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DOUBLE AFFINE HECKE ALGEBRAS AND CALOGERO-MOSER SPACES

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ABSTRACT. In this paper we prove that the spherical subalgebra $eH_{1,\tau}e$ of the double affine Hecke algebra $H_{1,\tau}$ is an integral Cohen-Macaulay algebra isomorphic to the center Z of $H_{1,\tau}$, and $H_{1,\tau}e$ is a Cohen-Macaulay $eH_{1,\tau}e$-module with the property $H_{1,\tau} = \mathrm{End}_{eH_{1,\tau}e}(H_{1,\tau}e)$ when τ is not a root of unity. In the case of the root system A_{n-1} the variety $\mathrm{Spec}(Z)$ is smooth and coincides with the completion of the configuration space of the Ruijenaars-Schneider system. It implies that the module $eH_{1,\tau}$ is projective and all irreducible finite dimensional representations of $H_{1,\tau}$ are isomorphic to the regular representation of the finite Hecke algebra.

INTRODUCTION

In his pioneering paper [1] Cherednik introduced double affine Hecke algebras. They played a crucial role in the proof of Macdonald conjectures [2, 3], and are currently a subject of active research. A double affine Hecke algebra attached to a root system R contains copies of the coweight and weight lattice of R, and thus can be informally viewed (for terminological convenience) as an "elliptic" object. We use the word "elliptic" because the double affine Hecke algebras studied here are closely related to the notion of the elliptic root system introduced by Saito [4, 5]. More precisely, as established in [6], the Hecke algebra associated to the elliptic root system is $H_{1,\tau}$. A double affine algebra has the trigonometric, respectively rational, degeneration, in which one, respectively both, lattices degenerate to a vector space.

The rational degeneration $H_{t,c}$ (called the rational Cherednik algebra) was recently studied in [7]. One of the main results of [7] is that for $t = 0$ the structure of the algebra $H_{t,c}$ has interesting connections with algebraic geometry. More specifically, the results of [7] for $t = 0$ can be summarized as follows.

1. The algebra $H = H_{0,c}$ is finite over its center Z, which is finitely generated. If χ is a generic central character, then the quotient H_χ of H by χ is simple. The unique irreducible representation of H with central character χ, as a representation of $\mathbb{C}[W]$, is isomorphic to the regular representation. Thus any irreducible H-module has dimension $\leq |W|$.

2. Let e be the symmetrizing idempotent in the group algebra of W. Then the natural homomorpism $Z \to eHe$ given by $z \to ze$ is an isomorphism. In particular, eHe is a commutative algebra. In addition, $Z = eHe$ carries a Poisson structure coming from the noncommutative deformation $eH_{t,c}e$ of $eH_{0,c}e$.

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3. Z is an integral Cohen-Macaulay domain. He is a Cohen-Macaulay module over $Z = eHe$, generically of rank $|W|$, and $H = \mathrm{End}_{eHe}(He)$.

4. Suppose R is the root system of the Lie algebra $\mathfrak{gl}_n$. Then the Poisson algebraic variety Spec$Z$ is smooth and symplectic. This variety is naturally isomorphic (as a symplectic variety) to the Calogero-Moser space CM introduced in [8]—the space of conjugacy classes of pairs (X, Y) of n-by-n matrices such that the matrix $[X, Y] + 1$ has rank 1, with the symplectic structure coming from the reduction procedure of [8] (this result depends on Wilson's theorem that the Calogero-Moser space is connected). In particular, 1 holds for any (not only generic) character χ; the module He over Z considered in 3 is projective, and thus corresponds to a vector bundle E over CM of dimension $n!$, and H is the endomorphism algebra of this vector bundle. Thus H in this case is an Azumaya algebra.

The Calogero-Moser space appearing in 4 was introduced in [8] as a completed configuration space of the Calogero-Moser classical integrable system. It recently found itself in the center of attention due to its interpretation as a deformation of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ [9] and as a noncommutative Hilbert scheme [10, 11]. In fact, the commutative analog of the vector bundle E (which is a vector bundle over $\mathrm{Hilb}_n(\mathbb{C}^2)$) is closely related to the $n!$ conjecture proved recently by Haiman.

The goal of this paper is to generalize the results 1–4 to the triginometric and elliptic cases. More specifically, we propose a modification of the approach of [7], in which all three cases (rational, trigonometric, and elliptic) can be treated uniformly. In fact, we treat mostly the elliptic case; the other two are analogous, and are discussed at the end in Section 7.

The structure of the paper is as follows.

In Section 1 we define the main characters of the paper—the double affine Hecke algebra $H_{q,\tau}$ for the root system of $\mathfrak{gl}_n$, and the corresponding Calogero-Moser space CM_τ, which is a completed configuration space of the Ruijsenaars-Shneider (RS) integrable system. Similarly to the rational case, this space should have an interpretation as a deformation and noncommutative version of the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^* \times \mathbb{C}^*)$ (in the trigonometric case $\mathbb{C}^* \times \mathbb{C}^*$ should be replaced with $\mathbb{C} \times \mathbb{C}^*$).

In Section 2, we prove that CM_τ is smooth and carries a symplectic structure (after [12, 13]); this symplectic structure can also be obtained by Quasi-Poisson reduction [14]. We also generalize Wilson's theorem by proving that CM_τ is connected.

In Section 3, we study the representation theory of $H_{q,\tau}$ for $q = 1$.

In Section 4, to every representation of $H_{1,\tau}$ which is regular as a representation of the finite Hecke algebra sitting in $H_{1,\tau}$, we attach a point on the space CM_τ.

In Section 5, we study the properties of the double affine Hecke algebra H for any root system R; in particular, we prove that the results 1–3 from [7] cited above hold in the elliptic case, with the group algebra $\mathbb{C}[W]$ replaced by the finite Hecke algebra $\mathbb{C}_\tau[W]$.

In Section 6, we use the results of Sections 2, 3 and 4 to prove the elliptic analogs of the results from [7] under item 4 above. Namely, we establish a symplectic isomorphism of the spectrum of the center Z of $H = H_{1,\tau}$ for $\mathfrak{gl}_N$ with the space CM_τ, which is the main result of the paper. In particular, Spec(Z) is smooth, and He is a vector bundle on it, such that the fibers are the regular representation of the finite Hecke algebra. Thus, H is the endomorphism algebra of this vector bundle, i.e. an Azumaya algebra.

In Section 7, we treat the rational and trigonometric case.

In a later publication, we plan to generalize the results of this paper to the case of the non-reduced root system $C^\vee C_n$. In this case, instead of a single parameter τ one has five independent parameters.

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1. Definitions

1.1. **Definition of the double affine Hecke algebra corresponding to** $GL(n,\mathbb{C})$. We denote this algebra by the symbol $H_{q,\tau}$. It is generated by the elements T_i, $1 \le i \le n-1$, π, $X_i^{\pm 1}$, $1 \le i \le n$ with relations

$$X_iX_j = X_jX_i, \quad (1 \le i,j \le n), \tag{1}$$

$$T_iX_iT_i = X_{i+1}, \quad (1 \le i < n), \tag{2}$$

$$T_iX_j = X_jT_i, \text{ if } j-i \neq 0,1 \tag{3}$$

$$[T_i,T_j] = 0, \text{ if } |i-j| > 1 \tag{4}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, (1 \le i < n), \tag{5}$$

$$\pi X_i = X_{i+1}\pi \quad (1 \le i \le n-1), \quad \pi X_n = q^{-1}X_1\pi, \tag{6}$$

$$\pi T_i = T_{i+1}\pi, (1 \le i \le n-2), \quad \pi^n T_i = T_i\pi^n, \quad (1 \le i \le n-1), \tag{7}$$

$$(T_i-\tau)(T_i+\tau^{-1}) = 0, \quad (1 \le i \le n). \tag{8}$$

Remark 1.1. To identify this definition with the standard definition from the papers of Cherednik one should replace τ by $t^{\frac{1}{2}}$ and q by $q^{\frac{1}{2}}$. Also, some definitions use the element $T_0 = \pi T_{n-1}\pi^{-1}$.

Remark 1.2. The double affine Hecke algebra corresponding to $SL(n,\mathbb{C})$ is a quotient of the subalgebra of $H_{q,\tau}$ generated by X_i/X_{i+1}, T_i, π, $1 \le i \le n-1$, by one extra relation:

$$\pi^n = 1.$$

1.2. **Definition of the Calogero-Moser space.** Let E be an n-dimensional vector space (over $\mathbb{C}$). We denote by the symbol CM'_τ the subset of $GL(E)\times GL(E)\times E\times E^*$ consisting of the elements (X,Y,U,V) satisfying the equation

$$X^{-1}Y^{-1}XY\tau^{-1} - \tau = U \otimes V. \tag{9}$$

Obviously it is an affine variety.

The group $GL(n,\mathbb{C}) = GL(E)$ acts on it by conjugation:

$$(X,Y,U,V) \to (gXg^{-1}, gYg^{-1}, gU, Vg^{-1}), \quad g \in GL(E).$$

Later we will show that this action is free if $\tau^{2i} \neq 1$ for $i = 1,\dots,n$. So the naive quotient by the action (i.e. the spectrum of the ring of $GL(E)$ invariant functions) yields an affine variety, and the quotient is nonsingular if CM'_τ is.

Definition. The quotient of CM'_τ by the action $GL(E)$ is called the Calogero-Moser space. We use the notation CM_τ for this space.

Below we always suppose that $\tau^{2i} \neq 1$ for $i = 1, \dots, n$.

2. Properties of the Calogero-Moser space

The goal of this section is to prove that CM_τ is a smooth irreducible algebraic variety of dimension $2n$. We also introduce coordinates on its dense subset. The methods of this section are analogous to the ones from the paper [10]. In principle, smoothness of CM_τ follows from the results of the paper [13], the authors of [13] use the moduli space of the vector bundles on the punctured torus. For convenience of the reader we give a direct elementary proof.

2.1. **Smoothness of the Calogero-Moser space.** First we prove a simple lemma on which all the following statements are based.

Lemma 2.1. *If* $(X, Y, U, V) \in CM'_\tau$ *and* $[A, X] = [A, Y] = 0$, $A \in \mathfrak{gl}(E)$, *then* $A = \lambda Id$ *for some* $\lambda \in \mathbb{C}$.

Proof. Let $W \subset E$ be a nonzero subspace which is invariant under the action of X, Y and A. We denote by $\bar{X}$ and $\bar{Y}$ the restriction of the operators X, Y to this subspace. It follows from equation (9) that there are two possibilities.

In the first case $W \subset V^\perp$, where $V^\perp$ is the notation for the annihilator. In this case (9) implies

$$\bar{X}^{-1}\bar{Y}^{-1}\bar{X}\bar{Y} = \tau^2 Id.$$

But the determinant of LHS is equal to 1, hence we get a contradiction.

In the second case $W \not\subseteq V^\perp$, $U \in W$. In this case (9) implies

$$\bar{X}^{-1}\bar{Y}^{-1}\bar{X}\bar{Y} - \tau U\bar{V} = \tau^2 Id,$$

where $0 \neq \bar{V}$ is the restriction of V to the subspace W. Since $\det(\bar{X}\bar{Y}\bar{X}^{-1}\bar{Y}^{-1}) = 1$, the last equation implies that there is a basis in W in which $\bar{X}^{-1}\bar{Y}^{-1}\bar{X}\bar{Y}$ is diagonal with the spectrum $\tau^2, \tau^2, \dots, \tau^2, \tau^{2-2k}$ where $k = \dim W$. But we know from equation (9) that the spectrum of $X^{-1}Y^{-1}XY$ is equal to $\tau^2, \tau^2, \dots, \tau^{2-2n}$. Thus we get $W = E$.

The fact that the only common nonzero invariant subspace of X, Y and A is the whole E immediately implies the statement of the lemma. Indeed, let λ be an eigenvalue of A, then the corresponding eigenspace W_λ is invariant under the action of X and Y, hence it coincides with E. □

Corollary 2.1. *The action of* $GL(E)$ *on* CM'_τ *is free.*

Lemma 2.2. CM'_τ *is smooth.*

Proof. Let us introduce the map Ψ: $GL(E) \times GL(E) \times E \times E^* \to \mathfrak{gl}(E)$:

$$\Psi(X, Y, U, V) = X^{-1}Y^{-1}XY - \tau U \otimes V.$$

It is enough to show that $d\Psi$ is epimorphic at a point $(X, Y, U, V) \in CM'_\tau$. Let $x, y \in \mathfrak{gl}(E)$, $u \in E, v \in E^*$ and $X(t) = Xe^{xt}$, $Y(t) = Ye^{yt}$, $U(t) = U + tu$,

$V(t) = V + tv$. Then

$$\begin{aligned} d\Psi_{(X,Y,U,V)}(x,y,u,v) &= \frac{d}{dt}(\Psi(X(t),Y(t),U(t),V(t))|_{t=0} \\ &= -xX^{-1}Y^{-1}XY + X^{-1}Y^{-1}XxY - X^{-1}yY^{-1}XY \\ &\quad + X^{-1}Y^{-1}XYy - \tau U\otimes v - \tau u\otimes V. \end{aligned}$$

If $d\Psi$ is not an epimorphism, then there exists $0\neq A\in\mathfrak{gl}(E)$ such that

$$tr(d\Psi_{(X,Y,U,V)}(x,y,u,v)A) = 0$$

for all $x, y \in \mathfrak{gl}(E)$, $u\in E, v\in E^*$. Using the cyclic invariance of the trace, we can rewrite the last condition in the form

$$\begin{aligned} &tr(x(YAX^{-1}Y^{-1}X - X^{-1}Y^{-1}XYA)) \\ &\qquad + tr(y(AX^{-1}Y^{-1}XY - Y^{-1}XYAX^{-1})) - \tau v(AU) - \tau VA(u) = 0. \end{aligned}$$

As the bilinear form $tr(xy)$ is nondegenerate, the last equation implies

$$YAX^{-1}Y^{-1}X - X^{-1}Y^{-1}XYA = 0, \tag{10}$$

$$AX^{-1}Y^{-1}XY - Y^{-1}XYAX^{-1} = 0, \tag{11}$$

$$AU = 0, \quad VA = 0. \tag{12}$$

These equations together with equation (9) imply $[A,X] = [A,Y] = 0$. Indeed, let us derive the first equation.

Multiplying on the right formula (9) by A we get

$$X^{-1}Y^{-1}XYA = \tau^2 A. \tag{13}$$

Hence

$$\tau^2 XAX^{-1} = Y^{-1}XYAX^{-1} = AX^{-1}Y^{-1}XY = A(\tau U\otimes V + \tau^2 Id) = \tau^2 A,$$

here the first equation uses (13), second (11), third (9) and fourth (12).

By the previous lemma $A = \lambda Id$ and finally from (12) we get $A = 0$. □

Corollary 2.2. *CM_τ is smooth algebraic variety, and all its irreducible components have dimension $2n$.*

2.2. **Local coordinates on CM_τ.** It is easy to see that matrices $X, Y \in \mathfrak{gl}(n,\mathbb{C})$,

$$X = \mathrm{diag}(\lambda_1,\ldots,\lambda_n), \tag{14}$$

$$Y_{ii} = q_i, \quad i = 1,\ldots,n, \tag{15}$$

$$Y_{ij} = \frac{(\tau^{-1}-\tau)q_i\lambda_j}{(\tau^{-1}\lambda_i - \tau\lambda_j)}, \quad 1\le i\neq j\le n, \tag{16}$$

satisfy the equation

$$rk(\tau^{-1}XY - \tau YX) = 1, \tag{17}$$

for all $\lambda\in(\mathbb{C}^*)^n$, $q\in(\mathbb{C}^*)^n$ such that $\tau\lambda_i\neq\tau^{-1}\lambda_j$ for $i\neq j$.

There is a well-known formula: if $M = (M_{ij})$, where $M_{ij} = (\lambda_i - \mu_j)^{-1}$, $1\le i,j\le n$, then

$$\det(M) = \frac{\prod_{i<j}(\lambda_i-\lambda_j)(\mu_j-\mu_i)}{\prod_{i,j}(\lambda_i-\mu_j)}.$$

To prove this formula one can proceed by the induction on n using the Gaussian method of calculation of the determinant for the step of the induction.

Applying the last formula to the matrix Y we see that $\det(Y)$ is nonzero if and only if $\lambda_i \neq \lambda_j$, $i \neq j$.

Let us denote by π'_{12}: $CM'_\tau \to GL(E) \times GL(E)$ the projection on the first two coordinates. The previous reasoning shows that $(X, Y) \in \pi'_{12}(CM'_\tau)$, for $\lambda \in (\mathbb{C}^*)^n \setminus D_\tau$, $q \in (\mathbb{C}^*)^n$ where

$$D_\tau = \{\lambda | \delta_\tau(\lambda) = \prod_{i \neq j} (1 - \lambda_i/\lambda_j)(\tau^{-1} - \tau\lambda_i/\lambda_j) = 0\}.$$

Now we can state

Proposition 2.1. *Let $(X, Y, U, V) \in CM'_\tau$ and X be diagonalizable with the distinct eigenvalues λ_i, $i = 1, \ldots, n$ such that $\tau\lambda_i \neq \tau^{-1}\lambda_j$. Then the $GL(n, \mathbb{C})$ orbit of (X, Y, U, V) contains a representative satisfying equations $V = \lambda^t$ and (14)–(16) for some $q \in (\mathbb{C}^*)^n$. Such a representative is unique up to (simultaneous) permutation of the parameters (λ_i, q_i).*

Proof. Equation (17) is equivalent to the system

$$\frac{(\tau^{-1}\lambda_i - \tau\lambda_j)Y_{ij}}{\tau^{-1} - \tau} = p_i s_j, \quad 1 \leq i, j \leq n, \tag{18}$$

if $X = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. If there exists i such that $s_i = 0$, then $Y_{ij} = 0$, $j = 1, \ldots, n$ and $\det(Y) = 0$. Thus we have $s_i \neq 0$. Analogously we get $p_i \neq 0$.

Let us fix a solution of (18) lying in the $GL(n, \mathbb{C})$ orbit of (X, Y, U, V). Putting $q_i = p_i s_i/\lambda_i$ we get the desired representative with X given by formula (14), Y by formulas (15) and (16), and $U = (\tau^{-1} - \tau)X^{-1}Y^{-1}q$. □

Let us denote by $\mathbf{U}' \subset CM'_\tau$ the subset consisting of the quadruples (X, Y, U, V) satisfying the conditions of the previous proposition and by $\mathbf{U} \subset CM_\tau$ the image of $\mathbf{U}'$ under the factorization by the action of $GL(n, \mathbb{C})$. The proposition together with Corollary 2.2 implies that (λ, q) are local coordinates on the open subset $\mathbf{U} \subset CM_\tau$. In the next section we show that this subset is dense.

2.3. Irreducibility of CM_τ.

In this subsection we prove

Proposition 2.2. *The variety CM_τ is irreducible.*

Let us consider the projection on the first component π'_1: $CM'_\tau \to GL(E)$. After taking the quotient by the action of $GL(E)$, this map becomes a map π_1: $CM_\tau \to JNF$, where JNF is a stack, but we can think about it as the set of Jordan normal forms of matrices (we do not need the stack structure).

Inside JNF there is an open part $\tilde{U}$ corresponding to diagonal matrices with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ such that $\lambda_i \neq \lambda_j$, $\tau^{-1}\lambda_i \neq \tau\lambda_j$ for $i \neq j$. The subset $\pi_1^{-1}(\tilde{U})$ was described in the previous section. It is obviously connected. If we show that $\dim \pi_1^{-1}(JNF \setminus \tilde{U}) < 2n$, then Corollary 2.2 implies the irreducibility.

Let us denote by $J_k(\lambda)$ the Jordan block of size k with the eigenvalue λ and by the symbol $J_{\vec{k}}(\lambda)$ the matrix $\mathrm{diag}(J_{k^1}(\lambda), \ldots, J_{k^t}(\lambda))$, $\vec{k} \in \mathbb{N}^t$ and $k^i \geq k^{i+1}$, $i = 1, \ldots, t-1$. Let us formulate without a proof an elementary statement from linear algebra.

Lemma 2.3. *The dimension of*

$$\mathrm{Stab}(J_{\vec{k}}(\lambda)) = \{X \in GL(n, \mathbb{C}) | [X, J_{\vec{k}}(\lambda)] = 0\}$$

is equal to $\sum_{1 \leq i,j \leq t} \min\{k^i, k^j\}$.

Let us denote by $J_{\mathbf{k}}(\lambda)$ the matrix

$$\mathrm{diag}(J_{\vec{k_1}}(\lambda), J_{\vec{k_2}}(\lambda\tau^{-2}), \dots, J_{\vec{k_r}}(\lambda\tau^{-2r})),$$

$\vec{k_i} \in \mathbb{N}^{t_i}$. We use the notation $|\vec{k_i}| = \sum_{j=1}^{t_i} k_i^j$, $|\mathbf{k}| = \sum_{j=1}^{r} |\vec{k_j}|$.

Let $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ be such that $\lambda_i/\lambda_j \neq \tau^{2c}$, $c \in \mathbb{Z}$, $|c| \leq n$ and

$$J = \mathrm{diag}(J_{\mathbf{k}_1}(\lambda_1), \dots, J_{\mathbf{k}_s}(\lambda_s)). \tag{19}$$

We denote by π'_{34}: $CM'_\tau \to \mathbb{C}^n \times \mathbb{C}^n$ the slightly modified projection on the last two components: $\pi'_{34}(X, Y, U, V) = (YXU, V)$. The fiber of the map π'_{34} over the point (U, V) of the subset $\hat{J} = \pi'_{34}((\pi'_1)^{-1}(J))$ consists of the points $(J, Y + F, J^{-1}(Y + F)^{-1}U, V)$ where F is an element of the kernel of the linear map:

$$S_J(F) = \tau^{-1}JF - \tau FJ, \quad F \in \mathfrak{gl}(E),$$

$Y + F$ is invertible, and $(J, Y, J^{-1}Y^{-1}U, V) \in CM'_\tau$. Obviously $(\pi'_{34})^{-1}(U, V)$ is a Zariski open nonempty subset inside $\ker(S_J)$ hence they have the same dimension.

First let us study the map S_J in the simple case when in the equation (19) we have $s = 1$ and $\mathbf{k}_1 = \mathbf{k} = (\vec{k}_1, \dots, \vec{k}_r)$, $\vec{k_i} \in \mathbb{N}^{d_i}$, $1 \leq i \leq r$. In this situation we denote by $F_{ij}^{st} \in \mathrm{Mat}(k_s^i, k_t^j)$, $1 \leq s, t \leq r$ the matrix with the entries $F_{ij;pq}^{st} = F_{p'q'}$, $p' = \sum_{l=1}^{s-1} |\vec{k_l}| + \sum_{l=1}^{i-1} k_s^l + p$, $q' = \sum_{l=1}^{t-1} |\vec{k_l}| + \sum_{l=1}^{j-1} k_s^l + q$. In these notations the following lemma holds

Lemma 2.4. *Let J be the matrix given by* (19) *with $s = 1$ and $\mathbf{k}_1 = \mathbf{k} = (\vec{k}_1, \dots, \vec{k}_r)$. Then $F \in \ker S_J$ if and only if*

$$F_{ij}^{st} = 0, \text{ if } t - s \neq 1, \tag{20}$$

$$F_{ij}^{s,s+1} = \Big(\sum_{l=0}^{k_s^i - 1} c_{ij;l}^s J_{k_s^i}^l(0)\Big) D_\tau^{k_s^i, k_{s+1}^j} \text{ if } k_s^i \leq k_{s+1}^j, \tag{21}$$

$$F_{ij}^{s,s+1} = D_\tau^{k_s^i, k_{s+1}^j} \Big(\sum_{l=0}^{k_{s+1}^j - 1} c_{ij;l}^s J_{k_{s+1}^j}^l(0)\Big) \text{ if } k_s^i > k_{s+1}^j, \tag{22}$$

where $c_{ij;l}^s \in \mathbb{C}$, $J_{k_s^i}^l(0)$ (and $J_{k_{s+1}^j}^l(0)$) is the l-th power of the Jordan block matrix, and $D^{k_s^i, k_{s+1}^j} \in \mathrm{Mat}(k_s^i, k_{s+1}^j)$ is given by the formula

$$D_{\tau;pq}^{k_s^i, k_{s+1}^j} = \delta_{p + k_{s+1}^j, q + k_s^i} \tau^{2p-2} \text{ if } k_s^i \leq k_{s+1}^j,$$

$$D_{\tau;pq}^{k_s^i, k_{s+1}^j} = \delta_{p,q} \tau^{2p-2} \text{ if } k_s^i > k_{s+1}^j.$$

Proof. The system of linear equations $S_J(F) = 0$ is equivalent to the collection of linear systems:

$$\tau J_{k_s^i}(\lambda\tau^{2-2s}) F_{ij}^{st} - \tau^{-1} F_{ij}^{st} J_{k_t^j}(\lambda\tau^{2-2t}) = 0, \quad 1 \leq s, t \leq r,$$

because J has a block structure. The equations for the entries of F_{ij}^{st} are of the simple form:

$$F_{ij;pq}^{st} \lambda(\tau^{1-2s} - \tau^{3-2t}) = \tau^{-1}(\delta_{p,k_i^s} - 1) F_{ij;p+1,q}^{st} - \tau(\delta_{q,1} - 1) F_{ij;p,q-1}^{st}. \tag{23}$$

First consider the case $t - s \neq 1$. Then $\tau^{1-2s} - \tau^{3-2t} \neq 0$ and equations (23) express the entries of l-th diagonal through the entries of $(l-1)$-th diagonal. It easy to see that in this case (23) implies $F^{st}_{ij;k^i_s,1} = 0$, that is, the first diagonal is zero. Moving from the left to the right we get that all the diagonals of F^{st}_{ij} are zero.

If $s+1 = t$, then equation (23) is a linear relation between the neighboring entries on the diagonal. It is easy to derive equations (21), (22) from this fact.

Indeed, let us consider the case $k^i_s \leq k^j_{s+1}$. Then equation (23) for $p = k^s_i$, $1 < q \leq k^s_i$ says $F^{s,s+1}_{ij;k^s_i,q-1} = 0$. Moving along the diagonal from the bottom to the top and using equation (23) we get that the first $k^s_i - 1$ diagonals of the matrix $F^{s,s+1}_{ij}$ are zero. For the rest of the diagonals equation (23) implies $F^{s,s+1}_{ij;p+1,q+p} = F^{s,s+1}_{ij;1,q}\tau^{2p}$. Putting $c^s_{ij;l} = F^{s,s+1}_{ij;1,l+k^j_{s+1}-k^i_s+1}$ we get equation (21). □

Obviously $Z \in \operatorname{Im} S_J$ if and only if $tr(ZF) = 0$ for all $F \in \ker \bar{S}_J$, $\bar{S}_J(F) = \tau JF - \tau^{-1}FJ$. The space $\ker \bar{S}_J$ has a description similar to the one of $\ker S_J$ (to get $\ker \bar{S}_J$ from $\ker S_J$ it is enough to change the order of the Jordan blocks in J) and one can easily derive

Corollary 2.3. *$Z \in \operatorname{Im} S_J$ if and only if following equations hold*

$$\sum_{l=0}^{u-1} Z^{s,s+1}_{ij;k^i_s-l,u-l}\tau^{2l} = 0, \quad u = 1, \dots, \min\{k^i_s, k^j_{s+1}\},$$

where $s = 1, \dots, r-1$.

The lowest nonzero diagonal of a rank one matrix contains only one nonzero entry. As $\hat{J} \subset \operatorname{Im} S_J \cap \{\text{matrices of rank 1}\}$ the following statement holds

Corollary 2.4. *$(U, V) \in \hat{J} = \pi'_{34}((\pi'_1)^{-1}(J))$ if and only if $Z = U \otimes V$ satisfies the equation*

$$Z^{s,s+1}_{ij;pq} = 0 \text{ if } p - q \geq \min\{0, k^i_s - k^j_{s+1}\}, \quad s = 1, \dots, r-1.$$

Lemma 2.4 gives us the formula for the dimension of the kernel

$$\dim \ker S_J = \sum_{s=1}^{r-1} \sum_{i,j} \min\{k^i_s, k^j_{s+1}\}.$$

We know that $GL(E)$ acts on CM'_τ freely. Hence if we want to estimate the dimension of the fiber of π_{34} over $\hat{J}$, we should estimate $\dim \operatorname{Stab}(J) - \dim \ker S_J$. This difference is positive:

Lemma 2.5. *Let $k_s \in \mathbb{N}^{d_s}$, $s = 1, \dots, r$, $k^i_s \geq k^{i+1}_s$, then the following inequality holds:*

$$\sum_{s=1}^{r} \sum_{i,j} \min\{k^i_s, k^j_s\} - \sum_{s=1}^{r-1} \sum_{i,j} \min\{k^i_s, k^j_{s+1}\} > 0,$$

if there exists s such that $k_s \neq 0$.

Proof. Because of the inequality $k^i_s \geq k^{i+1}_s$ we can rewrite LHS of the inequality in the form

$$\sum_{\nu=1} \left(\sum_{s=1}^{r} (x^\nu_s)^2 - \sum_{s=1}^{r-1} x^\nu_s x^\nu_{s+1} \right),$$
$$x^\nu_s = \#\{i \in \mathbb{N} | k^i_s \geq \nu\}.$$

But the first expression is a sum of positive definite quadratic forms. Thus we get the lemma. □

The following statement is crucial for estimating $\dim(\pi_1^{-1}(JNF \setminus \tilde{U}))$:

Proposition 2.3. *If J is given by* (19) *with $s = 1$ and $\mathbf{k}_1 = \mathbf{k} = (\vec{k}_1, \dots, \vec{k}_r)$, then $\dim \pi_1^{-1}(J) < 2n - 1$ when either $r > 1$ or $k_1^1 > 1$.*

Proof. In the case $r > 1$, Corollary 2.4 implies that $\dim \pi'_{34}((\pi'_1)^{-1}(J)) \leq 2n - 1$. The theorem on the dimension of the fibers and previous reasoning imply:

$$\dim \pi_1^{-1}(J) \leq \dim \pi'_{34}({\pi'}_1^{-1}(J)) + \dim \ker S_J - \dim \operatorname{Stab}(J).$$

Together with the inequality from Lemma 2.5 it proves the statement.

Another case (i.e. $\mathbf{k} = \vec{k}_1$) is even easier because in this case we have

$$\dim \pi_1^{-1}(J) \leq 2n - \dim \operatorname{Stab}(J) < 2n - 1.$$

□

The case in formula (19) $s > 1$, can be easily reduced to the previous case. For that let us introduce the embedding i_l: $\mathfrak{gl}(|\mathbf{k}_l|, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ and the projection pr_l: $\mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(|\mathbf{k}_l|, \mathbb{C})$: $i_l(Y)_{p',q'} = Y_{pq}$, $pr_l(Y)_{pq} = Y_{p',q'}$, $p' = p + \sum_{m=1}^{l-1} |\vec{k}_m|$, $q' = q + \sum_{m=1}^{l-1} |\vec{k}_m|$, $0 \leq p, q \leq |\vec{k}_l|$, and $i_l(Y)_{ij} = 0$ for the rest of the entries of $i_l(Y)$.

Using arguments analogous to the ones from Lemma 2.4 one gets

Lemma 2.6. *Let J be given by formula* (19)*. Then*

(1) $\ker S_J = \bigoplus_{l=1}^{s} i_l(\ker S_{J_{\mathbf{k}_l}}(\lambda_i))$,
(2) *for $l = 1, \dots, s$, $pr_l(\operatorname{Im} S_J) \subset \operatorname{Im} S_{J_{\mathbf{k}_l}}$.*

This lemma immediately implies

Proposition 2.4. *Let J be given by formula* (19)*, then there exists l, $1 \leq l \leq s$ such that $|\mathbf{k}_l| > 1$. Then $\dim \pi_1^{-1}(J) < 2n - s$.*

Thus we eventually achieve the goal of the subsection:

Proof of Proposition 2.2. Indeed Proposition 2.4 implies $\dim \pi_1^{-1}(JNF \setminus \tilde{U}) < 2n$. Hence by Corollary 2.2 $\pi_1^{-1}(JNF \setminus \tilde{U})$ lies inside the Zariski closure of $\pi_1^{-1}(\tilde{U})$. But $\pi_1^{-1}(\tilde{U})$ is irreducible. □

2.4. **The Poisson structure on the CM space.** In the paper [13] the Poisson structure on the space CM_τ was constructed. This Poisson structure on CM_τ yields the RS integrable system which is the relativistic analog of the trigonometric Calogero-Moser system.

On the open part $\mathbf{U}$ of CM_τ described in subsection 2.2 the Poisson bracket $\{\cdot, \cdot\}_{FR}$ takes the form (see Appendix of [13] for the proof):

$$\{\lambda_i, \lambda_j\}_{FR} = 0, \quad \{\lambda_j, q_i\}_{FR} = \lambda_i q_i \delta_{ij},$$
$$\{q_i, q_j\}_{FR} = \frac{(\tau^{-1} - \tau)^2 q_i q_j (\lambda_i + \lambda_j) \lambda_i \lambda_j}{(\tau^{-1}\lambda_i - \tau\lambda_j)(\tau^{-1}\lambda_j - \tau\lambda_i)(\lambda_i - \lambda_j)}.$$

Remark 2.1. The formulas in [13] contain a misprint, the authors lost the factor $(\tau^{-2} - 1)^2$ in the expression for $\{q_i, q_j\}_{FR}$.

Using the Hamiltonian reduction on the combinatorial model of the space of flat connections on the torus without a point, the authors of [13] prove that the Poisson structure $\{\cdot,\cdot\}_{FR}$ has a holomorphic extension from $\mathbf{U}$ to the whole CM_τ, and this Poisson structure is nondegenerate (i.e. CM_τ is a symplectic variety). Another way to see this Poisson structure is to use Quasi-Poisson reduction [14]. In this picture the Poisson structure is the result of the reduction of the natural Quasi-Poisson structure on the product $GL(n,\mathbb{C})\times GL(n,\mathbb{C})$ and it is immediate that this Poisson structure is symplectic.

3. Finite dimensional representations of $H_{1,\tau}$

In this subsection we construct a family of finite dimensional representations of $H_{1,\tau}$. Later we will show that this family forms an open dense set inside the space of all finite dimensional representations. The main tool of this section is the faithful representation of $H_{1,\tau}$ which is the quasiclassical limit of the standard realization of $H_{q,\tau}$ as a subring of the ring of reflection difference operators [2].

3.1. **Limit of the Lusztig-Demazure operators.** Let us introduce the ring $\tilde{R}=\mathbb{C}[P_1^{\pm1},\dots,P_n^{\pm1},X_1^{\pm1},\dots,X_n^{\pm1}]_{\delta(X)}\#S_n$, where the subscript $\delta(X)$ means localization by the ideal generated by $\delta(X)=\prod_{1\le i<j\le n}(1-X_i/X_j)$ and $\#$ is a notation for the smash product. Let us explain what the smash product is. For brevity we will use the notation $\mathbb{C}[P^{\pm1},X^{\pm1}]$ for the ring $\mathbb{C}[P_1^{\pm1},\dots,P_n^{\pm1},X_1^{\pm1},\dots,X_n^{\pm1}]$.

An element of the ring $\tilde{R}$ has the form $\sum_{w\in S_n}F_w(P,X)w$. The group S_n acts on the ring $R=\mathbb{C}[P^{\pm1},X^{\pm1}]_{\delta(X)}$ by the formulas

$$P_i^w=P_{w(i)},\quad X_i^w=X_{w(i)},$$

and

$$F(P,X)wF'(P,X)w'=F(P,X)(F')^w(P,X)ww'.$$

Proposition 3.1. [3] *The following formulas give an injective homomorphism of $H_{1,\tau}\to\tilde{R}$:*

$$X^\mu\mapsto X^\mu,$$

$$T_i\mapsto\tau s_i+\frac{\tau-\tau^{-1}}{X_i/X_{i+1}-1}(s_i-1),\quad i=1,\dots,n-1,$$

$$\pi\mapsto P_1^{-1}c,$$

where $s_i=(i,i+1)\in S_n$ is a transposition and $c\in S_n$ is a cyclic transformation: $c(i)=i+1, i=1,\dots,n-1$, $c(n)=1$.

The homomorphism from the proposition is a quasiclassical limit of the Lusztig-Demazure representation from Theorem 2.3, [3]. For brevity we call this homomorphism the Lusztig-Demazure representation.

Remark 3.1. Actually the paper [3] contains the proof for the case $q\neq1$ but leading term considerations used in the paper could be adapted for the case $q=1$. For example, one can take Lecture 5 from exposition [16] and get the proof in the case $q=1$ by mechanical replacement of operators $\tau(\lambda)$, $\lambda\in\mathbb{Z}^n$ by their quasiclassical limits P^λ. The operator $\tau(\lambda)$ from [16] acts on the ring of Laurent polynomials, it acts on the monomial X^μ by the formula $\tau(\lambda)(X^\mu)=q^{(\lambda,\mu)}X^\mu$, where $(\cdot,\cdot)$ is the standard scalar product.

Proposition 3.1 immediately implies

Corollary 3.1. *$H_{\delta_\tau(X)} \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \# S_n$, where*

$$\delta_\tau(X) = \prod_{i \neq j} (1 - X_i/X_j)(\tau^{-1} - \tau X_i/X_j).$$

3.2. **PBW theorem.** Let us introduce pairwise commutative elements of $Y_i \in H_{q,\tau}$:

$$Y_i = T_i \dots T_{n-1} \pi^{-1} T_1^{-1} \dots T_{i-1}^{-1}, \quad i = 1, \dots, n-1. \tag{24}$$

These elements satisfy the relations

$$T_i Y_{i+1} T_i = Y_i, \quad (1 \leq i < n), \tag{25}$$

$$T_i Y_j = Y_j T_i, \text{ if } j - i \neq 0, 1. \tag{26}$$

Using Y_i we can formulate the following PBW type result for $H_{q,\tau}$:

Proposition 3.2 ([3]). *Each element $h \in H_{q,\tau}$ can be uniquely presented in the form*

$$h = \sum_{w \in S_n} f_w(X) T_w g_w(Y),$$

$$h = \sum_{w \in S_n} g'_w(Y) T_w f'_w(X),$$

where f_w, f'_w, g_w, g'_w are polynomials and $T_w = T_{i_1} \dots T_{i_s}$ with $w = s_{i_1} \dots s_{i_s}$ being a reduced expression for $w \in S_n$.

3.3. **The representation $V_{\mu,\nu}$.** Let $(\mu, \nu) \in (\mathbb{C}^*)^{2n}$, $\delta_\tau(\nu) \neq 0$ and $\chi_{\mu,\nu} \simeq \mathbb{C}$ be a one-dimensional R-module (character): $\chi_{\mu,\nu}(R(P,X)) = R(\mu,\nu)$. We can induce a finite dimensional module $V_{\mu,\nu}$ from this module:

$$V_{\mu,\nu} = \tilde{R} \otimes_R \chi_{\mu,\nu}.$$

This module has a $\mathbb{C}$ basis $w \otimes 1$, $w \in S_n$, hence $\dim V_{\mu,\nu} = n!$.

Proposition 3.3. *If $\delta_\tau(\nu) \neq 0$, then the $H_{1,\tau}$-module $V_{\mu,\nu}$ is irreducible.*

Proof. The module $V_{\mu,\nu}$ has a natural $H_{\delta_\tau(X)} \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \# S_n$-module structure. The group S_n acts freely on the variety $\operatorname{Spec}(\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)})$ hence the algebra $H_{\delta_\tau(X)}$ is Morita equivalent to the algebra $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]^{S_n}_{\delta_\tau(X)}$. In particular, the module $V_{\mu,\nu}$ corresponds to the one-dimensional representation: $P \mapsto P(\mu,\nu)$. Thus $V_{\mu,\nu}$ is an irreducible $H_{\delta_\tau(X)}$-module and hence an irreducible H-module. □

3.4. **The action of the finite Hecke algebra.** The elements T_i, $i = 1, \dots, n-1$ generate an algebra of dimension $n!$ which is called the finite Hecke algebra. We will denote it by the symbol $\mathbb{C}_\tau[S_n]$.

Suppose that ν satisfies the inequality $\delta_\tau(\nu) \neq 0$. If e is the unit in S_n, then Corollary 3.1 and Proposition 3.2 imply that by the action of elements T_i we can get from the vector $\mathrm{e} \otimes 1$ the whole space $V_{\mu,\nu}$. Hence the map j: $\mathbb{C}_\tau[S_n] \to V_{\mu,\nu}$, $j(T_{i_1} \dots T_{i_k}) = T_{i_1} \dots T_{i_k} \mathrm{e} \otimes 1$ is an isomorphism of (left) $\mathbb{C}_\tau[S_n]$-modules.

Definition. We denote the subset of all finite dimensional irreducible $H_{1,\tau}$-modules which are regular $\mathbb{C}_\tau[S_n]$-modules by the symbol $\mathrm{Irrep}^{n!}$.

Let us denote the subset of $\mathrm{Irrep}^{n!}$ consisting of $V_{\mu,\nu}$ $\mu,\nu \in (\mathbb{C}^*)^n$, $\delta_\tau(\nu) \neq 0$ by $\mathcal{U}$. Later (see Corollary 6.2) we will show that all finite dimensional irreducible modules are from $\mathrm{Irrep}^{n!}$.

3.5. **The projective $GL(2,\mathbb{Z})$ action on double affine Hecke algebras.** One of the most important properties of the double affine Hecke algebra $H_{q,\tau}$ is the existence of a homomorphism from $GL(2,\mathbb{Z})$ to the group of outer automorphisms of $H_{q,\tau}$: $\mathrm{Out}(H_{q,\tau}) := \mathrm{Aut}(H_{q,\tau})/\mathrm{Int}(H_{q,\tau})$. This homomorphism was discovered by Cherednik [2] and he calls it projective action of $GL(2,\mathbb{Z})$. Below we use pairwise commutative elements $Y_i \in H_{q,\tau}$ defined by the formulas (24).

The group $GL(2,\mathbb{Z})$ is generated by the elements

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

These generators correspond to the maps

$$\varepsilon : X_i \mapsto Y_i, Y_i \mapsto X_i, T_i \mapsto T_i^{-1},$$
$$\sigma : X_i \mapsto X_i, Y_i \mapsto X_i Y_i q^{-1}, T_i \mapsto T_i,$$

where $\varepsilon : H_{q,\tau} \to H_{q^{-1},\tau^{-1}}$, $\sigma : H_{q,\tau} \to H_{q,\tau}$. The transformation ε is called the duality involution.

Using these transformations we can construct some finite dimensional representations. Indeed, if $\gamma \in GL(2,\mathbb{Z})$ is such that $\gamma(H_{1,\tau}) = H_{1,\tau'}$ and $\phi' : H_{1,\tau'} \to GL(V'_{\mu,\nu})$ is the corresponding representation of $H_{1,\tau'}$ (here τ' is either τ or τ^{-1}), then the map $\phi' \circ \gamma$ is a representation of $H_{1,\tau}$. We denote the set of such representations by $\gamma(\mathcal{U})$.

4. The map from $\mathrm{Irrep}^{n!}$ to CM_τ

In this section we construct a map Φ: $\mathrm{Irrep}^{n!} \to CM_\tau$. Later we will show that it is an isomorphism. Constructions of this section generalize constructions of section 11 of [7].

4.1. **Construction of the map.** Let us denote by $\mathbb{C}_\tau[S_{n-1}] \subset \mathbb{C}_\tau[S_n]$ the subalgebra generated by the elements $T_2, \dots, T_{n-1}$. It is the finite Hecke algebra of rank $n-2$. The element v of an $\mathbb{C}_\tau[S_n]$-module is said to be $\mathbb{C}_\tau[S_{n-1}]$ invariant if $T_i v = \tau v$ for all $i = 2, \dots, n-1$.

The $H_{1,\tau}$-module $V \in \mathrm{Irrep}^{n!}$ by definition is a regular $\mathbb{C}_\tau[S_n]$-module. Hence the space $V^{\mathbb{C}_\tau[S_{n-1}]}$ of $\mathbb{C}_\tau[S_{n-1}]$-invariants has dimension n. The relations inside $H_{1,\tau}$ and (26) imply that X_1 and Y_1 commute with the action of $\mathbb{C}_\tau[S_{n-1}]$. Thus if we fix a basis in V we get $X_1|_{V^{\mathbb{C}_\tau[S_{n-1}]}}, Y_1|_{V^{\mathbb{C}_\tau[S_{n-1}]}} \in GL(n,\mathbb{C})$. The following statement is a key statement of the section.

Proposition 4.1. *Let $V \in \mathrm{Irrep}^{n!}$. Then the operators $\bar{X}_1 = X_1|_{V^{\mathbb{C}_\tau[S_{n-1}]}}$, $\bar{Y}_1 = Y_1|_{V^{\mathbb{C}_\tau[S_{n-1}]}}$ satisfy the equation*

$$rk(\bar{X}_1\bar{Y}_1\bar{X}_1^{-1}\bar{Y}_1^{-1} - \tau^2 Id) = 1.$$

Obviously the space CM_τ is isomorphic to the quotient of the space of solutions of (17) by the action of $GL(n,\mathbb{C})$. Thus the last proposition proves that the map Φ: $\mathrm{Irrep}^{n!} \to CM_\tau$, $\Phi(V) = (\bar{X}_1, \bar{Y}_1)$ is well defined.

In the rest of the section we prove Proposition 4.1. It is done in two steps. First we prove

Lemma 4.1. *The elements* $X_1, Y_1 \in H_{1,\tau}$ *satisfy the relation*

$$X_1 Y_1 X_1^{-1} Y_1^{-1} = T_1^{-1} T_2^{-1} \dots T_{n-2}^{-1} T_{n-1}^{-2} T_{n-2}^{-1} \dots T_1^{-1}. \tag{27}$$

Proof. Indeed using formulas (24) and defining relations for DAHA we get

$$\begin{aligned} X_1 Y_1 X_1^{-1} Y_1^{-1} &= X_1 T_1 \dots T_{n-1} (\pi^{-1} X_1^{-1} \pi) T_{n-1}^{-1} \dots T_1^{-1} \\ &= X_1 T_1 \dots (T_{n-1} X_n^{-1}) T_{n-1}^{-1} \dots T_1^{-1} = X_1 T_1 \dots T_{n-2} (X_{n-1}^{-1} T_{n-1}^{-1}) T_{n-1}^{-1} \dots T_1 \\ &= T_1^{-1} T_2^{-1} \dots T_{n-1}^{-2} T_{n-2}^{-1} \dots T_1^{-1}. \end{aligned}$$

□

The last step is the analysis of the LHS of (27) using the quasiclassical limit $\tau \to 1$. It is done in the last subsection.

4.2. **The spectrum of** $\Theta = T_1^{-1} \dots T_{n-2}^{-1} T_{n-1}^{-2} T_{n-2}^{-1} \dots T_1^{-1}$**.** For a representation V from $\mathrm{Irrep}^{n!}$ there is an isomorphism $V \simeq \mathbb{C}_\tau[S_n]$ of left $\mathbb{C}_\tau[S_n]$-modules. Hence the right multiplication on $\mathbb{C}_\tau[S_n]$ induces a structure of a right $\mathbb{C}_\tau[S_n]$-module on V and as a consequence on $V^{\mathbb{C}_\tau[S_{n-1}]}$.

The right $\mathbb{C}_\tau[S_n]$-module $V^{\mathbb{C}_\tau[S_{n-1}]}$ is a sum of the $n-1$-dimensional vector representation and one-dimensional representation because it is true for $\tau = 1$ and for τ it is not a root of unity. Obviously, the operator Θ (acting by left multiplication) commutes with the right action of $\mathbb{C}_\tau[S_n]$. Hence by the Schur lemma, Θ acts as a constant on $\mathbb{C}_\tau[S_n]$-irreducible components of the right $\mathbb{C}_\tau[S_n]$-module $V^{\mathbb{C}_\tau[S_{n-1}]}$. That is, there exists a basis in the module in which Θ is diagonal and of the form $\mathrm{diag}(\lambda_1(\tau), \lambda_2(\tau), \dots, \lambda_2(\tau))$. Thus we only need to calculate $\lambda_1(\tau)$, $\lambda_2(\tau)$.

The module $V^{\mathbb{C}_\tau[S_{n-1}]}$ exists for all $\tau \neq 0$. As the operator Θ is invertible for all nonzero values of τ, we have $\lambda_1(\tau) \neq 0$, $\lambda_2(\tau) \neq 0$ and $\det(\Theta) = \lambda_1(\tau)(\lambda_2(\tau))^{n-1} = K\lambda^l$ for some integer l and $K \in \mathbb{C}^*$.

Let us consider

$$e = \sum_{w \in S_n} \tau^{l(w)} T_w / (\sum_{w \in W} \tau^{2l(w)}),$$

where $T_w = T_{i_1} \dots T_{i_{l(w)}}$ if $w = s_{i_1} \dots s_{i_{l(w)}}$ is a reduced expression for w. Then it is easy to see that $T_i e = e T_i = \tau e$ for $i = 1, \dots, n-1$ hence $e \in V^{\mathbb{C}_\tau[S_{n-1}]}$. As e spans the only copy of the trivial $\mathbb{C}_\tau[S_n]$-representation inside V, it spans the copy of the trivial $\mathbb{C}_\tau[S_n]$-representation inside $V^{\mathbb{C}_\tau[S_{n-1}]}$. Hence $\lambda_1(\tau) = \tau^{2-2n}$ because $\Theta e = \tau^{2-2n} e$. Combining the last observation with the conclusion from the last paragraph we get $\lambda_2(\tau) = C\tau^k$, for some integer k.

When $\tau = 1$, the algebra $\mathbb{C}_\tau[S_n]$ becomes the group algebra of S_n, and $\Theta = 1$. Thus we have $C = 1$. The calculation of k uses the quasiclassical limit reasoning.

If $\tau = e^h$, then we can write the expansion of T_i in terms of h,

$$T_i = s_i + h\tilde{s}_i + O(h^2), \quad i = 1, \dots, n-1,$$

where $s_i = (i, i+1)$ is a usual transposition. Relation (8) inside $H_{1,\tau}$ implies

$$s_i \tilde{s}_i + \tilde{s}_i s_i = 2 s_i, \quad i = 1, \dots, n-1.$$

Let us calculate the first nontrivial term $\tilde{\Theta}$ of the expansion of $\Theta = 1 + h\tilde{\Theta} + O(h^2)$:

$$\tilde{\Theta} = -\sum_{i=1}^{n-1} s_1 \dots s_{i-1} (\tilde{s}_i s_i + \tilde{s}_i s_i) s_{i-1} \dots s_1 = -2 \sum_{i=1}^{n-1} t_{1i},$$

where $t_{1i} = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1$ is a permutation of 1 and i.

The operator $\tilde{\Theta}/2$ acts on $\mathbb{C}[S_n]^{S_{n-1}}$ (by left multiplication) and in the basis $e_i = (\sum_{w' \in S_{n-1}} w')t_{1i}$ it has the matrix $J - Id$, $J_{ij} = 1$, $1 \le i, j \le n$. Hence $\mathrm{Spec}(\tilde{\Theta}/2) = (1-n, 1, \dots, 1)$. On the other hand, $\mathrm{Spec}(\tilde{\Theta}) = (2-2n, k, \dots, k)$. Thus $k = 2$ and we proved Proposition 4.1.

4.3. **The map Φ on the subset $\mathcal{U} \subset \mathrm{Irrep}^{n!}$.** It is possible to calculate $\Phi(V_{\mu,\nu})$ explicitly. Indeed let us fix a basis in $V_{\mu,\nu}^{\mathbb{C}_\tau[S_{n-1}]}$: $e_i = (\sum_{w' \in S_{n-1}} w')t_{1i}$, $i = 1, \dots, n$.

Proposition 4.2. *For the matrices of the operators $\bar{X}_1$ and $\bar{Y}_1$ written in the basis e_i the following equations hold:*

$$\bar{X}_1 = \mathrm{diag}(\nu_1, \dots, \nu_n),$$
$$\bar{Y}_{ii} = \mu_i \prod_{j \ne i} \frac{(\tau^{-1}\nu_j - \tau\nu_i)}{(\nu_j - \nu_i)}, \quad i = 1, \dots, n.$$

Proof. The first equation is obvious. The second formula is a result of direct calculation using formulas (24) for Y_1 and explicit formulas for T_i.

Indeed, we make this calculation for $i = 1$. The expansion of the product expression for Y_1 consists of the terms of the form $s_{i_1,j_1} \dots s_{i_r,j_r} c^{-1} F(X) P_1$, where $i_l < j_l$, $j_m < i_{m+1}$, $l = 1, \dots, r$, $m = 1, \dots, r-1$ and $F \in \mathbb{C}[X^{\pm 1}]_{\delta(X)}$. We know that $Y_1 e_1$ is a linear combination of e_i, $i = 1, \dots, n$. The terms of the expansion of $Y_1 e_1$ which contribute to the coefficient before e_1 satisfy the equation $s_{i_1,j_1} \dots s_{i_r,j_r} c^{-1}(1) = 1$. This is possible only in the case $r = 1$, $i_1 = 1, j_1 = n$. Thus rewriting T_i in the form

$$T_i = \frac{(\tau X_i - \tau^{-1} X_{i+1})}{X_i - X_{i+1}} s_i + \frac{X_{i+1}(\tau^{-1} - \tau)}{X_i - X_{i+1}},$$

we see that

$$Y_1 e_1 = \left(\prod_{i=1}^{n-1} \frac{(\tau X_i - \tau^{-1} X_{i+1})}{X_i - X_{i+1}} s_i \right) c^{-1} e_1 + r,$$

where r is a linear combination of e_j with $j > 1$. This formula immediately implies the last formula from the proposition for $i = 1$. □

It is actually not easy to compute all coefficients $\bar{Y}_1$ using explicit formulas for Y_1 and T_i but we do not need them. Because by Proposition 2.1, if the pair (X, Y) satisfies equation (17) and X is diagonal with eigenvalues satisfying the conditions of Proposition 2.1, then the corresponding $GL(E)$-orbit is uniquely determined by the diagonal elements of X and Y (because the stabilizer of X consists of diagonal matrices which do not change diagonal elements of Y and we can extract q from these elements). This reasoning implies

Corollary 4.1. *The map Φ is an isomorphism on the subset $\mathcal{U}$, and local coordinates λ, q on CM_τ are expressed through coordinates μ, ν on $\mathcal{U} \subset \mathrm{Irrep}^{n!}$ by the formulas*

$$\lambda_i = \nu_i, \quad q_i = \mu_i \prod_{j \ne i} \frac{(\tau^{-1}\nu_j - \tau\nu_i)}{(\nu_j - \nu_i)}.$$

5. Results on the general double affine Hecke algebra

Let $R = \{\alpha\}$ be a root system (possibly nonreduced) of type $A, BC, \dots, F, G$, W the Weyl group generated by the reflections s_α, $\alpha \in R$. The extended affine Weyl group $\tilde{W}$ is a semidirect product $W \ltimes P$, where P is a weight lattice (i.e. $b \in P$ if $2(b,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$ for all $\alpha \in R$).

The affine Hecke algebra $\hat{H}_\tau$ is a deformation of the group algebra $\mathbb{C}[\tilde{W}]$ with deformation parameters τ_α, $\tau_{w(\alpha)} = \tau_\alpha$, $\alpha \in R$, $w \in W$ (for the exact definition of the affine Hecke algebra see [15]). The double affine Hecke algebra $H_{q,\tau}$ is a nontrivial extension of the affine Hecke algebra $\hat{H}_\tau$ by the group algebra $\mathbb{C}[P^\vee]$ of the coweight lattice $P^\vee$ ($b \in P^\vee$ if $(b,\alpha) \in \mathbb{Z}$ for all $\alpha \in R$). This extension has one parameter q which is the shift parameter in the Lusztig-Demazure representation of this algebra. We consider algebras with $q = 1$ and we denote them by H. For the exact definition of the double affine Hecke algebra and formulas for the Lusztig-Demazure representation see the original paper [1] or survey [16].

We use the notation $\delta(X)$ for the Weyl denominator for the root system R. By $\mathbb{C}[X^{\pm 1}]$ we denote the group algebra of the weight lattice P lying inside the affine Hecke algebra $\hat{H}_\tau$ and by $\mathbb{C}[Y^{\pm 1}]$ we denote the group algebra $\mathbb{C}[P^\vee] \subset H$ which extends $\hat{H}_\tau$.

There is an injective homomorphism g: $H \to \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \# W$ via the quasiclassical Lusztig-Demazure operators. The formulas for the embedding are very similar to the formulas from the previous section. These formulas are quasiclassical limits of the Lusztig-Demazure operators from the papers [3] (in the case of reduced root systems) and [17] (in the case of nonreduced root systems).

Let $\mathbb{C}_\tau[W]$ be the corresponding finite Hecke algebra, and e the symmetrizer in $\mathbb{C}_\tau[W]$:

$$e = \sum_{w \in W} \tau^{l(w)} T_w / (\sum_{w \in W} \tau^{2l(w)}),$$

where $T_w = T_{i_1} \dots T_{i_{l(w)}}$ if $w = s_{i_1} \dots s_{i_{l(w)}}$ is a reduced expression for w.

In this section we will need the following PBW type result

Proposition 5.1 ([3]). *Each element $h \in H$ can be uniquely presented in the forms*

$$h = \sum_{w \in W} f_w(X) T_w g_w(Y),$$
$$h = \sum_{w \in W} g'_w(Y) T_w f'_w(X).$$

5.1. **Formulation of the theorem.** The goal of this section is to study the center Z of H and corresponding scheme $\mathrm{Spec}(Z)$. It turns out that Z is isomorphic to the subalgebra eHe and we can reduce the study of Z to the study of eHe.

We recall the definition of a Cohen-Macaulay variety.

Definition ([18]). A finitely generated commutative $\mathbb{C}$-algebra A is called *Cohen-Macaulay* if it contains a subalgebra of the form $\mathcal{O}(V)$ such that A is a free $\mathcal{O}(V)$-module of finite rank, and V is a smooth affine algebraic variety.

For the definition of a Cohen-Macaulay module see [19], (Chapter 4, p. 18). In this section we prove the following

Theorem 5.1. *For any double affine Hecke algebra H the following is true:*

(1) *eHe is commutative.*
(2) *$M = \mathrm{Spec}(eHe)$ is an irreducible Cohen-Macaulay and normal variety.*
(3) *The right eHe-module He is Cohen-Macaulay.*
(4) *The left action of H on He induces an isomorphism of algebras $H \simeq \mathrm{End}_{eHe}(He)$.*
(5) *The map $\eta : z \to ze$ is an isomorphism $Z \to eHe$. Thus, $M = \mathrm{Spec}\, Z$.*

We call the isomorphism η the Satake isomorphism (by analogy with [7]).

5.2. **Proof of Theorem 5.1.** Below we use the τ-deformed Weyl denominator

$$\delta_\tau(X) = \prod_{\alpha \in R} (1 - X^\alpha)(\tau_\alpha^{-1} - \tau_\alpha X^\alpha).$$

Lemma 5.1. (1) $H_{\delta_\tau(X)} \simeq \mathbb{C}[X^{\pm 1}, P^{\pm 1}]_{\delta_\tau(X)} \# W$.
(2) *The map h: $Z(H_{\delta_\tau(X)}) \to \mathbb{C}[P^{\pm 1}, X^{\pm 1}]^W_{\delta_\tau(X)} e$, induced by multiplication by e is an isomorphism.*
(3) *The left $H_{\delta_\tau(X)}$-action on $H_{\delta_\tau(X)}$ induces the isomorphism*

$$H_{\delta_\tau(X)} \simeq \mathrm{End}_{eH_{\delta_\tau(X)}e}(H_{\delta_\tau(X)}).$$

Proof. The first and second items of the lemma follow from the representation of H by the quasiclassical Lusztig-Demazure operators. The third item is equivalent to the isomorphism

$$\mathbb{C}[P^{\pm 1}, X^{\pm}]_{\delta_\tau(X)} \# W \simeq \mathrm{End}_{\mathbb{C}[P^{\pm 1}, X^{\pm 1}]^W_{\delta_\tau(X)}}(\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)}).$$

We will proceed analogously to the proof of Theorem 1.5 in [7].

If a: $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \to \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)}$ is $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]^W_{\delta_\tau(X)}$-linear, then it defines a $\mathbb{C}(P,X)^W$-linear map $\mathbb{C}(P,X) \to \mathbb{C}(P,X)$. The isomorphism $\mathbb{C}(P,X) \# W \simeq \mathrm{End}_{\mathbb{C}(P,X)^W}(\mathbb{C}(P,X))$ implies $a = \sum_{w \in W} a_w w$, $a_w \in \mathbb{C}(P,X)$. It is clear that the functions a_w are regular on $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \setminus \Delta$ where Δ is the subset of the points of $\mathbb{C}^n \times \mathbb{C}^n$ with a nontrivial stabilizer in W. But Δ has codimension 2, hence by Hartogs theorem $a_w \in \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)}$. □

Lemma 5.2. *Z contains $\mathbb{C}[X^{\pm 1}]^W$ and $\mathbb{C}[Y^{\pm 1}]^W$.*

Proof. $\mathbb{C}[X^{\pm 1}]^W$ clearly lies in the center of $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta(X)} \# W$, and therefore in the center of H. The fact that $\mathbb{C}[Y^{\pm 1}]^W$ is contained in Z follows from the existence of duality involution described in Theorem 2.3 in [20]. In the case of the root system A_{n-1} this morphism is the duality morphism described in subsection 3.5. This morphism maps the subalgebra $\mathbb{C}[Y^{\pm 1}]^W \subset H$ into subalgebra $\mathbb{C}[X^{\pm 1}]^W \subset H'$ where H' is DAHA corresponding to the dual root system. □

Lemma 5.3. *eHe is commutative, without zero divisors.*

Proof. Let us prove that the subalgebra $eH_{\delta_\tau(X)}e$ of $H_{\delta_\tau(X)} \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \# W$ is commutative and without zero divisors. Obviously it implies the statement.

An element $z \in H_{\delta_\tau(X)}$ has a unique representation in the form $z = \sum_{w \in W} Q_w T_w$; that is, $H_{\delta_\tau(X)}$ is isomorphic to $\mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \otimes \mathbb{C}_\tau[W]$ as a right $\mathbb{C}_\tau[W]$-module. If $z \in eHe$, then $zT_\alpha = \tau_\alpha z$ for all $\alpha \in R$ because $eT_\alpha = \tau_\alpha e$. Hence z is an $\mathbb{C}_\tau[W]$-invariant element of the right $\mathbb{C}_\tau[W]$-module $H_{\delta_\tau(X)} \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)} \otimes \mathbb{C}_\tau[W]$. As $\mathbb{C}(P,X) \otimes \mathbb{C}_\tau[W]$ is a regular $\mathbb{C}_\tau[W]$-module (over the field $\mathbb{C}(P,X)$),

$\mathbb{C}(P,X)\otimes e$ is a unique copy of the trivial representation. It implies that $z = Qe$, $Q \in \mathbb{C}[P^{\pm 1}, X^{\pm 1}]_{\delta_\tau(X)}$.

Finally, for $z = Qe \in eHe$ we have $(T_\alpha - \tau_\alpha)Qe = 0$. The simple calculation using the explicit expression for T_α yields

$$(T_\alpha - \tau_\alpha)Qe = P_\alpha(s_\alpha - 1)Qe = P_\alpha(s_\alpha(Q) - Q)e,$$

where $P_\alpha \in \mathbb{C}[X^{\pm 1}]_{\delta_\tau(X)}$ and α is a simple root. This implies $Q \in \mathbb{C}[P^{\pm 1}, X^{\pm 1}]^W_{\delta_\tau(X)}e$ and $eH_{\delta_\tau(X)}e \simeq \mathbb{C}[P^{\pm 1}, X^{\pm 1}]^W_{\delta_\tau(X)}$. □

The algebra H has a natural $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$-module structure: the element $p \otimes q$ acts on $x \in H$ by the formula $(p \otimes q)x = pxq$.

Lemma 5.4. *H is a projective finitely generated $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$-module.*

Proof. Let us first show that $\mathbb{C}[X^{\pm 1}]$ is a projective finitely generated $\mathbb{C}[X^{\pm 1}]^W$-module. Finite generation is clear, since W is a finite group. Also, it is well known that $\mathbb{C}[X^{\pm 1}]^W$ is a polynomial ring (it is generated by the characters of the fundamental representations of the corresponding simply connected group). Since $\mathbb{C}[X^{\pm 1}]$ is a regular ring, by Serre's theorem ([19], Chapter 4, p. 37, Proposition 22) $\mathbb{C}[X^{\pm 1}]$ must be locally free over $\mathbb{C}[X^{\pm 1}]^W$ (in fact, by the Steinberg-Pittie theorem [21] it is free, but we will not use it). For the same reasons $\mathbb{C}[Y^{\pm 1}]$ is locally free over $\mathbb{C}[Y^{\pm 1}]^W$.

Now the claim follows from the PBW factorization from Proposition 5.1 $H = \mathbb{C}[X^{\pm 1}] \otimes \mathbb{C}_\tau[W] \otimes \mathbb{C}[Y^{\pm 1}]$. □

Lemma 5.5. *He and eHe are projective finitely generated modules over $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$.*

Proof. The finite generation follows from the Hilbert-Noether lemma and Lemma 5.4. The projectivity is true because He and eHe are direct summands in H. □

Proof of Theorem 5.1. The first item follows form Lemma 5.3.

Proof of (2): $M = \mathrm{Spec}(eHe)$ is an irreducible affine variety by Lemma 5.3. As follows from Lemma 5.2 the elements $f(X)g(Y)e$, where $f \in \mathbb{C}[X^{\pm 1}]^W$ and $g \in \mathbb{C}[Y^{\pm 1}]^W$, form the commutative subalgebra $\mathcal{P}$. Obviously this subalgebra is polynomial. Hence to prove that M is Cohen-Macaulay it is sufficient to show that eHe is a locally free module of finite rank over its subalgebra $\mathcal{P} \simeq \mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$. But the module is projective and finitely generated by Lemma 5.5.

It is easy to see by localizing with respect to $e\delta_\tau(X)$ or $e\delta_\tau(Y)$ that M is smooth away from a codimension 2 subset. Indeed, by the first item of Lemma 5.1 after localizing with respect to $e\delta_\tau(X)$ the image of eHe under the injection g becomes $e\mathbb{C}[X,P]_{\delta_\tau(X)}e \simeq \mathbb{C}[X,P]^W_{\delta_\tau(X)}e$, which is the ring of regular functions on a smooth affine variety. The statement for the localization with respect to $e\delta_\tau(Y)$ follows from the existence of the duality involution discussed in the proof of Lemma 5.2. But an irreducible Cohen-Macaulay variety that is smooth outside of a codimension 2 subset is normal ([18], 2.2).

Proof of (3): eHe is finitely generated over $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$. Hence by Theorem 2.1 of [22] eH is Cohen-Macaulay over eHe if and only if it is Cohen-Macaulay over $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$.

We know that $He \simeq \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ as a $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$-module and He is projective over $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$. As $\mathbb{C}[X^{\pm 1}]^W \otimes \mathbb{C}[Y^{\pm 1}]^W$ is a polynomial ring, the module $\mathbb{C}[X^{\pm1}, Y^{\pm1}]$ is Cohen-Macaulay if and only if it is projective. So Lemma 5.5 implies the statement.

Proof of (4): We have an obvious homomorphism $f : H \to \operatorname{End}_{eHe} He$. It is clearly injective because it is injective after localization by the ideal $(\delta_\tau(X))$.

Let us denote $\operatorname{End}_{eHe}(He)$ by $\tilde{H}$. Regard $\tilde{H} \supset H$ as $\mathbb{C}[X^{\pm1}]^W \otimes \mathbb{C}[Y^{\pm1}]^W$-modules. $\tilde{H}$ is torsion free because He is a torsion free $\mathbb{C}[X^{\pm1}]^W \otimes \mathbb{C}[Y^{\pm1}]^W$-module (by the PBW theorem). As He is finitely generated over eHe, $\tilde{H}$ is a finitely generated $\mathbb{C}[X^{\pm1}]^W \otimes \mathbb{C}[Y^{\pm1}]^W$-module. Also, H is finitely generated projective, and $\tilde{H}/H$ is supported in codimension 2. Indeed, the last part of Lemma 5.1 implies that $H_{\delta_\tau(X)}$ is isomorphic to $\tilde{H}_{\delta_\tau(X)}$ as a $eH_{\delta_\tau(X)}e$-module. Similarly, the module $H_{\delta_\tau(Y)}$ is isomorphic to $\tilde{H}_{\delta_\tau(Y)}$ as a $eH_{\delta_\tau(Y)}e$-module because we can use (the same way as in the proof of Lemma 5.3) the duality involution.

The module $\tilde{H}$ represents some class in $\operatorname{Ext}^1(\tilde{H}/H, H)$, which must be zero since $\tilde{H}/H$ is finitely generated and lives in codimension 2 and H is projective. Thus, $\tilde{H} = H \oplus \tilde{H}/H$ and the summand $\tilde{H}/H$ is torsion. But $\tilde{H}$ is a torsion free eHe-module, hence $\tilde{H}/H = 0$ and $\tilde{H} = H$.

Proof of (5): It is clear that η is injective, by looking at the Lusztig-Demazure representation. Indeed the equation $ze = 0$ implies $zp = 0$ for any $p \in \mathbb{C}[X]^W$, hence by the PBW theorem $z = 0$.

It remains to show that η is surjective. Since eHe is commutative, every element $a \in eHe$ defines an endomorphism of He over eHe (by right multiplication). So by statement (4) a defines an element $z_a \in H$. This element commutes with H. Indeed the right multiplication by a is an endomorphism of the right eHe-module which commutes with left multiplication by elements of H hence by the forth part of the theorem $[z_a, h] = 0$ for all $h \in H$. For any $x \in H$, $z_a xe = xa$, so $xz_a e = xa$, i.e. $x(z_a e - a) = 0$. Since eHe has no zero divisors, we find $\eta(z_a) = a$, as desired. □

6. The results in the case of the root system A_{n-1}

In this section $H = H_{1,\tau}$ is the double Hecke algebra corresponding to $GL(n, \mathbb{C})$.

A point $(\mu, \nu) \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D_\tau)$ defines a $\mathbb{C}[P^{\pm1}, X^{\pm1}]^{S_n}_{\delta_\tau(X)}$-character $\chi_{(\mu,\nu)}$: $\chi_{(\mu,\nu)}(Q(P, X)) = Q(\mu, \nu)$. The embedding $Z \hookrightarrow Z_{\delta_\tau(X)} \simeq \mathbb{C}[P^{\pm1}, X^{\pm1}]^{S_n}_{\delta_\tau(X)}$ allows us to restrict this character to Z. We use the same notation for this character.

Lemma 6.1. *For any point $(\mu, \nu) \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D_\tau)$ we have*

$$He \otimes_{eHe} \chi_{(\mu,\nu)} \simeq V_{\mu,\nu}.$$

Proof. The H-module $V_{\mu,\nu}$ has a natural structure of an $H_{\delta_\tau(X)}$-module. Let us study finite dimensional irreducible $H_{\delta_\tau(X)}$-modules.

By Lemma 5.1 the ring $eH_{\delta_\tau(X)}e$ is a regular ring. As the action of S_n on $(\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D_\tau)$ is free, the ring $\mathbb{C}[P^{\pm1}, X^{\pm1}]_{\delta_\tau(X)} \simeq H_{\delta_\tau(X)}e$ is a projective $eH_{\delta_\tau(X)}e$-module and defines the vector bundle F over $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n \setminus D_\tau) = \operatorname{Spec}(eH_{\delta_\tau(X)}e)$. Hence by Lemma 5.1 $H_{\delta_\tau(X)} = \operatorname{End}(F)$ is an Azumaya algebra and by the basic property of Azumaya algebras any irreducible $H_{\delta_\tau(X)}$-module is of the form $H_{\delta_\tau(X)}e \otimes_{eH_{\delta_\tau(X)}e} \chi_{(\mu',\nu')}$ for some point $(\mu', \nu') \in (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D_\tau)$.

Obviously any irreducible $H_{\delta_\tau(X)}$-module is irreducible as an H-module. Also we have an obvious isomorphism of H-modules $H_{\delta_\tau(X)}e \otimes_{eH_{\delta_\tau(X)}e} \chi_{(\mu',\nu')} \simeq He \otimes_{eHe} \chi_{(\mu',\nu')}$. Thus the previous paragraph implies $V_{\mu,\nu} \simeq He \otimes_{eHe} \chi_{(\mu',\nu')}$. Comparing the action of the center on both sides yields the statement. □

The previous lemma implies that there is a map Υ from the open subset $\mathrm{Spec}(Z_{\delta_\tau(X)})$ of $\mathrm{Spec}(Z)$ to the CM space CM_τ: $\Upsilon(\mu,\nu) = \Phi(V_{\mu,\nu})$, where Φ is the map constructed in Section 4. As $\mathrm{Spec}(Z_{\delta_\tau(X)})$ is an open dense subset in $\mathrm{Spec}(Z)$, we can define a rational map $\Upsilon\colon \mathrm{Spec}(Z) \dashrightarrow CM_\tau$.

Theorem 6.1. *The map* $\Upsilon\colon \mathrm{Spec}(Z) \dashrightarrow CM_\tau$ *is a regular isomorphism of the algebraic varieties. In particular,* $\mathrm{Spec}(Z)$ *is smooth.*

Proof. The previous lemma and Corollary 4.1 imply that Υ is a regular isomorphism on $\mathrm{Spec}(Z_{\delta_\tau(X)})$. The duality involution from subsection 3.5 allows us to state the same for the open subset $\mathrm{Spec}(Z_{\delta_\tau(Y)})$.

Indeed, the duality involution ε maps the double affine Hecke algebra $H_{1,\tau}$ to $H_{1,\tau^{-1}}$ and it induces the map $\varepsilon_{CM}\colon\ CM_\tau \to CM_{\tau^{-1}},\ \varepsilon_{CM}(X,Y,U,V) = (Y,X,-Y^{-1}X^{-1}YXU,V)$. By the construction we have $\varepsilon_{CM}\circ\Upsilon = \Upsilon\circ\varepsilon$. Thus the restriction of the morphism $\varepsilon_{CM}^{-1}\circ\Upsilon\circ\varepsilon$ to $\mathrm{Spec}(Z_{\delta_\tau(Y)}$ is a regular isomorphism.

Now, we know from the Theorem 5.1 that $\mathrm{Spec}(Z)$ is normal. As the complement of $\mathrm{Spec}(Z_{\delta_\tau(X)}) \cup \mathrm{Spec}(Z_{\delta_\tau(Y)})$ has codimension 2 (because $\mathrm{Spec}(Z)$ is irreducible by Theorem 5.1), we can extend Υ to a regular map on the whole $\mathrm{Spec}(Z)$. The extended map is dominant because by Proposition 2.2 the variety CM_τ is irreducible.

Thus Υ is a regular birational map which is an isomorphism outside of the subset of codimension 2. But we know that CM_τ is smooth and $\mathrm{Spec}(Z)$ is normal, hence (by Theorem 5, Section 5, Chapter 2 of [23]) the map Υ^{-1} is regular and as a consequence is an isomorphism. □

Corollary 6.1. *He is a projective eHe-module.*

Proof. We proved for any R that He is a Cohen-Macaulay module over eHe. Since $M = \mathrm{Spec}(eHe)$ is smooth, the result follows from corollary 2 from chapter 4 of [19]. □

Thus He defines the vector bundle E on $\mathrm{Spec}(eHe)$, with fibers of the dimension $n!$.

Corollary 6.2. *For the double affine Hecke algebra $H = H_{1,\tau}$ the following is true:*

(1) *$H = \mathrm{End}\, E$ where E is a vector bundle over $\mathrm{Spec}(Z)$ i.e. H is an Azumaya algebra.*
(2) *Every irreducible representation of H is of the form $V_z = He\otimes_{eHe}\chi_z$, $z \in M = \mathrm{Spec}(Z)$.*
(3) *V_z has dimension $n!$ and is a regular representation of $\mathbb{C}_\tau[S_n]$.*

Proof. The first item follows from Theorem 5.1. The second item is a general property of Azumaya algebras. The third item follows from the fact that it is true for the generic point $z \in \mathrm{Spec}(Z)$. □

Remark 6.1. This corollary was proved in 2000 by Cherednik using the technique of the intertwiners [24].

The ring $Z \simeq eH_{1,\tau}e$ has a natural noncommutative deformation $eH_{q,\tau}e$. Hence this ring has a natural Poisson structure $\{\cdot,\cdot\}$. The variety CM_τ also has a Poisson structure described in subsection 2.4. It turns out that the isomorphism Φ respects these Poisson structures.

Theorem 6.2. *The isomorphism Φ is an isomorphism of Poisson varieties, that is the following formula holds:*

$$\{\cdot,\cdot\}_{FR} = \{\cdot,\cdot\}.$$

Proof. It is enough to prove that it is an isomorphism of Poisson varieties on the open set $\mathcal{U}$. For $q = e^h \neq 1$ we have an embedding g_q; $H_{q,\tau} \to \mathbb{D}_q \# S_n$ via Lusztig-Demazure reflection difference operators. Here $\mathbb{D}_q$ is a localization of the Weyl algebra with generators $X_i^{\pm 1}$, $\hat{P}_i^{\pm 1}$, $i = 1,\dots,n$ and relations:

$$[X_i, X_j] = 0, \quad [\hat{P}_i, \hat{P}_j] = 0, \quad X_j\hat{P}_i - q^{\delta_{ij}}\hat{P}_i X_j = 0,$$

by the ideal $(\delta_\tau(X))$. When $q = 1$, the noncommutative ring $\mathbb{D}_q$ becomes the commutative ring $\mathbb{C}[P^{\pm1}, X^{\pm1}]_{\delta_\tau(X)}$ and the corresponding Poisson structure on this ring is given by the formulas

$$\{X_i, X_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{X_i, P_j\} = \delta_{ij} X_i P_j.$$

The $H_{1,\tau}$-module $V_{\mu,\nu}$ has a natural $\mathbb{C}[P^{\pm1}, X^{\pm1}]_{\delta_\tau(X)}\# S_n$ structure. It is easy to see that in the basis $1 \otimes w$, $w \in W$ operators P_i, X_j are diagonal. In particular, $P_i(1\otimes \mathrm{e}) = \mu_i(1\otimes \mathrm{e})$ and $X_i(1\otimes \mathrm{e}) = \nu_i(1\otimes \mathrm{e})$, hence we have the following Poisson bracket on $\mathcal{U}$:

$$\{\nu_i, \nu_j\} = 0, \quad \{\mu_i, \mu_j\} = 0 \quad \{\nu_i, \mu_j\} = \delta_{ij}\nu_i\mu_j. \tag{28}$$

The comparison of the formulas for the Poisson bracket on $\mathbf{U} \subset CM_\tau$ from subsection 2.4 and explicit formulas for the map $\Phi|_{\mathcal{U}}$ from subsection 4.3 give the formula. Indeed, we can express the functions λ_i, q_k through the functions μ_s, ν_t and using (28) to calculate Poisson brackets $\{\lambda_i, \lambda_k\}$, $\{\lambda_i, q_k\}$, $\{q_i, q_k\}$. We give formulas for the last bracket:

$$\begin{aligned}
\{q_i, q_k\} &= q_i q_k \left(\nu_k \frac{\partial \ln(q_i)}{\partial \nu_k} - \nu_i \frac{\partial \ln(q_k)}{\partial \nu_i}\right) \\
&= q_i q_k \left(\nu_k\left(-\frac{\tau}{\tau^{-1}\nu_i - \tau\nu_k} + \frac{1}{\nu_i - \nu_k}\right) - \nu_i\left(-\frac{\tau}{\tau^{-1}\nu_k - \tau\nu_i} + \frac{1}{\nu_k - \nu_i}\right)\right) \\
&= \frac{(\tau^{-1} - \tau)^2 q_i q_k (\nu_k + \nu_i)\nu_i\nu_k}{(\nu_i - \nu_k)(\tau^{-1}\nu_k - \tau\nu_i)(\tau^{-1}\nu_i - \tau\nu_k)}.
\end{aligned}$$

□

7. The rational and trigonometric cases

In this section we explain how one can get an easier proof of the results of [7] on the rational double affine Hecke algebra. We also give the version of the results of the paper for the trigonometric Hecke algebra and explain how to modify the proof from the paper for this case.

We give the modifications of the results from the main body of the text only for the root system A_{n-1} but the rational (and trigonometric) version of Theorem 5.1 holds for any root system R (and the proof is analogous). Moreover, in the rational case we can replace the Weyl group W by a finite Coxeter group (see [7]). Proofs of these results repeat proofs for (nondegenerate) double affine Hecke algebras from the paper.

7.1. **Definition of the rational and trigonometric double affine Hecke algebras.** Below we give a definition of the rational and trigonometric double affine Hecke algebra.

Definition ([7])**.** The rational double affine Hecke algebra $H_{t,c}^{\text{rat}}$ is generated by elements s_{ij}, $1 \le i \ne j \le n$, x_i, y_j, $1 \le i, j \le n$. The elements s_{ij}, $1 \le i, j \le n$ generate the subalgebra inside $H_{t,c}^{\text{rat}}$ isomorphic to the group algebra of the symmetric group S_n, and s_{ij} corresponds to the transposition (ij). In addition, generators of $H_{t,c}^{\text{rat}}$ satisfy the relations

$$\begin{gathered}
x_i s_{ij} = s_{ij} x_j, \quad y_i s_{ij} = s_{ij} y_j, \quad 1 \le i, j \le n, \\
[x_k, s_{ij}] = 0, \quad [y_k, s_{ij}] = 0, \quad k \notin \{i, j\}, \quad 1 \le i, j, k \le n, \\
[y_i, x_j] = c s_{ij}, \quad 1 \le i \ne j \le n, \\
[x_i, x_j] = 0 = [y_i, y_j], \quad 1 \le i, j \le n, \\
[y_k, x_k] = t - c \sum_{i \ne k} s_{ik}, \quad 1 \le k \le n.
\end{gathered}$$

Definition. The trigonometric double affine Hecke algebra $H_{t,c}^{\text{trig}}$ is generated by elements s_{ij}, $1 \le i \ne j \le n$, $X_i^{\pm 1}, y_j$, $1 \le i, j \le n$. The elements s_{ij}, $1 \le i, j \le n$ generate the subalgebra inside $H_{t,c}^{\text{trig}}$ isomorphic to the group algebra of the symmetric group S_n, and s_{ij} corresponds to the transposition (ij). In addition, the generators of $H_{t,c}^{\text{trig}}$ satisfy the relations

$$\begin{gathered}
X_i s_{ij} = s_{ij} X_j, \quad 1 \le i, j \le n, \\
s_{ij} y_i - y_j s_{ij} = c \text{ if } j > i, \quad s_{ij} y_i - y_j s_{ij} = -c \text{ if } j < i, \\
[X_k, s_{ij}] = 0, \quad [y_k, s_{ij}] = 0 \text{ if } k \notin \{i, j\}, \quad 1 \le i, j, k \le n, \\
[X_i, X_j] = 0 = [y_i, y_j], \quad 1 \le i, j \le n, \\
X_j^{-1} y_i X_j - y_i = c s_{ij} \text{ if } j > i, \quad X_j^{-1} y_i X_j - y_i = X_i X_j^{-1} c s_{ij} \text{ if } j < i, \\
X_k^{-1} y_k X_k - y_k = t - c(\sum_{i<k} s_{ik} + \sum_{i>k} X_i X_k^{-1} s_{ik}), \quad 1 \le k \le n.
\end{gathered}$$

Remark 7.1. Let $\hat{H}$ be the $\mathbb{C}[c,t][[h]]$-algebra topologically generated (in the h-adic topology) by X_i, y_i, $s_{i,i+1}$ with $T_i = s_{i,i+1} e^{chs_{i,i+1}}$, $i = 1, \ldots, n-1$, $Y_i = e^{hy_i}$, X_i, $i = 1, \ldots, n$ satisfying the relations for the double affine Hecke algebra $H_{q,\tau}$, $q = e^{th}, \tau = e^{ch}$. It coincides with an appropriate completion of the double affine Hecke algebra $H_{q,\tau}$, in the h-adic topology. Moreover, one can show that $\hat{H}$ is flat over $\mathbb{C}[[h]]$ and $\hat{H}/h\hat{H} = H_{t,c}^{\text{trig}}$. Analogously, if $\hat{H}^{\text{trig}}$ is the $\mathbb{C}[c,t][[h]]$-algebra topologically generated by by s_{ij}, y_i, x_j, $1 \le i \le n$ with $s_{ij}, y_i, X_j = e^{hx_j}$, $i, j = 1, \ldots, n$, satisfying the relations for the trigonometric double affine Hecke algebra $H_{ht,hc}^{\text{trig}}$, then the algebra $\hat{H}^{\text{trig}}$ is flat over $\mathbb{C}[[h]]$ and $H_{t,c}^{\text{rat}} = \hat{H}^{\text{trig}}/h\hat{H}^{\text{trig}}$. Let us also mention that there is a direct limiting process from the double affine Hecke algebra to the rational double affine Hecke algebra [25].

7.2. **Representation by Dunkl operators.** Let $\mathcal{D}_t^{\text{rat}}$ be the localization of the n-dimensional Weyl algebra $\mathcal{A}_t^{\text{rat}}$ by the ideal generated by $\delta(x)$. The Weyl algebra $\mathcal{A}_t^{\text{rat}}$ is generated by elements x_i, p_i, $1 \le i \le n$ modulo relations

$$[x_i, x_j] = 0 = [p_i, p_j], \quad [x_i, p_j] = t\delta_{ij}, \quad 1 \le i, j \le n.$$

Let us denote by $\mathcal{D}_t^{\rm trig}$ the trigonometric version of algebra $\mathcal{D}_t^{\rm rat}$. This algebra is a localization by $(\delta(X))$ of the algebra $\mathcal{A}_t^{\rm trig}$ with generators $p_i, X_i^{\pm 1}$, $i = 1, \dots, n$ modulo relations

$$[X_i, X_j] = 0 = [p_i, p_j], \quad [X_i, p_j] = t\delta_{ij} X_i, \quad 1 \le i, j \le n. \tag{29}$$

It is easy to see that the ring $\mathcal{A}_t^{\rm trig}$ is isomorphic to the ring of differential operators on the torus $(\mathbb{C}^*)^n$.

Proposition 7.1. *The homorphisms* $g^{\rm rat}\colon H_{t,c}^{\rm rat} \to D_t^{\rm rat} \# S_n$, $g^{\rm trig}\colon H^{\rm trig} \to \mathcal{D}_t^{\rm trig} \# S_n$ *defined by the formulas*

$$g^{\rm rat}(y_i) = p_i + c \sum_{j \ne i} \frac{1}{x_i - x_j}(s_{ij} - 1),$$
$$g^{\rm rat}(x_i) = x_i, \quad g^{\rm rat}(w) = w,$$
$$g^{\rm trig}(y_i) = p_i + c \sum_{j<i} \frac{X_i}{X_i - X_j}(s_{ij} - 1) + c \sum_{j>i} \frac{X_j}{X_i - X_j}(s_{ij} - 1),$$
$$g^{\rm trig}(X_i) = X_i, \quad g^{\rm trig}(w) = w,$$

$(i = 1, \dots, n)$ *is injective.*

This proposition allows us to prove the PBW type result for these algebras.

7.3. **Calogero-Moser spaces.** Next we give a definition of the Calogero-Moser space in the rational and trigonometric cases. These spaces were defined by Kazhdan, Kostant and Sternberg in [8].

Let $CM'_{\rm rat}$ be the subset of $\mathfrak{gl}(n, \mathbb{C}) \times \mathfrak{gl}(n, \mathbb{C})$ consisting of the elements (x, y) satisfying the equation

$$rk([x, y] + Id) = 1.$$

By $CM'_{\rm trig} \subset GL(n, \mathbb{C}) \times \mathfrak{gl}(\mathfrak{n}, \mathbb{C})$ we denote the subset of pairs (X, y) satisfying

$$rk(X^{-1} y X - y + Id) = 1.$$

The group $GL(n, \mathbb{C})$ acts on the spaces $CM'_{\rm rat}$ and $CM'_{\rm trig}$ by conjugation. This action is free.

Definition. The quotient of $CM'_{\rm rat}$ $(CM'_{\rm trig})$ by the action of $GL(n, \mathbb{C})$ is called the rational (trigonometric) Calogero-Moser space. We use the notation $CM_{\rm rat}$ (respectively $CM_{\rm trig}$) for this space.

Proposition 7.2. *The rational (trigonometric) Calogero-Moser space* $CM_{\rm rat}(CM_{\rm trig})$ *is an irreducible smooth variety of dimension* $2n$.

For the rational Calogero-Moser space this statement is proved in section 1 of [10]. The proof in the trigonometric case almost identically repeats the proof in the rational case.

The Calogero-Moser spaces $CM_{\rm rat}$ and $CM_{\rm trig}$ are the configuration spaces for the rational and trigonometric integrable Calogero-Moser systems. The Poisson structures corresponding to these systems are the results of the Hamiltonian reduction of the natural Poisson structures on the spaces $\mathfrak{gl}(n, \mathbb{C}) \oplus \mathfrak{gl}^*(n, \mathbb{C})$ and $T^*GL(n, \mathbb{C})$ (see [26]).

7.4. **The main result for the rational and trigonometric double-affine Hecke algebras.** As we mentioned in the first subsection, the algebras $H^{\mathrm{rat}}_{0,c}$ $H^{\mathrm{trig}}_{0,c}$ are in some sense quasiclassical limits of the double-affine Hecke algebra $H_{1,\tau}$. Naturally, the theorems from the previous section have their rational and trigonometric analogs:

Theorem 7.1. *Let H be one of three described algebras: $H_{1,\tau}$, $H^{\mathrm{trig}}_{0,c}$, $H^{\mathrm{rat}}_{0,c}$, CM is the corresponding Calogero-Moser space, and e is the symmetrizer (in the finite Hecke algebra if $H = H_{1,q}$ and in the symmetric group otherwise). Then the following is true:*

(1) *The map h: $z \to ze$ is an isomorphism between $Z(H)$ and eHe.*
(2) $\mathrm{Spec}(Z(H))$ *is an irreducible smooth variety naturally isomorphic to CM.*
(3) *The Poisson structure on CM which comes from the noncommutative deformation $eH_{q,\tau}e$ ($eH^{\mathrm{trig}}_{t,c}e$, $eH^{\mathrm{rat}}_{t,c}e$ respectively) of eHe coincides (up to a constant) with the (quasi) Poisson structure on CM coming from the (quasi) Hamiltonian reduction.*
(4) *The left eHe-module He is projective and $H = \mathrm{End}_{eHe}(He)$.*

In particular, the algebras $H^{\mathrm{rat}}_{0,c}$ and $H^{\mathrm{trig}}_{0,c}$ are Azumaya algebras and for these algebras the statement of Corollary 6.2 holds with $\mathbb{C}_\tau[S_n]$ replaced by S_n.

The proof of the theorem in the case $H = H^{\mathrm{rat}}_{0,c}$ is completely parallel to the case $H = H_{1,\tau}$.

In the trigonometric case the only difficulty is that the group $GL(2,\mathbb{Z})$ does not act on $H^{\mathrm{trig}}_{0,c}$ and we do not have any analog of the duality involution. But instead of the duality transform one can use the faithful representation $\bar{g}^{\mathrm{trig}}$ of $H^{\mathrm{trig}}_{0,c}$. The representation $\bar{g}^{\mathrm{trig}}$ is the "bispectral dual" to g^{trig}; that is, the role of X_i, $1 \le i \le n$ is played by y_i, $1 \le i \le n$.

Let us describe the representation $\bar{g}^{\mathrm{trig}}$. The homomorphism $\bar{g}^{\mathrm{trig}} : H^{\mathrm{trig}}_{t,c} \to \mathbb{C}[P^{\pm 1}, y]_{\delta(y)}\# S_n$ is defined by the formulas

$$s_{i,i+1} \mapsto \bar{T}_i = s_{i,i+1} + \frac{c}{y_i - y_{i+1}}(s_{i,i+1} - 1), \quad 1 \le i \le n-1,$$
$$y_i \mapsto y_i, \quad 1 \le i \le n,$$
$$X_i \mapsto \bar{T}_i \dots \bar{T}^{-1}_{n-1} w P_1 \bar{T}^{-1}_1 \dots \bar{T}_{i-1}, \quad 1 \le i \le n.$$

where $w \in S_n$, $w(1) = n$, $w(i) = i-1$, $i = 2, \dots, n$.

Remark 7.2. It may appear that one can obtain some of our results from the rational case by a naive deformation argument. However, it is not clear how to do it, since the variety $\mathrm{Spec}(Z)$ is not compact, and when it is deformed, there is a priori a possibility of singularities arriving from infinity.

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