcal{I}_{\leq_L \mathfrak{L}}/\mathcal{I}_{<_L \mathfrak{L}}$. This is an \mathcal{H}_n -module which is free over A .

We denote by $A \rightarrow \mathbb{Z}, a \mapsto \hat{a}$ the unique morphism of \mathbb{Z} -algebras such that $\hat{V} = \hat{v} = 1$. Let \mathfrak{a} denote the kernel of this morphism: it is generated by $v - 1$ and $V - 1$. Then $A/\mathfrak{a} \simeq \mathbb{Z}$ and the \mathbb{Z} -algebra $\mathcal{H}_n/\mathfrak{a}\mathcal{H}_n$ may be identified with the group algebra $\mathbb{Z}W_n$. Then $\hat{\mathcal{V}}_{\mathfrak{L}} = \mathcal{V}_{\mathfrak{L}}/\mathfrak{a}\mathcal{V}_{\mathfrak{L}}$ is a $\mathbb{Z}W_n$ -module which is free over \mathbb{Z} . By the computational argument of the proof of Theorem 7.7, we easily get:

Proposition 7.9. *With the above notation we have:*

- (a) *If \mathfrak{L} is a left cell of W_n , then $K \otimes_A \mathcal{V}_{\mathfrak{L}}$ is an irreducible \mathcal{H}_n^K -module and $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathcal{V}}_{\mathfrak{L}}$ is an irreducible $\mathbb{Q}W_n$ -module.*
- (b) *Every irreducible \mathcal{H}_n^K -module (resp. $\mathbb{Q}W_n$ -module) is isomorphic to $K \otimes_A \mathcal{V}_{\mathfrak{L}}$ (resp. $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{\mathcal{V}}_{\mathfrak{L}}$) for some left cell \mathfrak{L} .*

Corollary 7.10. *An \mathcal{H}_n^K -module is constructible if and only if it is isomorphic to $K \otimes_A \mathcal{V}_{\mathfrak{L}}$ for some left cell \mathfrak{L} of W_n .*

Proof. This follows from [4, Prop. 5.2] (for the definition of a constructible module for \mathcal{H}_n^K , the reader may refer to [17, §22]). □

We conclude by giving an explicit description of the irreducible character of \mathcal{H}_n^K (or W_n) defined by a left cell. We need some notation.

We denote by ε_n the sign character of W_n (it is defined by $\varepsilon_n(t) = \varepsilon_n(s_1) = \dots = \varepsilon_n(s_{n-1}) = -1$). If λ is a partition of n , we denote by 1_λ (resp. ε_λ) the restriction to \mathfrak{S}_λ of the trivial character (resp. of ε_n) to \mathfrak{S}_λ . We then define $\chi_\lambda \in \text{Irr}\mathfrak{S}_n$ to be the unique common irreducible component of $\text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1_\lambda$ and $\text{Ind}_{\mathfrak{S}_{\lambda^*}}^{\mathfrak{S}_n} \varepsilon_{\lambda^*}$. We denote by χ_λ^+ the irreducible character of W_n obtained by the composition of χ_λ with the surjective morphism $W_n \rightarrow \mathfrak{S}_n$.

If $0 \leq l \leq n$, we denote by $\theta_{l,n-l}$ the linear character of the normal subgroup $N_n = \langle t_1, t_2, \dots, t_n \rangle$ of W_n such that

$$\theta_{l,n-l}(t_i) = \begin{cases} 1 & \text{if } i \leq l, \\ -1 & \text{if } i \geq l + 1. \end{cases}$$

The stabilizer of $\theta_{l,n-l}$ in \mathfrak{S}_n is $\mathfrak{S}_{l,n-l}$. We denote by $\tilde{\theta}_{l,n-l}$ the linear character of $\mathfrak{S}_{l,n-l}.N_n$ whose restriction to N_n is $\theta_{l,n-l}$ and which is trivial on $\mathfrak{S}_{l,n-l}$. Now, if λ is a partition of n , we denote by χ_λ^- the irreducible character $\chi_\lambda^+ \cdot \tilde{\theta}_{0,n}$ of W_n .

We say that an irreducible character χ of W_n has weight l if $\theta_{l,n-l}$ occurs in $\text{Res}_{N_n}^{W_n} \chi$. By Clifford theory, the weight of an irreducible character of W_n is uniquely determined.

If (λ, μ) is a bipartition of n , we denote by $\chi_\lambda^+ \boxtimes \chi_\mu^-$ the irreducible character of $\mathfrak{S}_{|\lambda|, |\mu|} \cdot N_n \simeq W_{|\lambda|} \times W_{|\mu|}$ obtained by the (external) tensor product of $\chi_\lambda^+ \in \text{Irr}W_{|\lambda|}$ and $\chi_\mu^- \in \text{Irr}W_{|\mu|}$. We then set

$$\chi_{\lambda, \mu}^W = \text{Ind}_{\mathfrak{S}_{|\lambda|, |\mu|} \cdot N_n}^{W_n} (\chi_\lambda^+ \boxtimes \chi_\mu^-).$$

By Clifford theory, $\chi_{\lambda, \mu}^W$ is an irreducible character of W_n and the map $(\lambda, \mu) \mapsto \chi_{\lambda, \mu}^W$ is a bijection between the set of bipartitions of n and $\text{Irr}W_n$. Note that $\chi_{\lambda, \mu}^W$ has weight $|\lambda|$. If λ is a partition of n , then $\chi_{\lambda, 0}^W = \chi_\lambda^+$ and $\chi_{0, \lambda}^W = \chi_\lambda^-$.

Let us now talk about irreducible characters of Hecke algebras. We denote by $\mathcal{H}(\mathfrak{S}_n)$ the sub- A -algebra of \mathcal{H}_n generated by $T_{s_1}, T_{s_2}, \dots, T_{s_{n-1}}$. We set $\mathcal{H}^K(\mathfrak{S}_n) = K \otimes_A \mathcal{H}(\mathfrak{S}_n)$. If λ is a partition of n , we denote by $\chi_\lambda^{\mathcal{H}}$ the unique irreducible character of $\mathcal{H}^K(\mathfrak{S}_n)$ such that $\widehat{\chi_\lambda^{\mathcal{H}}}(T_w) = \chi_\lambda(w)$ for every $w \in \mathfrak{S}_n$. If (λ, μ) is a bipartition of n , we denote by $\chi_{\lambda, \mu}^{\mathcal{H}}$ the unique irreducible character of \mathcal{H}_n such that $\widehat{\chi_{\lambda, \mu}^{\mathcal{H}}}(T_w) = \chi_{\lambda, \mu}^W(w)$ for every $w \in W_n$.

Finally, if \mathfrak{L} is a left cell, we define the *shape* of \mathfrak{L} to be the bipartition (λ, μ) such that $B_n^+(w)$ has shape λ and $B_n^-(w)$ has shape μ for every $w \in \mathfrak{L}$.

Proposition 7.11. *Let \mathfrak{L} be a left cell of shape (λ, μ) . Then the irreducible character of \mathcal{H}_n^K afforded by $K \otimes_A \mathcal{V}_\mathfrak{L}$ is $\chi_{\mu, \lambda}^{\mathcal{H}}$.*

Proof. By Proposition 7.2, we may (and we will) assume that $b_w = 1$ for every $w \in \mathfrak{L}$. We set $l = |\mu|$, so that $|\lambda| = n - l$. We denote by χ the irreducible character of W_n afforded by $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{V}}_\mathfrak{L}$. We denote by $\mathcal{H}_{l, n-l}$ the sub- A -algebra of \mathcal{H}_n generated by $(T_s)_{s \in S_{l, n-l}}$. It is a Hecke algebra for the Weyl group $W_{l, n-l} \simeq W_l \times \mathfrak{S}_{n-l}$. We also set $\mathcal{H}_{l, n-l}^K = K \otimes_A \mathcal{H}_{l, n-l}$.

Let \mathfrak{L}' (resp. \mathfrak{L}'') be the left cell of \mathfrak{S}_l (resp. $\mathfrak{S}_{[l+1, n]}$) such that $\sigma'_w \in \mathfrak{L}'$ (resp. $\sigma''_w \in \mathfrak{L}''$) for every $w \in \mathfrak{L}$ (see Theorem 7.7). If $w \in \mathfrak{L}$, we denote by c_w the image of $C_w \in \mathcal{I}_{\leq \mathfrak{L}}$ in $\mathcal{V}_\mathfrak{L} = \mathcal{I}_{\leq \mathfrak{L}} / \mathcal{I}_{< \mathfrak{L}}$.

Then \mathfrak{L}' has shape μ and \mathfrak{L}'' has shape λ . We set

$$\mathfrak{M} = \{w \in \mathfrak{L} \mid a_w = 1\}.$$

We denote by $M_\mathfrak{L}$ the A -submodule of $\mathcal{V}_\mathfrak{L}$ generated by $(c_w)_{w \in \mathfrak{M}}$. If $w \in \mathfrak{M}$, then $C_w = C_{w_l \sigma'_w} C_{\sigma''_w}$. This shows that $M_\mathfrak{L}$ is a sub- $\mathcal{H}_{l, n-l}$ -module isomorphic to $\mathcal{V}_{w_l \mathfrak{L}'} \boxtimes_A \mathcal{V}_{\mathfrak{L}''}$ through the canonical isomorphism $\mathcal{H}_{l, n-l} \simeq \mathcal{H}_l \otimes_A \mathcal{H}(\mathfrak{S}_{[l+1, n]})$ (note that \mathfrak{L}' is also a left cell of W_l).

But, the character of W_l afforded by $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{V}}_{w_l \mathfrak{L}'}$ is equal to the product of ε_l with the character afforded by $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{V}}_{\mathfrak{L}'}$. Since $C_t C_w = C_{tw}$ for every $w \in \mathfrak{S}_n$, we get that $C_t = T_t + V^{-1}$ acts as 0 on $\mathcal{V}_{\mathfrak{L}'}$. So the character of W_n afforded by $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{V}}_{\mathfrak{L}'}$ is χ_{μ}^- . In other words, the character of W_l afforded by $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathcal{V}}_{w_l \mathfrak{L}'}$ is $\varepsilon_l \cdot \chi_{\mu}^- = \chi_{\mu}^+$. Therefore,

$$(*) \quad \langle \text{Res}_{W_{l, n-l}}^{W_n} \chi, \chi_{\mu}^+ \boxtimes \chi_{\lambda}^* \rangle_{W_{l, n-l}} \neq 0.$$

This shows that the weight of χ is greater than or equal to l .

But, if we use the same argument for the left cell $w_n \mathfrak{L}$ of W_n , we get that the weight of $\varepsilon_n \cdot \chi$ is greater than or equal to $n - l$. Therefore, the weight of χ is equal to l . By (*), χ is an irreducible constituent of weight l of

$$\text{Ind}_{\mathfrak{S}_{l, n-l} \cdot N_n}^{W_n} (\chi_{\mu}^+ \boxtimes (\text{Ind}_{\mathfrak{S}_{n-l}}^{W_{n-l}} \chi_{\lambda}^*)).$$

But, $\chi_{\lambda^*}^-$ is the unique irreducible constituent of $\text{Ind}_{\mathfrak{S}_{n-l}}^{W_{n-l}} \chi_{\lambda^*}$ of weight 0. Therefore, χ is an irreducible constituent of

$$\text{Ind}_{\mathfrak{S}_{l,n-l} \cdot N_n}^{W_n} \left(\chi_{\mu}^+ \boxtimes \chi_{\lambda^*}^- \right).$$

Consequently, $\chi = \chi_{\mu, \lambda^*}^W$. \square

7.12. Final remarks. We conclude by mentioning three further developments, that will make the object of a forthcoming paper.

1. The generalized Robinson-Schensted correspondence described in §3 can be extended to the complex reflection groups $G(e, 1, n) = (\mathbb{Z}/e\mathbb{Z}) \wr \mathfrak{S}_n$ (see [10]). RS-cell representations of the corresponding Ariki-Koike algebra can be constructed as in Proposition 7.11.

2. In the asymptotic case that we have studied here, the Kazhdan-Lusztig basis proves to be a *cellular basis* for \mathcal{H}_n in the sense of Graham-Lehrer [9, Definition 1.1]. We intend to investigate the links between the Kazhdan-Lusztig basis and the cellular basis constructed by Graham and Lehrer [9, §5].

3. We expect that our explicit results will help to prove some of the Lusztig's conjectures on the \mathbf{a} -function [17, §14] in this asymptotic context.

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