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ON TIGHT MONOMIALS IN QUANTIZED ENVELOPING ALGEBRAS

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ABSTRACT. In this paper, the author studies when some monomials are in the canonical basis of the quantized enveloping algebra corresponding to a simply laced semisimple finite dimensional complex Lie algebra.

0. INTRODUCTION

To any graph Γ, Lusztig has associated in [L1] and [L2] an algebra $\mathbf{U}^-$ over $\mathbf{Z}[v, v^{-1}]$ provided with a canonical basis $\mathbf{B}$. In the case that Γ is the Dynkin graph of a simply laced semisimple finite dimensional complex Lie algebra $\mathfrak{g}$, then $\mathbf{U}^-$ is the negative part of the corresponding quantized enveloping algebra $\mathbf{U}$ and $\mathbf{B}$, the canonical basis (or crystal basis).

The simplest elements in $\mathbf{U}^-$ are certain elements $F_i^{(a)}$, where i is a vertex of Γ and $a \in \mathbf{N}$. In this paper, we study when some monomials in the $F_i^{(a)}$'s are in $\mathbf{B}$. These monomials are said to be tight in that case.

In Section 1, we first recall the approach of Lusztig to this question as presented in [L2]. This comes down to studying a quadratic form $\bar{Q}_{\Omega,\mathbf{i}}$ where Ω is a quiver whose graph is Γ and $\mathbf{i} = (i_1, i_2, \ldots, i_m)$, a sequence of vertices of Γ. This is explored in more detail in Sections 2 and 3. The nicest case is when Γ is loop free. This is studied in Section 3. In Section 4, we give criteria for tightness and semi-tightness. In Sections 5 and 6, we give many examples of tight and semi-tight monomials. Some of these were already studied by Lusztig in [L2] and by Marsh in [M]. We present these examples using our approach. Finally in Section 7, we consider the case where Γ is a Dynkin graph of a simply laced semisimple finite dimensional complex Lie algebra $\mathfrak{g}$ of small rank and $\mathbf{i}$ is the reduced expression for the longest element of the Weyl group of $\mathfrak{g}$. Some of these were also studied by Lusztig in [L2] and by Marsh in [M]. In our approach, there is a unit form $Q_{\mathbf{i}}^+$ that we need to study. We do this using results of the theory of representations of algebras. They are presented in Section 6.

Our motivation was a question of Lusztig presented in Section 16 of [L2]. We recall this in the first part of Section 7. We still don't know the answer to this question, but our hope is that this article will be useful in the search of a solution.

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1. Notations and results of Lusztig

1.1. We will now recall the notation and results of [L2]. Fix a finite graph Γ with a set of vertices $I \neq \emptyset$ and a set of edges H. We assume given an orientation Ω for our graph, i.e., two maps $h \mapsto h'$ and $h \mapsto h''$ from H to I such that the ends of h are h', h''. In this way we get a quiver that we will also denote Ω. The graph Γ could have loops, that is, elements $h \in H$ such that $h' = h''$. If there are no loops, then we say that our graph is loop free. This property does not depend on Ω, but only on Γ.

In [L2], Lusztig has associated to this graph an algebra $\mathbf{U}^-$ over $\mathbf{Z}[v, v^{-1}]$ (v is an indeterminate) provided with a canonical basis $\mathbf{B}$. Our graph could have loops and this construction extends the one of [L1]. The simplest elements in $\mathbf{U}^-$ are certain elements of $F_i^{(a)}$ for various $i \in I$ and $a \in \mathbf{N}$.

A monomial in $\mathbf{U}^-$ is an element of the form

(a) $$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \dots F_{i_m}^{(a_m)}$$

where $i_1, i_2, \dots, i_m \in I$ and $a_1, a_2, \dots, a_m \in \mathbf{N}$. Such a monomial is said to be tight (resp. semi-tight) if it belongs to $\mathbf{B}$ (resp. is a linear combination of elements of $\mathbf{B}$ with constant coefficients, necessarily in $\mathbf{N}$).

1.2. For the rest of this article, we fix $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbf{N}^m$ and a sequence $\mathbf{i} = (i_1, i_2, \dots, i_m)$ in I.

1.3. For $i \in I$, define $Z(i) = \{1 \leq k \leq m \mid i_k = i\}$. Let $\mathcal{X} = \{(i, p, q) \mid i \in I,\ p, q, \in Z(i) \text{ and } 1 \leq p < q \leq m\}$, $\mathcal{Y} = \{(i, p, q) \in \mathcal{X} \mid \ \exists r \in \ Z(i) \text{ such that } p < r < q\}$ and $\mathcal{Z} = \mathcal{X} \setminus \mathcal{Y}$. When we want to specify $\mathbf{i}$, we write $\mathcal{X}(\mathbf{i})$, $\mathcal{Y}(\mathbf{i})$, $\mathcal{Z}(\mathbf{i})$ rather than $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$.

1.4. Let P' be the real vector space with coordinate functions $z_i^{p,q}$ indexed by the triples (i, p, q) such that $i \in I$ and $p, q \in Z(i)$. Define $z_i^{p,q}$ to be the 0 linear function whenever $i \in I$ and $p, q \in [1, m]$ are not both contained in $Z(i)$.

For $s \in [1, m]$, define the linear forms on P':

$$\xi_s(z) = \sum_r z_i^{r,s} \quad \text{and} \quad \xi'_s(z) = \sum_r z_i^{s,r} \quad \text{for } z = (z_i^{p,q}) \in P'.$$

Let P be the subspace of P' defined by

$$P = \bigcap_{s=1}^{m} \operatorname{Ker}(\xi_s - \xi'_s).$$

1.5. Let $T_{\mathbf{a}}$ be the finite subset of P' consisting of $z = (z_i^{p,q})$ with coordinates in $\mathbf{N}$ such that $\xi_s(z) = \xi'_s(z) = a_s, \quad \forall s \in [1, m]$. Clearly $T_{\mathbf{a}} \subset P$.

There is a distinguished vector $z_{\mathbf{a}} \in T_{\mathbf{a}}$; it has coordinates $z_i^{p,q} = 0$ if $p \neq q$, and $z_i^{p,p} = a_p$ if $p \in Z(i)$.

1.6. Consider the quadratic form $Q_{\Omega,\mathbf{i}}$ on P' defined by

$$Q_{\Omega,\mathbf{i}}(z) = \sum_{h \in H} \sum_{\substack{r \leq p \\ q < s}} z_{h'}^{p,q}\, z_{h''}^{r,s} - \sum_{i \in I} \sum_{\substack{r \leq p \\ q < s}} z_i^{p,q}\, z_i^{r,s}.$$

Obviously $Q_{\Omega,\mathbf{i}}(z_{\mathbf{a}}) = 0$.

Theorem 1.7 (Lusztig). (a) *If the quadratic form $Q_{\Omega,\mathbf{i}}$ takes only values > 0 on $T_{\mathbf{a}} \setminus \{z_{\mathbf{a}}\}$, then the monomial* 1.1 (a) *is tight.*

(b) *If the quadratic form $Q_{\Omega,\mathbf{i}}$ takes only values ≥ 0 on $T_{\mathbf{a}}$, then the monomial* 1.1 (a) *is semi-tight.*

Proof. See Theorem 6 in [L2]. □

We will use the same machinery as Lusztig in [L2] to study $Q_{\Omega,\mathbf{i}}$ on $T_{\mathbf{a}}$. The idea is to get an expression for the quadratic form $Q_{\Omega,\mathbf{i}}$ by eliminating the variables $z_i^{p,p}$.

1.8. Let V' be the real vector space with coordinate functions $w_i^{p,q}$ indexed by triples (i,p,q) such that $i \in I$ and $p,q \in Z(i)$ satisfy $p \neq q$. Define $w_i^{p,q}$ to be the 0 linear function whenever $i \in I$ and $p \neq q \in [1,m]$ are not both contained in $Z(i)$.

Let V be the subspace of V' such that

(a) $$\sum_{\substack{r,s \\ r \leq p < s}} w_i^{r,s} = \sum_{\substack{r,s \\ r \leq p < s}} w_i^{s,r} \qquad \text{for all } p \in [1,m].$$

Let V^+ be the set of all $w = (w_i^{p,q}) \in V$ such that $w_i^{p,q} \in \mathbf{N}$ for all (i,p,q) with $i \in I$ and $p \neq q \in Z(i)$.

Lemma 1.9. *V is the subspace of V' such that*

(a) $$\sum_{\substack{s \\ s \neq p}} w_i^{p,s} = \sum_{\substack{s \\ s \neq p}} w_i^{s,p} \qquad \text{for all } p \in [1,m].$$

Proof. Let $w = (w_i^{p,q}) \in V$. If $p = 1$ in 1.8 (a), then

$$\sum_{\substack{s \\ s \neq 1}} w_i^{1,s} = \sum_{\substack{r,s \\ r \leq 1 < s}} w_i^{r,s} = \sum_{\substack{r,s \\ r \leq 1 < s}} w_i^{s,r} = \sum_{\substack{s \\ s \neq 1}} w_i^{s,1}.$$

This is the first equation of Lemma 1.9 (a), the one corresponding to $p = 1$.

If $p > 1$, then

$$\begin{aligned} \sum_{\substack{s \\ p < s}} w_i^{p,s} - \sum_{\substack{r \\ r < p}} w_i^{r,p} &= \sum_{\substack{r,s \\ r \leq p < s}} w_i^{r,s} - \sum_{\substack{r,s \\ r \leq (p-1) < s}} w_i^{r,s} \\ &= \sum_{\substack{r,s \\ r \leq p < s}} w_i^{s,r} - \sum_{\substack{r,s \\ r \leq (p-1) < s}} w_i^{s,r} = \sum_{\substack{r \\ p < r}} w_i^{r,p} - \sum_{\substack{s \\ s < p}} w_i^{p,s}. \end{aligned}$$

Consequently,

$$\sum_{\substack{s \\ s \neq p}} w_i^{p,s} = \sum_{\substack{r \\ r \neq p}} w_i^{r,p}.$$

This is the pth equation of Lemma 1.9 (a) for $p > 1$. We have thus shown that V is included in the subspace of V' defined by Lemma 1.9 (a).

If $w = (w_i^{p,q})$ is an element of V' defined by Lemma 1.9 (a), we want to prove that $w \in V$. We have

$$\sum_{\substack{s \\ s \neq p'}} w_i^{p',s} = \sum_{\substack{s \\ s \neq p'}} w_i^{s,p'} \quad \Rightarrow \quad \sum_{\substack{s \\ p' < s}} w_i^{p',s} - \sum_{\substack{r \\ r < p'}} w_i^{r,p'} = \sum_{\substack{r \\ p' < r}} w_i^{r,p'} - \sum_{\substack{s \\ s < p'}} w_i^{p',s}$$

for all $p' \in [1,m]$. Thus because

$$\sum_{\substack{p' \\ p' \le p}} \left[\sum_{\substack{s \\ p' < s}} w_i^{p',s} - \sum_{\substack{r \\ r < p'}} w_i^{r,p'} \right] = \sum_{\substack{p' \\ p' \le p}} \left[\sum_{\substack{r \\ p' < r}} w_i^{r,p'} - \sum_{\substack{s \\ s < p'}} w_i^{p',s} \right],$$

we get that

$$\sum_{\substack{r,s \\ r \le p < s}} w_i^{r,s} = \sum_{\substack{r,s \\ r \le p < s}} w_i^{s,r}.$$

This is the pth equation of 1.8 (a) defining V. □

1.10. Let $\mu_{\mathbf{a}} : V' \to P'$ be the affine linear map defined by $w = (w_i^{p,q}) \mapsto z = (z_i^{p,q})$ where $z_i^{p,q} = w_i^{p,q}$ if $p \neq q$ and $z_i^{p,p} = a_p - \sum_{q, p \neq q} w_i^{p,q}$.

Lemma 1.11. (a) *$\mu_{\mathbf{a}}$ is injective and carries V into P.*

(b) *Any point $z \in T_{\mathbf{a}}$ is the image of a unique point in V^+ under $\mu_{\mathbf{a}}$. In particular, $\mu_{\mathbf{a}}(0) = z_{\mathbf{a}}$.*

(c) *$\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$ is the set of $w = (w_i^{p,q}) \in V^+$ such that we have $\sum_{q,p\neq q} w_i^{p,q} = \sum_{q,p \neq q} w_i^{q,p} \le a_p$ for all $p \in [1,m]$.*

Proof. (a) It is obvious that $\mu_{\mathbf{a}}$ is injective. If $w \in V$, then we want to prove that $z = \mu_{\mathbf{a}}(w) \in P$. We have

$$\xi_s(z) = \sum_r z_i^{r,s} = \sum_{\substack{r \\ r \neq s}} w_i^{r,s} + \left(a_s - \sum_{\substack{t \\ t \neq s}} w_i^{s,t} \right)$$

and

$$\xi'_s(z) = \sum_r z_i^{s,r} = \sum_{\substack{r \\ r \neq s}} w_i^{s,r} + \left(a_s - \sum_{\substack{t \\ t \neq s}} w_i^{s,t} \right).$$

Because $w \in V$ and by Lemma 1.9, $\xi_s(z) = \xi'_s(z)$ for all $s \in [1,m]$. Thus $z \in P$.

(b) For $z = (z_i^{p,q}) \in T_{\mathbf{a}}$, let $w = (w_i^{p,q})$ be defined by $w_i^{p,q} = z_i^{p,q}$ for $p \neq q$. Then $\mu_{\mathbf{a}}(w) = z$. In fact, we just have to compute the (i,p,p)-component

$$(\mu_{\mathbf{a}}(w))_i^{p,p} = a_p - \sum_{\substack{q \\ p \neq q}} w_i^{p,q} = a_p - \sum_{\substack{q \\ p \neq q}} z_i^{p,q} = a_p - \xi'_p(z) + z_i^{p,p} = z_i^{p,p}$$

because $\xi'_p(z) = a_p$. Obviously $\mu_{\mathbf{a}}(0) = z_{\mathbf{a}}$.

(c) If $w = (w_i^{p,q}) \in V'$ is such that $\mu_{\mathbf{a}}(w) \in T_{\mathbf{a}}$, then $w_i^{p,q} \in \mathbf{N}$ for $p \neq q$. By (a), we have $\sum_{q,p\neq q} w_i^{p,q} = \sum_{q,p\neq q} w_i^{q,p}$. We also have $a_p - \sum_{q,p\neq q} w_i^{p,q} \ge 0$. So we have just proved that

$$\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \subseteq \left\{ w = (w_i^{p,q}) \,\middle|\, w_i^{p,q} \in \mathbf{N} \text{ and } \sum_{\substack{q \\ p \neq q}} w_i^{p,q} = \sum_{\substack{q \\ p \neq q}} w_i^{q,p} \le a_p \right\}.$$

The reverse inclusion is easy to get. In fact,

$$\xi_p(\mu_{\mathbf{a}}(w)) = \sum_{\substack{q \\ q\neq p}} w_i^{q,p} + a_p - \sum_{\substack{q \\ p\neq q}} w_i^{p,q} = a_p$$

and

$$\xi'_p(\mu_{\mathbf{a}}(w)) = \sum_{\substack{q \\ q\neq p}} w_i^{p,q} + a_p - \sum_{\substack{q \\ p\neq q}} w_i^{p,q} = a_p.$$

□

1.12. Let $\bar{Q}_{\Omega,\mathbf{i}}$ be the quadratic form on V' defined by

$$\begin{aligned}\bar{Q}_{\Omega,\mathbf{i}}(w) = &\sum_{i\in I} \sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ r\leq p<s\leq q}} w_i^{p,q}\, w_i^{r,s} + \sum_{i\in I} \sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ q<s\leq p<r}} w_i^{p,q}\, w_i^{r,s}\\ &- \sum_{h\in H} \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\leq p<s\leq q}} w_{h'}^{p,q}\, w_{h''}^{r,s} - \sum_{h\in H} \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s\leq p<r}} w_{h'}^{p,q}\, w_{h''}^{r,s}.\end{aligned}$$

We will again denote by $\bar{Q}_{\Omega,\mathbf{i}}$ the restriction of $\bar{Q}_{\Omega,\mathbf{i}}$ on V.

1.13. For $s < s'$ in $[1, m]$ such that $i_s = i_{s'} = i$ and $p \notin Z(i)$ whenever $s < p < s'$, we set

$$N(s, s') = (l_i - 1)(a_s + a_{s'}) + \sum_{\substack{j\in I\\ j\neq i}} \sum_{\substack{p\in Z(j)\\ s<p<s'}} e_{i,j}\, a_p$$

where l_i is the number of edges joining i with i (loops) and $e_{i,j}$ is the number of (unoriented) edges joining i with j.

More generally, given $s < s'$ in $[1, m]$ such that $i_s = i_{s'} = i$, we set $N(s, s') = N(s_0, s_1) + N(s_1, s_2) + \cdots + N(s_{k-1}, s_k)$ where $s = s_0 < s_1 < \cdots < s_k = s'$ are the elements of $Z(i) \cap [s, s']$ in increasing order.

Let $L_{\mathbf{i},\mathbf{a}}$ be the linear form on V' defined by

$$L_{\mathbf{i},\mathbf{a}}(w) = \sum_{i\in I} \sum_{\substack{r,s\in Z(i)\\ s<r}} N(s, r) w_i^{r,s}$$

and denote again by $L_{\mathbf{i},\mathbf{a}}$ the restriction of $L_{\mathbf{i},\mathbf{a}}$ to V.

Proposition 1.14 (Lusztig). (a) *For any $w \in V$, we have $Q_{\Omega,\mathbf{i}}(\mu_{\mathbf{a}}(w)) = \bar{Q}_{\Omega,\mathbf{i}}(w) + L_{\mathbf{i},\mathbf{a}}(w)$.*

(b) *If the non-homogeneous quadratic form $\bar{Q}_{\Omega,\mathbf{i}} + L_{\mathbf{i},\mathbf{a}}$ takes only values > 0 on $\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \setminus \{0\}$, then the monomial* 1.1 (a) *is tight.*

(c) *If the non-homogeneous quadratic form $\bar{Q}_{\Omega,\mathbf{i}} + L_{\mathbf{i},\mathbf{a}}$ takes only values ≥ 0 on $\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$, then the monomial* 1.1 (a) *is semi-tight.*

Proof. (a) is Lemma 9 (c) in [L2]. (b) and (c) follow from (a) and Theorem 1.7. □

2. Study of $\bar{Q}_{\Omega,\mathbf{i}}$ on V

We will first study V by decomposing it as a direct sum $V_1 \oplus V_{-1}$ of two subspaces V_1 and V_{-1}. In the last part of this section, we will show that the quadratic form $\bar{Q}_{\Omega,\mathbf{i}}$ can be decomposed with respect to the direct sum $V_1 \oplus V_{-1}$ and that $\bar{Q}_{\Omega,\mathbf{i}}$ is also independent of Ω.

2.1. If $w = (w_i^{p,q})$, then we write ${}^{\mathrm{tr}}w = (w_i^{q,p})$ for the transpose of w. If $i \in I$, then the cardinality $|\{k \in [1,m] \mid i_k = i\}|$ will be denoted $n_{\mathbf{i}}(i)$.

Proposition 2.2. (a) *If $w \in V$ (resp. V^+, $T_{\mathbf{a}}$), then ${}^{\mathrm{tr}}w \in V$ (resp. V^+, $T_{\mathbf{a}}$).*

(b) *$V = V_1 \oplus V_{-1}$ where $V_\epsilon = \{w \in V \mid {}^{\mathrm{tr}}w = \epsilon w\}$ for $\epsilon \in \{-1,1\}$.*

(c) *V_1 is the subspace of symmetric matrices in V'. For $(i,p,q) \in \mathcal{X}$, let $u(i,p,q)$ denote the matrix whose only nonzero entries are at the positions (i,p,q) and (i,q,p) and these nonzero entries are equal to $1/2$. Then the set $\mathcal{B}_1$ of matrices $u(i,p,q)$ with $(i,p,q) \in \mathcal{X}$ is a basis of V_1 and the dimension of V_1 is $\dim(V_1) = \sum_{i\in I} n_{\mathbf{i}}(i)(n_{\mathbf{i}}(i)-1)/2$.*

(d) *V_{-1} is the subspace of skew symmetric matrices $(w_i^{p,q})$ in V' such that we have $\sum_{q,q\neq p} w_i^{p,q} = \sum_{q,q\neq p} w_i^{q,p} = 0$ whenever $p \in [1,m]$. For $(i,p,q) \in \mathcal{Y}$, let $v(i,p,q)$ denote the matrix whose entries are given by $w_i^{p,q} = 1/2$, $w_i^{q,p} = -1/2$, $w_i^{p_k,p_{k+1}} = -1/2$, $w_i^{p_{k+1},p_k} = 1/2$ where $k = 0,1,\ldots,r-1$ and $p = p_0 < p_1 < \cdots < p_r = q$ are the elements of $Z(i)$ between p and q in increasing order and all other entries of $v(i,p,q)$ are 0. Then the set $\mathcal{B}_{-1}$ of matrices $v(i,p,q)$ with $(i,p,q) \in \mathcal{Y}$ is a basis of V_{-1} and the dimension of V_{-1} is $\dim(V_{-1}) = \sum_{i\in I}(n_{\mathbf{i}}(i)-1)(n_{\mathbf{i}}(i)-2)/2$.*

(e) *$\dim(V) = \sum_{i\in I}(n_{\mathbf{i}}(i)-1)^2$.*

Proof. (a) is obvious for V and V^+ by Lemma 1.9. For $T_{\mathbf{a}}$, it follows from its definition.

(b) This is simply because $w \in V$ can be written as

$$w = \frac{(w + {}^{\mathrm{tr}}w)}{2} + \frac{(w - {}^{\mathrm{tr}}w)}{2} \quad \text{with } \frac{(w + \epsilon\, {}^{\mathrm{tr}}w)}{2} \in V_\epsilon \quad \text{for } \epsilon \in \{-1,1\}.$$

We also have $V_1 \cap V_{-1} = \{0\}$. Thus $V = V_1 \oplus V_{-1}$.

(c) Denote by Sym: the space of symmetric matrices. We want to prove that $V_1 = \mathrm{Sym} \cap V'$. It is easy to check that $V_1 \subseteq \mathrm{Sym} \cap V'$. If $w = (w_i^{p,q}) \in \mathrm{Sym} \cap V'$, then

$$\sum_{\substack{q \\ q\neq p}} w_i^{p,q} = \sum_{\substack{q \\ q\neq p}} w_i^{q,p} \quad \text{for all } p \in [1,m].$$

Thus $w \in V$ by Lemma 1.9. Because ${}^{\mathrm{tr}}w = w$, we get $w \in V_1$. It is easy to prove that $\{u(i,p,q) \mid (i,p,q) \in \mathcal{X}\}$ is a basis of V_1. From this, we get $\dim(V_1) = \sum_{i\in I} n_{\mathbf{i}}(i)(n_{\mathbf{i}}(i)-1)/2$.

(d) Denote by Skew: the space of skew symmetric matrices and by Skew_0: the subspace of Skew consisting of the matrices $w = (w_i^{p,q})$ such that

$$\sum_{\substack{q \\ q\neq p}} w_i^{p,q} = \sum_{\substack{q \\ q\neq p}} w_i^{q,p} = 0 \quad \text{for all } p \in [1,m].$$

We must prove that $V_{-1} = \mathrm{Skew}_0 \cap V'$. It is obvious that $\mathrm{Skew}_0 \cap V' \subseteq V_{-1}$. If $w = (w_i^{p,q}) \in V_{-1}$, then w is skew symmetric and is an element of V'. We also have

$$\sum_{\substack{q\\ q\neq p}} w_i^{p,q} = \sum_{\substack{q\\ q\neq p}} w_i^{q,p} = -\sum_{\substack{q\\ q\neq p}} w_i^{p,q} \quad\Rightarrow\quad \sum_{\substack{q\\ q\neq p}} w_i^{p,q} = \sum_{\substack{q\\ q\neq p}} w_i^{q,p} = 0$$

because $w \in V \cap \mathrm{Skew}$. Consequently, $w \in \mathrm{Skew}_0 \cap V'$ and $V_{-1} \subseteq \mathrm{Skew}_0 \cap V'$.

We now want to prove that $\{v(i,p,q) \mid (i,p,q) \in \mathcal{Y}\}$ is a basis of V_{-1}. It is easy to check that these vectors belong to V_{-1} and are linearly independent. We must prove that they span V_{-1}. Let $w = (w_i^{p,q}) \in V_{-1}$. We will prove that

$$w = \sum_{(i,p,q)\in\mathcal{Y}} 2w_i^{p,q}\ v(i,p,q).$$

Note that $\bar{w} = (\bar{w}_i^{p,q}) = w - \sum_{(i,p,q)\in\mathcal{Y}} 2w_i^{p,q}\ v(i,p,q)$ is an element of V_{-1} such that $\bar{w}_i^{p,q} = 0$ whenever $(i,p,q) \in \mathcal{Y}$. But this implies that $\bar{w} = 0$. In fact, if there are $p \neq q$ both belonging to $Z(i)$ such that $\bar{w}_i^{p,q} \neq 0$, then we can, by considering $\bar{w}_i^{q,p} \neq 0$ if needed, assume that $p < q$ and $(i,p,q) \in \mathcal{X}$. By our definition of $\bar{w}$, we have that $(i,p,q) \in \mathcal{Z}$. Among all such triples $(i,p,q) \in \mathcal{Z}$ with $\bar{w}_i^{p,q} \neq 0$, take one for which p is minimal. For this p, we claim that $\bar{w}_i^{p,r} = 0$ for all $r \in Z(i)$ and $r \neq q$. In fact, if $r < p$ and we assume that $\bar{w}_i^{p,r} \neq 0$, then $-\bar{w}_i^{r,p} \neq 0$ and this contradicts our choice of p. So $\bar{w}_i^{p,r} = 0$ when $r < p$. If $r = p$, then $\bar{w}_i^{p,r} = 0$ by definition of V. If $p < r$ and $r \neq q$, then $r > q$, $(i,p,r) \in \mathcal{Y}$ because $(i,p,q) \in \mathcal{Z}$, and $\bar{w}_i^{p,r} = 0$ from our construction of $\bar{w}$. From this we get a contradiction because $0 = \sum_{r, r\neq p} \bar{w}_i^{p,r} = \bar{w}_i^{p,q}$.

From this we can conclude that $\{v(i,p,q) \mid (i,p,q) \in \mathcal{Y}\}$ is a basis of V_{-1}. We get easily that $\dim(V_{-1}) = \sum_{i\in I} (n_{\mathbf{i}}(i) - 1)(n_{\mathbf{i}}(i) - 2)/2$.

(e) follows from (c) and (d). □

Lemma 2.3. *Let $w \in V$. Write w with respect to the basis $\mathcal{B}_1 \cup \mathcal{B}_{-1}$ of V:*

(a) $$w = (w_i^{p,q}) = \sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)}\ u(i,p,q) + \sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)}\ v(i,p,q).$$

Then $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$ if and only if the following six conditions are satisfied:

(1) $x_{(i,p,q)} \in \mathbf{N}$ *for all* $(i,p,q) \in \mathcal{X}$,
(2) $y_{(i,p,q)} \in \mathbf{Z}$ *and* $y_{(i,p,q)} \equiv x_{(i,p,q)} \pmod 2$ *for all* $(i,p,q) \in \mathcal{Y}$,
(3) $|y_{(i,p,q)}| \leq x_{(i,p,q)}$ *for all* $(i,p,q) \in \mathcal{Y}$,
(4) $\sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\leq p<q\leq s}} y_{(i,r,s)} \equiv x_{(i,p,q)} \pmod 2$ *for all* $(i,p,q) \in \mathcal{Z}$,
(5) $\left|\sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\leq p<q\leq s}} y_{(i,r,s)}\right| \leq x_{(i,p,q)}$ *for all* $(i,p,q) \in \mathcal{Z}$,
(6) $\sum_{q,\ (i,p,q)\in\mathcal{X}} x_{(i,p,q)} + \sum_{q,\ (i,q,p)\in\mathcal{X}} x_{(i,q,p)} \leq 2\, a_p$ *for all* $p \in [1,m]$.

For now on, we will denote the set of pairs (x,y) with $x = (x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$ and $y = (y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$ satisfying the above six conditions by $\tilde{T}_{\mathbf{a}}$.

Proof. We note that $(x_{(i,p,q)} + y_{(i,p,q)})/2 = w_i^{p,q}$ and $(x_{(i,p,q)} - y_{(i,p,q)})/2 = w_i^{q,p}$ for all $(i,p,q) \in \mathcal{Y}$. Consequently, $x_{(i,p,q)} = w_i^{p,q} + w_i^{q,p}$ and $y_{(i,p,q)} = w_i^{p,q} - w_i^{q,p}$ for all $(i,p,q) \in \mathcal{Y}$.

If $(i,p,q) \in \mathcal{Z}$, then

$$w_i^{p,q} = \frac{1}{2}\left[x_{(i,p,q)} - \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right] \text{ and } w_i^{q,p} = \frac{1}{2}\left[x_{(i,p,q)} + \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right].$$

Consequently, $x_{(i,p,q)} = w_i^{p,q} + w_i^{q,p}$ if $(i,p,q) \in \mathcal{Z}$.

If $w = (w_i^{p,q}) \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$, then $w_i^{p,q} \in \mathbf{N}$ whenever $i \in I$ and $p \neq q \in Z(i)$ and we also have $\sum_{q,p\neq q} w_i^{p,q} = \sum_{q,p\neq q} w_i^{q,p} \le a_p$ for all $p \in [1,m]$. From this, we get easily the six conditions as follows. Because $x_{(i,p,q)} = w_i^{p,q} + w_i^{q,p} \in \mathbf{N}$ if $(i,p,q) \in \mathcal{X}$, we must have condition 1. Because $y_{(i,p,q)} = w_i^{p,q} - w_i^{q,p} \in \mathbf{Z}$ and $y_{(i,p,q)} \equiv x_{(i,p,q)} \pmod 2$ if $(i,p,q) \in \mathcal{Y}$, we must have condition 2. Because $(x_{(i,p,q)} + y_{(i,p,q)}) = 2w_i^{p,q} \ge 0$ and $(x_{(i,p,q)} - y_{(i,p,q)}) = 2w_i^{q,p} \ge 0$ if $(i,p,q) \in \mathcal{Y}$, then $|y_{(i,p,q)}| \le x_{(i,p,q)}$ if $(i,p,q) \in \mathcal{Y}$ and condition 3 is verified. Because we have

$$\left(x_{(i,p,q)} - \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right) = 2w_i^{p,q} \ge 0$$

and

$$\left(x_{(i,p,q)} + \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right) = 2w_i^{q,p} \ge 0,$$

if $(i,p,q) \in \mathcal{Z}$, then both conditions 4 and 5 follow easily.

$\sum_{q,q\neq p} w_i^{p,q}$ is the sum of the entries of the pth row of w. Note that the sum of the entries of the pth row of $u(i,p',q')$ is

$$\begin{cases} 1/2, & \text{if } (i,p',q') = (i,p,q) \in \mathcal{X} \text{ for some } q; \\ 1/2, & \text{if } (i,p',q') = (i,q,p) \in \mathcal{X} \text{ for some } q; \\ 0 & \text{otherwise.} \end{cases}$$

Note also that the sum of the entries of the pth row of $v(i,p',q')$ is 0 for all $(i,p',q') \in \mathcal{Y}$, because $v(i,p',q') \in \mathrm{Skew}_0$. From this we get

$$\sum_{q,q\neq p} w_i^{p,q} = \frac{1}{2}\sum_{q,(i,p,q)\in\mathcal{X}} x_{(i,p,q)} + \frac{1}{2}\sum_{q,(i,q,p)\in\mathcal{X}} x_{(i,q,p)} \le a_p$$

if $1 \le p \le m$, and condition 6 is verified.

Conversely, if the six conditions above are satisfied, then we must show that $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$. If $(i,p,q) \in \mathcal{Y}$, then $w_i^{p,q} = (x_{(i,p,q)} + y_{(i,p,q)})/2 \in \mathbf{N}$ and $w_i^{q,p} = (x_{(i,p,q)} - y_{(i,p,q)})/2 \in \mathbf{N}$ by conditions 1, 2 and 3. If $(i,p,q) \in \mathcal{Z}$, then

$$w_i^{p,q} = \frac{1}{2}\left[x_{(i,p,q)} - \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right] \in \mathbf{N}$$

and

$$w_i^{q,p} = \frac{1}{2}\left[x_{(i,p,q)} + \sum_{\substack{(i,r,s)\in\mathcal{Y}\\ r\le p<q\le s}} y_{(i,r,s)}\right] \in \mathbf{N}$$

by conditions 1, 2, 4 and 5.

As we saw above

$$\sum_{\substack{q\\ q\neq p}} w_i^{p,q} = \frac{1}{2}\sum_{\substack{q\\ (i,p,q)\in\mathcal{X}}} x_{(i,p,q)} + \frac{1}{2}\sum_{\substack{q\\ (i,q,p)\in\mathcal{X}}} x_{(i,q,p)} \le a_p$$

by condition 6. We also have that $\sum_{q,q\neq p} w_i^{p,q} = \sum_{q,q\neq p} w_i^{q,p}$. From all of this and Lemma 1.11 (c), we see that $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$. □

Theorem 2.4. *Let $w \in V$ and write $w = u + v$ with $u \in V_1$ and $v \in V_{-1}$. Then $\bar{Q}_{\Omega,\mathbf{i}}(w) = \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(v)$.*

Proof. We have $w_i^{p,q} = u_i^{p,q} + v_i^{p,q}$ for $i \in I$ and $p \neq q \in Z(i)$. To prove that $\bar{Q}_{\Omega,\mathbf{i}}(w) = \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(v)$, we must consider

$$\sum_i \sum_{\substack{p,q,r,s,\in Z(i)\\ r\le p<s\le q}} (u_i^{p,q}+v_i^{p,q})(u_i^{r,s}+v_i^{r,s}) + \sum_i \sum_{\substack{p,q,r,s,\in Z(i)\\ q<s\le p<r}} (u_i^{p,q}+v_i^{p,q})(u_i^{r,s}+v_i^{r,s})$$
$$-\sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\le p<s\le q}} (u_{h'}^{p,q}+v_{h'}^{p,q})(u_{h''}^{r,s}+v_{h''}^{r,s}) - \sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s\le p<r}} (u_{h'}^{p,q}+v_{h'}^{p,q})(u_{h''}^{r,s}+v_{h''}^{r,s})$$

and show that it is equal to

$$\sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ r\le p<s\le q}} u_i^{p,q}u_i^{r,s} + \sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ q<s\le p<r}} u_i^{p,q}u_i^{r,s} - \sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\le p<s\le q}} u_{h'}^{p,q}u_{h''}^{r,s}$$
$$-\sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s\le p<r}} u_{h'}^{p,q}u_{h''}^{r,s} + \sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ r\le p<s\le q}} v_i^{p,q}v_i^{r,s} + \sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ q<s\le p<r}} v_i^{p,q}v_i^{r,s}$$
$$-\sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\le p<s\le q}} v_{h'}^{p,q}v_{h''}^{r,s} - \sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s\le p<r}} v_{h'}^{p,q}v_{h''}^{r,s}.$$

It is enough to prove that

(a) $$\sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ r\le p<s\le q}} (u_i^{p,q}v_i^{r,s}+v_i^{p,q}u_i^{r,s}) + \sum_i \sum_{\substack{p,q,r,s\in Z(i)\\ q<s\le p<r}} (u_i^{p,q}v_i^{r,s}+v_i^{p,q}u_i^{r,s}) = 0$$

and

$$\text{(b)}\qquad \sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\le p<s\le q}} (u_{h'}^{p,q}v_{h''}^{r,s}+v_{h'}^{p,q}u_{h''}^{r,s}) + \sum_h \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s\le p<r}} (u_{h'}^{p,q}v_{h''}^{r,s}+v_{h'}^{p,q}u_{h''}^{r,s}) = 0.$$

We will start with the equation (a) and use the fact that u is symmetric and v belongs to Skew_0. Thus the left-hand side of equation (a) is equal to

$$\sum_i \left[\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ r\le p<s\le q}} u_i^{p,q}v_i^{r,s} + \sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ p\le r<q\le s}} u_i^{p,q}v_i^{r,s} - \sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ p<r\le q<s}} u_i^{p,q}v_i^{r,s} - \sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ r<p\le s<q}} u_i^{p,q}v_i^{r,s} \right]$$
$$= \sum_{i,p,q} u_i^{p,q} \left[\sum_{r<p} (v_i^{r,q}-v_i^{r,p}) + \left(\sum_{\substack{r=p\\ p<s}} v_i^{p,s} \right) + v_i^{p,q} + \left(\sum_{p<r<q} v_i^{r,q} \right) - \left(\sum_{\substack{r=q\\ q<s}} v_i^{q,s} \right) \right]$$
$$= \sum_{i,p,q} u_i^{p,q}\,[(\text{sum of the entries in line } p \text{ of } v) + (\text{sum of entries in column } q \text{ of } v)]$$
$$= 0.$$

We now consider equation (b) and use again the fact that u is symmetric and v belongs to Skew_0. The left-hand side of equation (b) can be split into two sums; one sum is when h runs over H with $h' \neq h''$ and the other sum is when h runs over H with $h' = h''$. For the first sum, we get

$$\sum_{\substack{h\\ h'\neq h''}} \left[\sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r\le p<s\le q}} u_{h'}^{p,q}v_{h''}^{r,s} + \sum_{\substack{p,q\in Z(h'')\\ r,s\in Z(h')\\ p\le r<q\le s}} u_{h''}^{p,q}v_{h'}^{r,s} - \sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ p<r\le q<s}} u_{h'}^{p,q}v_{h''}^{r,s} - \sum_{\substack{p,q\in Z(h'')\\ r,s\in Z(h')\\ r<p\le s<q}} u_{h''}^{p,q}v_{h'}^{r,s} \right]$$
$$= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q} \left[\sum_{\substack{h\\ h'=i}} \left[\sum_{\substack{r,s\in Z(h'')\\ r\le p<s\le q}} v_{h''}^{r,s} - \sum_{\substack{r,s\in Z(h'')\\ p<r\le q<s}} v_{h''}^{r,s} \right] \right]$$
$$+ \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q} \left[\sum_{\substack{h\\ h''=i}} \left[\sum_{\substack{r,s\in Z(h')\\ p\le r<q\le s}} v_{h'}^{r,s} - \sum_{\substack{r,s\in Z(h')\\ r<p\le s<q}} v_{h'}^{r,s} \right] \right]$$
$$= 0.$$

This last sum is zero because we have that $h' \neq h''$ for these h and, if we denote $e^{i\to j} = |\{h \in H \mid h' = i, h'' = j\}|$ and $e^{i\leftarrow j} = |\{h \in H \mid h' = j, h'' = i\}|$, then

$$\begin{aligned}
&\sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{h\\ h'=i}}\left[\sum_{\substack{r,s\in Z(h'')\\ r\le p<s\le q}} v_{h''}^{r,s} - \sum_{\substack{r,s\in Z(h'')\\ p<r\le q<s}} v_{h''}^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\to j}\left[\sum_{\substack{r,s\in Z(j)\\ r<p<s<q}} v_j^{r,s} - \sum_{\substack{r,s\in Z(j)\\ p<r<q<s}} v_j^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\to j}\left[\sum_{\substack{r,s\in Z(j)\\ r<p<s<q}} v_j^{r,s} + \sum_{\substack{r,s\in Z(j)\\ p<s<q<r}} v_j^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\to j}\left[-\sum_{\substack{r,s\in Z(j)\\ p<r<q\\ p<s<q}} v_j^{r,s}\right]\right] = 0
\end{aligned}$$

because v is skew symmetric, the sum of the entries of any column of v is 0 and finally the sum of the entries of any principal submatrix of a skew symmetric matrix is 0. Similarly,

$$\begin{aligned}
&\sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{h\\ h''=i}}\left[\sum_{\substack{r,s\in Z(h')\\ p\le r<q\le s}} v_{h'}^{r,s} - \sum_{\substack{r,s\in Z(h')\\ r<p\le s<q}} v_{h'}^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\leftarrow j}\left[\sum_{\substack{r,s\in Z(j)\\ p<r<q<s}} v_j^{r,s} - \sum_{\substack{r,s\in Z(j)\\ r<p<s<q}} v_j^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\leftarrow j}\left[\sum_{\substack{r,s\in Z(j)\\ p<r<q<s}} v_j^{r,s} + \sum_{\substack{r,s\in Z(j)\\ s<p<r<q}} v_j^{r,s}\right]\right]\\
&= \sum_{\substack{i\\ p,q\in Z(i)\\ p<q}} u_i^{p,q}\left[\sum_{\substack{j\\ j\neq i}} e^{i\leftarrow j}\left[-\sum_{\substack{r,s\in Z(j)\\ p<r<q\\ p<s<q}} v_j^{r,s}\right]\right] = 0
\end{aligned}$$

because v is skew symmetric, the sum of the entries of any row of v is 0 and finally the sum of the entries of any principal submatrix of a skew symmetric matrix is 0.

For the second sum in the left-hand side of equation (b), the one where h runs over H with $h' = h''$, denote by $l_i = |\{h \in H \mid h' = h'' = i\}|$ for $i \in I$, then we get

$$\sum_{\substack{h \\ h'=h''}} \left[\sum_{\substack{p,q\in Z(h') \\ r,s\in Z(h'') \\ r\le p<s\le q}} u_{h'}^{p,q} v_{h''}^{r,s} + \sum_{\substack{p,q\in Z(h'') \\ r,s\in Z(h') \\ p\le r<q\le s}} u_{h''}^{p,q} v_{h'}^{r,s} - \sum_{\substack{p,q\in Z(h') \\ r,s\in Z(h'') \\ p<r\le q<s}} u_{h'}^{p,q} v_{h''}^{r,s} - \sum_{\substack{p,q\in Z(h'') \\ r,s\in Z(h') \\ r<p\le s<q}} u_{h''}^{p,q} v_{h'}^{r,s} \right]$$

$$= \sum_{\substack{i \\ p,q\in Z(i) \\ p<q}} l_i u_i^{p,q} \left[\sum_{\substack{r,s\in Z(i) \\ r\le p<s\le q}} v_i^{r,s} + \sum_{\substack{r,s\in Z(i) \\ p\le r<q\le s}} v_i^{r,s} - \sum_{\substack{r,s\in Z(i) \\ p<r\le q<s}} v_i^{r,s} - \sum_{\substack{r,s\in Z(i) \\ r<p\le s\le q}} v_i^{r,s} \right]$$

$$= \sum_{\substack{i \\ p,q\in Z(i) \\ p<q}} l_i u_i^{p,q} [(\text{sum of entries in line } p \text{ of } v) + (\text{sum of entries in line } q \text{ of } v)]$$

$$= 0$$

as in the case of equation (a).

Thus we have proved $\bar{Q}_{\Omega,\mathbf{i}}(w) = \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(v)$. □

Corollary 2.5. *The quadratic form $\bar{Q}_{\Omega,\mathbf{i}}$ on V is independent of Ω.*

Proof. Let Ω and Ω' be two quivers for the graph Γ. We have four maps $h \mapsto h'(\Omega)$, $h \mapsto h''(\Omega)$, $h \mapsto h'(\Omega')$ and $h \mapsto h''(\Omega')$ from $H \to I$ corresponding to these two quivers. It is enough to show the result assuming that Ω and Ω' are such that $h'(\Omega) = h'(\Omega')$ and $h''(\Omega) = h''(\Omega')$ for all $h \in H$ except for exactly one edge h_0 for which we have $h_0'(\Omega) = h_0''(\Omega')$, $h_0''(\Omega) = h_0'(\Omega')$ and $h_0'(\Omega) \neq h_0''(\Omega)$. In other words, all the edges have the same orientation in Ω and Ω' except for h_0 which has opposite orientation in Ω and Ω' and h_0 is an edge between distinct vertices. Let $w \in V$. Consider $\bar{Q}_{\Omega,\mathbf{i}}(w)$ (resp. $\bar{Q}_{\Omega',\mathbf{i}}(w)$) the quadratic form associated to $\mathbf{i}$ and Ω (resp. to $\mathbf{i}$ and Ω'). Write $w = u + v$ with $u \in V_1$ and $v \in V_{-1}$. Because u is symmetric, then we get easily that $\bar{Q}_{\Omega,\mathbf{i}}(u)$ is equal to

$$\sum_i \sum_{\substack{p,q\in Z(i) \\ r,s\in Z(i) \\ r\le p<s\le q}} u_i^{p,q} u_i^{r,s} + \sum_i \sum_{\substack{p,q\in Z(i) \\ r,s\in Z(i) \\ r<p\le s<q}} u_i^{p,q} u_i^{r,s} - \sum_{h\in H} \sum_{\substack{p,q\in Z(h'(\Omega)) \\ r,s\in Z(h''(\Omega)) \\ r\le p<s\le q}} u_{h'(\Omega)}^{p,q} u_{h''(\Omega)}^{r,s}$$
$$- \sum_{h\in H} \sum_{\substack{p,q\in Z(h''(\Omega)) \\ r,s\in Z(h'(\Omega)) \\ r<p\le s<q}} u_{h''(\Omega)}^{p,q} u_{h'(\Omega)}^{r,s}$$

while $\bar{Q}_{\Omega',\mathbf{i}}(u)$ is equal to

$$\sum_i \sum_{\substack{p,q\in Z(i) \\ r,s\in Z(i) \\ r\le p<s\le q}} u_i^{p,q} u_i^{r,s} + \sum_i \sum_{\substack{p,q\in Z(i) \\ r,s\in Z(i) \\ r<p\le s<q}} u_i^{p,q} u_i^{r,s} - \sum_{h\in H} \sum_{\substack{p,q\in Z(h'(\Omega')) \\ r,s\in Z(h''(\Omega')) \\ r\le p<s\le q}} u_{h'(\Omega')}^{p,q} u_{h''(\Omega')}^{r,s}$$
$$- \sum_{h\in H} \sum_{\substack{p,q\in Z(h''(\Omega')) \\ r,s\in Z(h'(\Omega')) \\ r<p\le s<q}} u_{h''(\Omega')}^{p,q} u_{h'(\Omega')}^{r,s}.$$

For the rest of this proof, we will write $h_0' = h_0'(\Omega) = h_0''(\Omega')$ and $h_0'' = h_0''(\Omega) = h_0'(\Omega')$. Consequently, $\bar{Q}_{\Omega',\mathbf{i}}(u) - \bar{Q}_{\Omega,\mathbf{i}}(u)$ is equal to

$$\begin{aligned}
&\sum_{\substack{p,q\in Z(h_0')\\ r,s\in Z(h_0'')\\ r\le p<s\le q}} u_{h_0'}^{p,q}u_{h_0''}^{r,s} + \sum_{\substack{p,q\in Z(h_0'')\\ r,s\in Z(h_0')\\ r<p\le s<q}} u_{h_0''}^{p,q}u_{h_0'}^{r,s} - \sum_{\substack{p,q\in Z(h_0'')\\ r,s\in Z(h_0')\\ r\le p<s\le q}} u_{h_0''}^{p,q}u_{h_0'}^{r,s} - \sum_{\substack{p,q\in Z(h_0')\\ r,s\in Z(h_0'')\\ r<p\le s<q}} u_{h_0'}^{p,q}u_{h_0''}^{r,s}\\
&= \sum_{\substack{p,q\in Z(h_0')\\ r,s\in Z(h_0'')\\ r<p<s<q}} u_{h_0'}^{p,q}u_{h_0''}^{r,s} + \sum_{\substack{p,q\in Z(h_0'')\\ r,s\in Z(h_0')\\ r<p<s<q}} u_{h_0''}^{p,q}u_{h_0'}^{r,s} - \sum_{\substack{p,q\in Z(h_0'')\\ r,s\in Z(h_0')\\ r<p<s<q}} u_{h_0''}^{p,q}u_{h_0'}^{r,s} - \sum_{\substack{p,q\in Z(h_0')\\ r,s\in Z(h_0'')\\ r<p<s<q}} u_{h_0'}^{p,q}u_{h_0''}^{r,s}\\
&= 0
\end{aligned}$$

because $h_0' \neq h_0''$.

We can use a similar argument to show that $\bar{Q}_{\Omega',\mathbf{i}}(v) - \bar{Q}_{\Omega,\mathbf{i}}(v) = 0$ whenever $v \in V_{-1}$. From Theorem 2.4, we can conclude that $\bar{Q}_{\Omega,\mathbf{i}} = \bar{Q}_{\Omega',\mathbf{i}}$ on V. □

3. The case of loop free graph Γ

3.1. For the rest of this article, we will assume that Γ is loop free. As we have seen in the previous section, the quadratic form $\bar{Q}_{\Omega,\mathbf{i}}$ on V is independent of the quiver Ω. We will denote this quadratic form on V by $\bar{Q}_{\mathbf{i}}$. We have also seen that $\bar{Q}_{\mathbf{i}}(u+v) = \bar{Q}_{\mathbf{i}}(u) + \bar{Q}_{\mathbf{i}}(v)$ where $u \in V_1$ and $v \in V_{-1}$. Because $V = V_1 \oplus V_{-1}$, we see that to analyse $\bar{Q}_{\mathbf{i}}$ on V, it is enough to consider its restriction on V_1 and V_{-1}. This is what we will do in this section.

The fact that Γ is loop free has a consequence that the restriction of $\bar{Q}_{\mathbf{i}}$ on V_1 (resp. V_{-1}) will be expressed as an integral unit form (resp. integral quadratic form) and this will enable us to use results from the theory of representation of algebras. Because all the graphs Γ we will consider in later sections are loop free, we have decided to study $\bar{Q}_{\mathbf{i}}$ in this context.

More precisely, if we write $u'(i,p,q)$ for $2u(i,p,q)$ when $(i,p,q) \in \mathcal{X}$ and $v'(i,p,q)$ for $2v(i,p,q)$ when $(i,p,q) \in \mathcal{Y}$, then we will describe

$$\bar{Q}_{\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)}u'(i,p,q)\right) \quad \text{and} \quad \bar{Q}_{\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)}v'(i,p,q)\right).$$

3.2. $Q_{\mathbf{i}}^{+}(x)$ will denote the quadratic form on V_1 defined by

$$Q_{\mathbf{i}}^{+}(x) = \bar{Q}_{\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)}u'(i,p,q)\right)$$

where (x) is equal to $(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$ and $Q_{\mathbf{i}}^{-}(y)$ will denote the quadratic form on V_{-1} defined by

$$Q_{\mathbf{i}}^{-}(y) = \bar{Q}_{\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)}v'(i,p,q)\right)$$

where (y) is equal to $(y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$.

Proposition 3.3. *With the hypothesis of* 3.1 *and the notation of* 3.2, $Q^+_{\mathbf{i}}(x)$ *is equal to*

$$\sum_{(i,p,q)\in\mathcal{X}} x^2_{(i,p,q)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,p,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,p,s)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,q)\in\mathcal{X}\\p<r<q}} x_{(i,p,q)}x_{(i,r,q)}$$
$$+ \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,q,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,q,s)} + 2\sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,s)\in\mathcal{X}\\p<r<q<s}} x_{(i,p,q)}x_{(i,r,s)} - \sum_{\substack{(i,p,q)\in\mathcal{X}\\(j,r,s)\in\mathcal{X}\\i\neq j\\p<r<q<s}} e_{i,j}\, x_{(i,p,q)}x_{(j,r,s)}$$

where $e_{i,j}$ *is the number of (unoriented) edges joining* i *and* j.

Proof. Choose a quiver Ω for the graph Γ. So we have two maps $h \mapsto h'$ and $h \mapsto h''$ from H to I. If we write $u = (u_i^{p,q}) = \sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)}u'(i,p,q)$, then

$$\text{(a)} \qquad u_i^{p,q} = \begin{cases} x_{(i,p,q)}, & \text{if } i\in I, p,q\in Z(i) \text{ and } p<q,\\ x_{(i,q,p)}, & \text{if } i\in I, p,q\in Z(i) \text{ and } q<p,\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$Q^+_{\mathbf{i}}(x) = \bar{Q}_{\Omega,\mathbf{i}}(u) = \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\r\le p<s\le q}} u_i^{p,q}u_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\q<s\le p<r}} u_i^{p,q}u_i^{r,s}$$
$$- \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\r,s\in Z(h'')\\r<p<s<q}} u_{h'}^{p,q}u_{h''}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\r,s\in Z(h'')\\q<s<p<r}} u_{h'}^{p,q}u_{h''}^{r,s}.$$

because by our hypothesis , $h' \neq h''$ for $h \in H$. From this and the fact that u is symmetric, we get that $Q^+_{\mathbf{i}}(x)$ is equal to

$$Q^+_{\mathbf{i}}(x) = \bar{Q}_{\Omega,\mathbf{i}}(u) = \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\p\le r<q\le s}} u_i^{p,q}u_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\q<s\le p<r}} u_i^{q,p}u_i^{s,r}$$
$$- \sum_{h\in H}\sum_{\substack{p,q\in Z(h'')\\r,s\in Z(h')\\p<r<q<s}} u_{h''}^{p,q}u_{h'}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\r,s\in Z(h'')\\q<s<p<r}} u_{h'}^{q,p}u_{h''}^{s,r}$$
$$= \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\p\le r<q\le s}} u_i^{p,q}u_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\r,s\in Z(i)\\p<r\le q<s}} u_i^{p,q}u_i^{r,s}$$
$$- \sum_{h\in H}\sum_{\substack{p,q\in Z(h'')\\r,s\in Z(h')\\p<r<q<s}} u_{h''}^{p,q}u_{h'}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\r,s\in Z(h'')\\p<r<q<s}} u_{h'}^{p,q}u_{h''}^{r,s}.$$

We can now consider the different possibilities for the intersection $\{p,q\}\cap\{r,s\}$ in the two summations above where $p,q,r,s \in Z(i)$. We get the following possibilities:

(1) $p = r, q = s$ in the first sum;
(2) $p = r < q < s$ in the first sum;

(3) $p < r < q = s$ in the first sum;
(4) $p < r = q < s$ in the second sum; and
(5) $p < r < q < s$ in the first and second sums.

We can also use equation 3.3 (a). Thus

$$\begin{aligned}
Q^+_{\mathbf{i}}(x) = {} & \sum_{(i,p,q)\in\mathcal{X}} x^2_{(i,p,q)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,p,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,p,s)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,q)\in\mathcal{X}\\p<r<q}} x_{(i,p,q)}x_{(i,r,q)} \\
& + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,q,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,q,s)} + 2\sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,s)\in\mathcal{X}\\p<r<q<s}} x_{(i,p,q)}x_{(i,r,s)} \\
& - \sum_{h\in H}\sum_{\substack{p,q\in Z(h'')\\r,s\in Z(h')\\p<r<q<s}} x_{(h'',p,q)}x_{(h',r,s)} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\r,s\in Z(h'')\\p<r<q<s}} x_{(h',p,q)}x_{(h'',r,s)} \\
= {} & \sum_{(i,p,q)\in\mathcal{X}} x^2_{(i,p,q)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,p,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,p,s)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,q)\in\mathcal{X}\\p<r<q}} x_{(i,p,q)}x_{(i,r,q)} \\
& + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,q,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,q,s)} + 2\sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,s)\in\mathcal{X}\\p<r<q<s}} x_{(i,p,q)}x_{(i,r,s)} \\
& - \sum_{\substack{(i,p,q)\in\mathcal{X}\\(j,r,s)\in\mathcal{X}\\i\neq j\\p<r<q<s}} \left(\left|\left\{h\in H \mid \{h',h''\}=\{i,j\}\right\}\right|\right) x_{(i,p,q)}x_{(j,r,s)} \\
= {} & \sum_{(i,p,q)\in\mathcal{X}} x^2_{(i,p,q)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,p,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,p,s)} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,q)\in\mathcal{X}\\p<r<q}} x_{(i,p,q)}x_{(i,r,q)} \\
& + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,q,s)\in\mathcal{X}\\p<q<s}} x_{(i,p,q)}x_{(i,q,s)} + 2\sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,s)\in\mathcal{X}\\p<r<q<s}} x_{(i,p,q)}x_{(i,r,s)} \\
& - \sum_{\substack{(i,p,q)\in\mathcal{X}\\(j,r,s)\in\mathcal{X}\\i\neq j\\p<r<q<s}} e_{ij}\, x_{(i,p,q)}x_{(j,r,s)}
\end{aligned}$$

□

3.4. Let $\kappa_{\mathbf{i}} : [1,m] \to \mathbf{N}$: be the function defined by $\kappa_{\mathbf{i}}(p) = |\{1 \le k \le p \mid i_k = i_p\}|$.

Proposition 3.5. *With the hypothesis of* 3.1 *and the notation of* 3.2*, the quadratic form* $Q^-_{\mathbf{i}}(y)$ *is equal to*

$$\sum_{(i,p,q)\in\mathcal{Y}} 2\big(\kappa_{\mathbf{i}}(q)-\kappa_{\mathbf{i}}(p)-1\big)\, y_{(i,p,q)}^2$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,p,s)\in\mathcal{Y}\\p<q<s}} 4\big(\kappa_{\mathbf{i}}(q)-\kappa_{\mathbf{i}}(p)-1\big)\, y_{(i,p,q)}\, y_{(i,p,s)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,q)\in\mathcal{Y}\\p<r<q}} 4\big(\kappa_{\mathbf{i}}(q)-\kappa_{\mathbf{i}}(r)-1\big)\, y_{(i,p,q)}\, y_{(i,r,q)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,s)\in\mathcal{Y}\\p<r<q<s}} \big(4\big(\kappa_{\mathbf{i}}(q)-\kappa_{\mathbf{i}}(r)\big)-2\big)\, y_{(i,p,q)}\, y_{(i,r,s)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,s)\in\mathcal{Y}\\p<r<s<q}} 4\big(\kappa_{\mathbf{i}}(s)-\kappa_{\mathbf{i}}(r)-1\big)\, y_{(i,p,q)}\, y_{(i,r,s)}$$
$$-\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(j,r,s)\in\mathcal{Y}\\i\neq j\\p<r<q<s}} e_{i,j}\, \alpha_{(i,p,q),(j,r,s)}\, y_{(i,p,q)}\, y_{(j,r,s)}$$
$$-\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(j,r,s)\in\mathcal{Y}\\i\neq j\\p<r<s<q}} e_{i,j}\, \beta_{(i,p,q),(j,r,s)}\, y_{(i,p,q)}\, y_{(j,r,s)}$$

where $e_{i,j}$ is the number of (unoriented) edges joining i and j, $\alpha_{(i,p,q),(j,r,s)}$ is equal to

$$\begin{aligned}
&|\{((i,p',q'),(j,r',s'))\in\mathcal{Z}\times\mathcal{Z}\mid p\le p'<r'<q'<s'\le s \text{ and } r\le r'<q'\le q\}|\\
&\quad+|\{((i,p',q'),(j,r',s'))\in\mathcal{Z}\times\mathcal{Z}\mid r\le r'<p'<s'<q'\le q\}|\\
&\quad-|\{(i,p',q')\in\mathcal{Z}\mid p\le p'<r<q'\le q\}|\\
&\quad-|\{(j,r',s')\in\mathcal{Z}\mid r\le r'<q<s'\le s\}|+1
\end{aligned}$$

and $\beta_{(i,p,q),(j,r,s)}$ is equal to

$$\begin{aligned}
&|\{((i,p',q'),(j,r',s'))\in\mathcal{Z}\times\mathcal{Z}\mid p\le p'<r'<q'<s'\le s \text{ and } r\le r'\}|\\
&\quad+|\{((i,p',q'),(j,r',s'))\in\mathcal{Z}\times\mathcal{Z}\mid r\le r'<p'<s'<q'\le q \text{ and } s'\le s\}|\\
&\quad-|\{(i,p',q')\in\mathcal{Z}\mid p\le p'<r<q'<s\}|\\
&\quad-|\{(i,p',q')\in\mathcal{Z}\mid r<p'<s<q'\le q\}|
\end{aligned}$$

Proof. Choose a quiver Ω for the graph Γ. So we have two maps $h \mapsto h'$ and $h \mapsto h''$ from H to I. If we write $v = (v_i^{p,q}) = \sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v'(i,p,q)$, then

$$\text{(a)}\quad v_i^{p,q} = \begin{cases} y_{(i,p,q)}, & \text{if } i\in I, p,q\in Z(i),\ p<q \text{ and } (i,p,q)\in\mathcal{Y},\\ -y_{(i,q,p)}, & \text{if } i\in I, p,q\in Z(i),\ q<p \text{ and } (i,q,p)\in\mathcal{Y},\\ -\sum_{\substack{(i,p',q')\in\mathcal{Y}\\ p'\le p<q\le q'}} y_{(i,p',q')}, & \text{if } i\in I, p,q\in Z(i),\ p<q \text{ and } (i,p,q)\in\mathcal{Z},\\ \sum_{\substack{(i,p',q')\in\mathcal{Y}\\ p'\le q<p\le q'}} y_{(i,p',q')}, & \text{if } i\in I, p,q\in Z(i),\ q<p \text{ and } (i,q,p)\in\mathcal{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$\begin{aligned} Q_{\mathbf{i}}^{-}(y) = \bar{Q}_{\Omega,\mathbf{i}}(v) = &\sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ r\le p<s\le q}} v_i^{p,q}v_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ q<s\le p<r}} v_i^{p,q}v_i^{r,s}\\ &-\sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ r<p<s<q}} v_{h'}^{p,q}v_{h''}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s<p<r}} v_{h'}^{p,q}v_{h''}^{r,s}\end{aligned}$$

because by our hypothesis, $h' \neq h''$ for $h \in H$. From this and the fact that v is skew symmetric, we get that $Q_{\mathbf{i}}^{-}(y)$ is equal to

$$\begin{aligned} Q_{\mathbf{i}}^{-}(y) = \bar{Q}_{\Omega,\mathbf{i}}(v) = &\sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ p\le r<q\le s}} v_i^{p,q}v_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ q<s\le p<r}} v_i^{q,p}v_i^{s,r}\\ &-\sum_{h\in H}\sum_{\substack{p,q\in Z(h'')\\ r,s\in Z(h')\\ p<r<q<s}} v_{h''}^{p,q}v_{h'}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ q<s<p<r}} v_{h'}^{q,p}v_{h''}^{s,r}\\ = &\sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ p\le r<q\le s}} v_i^{p,q}v_i^{r,s} + \sum_{i\in I}\sum_{\substack{p,q\in Z(i)\\ r,s\in Z(i)\\ p<r\le q<s}} v_i^{p,q}v_i^{r,s}\\ &-\sum_{h\in H}\sum_{\substack{p,q\in Z(h'')\\ r,s\in Z(h')\\ p<r<q<s}} v_{h''}^{p,q}v_{h'}^{r,s} - \sum_{h\in H}\sum_{\substack{p,q\in Z(h')\\ r,s\in Z(h'')\\ p<r<q<s}} v_{h'}^{p,q}v_{h''}^{r,s}.\end{aligned}$$

We can now consider the different possibilities for the intersection $\{p,q\}\cap\{r,s\}$ in the two summations above where $p,q,r,s\in Z(i)$. We get the following possibilities:

(1) $p=r, q=s$ in the first sum;
(2) $p=r<q<s$ in the first sum;
(3) $p<r<q=s$ in the first sum;
(4) $p<r=q<s$ in the second sum; and
(5) $p<r<q<s$ in the first and second sums.

We get that

$$Q_{\mathbf{i}}^{-}(y) = \sum_{(i,p,q)\in\mathcal{X}} (v_i^{p,q})^2 + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,p,s)\in\mathcal{X}\\p<q<s}} v_i^{p,q}v_i^{p,s} + \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,q)\in\mathcal{X}\\p<r<q}} v_i^{p,q}v_i^{r,q}$$
$$+ \sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,q,s)\in\mathcal{X}\\p<q<s}} v_i^{p,q}v_i^{q,s} + 2\sum_{\substack{(i,p,q)\in\mathcal{X}\\(i,r,s)\in\mathcal{X}\\p<r<q<s}} v_i^{p,q}v_i^{r,s} - \sum_{h\in H}\sum_{\substack{(h'',p,q)\in\mathcal{X}\\(h',r,s)\in\mathcal{X}\\p<r<q<s}} v_{h''}^{p,q}v_{h'}^{r,s}$$
$$- \sum_{h\in H}\sum_{\substack{(h',p,q)\in\mathcal{X}\\(h'',r,s)\in\mathcal{X}\\p<r<q<s}} v_{h'}^{p,q}v_{h''}^{r,s}.$$

To complete now, we must use equation 3.5 (a). We replace all of these components by the appropriate expression, $Q_{\mathbf{i}}^{-}(y)$ is equal to

$$\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)}^2 + \sum_{(i,p,q)\in\mathcal{Z}} \left[-\sum_{\substack{(i,p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(i,p',q')}\right]^2 + \sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,p,s)\in\mathcal{Y}\\p<q<s}} y_{(i,p,q)}y_{(i,p,s)}$$
$$- \sum_{\substack{(i,p,q)\in\mathcal{Z}\\(i,p,s)\in\mathcal{Y}\\p<q<s}} \left[\sum_{\substack{(i,p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(i,p',q')}\right] y_{(i,p,s)} + \sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,q)\in\mathcal{Y}\\p<r<q}} y_{(i,p,q)}y_{(i,r,q)}$$
$$- \sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,q)\in\mathcal{Z}\\p<r<q}} y_{(i,p,q)} \left[\sum_{\substack{(i,r',q')\in\mathcal{Y}\\r'\leq r<q\leq q'}} y_{(i,r',q')}\right] + \sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,q,s)\in\mathcal{Y}\\p<q<s}} y_{(i,p,q)}y_{(i,q,s)}$$
$$- \sum_{\substack{(i,p,q)\in\mathcal{Z}\\(i,q,s)\in\mathcal{Y}\\p<q<s}} \left[\sum_{\substack{(i,p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(i,p',q')}\right] y_{(i,q,s)} - \sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,q,s)\in\mathcal{Z}\\p<q<s}} y_{(i,p,q)} \left[\sum_{\substack{(i,q',s')\in\mathcal{Y}\\q'\leq q<s\leq s'}} y_{(i,q',s')}\right]$$
$$+ \sum_{\substack{(i,p,q)\in\mathcal{Z}\\(i,q,s)\in\mathcal{Z}\\p<q<s}} \left[\sum_{\substack{(i,p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(i,p',q')}\right] \left[\sum_{\substack{(i,q'',s')\in\mathcal{Y}\\q''\leq q<s\leq s'}} y_{(i,q'',s')}\right] + 2\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,s)\in\mathcal{Y}\\p<r<q<s}} y_{(i,p,q)}y_{(i,r,s)}$$
$$- \sum_{h\in H}\sum_{\substack{(h'',p,q)\in\mathcal{Y}\\(h',r,s)\in\mathcal{Y}\\p<r<q<s}} y_{(h'',p,q)}y_{(h',r,s)} + \sum_{h\in H}\sum_{\substack{(h'',p,q)\in\mathcal{Z}\\(h',r,s)\in\mathcal{Y}\\p<r<q<s}} \left[\sum_{\substack{(h'',p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(h'',p',q')}\right] y_{(h',r,s)}$$

$$+\sum_{h\in H}\sum_{\substack{(h'',p,q)\in\mathcal{Y}\\(h',r,s)\in\mathcal{Z}\\p<r<q<s}} y_{(h'',p,q)}\left[\sum_{\substack{(h',r',s')\in\mathcal{Y}\\r'\leq r<s\leq s'}} y_{(h',r',s')}\right]$$

$$-\sum_{h\in H}\sum_{\substack{(h'',p,q)\in\mathcal{Z}\\(h',r,s)\in\mathcal{Z}\\p<r<q<s}}\left[\sum_{\substack{(h'',p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(h'',p',q')}\right]\left[\sum_{\substack{(h',r',s')\in\mathcal{Y}\\r'\leq r<s\leq s'}} y_{(h',r',s')}\right]$$

$$-\sum_{h\in H}\sum_{\substack{(h',p,q)\in\mathcal{Y}\\(h'',r,s)\in\mathcal{Y}\\p<r<q<s}} y_{(h',p,q)}y_{(h'',r,s)}+\sum_{h\in H}\sum_{\substack{(h',p,q)\in\mathcal{Z}\\(h'',r,s)\in\mathcal{Y}\\p<r<q<s}}\left[\sum_{\substack{(h',p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(h',p',q')}\right]y_{(h'',r,s)}$$

$$+\sum_{h\in H}\sum_{\substack{(h',p,q)\in\mathcal{Y}\\(h'',r,s)\in\mathcal{Z}\\p<r<q<s}} y_{(h',p,q)}\left[\sum_{\substack{(h'',r',s')\in\mathcal{Y}\\r'\leq r<s\leq s'}} y_{(h'',r',s')}\right]$$

$$-\sum_{h\in H}\sum_{\substack{(h',p,q)\in\mathcal{Z}\\(h'',r,s)\in\mathcal{Z}\\p<r<q<s}}\left[\sum_{\substack{(h',p',q')\in\mathcal{Y}\\p'\leq p<q\leq q'}} y_{(h',p',q')}\right]\left[\sum_{\substack{(h'',r',s')\in\mathcal{Y}\\r'\leq r<s\leq s'}} y_{(h'',r',s')}\right].$$

Expanding and regrouping the terms of this summation, we get that $Q^-_{\mathbf{i}}(y)$ is equal to

$$\sum_{(i,p,q)\in\mathcal{Y}} A_{p,q}\, y^2_{(i,p,q)}+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,p,s)\in\mathcal{Y}\\p<q<s}} B_{p,q,p,s}\, y_{(i,p,q)}y_{(i,p,s)}+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,q)\in\mathcal{Y}\\p<r<q}} C_{p,q,r,q}\, y_{(i,p,q)}y_{(i,r,q)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,q,s)\in\mathcal{Y}\\p<q<s}} D_{p,q,q,s}\, y_{(i,p,q)}y_{(i,q,s)}+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,s)\in\mathcal{Y}\\p<r<q<s}} E_{p,q,r,s}\, y_{(i,p,q)}y_{(i,r,s)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(i,r,s)\in\mathcal{Y}\\p<r<s<q}} F_{p,q,r,s}\, y_{(i,p,q)}y_{(i,r,s)}+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(j,r,s)\in\mathcal{Y}\\i\neq j\\p<r<q<s}} A'_{p,q,r,s}\, y_{(i,p,q)}y_{(j,r,s)}$$
$$+\sum_{\substack{(i,p,q)\in\mathcal{Y}\\(j,r,s)\in\mathcal{Y}\\i\neq j\\p<r<s<q}} B'_{p,q,r,s}\, y_{(i,p,q)}y_{(j,r,s)}$$

where

$$
\begin{aligned}
A_{p,q} = {} & 1 + |\{(i,p',q') \in \mathcal{Z} \mid p < p' < q' < q\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le q\}|;
\end{aligned}
$$

$$
\begin{aligned}
B_{p,q,p,s} = {} & 1 + |\{(i,p',q') \in \mathcal{Z} \mid p < p' < q' \le q\}| \\
& + |\{(i,p',q') \in \mathcal{Z} \mid p < p' < q' < q\}| - |\{(i,q,s') \in \mathcal{Z} \mid q < s' \le s\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le q\}| \\
& + |\{((i,p',q'),(i,q's')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le s \text{ and } q' \le q\}|;
\end{aligned}
$$

$$
\begin{aligned}
C_{p,q,r,q} = {} & 1 + |\{(i,p',q') \in \mathcal{Z} \mid r < p' < q' < q\}| \\
& + |\{(i,p',q') \in \mathcal{Z} \mid r \le p' < q' < q\}| - |\{(i,p',r) \in \mathcal{Z} \mid p \le p' < r\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid r \le p' < q' < s' \le q\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le q \text{ and } r \le q' \le q\}|;
\end{aligned}
$$

$$
\begin{aligned}
D_{p,q,q,s} = {} & 1 - |\{(i,p',q) \in \mathcal{Z} \mid p < p' < q\}| - |\{(i,q,s') \in \mathcal{Z} \mid q < s' < s\}| \\
& + |\{((i,p',q),(i,q,s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q < s' \le s\}|;
\end{aligned}
$$

$$
\begin{aligned}
E_{p,q,r,s} = {} & 2|\{(i,p',q') \in \mathcal{Z} \mid r \le p' < q' \le q\}| - |\{(i,r,r') \in \mathcal{Z} \mid r < r' \le q\}| \\
& - |\{(i,p',q) \in \mathcal{Z} \mid r \le p' < q\}| - |\{(i,p',r) \in \mathcal{Z} \mid p \le p' < r\}| \\
& - |\{(i,q,s') \in \mathcal{Z} \mid q < s' \le s\}| + 2 \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid r \le p' < q' < s' \le q\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le s \text{ and } r \le q' \le q\}|;
\end{aligned}
$$

$$
\begin{aligned}
F_{p,q,r,s} = {} & 2|\{(i,p',q') \in \mathcal{Z} \mid r \le p' < q' \le s\}| - |\{(i,r,r') \in \mathcal{Z} \mid r < r' < s\}| \\
& - |\{(i,r',s) \in \mathcal{Z} \mid r < r' < s\}| - |\{(i,p',r) \in \mathcal{Z} \mid p \le p' < r\}| \\
& - |\{(i,s,s') \in \mathcal{Z} \mid s < s' \le q\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid r \le p' < q' < s' \le q \text{ and } r \le q' \le s\}| \\
& + |\{((i,p',q'),(i,q',s')) \in \mathcal{Z} \times \mathcal{Z} \mid p \le p' < q' < s' \le s \text{ and } r \le q' \le s\}|;
\end{aligned}
$$

$$
\begin{aligned}
A'_{p,q,r,s} = {} & -|\{h \in H \mid h' = j, h'' = i\}| \\
& + \left| \left\{ (h,p',q') \,\middle|\, \begin{array}{l} h \in H, 1 \le p', q' \le m, h' = j, h'' = i, \\ (i,p',q') \in \mathcal{Z}, p \le p' < r < q' \le q \end{array} \right\} \right| \\
& + \left| \left\{ (h,r',s') \,\middle|\, \begin{array}{l} h \in H, 1 \le r', s' \le m, h' = j, h'' = i \\ (j,r',s') \in \mathcal{Z}, r \le r' < q < s' \le s \end{array} \right\} \right| \\
& - \left| \left\{ (h,p',q',r',s') \,\middle|\, \begin{array}{l} h \in H, 1 \le p', q', r', s' \le m, h' = j, h'' = i \\ (i,p',q') \in \mathcal{Z}, (j,r',s') \in \mathcal{Z} \\ p \le p' < r' < q' < s' \le s, r \le r' < q' \le q \end{array} \right\} \right| \\
& - \left| \left\{ (h,p',q',r',s') \,\middle|\, \begin{array}{l} h \in H, 1 \le p', q', r', s' \le m, h' = i, h'' = j \\ (i,r',s') \in \mathcal{Z}, (j,p',q') \in \mathcal{Z} \\ r \le p' < r' < q' < s' \le q \end{array} \right\} \right|
\end{aligned}
$$

$$
\begin{aligned}
&- |\{h \in H \mid h' = i, h'' = j\}| \\
&+ \left|\left\{(h,p',q') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q' \le m, h' = i, h'' = j, \\ (i,p',q') \in \mathcal{Z}, p \le p' < r < q' \le q \end{matrix}\right\}\right| \\
&+ \left|\left\{(h,r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le r', s' \le m, h' = i, h'' = j \\ (j,r',s') \in \mathcal{Z}, r \le r' < q < s' \le s \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = i, h'' = j \\ (i,p',q') \in \mathcal{Z}, (j,r',s') \in \mathcal{Z} \\ p \le p' < r' < q' < s' \le s, r \le r' < q' \le q \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = j, h'' = i \\ (i,r',s') \in \mathcal{Z}, (j,p',q') \in \mathcal{Z} \\ r \le p' < r' < q' < s' \le q \end{matrix}\right\}\right|
\end{aligned}
$$

$$
\begin{aligned}
B'_{p,q,r,s} = &\left|\left\{(h,p',q') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q' \le m, h' = j, h'' = i \\ (i,p',q') \in \mathcal{Z}, p \le p' < r < q' < s \end{matrix}\right\}\right| \\
&+ \left|\left\{(h,r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le r', s' \le m, h' = i, h'' = j \\ (i,r',s') \in \mathcal{Z}, r < r' < s < s' \le q \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = j, h'' = i \\ (i,p',q') \in \mathcal{Z}, (j,r',s') \in \mathcal{Z} \\ p \le p' < r' < q' < s' \le s, r \le r' \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = i, h'' = j \\ (i,r',s') \in \mathcal{Z}, (j,p',q') \in \mathcal{Z} \\ r \le p' < r' < q' < s' \le q, q' \le s \end{matrix}\right\}\right| \\
&+ \left|\left\{(h,p',q') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q' \le m, h' = i, h'' = j \\ (i,p',q') \in \mathcal{Z}, p \le p' < r < q' < s \end{matrix}\right\}\right| \\
&+ \left|\left\{(h,r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le r', s' \le m, h' = j, h'' = i \\ (i,r',s') \in \mathcal{Z}, r < r' < s < s' \le q \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = i, h'' = j \\ (i,p',q') \in \mathcal{Z}, (j,r',s') \in \mathcal{Z} \\ p \le p' < r' < q' < s' \le s, r \le r' \end{matrix}\right\}\right| \\
&- \left|\left\{(h,p',q',r',s') \,\middle|\, \begin{matrix} h \in H, 1 \le p', q', r', s' \le m, h' = j, h'' = i \\ (i,r',s') \in \mathcal{Z}, (j,p',q') \in \mathcal{Z} \\ r \le p' < r' < q' < s' \le q, q' \le s \end{matrix}\right\}\right|.
\end{aligned}
$$

We can easily compute the cardinality of the sets appearing in each of the expressions for the coefficients in $Q^-_{\mathbf{i}}(y)$ and get the proposition. We will leave this part to the reader. □

Remark 3.6. Propositions 3.3 and 3.5 show that both $Q_{\mathbf{i}}^{+}(x)$ and $Q_{\mathbf{i}}^{-}(y)$ are integral quadratic forms, i.e., the coefficients in the formula for $Q_{\mathbf{i}}^{+}(x)$ (resp. $Q_{\mathbf{i}}^{-}(y)$) relative to the coordinates $(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$ (resp. $(y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$) are integers. Moreover, $Q_{\mathbf{i}}^{+}(x)$ is also a unit form. This means that the coefficients of the $x_{(i,p,q)}^2$ in the formula for $Q_{\mathbf{i}}^{+}(x)$ are all equal to 1. Unit forms have been extensively studied. See for example [Ri].

We will denote the restriction of $Q_{\mathbf{i}}^{+}(x)$ (resp. $Q_{\mathbf{i}}^{-}(y)$) on $\mathbf{Z}^{|\mathcal{X}|}$ (resp. $\mathbf{Z}^{|\mathcal{Y}|}$) also by $Q_{\mathbf{i}}^{+}(x)$ (resp. $Q_{\mathbf{i}}^{-}(y)$).

4. Criteria for tightness and semi-tightness

In this section, we will give criteria for a monomial to be tight or semi-tight.

4.1. Let $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ be the integral quadratic form defined by

$$\mathcal{Q}(z) = \sum_{i=1}^{n} b_{i,i} z_i^2 + \sum_{1\le i<j\le n} b_{i,j} z_i z_j$$

for $(z) = (z_i)_{1\le i\le n}$ and where $b_{i,i}, b_{i,j} \in \mathbf{Z}$ for all i, j. To $\mathcal{Q}$, we can associate the bilinear form $B_{\mathcal{Q}} : \mathbf{Z}^n \times \mathbf{Z}^n \to \mathbf{Z}$ defined by $B_{\mathcal{Q}}(z, z') = \mathcal{Q}(z + z') - \mathcal{Q}(z) - \mathcal{Q}(z')$ for all $z, z' \in \mathbf{Z}^n$. We get easily that $\mathcal{Q}(z) = B_{\mathcal{Q}}(z, z)/2$.

We can also associate to $\mathcal{Q}$ the symmetric matrix $A(\mathcal{Q})$ (relative to the coordinates $(z_i)_{1\le i\le n}$) defined by

$$A(\mathcal{Q}) = (a_{i,j})_{1\le i,j\le n} \quad \text{with} \quad a_{i,j} = \begin{cases} 2b_{i,i} & \text{if } 1\le i\le n; \\ b_{i,j} & \text{if } 1\le i<j\le n; \\ b_{j,i} & \text{if } 1\le j<i\le n. \end{cases}$$

For the rest of this paper, $A_{\mathbf{i}}^{+}$ (resp. $A_{\mathbf{i}}^{-}$) will denote the symmetric matrix $A(Q_{\mathbf{i}}^{+})$ (resp. $A(Q_{\mathbf{i}}^{-})$) corresponding to the integral quadratic form $Q_{\mathbf{i}}^{+} : \mathbf{Z}^{|\mathcal{X}|} \to \mathbf{Z}$ (resp. $Q_{\mathbf{i}}^{-} : \mathbf{Z}^{|\mathcal{Y}|} \to \mathbf{Z}$) defined by $x \mapsto Q_{\mathbf{i}}^{+}(x)$ (resp. $y \mapsto Q_{\mathbf{i}}^{-}(y)$) for $x = (x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$ (resp. $y = (y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$) where the formula for $Q_{\mathbf{i}}^{+}(x)$ (resp. $Q_{\mathbf{i}}^{-}(y)$) is given in Proposition 3.3 (resp. 3.5). Here the coordinates $x_{(i,p,q)}$ (resp. $y_{(i,p,q)}$) for $(i,p,q) \in \mathcal{X}$ (resp. $(i,p,q) \in \mathcal{Y}$) are ordered lexicographically.

4.2. An integral quadratic form $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ is said to be positive definite (resp. non-negative definite) if and only if $\mathcal{Q}(z) > 0$ for all $z \in \mathbf{Z}^n$, $z \ne 0$ (resp. $\mathcal{Q}(z) \ge 0$ for all $z \in \mathbf{Z}^n$).

4.3. An integral quadratic form $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ is said to be weakly positive (resp. weakly non-negative) if and only if $\mathcal{Q}(z) > 0$ (resp. $\mathcal{Q}(z) \ge 0$) for all $z \in \mathbf{N}^n$, $z \ne 0$ (resp. for all $z \in \mathbf{N}^n$).

4.4. Let $\mathcal{C}(\mathbf{i})$ be the set of $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbf{N}^m$ such that for all $s < s' \in [1, m]$ with $i_s = i_{s'} = i$ and $p \notin Z(i)$ whenever $s < p < s'$, we have

$$N(s, s') = -(a_s + a_{s'}) + \sum_{\substack{j\in I\\ j\ne i}} \sum_{\substack{p\in Z(j)\\ s<p<s'}} e_{i,j} a_p \ge 0.$$

Recall that $e_{i,j}$ is the number of (unoriented) edges joining i to j, $\mathbf{i} = (i_1, i_2, \dots, i_m)$ and that we assume that our graph Γ is loop free.

$\mathcal{C}^\circ(\mathbf{i})$ will denote the subset of $\mathcal{C}(\mathbf{i})$ consisting of the elements $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ such that for all $s < s' \in [1, m]$ with $i_s = i_{s'} = i$ and $p \notin Z(i)$ whenever $s < p < s'$, we have

$$N(s, s') = -(a_s + a_{s'}) + \sum_{\substack{j \in I \\ j \neq i}} \sum_{\substack{p \in Z(j) \\ s < p < s'}} e_{i,j} a_p > 0.$$

In other words, $\mathcal{C}^\circ(\mathbf{i})$ is the interior of $\mathcal{C}(\mathbf{i})$.

Proposition 4.5. *Let $Q^+_{\mathbf{i}}$ be the unit form defined by*

$$Q^+_{\mathbf{i}} : \mathbf{Z}^{|\mathcal{X}|} \to \mathbf{Z}, \quad x = (x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}} \mapsto Q^+_{\mathbf{i}}(x)$$

as in Proposition 3.3*, $Q^-_{\mathbf{i}}$ the integral quadratic form defined by*

$$Q^-_{\mathbf{i}} : \mathbf{Z}^{|\mathcal{Y}|} \to \mathbf{Z}, \quad y = (y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}} \mapsto Q^-_{\mathbf{i}}(y)$$

as in Proposition 3.5 *and $L^+_{\mathbf{i},\mathbf{a}} : \mathbf{Z}^{|\mathcal{X}|} \to \mathbf{Z}$ the linear map defined by*

$$L^+_{\mathbf{i},\mathbf{a}}(x) = \sum_{(i,p,q)\in\mathcal{X}} N(p,q)\, x_{(i,p,q)} \qquad \text{for } x = (x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}.$$

Recall that $N(p, q)$ has been defined in 1.13.

(a) *If $Q^+_{\mathbf{i}}(x) + Q^-_{\mathbf{i}}(y) + 2L^+_{\mathbf{i},\mathbf{a}}(x) > 0$ for all $(x, y) \in \tilde{T}_{\mathbf{a}}$ with $x \neq 0$, then the monomial* 1.1 (a) *is tight. Recall that $\tilde{T}_{\mathbf{a}}$ has been defined in Lemma* 2.3.

(b) *If $Q^+_{\mathbf{i}}(x) + Q^-_{\mathbf{i}}(y) + 2L^+_{\mathbf{i},\mathbf{a}}(x) \geq 0$ for all $(x, y) \in \tilde{T}_{\mathbf{a}}$, then the monomial* 1.1 (a) *is semi-tight.*

(c) *If $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathcal{C}(\mathbf{i})$, the unit form $Q^+_{\mathbf{i}}$ is weakly non-negative and the integral quadratic form $Q^-_{\mathbf{i}}$ is non-negative, then the monomial* 1.1 (a) *is semi-tight.*

(d) *If $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathcal{C}(\mathbf{i})$, the unit form $Q^+_{\mathbf{i}}$ is weakly positive and the integral quadratic form $Q^-_{\mathbf{i}}$ is non-negative, then the monomial* 1.1 (a) *is tight.*

(e) *If $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathcal{C}^\circ(\mathbf{i})$, the unit form $Q^+_{\mathbf{i}}$ is weakly non-negative and the integral quadratic form $Q^-_{\mathbf{i}}$ is non-negative, then the monomial* 1.1 (a) *is tight.*

Proof. Choose a quiver Ω for the graph Γ. By Proposition 1.14, the monomial 1.1 (a) is tight (resp. semi-tight) if $\bar{Q}_{\Omega,\mathbf{i}} + L_{\mathbf{i},\mathbf{a}}$ takes only values > 0 (resp. ≥ 0) on $\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \setminus \{0\}$ (resp. $\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$). So we have to study $\bar{Q}_{\Omega,\mathbf{i}} + L_{\mathbf{i},\mathbf{a}}$.

If $w = u + v$ where

$$u = \sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u(i,p,q) \in V_1 \quad \text{and} \quad v = \sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v(i,p,q) \in V_{-1},$$

then $\bar{Q}_{\Omega,\mathbf{i}}(w) + L_{\mathbf{i},\mathbf{a}}(w) = \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(v) + L_{\mathbf{i},\mathbf{a}}(u) + L_{\mathbf{i},\mathbf{a}}(v)$ because of Theorem 2.4 and the fact that $L_{\mathbf{i},\mathbf{a}}$ is linear. With the notation of 3.1 and 3.2, we can evaluate each of these terms:

$$\bar{Q}_{\Omega,\mathbf{i}}(u) = \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u(i,p,q)\right) = \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} \left[\frac{1}{2}\right] u'(i,p,q)\right)$$
$$= \left[\frac{1}{4}\right] \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u'(i,p,q)\right) = \left[\frac{1}{4}\right] Q^+_{\mathbf{i}}(x)$$

where $(x) = (x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$.

$$\bar{Q}_{\Omega,\mathbf{i}}(v) = \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v(i,p,q)\right) = \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} \left[\frac{1}{2}\right] v'(i,p,q)\right)$$
$$= \left[\frac{1}{4}\right] \bar{Q}_{\Omega,\mathbf{i}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v'(i,p,q)\right) = \left[\frac{1}{4}\right] Q^{-}_{\mathbf{i}}(y)$$

where $(y) = (y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$.

$$L_{\mathbf{i},\mathbf{a}}(u) = L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u(i,p,q)\right) = L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} \left[\frac{1}{2}\right] u'(i,p,q)\right)$$
$$= \left[\frac{1}{2}\right] L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u'(i,p,q)\right) = \left[\frac{1}{2}\right]\left(\sum_{(i,p,q)\in\mathcal{X}} N(p,q) x_{(i,p,q)}\right).$$

This follows from the definition of $L_{\mathbf{i},\mathbf{a}}$ and the fact that the matrix $\bar{u} = 2u = (\bar{u}_i^{p,q}) = \sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u'(i,p,q)$ is symmetric and

$$\bar{u}_i^{p,q} = \begin{cases} x_{(i,p,q)}, & \text{if } i \in I, p,q \in Z(i) \text{ and } p < q, \\ x_{(i,q,p)}, & \text{if } i \in I, p,q \in Z(i) \text{ and } q < p, \\ 0, & \text{otherwise.} \end{cases}$$

So $L_{\mathbf{i},\mathbf{a}}(u) = L^{+}_{\mathbf{i},\mathbf{a}}(x)/2$ with the above notation.

$$L_{\mathbf{i},\mathbf{a}}(v) = L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v(i,p,q)\right) = L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} \left[\frac{1}{2}\right] v'(i,p,q)\right)$$
$$= \left[\frac{1}{2}\right] L_{\mathbf{i},\mathbf{a}}\left(\sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v'(i,p,q)\right).$$

The matrix $\bar{v} = 2v = (\bar{v}_i^{p,q}) = \sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v'(i,p,q)$ is skew symmetric and

$$\bar{v}_i^{p,q} = \begin{cases} y_{(i,p,q)}, & \text{if } i \in I, p,q \in Z(i),\ p < q \text{ and } (i,p,q) \in \mathcal{Y}, \\ -y_{(i,q,p)}, & \text{if } i \in I, p,q \in Z(i),\ q < p \text{ and } (i,q,p) \in \mathcal{Y}, \\ -\sum_{\substack{(i,p',q')\in\mathcal{Y} \\ p'\le p<q\le q'}} y_{(i,p',q')}, & \text{if } i \in I, p,q \in Z(i),\ p < q \text{ and } (i,p,q) \in \mathcal{Z}, \\ \sum_{\substack{(i,p',q')\in\mathcal{Y} \\ p'\le q<p\le q'}} y_{(i,p',q')}, & \text{if } i \in I, p,q \in Z(i),\ q < p \text{ and } (i,q,p) \in \mathcal{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned}
L_{\mathbf{i},\mathbf{a}}(v) &= \frac{1}{2}\left(\sum_{i\in I}\sum_{\substack{r,s\in Z(i)\\ s<r}} N(s,r)\bar{v}_i^{r,s}\right)\\
&= \frac{1}{2}\left[-\left(\sum_{(i,p,q)\in\mathcal{Y}} N(p,q)y_{(i,p,q)}\right)+\sum_{(i,p,q)\in\mathcal{Z}} N(p,q)\left(\sum_{\substack{(i,p',q')\in\mathcal{Y}\\ p'\le p<q\le q'}} y_{(i,p',q')}\right)\right]\\
&= \frac{1}{2}\sum_{(i,p,q)\in\mathcal{Y}}\left[\left(\sum_{\substack{(i,p',q')\in\mathcal{Z}\\ p\le p'<q'\le q}} N(p',q')\right)-N(p,q)\right]y_{(i,p,q)}=0
\end{aligned}$$

because of the definitions of $N(s,s')$ and $\mathcal{Z}$.

With the above notation, we get that $\bar{Q}_{\Omega,\mathbf{i}}(w)+L_{\mathbf{i},\mathbf{a}}(w)>0$ (resp. ≥ 0) if and only if $Q^+_{\mathbf{i}}(x)+Q^-_{\mathbf{i}}(y)+2L^+_{\mathbf{i},\mathbf{a}}(x)>0$ (resp. ≥ 0). We can now prove the proposition.

(a) and (b) By Lemma 2.3, $w\in\mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}})$ if and only if $(x,y)\in\tilde{T}_{\mathbf{a}}$ with $x=(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}$ and $y=(y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}$ as above. Moreover, $w=0$ if and only if $x=0$ by condition 3 in Lemma 2.3. (a) and (b) follow easily from Proposition 1.14.

(c) $Q^+_{\mathbf{i}}(x)\geq 0$ for all $(x)=(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}\in\mathbf{N}^{|\mathcal{X}|}$ because $Q^+_{\mathbf{i}}$ is weakly non-negative.

If $\mathbf{a}\in\mathcal{C}(\mathbf{i})$, then $N(p,q)\geq 0$ for all $(i,p,q)\in\mathcal{X}$ by the definition of $N(s,s')$ given in 1.13 and the definition of $\mathcal{C}(\mathbf{i})$. Consequently, $L^+_{\mathbf{i},\mathbf{a}}(x)\geq 0$ for all $(x)=(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}\in\mathbf{N}^{|\mathcal{X}|}$.

We have that $Q^-_{\mathbf{i}}(y)\geq 0$ for all $(y)=(y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}\in\mathbf{Z}^{|\mathcal{Y}|}$, because $Q^-_{\mathbf{i}}$ is non-negative.

Finally, we get that $Q^+_{\mathbf{i}}(x)+Q^-_{\mathbf{i}}(y)+2L^+_{\mathbf{i},\mathbf{a}}(x)\geq 0$ for all $(x,y)\in\tilde{T}_{\mathbf{a}}$ and from (b), we can conclude that the monomial 1.1(a) is semi-tight.

(d) By our proof of (c), we have that $Q^+_{\mathbf{i}}(x)\geq 0$, $Q^-_{\mathbf{i}}(y)\geq 0$ and $L^+_{\mathbf{i},\mathbf{a}}(x)\geq 0$ whenever $(x,y)\in\tilde{T}_{\mathbf{a}}$.

If we now assume that $Q^+_{\mathbf{i}}(x)+Q^-_{\mathbf{i}}(y)+2L^+_{\mathbf{i},\mathbf{a}}(x)=0$, then we must have $Q^+_{\mathbf{i}}(x)=0$, $Q^-_{\mathbf{i}}(y)=0$ and $L^+_{\mathbf{i},\mathbf{a}}(x)=0$ where $(x)=(x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}}\in\mathbf{N}^{|\mathcal{X}|}$ and $(y)=(y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}}\in\mathbf{Z}^{|\mathcal{Y}|}$ are such that $(x,y)\in\tilde{T}_{\mathbf{a}}$. Because $Q^+_{\mathbf{i}}$ is weakly positive, then $x=0$. Because of condition 3 in the definition of $\tilde{T}_{\mathbf{a}}$, we get that $y=0$. Consequently, from our observation above, we can conclude that the monomial 1.1 (a) is tight.

(e) By (c), the monomial 1.1 (a) is semi-tight. By our hypothesis, we have $Q^+_{\mathbf{i}}(x)\geq 0$, $Q^-_{\mathbf{i}}(y)\geq 0$ and $L^+_{\mathbf{i},\mathbf{a}}(x)\geq 0$ whenever $(x,y)\in\tilde{T}_{\mathbf{a}}$. If we now assume that $Q^+_{\mathbf{i}}(x)+Q^-_{\mathbf{i}}(y)+2L^+_{\mathbf{i},\mathbf{a}}(x)=0$, then we must have that $L^+_{\mathbf{i},\mathbf{a}}(x)=0$. Because $\mathbf{a}\in\mathcal{C}^\circ(\mathbf{i})$, then all the coefficients $N(p,q)$ of $L^+_{\mathbf{i},\mathbf{a}}(x)$ are >0. From this, we get that $x=0$ because $x\in\mathbf{N}^{|\mathcal{X}|}$. By condition 3 in the definition of $\tilde{T}_{\mathbf{a}}$, we get that $y=0$. Consequently, we can conclude from (a) that the monomial 1.1 (a) is tight. □

5. Some examples

In this section, we will give examples of tight and semi-tight monomials for different algebras $\mathbf{U}^-$. In some of our examples below, the graph Γ will be the Dynkin graph of a simple simply laced finite dimensional complex Lie algebra $\mathfrak{g}$ and, in these cases, we will use the same notation for the Dynkin graph as the one in Bourbaki [B].

The next three examples were considered by Lusztig in [L2]. We will study in Section 7 the case when Γ is the Dynkin graph of a simple simply laced finite dimensional complex Lie algebra $\mathfrak{g}$ of small rank and $\mathbf{i}$ is a reduced expression of the longest element of the Weyl group of $\mathfrak{g}$. These three examples are particular cases of this.

Example 5.1. Let Γ be the Dynkin graph of type A_2 and $\mathbf{i} = (1, 2, 1)$. Then $\dim V_1 = 1$, $V_{-1} = 0$ and $V = V_1$. We get that $A^+_{\mathbf{i}} = (2)$. Because $Q^+_{\mathbf{i}}$ is positive definite and $\dim(V_{-1}) = 0$, we get from Proposition 4.5 that the monomial $F_1^{(a_1)} F_2^{(a_2)} F_1^{(a_3)}$ is tight when $a_1, a_2, a_3 \in \mathbf{N}$ and $a_2 \geq a_1 + a_3$.

Example 5.2. Let Γ be the Dynkin graph of type A_3 and $\mathbf{i} = (1, 2, 1, 3, 2, 1)$. Then $\dim(V_1) = 4$, $\dim(V_{-1}) = 1$ and $\dim(V) = 5$. We get that

$$A^+_{\mathbf{i}} = \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

and $A^-_{\mathbf{i}} = (4)$. By computing the primary principal minors of $A^+_{\mathbf{i}}$, we get that $Q^+_{\mathbf{i}}$ is positive definite. In fact, the eigenvalues of $A^+_{\mathbf{i}}$ are 1, 2, $(5 + \sqrt{17})/2$ and $(5 - \sqrt{17})/2$. Obviously $Q^-_{\mathbf{i}}$ is positive definite. Using Proposition 4.5, we get that the monomial $F_1^{(a_1)} F_2^{(a_2)} F_1^{(a_3)} F_3^{(a_4)} F_2^{(a_5)} F_1^{(a_6)}$ is tight when $a_i \in \mathbf{N}$ for $i = 1, \dots, 6$ and

$$a_2 \geq a_1 + a_3, \quad a_3 + a_4 \geq a_2 + a_5 \quad \text{and} \quad a_5 \geq a_3 + a_6.$$

Example 5.3. Let Γ be the Dynkin graph of type A_3 and $\mathbf{i} = (2, 1, 3, 2, 1, 3)$. Then $\dim(V_1) = 3$, $\dim(V_{-1}) = 0$ and $\dim(V) = 3$. We get that

$$A^+_{\mathbf{i}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

By computing the primary principal minors of $A^+_{\mathbf{i}}$, we get that $Q^+_{\mathbf{i}}$ is positive definite. In fact, the eigenvalues of $A^+_{\mathbf{i}}$ are 2, $(2 + \sqrt{2})$ and $(2 - \sqrt{2})$. Using Proposition 4.5, we get that the monomial $F_2^{(a_1)} F_1^{(a_2)} F_3^{(a_3)} F_2^{(a_4)} F_1^{(a_5)} F_3^{(a_6)}$ is tight when $a_i \in \mathbf{N}$ for $i = 1, \dots, 6$ and

$$a_2 + a_3 \geq a_1 + a_4, \quad a_4 \geq a_2 + a_5 \quad \text{and} \quad a_4 \geq a_3 + a_6.$$

The previous example and the example of Lusztig in the case of affine A_1 presented in section 12 of [L2] can be generalized. This is done next.

Proposition 5.4. *Let Γ be the Dynkin graph or the extended Dynkin graph of a simply laced simple finite dimensional complex Lie algebra $\mathfrak{g}$ of rank n. Fix a total order $\{j_1, j_2, \dots, j_n\}$ on the set I of vertices of Γ. Let $\mathbf{i}$ be the sequence*

$\mathbf{i} = (i_1, i_2, \ldots, i_{2n})$ of length $m = 2n$ such that $i_k = i_{k+n} = j_k$ for $1 \le k \le n$ and let $\mathbf{a} = (a_1, a_2, \ldots, a_{2n}) \in \mathbf{N}^{2n}$. If $\mathbf{a} \in \mathcal{C}(\mathbf{i})$, then the monomial

$$F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \ldots F_{i_{2n}}^{(a_{2n})} \tag{a}$$

is semi-tight. Moreover, if Γ is the Dynkin graph of $\mathfrak{g}$, then the monomial 5.4 (a) *is tight; while if Γ is the extended Dynkin graph of $\mathfrak{g}$ and at least one of the inequalities defining $\mathcal{C}(\mathbf{i})$ in* 4.4 *is strict, then again the monomial* 5.4 (a) *is tight.*

Proof. We can note that $\dim(V_{-1}) = 0$. For each $i \in I$, there exists a unique pair $p(i), q(i)$ such that $1 \le p(i) \le n < q(i) \le 2n$, $q(i) = p(i) + n$ and $i_{p(i)} = i_{q(i)} = i$. With this notation, $\mathcal{X} = \{(i, p(i), q(i)) \mid i \in I\}$. By Proposition 3.3, we get easily that

$$Q_{\mathbf{i}}^{+}(x) = \sum_{i \in I} x_{(i,p(i),q(i))}^2 - \sum_{\{i,j\} \in H} e_{i,j}\; x_{(i,p(i),q(i))} x_{(j,p(j),q(j))}$$

for $x = (x_{(i,p(i),q(i))})_{(i,p(i),q(i)) \in \mathcal{X}}$. If Γ is not the graph of A_1 affine, then $e_{i,j} = 1$ for all $i, j \in I$ such that $\{i, j\} \in H$. If Γ is the graph of A_1 affine, then $|I| = 2$ and $e_{i,j} = 2$ when $i \ne j$. It is well known that this quadratic form $Q_{\mathbf{i}}^{+}$ is positive definite when Γ is the Dynkin graph of $\mathfrak{g}$ and it is non-negative definite when Γ is the extended Dynkin graph of $\mathfrak{g}$. (See for example chapter 1 in [Ri].) Because of Proposition 4.5 (c), we get that the monomial 5.4 (a) is semi-tight when Γ is the Dynkin graph or the extended Dynkin graph of $\mathfrak{g}$.

If Γ is the Dynkin graph of $\mathfrak{g}$, then we get that the monomial 5.4 (a) is tight by Proposition 4.5 (d).

If Γ is the extended Dynkin graph of $\mathfrak{g}$, then $Q_{\mathbf{i}}^{+}(x)$ is non-negative definite and it is well known that there exists a vector $\tilde{x} \in \mathbf{N}^n$ whose components are all > 0 such that the radical of $Q_{\mathbf{i}}^{+}$ is $\{x \in \mathbf{Z}^n \mid Q_{\mathbf{i}}^{+}(x) = 0\} = \mathbf{Z}\tilde{x}$. (See again chapter 1 in [Ri].) If $Q_{\mathbf{i}}^{+}(x) = 0$ and $L_{\mathbf{i}}^{+}(x) = 0$ for some $x \in \mathbf{N}^n$, then $x = k\tilde{x}$ for some $k \in \mathbf{N}$ and $L_{\mathbf{i}}^{+}(x) = kL_{\mathbf{i}}^{+}(\tilde{x}) = 0$. If at least one of the inequalities defining $\mathcal{C}(\mathbf{i})$ is strict, then $L_{\mathbf{i}}^{+}(\tilde{x}) > 0$ because all the components of $\tilde{x}$ are > 0 and all the coefficients of $L_{\mathbf{i}}^{+}$ are ≥ 0 with at least one > 0. Consequently, $k = 0$ and $x = 0$. Using the same argument as the one used in the proof of Proposition 4.5 (d), we can conclude that $y = 0$ and consequently $Q_{\mathbf{i}}^{+}(x) + Q_{\mathbf{i}}^{-}(y) + 2L_{\mathbf{i},\mathbf{a}}^{+}(x) > 0$ for $(x, y) \in \tilde{T}_{\mathbf{a}}$, $x \ne 0$. So the monomial 5.4 (a) is tight when at least one of the inequalities defining $\mathcal{C}(\mathbf{i})$ is strict. □

The next results will be used in Section 7.

5.5. Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ and $\mathbf{j} = (i_1, j_2, \ldots, j_m)$ be two sequences in I. We say that $\mathbf{i}$ and $\mathbf{j}$ are related by a commutation if and only if there exists an integer a such that $1 \le a \le (m-1)$, $i_a \ne i_{(a+1)}$,

$$j_k = \begin{cases} i_{a+1}, & \text{if } k = a, \\ i_a, & \text{if } k = a+1, \\ i_k, & \text{if } k \ne a, (a+1), \end{cases}$$

and $\{i_a, i_{(a+1)}\}$ is not an edge in Γ.

In this case $\phi_{\mathbf{i}}^{\mathbf{j}}$ will denote the function $\phi_{\mathbf{i}}^{\mathbf{j}} : \mathcal{X}(\mathbf{i}) \to \mathcal{X}(\mathbf{j})$ defined by

$$\phi_{\mathbf{i}}^{\mathbf{j}}(i,p,q) = \begin{cases} (i, a+1, q), & \text{if } p = a < q, \\ (i, p, a+1), & \text{if } p < q = a, \\ (i, a, q), & \text{if } p = (a+1) < q, \\ (i, p, a), & \text{if } p < q = (a+1), \\ (i, p, q), & \text{if } \{p,q\} \cap \{a, a+1\} = \emptyset. \end{cases}$$

Note that $|\{p,q\} \cap \{a,a+1\}| \leq 1$ above, because $(i,p,q) \in \mathcal{X}(\mathbf{i})$ and $i_a \neq i_{a+1}$.

It is easy to see that $\phi_{\mathbf{i}}^{\mathbf{j}}$ is a bijection whose inverse is $\phi_{\mathbf{j}}^{\mathbf{i}}$ and such that $\phi_{\mathbf{i}}^{\mathbf{j}}(\mathcal{Y}(\mathbf{i})) = \mathcal{Y}(\mathbf{j})$ and $\phi_{\mathbf{i}}^{\mathbf{j}}(\mathcal{Z}(\mathbf{i})) = \mathcal{Z}(\mathbf{j})$.

Lemma 5.6. *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two sequences in* I *of length* m *related by a commutation as in* 5.5 *and denote the bijection* $\phi_{\mathbf{i}}^{\mathbf{j}} : \mathcal{X}(\mathbf{i}) \to \mathcal{X}(\mathbf{j})$ *by* ϕ.

(a) *If* $T_{\mathbf{i}}^{\mathbf{j}} : \mathbf{Z}^{|\mathcal{X}(\mathbf{i})|} \to \mathbf{Z}^{|\mathcal{X}(\mathbf{j})|}$ *is the linear map defined by* $T_{\mathbf{i}}^{\mathbf{j}}\big((x_{(i,p,q)})_{(i,p,q)\in\mathcal{X}(\mathbf{i})}\big) = (x'_{(i',p',q')})_{(i',p',q')\in\mathcal{X}(\mathbf{j})}$ *where* $x'_{(i',p',q')} = x_{\phi^{-1}(i',p',q')}$ *for all* $(i',p',q') \in \mathcal{X}(\mathbf{j})$, *then* $T_{\mathbf{i}}^{\mathbf{j}}$ *is an isomorphism whose inverse is* $T_{\mathbf{j}}^{\mathbf{i}}$ *and such that* $T_{\mathbf{i}}^{\mathbf{j}}(\mathbf{N}^{|\mathcal{X}(\mathbf{i})|}) = \mathbf{N}^{|\mathcal{X}(\mathbf{j})|}$ *and* $Q_{\mathbf{i}}^{+} = Q_{\mathbf{j}}^{+} \circ T_{\mathbf{i}}^{\mathbf{j}}$. *In particular,* $Q_{\mathbf{i}}^{+}$ *is weakly positive (resp. weakly non-negative, positive definite, non-negative definite) if and only if* $Q_{\mathbf{j}}^{+}$ *is weakly positive (resp. weakly non-negative, positive definite, non-negative definite). In the case that* $Q_{\mathbf{i}}^{+}$ *is weakly positive, then*

$$|\{x \in \mathbf{N}^{|\mathcal{X}(\mathbf{i})|} \mid Q_{\mathbf{i}}^{+}(x) = 1\}| = |\{x \in \mathbf{N}^{|\mathcal{X}(\mathbf{j})|} \mid Q_{\mathbf{j}}^{+}(x) = 1\}|$$

(b) *If* $\bar{T}_{\mathbf{i}}^{\mathbf{j}} : \mathbf{Z}^{|\mathcal{Y}(\mathbf{i})|} \to \mathbf{Z}^{|\mathcal{Y}(\mathbf{j})|}$ *is the linear map defined by* $\bar{T}_{\mathbf{i}}^{\mathbf{j}}\big((y_{(i,p,q)})_{(i,p,q)\in\mathcal{Y}(\mathbf{i})}\big) = (y'_{(i',p',q')})_{(i',p',q')\in\mathcal{Y}(\mathbf{j})}$ *where* $y'_{(i',p',q')} = y_{\phi^{-1}(i',p',q')}$ *for all* $(i',p',q') \in \mathcal{Y}(\mathbf{j})$, *then* $\bar{T}_{\mathbf{i}}^{\mathbf{j}}$ *is an isomorphism whose inverse is* $\bar{T}_{\mathbf{j}}^{\mathbf{i}}$ *and such that* $Q_{\mathbf{i}}^{-} = Q_{\mathbf{j}}^{-} \circ \bar{T}_{\mathbf{i}}^{\mathbf{j}}$. *In particular,* $Q_{\mathbf{i}}^{-}$ *is positive definite (resp. non-negative definite) if and only if* $Q_{\mathbf{j}}^{-}$ *is positive definite (resp. non-negative definite).*

(c) *If* $\psi_{\mathbf{i}}^{\mathbf{j}} : \mathbf{Z}^m \to \mathbf{Z}^m$ *is the linear map defined by* $\phi_{\mathbf{i}}^{\mathbf{j}}(z_1, \ldots, z_m) = (z'_1, \ldots, z'_m)$ *where*

$$z'_k = \begin{cases} z_{a+1}, & \text{if } k = a, \\ z_a, & \text{if } k = a+1, \\ z_k, & \text{if } k \neq a, a+1, \end{cases}$$

then $\psi_{\mathbf{i}}^{\mathbf{j}}$ *is an isomorphism whose inverse is* $\psi_{\mathbf{j}}^{\mathbf{i}}$ *and* $\psi_{\mathbf{i}}^{\mathbf{j}}(\mathbf{N}^m) = \mathbf{N}^m$ *and* $\psi_{\mathbf{i}}^{\mathbf{j}}(\mathcal{C}(\mathbf{i})) = \mathcal{C}(\mathbf{j})$.

Proof. It is easy to prove both (a) and (b) using the formulae in Propositions 3.3 and 3.5. (c) follows also easily from the definition of $\mathcal{C}(\mathbf{i})$. These arguments are left to the reader. □

5.7. Let $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ and $\mathbf{j} = (i_1, j_2, \ldots, j_m)$ be two sequences in I. We write $\mathbf{i} \sim \mathbf{j}$ if and only if there exists a sequence $\mathbf{i} = \mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_n = \mathbf{j}$ of sequences in I such that, for each $0 \leq k \leq (n-1)$, $\mathbf{i}_k$, $\mathbf{i}_{(k+1)}$ are related by a commutation.

Clearly $\sim$ is an equivalence relation and the equivalence classes of $\sim$ are called commutation classes.

Corollary 5.8. *Let* $\mathbf{i}$, $\mathbf{j}$ *be two sequences in* I *such that* $\mathbf{i} \sim \mathbf{j}$.

(a) $Q_{\mathbf{i}}^+$ *is weakly positive (resp. weakly non-negative, positive definite, non-negative definite) if and only if* $Q_{\mathbf{j}}^+$ *is weakly positive (resp. weakly non-negative, positive definite, non-negative definite). Moreover, in the case that* $Q_{\mathbf{i}}^+$ *is weakly positive, then*

$$|\{z \in \mathbf{N}^{|\mathcal{X}(\mathbf{i})|} \mid Q_{\mathbf{i}}^+(z) = 1\}| = |\{z \in \mathbf{N}^{|\mathcal{X}(\mathbf{j})|} \mid Q_{\mathbf{j}}^+(z) = 1\}|.$$

(b) $Q_{\mathbf{i}}^-$ *is positive definite (resp. non-negative definite) if and only if* $Q_{\mathbf{j}}^-$ *is positive definite (resp. non-negative definite).*

(c) *There exists an isomorphism* $\psi : \mathbf{Z}^m \to \mathbf{Z}^m$ *such that* $\psi(\mathbf{N}^m) = \mathbf{N}^m$ *and* $\psi(\mathcal{C}(\mathbf{i})) = \mathcal{C}(\mathbf{j})$.

Proof. This is simply an easy application of Lemma 5.6. □

6. Use of criteria from representation theory of algebras

We will need for the next examples a way of knowing when an integral unit form is weakly positive. We will now describe the algorithm of de la Peña presented in [P] to decide if an integral unit form is weakly positive.

6.1. Let $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ be an integral unit form. We start with $C_1 = \{e_i \mid 1 \le i \le n\}$, the standard basis of $\mathbf{Z}^n$, i.e. e_i is the unique element of $\mathbf{Z}^n$ whose coordinates are all 0 except for its ith component which is equal to 1. We define inductively a procedure for constructing a new set C_{a+1} from C_a. The procedure could fail and in that case C_{a+1} is not defined and the procedure stops, indicating that $\mathcal{Q}$ is not weakly positive. Otherwise, it continues.

Assume that $C_a = \{z^1, z^2, \ldots, z^k\} \subset \mathbf{Z}^n$ and the procedure has not failed (to be defined subsequently). We now construct C_{a+1} as follows. Let $z^j = (z_1^j, z_2^j, \ldots, z_n^j)$. If either

(a) there is some $1 \le i \le n$ such that $B_{\mathcal{Q}}(z^j, e_i) \le -2$, or
(b) there is some $1 \le i \le n$ such that $z_i^j \ge 7$,

then the procedure is said to fail. If the procedure has not failed (in other words, (a) and (b) have not occurred for any $z^j \in C_a$), let $R_a \subseteq C_a$ be those z^j with the property that there is some $1 \le i \le n$ such that $B_{\mathcal{Q}}(z^j, e_i) = -1$. If $R_a = \emptyset$, then set $C_{a+1} = \emptyset$ and the procedure is said to be successful. If $R_a \ne \emptyset$, then set $C_{a+1} = \{z^j + e_i \mid z^j \in R_a \text{ and } e_i \text{ is such that } B_{\mathcal{Q}}(z^j, e_i) = -1\}$.

Because of (b) above, this procedure will stop after a finite number of steps.

Proposition (de la Peña). *The integral unit form* $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ *is weakly positive if and only if the above procedure is successful. Moreover, if the procedure is successful with* $C_{a+1} = \emptyset$*, then* $C_1 \cup C_2 \cup \cdots \cup C_a$ *is the set* $\{z \in \mathbf{N}^n \mid \mathcal{Q}(z) = 1\}$ *of positive roots of* $\mathcal{Q}$.

Proof. See [P]. □

The next example is presented in Proposition 15 of [L2] and is a particular case of Proposition 4.1 in [M]. In our case, we can lift the conditions of strict inequalities in the statement of the proposition of Lusztig for the monomial to be tight.

Example 6.2. Let Γ be the Dynkin graph of type A_4 and let $\mathbf{i}$ be the sequence $(1, 3, 2, 4, 1, 3, 2, 4, 1, 3)$. Then $\dim(V_1) = 8$, $\dim(V_{-1}) = 2$ and $\dim(V) = 10$. We get that

$$A_{\mathbf{i}}^{+} = \begin{pmatrix} 2 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{i}}^{-} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

The integral unit form $Q_{\mathbf{i}}^{+}$ is weakly positive. We get this by using the algorithm of de la Peña presented in 6.1. In fact, we get that the procedure is successful and that

$$\begin{aligned} C_1 &= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}, \\ C_2 &= \{(e_1+e_4), (e_3+e_4), (e_4+e_5), (e_4+e_7), (e_5+e_8), (e_7+e_8)\}, \\ C_3 &= \left\{ \begin{matrix} (e_1+e_4+e_5), (e_1+e_4+e_7), (e_3+e_4+e_5), \\ (e_3+e_4+e_7), (e_4+e_5+e_8), (e_4+e_7+e_8) \end{matrix} \right\}, \\ C_4 &= \left\{ \begin{matrix} (e_1+e_4+e_5+e_8), (e_1+e_4+e_7+e_8), (e_3+e_4+e_5+e_8), \\ (e_3+e_4+e_7+e_8), (e_4+e_5+e_7+e_8) \end{matrix} \right\}, \\ C_5 &= \{(e_1+e_4+e_5+e_7+e_8), (e_3+e_4+e_5+e_7+e_8)\}, \\ C_6 &= \{(e_1+2e_4+e_5+e_7+e_8), (e_3+2e_4+e_5+e_7+e_8)\}, \text{ and} \\ C_7 &= \{(e_1+e_3+2e_4+e_5+e_7+e_8)\}. \end{aligned}$$

Obviously, $Q_{\mathbf{i}}^{-}$ is positive definite. Note that $Q_{\mathbf{i}}^{+}$ is not positive definite. In fact, the eigenvalues of $A_{\mathbf{i}}^{+}$ are 1, 1, 2, 2, $(2+\sqrt{5})$, $(2-\sqrt{5})$ $(3+\sqrt{5})$ and $(3-\sqrt{5})$.

Using Proposition 4.5, we get that the monomial

$$F_1^{(a_1)} F_3^{(a_2)} F_2^{(a_3)} F_4^{(a_4)} F_1^{(a_5)} F_3^{(a_6)} F_2^{(a_7)} F_4^{(a_8)} F_1^{(a_9)} F_3^{(a_{10})}$$

is tight when $a_i \in \mathbf{N}$ for $i = 1, \ldots, 10$ and

$$\begin{aligned} &a_3 \geq a_1 + a_5, \quad a_3 + a_4 \geq a_2 + a_6, \quad a_5 + a_6 \geq a_3 + a_7, \\ &a_6 \geq a_4 + a_8, \qquad\quad a_7 \geq a_5 + a_9, \quad a_7 + a_8 \geq a_6 + a_{10}. \end{aligned}$$

There are determinantal criteria to test if an integral quadratic form $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ is weakly positive (resp. weakly non-negative). We will now present a criteria for weakly non-negative due to Keller (see [K]) and one for weakly positive due to Motzkin (see [Mo]).

Proposition 6.3. *Let $\mathcal{Q} : \mathbf{Z}^n \to \mathbf{Z}$ be an integral quadratic form and $A(\mathcal{Q})$, its corresponding symmetric matrix. $\mathcal{Q}$ is not weakly non-negative (resp. weakly positive) if and only if there is a principal submatrix D of $A(\mathcal{Q})$ with* $\det(D) < 0$ *(resp. ≤ 0) for which the cofactors of the last column are non-negative (resp. positive).*

Proof. Denote the extension of $\mathcal{Q}$ to $\mathbf{R}^n$ by $\mathcal{Q}_{\mathbf{R}} : \mathbf{R}^n \to \mathbf{R}$. We have $A(\mathcal{Q}) = A(\mathcal{Q}_{\mathbf{R}})$. If $\mathcal{Q}$ is weakly non-negative (resp. weakly positive), then $\mathcal{Q}_{\mathbf{R}}(x) \geq 0$ (resp. > 0) when $x = (x_1, x_2, \ldots, x_n) \in \mathbf{Q}^n$, $x_i \geq 0$ for all $1 \leq i \leq n$ and $x \neq 0$. By continuity, if $\mathcal{Q}$ is weakly positive or weakly non-negative, we get that $\mathcal{Q}_{\mathbf{R}}$ is weakly non-negative. Obviously, if $\mathcal{Q}_{\mathbf{R}}$ is weakly non-negative (resp. weakly positive), then $\mathcal{Q}$ is weakly non-negative (resp. weakly positive).

In the proof of a proposition of Drozd in section 1.1 of [Ri], Ringel showed that if $\mathcal{Q}$ is weakly positive, then $\mathcal{Q}_{\mathbf{R}}$ is weakly positive. Consequently, $\mathcal{Q}$ is weakly non-negative (resp. weakly positive) if and only if $\mathcal{Q}_{\mathbf{R}}$ is weakly non-negative (resp. weakly positive).

Then the proposition follows by theorem 4.7 in [V] and the fact that $A(\mathcal{Q}) = A(\mathcal{Q}_{\mathbf{R}})$. In [V], weakly non-negative is called copositive and weakly positive is called strictly copositive. □

Example 6.4. Let Γ be the Dynkin graph of type D_4 and let $\mathbf{i}$ be the sequence $(2,1,3,4,2,1,3,4,2,1,3,4)$. Then $\dim(V_1) = 12$, $\dim(V_{-1}) = 4$ and $\dim(V) = 16$. We get that the symmetric matrix $A^+_{\mathbf{i}}$ is then

$$A^+_{\mathbf{i}} = \begin{pmatrix} 2 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 2 & 1 & 0 & -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 1 & 2 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of $A^+_{\mathbf{i}}$ are $(4+2\sqrt{3})$, $(4-2\sqrt{3})$, 4, $(1+\sqrt{3})$, $(1-\sqrt{3})$, 4, $(1+\sqrt{3})$, $(1-\sqrt{3})$, 1, 1, 1 and 1. So $Q^+_{\mathbf{i}}$ is not positive definite nor non-negative definite. Using the algorithm of de la Peña, we see that $z = e_1 + e_4 + e_6 + e_7 \in C_4$ and $B_{\mathcal{Q}^+_i}(z, e_{10}) = -2$. This means that the algorithm fails. By this test, $Q^+_{\mathbf{i}}$ is not weakly positive.

Using the criteria in Proposition 6.3, it is possible to test if this form $Q^+_{\mathbf{i}}$ is weakly non-negative. There are $2^{12} - 1 = 4095$ principal submatrices that had to be tested. Of these, 1134 had a negative determinant. In each case, the last column of the inverse matrix had a positive entry. Hence $\mathcal{Q}^+_{\mathbf{i}}$ is weakly non-negative. These computations were performed with the help of a computer.

The symmetric matrix $A^-_{\mathbf{i}}$ is

$$A^-_{\mathbf{i}} = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & -2 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix}.$$

Its eigenvalues are $(4+2\sqrt{3})$, $(4-2\sqrt{3})$, 4 and 4. Thus $Q^-_{\mathbf{i}}$ is positive definite.

Using Proposition 4.5, we get that the monomial

$$\text{(a)} \qquad F_2^{(a_1)} F_1^{(a_2)} F_3^{(a_3)} F_4^{(a_4)} F_2^{(a_5)} F_1^{(a_6)} F_3^{(a_7)} F_4^{(a_8)} F_2^{(a_9)} F_1^{(a_{10})} F_3^{(a_{11})} F_4^{(a_{12})}$$

is semi-tight when $a_i \in \mathbf{N}$ for $i = 1, 2, \ldots, 12$ and

$$a_2 + a_3 + a_4 \geq a_1 + a_5, \qquad a_5 \geq a_2 + a_6, \qquad a_5 \geq a_3 + a_7,$$
$$a_5 \geq a_4 + a_8, \qquad a_6 + a_7 + a_8 \geq a_5 + a_9, \qquad a_9 \geq a_6 + a_{10},$$
$$a_9 \geq a_7 + a_{11}, \qquad a_9 \geq a_8 + a_{12}.$$

We can prove more. If at least one of the following inequalities,

$$a_2 + a_3 + a_4 \geq a_1 + a_5 \quad \text{and} \quad a_6 + a_7 + a_8 \geq a_5 + a_9$$

is strict and at least one of the following inequalities,

$$\begin{array}{lll} a_5 \geq a_2 + a_6, & a_5 \geq a_3 + a_7, & a_5 \geq a_4 + a_8, \\ a_9 \geq a_6 + a_{10}, & a_9 \geq a_7 + a_{11}, & a_9 \geq a_8 + a_{12}. \end{array}$$

is strict, then the monomial 6.4 (a) is tight.

In fact, we have that $Q^+_{\mathbf{i}}(x) \geq 0$, $Q^-_{\mathbf{i}}(y) \geq 0$ and $L^+_{\mathbf{i},\mathbf{a}}(x) \geq 0$ for all pairs $(x, y) \in \tilde{T}_{\mathbf{a}}$. So if $Q^+_{\mathbf{i}}(x) + Q^-_{\mathbf{i}}(y) + 2L^+_{\mathbf{i},\mathbf{a}}(x) = 0$ for $(x, y) \in \tilde{T}_{\mathbf{a}}$, then $Q^+_{\mathbf{i}}(x) = 0$, $Q^-_{\mathbf{i}}(y) = 0$ and $L^+_{\mathbf{i},\mathbf{a}}(x) = 0$. Under the above strict inequalities, there exist $s < s'$ and $t < t'$ belonging to $[1, 12]$ such that $i_s = i_{s'} = 2$, $i_t = i_{t'} = i \neq 2$, $i_p \neq 2$ whenever $s < p < s'$ and $i_{p'} \neq i$ whenever $t < p' < t'$ with

$$\sum_{\substack{j \in I \\ j \neq 2}} \sum_{\substack{p \in Z(j) \\ s < p < s'}} a_p > (a_s + a_{s'}) \quad \text{and} \quad \sum_{\substack{p' \in Z(2) \\ t < p' < t'}} a_{p'} > (a_t + a_{t'}).$$

This implies that $N(u, u') > 0$ if either $i_u = i_{u'} = 2$ with $u \leq s < s' \leq u'$ or $i_u = i_{u'} = i$ with $u \leq t < t' \leq u'$. Because $L^+_{\mathbf{i},\mathbf{a}}(x) = \sum_{(i,p,q) \in \mathcal{X}} N(p, q)\, x_{(i,p,q)} = 0$, then $x_{(i,p,q)} = 0$ whenever $i_p = i_q = 2$ with $p \leq s < s' \leq q$ or $i_p = i_q = i$ with $p \leq t < t' \leq q$. If we restrict $Q^+_{\mathbf{i}}$ to the submodule of $\mathbf{Z}^{|\mathcal{X}|}$ where $x_{(i,p,q)} = 0$ whenever $i_p = i_q = 2$ with $p \leq s < s' \leq q$ or $i_p = i_q = i$ with $p \leq t < t' \leq q$, then we get that this restriction is weakly positive using Proposition 6.1. For this we must test all the possible cases. We can conclude that $x = 0$. By condition 3 of Lemma 2.3, we get that $y = 0$. Finally, we can conclude that the monomial 6.4 (a) is tight by Proposition 4.5.

Using the same type of argument, we can prove that the monomial 6.4 (a) is tight if either $(a_5 > (a_2+a_6)$ and $a_9 > (a_6+a_{10}))$, or if $(a_5 > (a_2+a_6)$ and $a_9 > (a_7+a_{11}))$, or if $(a_5 > (a_2+a_6)$ and $a_9 > (a_8+a_{12}))$, or if $(a_9 > (a_6+a_{10})$ and $a_5 > (a_3+a_7))$, or if $(a_9 > (a_6 + a_{10})$ and $a_5 > (a_4 + a_8))$, or if $((a_2 + a_3 + a_4) > (a_1 + a_5)$ and $(a_6 + a_7 + a_8) > (a_5 + a_9))$, or if $(a_5 > (a_3 + a_7)$ and $a_9 > (a_7 + a_{11}))$, or if $(a_5 > (a_3 + a_7)$ and $a_9 > (a_8 + a_{12}))$, or if $(a_9 > (a_7 + a_{11})$ and $a_5 > (a_4 + a_8))$, or if $(a_5 > (a_4 + a_8)$ and $a_9 > (a_8 + a_{12}))$.

We will conclude this section with a last example. This is proved by proceeding as above.

Example 6.5. Let Γ be the Dynkin graph of type A_5 and let $\mathbf{i}$ be the sequence $(1, 2, 1, 3, 2, 1, 4, 3, 2, 1, 5, 4, 3, 2, 1)$. Then $\dim(V_1) = 20$, $\dim(V_{-1}) = 10$ and consequently $\dim(V) = 30$.

$Q^+_{\mathbf{i}}$ is not positive definite, not non-negative definite nor weakly positive. But we can use the determinantal criterion of Keller to show that $Q^+_{\mathbf{i}}$ is weakly non-negative. $Q^-_{\mathbf{i}}$ is not positive definite, but it is non-negative definite. Using Proposition 4.5, we get that the monomial

$$F_1^{(a_1)} F_2^{(a_2)} F_1^{(a_3)} F_3^{(a_4)} F_2^{(a_5)} F_1^{(a_6)} F_4^{(a_7)} F_3^{(a_8)} F_2^{(a_9)} F_1^{(a_{10})} F_5^{(a_{11})} F_4^{(a_{12})} F_3^{(a_{13})} F_2^{(a_{14})} F_1^{(a_{15})}$$

is semi-tight when $a_i \in \mathbf{N}$ for $i = 1, 2, \ldots, 15$ and

$$a_2 \geq a_1 + a_3, \qquad a_3 + a_4 \geq a_2 + a_5, \qquad a_5 \geq a_3 + a_6,$$
$$a_5 + a_7 \geq a_4 + a_8, \qquad a_6 + a_8 \geq a_5 + a_9, \qquad a_9 \geq a_6 + a_{10},$$
$$a_8 + a_{11} \geq a_7 + a_{12}, \qquad a_9 + a_{12} \geq a_8 + a_{13} \qquad a_{10} + a_{13} \geq a_9 + a_{14},$$
$$a_{14} \geq a_{10} + a_{15}.$$

Moreover, if $a_2 > (a_1 + a_3)$ and at least one of the following inequalities,

$$a_5 \geq a_3 + a_6, \qquad a_9 \geq (a_6 + a_{10}), \qquad (a_3 + a_4) \geq (a_2 + a_5),$$
$$(a_{10} + a_{13}) \geq (a_9 + a_{14}), \qquad (a_5 + a_7) \geq (a_4 + a_8), \qquad (a_9 + a_{12}) \geq (a_8 + a_{13})$$

is strict; or if $a_5 > (a_3 + a_6)$ and at least one of the following inequalities,

$$a_9 \geq (a_6 + a_{10}), \qquad a_{14} \geq (a_{10} + a_{15}), \qquad (a_3 + a_4) \geq (a_2 + a_5),$$
$$(a_6 + a_8) \geq (a_5 + a_9), \qquad (a_9 + a_{12}) \geq (a_8 + a_{13}), \qquad (a_8 + a_{11}) \geq (a_7 + a_{12})$$

is strict; or if $a_9 > (a_6 + a_{10})$ and at least one of the following inequalities,

$$a_{14} \geq (a_{10} + a_{15}), \qquad (a_6 + a_8) \geq (a_5 + a_9), \qquad (a_{10} + a_{13}) \geq (a_9 + a_{14}),$$
$$(a_5 + a_7) \geq (a_4 + a_8), \qquad (a_8 + a_{11}) \geq (a_7 + a_{12})$$

is strict; or if $a_{14} > (a_{10} + a_{15})$ and at least one of the following inequalities,

$$(a_3 + a_4) \geq (a_2 + a_5), \qquad (a_{10} + a_{13}) \geq (a_9 + a_{14}),$$
$$(a_5 + a_7) \geq (a_4 + a_8), \qquad (a_9 + a_{12}) \geq (a_8 + a_{12})$$

is strict; or if $(a_3 + a_4) > (a_2 + a_5)$ and at least one of the inequalities,

$$(a_6 + a_8) \geq (a_5 + a_9), \qquad (a_{10} + a_{13}) \geq (a_9 + a_{14}),$$
$$(a_5 + a_7) \geq (a_4 + a_8), \qquad (a_8 + a_{11}) \geq (a_7 + a_{12})$$

is strict; or if $(a_6 + a_8) > (a_5 + a_9)$ and at least one of the inequalities,

$$(a_{10} + a_{13}) \geq (a_9 + a_{14}), \qquad (a_5 + a_7) \geq (a_4 + a_8) \qquad (a_9 + a_{12}) \geq (a_8 + a_{13})$$

is strict; or if $(a_{10} + a_{13}) > (a_9 + a_{14})$ and at least one of the inequalities,

$$(a_9 + a_{12}) \geq (a_8 + a_{13}) \qquad (a_8 + a_{11}) \geq (a_7 + a_{12})$$

is strict; or if $(a_5 + a_7) > (a_4 + a_8)$ and at least one of the inequalities,

$$(a_9 + a_{12}) \geq (a_8 + a_{13}) \qquad (a_8 + a_{11}) \geq (a_7 + a_{12})$$

is strict; or if $((a_9 + a_{12}) > (a_8 + a_{13})$ and $(a_8 + a_{11}) > (a_7 + a_{12}))$, then the above monomial is tight.

7. The case of Dynkin graph Γ of small rank

In this section, we will consider the case where Γ is a Dynkin graph of a simply laced semisimple finite dimensional complex Lie algebra $\mathfrak{g}$ of small rank and the sequence $\mathbf{i}$ is a reduced expression for the longest element w_0 of the Weyl group W of $\mathfrak{g}$ (relative to a choice of a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$). $R^+(\Gamma)$ will denote the set of positive roots of $\mathfrak{g}$ relative to a choice of $\mathfrak{h}$ and $\mathfrak{b}$ containing $\mathfrak{h}$.

Lusztig has asked in section 16 of [L2] under what circumstances is the monomial 1.1 (a) tight or semi-tight when $\mathbf{a} \in \mathcal{C}(\mathbf{i})$ for Γ and $\mathbf{i}$ as above? He also answered this question for all reduced expressions $\mathbf{i}$ of w_0 when Γ is the Dynkin graph of type

A_1, A_2 and A_3. Marsh has also considered the same question in [M] for all reduced expressions $\mathbf{i}$ of w_0 when Γ is the Dynkin graph of type A_4.

In the case where Γ is the Dynkin graph of a simply laced finite dimensional complex Lie algebra, Reineke has given in [Re] a sufficient and necessary condition for the monomial 1.1 (a) to be tight. He also gave an example in the case A_6 to illustrate the fact that even if $\mathbf{a} \in \mathcal{C}(\mathbf{i})$, then the monomial 1.1 (a) is not necessarily tight.

We will now recall this criterion of Reineke. We will also describe the behavior of $Q_{\mathbf{i}}^+$ and $Q_{\mathbf{i}}^-$ in the case A_n ($n = 1, 2, 3, 4$) and D_4. From this, we get the results of Lusztig and Marsh in the case A_n.

Proposition 7.1 (Reineke). *Let Γ be the graph attached to a symmetric Cartan datum (See* 1.1.1, 2.1.3 *and* 14.1.3 *in* [L3]*). Then the monomial* 1.1 (a) *is tight if and only if $Q_{\Omega,\mathbf{i}}(z) + Q_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}z) > 0$ for all $z \in T_{\mathbf{a}} \setminus \{z_{\mathbf{a}}\}$. Here ${}^{\mathrm{tr}}z$ is the transpose of z.*

Proof. This is a simple consequence of Theorem 3.2 and Lemma 3.3 of [Re]. □

Corollary 7.2. *Let Γ be the Dynkin graph or the extended Dynkin graph of a simply laced finite dimensional complex Lie algebra $\mathfrak{g}$. Then the monomial* 1.1 (a) *is tight if and only if $Q_{\mathbf{i}}^+(x) + Q_{\mathbf{i}}^-(y) + 2L_{\mathbf{i},\mathbf{a}}^+(x) > 0$ for all pairs (x, y) in $\tilde{T}_{\mathbf{a}}$ with $x \neq 0$.*

Proof. If Γ is the Dynkin graph or the extended Dynkin graph of a simply laced finite dimensional complex Lie algebra $\mathfrak{g}$, then Γ is the graph attached to a symmetric Cartan datum and Proposition 7.1 can be applied. It is easy to see that if $w \in V$, then ${}^{\mathrm{tr}}\mu_{\mathbf{a}}(w) = \mu_{\mathbf{a}}({}^{\mathrm{tr}}w)$. By Lemma 1.11 and Proposition 1.14, $z \in T_{\mathbf{a}} \setminus \{z_{\mathbf{a}}\}$ is the image under $\mu_{\mathbf{a}}$ of a unique $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \setminus \{0\}$ and

$$\begin{aligned} Q_{\Omega,\mathbf{i}}(z) + Q_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}z) &= Q_{\Omega,\mathbf{i}}(\mu_{\mathbf{a}}(w)) + Q_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}\mu_{\mathbf{a}}(w)) \\ &= Q_{\Omega,\mathbf{i}}(\mu_{\mathbf{a}}(w)) + Q_{\Omega,\mathbf{i}}(\mu_{\mathbf{a}}({}^{\mathrm{tr}}w)) \\ &= \bar{Q}_{\Omega,\mathbf{i}}(w) + \bar{Q}_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}w) + L_{\mathbf{i},\mathbf{a}}(w) + L_{\mathbf{i},\mathbf{a}}({}^{\mathrm{tr}}w). \end{aligned}$$

Consequently, $Q_{\Omega,\mathbf{i}}(z) + Q_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}z) > 0$ for all $z \in T_{\mathbf{a}} \setminus \{z_{\mathbf{a}}\}$ if and only if $\bar{Q}_{\Omega,\mathbf{i}}(w) + \bar{Q}_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}w) + L_{\mathbf{i},\mathbf{a}}(w) + L_{\mathbf{i},\mathbf{a}}({}^{\mathrm{tr}}w) > 0$ for all $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \setminus \{0\}$.

If we write $w = u + v$ with

$$u = \sum_{(i,p,q)\in\mathcal{X}} x_{(i,p,q)} u(i,p,q) \in V_1 \quad \text{and} \quad v = \sum_{(i,p,q)\in\mathcal{Y}} y_{(i,p,q)} v(i,p,q) \in V_{-1},$$

then we get that $\bar{Q}_{\Omega,\mathbf{i}}(w) + \bar{Q}_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}w) + L_{\mathbf{i},\mathbf{a}}(w) + L_{\mathbf{i},\mathbf{a}}({}^{\mathrm{tr}}w)$ is equal to

$$\begin{aligned} &\bar{Q}_{\Omega,\mathbf{i}}(u+v) + \bar{Q}_{\Omega,\mathbf{i}}(u-v) + L_{\mathbf{i},\mathbf{a}}(u+v) + L_{\mathbf{i},\mathbf{a}}(u-v) \\ &= \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(v) + \bar{Q}_{\Omega,\mathbf{i}}(u) + \bar{Q}_{\Omega,\mathbf{i}}(-v) + L_{\mathbf{i},\mathbf{a}}(u) + L_{\mathbf{i},\mathbf{a}}(v) + L_{\mathbf{i},\mathbf{a}}(u) - L_{\mathbf{i},\mathbf{a}}(v) \\ &= 2\bar{Q}_{\Omega,\mathbf{i}}(u) + 2\bar{Q}_{\Omega,\mathbf{i}}(v) + 2L_{\mathbf{i},\mathbf{a}}(u) \\ &= \frac{2}{4}Q_{\mathbf{i}}^+(x) + \frac{2}{4}Q_{\mathbf{i}}^-(y) + \frac{2}{2}L_{\mathbf{i},\mathbf{a}}^+(x) \end{aligned}$$

as in Proposition 4.5. Consequently, $\bar{Q}_{\Omega,\mathbf{i}}(w) + \bar{Q}_{\Omega,\mathbf{i}}({}^{\mathrm{tr}}w) + L_{\mathbf{i},\mathbf{a}}(w) + L_{\mathbf{i},\mathbf{a}}({}^{\mathrm{tr}}w) > 0$ if and only if $Q_{\mathbf{i}}^+(x) + Q_{\mathbf{i}}^-(y) + 2L_{\mathbf{i},\mathbf{a}}^+(x) > 0$.

From Lemma 2.3, we get that $w \in \mu_{\mathbf{a}}^{-1}(T_{\mathbf{a}}) \setminus \{0\}$ if and only if the pair $(x, y) \in \tilde{T}_{\mathbf{a}}$ with $x \neq 0$. From these observations, we get easily the corollary. □

We will now recall a result due to Matsumoto and Tits.

7.3. Let $\mathcal{S}$ be the set of all sequences $(i_1, i_2, \ldots, i_q)$ in I such that $s_{i_1} s_{i_2} \ldots s_{i_q}$ is a reduced expression in the Weyl group (W, S) of Γ. We can regard $\mathcal{S}$ as the vertices of a graph. Two vertices $(i_1, i_2, \ldots, i_q)$ and $(i'_1, i'_2, \ldots, i'_{q'})$ are joined by an edge if one is obtained from the other by replacing m consecutive entries of the form $i, j, i, j, \ldots$ by the m entries $j, i, j, i, \ldots$; here $i \neq j$ and $m = m_{i,j} < \infty$ is the order of $s_i s_j$ in W. We write $(i_1, i_2, \ldots, i_q) \approx (i'_1, i'_2, \ldots, i'_{q'})$ if and only if $(i_1, i_2, \ldots, i_q)$, $(i'_1, i'_2, \ldots, i'_{q'})$ are in the same connected component of $\mathcal{S}$.

Theorem 7.4 (Matsumoto-Tits). *Let $(i_1, i_2, \ldots, i_q)$ and $(j_1, j_2, \ldots, j_q)$ in $\mathcal{S}$ be such that $s_{i_1} s_{i_2} \ldots s_{i_q} = s_{j_1} s_{j_2} \ldots s_{j_q} = w \in W$. Then $(i_1, i_2, \ldots, i_q) \approx (j_1, j_2, \ldots, j_q)$.*

Proof. See, for example, Theorem 1.9 in [L4]. □

Theorem 7.5. *Let Γ be the Dynkin graph of type A_n and $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ a sequence in I such that $s_{i_1} s_{i_2} \ldots s_{i_m}$ is a reduced expression of the longest element w_0 of the Weyl group of Γ.*

(a) *If $n = 2$, then $Q^+_{\mathbf{i}}$ is positive definite and* $\dim(V_{-1}) = 0$. *Also the number of positive roots of $Q^+_{\mathbf{i}}$ is $1 = |R^+(A_1)|$.*
(b) *If $n = 3$, then $Q^+_{\mathbf{i}}$ is positive definite. If $\mathbf{i}$ is not in the commutation class of $(2,1,3,2,1,3)$ or $(1,3,2,1,3,2)$, then $Q^-_{\mathbf{i}}$ is positive definite; while if $\mathbf{i}$ is in the commutation class of $(2,1,3,2,1,3)$ or of $(1,3,2,1,3,2)$, then $\dim(V_{-1}) = 0$. Also the number of positive roots of $Q^+_{\mathbf{i}}$ is $6 = |R^+(A_3)|$.*
(c) *If $n = 4$, then $Q^+_{\mathbf{i}}$ is weakly positive and $Q^-_{\mathbf{i}}$ is positive definite. Also the number of positive roots of $Q^+_{\mathbf{i}}$ is $30 = |R^+(D_6)|$.*
(d) **(Lusztig-Marsh)** *If $n \leq 4$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathcal{C}(\mathbf{i})$, then the monomial* 1.1 (a) *is tight.*

Proof. Because of Corollary 5.8, it is enough to verify (a), (b), (c) for a representative of each commutation class of reduced expressions of w_0. We can get a list of all reduced expressions of w_0 and also of all the representatives of the commutation classes of reduced expressions of w_0 using Theorem 7.4 of Matsumoto-Tits starting with a reduced expression of w_0 and using braid relations.

(a) In the case A_2, there are two reduced expressions: $(1,2,1)$ and $(2,1,2)$ and each is in a different commutation class. Then $\dim(V_1) = 1$, $A^+_{\mathbf{i}} = (2)$, $Q^+_{\mathbf{i}}$ is positive definite, $\dim(V_{-1}) = 0$ and the number of positive roots is 1 for each representative $\mathbf{i}$ of the commutation classes of w_0.

(b) In the case A_3, there are 16 reduced expressions for w_0 and these are partitioned into 8 different commutation classes. Representatives for these 8 commutation classes are: $(1,2,1,3,2,1)$, $(1,2,3,2,1,2)$, $(1,3,2,1,3,2)$, $(3,2,1,2,3,2)$, $(3,2,3,1,2,3)$, $(2,3,2,1,2,3)$, $(2,1,3,2,1,3)$ and $(2,1,2,3,2,1)$. Using Propositions 3.3 and 3.5, we can compute $A^+_{\mathbf{i}}$, $A^-_{\mathbf{i}}$ in each case and test for positive definiteness of $Q^+_{\mathbf{i}}$ and $Q^-_{\mathbf{i}}$ and non-negative definiteness of $Q^-_{\mathbf{i}}$. We get (b) this way. We can compute the number of positive roots of $Q^+_{\mathbf{i}}$ using for example Proposition 6.1.

(c) In the case A_4, there are 768 reduced expressions for w_0 and these are partitioned into 62 different commutation classes. Using a computer, we can test each of the representatives of the commutation classes of w_0. We get that $Q^+_{\mathbf{i}}$ is weakly positive, $Q^-_{\mathbf{i}}$ is positive definite and the number of positive roots of $Q^+_{\mathbf{i}}$ is 30 by using Proposition 6.1 for each of these representatives $\mathbf{i}$.

(d) follows by (a), (b), (c) and Proposition 4.5 (d). □

Remark 7.6. The fact that in the case A_n ($n = 2, 3, 4$), the weakly positive unit forms $Q_{\mathbf{i}}^+$ have the same number of positive roots is mysterious to the author. The vector spaces V_1 for these different reduced expressions do not all have the same dimension and it is not clear how the unit forms $Q_{\mathbf{i}}^+$ are related. For example in the case A_3, if $\mathbf{i} = (1, 3, 2, 1, 3, 2)$, then $\dim(V_1) = 3$,

$$A_{\mathbf{i}}^+ = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the positive roots of $Q_{\mathbf{i}}^+$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$ and $(1, 1, 1)$; while if $\mathbf{i} = (1, 2, 1, 3, 2, 1)$, then $\dim(V_1) = 4$,

$$A_{\mathbf{i}}^+ = \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

and the positive roots of $Q_{\mathbf{i}}^+$ are $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 0, 0, 1)$ and $(0, 0, 1, 1)$.

Robert Marsh has noticed that the number of positive roots for the different $Q_{\mathbf{i}}^+$ is the number of positive roots of the cluster algebra corresponding to the quantized enveloping algebra of type A_n ($n = 2, 3, 4$) (See for example [Z] or [FZ]). These cluster algebras are of finite type and we can associate to them a Dynkin graph. This correspondence associates to $\Gamma = A_2$ the cluster algebra $\mathbf{C}[N_-\backslash SL_3]$ of type A_1; to $\Gamma = A_3$, the cluster algebra $\mathbf{C}[N_-\backslash SL_4]$ of type A_3; and to $\Gamma = A_4$, the cluster algebra $\mathbf{C}[N_-\backslash SL_5]$ of type D_6. Here N_- denotes in each case the subgroup of lower triangular unipotent matrices of SL_n ($n = 3, 4, 5$).

Theorem 7.7. *Let Γ be the Dynkin graph of type D_4 and $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ be a sequence in I such that $s_{i_1} s_{i_2} \ldots s_{i_m}$ is a reduced expression of the longest element w_0 of the Weyl group of Γ. Then*

(a) *$Q_{\mathbf{i}}^+$ is weakly non-negative.*
(b) *$Q_{\mathbf{i}}^-$ is non-negative definite. Moreover, if the commutation class of $\mathbf{i}$ does not consist of only $\mathbf{i}$, then $Q_{\mathbf{i}}^-$ is positive definite.*
(c) *If $\mathbf{a} \in \mathcal{C}(\mathbf{i})$, then the monomial* 1.1 (a) *is semi-tight.*
(d) *If $\mathbf{a} \in \mathcal{C}^\circ(\mathbf{i})$, then the monomial* 1.1 (a) *is tight.*

Proof. (a) and (b). Because of Corollary 5.8, it is enough to verify (a) and (b) for a representative of each commutation class of reduced expressions of w_0. We can get a list of all the reduced expressions of w_0 and also of representatives of each commutation class of reduced expressions of w_0 using the proposition of Matsumoto-Tits starting with a reduced expression and using braid relations. There are 2316 reduced expressions of w_0 and these are partitioned into 182 commutation classes. For each of the representatives $\mathbf{i}$, we can test that $Q_{\mathbf{i}}^+$ is weakly non-negative using Proposition 6.3. Note that we can also see that $Q_{\mathbf{i}}^+$ is not weakly positive using the algorithm of de la Peña.

We can also test that $Q_{\mathbf{i}}^-$ is non-negative definite. If the cardinality of the commutation class of $\mathbf{i}$ is > 1, then $Q_{\mathbf{i}}^-$ is positive definite. This is obtained by computing the determinant of the primary principal submatrices of $A_{\mathbf{i}}^-$ and showing that they are > 0. If the cardinality of the commutation class of $\mathbf{i}$ is 1, then $Q_{\mathbf{i}}^-$ is not positive definite. In fact, $A^-_{\mathbf{i}}$ is singular and computing the determinant of the principal submatrices of $A^-_{\mathbf{i}}$ and showing that they are ≥ 0 to conclude that $Q^-_{\mathbf{i}}$ is non-negative definite. This proves (a) and (b).

(c) By (a), (b) and Proposition 4.5 (c), we can conclude that the monomial 1.1 (a) is semi-tight.

(d) By (a), (b) and Proposition 4.5 (e), we can conclude that the monomial 1.1 (a) is tight. □

We will close this article with an example to show that even if $\mathbf{a} \in \mathcal{C}(\mathbf{i})$, then the monomial 1.1 (a) is not necessarily tight. Reineke has given such an example in the case that $\Gamma = A_6$ in [Re].

Example 7.8. Let Γ be the graph of A_1 affine. This is the loop free graph with two vertices: i, j such that $e_{i,j} = 2$. Take $\mathbf{i} = (i,j,i,j,i)$. Then $\dim(V_1) = 4$, $\dim(V_{-1}) = 1$, $\mathcal{X} = \{(i,1,3),(i,1,5),(i,3,5),(j,2,4)\}$ and $\mathcal{Y} = \{(i,1,5)\}$.

If $x = (x_{(i,1,3)}, x_{(i,1,5)}, x_{(i,3,5)}, x_{j,2,4)})$, then

$$\begin{aligned} Q^+_{\mathbf{i}}(x) &= x^2_{(i,1,3)} + x^2_{(i,1,5)} + x^2_{(i,3,5)} + x^2_{(j,2,4)} + x_{(i,1,3)}x_{(i,1,5)} + x_{(i,1,3)}x_{(i,3,5)} \\ &\quad + x_{(i,1,5)}x_{(i,3,5)} - 2x_{(i,1,3)}x_{(j,2,4)} - 2x_{(i,3,5)}x_{(j,2,4)} \end{aligned}$$

and

$$\begin{aligned} L^+_{\mathbf{i},\mathbf{a}}(x) &= (2a_2 - a_1 - a_3)x_{(i,1,3)} + (2a_2 + 2a_4 - a_1 - 2a_3 - a_5)x_{(i,1,5)} \\ &\quad + (2a_4 - a_3 - a_5)x_{(i,3,5)} + (2a_3 - a_2 - a_4)x_{(j,2,4)}. \end{aligned}$$

If $y = (y_{(i,1,5)})$, then $Q^-_{\mathbf{i}}(y) = 2y^2_{(i,1,5)}$.

We have also

$$\mathcal{C}(\mathbf{i}) = \{(a_1,a_2,a_3,a_4,a_5) \in \mathbf{N}^5 \mid 2a_2 \geq (a_1 + a_3), 2a_3 \geq (a_2 + a_4), 2a_4 \geq (a_3 + a_5)\}.$$

Let $\mathbf{a} = (6,8,9,9,8)$. Clearly $\mathbf{a} \in \mathcal{C}(\mathbf{i})$; in fact, $\mathbf{a} \in \mathcal{C}^\circ(\mathbf{i})$. For $x = (8,0,10,16)$ and $y = (0)$, then $(x,y) \in \tilde{T}_{\mathbf{a}}$, $Q^+_{\mathbf{i}}(x) + Q^-_{\mathbf{i}}(y) + 2L^+_{\mathbf{i},\mathbf{a}}(x) = -8$ and we can conclude from Corollary 7.2 that the monomial $F_i^{(6)}F_j^{(8)}F_i^{(9)}F_j^{(9)}F_i^{(8)}$ is not tight.

References

[B] N. Bourbaki, *Groupes et algèbres de Lie* (Hermann, ed.), Ch 4-6, Paris, 1968. MR 0240238 (39:1590)

[FZ] S. Fomin and A. Zelevinsky, *Cluster algebras* II*: Finite type classification*, Invent. Math. **154** (2003), 63–121. MR 2004457

[K] E.L. Keller, *The general quadratic optimization problem*, Math. Programming **5** (1973), 311–337. MR 0376147 (51:12333)

[L1] G. Lusztig, *Quivers, perverse sheaves and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421. MR 1088333 (91m:17018)

[L2] ———, *Tight monomials in quantized enveloping algebras*, Israel Math. Conf. Proc. **7** (1993), 117–132. MR 1261904 (95i:17016)

[L3] ———, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser, Boston, 1993. MR 1227098 (94m:17016)

[L4] ———, *Hecke Algebras with Unequal Parameters*, CRM monograph series, vol. 18, Amer. Math. Soc., 2003. MR 1974442

[M] R. Marsh, *More tight monomials in quantized enveloping algebras*, J.Algebra **204** (1998), 711–732. MR 1624436 (99j:17023)

[Mo] T.S. Motzkin, *Signs of minors*, Inequalities (O.Shisha, ed.), Academic Press, New York, 1967, pp. 129–140. MR 0223384 (36:6432)

[P] J.A. de la Peña, *On the representation type of one-point extensions of tame concealed algebras*, Manuscripta Math. **61** (1988), 183–194. MR 0943535 (89h:16022)

[Re] M. Reineke, *Monomials in canonical bases of quantum groups and quadratic forms*, J. Pure Appl. Algebra **157** (2001), 301–309. MR 1812057 (2002e:17022)

[Ri] C. M. Ringel, *Tame algebras and integral quadratic forms* (Springer-Verlag, ed.), Lecture Notes in Math., vol. 1099, Berlin-Heidelberg-New York, 1984. MR 0774589 (87f:16027)

[V] H. Väliaho, *Criteria for copositive matrices*, Linear Algebra Appl. **81** (1986), 19–34. MR 0852891 (87j:15040)

[Z] A. Zelevinsky, *From Littlewood-Richardson coefficients to cluster algebras in three lectures*, Symmetric Functions 2001: Surveys of Developments and Perspectives (S.Fomin, ed.), NATO Sci. Ser. II Math. Phys. Chem., vol. 74, Kluwer Acad. Publ., Dordrecht, 2002, pp. 253–273.

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