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CHARACTER SHEAVES ON DISCONNECTED GROUPS, V

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Abstract. We prove orthogonality formulas for the characteristic functions of certain complexes on a connected component of a reductive group.

Introduction

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G.

Section 23 is a generalization of results in [L3, II, §7]. It is a preparation for the proof of the orthogonality formulas for certain characteristic functions in Section 24 which generalize those in [L3, II, §9, §10]. Section 25 describes the cohomology sheaves of a class of complexes which includes the admissible complexes on G. In particular, we show that these cohomology sheaves restricted to any stratum of G are local systems of a particular kind. In the connected case this reduces to a strengthening of [L3, III, 14.2(a)]. In Section 26 we give a variant of the definition of parabolic character sheaves in [L10] in terms of admissibile complexes. Note that even if one is only interested in parabolic character sheaves of connected groups, one cannot avoid using the theory of character sheaves on disconnected groups. In Section 27 we discuss the induction functor. The present treatment differs from that in the connected case, given in [L3, I, §4].

We adhere to the notation of [L9]. Here is some additional notation. If D, H are subsets of a group, we set $N_D H = \{\gamma \in D; \gamma H \gamma^{-1} = H\}$. If $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$, $F : Y \to Y$ is the Frobenius map for an $\mathbf{F}_q$-rational structure on an algebraic variety Y, $\mathcal{E}$ is a local system on Y and $\epsilon : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is an isomorphism, we denote by $\check{\epsilon} : F^*\check{\mathcal{E}} \to \check{\mathcal{E}}$ the unique isomorphism such that for any $y \in Y$, $\check{\epsilon} : \check{\mathcal{E}}_{F(y)} \to \check{\mathcal{E}}_y$ is the transpose inverse to $\epsilon : \mathcal{E}_{F(y)} \to \mathcal{E}_y$. If X is an algebraic variety, X' is a closed irreducible subvariety of X, $\mathcal{F}$ is a local system on an open dense smooth subvariety X'_0 of X' and $A \in \mathcal{D}(X)$ is $IC(X', \mathcal{F})$ extended by 0 on $X - X'$, let $\check{A} = IC(X', \check{\mathcal{F}})$ extended by 0 on $X - X'$. We have $\check{A} = \mathfrak{D}(A)[-2\dim X']$. If X, X', X'_0 are defined over $\mathbf{F}_q$ with Frobenius map F and $\alpha : F^*A \xrightarrow{\sim} A$ is an isomorphism which restricts to $\epsilon : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ over X_0, we denote by $\check{\alpha} : F^*\check{A} \to \check{A}$ the unique isomorphism which restricts to $\check{\epsilon} : F^*\check{\mathcal{F}} \xrightarrow{\sim} \check{\mathcal{F}}$ over X_0.

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If $f : X \to Y$ is a smooth morphism between algebraic varieties with connected fibres of dimension δ, we set $f^{\bigstar}A = f^*A[\delta]$ for any $A \in \mathcal{D}(Y)$. Let $\mathcal{D}(X)^{\le 0}$ be as in [L3, I, 1.3]. Let $\mathcal{M}(X)$ be the category of perverse sheaves on X.

If D is a connected component of G, a simple perverse sheaf A on D is said to be admissible if A, regarded as a simple perverse sheaf on G, zero on $G - D$, is admissible in the sense of 6.7.

Contents

23. Strongly cuspidal local systems

23.1. Let S be an isolated stratum of G and let $\mathcal{E} \in \mathcal{S}(S)$. Let $K_0 = IC(\bar{S}, \mathcal{E})$. Let D be the connected component of G that contains S.

If P is a parabolic of G^0 such that $S \subset N_G P$ and R is a U_P-coset in $N_G P$, let

$$d_R = \dim S - \dim {}^D\mathcal{Z}^0_{G^0} - \dim(P/U_P - \text{orbit of } R/U_P \text{ in } N_G P/U_P).$$

We show that conditions (i) and (ii) below are equivalent:

(i) *$\mathcal{E}$ is a cuspidal local system;*

(ii) *for any P, R as above such that $P \ne G^0$ and any irreducible component $\mathfrak{C}$ of $S \cap R$ of dimension $d_R/2$, the restriction of $\check{\mathcal{E}}$ to some/any smooth open dense subset of $\mathfrak{C}$ has no direct summand $\bar{\mathbf{Q}}_l$.*

Let $\mathfrak{C}_0$ be a smooth open dense subset of $\mathfrak{C}$. We have $H^{d_R}_c(\mathfrak{C}, \mathcal{E}) \cong H^{d_R}_c(\mathfrak{C}_0, \mathcal{E})$ and, by Poincaré duality, the last vector space is isomorphic to $H^0(\mathfrak{C}_0, \check{\mathcal{E}})$, a vector space whose dimension is the multiplicity of $\bar{\mathbf{Q}}_l$ in a decomposition of $\check{\mathcal{E}}|_{\mathfrak{C}_0}$ as a direct sum of irreducible local systems. It remains to note that, by 6.2, $H^{d_R}_c(S \cap R, \mathcal{E}) \cong \bigoplus_{\mathfrak{C}} H^{d_R}_c(\mathfrak{C}, \mathcal{E})$ where $\mathfrak{C}$ runs over the irreducible components of $S \cap R$ of dimension $d_R/2$.

Since condition (ii) for $\mathcal{E}$ is equivalent to condition (ii) for $\check{\mathcal{E}}$, we deduce:

(a) *$\mathcal{E}$ is cuspidal if and only if $\check{\mathcal{E}}$ is cuspidal.*

23.2. Let $S, \mathcal{E}, D, P$ be as in 23.1; assume that $P \ne G^0$. We show:

(a) *if R, d_R are as in 23.1 and $H^i_c((\bar{S} - S) \cap R, K_0) \ne 0$, then $i < d_R$.*

Indeed, there exist i', i'' and a stratum $S_1 \subset \bar{S} - S$ with $i = i' + i''$ and $H^{i'}_c(S_1 \cap R, \mathcal{H}^{i''}K_0) \ne 0$. By 6.2 we have $\dim(S_1 \cap R) \le (\dim S_1 - \dim S + d_R)/2$. Since $i' \le 2\dim(S_1 \cap R)$ we have $i' \le \dim S_1 - \dim S + d_R$. Since $\mathcal{H}^{i''}K_0 \ne 0$ at all points of S_1, we have $i'' < \dim S - \dim S_1$. Hence

$$i = i' + i'' < \dim S_1 - \dim S + d_R + \dim S - \dim S_1$$

and $i < d_R$, as required.

Next we show that:

(b) *if R, d_R are as in 23.1, the natural map $H^i_c(S \cap R, \mathcal{E}) \xrightarrow{j_i} H^i_c(\bar{S} \cap R, K_0)$ is an isomorphism for $i \ge d_R$.*

Indeed, from (a) we see that j_i is surjective for $i = d_R$ and an isomorphism for $i > d_R$. Moreover, for $i = d_R$, the kernel of j_i equals the image of the natural map $f : H^{d_R-1}_c((\bar{S} - S) \cap R, K_0) \to H^{d_R}_c(S \cap R, \mathcal{E})$. It is enough to show that

$f = 0$. We argue as in the proof of 8.3(b). We may assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$, that G has a fixed $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$, that P, R and any stratum in $\bar{S}$ are defined over $\mathbf{F}_q$ and that we are given an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which makes $\mathcal{E}$ into a local system of pure weight 0. Then $H_c^{d_R-1}((\bar{S}-S)\cap R, K_0), H_c^{d_R}(S\cap R, \mathcal{E})$ have natural Frobenius endomorphisms compatible with f. To show that $f = 0$ it is enough to show that

$(*)$ $H_c^{d_R}(S\cap R, \mathcal{E})$ is pure of weight d_R;

$(**)$ $H_c^{d_R-1}((\bar{S}-S)\cap R, K_0)$ is mixed of weight $\le d_R - 1$.

Now $(*)$ is clear since $\dim(S\cap R) \le d_R/2$ (see 6.2). We prove $(**)$. As in the proof of (a), it is enough to show that for any stratum $S_1 \subset \bar{S} - S$ and any i', i'' such that $i' + i'' = d_R - 1$, $H_c^{i'}(S_1 \cap R, \mathcal{H}^{i''}K_0)$ is mixed of weight $\le d_R - 1$. By Gabber's theorem [BBD, 5.3.2], the local system $\mathcal{H}^{i''}K_0$ on S_1 is mixed of weight $\le i''$. Using Deligne's theorem [BBD, 5.1.14(i)] we deduce that $H_c^{i'}(S_1\cap R, \mathcal{H}^{i''}K_0)$ is mixed of weight $\le i' + i'' = d_R - 1$. This completes the proof of (b).

We show that for $S, \mathcal{E}, P$ conditions (i) and (ii) below are equivalent:

(i) *for any U_P-coset R in N_GP we have $H_c^{d_R}(S\cap R, \mathcal{E}) = 0$, d_R as in 23.1;*

(ii) *for any i, the set $\mathcal{X}_i = \{R \in N_GP/U_P; H_c^i(\bar{S}\cap R, K_0) \ne 0\}$ has dimension $< \dim S - i$.*

Consider the set of all U_P-cosets R in N_GP such that $\bar{S}\cap R \ne \emptyset$. On this set we have a ${}^D\mathcal{Z}^0_{G^0} \times P/U_P$-action $(z,p) : R \mapsto zpRp^{-1}$ which has only finitely many orbits and has some orbit of dimension $\dim S - d_R$. Also, this action leaves stable each of the subsets $\mathcal{X}_i$ in (ii).

Assume that (ii) holds. Let $R \in N_GP/U_P$. Assume that $H_c^{d_R}(S\cap R, \mathcal{E}) \ne 0$. By (b) we have $R \in \mathcal{X}_i$ with $i = d_R$. Hence the orbit of R is contained in $\mathcal{X}_i$. It follows that $\dim \mathcal{X}_i \ge \dim S - d_R$ contradicting (ii). Thus, (i) holds.

Conversely, assume that (i) holds. To establish (ii), we assume that i is such that $\mathcal{X}_i \ne \emptyset$; it is enough to show that for any ${}^D\mathcal{Z}^0_{G^0} \times P/U_P$-orbit η in $\mathcal{X}_i$ we have $\dim \eta < \dim S - i$. If $R \in \eta$ we have $\dim \eta = \dim S - d_R$. Hence it is enough to show that $\dim S - d_R < \dim S - i$ or that $i < d_R$. Assume that $i \ge d_R$. Since $\mathcal{X}_i \ne \emptyset$, we have $H_c^i(\bar{S}\cap R, K_0) \ne 0$ for some R hence by (b) we have $H_c^i(S\cap R, \mathcal{E}) \ne 0$. If $i > d_R$, this is impossible, by 6.2. If $i = d_R$, this is impossible since (i) holds. Thus we have $i < d_R$ and (ii) holds.

23.3. In the setup of 23.1 we say that $\mathcal{E}$ is *clean* if $IC(\bar{S}, \mathcal{E})|_{\bar{S}-S} = 0$. We say that $\mathcal{E} \in \mathcal{S}(S)$ is *strongly cuspidal* if for any parabolic P of G^0 such that $P \ne G^0, S \subset N_GP$ and any U_P-coset R in N_GP we have $H_c^i(\bar{S}\cap R, K_0) = 0$ for all i.

From the equivalence of (i) and (ii) in 23.2 we see that:

if $\mathcal{E}$ is strongly cuspidal, then $\mathcal{E}$ is cuspidal.

If $\mathcal{E}$ is assumed to be clean, then the condition that $\mathcal{E}$ is strongly cuspidal is equivalent to the following condition: for any parabolic P of G^0 such that $P \ne G^0, S \subset N_GP$ and any U_P-coset R in N_GP we have $H_c^i(S\cap R, \mathcal{E}) = 0$ for all i.

Let D be a connected component of G and let P be a parabolic of G^0 such that $N_DP \ne \emptyset$. Let $\pi' : N_DP \to N_DP/U_P$ be the obvious map. Now N_DP/U_P is a connected component of N_GP/U_P. Let $i : N_DP \to D$ be the inclusion. Let $\alpha = \dim U_P$. Define a functor $\mathrm{res}_D^{N_DP/U_P} : \mathcal{D}(D) \to \mathcal{D}(N_DU_P/U_P)$ by

$$\mathrm{res}_D^{N_DP/U_P} A = \pi'_! i^* A(\alpha).$$

Let L be a Levi of P. Let $G' = N_G P \cap N_G L$, a reductive group with $G'^0 = L$. Let $D' = G' \cap D$, a connected component of G'. Define a homomorphism $\pi : N_G P \to G'$ by $\pi(z\omega) = z$ for $z \in N_G P \cap N_G L$, $\omega \in U_P$ (see 1.26). The restriction of π to $N_D P \to D'$ is denoted again by π. We may identify $N_G P/U_P = G'$, $N_D P/U_P = D'$ via π. Then $\mathrm{res}_D^{N_D P/U_P}$ becomes the functor

$$\mathrm{res}_D^{D'} : \mathcal{D}(D) \to \mathcal{D}(D')$$

given by $\mathrm{res}_D^{D'} A = \pi_! i^* A(\alpha)$.

Let A be a perverse sheaf on D. We say that A is *cuspidal* if

$$\mathrm{res}_D^{D'} A[-1] \in \mathcal{D}(D')^{\le 0}$$

for any P, L, D' as above with $P \ne G^0$. We say that A is *strongly cuspidal* if

$$\mathrm{res}_D^{D'}(A) = 0$$

for any P, L, D' as above with $P \ne G^0$. Clearly, if A is strongly cuspidal, then it is cuspidal.

Let $S, \mathcal{E}$ be as in 23.1. Let $A = IC(\bar{S}, \mathcal{E})[\dim S]$ regarded as a perverse sheaf on D, zero on $D - \bar{S}$. Clearly, A is strongly cuspidal if and only if $\mathcal{E}$ is a strongly cuspidal local system. We show:

(a) *A is cuspidal if and only if $\mathcal{E}$ is a cuspidal local system.*

By the equivalence of (i) and (ii) in 23.2, the condition that $\mathcal{E}$ is a cuspidal local system is that, for any P, L, D' be as above with $P \ne G^0$ and any $j \in \mathbf{Z}$ we have

$$\dim(\mathrm{supp}\mathcal{H}^j(\mathrm{res}_D^{D'} A[-\dim S])) < \dim S - j$$

or equivalently $\dim(\mathrm{supp}\mathcal{H}^j(\mathrm{res}_D^{D'} A)) < -j$, that is, $\dim(\mathrm{supp}\mathcal{H}^j(\mathrm{res}_D^{D'} A[-1])) \le -j$. This is the same as $\mathrm{res}_D^{D'} A[-1] \in \mathcal{D}(D')^{\le 0}$. This proves (a).

23.4. Let $\mathcal{E} \in \mathcal{S}(S)$. Let $s \in S_s$ and let $G' = Z_G(s)$. Let $\mathbf{c}$ be a G'^0-conjugacy class in $\mathcal{V} = \{v \in G'; v \text{ unipotent}, sv \in S\}$. Let δ be the connected component of G' that contains $\mathbf{c}$. Let $\mathcal{E}'$ be the inverse image of $\mathcal{E}$ under ${}^\delta\mathcal{Z}^0_{G'^0}\mathbf{c} \to S, g \mapsto sg$. Then $\mathcal{E}' \in \mathcal{S}(S')$ where $S' = {}^\delta\mathcal{Z}^0_{G'^0}\mathbf{c}$. Let $K_0 = IC(\bar{S}, \mathcal{E}), K'_0 = IC(\bar{S}', \mathcal{E}')$. We show:

(a) *$\mathcal{E}$ is clean (with respect to G) if and only if $\mathcal{E}'$ is clean (with respect to G').*

Let $\mathcal{E}''$ be the inverse image of $\mathcal{E}$ under $\mathcal{V} \to S, v \mapsto sv$. Since $\mathcal{V}$ is a smooth equidimensional variety (it admits a transitive action of an algebraic group), the complex $IC(\bar{\mathcal{V}}, \mathcal{E}'') \in \mathcal{D}(\bar{\mathcal{V}})$ is well defined.

Using 1.22 we see that $\pi : \bar{S} \to S_s, g \mapsto g_s$ is a morphism of algebraic varieties. Now ${}^D\mathcal{Z}^0_{G^0} \times G^0$ acts on $\bar{S}$ and S_c compatibly with π so that the action on S_c is transitive. Since the fibre $\pi^{-1}(s)$ may be identified with $\bar{\mathcal{V}}$, we see that $IC(\bar{\mathcal{V}}, \mathcal{E}'') = h^* K_0$ where $h : \bar{\mathcal{V}} \to \bar{S}, v \mapsto sv$. It follows that $\mathcal{E}$ is clean if and only if $IC(\bar{\mathcal{V}}, \mathcal{E}'')|_{\bar{\mathcal{V}}-\mathcal{V}} = 0$. We have $\mathcal{V} = \bigsqcup_{i\in[1,m]} \mathbf{c}_i$ where $\mathbf{c}_i$ are G'^0-conjugacy classes. Hence $IC(\bar{\mathcal{V}}, \mathcal{E}'') = \bigoplus_{i\in[1,m]} IC(\bar{\mathbf{c}}_i, \mathcal{E}''|_{\bar{\mathbf{c}}_i})$. Thus, $IC(\bar{\mathcal{V}}, \mathcal{E}'')|_{\bar{\mathcal{V}}-\mathcal{V}} = 0$ if and only if $IC(\bar{\mathbf{c}}_i, \mathcal{E}''|_{\bar{\mathbf{c}}_i})|_{\bar{\mathbf{c}}_i - \mathbf{c}_i} = 0$ for all i. By the homogeneity of $\mathcal{V}$ this is equivalent to the condition that $IC(\bar{\mathbf{c}}, \mathcal{E}''|_{\bar{\mathbf{c}}})|_{\bar{\mathbf{c}}-\mathbf{c}} = 0$. (We have $\mathbf{c} = \mathbf{c}_i$ for some i.) This last condition is equivalent to the condition that $K'_0|_{\bar{S}'-\bar{S}'} = 0$. This proves (a).

We show:

(b) *If $\mathcal{E}$ is strongly cuspidal (with respect to G), then $\mathcal{E}'$ is strongly cuspidal (with respect to G').*

Let Q be a parabolic of G'^0 such that $Q \neq G'^0, S' \subset N_{G'}Q$. We must show that for any $z \in {}^\delta\mathcal{Z}^0_{G'^0}, u \in \bar{\mathbf{c}} \cap Q$ we have $H^i_c(zuU_Q, K'_0) = 0$ for all i. We may assume that $z = 1$ and we must show that $H^i_c(uU_Q \cap \bar{\mathbf{c}}, K'_0) = 0$ for all i. Since $H^i_c(uU_Q \cap \bar{\mathbf{c}}, K'_0)$ is a direct summand of $H^i_c(uU_Q \cap \bar{\mathcal{V}}, IC(\bar{\mathcal{V}}, \mathcal{E}''))$, it is enough to show that $H^i_c(uU_Q \cap \bar{\mathcal{V}}, IC(\bar{\mathcal{V}}, \mathcal{E}'')) = 0$ for all i. Since $IC(\bar{\mathcal{V}}, \mathcal{E}'') = h^* K_0$, it is enough to show that $H^i_c(suU_Q \cap s\bar{\mathcal{V}}, K_0) = 0$ for all i. We have $suU_Q \cap s\bar{\mathcal{V}} = suU_Q \cap \bar{S}$ hence it is enough to show that $H^i_c(suU_Q \cap \bar{S}, K_0) = 0$ for all i. By 1.18(a) we can find a parabolic P of G^0 such that $P \cap G'^0 = Q$ and $su \in N_G P$. Clearly, $P \neq G^0$. Let $f : suU_P \cap \bar{S} \to sU_P \cap S_s$ be the restriction of $\pi : \bar{S} \to S_s$. Now U_P acts by conjugation on $suU_P \cap \bar{S}$ and on $sU_P \cap S_s$ compatibly with f; moreover, this action is transitive on $sU_P \cap S_s$ (see 19.3(a)). We have $f^{-1}(s) = suU_Q \cap \bar{S}$ hence we must only show that $H^i_c(f^{-1}(s), K_0) = 0$ for all i. The Leray spectral sequence of f is:

$$E_2^{p,q} = H^p_c(sU_P \cap S_s, \mathcal{H}^q f_! K_0) \implies H^{p+q}_c(suU_P \cap \bar{S}, K_0).$$

The last vector space is zero since $\mathcal{E}$ is strongly cuspidal. Thus, $E_\infty^{p,q} = 0$ for all p, q. Now $\mathcal{H}^q f_! K_0$ is a U_P-equivariant local system on $sU_P \cap S_s$ and $sU_P \cap S_s \cong U_P/U_{P'}$ is an affine space. Hence $E_2^{p,q} = 0$ for $p \neq 2 \dim U_P/U_{P'}$. This implies that $E_2^{p,q} = E_\infty^{p,q}$ for all p, q; it follows that $E_2^{p,q} = 0$ for all p, q so that $\mathcal{H}^q f_! K_0 = 0$ for all q. Taking the stalk at s we see that $H^q_c(f^{-1}(s), K_0) = 0$ for all q and (b) is proved.

We show:

(c) *Assume that $S = {}^D\mathcal{Z}^0_{G^0}\mathbf{c}$ where $\mathbf{c}$ is a unipotent G^0-conjugacy class. Let $\mathcal{L} \in \mathcal{S}({}^D\mathcal{Z}^0_{G^0})$ be a local system of rank 1. If $\mathcal{E}$ is strongly cuspidal, then $\mathcal{E} \otimes (\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)$ is strongly cuspidal.*

We have $IC(\bar{S}, \mathcal{E} \otimes (\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)) = K_0 \otimes (\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)$ since $\bar{S} \cong {}^D\mathcal{Z}^0_{G^0} \times \bar{\mathbf{c}}$. Let P be a parabolic of G^0 such that $P \neq G^0, S \subset N_G P$. Let $z \in {}^D\mathcal{Z}^0_{G^0}, u \in \bar{\mathbf{c}}$. We know that $H^i_c(zuU_P \cap \bar{S}, K_0) = 0$. We must show that $H^i_c(zuU_P \cap \bar{S}, K_0 \otimes (\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)) = 0$. It is enough to show that $(\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)|_{zuU_P \cap \bar{S}} \cong \bar{\mathbf{Q}}_l$. This follows from the fact that $(\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)|_{zuU_P \cap \bar{S}} = \mathcal{L}_z \otimes \bar{\mathbf{Q}}_l$ where $\mathcal{L}_z$ is the stalk of $\mathcal{L}$ at z.

This argument shows also that, in the setup of (c):

(d) *if $\mathcal{E}$ is clean, then $\mathcal{E} \otimes (\mathcal{L} \boxtimes \bar{\mathbf{Q}}_l)$ is clean.*

23.5. Let S, D be as in 23.1. Assume that there exists a non-zero cuspidal local system in $\mathcal{S}(S)$. Let $\mathcal{E}', \mathcal{E}'' \in \mathcal{S}(S)$ be such that the local system $\mathcal{E}' \otimes \mathcal{E}''$ has no direct summand isomorphic to $\bar{\mathbf{Q}}_l$. We show that

(a) $H^i_c(S, \mathcal{E}' \otimes \mathcal{E}'') = 0$ *for all* i.

It is enough to show that, if $\mathcal{E} \in \mathcal{S}(S)$ is irreducible and $\mathcal{E} \not\cong \bar{\mathbf{Q}}_l$, then $H^i_c(S, \mathcal{E}) = 0$ for all i. Let $H = {}^D\mathcal{Z}^0_{G^0} \times G^0$. We can find $n \in \mathbf{N}^*_{\mathbf{k}}$ such that $\mathcal{E}$ is equivariant for the transitive H-action $(z, x) : g \mapsto xz^n gx^{-1}$ on S. Let $y \in S$ and let H_y be the stabilizer of y in H for this action. Let $\tilde{S} = H/H^0_y$. Define $f : \tilde{S} \to S$ by $f(z, x) = xz^n yx^{-1}$ (a principal H_y/H^0_y-covering where H_y acts on $\tilde{S}$ by right multiplication). Now $\mathcal{E}$ is a direct summand of the local system $f_! \bar{\mathbf{Q}}_l$. It is enough to show that $H^i_c(S, f_! \bar{\mathbf{Q}}_l) = H^i_c(S, \bar{\mathbf{Q}}_l)$ for all i or equivalently that H_y/H^0_y acts trivially on $H^i_c(\tilde{S}, \bar{\mathbf{Q}}_l)$. Let $\bar{H} = {}^D\mathcal{Z}^0_{G^0} \times (G^0/{}^D\mathcal{Z}^0_{G^0})$. Define $f' : \bar{H} \to \tilde{S}$ by $(z, x{}^D\mathcal{Z}^0_{G^0}) \mapsto (z, x)H^0_y$. Now f' is a fibration with fibres isomorphic to $H^0_y/(\{1\} \times {}^D\mathcal{Z}^0_{G^0}) \cong Z_G(y)^0/{}^D\mathcal{Z}^0_{G^0}$ which by 10.2 is isomorphic to an affine space of dimension say a. Hence we have $H^i_c(\tilde{S}, \bar{\mathbf{Q}}_l) = H^{i+2a}_c(\bar{H}, \bar{\mathbf{Q}}_l)$. Also the H_y/H^0_y-action on $\tilde{S}$ is compatible under f' with the $H_y/(\{1\} \times {}^D\mathcal{Z}^0_{G^0})$-action on $\bar{H}$ by right multiplication.

It is enough to show that $H_y/(\{1\}\times {}^D\mathcal{Z}^0_{G^0})$ acts trivially on $H_c^{i+2a}(\bar{H},\bar{\mathbf{Q}}_l)$. This follows from the fact that the $H_y/(\{1\}\times {}^D\mathcal{Z}^0_{G^0})$-action on $\bar{H}$ is the restriction of an action of the connected group $H/(\{1\}\times {}^D\mathcal{Z}^0_{G^0})$ and a connected group must act trivially in cohomology. This proves (a).

23.6. Let $(L',S'),(L'',S'')\in\mathbf{A}$. Assume that $\mathcal{E}'\in\mathcal{S}(S')$ and $\mathcal{E}''\in\mathcal{S}(S'')$ are strongly cuspidal relative to N_GL',N_GL'' respectively. Let $K_0'=IC(\bar{S}',\mathcal{E}')$, $K_0''=IC(\bar{S}'',\mathcal{E}'')$. We regard K_0' (resp. K_0'') as a complex on N_GL' (resp. N_GL'') zero outside $\bar{S}'$ (resp. $\bar{S}''$). To $L',S',\mathcal{E}'$ (resp. $L'',S'',\mathcal{E}''$) we attach $\mathfrak{K}'\in\mathcal{D}(\bar{Y}_{L',S'})$ (resp. $\mathfrak{K}''\in\mathcal{D}(\bar{Y}_{L'',S''})$) in the same way as $\mathfrak{K}=IC(\bar{Y}_{L,S},\pi_!\tilde{\mathcal{E}})$ was attached to $L,S,\mathcal{E}$ in 5.6. We regard $\mathfrak{K}',\mathfrak{K}''$ as complexes on G, zero outside $\bar{Y}_{L',S'},\bar{Y}_{L'',S''}$ respectively.

Proposition 23.7. *Assume that for any $n\in G^0$ such that $n^{-1}L'n=L''$, hence $n^{-1}N_GL'n=N_GL''$, we have $H_c^i(N_GL',K_0'\otimes\mathrm{Ad}(n^{-1})^*K_0'')=0$ for all i. (This condition is automatically verified if L',L'' are not G^0-conjugate.) Then $H_c^i(G,\mathfrak{K}'\otimes\mathfrak{K}'')=0$ for all i.*

The proof is quite similar to (but simpler than) that in 7.8. Let P' (resp. P'') be a parabolic of G^0 with Levi L' (resp. L'') such that $S'\subset N_GP'$ (resp. $S''\subset N_GP''$). Let $X',X'',K'\in\mathcal{D}(X'),K''\in\mathcal{D}(X''),\mathfrak{Z}$ be as in 7.4. We may regard $\mathfrak{Z}$ as a subvariety of $X'\times X''$ via the imbedding $(g,x'P',x''P'')\mapsto((g,x'P'),(g,x''P''))$. The inverse image of $K'\boxtimes K''\in\mathcal{D}(X'\times X'')$ under this imbedding is a complex $\tilde{K}\in\mathcal{D}(\mathfrak{Z})$. Using a description of $\mathfrak{K}',\mathfrak{K}''$ as in 5.7 we see that $\mathfrak{K}'\otimes\mathfrak{K}''=(pr_1)_!\tilde{K}$ where $pr_1:\mathfrak{Z}\to G$ is the first projection. It follows that $H_c^i(G,\mathfrak{K}'\otimes\mathfrak{K}'')=H_c^i(\mathfrak{Z},\tilde{K})$ for all i. Hence it is enough to show that $H_c^i(\mathfrak{Z},\tilde{K})=0$ for all i. For any G^0-orbit E on $G^0/P'\times G^0/P''$ let ${}^E\mathfrak{Z}=\{(g,x'P',x''P'')\in\mathfrak{Z};(x'P',x''P'')\in E\}$. Using the partition of $\mathfrak{Z}$ into the finitely many locally closed subvarieties ${}^E\mathfrak{Z}$ we see that it is enough to show that $H_c^i({}^E\mathfrak{Z},\tilde{K})=0$ for all i and any E as above. Using the spectral sequence of the fibration $pr_{23}:{}^E\mathfrak{Z}\to E$, $(g,x'P',x''P'')\mapsto(x'P',x''P'')$ we see that it is enough to show that for any $(x',x'')\in G^0\times G^0$ such that $(x'P',x''P'')\in E$ we have $H_c^i(\mathcal{V},\tilde{K})=0$ for all i where $\mathcal{V}$ is the fibre of pr_{23} at $(x'P',x''P'')$. We identify $\mathcal{V}=\{g\in G;x'^{-1}gx'\in\bar{S}'U_{P'},x''^{-1}gx''\in\bar{S}''U_{P''}\}$ and define $j:\mathcal{V}\to\bar{S}'\times\bar{S}''$ by

$$j(g)=(\bar{S}'-\text{component of }x'^{-1}gx',\bar{S}''-\text{component of }x''^{-1}gx'').$$

Then $\tilde{K}|_{\mathcal{V}}$ may be identified with $j^*(K_0'\boxtimes K_0'')$ and we must show that
$H_c^i(\mathcal{V},j^*(K_0'\boxtimes K_0''))=0$ for all i.
Let $Q',Q'',M',M'',\Sigma',\Sigma'',\mathcal{F}',\mathcal{F}''$ be as in 7.8. Let $\hat{K}_0'=IC(\bar{\Sigma}',\mathcal{F}')$, $\hat{K}_0''=IC(\bar{\Sigma}'',\mathcal{F}'')$. As in 7.8, $\mathcal{V}$ is fibred over

$$\mathcal{V}_1=\{(u'',u',z)\in(M'\cap U_{Q''})\times(M''\cap U_{Q'})\times(\tilde{M}'\cap\tilde{M}'');zu''\in\bar{\Sigma}',zu'\in\bar{\Sigma}''\}$$

with all fibres isomorphic to $U_{Q'}\cap U_{Q''}$. Since $U_{Q'}\cap U_{Q''}$ is an affine space we see that it is enough to show that $H_c^i(\mathcal{V}_1,\bar{j}^*(\hat{K}_0'\boxtimes\hat{K}_0''))=0$ for all i where $\bar{j}:\mathcal{V}_1\to\bar{\Sigma}'\times\bar{\Sigma}''$ is defined by $\bar{j}(u'',u',z)=(zu'',zu')$.

Assume first that Q',Q'' have no common Levi. Let $\tilde{p}_3:\mathcal{V}_1\to\tilde{M}'\cap\tilde{M}''$ be the third projection. It is enough to show that for any $z\in\tilde{M}'\cap\tilde{M}''$ we have $H_c^i(\tilde{p}_3^{-1}(z),\bar{j}^*(\hat{K}_0'\boxtimes\hat{K}_0''))=0$ for all i. Now $\tilde{p}_3^{-1}(z)$ is a product $R'\times R''$ where R' (resp. R'') is the set of all elements in $(\tilde{M}'\cap N_GQ'')\cap\bar{\Sigma}'$ (resp. $(\tilde{M}''\cap N_GQ')\cap\bar{\Sigma}''$) whose image under $\tilde{M}'\cap N_GQ''\to\tilde{M}'\cap\tilde{M}''$ (resp. $\tilde{M}''\cap N_GQ'\to\tilde{M}'\cap\tilde{M}''$) is equal to z. We are reduced to showing that $H_c^{i'}(R',\hat{K}_0')\otimes H_c^{i''}(R'',\hat{K}_0'')=0$ for all

i', i''. Since Q', Q'' have no common Levi, we see that either $M' \cap Q''$ is a proper parabolic of M' or $M'' \cap Q'$ is a proper parabolic of M''. In the first case we have $H_c^{i'}(R', \hat{K}'_0) = 0$ for all i' since $\mathcal{F}'$ is strongly cuspidal in $\mathcal{S}(\Sigma')$. In the second case we have $H_c^{i''}(R'', \hat{K}''_0) = 0$ for all i'' since $\mathcal{F}''$ is strongly cuspidal in $\mathcal{S}(\Sigma'')$. Thus the desired vanishing result holds in our case.

Assume next that Q', Q'' have a common Levi. Then $M' = M''$, $M' \cap U_{Q''} = \{1\}$, $M'' \cap U_{Q'} = \{1\}$ and we may identify $\mathcal{V}_1 = \bar{\Sigma}' \cap \bar{\Sigma}''$. We must only show that $H_c^i(\bar{\Sigma}' \cap \bar{\Sigma}'', \hat{K}'_0 \otimes \hat{K}''_0)) = 0$ for all i. As in 7.8 we can find $v' \in U_{P'}, v'' \in U_{P''}$ such that, setting $n = v'^{-1} x'^{-1} x'' v'' \in G^0$, we have $n^{-1} L' n = L''$. Then the desired vanishing result is equivalent to $H_c^i(N_G L', K'_0 \otimes \mathrm{Ad}(n^{-1})^* K''_0) = 0$ for all i, which is part of our assumptions. The proposition is proved.

24. Orthogonality

24.1. In this section we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$.

24.2. Let L be a Levi of a parabolic of G^0. Let δ be a connected component of $N_G L$ such that $\delta \subset N_G^{\bullet} L$. Assume that $F(L) = L, F(\delta) = \delta$ and that the F-stable torus $\mathcal{Z} = {}^{\delta}\mathcal{Z}_L^0$ is $\mathbf{F}_q$-split, that is, $F(z) = z^q$ for all $z \in \mathcal{Z}$. We show that

(a) *there exists a parabolic P of G^0 with Levi L such that $F(P) = P$.*

We can find $\chi \in \mathrm{Hom}(\mathbf{k}^*, \mathcal{Z})$ such that $Z_{G^0}(\chi(\mathbf{k}^*)) = Z_{G^0}(\mathcal{Z})$. Let $g \in \delta$. Define $\chi', \chi'' \in \mathrm{Hom}(\mathbf{k}^*, \mathcal{Z})$ by $\chi'(a) = g\chi(a)g^{-1}$, $\chi''(a) = F(\chi(a^{q^{-1}}))$. To χ, χ', χ'' we attach parabolics $P_\chi, P_{\chi'}, P_{\chi''}$ of G^0 as in 1.16. Let $P = P_\chi$. From the definitions we have $P_{\chi'} = gPg^{-1}$, $P_{\chi''} = F(P)$. Since $\chi(\mathbf{k}^*) \subset Z_{G^0}(g)$ we have $\chi' = \chi$. Since $\mathcal{Z}$ is $\mathbf{F}_q$-split, we have $\chi''(a) = \chi(a^{q^{-1}})^q = \chi(a)$ for any a hence $\chi'' = \chi$. Thus, $gPg^{-1} = P, F(P) = P$. It follows that $\delta \subset N_G P$. Now $Z_{G^0}(\chi(\mathbf{k}^*))$ is a Levi of P and $Z_{G^0}(\mathcal{Z}) = L$ by 1.10(a). Hence L is a Levi of P. This proves (a).

24.3. Let $(L', S'), (L'', S'') \in \mathbf{A}$. Assume that $\mathcal{E}' \in \mathcal{S}(S')$ and $\mathcal{E}'' \in \mathcal{S}(S'')$ are strongly cuspidal relative to $N_G L', N_G L''$ respectively. Assume that $F(L') = L', F(S') = S', F(L'') = L'', F(S'') = S''$ and that we are given isomorphisms $\epsilon' : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}', \epsilon'' : F^*\mathcal{E}'' \xrightarrow{\sim} \mathcal{E}''$. Let $\mathfrak{K}', \mathfrak{K}''$ be as in 23.6 and let $\phi' : F^*\mathfrak{K}' \xrightarrow{\sim} \mathfrak{K}'$, $\phi'' : F^*\mathfrak{K}'' \xrightarrow{\sim} \mathfrak{K}''$ be the isomorphisms induced by ϵ', ϵ''. Let δ' (resp. δ'') be the connected component of $N_G L'$ (resp. $N_G L''$) that contains S' (resp. S''). Assume that either L', L'' are not G^0-conjugate or that $\mathcal{E}'$ and $\mathcal{E}''$ are clean (relative to $N_G L', N_G L''$). Let $\Theta = \{n \in G^{0F}; n^{-1} L' n = L'', n^{-1} S' n = S''\}$. With these assumptions we state:

Lemma 24.4.

$$\begin{aligned} &|G^{0F}|^{-1} \sum_{g \in G^F} \chi_{\mathfrak{K}', \phi'}(g) \chi_{\mathfrak{K}'', \phi''}(g) \\ \text{(a)} \qquad &= |L'^F|^{-1} |L''^F|^{-1} \sum_{n \in \Theta} \sum_{y \in S'^F} \chi_{\mathcal{E}', \epsilon'}(y) \chi_{\mathcal{E}'', \epsilon''}(n^{-1} y n). \end{aligned}$$

The proof is given in 24.7–24.12.

24.5. Let $(L',S'),(L'',S'')\in\mathbf{A}$. Let δ' (resp. δ'') be the connected component of N_GL' (resp. N_GL'') that contains S' (resp. S''). Assume that $S'={}^{\delta'}\mathcal{Z}^0_{L'}\mathbf{c}', S''={}^{\delta''}\mathcal{Z}^0_{L''}\mathbf{c}''$ where $\mathbf{c}'$ is a unipotent L'-conjugacy class and $\mathbf{c}''$ is a unipotent L''-conjugacy class. Let $\mathcal{F}'$ (resp. $\mathcal{F}''$) be an L'- (resp. L''-) equivariant local system on $\mathbf{c}'$ (resp. $\mathbf{c}''$) such that $\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}'$ (resp. $\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}''$) is a strongly cuspidal local system relative to N_GL' (resp. N_GL''). Assume that $F(L')=L'$, $F(\mathbf{c}')=\mathbf{c}'$, $F(L'')=L''$, $F(\mathbf{c}'')=\mathbf{c}''$ and that we are given isomorphisms $\bar\epsilon':F^*\mathcal{F}'\xrightarrow{\sim}\mathcal{F}'$, $\bar\epsilon'':F^*\mathcal{F}''\xrightarrow{\sim}\mathcal{F}''$. Assume that either L',L'' are not G^0-conjugate or that $\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}'$ and $\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}''$ are clean (relative to N_GL',N_GL''). Let Θ be as in 24.3. With these assumptions we state:

Lemma 24.6.

$$|G^{0F}|^{-1}\sum_{u\in G^F;unip.}Q_{L',G,\mathbf{c}',\mathcal{F}',\bar\epsilon'}(u)Q_{L'',G,\mathbf{c}'',\mathcal{F}'',\bar\epsilon''}(u)$$

$$=|L'^F|^{-1}|L''^F|^{-1}\sum_{n\in\Theta}\sum_{y\in\mathbf{c}'^F}\chi_{\mathcal{F}',\bar\epsilon'}(y)\chi_{\mathcal{F}'',\bar\epsilon''}(n^{-1}yn). \tag{a}$$

The proof is given (together with that of Lemma 24.4) in 24.7-24.12.

24.7. We prove Lemma 24.4 assuming that L',L'' are not G^0-conjugate. Let $\Phi:H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')\xrightarrow{\sim}H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')$ be the composition

$$H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')\xrightarrow{\sim}H^i_c(G,F^*\mathfrak{K}'\otimes F^*\mathfrak{K}'')\xrightarrow{\sim}H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')$$

(the first map is induced by $F:G\to G$, the second map is induced by $\phi'\otimes\phi''$). By the Grothendieck trace formula, the left-hand side of 24.4(a) is equal to $\sum_i(-1)^i\mathrm{tr}(\Phi,H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}''))$. This is zero since $H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')=0$ for all i, by 23.7. The right-hand side of 24.4(a) is also zero, by our assumption. Thus Lemma 24.4 is proved in the present case.

24.8. We prove Lemma 24.4 under the following assumptions:

$S'={}^{\delta'}\mathcal{Z}^0_{L'}\mathbf{c}', S''={}^{\delta''}\mathcal{Z}^0_{L''}\mathbf{c}''$ where $\delta',\delta',\mathbf{c}',\mathbf{c}''$ are as in 24.5;

for some/any $c'\in\mathbf{c}'$, $\mathcal{E}'|_{{}^{\delta'}\mathcal{Z}^0_{L'}c'}$ has no direct summand isomorphic to $\bar{\mathbf{Q}}_l$;

for some/any $c''\in\mathbf{c}''$, $\mathcal{E}''|_{{}^{\delta''}\mathcal{Z}^0_{L''}c''}$ is isomorphic to $\bar{\mathbf{Q}}_l^N$ for some N.

By 24.7 we may assume that L,L' are G^0-conjugate. Then $\mathcal{E}',\mathcal{E}''$ are clean. We show that the left-hand side of 24.4(a) is zero. As in 24.7 it is enough to show that $H^i_c(G,\mathfrak{K}'\otimes\mathfrak{K}'')=0$ for all i. Using 23.7, it is enough to show that for any $n\in G^0$ such that $n^{-1}L'n=L''$ we have $H^i_c(N_GL',K'_0\otimes\mathrm{Ad}(n^{-1})^*K''_0)=0$ for all i. The last equality is clear if $n^{-1}S'n\ne S''$ (in this case $n^{-1}S'n\cap S''=\emptyset$ and we have $K'_0\otimes\mathrm{Ad}(n^{-1})^*K''_0=0$ since $\mathcal{E}',\mathcal{E}''$ are clean). Assume now that $n^{-1}S'n=S''$. Then $K'_0\otimes\mathrm{Ad}(n^{-1})^*K''_0$ is the local system $\mathcal{E}'\otimes\mathrm{Ad}(n^{-1})^*\mathcal{E}''$ on S' extended by 0 on $N_GL'-S'$. Hence it is enough to show that $H^i_c(S',\mathcal{E}'\otimes\mathrm{Ad}(n^{-1})^*\mathcal{E}'')=0$ for all i. This follows from 23.5(a) since, by our assumption, the local system $\mathcal{E}'\otimes\mathrm{Ad}(n^{-1})^*\mathcal{E}''$ has no direct summand isomorphic to $\bar{\mathbf{Q}}_l$. Next we show that the right-hand side of 24.4(a) is zero. It is enough to show that for any $n\in\Theta$ the sum $\sum_{y\in S'^F}\chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n^{-1}yn)$ is zero. By the Grothendieck trace formula, this sum is an alternating sum of traces of the Frobenius map on $H^i_c(S',\mathcal{E}'\otimes\mathrm{Ad}(n^{-1})^*\mathcal{E}'')$. As we have seen above, this vector space is zero. Thus Lemma 24.4 is proved in the present case.

24.9. We prove Lemma 24.4 under the following assumption: there exist parabolics P', P'' of G^0 with Levi L', L'' respectively such that $S' \subset N_G P', S'' \subset N_G P''$, $F(P') = P', F(P'') = P''$.

By 24.7 we may assume that L, L' are G^0-conjugate. Then $\mathcal{E}', \mathcal{E}''$ are clean. Define $X'_{S'}, \bar{\mathcal{E}}'$ (resp. $X''_{S''}, \bar{\mathcal{E}}''$) in terms of $L', P', S', \mathcal{E}'$ (resp. $L'', P'', S'', \mathcal{E}''$) in the same way as $X_S, \bar{\mathcal{E}}$ were defined in terms of $L, P, S, \mathcal{E}$ in 5.6. Using 5.7 and the cleanness of $\mathcal{E}', \mathcal{E}''$ we see that $\mathfrak{K}' = f'_! \bar{\mathcal{E}}', \mathfrak{K}'' = f''_! \bar{\mathcal{E}}''$ where $f' : X'_{S'} \to G, f'' : X''_{S''} \to G$ are given by the first projection. Note that $X'_{S'}, X''_{S''}, f', f''$ are naturally defined over $\mathbf{F}_q$ and there are obvious isomorphisms $F^*\bar{\mathcal{E}}' \xrightarrow{\sim} \bar{\mathcal{E}}'$, $F^*\bar{\mathcal{E}}'' \xrightarrow{\sim} \bar{\mathcal{E}}''$ induced by ϵ', ϵ''. It follows that

$$\chi_{\mathfrak{K}',\phi'}(g) = \sum_{x'P'^F \in G^{0F}/P'^F; x'^{-1}gx' \in S'U_{P'}} \chi_{\mathcal{E}',\epsilon'}(\pi'(x'^{-1}gx')),$$

$$\chi_{\mathfrak{K}'',\phi''}(g) = \sum_{x''P''^F \in G^{0F}/P''^F; x''^{-1}gx'' \in S''U_{P''}} \chi_{\mathcal{E}'',\epsilon''}(\pi''(x''^{-1}gx'')),$$

where $\pi' : (S'U_{P'})^F \to S'^F, \pi'' : (S''U_{P''})^F \to S''^F$ are the obvious projections (see 1.26). Hence the left-hand side of 24.4(a) is

(a) $$|G^{0F}|^{-1}|P'^F|^{-1}|P''^F|^{-1} \sum_{x',x'' \in G^{0F}} h(x', x'')$$

where

$$h(x', x'') = \sum_{g \in V^F} \chi_{\mathcal{E}',\epsilon'}(\pi'(x'^{-1}gx'))\chi_{\mathcal{E}'',\epsilon''}(\pi''(x''^{-1}gx''))$$

and $V = \{g \in G; x'^{-1}gx' \in S'U_{P'}, x''^{-1}gx'' \in S''U_{P''}\}$.

Assume first that $x'P'x'^{-1}, x''P''x''^{-1}$ have no common Levi. In this case we show that $h(x', x'') = 0$. By the Grothendieck trace formula, $h(x', x'')$ is equal to an alternating sum of traces of Frobenius on $H^i_c(\mathcal{V}, \tilde{K})$ (notation as in the proof of 23.7). By the proof of 23.7, in our case we have $H^i_c(\mathcal{V}, \tilde{K}) = 0$ for all i. (The relevant part of the proof of 23.7 does not make use of the assumptions in the first sentence of 23.7; it only uses the strong cuspidality of $\mathcal{E}', \mathcal{E}''$.) Thus $h(x', x'') = 0$, as desired.

Assume now that $x', x'' \in G^{0F}$ are such that $Q' = x'P'x'^{-1}, Q'' = x''P''x''^{-1}$ have a common Levi $M' = M''$ (we may assume that $M' = M''$ is F-stable). We can find $v' \in U^F_{P'}, v'' \in U^F_{P''}$ such that, setting $n = v'^{-1}x'^{-1}x''v'' \in G^{0F}$, we have $n^{-1}L'n = L''$. Also, n is uniquely determined by x', x''. Let Σ', Σ'' be strata of $N_G M' = N_G M''$ as in 7.8. As in 7.8, we have a natural map $V \to \Sigma' \cap \Sigma''$ with fibres isomorphic to $U_{Q'} \cap U_{Q''} \cong U_{P'} \cap U_{nP''n^{-1}}$. In particular, $V = \emptyset$ unless $\Sigma' \cap \Sigma'' \neq \emptyset$ or equivalently, $\Sigma' = \Sigma''$ or equivalently, $n^{-1}S'n = S''$. If this last condition is satisfied, we have

$$h(x', x'') = |U^F_{P'} \cap U^F_{nP''n^{-1}}| \sum_{y \in S'^F} \chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n^{-1}yn).$$

It is then enough to show that

$$\begin{aligned}&\sharp((x', x'') \in G^{0F} \times G^{0F}; x'^{-1}x'' \in U^F_{P'}nU^F_{P''})\\ &= |G^{0F}||P'^F||P''^F||L'^F|^{-1}|L''^F|^{-1}|U^F_{P'} \cap U^F_{nP''n^{-1}}|.\end{aligned}$$

This is immediate. Thus Lemma 24.4 is proved in the present case.

24.10. We show that Lemma 24.4 holds for G under the assumption that Lemma 24.6 holds when G is replaced by $Z_G(s)$ for any semisimple element s of G^F. We evaluate the left-hand side of 24.4(a) using the "character formula" 16.14. We have

$$|G^{0F}|^{-1}\sum_{g\in G^F}\chi_{\mathfrak{K}',\phi'}(g)\chi_{\mathfrak{K}'',\phi''}(g)=|G^{0F}|^{-1}\sum_{s\in G^F\text{semis.}}\sum_{\substack{x',x''\in G^{0F}\\ x'^{-1}sx'\in S'_s\\ x''^{-1}sx''\in S''_s\\ \mathbf{d}',\mathbf{d}''}}$$
$$|Z_G(s)^{0F}|^{-2}|L'^{\,F}_{x'}||L'^F|^{-1}|L''^{\,F}_{x''}||L''^F|^{-1}f(s,x',x'',\mathbf{d}',\mathbf{d}'')$$

where
$L'_{x'}=x'L'x'^{-1}\cap Z_G(s)^0$, $L''_{x''}=x''L''x''^{-1}\cap Z_G(s)^0$,
$\mathbf{d}'$ runs over the set of F-stable $L'_{x'}$-conjugacy classes contained in $\{v\in Z_G(s); v \text{ unipotent}, x'^{-1}svx'\in S'\}$,
$\mathbf{d}''$ runs over the set of F-stable $L''_{x''}$-conjugacy classes contained in $\{v\in Z_G(s); v \text{ unipotent}, x''^{-1}svx''\in S''\}$,

$$f(s,x',x'',\mathbf{d}',\mathbf{d}'')=\sum_{\substack{u\in Z_G(s)^F\\ u\text{unip.}}}Q_{L'_{x'},Z_G(s),\mathbf{d}',\mathcal{F}'_{x'},\epsilon'_{x'}}(u)Q_{L''_{x''},Z_G(s),\mathbf{d}'',\mathcal{F}''_{x''},\epsilon''_{x''}}(u).$$

Here $\mathcal{F}'_{x'}$ is the inverse image of $\mathcal{E}'$ under $\mathbf{d}'\to S', v\mapsto x'^{-1}svx'$, $\mathcal{F}''_{x''}$ is the inverse image of $\mathcal{E}''$ under $\mathbf{d}''\to S'', v\mapsto x''^{-1}svx''$, and $\epsilon'_{x'}:F^*\mathcal{F}'_{x'}\xrightarrow{\sim}\mathcal{F}'_{x'}$, $\epsilon''_{x''}:F^*\mathcal{F}''_{x''}\xrightarrow{\sim}\mathcal{F}''_{x''}$ are induced by ϵ',ϵ''. Using our assumption we have

$$f(s,x',x'',\mathbf{d}',\mathbf{d}'')=|Z_G(s)^{0F}||L'^{\,F}_{x'}|^{-1}|L''^{\,F}_{x''}|^{-1}$$
(a)
$$\times\sum_{\substack{n\in Z_G(s)^{0F}\\ n^{-1}L'_{x'}n=L''_{x''}\\ n^{-1}\mathbf{d}'n=\mathbf{d}''}}\sum_{v\in\mathbf{d}'^F}\chi_{\mathcal{F}'_{x'},\epsilon'_{x'}}(v)\chi_{\mathcal{F}''_{x''},\epsilon''_{x''}}(n^{-1}vn).$$

To be able to apply our assumption, we use 23.4(a),(b). We also use the following fact.

For s,x',x'' as above and for $n\in Z_G(s)^0$, the condition $n^{-1}L'_{x'}n=L''_{x''}$ is equivalent to $n^{-1}x'Lx'^{-1}n=x''Lx''$.

Indeed, since S' is an isolated stratum of N_GL' and $x'^{-1}sx'\in S_s$, we see using 18.2 that $x'^{-1}sx'$ is isolated in N_GL'. Hence s is isolated in $N_G(x'L'x'^{-1})$. It follows that s is isolated in $N_G(n^{-1}x'L'x'^{-1}n)$. Similarly, s is isolated in $N_G(x''L''x''^{-1})$. By the injectivity of the map a in 21.3 (for s instead of g) we see that

$$n^{-1}x'L'x'^{-1}n\cap Z(s)^0=x''L''x''^{-1}\cap Z_G(s)^0\leftrightarrow n^{-1}x'L'x'^{-1}n=x''L''x''^{-1},$$

that is, $n^{-1}L'_{x'}n=L''_{x''}\leftrightarrow n^{-1}x'L'x'^{-1}n=x''L''x''^{-1}$, as required. We have

$$\chi_{\mathcal{F}'_{x'},\epsilon'_{x'}}(v)=\chi_{\mathcal{E}',\epsilon'}(x'^{-1}svx'),\quad \chi_{\mathcal{F}''_{x''},\epsilon''_{x''}}(n^{-1}vn)=\chi_{\mathcal{E}'',\epsilon''}(x''^{-1}sn^{-1}vnx'').$$

Note also that the condition $n^{-1}\mathbf{d}'n = \mathbf{d}''$ implies $n^{-1}x'S'x'^{-1}n = x''S''x''^{-1}$. We see that

$$
\begin{aligned}
&|G^{0F}|^{-1}\sum_{g\in G^F}\chi_{\mathfrak{K}',\phi'}(g)\chi_{\mathfrak{K}'',\phi''}(g)\\
&=|G^{0F}|^{-1}|L'^F|^{-1}|L''^F|^{-1}\sum_{\substack{s\in G^F \text{semis.}\\ x',x''\in G^{0F},\mathbf{d}'\\ x'^{-1}sx'\in S'_s\\ x''^{-1}sx''\in S''_s}}|Z_G(s)^{0F}|^{-1}\\
&\sum_{\substack{n\in Z_G(s)^{0F}\\ n^{-1}x'L'x'^{-1}n=x''L''x''^{-1}\\ n^{-1}x'S'x'^{-1}n=x''S''x''^{-1}}}\ \sum_{v\in\mathbf{d}'^F}\chi_{\mathcal{E}',\epsilon'}(x'^{-1}svx')\chi_{\mathcal{E}'',\epsilon''}(x''^{-1}sn^{-1}vnx'').
\end{aligned}
$$

We now make the change of variable $(x',x'',n)\mapsto(x',n,n')$, $n'^{-1}=x''^{-1}n^{-1}x'$. The condition $n^{-1}x'L'x'^{-1}n = x''L''x''^{-1}$ becomes $n'^{-1}L'n' = L''$; the condition $n^{-1}x'S'x'^{-1}n = x''S''x''^{-1}$ becomes $n'^{-1}S'n' = S''$ (thus, $n'\in\Theta$). The condition $x''^{-1}sx''\in S''_s$ becomes $n'^{-1}x'^{-1}nsn^{-1}x'n'\in n'^{-1}S'_sn'$, that is, $x'^{-1}sx'\in S'_s$. Our sum becomes

$$
\begin{aligned}
&|G^{0F}|^{-1}|L'^F|^{-1}|L''^F|^{-1}\sum_{\substack{s\in G^F\text{semis.}\\ x'\in G^{0F}\\ x'^{-1}sx'\in S'_s}}|Z_G(s)^{0F}|^{-1}\\
&\sum_{\substack{n\in Z_G(s)^{0F}\\ n'\in\Theta}}\ \sum_{\substack{v\in Z_G(s)^F;v\text{ unip.}\\ x'^{-1}svx'\in S'}}\chi_{\mathcal{E}',\epsilon'}(x'^{-1}svx')\chi_{\mathcal{E}'',\epsilon''}(n'^{-1}x'^{-1}svx'n').
\end{aligned}
$$

By the change of variable $(s,x',v)\mapsto(s',x',v')$ where $s'=x'^{-1}sx'\in S'_s$, $v'=x'^{-1}vx'\in Z_G(s')\cap s'^{-1}S'$ our sum becomes

$$
\begin{aligned}
&|L'^F|^{-1}|L''^F|^{-1}\sum_{\substack{s'\in S'^F_s\\ n'\in\Theta}}\ \sum_{\substack{v'\in Z_G(s')^F;v'\text{unip.}\\ v'\in s'^{-1}S'}}\chi_{\mathcal{E}',\epsilon'}(s'v')\chi_{\mathcal{E}'',\epsilon''}(n'^{-1}s'v'n')\\
&=|L'^F|^{-1}|L''^F|^{-1}\sum_{n'\in\Theta}\sum_{y\in S'^F}\chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n'^{-1}yn'),
\end{aligned}
$$

as required.

24.11. We show that Lemma 24.6 holds for G under the assumption that Lemma 24.6 holds when G is replaced by $Z_G(s)$ for any semisimple element s of G^F such that $\dim Z_G(s) < \dim G$ (that is, $s\notin Z_G(G^0)$).

Let $\mathcal{L}'\in\mathcal{S}({}^{\delta'}\mathcal{Z}^0_{L'})$, $\mathcal{L}''\in\mathcal{S}({}^{\delta''}\mathcal{Z}^0_{L''})$ be local systems of rank 1 with given isomorphisms $\iota' : F^*\mathcal{L}'\xrightarrow{\sim}\mathcal{L}'$, $\iota'' : F^*\mathcal{L}''\xrightarrow{\sim}\mathcal{L}''$. Let $\mathcal{E}'=\mathcal{L}'\boxtimes\mathcal{F}'\in\mathcal{S}(S')$, $\mathcal{E}''=\mathcal{L}''\boxtimes\mathcal{F}''\in\mathcal{S}(S'')$. Let $\epsilon'=\iota'\boxtimes\bar{\epsilon}' : F^*\mathcal{E}'\xrightarrow{\sim}\mathcal{E}'$, $\epsilon''=\iota''\boxtimes\bar{\epsilon}'' : F^*\mathcal{E}''\xrightarrow{\sim}\mathcal{E}''$. Note that $\mathcal{E}',\mathcal{E}''$ are strongly cuspidal by 23.4(c) and that, if L',L'' are not G^0-conjugate, then $\mathcal{E}',\mathcal{E}''$ are clean by 23.4(d). For this $\mathcal{E}',\mathcal{E}''$ we can still try to carry out the argument in 24.10 but now we can only use 24.6(a) for s such that

$s\notin Z_G(G^0)$. We obtain:

$$
\begin{aligned}
&|G^{0F}|^{-1}\sum_{g\in G^F}\chi_{\mathfrak{K}',\phi'}(g)\chi_{\mathfrak{K}'',\phi''}(g)\\
&-|L'^F|^{-1}|L''^F|^{-1}\sum_{n\in\Theta}\sum_{y\in S'^F}\chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n^{-1}yn)\\
&=|G^{0F}|^{-1}\sum_{s\in Z_G(G^0)^F\text{semis.}}\ \sum_{\substack{x',x''\in G^{0F},\mathbf{d}',\mathbf{d}''\\ x'^{-1}sx'\in S'_s\\ x''^{-1}sx''\in S''_s}}\\
&|Z_G(s)^{0F}|^{-2}|L'_{x'}{}^F||L'^F|^{-1}|L''_{x''}{}^F||L''^F|^{-1}\\
&(\sum_{u\in Z_G(s)^F;\text{unip.}}Q_{L'_{x'},Z_G(s),\mathbf{d}',\mathcal{F}'_{x'},\epsilon'_{x'}}(u)Q_{L''_{x''},Z_G(s),\mathbf{d}'',\mathcal{F}''_{x''},\epsilon''_{x''}}(u)\\
&-|Z_G(s)^{0F}||L'_{x'}{}^F|^{-1}|L''_{x''}{}^F|^{-1}\sum_{\substack{n\in Z_G(s)^{0F}\\ n^{-1}L'_{x'}n=L''_{x''}\\ n^{-1}\mathbf{d}'n=\mathbf{d}''}}\\
\text{(a)}\qquad&\sum_{v\in\mathbf{d}'^F}\chi_{\mathcal{F}'_{x'},\epsilon'_{x'}}(v)\chi_{\mathcal{F}''_{x''},\epsilon''_{x''}}(n^{-1}vn))
\end{aligned}
$$

(notation of 24.10.) For each s,x',x'' in the right-hand side of (a) we have $s=x'^{-1}sx'\in{}^{\delta'}\mathcal{Z}^0_{L'}$. Similarly, $s\in{}^{\delta''}\mathcal{Z}^0_{L''}$. In particular, we have $s\in G^0$ hence $s\in\mathcal{Z}_{G^0}$ and $Z_G(s)^0=G^0$, $L'_{x'}=x'L'x'^{-1}$, $L''_{x''}=x''L''x''^{-1}$. Also we necessarily have $\mathbf{d}'=x'\mathbf{c}'x'^{-1}$, $\mathbf{d}''=x''\mathbf{c}''x''^{-1}$. We see that the right-hand side of (a) is

$$
\begin{aligned}
&|G^{0F}|^{-1}\sum_{s\in\mathcal{Z}_{G^0}\cap{}^{\delta'}\mathcal{Z}^0_{L'}\cap{}^{\delta''}\mathcal{Z}^0_{L''}}\ \sum_{x',x''\in G^{0F}}\\
&|G^{0F}|^{-2}(\sum_{u\in Z_G(s)^F;\text{unip.}}Q_{L',Z_G(s),\mathbf{c}',\mathcal{F}',\bar\epsilon'}(u)Q_{L'',Z_G(s),\mathbf{c}'',\mathcal{F}'',\bar\epsilon''}(u)\\
&-|G^{0F}||L'^F|^{-1}|L''^F|^{-1}\sum_{\substack{n\in G^{0F}\\ n^{-1}x'L'x'^{-1}n=x''L''x''^{-1}\\ n^{-1}x'\mathbf{c}'x'^{-1}n=\mathbf{c}''}}\ \sum_{v\in\mathbf{c}'^F}\chi_{\mathcal{F}',\bar\epsilon'}(v)\chi_{\mathcal{F}'',\bar\epsilon''}(x''^{-1}n^{-1}x'vn))
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&|G^{0F}|^{-1}\sum_{s\in\mathcal{Z}_{G^0}\cap{}^{\delta'}\mathcal{Z}^0_{L'}\cap{}^{\delta''}\mathcal{Z}^0_{L''}}\\
&(\sum_{u\in Z_G(s)^F;\text{unip.}}Q_{L',Z_G(s),\mathbf{c}',\mathcal{F}',\bar\epsilon'}(u)Q_{L'',Z_G(s),\mathbf{c}'',\mathcal{F}'',\bar\epsilon''}(u)\\
\text{(b)}\qquad&-|G^{0F}||L'^F|^{-1}|L''^F|^{-1}\sum_{n'\in\Theta}\sum_{v\in\mathbf{c}'^F}\chi_{\mathcal{F}',\bar\epsilon'}(v)\chi_{\mathcal{F}'',\bar\epsilon''}(n'^{-1}vn')).
\end{aligned}
$$

Let D' (resp. D'') be the connected component of G that contains δ' (resp. δ'') For each s in the sum we have $D'\subset Z_G(s)$, $D''\subset Z_G(s)$; moreover, $Q_{L',Z_G(s),\mathbf{c}',\mathcal{F}',\bar\epsilon'}(u)=$

0 unless $u \in D'$ and $Q_{L'',Z_G(s),\mathbf{c}'',\mathcal{F}'',\bar{\epsilon}''}(u) = 0$ unless $u \in D''$. We see that

$$\sum_{u \in Z_G(s)^F;\text{unip.}} Q_{L',Z_G(s),\mathbf{c}',\mathcal{F}',\bar{\epsilon}'}(u) Q_{L'',Z_G(s),\mathbf{c}'',\mathcal{F}'',\bar{\epsilon}''}(u)$$
$$= \sum_{u \in G^F;\text{unip.}} Q_{L',G,\mathbf{c}',\mathcal{F}',\bar{\epsilon}'}(u) Q_{L'',G,\mathbf{c}'',\mathcal{F}'',\bar{\epsilon}''}(u).$$

Hence (a) becomes

$$|G^{0F}|^{-1} \sum_{g \in G^F} \chi_{\mathfrak{K}',\phi'}(g) \chi_{\mathfrak{K}'',\phi''}(g)$$
$$- |L'^F|^{-1} |L''^F|^{-1} \sum_{n \in \Theta} \sum_{y \in S'^F} \chi_{\mathcal{E}',\epsilon'}(y) \chi_{\mathcal{E}'',\epsilon''}(n^{-1} y n)$$
$$= |G^{0F}|^{-1} \sharp(\mathcal{Z}_{G^0} \cap {}^{\delta'}\mathcal{Z}^0_{L'} \cap {}^{\delta''}\mathcal{Z}^0_{L''})$$
$$(\sum_{u \in G^F;\text{unip.}} Q_{L',G,\mathbf{c}',\mathcal{F}',\bar{\epsilon}'}(u) Q_{L'',G,\mathbf{c}'',\mathcal{F}'',\bar{\epsilon}''}(u)$$
$$- |G^{0F}||L'^F|^{-1}|L''^F|^{-1} \sum_{n' \in \Theta} \sum_{v \in \mathbf{c}'^F} \chi_{\mathcal{F}',\bar{\epsilon}'}(v) \chi_{\mathcal{F}'',\bar{\epsilon}''}(n'^{-1} v n')). \tag{c}$$

Hence to prove the equality 24.6(a) it is enough to show that the left-hand side of (c) is zero. In order to do so, we are free to choose $\mathcal{L}', \mathcal{L}'', \iota', \iota''$ in a convenient way.

Assume first that ${}^{\delta'}\mathcal{Z}^{0F}_{L'} \neq \{1\}$. Then we can find a non-trivial character $\theta' : {}^{\delta'}\mathcal{Z}^{0F}_{L'} \to \bar{\mathbf{Q}}_l^*$ and $\mathcal{L}', \iota'$ as above such that $\chi_{\mathcal{L}',\iota'} = \theta'$. We have $\mathcal{L}' \ncong \bar{\mathbf{Q}}_l$. Let $\mathcal{L}'' = \bar{\mathbf{Q}}_l$ and take any ι''. With these choices, 24.8 shows that the left-hand side of (c) is zero.

Assume next that ${}^{\delta''}\mathcal{Z}^{0F}_{L''} \neq \{1\}$. Since L', L'' play a symmetrical role, we see as in the previous paragraph that the left-hand side of (c) is zero.

Finally, assume that ${}^{\delta'}\mathcal{Z}^{0F}_{L'} = \{1\}$ and ${}^{\delta''}\mathcal{Z}^{0F}_{L''} = \{1\}$. Then the F-stable tori ${}^{\delta'}\mathcal{Z}^0_{L'}, {}^{\delta''}\mathcal{Z}^0_{L''}$ are necessarily $\mathbf{F}_q$-split and $q = 2$. Using 24.2(a) we can find parabolics P', P'' of G^0 with Levi L', L'' such that $F(P') = P', F(P'') = P''$. Then 24.9 shows that the left-hand side of (c) is zero. Thus Lemma 24.6 is proved in the present case.

24.12. Clearly, the arguments in 24.10, 24.11 provide an inductive proof of Lemmas 24.4, 24.6.

24.13. In the setup of 24.3, assume that $\mathcal{E}', \mathcal{E}''$ are irreducible. Let

$$\Theta(\mathcal{E}', \mathcal{E}'') = \{n \in G^{0F}; n^{-1} L' n = L'', n^{-1} S' n = S'', \mathrm{Ad}(n^{-1})^* \mathcal{E}'' \cong \check{\mathcal{E}}'\}.$$

If $n \in \Theta(\mathcal{E}', \mathcal{E}'')$, the local system $\hat{\mathcal{E}} = \mathcal{E}' \otimes \mathrm{Ad}(n^{-1})^* \mathcal{E}''$ is canonically of the form $\bar{\mathbf{Q}}_l \oplus \hat{\mathcal{E}}_1$ where $\hat{\mathcal{E}}_1 \in \mathcal{S}(S)$ has no direct summand $\bar{\mathbf{Q}}_l$. The isomorphisms ϵ', ϵ'' induce an isomorphism $F^* \hat{\mathcal{E}} \xrightarrow{\sim} \hat{\mathcal{E}}$ which respects the summand $\bar{\mathbf{Q}}_l$ and induces on it $\zeta(n)$ times the obvious isomorphism $F^* \bar{\mathbf{Q}}_l \xrightarrow{\sim} \bar{\mathbf{Q}}_l$. Here $\zeta(n) \in \bar{\mathbf{Q}}_l^*$ is well defined.

We show that, if $n \in \Theta(\mathcal{E}', \mathcal{E}'')$ and $n_0 \in G^{0F}$, $n_0^{-1} L' n_0 = L'$, $n_0^{-1} S' n_0 = S'$, $\mathrm{Ad}(n_0^{-1})^* \mathcal{E}' \cong \mathcal{E}'$, then $n_0 n \in \Theta(\mathcal{E}', \mathcal{E}'')$ satisfies

(a) $\zeta(n_0 n) = \eta(n_0^{-1}) \zeta(n)$

where $\eta(n_0^{-1})$ is as in 21.6. Let $\alpha : \mathrm{Ad}(n_0^{-1})^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ be an isomorphism. We have an isomorphism

$$\begin{aligned}&\mathrm{Ad}(n_0^{-1})^*(\mathcal{E}' \otimes \mathrm{Ad}(n^{-1})^*\mathcal{E}'')\\&= \mathrm{Ad}(n_0^{-1})^*\mathcal{E}' \otimes \mathrm{Ad}(n^{-1}n_0^{-1})^*\mathcal{E}'' \xrightarrow{\alpha\otimes} \mathcal{E}' \otimes \mathrm{Ad}(n^{-1}n_0^{-1})^*\mathcal{E}''\end{aligned}$$

which must carry the summand $\mathrm{Ad}(n_0^{-1})^*(\bar{\mathbf{Q}}_l)$ to the summand $\bar{\mathbf{Q}}_l$. Let $x \in S'^F$. Let $e'_i \in \mathcal{E}'_{n_0^{-1}xn_0}, e''_i \in \mathcal{E}''_{n^{-1}n_0^{-1}xn_0n}$ be such that $\sum_i e'_i \otimes e''_i$ belongs to the stalk of the summand $\bar{\mathbf{Q}}_l$ of $\mathrm{Ad}(n_0^{-1})^*\mathcal{E}'\otimes\mathrm{Ad}(n^{-1}n_0^{-1})^*\mathcal{E}''$ at x. Then $\sum_i \alpha(e'_i)\otimes e''_i$ belongs to the stalk of the summand $\bar{\mathbf{Q}}_l$ of $\mathcal{E}' \otimes \mathrm{Ad}(n^{-1}n_0^{-1})^*\mathcal{E}''$ at x. By definition,

$$\sum_i \epsilon'(e'_i) \otimes \epsilon''(e''_i) = \zeta(n) \sum_i e'_i \otimes e''_i, \sum_i \epsilon'(\alpha(e'_i)) \otimes \epsilon''(e''_i) = \zeta(n_0n) \sum_i \alpha(e'_i) \otimes e''_i,$$

$$\epsilon'(\alpha(e'_i)) = \eta(n_0^{-1})\alpha(\epsilon'(e'_i)).$$

We deduce

$$\begin{aligned}\sum_i \epsilon'(\alpha(e'_i)) \otimes \epsilon''(e''_i) =&\eta(n_0^{-1})(\alpha \otimes 1)(\sum_i \epsilon'(e'_i) \otimes \epsilon''(e''_i))\\ &= \eta(n_0^{-1})\zeta(n)(\alpha \otimes 1)(\sum_i e'_i \otimes e''_i)\end{aligned}$$

and (a) follows.

From (a) we deduce:

(b) *if $(L', S', \mathcal{E}')$ is effective (see 21.6), then $\zeta : \Theta(\mathcal{E}', \mathcal{E}'') \to \bar{\mathbf{Q}}_l^*$ is constant.*

Lemma 24.14. *Assume that $n \in \Theta$. Then $\sum_{y\in S'^F} \chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n^{-1}yn)$ equals*

$$\zeta(n)q^{\dim S'-\dim L'}|L'^F|$$

if $n \in \Theta(\mathcal{E}', \mathcal{E}'')$ and 0, otherwise.

In the following proof we write S instead of S' and δ for the connected component of N_GL' that contains S. By the Grothendieck trace formula, our sum is an alternating sum of traces of the Frobenius map on $H^i_c(S, \hat{\mathcal{E}})$ where $\hat{\mathcal{E}} = \mathcal{E}' \otimes \mathrm{Ad}(n^{-1})^*\mathcal{E}''$. If $n \notin \Theta(\mathcal{E}', \mathcal{E}'')$, then $\hat{\mathcal{E}}$ has no direct summand isomorphic to $\bar{\mathbf{Q}}_l$, hence by 23.5(a), $H^i_c(S', \hat{\mathcal{E}}) = 0$. Thus we may assume that $n \in \Theta(\mathcal{E}', \mathcal{E}'')$. Then we have canonically $\hat{\mathcal{E}} = \bar{\mathbf{Q}}_l \oplus \hat{\mathcal{E}}_1$ as in 24.13. As in the proof of 23.5(a) we have $H^i_c(S, \hat{\mathcal{E}}_1) = 0$ for all i, hence $H^i_c(S, \hat{\mathcal{E}}) = H^i_c(S, \bar{\mathbf{Q}}_l)$. By the definition of $\zeta(n)$ in 24.13 we see that the sum in the lemma is equal to $\zeta(n)\sum_i(-1)^i\mathrm{tr}(F^*, H^i_c(S, \bar{\mathbf{Q}}_l))$. Let $f : \tilde{S} \to S$ be as in 23.5 (for a fixed $y \in S^F$ and for N_GL', δ instead of G, D). Then $\tilde{S}, f$ are defined over $\mathbf{F}_q$ and from the proof of 23.5 we see that $\mathrm{tr}(F^*, H^i_c(S, \bar{\mathbf{Q}}_l)) = \mathrm{tr}(F^*, H^i_c(\tilde{S}, \bar{\mathbf{Q}}_l))$. Hence our sum is equal to

$$\zeta(n) \sum_i (-1)^i\mathrm{tr}(F^*, H^i_c(\tilde{S}, \bar{\mathbf{Q}}_l)) = \zeta(n)|\tilde{S}^F| = \zeta(n)|^\delta\mathcal{Z}^{0F}_{L'}||L'^F||Z_{L'}(y)^{0F}|^{-1}.$$

By 10.2, $Z_{L'}(y)^0/^\delta\mathcal{Z}^0_{L'}$ is a (connected) unipotent group. Hence

$$|^\delta\mathcal{Z}^{0F}_{L'}||Z_{L'}(y)^{0F}|^{-1} = q^{\dim {}^\delta\mathcal{Z}^0_{L'}-\dim Z_{L'}(y)^0} = q^{\dim S-\dim L'}.$$

The lemma is proved.

Proposition 24.15. *In the setup of 24.3 and assuming that $\mathcal{E}',\mathcal{E}''$ are irreducible and $(L',S',\mathcal{E}'),(L'',S'',\mathcal{E}'')$ are effective, the sum*

$$|G^{0F}|^{-1}\sum_{g\in G^F}\chi_{\mathfrak{K}',\phi'}(g)\chi_{\mathfrak{K}'',\phi''}(g)$$

is 0 *if $\Theta(\mathcal{E}',\mathcal{E}'')=\emptyset$ and is equal to*

$$\zeta q^{\dim S'-\dim L'}|\Theta(\mathcal{E}',\mathcal{E}'')|/|L'^F|$$

where $\zeta=\zeta(n)$ (see 24.13) for any $n\in\Theta(\mathcal{E}',\mathcal{E}'')$ if $\Theta(\mathcal{E}',\mathcal{E}'')\neq\emptyset$.

This follows from the results in 24.4, 24.13, 24.14.

Proposition 24.16. *In the setup of 24.5 let $\mathcal{E}'=\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}'\in\mathcal{S}(S')$, $\mathcal{E}''=\bar{\mathbf{Q}}_l\boxtimes\mathcal{F}''\in\mathcal{S}(S'')$ and let $\epsilon'=1\boxtimes\bar{\epsilon}':F^*\mathcal{E}'\xrightarrow{\sim}\mathcal{E}'$, $\epsilon''=1\boxtimes\bar{\epsilon}'':F^*\mathcal{E}''\xrightarrow{\sim}\mathcal{E}''$. Assume that $\mathcal{F}',\mathcal{F}''$ are irreducible. Then the sum*

$$|G^{0F}|^{-1}\sum_{u\in G^F;unip.}Q_{L',G,\mathbf{c}',\mathcal{F}',\bar{\epsilon}'}(u)Q_{L'',G,\mathbf{c}'',\mathcal{F}'',\bar{\epsilon}''}(u)$$

is 0 *if $\Theta(\mathcal{E}',\mathcal{E}'')=\emptyset$ and is equal to*

$$\zeta q^{\dim S-\dim L'}|\Theta(\mathcal{E}',\mathcal{E}'')||L'^F|^{-1}|^{\delta'}\mathcal{Z}_{L'}^{0F}|^{-1}$$

where $\zeta=\zeta(n)$ (defined as in 24.13 in terms of ϵ',ϵ'') for any $n\in\Theta(\mathcal{E}',\mathcal{E}'')$ if $\Theta(\mathcal{E}',\mathcal{E}'')\neq\emptyset$.

If $n\in\Theta$ (see 24.3) we have clearly

$$\sum_{y\in\mathbf{c}'^F}\chi_{\mathcal{F}',\bar{\epsilon}'}(y)\chi_{\mathcal{F}'',\bar{\epsilon}''}(n^{-1}yn)=|^{\delta'}\mathcal{Z}_{L'}^{0F}|^{-1}\sum_{y\in S'^F}\chi_{\mathcal{E}',\epsilon'}(y)\chi_{\mathcal{E}'',\epsilon''}(n^{-1}yn).$$

The last sum can be evaluated using the results in 24.13, 24.14. (In our case, $(L',S',\mathcal{E}'),(L'',S',\mathcal{E}'')$ are automatically effective; see 21.8.) We introduce this in Lemma 24.6. The proposition follows.

24.17. Let $A\in\mathfrak{A}(G)$ (see 21.18) and let $\alpha:F^*A\xrightarrow{\sim}A$ be an isomorphism. We have $A=IC(\bar{Y},\mathcal{A})$ (extended by 0 on $G-\bar{Y}$) where $\bar{Y}$ is the closure of a stratum $Y_{L,S}$, $(L,S)\in\mathbf{A}$ and $\mathcal{A}$ is an irreducible local system on $\bar{Y}$ which is a direct summand of $\pi_!\tilde{\mathcal{E}}$ (here $\mathcal{E}\in\mathcal{S}(S)$ is irreducible cuspidal and $\pi,\tilde{\mathcal{E}}$ are as in 5.6). By 21.19, 21.20 we can assume that $F(L)=L,F(S)=S$ and $F^*\mathcal{E}\cong\mathcal{E}$. We consider also $A'\in\mathfrak{A}(G)$ and $\alpha':F^*A'\xrightarrow{\sim}A'$. Let $L',S',\mathcal{E}',\mathcal{A}',\bar{Y}',\pi'_!\tilde{\mathcal{E}}'$ play the same role for A' as $L,S,\mathcal{E},\mathcal{A},\bar{Y},\pi_!\tilde{\mathcal{E}}$ for A. (In particular, $A'=IC(\bar{Y}',\mathcal{A}')$ extended by 0 on $G-\bar{Y}'$, $F(L')=L',F(S')=S',F^*\mathcal{E}'\cong\mathcal{E}'$.) We fix $\epsilon:F^*\mathcal{E}\xrightarrow{\sim}\mathcal{E},\epsilon':F^*\mathcal{E}'\xrightarrow{\sim}\mathcal{E}'$. Let

$$\tilde{\Gamma},\Gamma,\mathfrak{K}^w,\phi^w,n_w,g_w,L^w,S^w,\mathcal{E}^w,\epsilon^w,r,\mathbf{E},b_w,\iota_i,V_i,\mathfrak{K}_i,\phi_i$$

be associated with $L,S,\mathcal{E},\epsilon$ as in 21.6 and let

$$\tilde{\Gamma}',\Gamma',\mathfrak{K}'^{w'},\phi'^{w'},n'_{w'},g'_{w'},L'^{w'},S'^{w'},\mathcal{E}'^{w'},\epsilon'^{w'},r',\mathbf{E}',b'_{w'},\iota'_{i'},V'_{i'},\mathfrak{K}'_{i'},\phi'_{i'}$$

be associated in an analogous way to $L',S',\mathcal{E}',\epsilon'$. From 21.6(c) we have

(a) $\chi_{\mathfrak{K}^w,\phi^w}=\sum_{i\in[1,r]}\mathrm{tr}(b_w\iota_i,V_i)\chi_{\mathfrak{K}_i,\phi_i}$

for any effective $w\in\Gamma$. We multiply both sides of (a) by $|\Gamma|^{-1}\mathrm{tr}(\iota_j^{-1}b_w^{-1},V_j)$ and sum over all effective $w\in\Gamma$. Using 20.4(c) we obtain

$$\chi_{\mathfrak{K}_j,\phi_j}=|\Gamma|^{-1}\sum_{w\in\Gamma;\text{eff.}}\mathrm{tr}(\iota_j^{-1}b_w^{-1},V_j)\chi_{\mathfrak{K}^w,\phi^w}$$

for any $j \in [1,r]$. Similarly,

$$\chi_{\mathfrak{K}'_{j'},\phi'_{j'}} = |\Gamma'|^{-1} \sum_{w'\in\Gamma';\text{eff.}} \operatorname{tr}(\iota'_{j'}{}^{-1} b'_{w'}{}^{-1}, V'_{j'}) \chi_{\mathfrak{K}'^{w'},\phi'^{w'}}$$

for any $j' \in [1,r']$. It follows that

$$|G^{0F}|^{-1} \sum_{g\in G^F} \chi_{\mathfrak{K}_j,\phi_j}(g)\chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(g)$$
$$= |\Gamma|^{-1}|\Gamma'|^{-1} \sum_{w\in\Gamma;\text{eff.}} \sum_{w'\in\Gamma';\text{eff.}} \operatorname{tr}(\iota_j^{-1} b_w^{-1}, V_j)\operatorname{tr}(\iota'_{j'}{}^{-1} b'_{w'}{}^{-1}, V'_{j'})$$
$$\text{(b)} \qquad \times |G^{0F}|^{-1} \sum_{g\in G^F} \chi_{\mathfrak{K}^w,\phi^w}(g)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(g)$$

for any $j \in [1,r], j' \in [1,r']$.

Proposition 24.18. *Assume that $\check{\mathcal{E}}, \mathcal{E}'$ are strongly cuspidal. Assume also that either L, L' are not G^0-conjugate or $\check{\mathcal{E}}, \mathcal{E}'$ are clean. Then*

$$|G^{0F}|^{-1} \sum_{g\in G^F} \chi_{A,\alpha}(g)\chi_{A',\alpha'}(g)$$

is 0 *if $A' \not\cong \check{A}$ and is $q^{\dim S - \dim L}$ if $A' = \check{A}$ and $\alpha' = \check{\alpha}$.*

We may assume that $A = \mathfrak{K}_j, \alpha = \phi_j, A' = \mathfrak{K}'_{j'}, \alpha' = \phi'_{j'}$ for some $j \in [1,r]$, $j' \in [1,r']$. If $(L,S,\check{\mathcal{E}}),(L',S',\mathcal{E}')$ are not G^0-conjugate, the result follows from 24.17(b) since by 24.15 we have

$$|G^{0F}|^{-1} \sum_{g\in G^F} \chi_{\mathfrak{K}^w,\phi^w}(g)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(g) = 0$$

for any effective w, w'.

Thus we may assume that $(L,S,\check{\mathcal{E}}),(L',S',\mathcal{E}')$ are G^0-conjugate. We may also assume that $L = L', S = S', \check{\mathcal{E}} = \mathcal{E}', \epsilon' = \check{\epsilon}$. We have $\tilde{\Gamma}' = \tilde{\Gamma}, \Gamma' = \Gamma$, $\tilde{\mathcal{E}}' = (\tilde{\mathcal{E}})\check{}$, $\pi'_!\tilde{\mathcal{E}}' = (\pi_!\tilde{\mathcal{E}})\check{}$. Taking transpose we may identify $\mathbf{E}' = \mathbf{E}$ as vector spaces but with opposed algebra structures. Then $\mathbf{E}'_w = \mathbf{E}_{w^{-1}}$ for any $w \in \Gamma$ and we may assume that $b'_w \in \mathbf{E}'_w$ corresponds to $b_w^{-1} \in \mathbf{E}_{w^{-1}}$. The simple $\mathbf{E}'$ modules are $V'_i = V_i^*$ (dual space) with $\mathbf{E}'$-action given by taking transpose. We take ι'_i to be the transpose inverse of ι_i. We have $r' = r, \mathfrak{K}'_i = (\mathfrak{K}_i)\check{}, \phi'_i = (\phi_i)\check{}$. We take $n'_w = n_w, g'_w = g_w$. Then $L'^w = L^w, S'^w = S^w, \mathcal{E}'^w = (\mathcal{E}^w)\check{}$, $\epsilon'^w = (\epsilon^w)\check{}$. Moreover, for $w \in \Gamma$, $(L^w, S^w, \mathcal{E}^w)$ is effective if and only if $(L^w, S^w, (\mathcal{E}^w)\check{})$ is effective.

Let $w, w' \in \Gamma$. We compute

$$a_{w,w'} = \sharp(n \in G^{0F}; n^{-1}L^w n = L^{w'}, n^{-1}S^w n = S^{w'}, \operatorname{Ad}(n^{-1})^*(\mathcal{E}^{w'})\check{} \cong (\mathcal{E}^w)\check{}).$$

Setting $g_w^{-1} n g_{w'} = \mathbf{n}$ we see that

$$a_{w,w'} = \sharp(\mathbf{n} \in \tilde{\Gamma}; F(g_w \mathbf{n} g_{w'}^{-1}) = g_w \mathbf{n} g_{w'}^{-1}) = \sharp(\mathbf{n} \in \tilde{\Gamma}; n_w^{-1}F(\mathbf{n})n_{w'} = \mathbf{n})$$
$$= \sum_{\substack{y\in\Gamma \\ w^{-1}F(y)w'=y}} |\Delta_y^{F_0}|$$

where Δ_y is the L-coset in $\tilde{\Gamma}$ defined by y and $F_0 : \Delta_y \to \Delta_y$ is $F_0(\mathbf{n}) = n_w^{-1}F(\mathbf{n})n_{w'}$. Now L^w acts freely on Δ_y by $l_1 : \mathbf{n} \mapsto l_1 * \mathbf{n} = g_w^{-1} l_1 g_w \mathbf{n}$; this action satisfies $F_0(l_1 * \mathbf{n}) = F(l_1) * F_0(\mathbf{n})$. It follows that $|\Delta_y^{F_0}| = |L^{wF}|$ and

$a_{w,w'}|L^{wF}|^{-1} = \sharp(y \in \Gamma; w^{-1}F(y)w' = y)$. Using this and 24.15 we see that for any effective $w, w' \in \Gamma$:

$$\begin{aligned}&|G^{0F}|^{-1} \sum_{g \in G^F} \chi_{\mathfrak{K}^w,\phi^w}(g)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(g)\\ &= q^{\dim S - \dim L} a_{w,w'}|L^{wF}|^{-1} = \zeta_{w,w'} q^{\dim S - \dim L} \sharp(y \in \Gamma; w^{-1}F(y)w' = y).\end{aligned}$$

Here $\zeta_{w,w'} \in \bar{\mathbf{Q}}_l^*$ is defined as follows (assuming that $\{y \in \Gamma; w^{-1}F(y)w' = y\} \neq \emptyset$): for any $n \in G^{0F}$ such that

$$n^{-1}L^w n = L^{w'}, n^{-1}S^w n = S^{w'}, \mathrm{Ad}(n^{-1})^*(\mathcal{E}^{w'})\check{} \cong (\mathcal{E}^w)\check{},$$

$\epsilon^w, (\epsilon^{w'})\check{}$ induce an isomorphism $F^*(\mathcal{E}^w \otimes \mathrm{Ad}(n^{-1})^*(\mathcal{E}^{w'})\check{}) \to \mathcal{E}^w \otimes \mathrm{Ad}(n^{-1})^*(\mathcal{E}^{w'})\check{}$ which on the summand $\bar{\mathbf{Q}}_l$ of $\mathcal{E}^w \otimes \mathrm{Ad}(n^{-1})^*(\mathcal{E}^{w'})\check{}$ is $\zeta_{w,w'}$ times the obvious isomorphism $F^*\bar{\mathbf{Q}}_l \xrightarrow{\sim} \bar{\mathbf{Q}}_l$.

For n as above we set $g_w^{-1} n g_{w'} = \mathbf{n} \in \tilde{\Gamma}$ and let y be the image of $\mathbf{n}$ in Γ. Let $g \in S$, let (e_h) be a basis of $\mathcal{E}_g$ and let $(\check{e}_h)$ be the dual basis of $\check{\mathcal{E}}_g$. We have an isomorphism $b_y : \mathrm{Ad}(\mathbf{n}^{-1})^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Taking transpose inverse we obtain an isomorphism ${}^t b_y^{-1} : \mathrm{Ad}(\mathbf{n}^{-1})^*\check{\mathcal{E}} \xrightarrow{\sim} \check{\mathcal{E}}$. This restricts to ${}^t b_y^{-1} : \check{\mathcal{E}}_{\mathbf{n}^{-1}g\mathbf{n}} \xrightarrow{\sim} \check{\mathcal{E}}_g$ and $\sum_h e_h \otimes {}^t b_y(\check{e}_h)$ belongs to the stalk of the summand $\bar{\mathbf{Q}}_l$ of $\mathcal{E} \otimes \mathrm{Ad}(\mathbf{n}^{-1})^*\check{\mathcal{E}}$ at g. We have

$$\mathcal{E}^w \otimes \mathrm{Ad}(g_{w'}\mathbf{n}^{-1}g_w^{-1})^*(\mathcal{E}^{w'})\check{} = \mathrm{Ad}(g_w^{-1})^*(\mathcal{E} \otimes \mathrm{Ad}(\mathbf{n}^{-1})^*\check{\mathcal{E}}),$$

hence $\sum_h e_h \otimes {}^t b_y(\check{e}_h)$ belongs to the stalk of the summand $\bar{\mathbf{Q}}_l$ of

$$\mathcal{E}^w \otimes \mathrm{Ad}(g_{w'}\mathbf{n}^{-1}g_w^{-1})^*(\mathcal{E}^{w'})\check{}$$

at $g_w g g_w^{-1} \in S^w$. Assuming that $g_w g g_w^{-1} \in S^{wF}$ we see from the definitions that

$$\sum_h \epsilon^w(e_h) \otimes \check{\epsilon}^{w'}({}^t b_y(\check{e}_h)) = \zeta_{w,w'} \sum_h e_h \otimes {}^t b_y(\check{e}_h) \in \mathcal{E}_g \otimes \check{\mathcal{E}}_{\mathbf{n}^{-1}g\mathbf{n}},$$

hence

$$\sum_h \epsilon(b_w e_h) \otimes \check{\epsilon}({}^t b_{w'}^{-1}{}^t b_y(\check{e}_h)) = \zeta_{w,w'} \sum_h e_h \otimes {}^t b_y(\check{e}_h) \in \mathcal{E}_g \otimes \check{\mathcal{E}}_{\mathbf{n}^{-1}g\mathbf{n}}.$$

Applying $\epsilon^{-1} \otimes \check{\epsilon}^{-1}$ to both sides and using $\check{\epsilon}^{-1}{}^t b_y = {}^t(\iota^{-1}(b_y))\check{\epsilon}^{-1}$ we obtain

$$\sum_h b_w e_h \otimes {}^t b_{w'}^{-1}{}^t b_y(\check{e}_h) = \zeta_{w,w'} \sum_h \epsilon^{-1}(e_h) \otimes {}^t(\iota^{-1}(b_y))\check{\epsilon}^{-1}(\check{e}_h).$$

Setting $e'_h = \epsilon^{-1}(e_h), \check{e}'_h = \check{\epsilon}^{-1}(\check{e}_h)$ we see that $(e'_h), (\check{e}'_h)$ are dual bases of $\mathcal{E}_{F(g)}$, $\check{\mathcal{E}}_{F(g)}$ and we have

$$\sum_h b_w e_h \otimes {}^t b_{w'}^{-1}{}^t b_y(\check{e}_h) = \zeta_{w,w'} \sum_h e'_h \otimes {}^t(\iota^{-1}(b_y))\check{e}'_h \in \mathcal{E}_{F(g)} \otimes \check{\mathcal{E}}_{F(\mathbf{n}^{-1}g\mathbf{n})}.$$

Applying $1 \otimes {}^t(\iota^{-1}(b_y))^{-1}$ to both sides gives

$$\sum_h b_w e_h \otimes {}^t(\iota^{-1}(b_y))^{-1}{}^t b_{w'}^{-1}{}^t b_y(\check{e}_h)) = \zeta_{w,w'} \sum_h e'_h \otimes \check{e}'_h \in \mathcal{E}_{F(g)} \otimes \check{\mathcal{E}}_{F(g)}.$$

Let $e''_h = b_w e_h, \check{e}''_h = {}^t b_w^{-1}\check{e}_h$. Then $(e''_h), (\check{e}''_h)$ are dual bases of

$$\mathcal{E}_{n_w g n_w^{-1}} = \mathcal{E}_{F(g)}, \check{\mathcal{E}}_{n_w g n_w^{-1}} = \check{\mathcal{E}}_{F(g)},$$

hence $\sum_h e'_h \otimes \check{e}'_h = \sum_h e''_h \otimes \check{e}''_h$. We see that

$$\sum_h b_w e_h \otimes {}^t(\iota^{-1}(b_y))^{-1} {}^t b_{w'}^{-1} {}^t b_y(\check{e}_h) = \zeta_{w,w'} \sum_h b_w e_h \otimes {}^t b_w^{-1} \check{e}_h.$$

This shows that

$${}^t(\iota^{-1}(b_y))^{-1} {}^t b_{w'}^{-1} {}^t b_y(\check{e}_h) = \zeta_{w,w'} {}^t b_w^{-1} \check{e}_h \in \check{\mathcal{E}}_{n_w g n_w^{-1}}$$

for all h. Thus ${}^t(\iota^{-1}(b_y))^{-1} {}^t b_{w'}^{-1} {}^t b_y = \zeta_{w,w'} {}^t b_w^{-1}$ as linear maps $\check{\mathcal{E}}_{n_w g n_w^{-1}} \to \check{\mathcal{E}}_g$ hence $\iota^{-1}(b_y) b_{w'} b_y^{-1} = \zeta_{w,w'}^{-1} b_w$ as linear maps $\mathcal{E}_{n_w g n_w^{-1}} \to \mathcal{E}_g$. It follows that

$$\iota^{-1}(b_y) b_{w'} b_y^{-1} = \zeta_{w,w'}^{-1} b_w \in \mathbf{E}$$

for any $y \in \Gamma$ such that $w^{-1} F(y) w' = y$. In our case, 24.17(b) becomes

$$\begin{aligned}
&|G^{0F}|^{-1} \sum_{g \in G^F} \chi_{\mathfrak{K}_j,\phi_j}(g) \chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(g) = |\Gamma|^{-2} \sum_{w,w' \in \Gamma;\text{eff.}} \mathrm{tr}(b_w^{-1} \iota_j^{-1}, V_j) \\
&\times \mathrm{tr}(\zeta_{w,w'} b_{w'} \iota_{j'}, V_{j'}) q^{\dim S - \dim L} \sharp(y \in \Gamma; w^{-1}F(y)w' = y) \\
&= q^{\dim S - \dim L} |\Gamma|^{-2} \sum_{w \in \Gamma;\text{eff.}} \sum_{y \in \Gamma} \mathrm{tr}(b_w^{-1} \iota_j^{-1}, V_j) \mathrm{tr}(\iota^{-1}(b_y^{-1}) b_w b_y \iota_{j'}, V_{j'}).
\end{aligned}$$

We have

$$\begin{aligned}
&\mathrm{tr}(\iota^{-1}(b_y^{-1}) b_w b_y \iota_{j'}, V_{j'}) = \mathrm{tr}(\iota_{j'} \iota^{-1}(b_y^{-1}) b_w b_y, V_{j'}), \\
&\mathrm{tr}(b_y^{-1} \iota_{j'} b_w b_y, V_{j'}) = \mathrm{tr}(\iota_{j'} b_w, V_{j'}),
\end{aligned}$$

hence

$$\begin{aligned}
&|G^{0F}|^{-1} \sum_{g \in G^F} \chi_{\mathfrak{K}_j,\phi_j}(g) \chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(g) \\
&= q^{\dim S - \dim L} |\Gamma|^{-1} \sum_{w \in \Gamma;\text{eff.}} \mathrm{tr}(b_w^{-1} \iota_j^{-1}, V_j) \mathrm{tr}(\iota_{j'} b_w, V_{j'})
\end{aligned}$$

which by 20.4(c),(b) equals $q^{\dim S - \dim L}$ if $j = j'$ and 0, otherwise. The proposition is proved.

24.19. Assume that we are in the setup of 24.17. Let δ (rep. δ') be the connected component of $N_G L$ (resp. $N_G L'$) that contains S (resp. S'). Assume that $S = {}^\delta\mathcal{Z}_L^0 \mathbf{c}, S' = {}^{\delta'}\mathcal{Z}_{L'}^0 \mathbf{c}'$ where $\mathbf{c}$ is a unipotent L-conjugacy class and $\mathbf{c}'$ is a unipotent L'-conjugacy class. Assume that $\mathcal{E} = \bar{\mathbf{Q}}_l \boxtimes \mathcal{F}, \mathcal{E}' = \bar{\mathbf{Q}}_l \boxtimes \mathcal{F}'$ where $\mathcal{F}$ (resp. $\mathcal{F}'$) is an irrreducible L- (resp. L'-)equivariant local system on $\mathbf{c}$ (resp. $\mathbf{c}'$). Let $\bar{\epsilon} : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ (resp. $\bar{\epsilon}' : F^*\mathcal{F}' \xrightarrow{\sim} \mathcal{F}'$) be the restriction of ϵ (resp. ϵ'). For $w \in \Gamma$ let δ^w be the connected component of $N_G(L^w)$ that contains S^w. Let $j \in [1, r], j' \in [1, r']$. We can state the following variant of Proposition 24.18.

Proposition 24.20. *Assume that $\check{\mathcal{E}}, \mathcal{E}'$ are strongly cuspidal. Assume also that either L, L' are not G^0-conjugate or $\check{\mathcal{E}}, \mathcal{E}'$ are clean.*

(a) If $(L, \mathbf{c}, \check{\mathcal{F}}), (L', \mathbf{c}', \mathcal{F}')$ are not G^0-conjugate, then $|G^{0F}|^{-1} \sum_{u \in G^F;unip.} \chi_{\mathfrak{K}_j,\phi_j}(u) \chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(u) = 0.$

(b) Assume that $L=L', \mathbf{c}=\mathbf{c}', \check{\mathcal{F}}=\mathcal{F}', \check{\vec{\epsilon}}=\vec{\epsilon}', \mathfrak{K}'_{j'}=(\mathfrak{K}_{j'})\check{}, \phi'_{j'}=(\phi_{j'})\check{}$*. Then*

$$|G^{0F}|^{-1}\sum_{u\in G^F;unip.}\chi_{\mathfrak{K}_j,\phi_j}(u)\chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(u)$$
$$=|\Gamma|\sum_{w\in\Gamma}\mathrm{tr}(\iota_j^{-1}b_w^{-1},V_j)\mathrm{tr}(b_w\iota_{j'},V_{j'})|^{\delta^w}\mathcal{Z}^{0F}_{L^w}|^{-1}q^{\dim S-\dim L}.$$

As in 24.17 we have

$$|G^{0F}|^{-1}\sum_{u\in G^F;\mathrm{unip.}}\chi_{\mathfrak{K}_j,\phi_j}(u)\chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(u)$$
$$=|\Gamma|^{-1}|\Gamma'|^{-1}\sum_{w\in\Gamma}\sum_{w'\in\Gamma'}\mathrm{tr}(\iota_j^{-1}b_w^{-1},V_j)\mathrm{tr}({\iota'_{j'}}^{-1}{b'_{w'}}^{-1},V'_{j'})$$
$$\times|G^{0F}|^{-1}\sum_{u\in G^F;\mathrm{unip.}}\chi_{\mathfrak{K}^w,\phi^w}(u)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(u).$$

(All elements of Γ,Γ' are effective in this case.) In the setup of (a), we have

$$|G^{0F}|^{-1}\sum_{u\in G^F;\mathrm{unip.}}\chi_{\mathfrak{K}^w,\phi^w}(u)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(u)=0$$

by 24.16 and (a) follows.

In the setup of (b) we have $\Gamma=\Gamma'$. As in the proof of 24.18 we have (using 24.16 instead of 24.15):

$$|G^{0F}|^{-1}\sum_{u\in G^F;\mathrm{unip.}}\chi_{\mathfrak{K}^w,\phi^w}(u)\chi_{\mathfrak{K}'^{w'},\phi'^{w'}}(u)$$
$$=\zeta_{w,w'}q^{\dim S-\dim L}|^{\delta^w}\mathcal{Z}^{0F}_{L^w}|^{-1}\sharp(y\in\Gamma;w^{-1}F(y)w'=y).$$

Hence

$$|G^{0F}|^{-1}\sum_{u\in G^F;\mathrm{unip.}}\chi_{\mathfrak{K}_j,\phi_j}(u)\chi_{\mathfrak{K}'_{j'},\phi'_{j'}}(u)=|\Gamma|^{-2}\sum_{w,w'\in\Gamma}\mathrm{tr}(\iota_j^{-1}b_w^{-1},V_j)\mathrm{tr}(\iota_{j'}b_{w'},V_{j'})$$
$$\times\zeta_{w,w'}q^{\dim S-\dim L}|({}^{\delta^w}\mathcal{Z}^0_{L^w})^F|^{-1}\sharp(y\in\Gamma;w^{-1}F(y)w'=y).$$

As in the proof of 24.18 we may replace $\mathrm{tr}(\zeta_{w,w'}\iota_{j'}b'_{w'},V_{j'})$ by $\mathrm{tr}(\iota_{j'}b_w,V_{j'})$ and (b) follows.

25. Properties of cohomology sheaves

25.1. Let D be a connected component of G. Let $(L,S),(L',S')\in\mathbf{A}$ with $S\subset D, S'\subset D$. Let $\pi:\tilde{Y}_{L,S}\to Y_{L,S}=Y$ be as in 3.13 and let $\pi':\tilde{Y}_{L',S'}\to Y_{L',S'}$ be the analogous map. For any $\mathcal{E}\in\mathcal{S}(S)$, the local system $\pi_!\tilde{\mathcal{E}}$ on Y is defined as in 5.6; similarly, for any $\mathcal{E}'\in\mathcal{S}(S')$, the local syatem $\pi'_!\tilde{\mathcal{E}}'$ on $Y_{L',S'}$ is defined.

Proposition 25.2. *Let* $\mathcal{E}'\in\mathcal{S}(S')$*. Let* $\mathfrak{K}\in\mathcal{D}(D)$ *be* $IC(\bar{Y}_{L',S'},\tilde{\mathcal{E}}')$*, extended by zero on* $D-\bar{Y}_{L',S'}$*. Let* $A\in\mathcal{D}(D)$ *be a direct summand of* $\mathfrak{K}$*. Then for any* $i\in\mathbf{Z}$*, there exists* $\mathcal{E}\in\mathcal{S}(S)$ *such that the constructible sheaf* $\mathcal{H}^iA|_Y$ *is a local system isomorphic to a direct summand of the local system* $\pi_!\tilde{\mathcal{E}}$*.*

The proof is given in 25.9.

25.3. Let $\mathcal{Z} = {}^D\mathcal{Z}^0_{G^0}$. We show:

(a) *Let $\mathcal{E}$ be a G^0-equivariant local system on an isolated stratum C of D. Assume that there exists $n \in \mathbf{N}^*_{\mathbf{k}}$ and a $\mathcal{Z}$-orbit F in C such that $\mathcal{E}|_F$ is $\mathcal{Z}$-equivariant for the action $z : f \mapsto z^n f$ on F. Then $\mathcal{E} \in \mathcal{S}(C)$.*

We may assume that G is generated by D and that $\mathcal{E}$ is indecomposable. We have a canonical direct sum decomposition of $\mathcal{E}$ such that each summand restricted to any $\mathcal{Z}$-orbit in C is an isotypical local system isomorphic to a direct sum of copies of a fixed $\mathcal{Z}$-equivariant local system of rank 1. Since $\mathcal{E}$ is indecomposable, it is equal to one of these summands. Thus, if F is a $\mathcal{Z}$-orbit in C, we may assume that $\mathcal{E}|_F = \mathcal{L}^{\oplus k}$ where $\mathcal{L}$ is a $\mathcal{Z}$-equivariant local system of rank 1 on F. As in 5.3 we can find $\mathcal{E}_1 \in \mathcal{S}(C)$ of rank 1 such that $\mathcal{E}_1|_F \cong \mathcal{L}$. Let $\underline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{E}_1^*$. Then $\underline{\mathcal{E}}$ is a G^0-equivariant local system on C such that $\underline{\mathcal{E}}|_F \cong \bar{\mathbf{Q}}_l^{\oplus k}$. Since G^0 permutes transitively the fibres of $C \to C'$ (as in 5.3) we see that the restriction of $\underline{\mathcal{E}}$ to any fibre of $C \to C'$ is $\cong \bar{\mathbf{Q}}_l^{\oplus k}$. As in 5.3, we see that there is a well defined local system $\mathcal{E}'$ on C' whose inverse image under $C \to C'$ is $\underline{\mathcal{E}}$. Moreover, $\mathcal{E}'$ is automatically G^0-equivariant. It follows that $\underline{\mathcal{E}} \in \mathcal{S}(C)$. Since $\mathcal{E} = \underline{\mathcal{E}} \otimes \mathcal{E}_1$ and $\mathcal{E}_1 \in \mathcal{S}(C)$, we see that $\mathcal{E} \in \mathcal{S}(C)$.

25.4. Let $n \geq 1$ be such that $\mathcal{E}' \in \mathcal{S}_n(S')$. From the definitions we see that

(a) $\mathfrak{K}$ is $\mathcal{Z} \times G^0$-equivariant for the action $(z, x) \mapsto g \mapsto xz^n gx^{-1}$ on D.

Let $\mathcal{H} = \mathcal{H}^i\mathfrak{K}$, a constructible sheaf on D. Let δ be the connected component of $N_G L$ such that $S \subset \delta$. Let $\mathcal{T} = {}^\delta\mathcal{Z}^0_L$.

We have a canonical map $S \to S_s$, $g \mapsto g_s$ and S_s is a single orbit for the $\mathcal{T} \times L$-action $(z, x) : y \mapsto xzyx^{-1}$. Let $\mathbf{s}$ be the set of $\mathcal{T}$-cosets in S_s. Then L acts transitively on $\mathbf{s}$. We fix a $\mathcal{T}$-coset τ on S_s. Let $R = \{g \in S; g_s \in \tau\}$, $R^* = \{g \in S^*; g_s \in \tau\} = R \cap S^*$. Then R^* is open in R. By the proof of 3.11, R^* is dense in R. Now R is a single orbit for the group $\mathcal{T} \times N_L\tau$ acting by $(z, l) : g \mapsto lzgl^{-1}$. Hence R is smooth, equidimensional. Let $s \in \tau$. Then $Z_L(s)^0$ is independent of the choice of s: it is $N_L(\tau)^0 = Z_L(\tau)^0$. Hence R is a union of finitely many orbits of $\mathcal{T} \times Z_L(\tau)^0$.

Lemma 25.5. *Let $s \in \tau$. Then*

(a) $R \subset Z_G(s)$.

(b) Any connected component of $s^{-1}R$ contains some unipotent element.

We can find $y \in S$ with $y_s = s$. We have $y \in R$ and $ys = sy$. Let $y' \in R$. We have $y' = lzyl^{-1}$ where $l \in L, l^{-1}sl = z's$, $z, z' \in \mathcal{T}$. We must show that $sy' = y's$, or that $slzyl^{-1} = lzyl^{-1}s$, or that $l^{-1}slzy = zyl^{-1}sl$, or that $z'szy = zyz's$, or that $z'zsy = zz'ys$, or that $sy = ys$. This proves (a). We prove (b). Let $g \in R$. Then $g_s = zs$ for some $z \in \mathcal{T}$. Let $g' = z^{-1}g$. Then $g'_s = z^{-1}g_s = s$. Hence $g' = su$ where u is unipotent in $Z_G(s)$. Now $\mathcal{T}R = R$ hence g', g are in the same connected component of R. Hence the connected component of g in R contains an element g' such that $s^{-1}g'$ is unipotent. This proves (b).

Lemma 25.6. *There exists a local system $\mathcal{F}$ on R and $n \in \mathbf{N}^*_{\mathbf{k}}$ such that $\mathcal{F}$ is $\mathcal{T} \times Z_L(\tau)^0$-equivariant for the action $(z, x) \mapsto g \mapsto xz^n gx^{-1}$ and $\mathcal{F}|_{R^*} \cong \mathcal{H}|_{R^*}$.*

Let R_1 be a connected component of R which is a $\mathcal{T} \times Z_L(s)^0$-orbit in R. Let $R_1^* = R^* \cap R$, an open dense subset of R_1. It is enough to show:

there exists a local system $\mathcal{F}_1$ on R_1 and $n \in \mathbf{N}^*_{\mathbf{k}}$ such that $\mathcal{F}_1$ is $\mathcal{T} \times Z_L(\tau)^0$-equivariant for the action $(z, x) \mapsto g \mapsto xz^n gx^{-1}$ and $\mathcal{F}_1|_{R_1^*} \cong \mathcal{H}|_{R_1^*}$.

Let τ^* be the set of all $s \in \tau$ such that $s = y_s$ for some $y \in S^*$. By the proof of 3.11 we have $\tau^* \neq \emptyset$. Let $s \in \tau^*$. Let δ_1 be the connected component of $Z_{N_G L}(s)$ such that $R_1 \subset s\delta_1$. By 25.5(b), δ_1 contains some unipotent element. By 16.12(b) we can find an open subset $\mathcal{U}$ of δ_1 containing all unipotents of δ_1, such that $\epsilon^*(\mathfrak{K}|_{s\mathcal{U}})$ is isomorphic to the restriction to $\mathcal{U}$ of $\tilde{K}$ where $\tilde{K}$ is a direct sum of finitely many objects $\tilde{K}^k \in \mathcal{D}(\delta_1)$ each one of the same type as $\mathfrak{K}$ (for $Z_G(s)$ instead of G). (Here $\epsilon : \mathcal{U} \to s\mathcal{U}$ is $g \mapsto sg$.) By an analogue of 25.4(a), each $\tilde{K}^k$ is ${}^{\delta_1}\mathcal{Z}^0_{Z_G(s)^0} \times Z_G(s)^0$-equivariant hence $\mathcal{T} \times Z_L(\tau)^0$-equivariant for an action as in the lemma. (We have $Z_G(s)^0 = Z_L(s)^0 = Z_L(\tau)^0$, since $s \in \tau^*$ and ${}^{\delta_1}\mathcal{Z}^0_{Z_G(s)^0} = {}^{\delta_1}\mathcal{Z}^0_{Z_L(s)^0} = T_{N_G L}(y) = {}^{\delta}\mathcal{Z}^0_L = \mathcal{T}$, since $y \in R_1, y_s = s$ is isolated in $N_G L$.) Then $\mathcal{H}^i\tilde{K}$ is a $\mathcal{T} \times Z_L(\tau)^0$-equivariant constructible sheaf on δ_1 whose restriction to $\mathcal{U}$ is $\cong \mathcal{H}^i(\epsilon^*(\mathfrak{K}|_{s\mathcal{U}}))$. Let $\mathcal{F}_1$ be the inverse image of $\mathcal{H}^i\tilde{K}|_{s^{-1}R_1}$ under $R_1 \to s^{-1}R_1, g \mapsto s^{-1}g$. This is a $\mathcal{T} \times Z_L(\tau)^0$-equivariant constructible sheaf on R_1. Since R_1 is a single orbit, we see that $\mathcal{F}_1$ is a $\mathcal{T} \times Z_L(\tau)^0$-equivariant local system on R_1. We have $\mathcal{F}_1|_{\mathcal{V}} \cong \mathcal{H}|_{\mathcal{V}}$ where $\mathcal{V} = R_1 \cap s\mathcal{U}$, an open subset of R_1 containing $\{g \in R_1; g_s = s\}$. We summarize: for any $s \in \tau^*$ there exists an open subset $\mathcal{V}(s)$ of R_1 and a $\mathcal{T} \times Z_L(\tau)^0$-equivariant local system $\mathcal{F}_s$ on R_1 such that $\{g \in R_1; g_s = s\} \subset \mathcal{V}(s)$ and $\mathcal{F}_s|_{\mathcal{V}(s)} \cong \mathcal{H}|_{\mathcal{V}(s)}$. If $g \in R_1^*$, then $g \in R_1$ and $g_s \in \tau^*$ hence $g \in \mathcal{V}(g_s)$. Thus $R_1^* \subset R'$ where $R' = \bigcup_{s \in \tau^*} \mathcal{V}(s) \subset R_1$. The constructible sheaf $\mathcal{H}|_{R'}$ is a local system when restricted to any of the open sets $\mathcal{V}(s)$ that cover R'. Hence $\mathcal{H}|_{R'}$ is a local system. Let $s \in \tau^*$ be such that $\mathcal{V}(s) \neq \emptyset$. The local systems $\mathcal{H}|_{R'}, \mathcal{F}_s|_{R'}$ have isomorphic restrictions to $\mathcal{V}(s)$ hence $\mathcal{H}|_{R'} \cong \mathcal{F}_s|_{R'}$. (Note that R' is smooth, irreducible, since R_1 is smooth, irreducible. Hence $\mathcal{H}|_{R'} = IC(R', \mathcal{H}|_{\mathcal{V}(s)}) \cong IC(R', \mathcal{F}_s|_{\mathcal{V}(s)}) = \mathcal{F}_s|_{R'}$.) Since $R_1^* \subset R'$, it follows that $\mathcal{H}|_{R_1^*} \cong \mathcal{F}_s|_{R_1^*}$. We set $\mathcal{F} = \mathcal{F}_s$. Then $\mathcal{F}$ has the required properties. The lemma is proved.

Lemma 25.7. *There exists $\mathcal{E} \in \mathcal{S}(S)$ such that $\mathcal{E}|_{S^*} \cong \mathcal{H}|_{S^*}$.*

Define $\rho : S \to \mathbf{s}$ by $g \mapsto (\mathcal{T} - \text{coset of } g_s)$. Let $\rho' : S^* \to \mathbf{s}$ be the restriction of ρ. Then ρ, ρ' are L-equivariant maps with L acting transitively on $\mathbf{s}$ and R (resp. R^*) is a fibre of ρ (resp. ρ'). Since $\mathcal{H}|_{S^*}$ is an L-equivariant constructible sheaf whose restriction to any fibre of ρ' is a local system (see 25.6), we see that $\mathcal{H}|_{S^*}$ is an L-equivariant local system. Hence $\underline{K} := IC(S, \mathcal{H}|_{S^*}) \in \mathcal{D}(S)$ is well defined. Let $\underline{K}' = IC(R, \mathcal{H}|_{R^*})$. Using the L-homogeneity of $\mathbf{s}$ we see that for any $j \in \mathbf{Z}$ we have $\mathcal{H}^j\underline{K}|_R = \mathcal{H}^j\underline{K}'$ and $\mathcal{H}^j\underline{K}$ is an L-equivariant constructible sheaf. From 25.6 we see that $\underline{K}'$ is a local system on R. In particular, $\mathcal{H}^j\underline{K}' = 0$ for $j > 0$. Since for $j > 0$, $\mathcal{H}^j\underline{K}$ is an L-equivariant constructible sheaf on S whose restriction to R is 0, we see that $\mathcal{H}^j\underline{K} = 0$. Similarly, since $\mathcal{H}^0\underline{K}$ is an L-equivariant constructible sheaf on S whose restriction to R is a local system, we see that $\mathcal{H}^0\underline{K}$ is a local system. Thus, $\underline{K} = \mathcal{H}^0\underline{K}$ is a local system $\mathcal{E}$ on S whose restriction to S^* is $\mathcal{H}|_{S^*}$. Since $\mathcal{E} = IC(S, \mathcal{H}|_{S^*})$ and $\mathcal{H}|_{S^*}$ is L-equivariant, we see that $\mathcal{E}$ is L-equivariant. Since $\mathcal{E}|_R = \underline{K}'$ and $\underline{K}'$ is $\mathcal{T}$-equivariant, we may apply 25.3(a) with G, C replaced by $N_G L, S$ and we see that $\mathcal{E} \in \mathcal{S}(S)$. The lemma is proved.

Lemma 25.8. *Let $\tilde{Y} = \tilde{Y}_{L,S}$. Define $\xi : \tilde{Y} \to D$ by $\xi(g, xL) = g$. Then $\xi^*\mathcal{H} \cong \tilde{\mathcal{E}}$ where $\tilde{\mathcal{E}}$ is defined in terms of $\mathcal{E} \in \mathcal{S}(S)$ of Lemma 25.7.*

Let $\hat{Y} = \{(g, x) \in D \times G^0; x^{-1}gx \in S^*\}$. Define $b : \hat{Y} \to S$ and $b' : \hat{Y} \to S^*$ by $(g, x) \mapsto x^{-1}gx$. Define $c : \hat{Y} \to D$ by $c(g, x) = g$. By the definition of $\mathcal{E}$ we have $b^*\mathcal{E} = b'^*(\mathcal{E}|_{S^*}) \cong b'^*(\mathcal{H}|_{S^*})$. Define $r : \hat{Y} \to G^0 \times D$ by $(g,x) \mapsto (x, x^{-1}gx)$. Define $p_1, p_2 : G^0 \times D \to D$ by $p_1(x,d) = d, p_2(x,d) = xdx^{-1}$. Since $\mathcal{H}$ is a G^0-equivariant constructible sheaf on D, we have $p_1^*\mathcal{H} \cong p_2^*\mathcal{H}$, hence $r^*p_1^*\mathcal{H} \cong r^*p_2^*\mathcal{H}$. Thus, $b'^*(\mathcal{H}|_{S^*}) \cong c^*\mathcal{H}$. It follows that $c^*\mathcal{H} \cong b^*\mathcal{E}$. As in 5.6, define $a : \hat{Y} \to \tilde{Y}$ by $(g,x) \mapsto (g, xL)$. We have $c^*\mathcal{H} = a^*(\xi^*\mathcal{H}), b^*\mathcal{E} = a^*\tilde{\mathcal{E}}$. Hence $a^*(\xi^*\mathcal{H}) \cong a^*\tilde{\mathcal{E}}$. Since $a^*\tilde{\mathcal{E}}$ is a local system, we see that $a^*(\xi^*\mathcal{H})$ is a local system. Since a is a principal L-bundle it follows that $\xi^*\mathcal{H}$ is a local system and that $\xi^*\mathcal{H} \cong \tilde{\mathcal{E}}$. The lemma is proved.

25.9. We now prove Proposition 25.2. We may assume that $A = \mathfrak{K}$. Let $j : Y \to D$ be the inclusion. With notation in 25.8 we have $\pi^*j^*\mathcal{H} = \xi^*\mathcal{H}$ hence $\pi^*j^*\mathcal{H} = \tilde{\mathcal{E}}$. Since π is a finite unramified covering and $\tilde{\mathcal{E}}$ is a local system, it follows that $j^*\mathcal{H}$ is a local system. Since $\bar{\mathbf{Q}}_l$ is a direct summand of $\pi_!\pi^*\bar{\mathbf{Q}}_l$, we see that $j^*\mathcal{H}$ is a direct summand of $(j^*\mathcal{H}) \otimes (\pi_!\pi^*\bar{\mathbf{Q}}_l) = \pi_!\pi^*(j^*\mathcal{H}) = \pi_!\tilde{\mathcal{E}}$. Proposition 25.2 is proved.

Lemma 25.10. *Let $(L,S), (L',S') \in \mathbf{A}$. Assume that $Y_{L',S'} \subset \bar{Y}_{L,S}$.*

(a) For any L-conjugacy class C in S and any L'-conjugacy class C' in S' we have $\dim L' - \dim C' \geq \dim L - \dim C$.

(b) For any G^0-conjugacy class $\mathbf{c}$ in $Y_{L,S}$ and any G^0-conjugacy class $\mathbf{c}'$ in $Y_{L',S'}$ we have $\dim \mathbf{c}' \leq \dim \mathbf{c}$.

We prove (a). By 7.2(c) we may assume that $L' = G^0$. Then S' is as isolated stratum of G and $S' \subset \bar{Y}$ where $Y = Y_{S,L}$. Let $\mathbf{a} = \mathbf{a}_{L,S}, \sigma : \bar{Y} \to \mathbf{a}$ be as in 7.2. Set $a = \sigma(C') \in \mathbf{a}$. By 7.16(b), $\bar{Y}^a$ has pure dimension $\dim G^0/L + \dim C$. Since $C' \subset \bar{Y}^a$ we have $\dim C' \leq \dim G^0/L + \dim C$, as required.

We prove (b). Let $g \in \mathbf{c}, g' \in \mathbf{c}'$. We can assume that $g \in S^*, g' \in S'^*$. Let C be the L-conjugacy class of g; let C' be the L'-conjugacy class of g'. By the definition of S^* we have $Z_G(g)^0 = Z_L(g)^0$ hence $\dim Z_{G^0}(g) = \dim Z_L(g)$ so that $\dim \mathbf{c} = \dim G^0/L + \dim C$. Similarly, $\dim \mathbf{c}' = \dim G^0/L' + \dim C'$. Hence (b) follows from (a).

25.11. For $(L,S) \in \mathbf{A}$ let $\mathfrak{T}_{L,S}$ be the union of all G^0-conjugacy classes in $\bar{Y}_{L,S}$ whose dimension equals the dimension of some/any G^0-conjugacy class in $Y_{L,S}$. (See 3.4.) From 3.15 and 3.4 we see that $\mathfrak{T}_{L,S}$ is a union of strata of G. Clearly,

(a) $Y_{L,S} \subset \mathfrak{T}_{L,S} \subset \bar{Y}_{L,S}$.

Lemma 25.12. *For $(L,S) \in \mathbf{A}$, $\mathfrak{T}_{L,S}$ is open dense in $\bar{Y}_{L,S}$.*

The fact that it is dense follows from 25.11(a). It remains to show that $\bar{Y}_{L,S} - \mathfrak{T}_{L,S}$ is closed in $\bar{Y}_{L,S}$. Since $\bar{Y}_{L,S}$ is a union of strata (see 3.15) and $\mathfrak{T}_{L,S}$ is a union of strata (by definition) it is enough to verify the following statement:

$$Y_{L'',S''} \subset \bar{Y}_{L,S} - \mathfrak{T}_{L,S}, Y_{L',S'} \subset \bar{Y}_{L'',S''} \implies Y_{L',S'} \subset \bar{Y}_{L,S} - \mathfrak{T}_{L,S}.$$

Let $\mathbf{c}$ (resp. $\mathbf{c}', \mathbf{c}''$) be a G^0-conjugacy class in $Y_{L,S}$ (resp. $Y_{L',S'}$, $Y_{L'',S''}$). Using Lemma 25.10 we see that $\dim \mathbf{c}'' < \dim \mathbf{c}$, $\dim \mathbf{c}' \leq \dim \mathbf{c}''$. Hence $\dim \mathbf{c}' < \dim \mathbf{c}$ and $Y_{L',S'} \subset \bar{Y}_{L,S} - \mathfrak{T}_{L,S}$. The lemma is proved.

25.13. Let $(L,S) \in \mathbf{A}$ and let P be a parabolic of G^0 with Levi L such that $S \subset N_GP$. Let $\mathfrak{T} = \mathfrak{T}_{L,S}$. Let $\psi : X \to \bar{Y}_{L,S}$ be as in 3.14. For any stratum S' of $N_GP \cap N_GL$ such that $S' \subset \bar{S}$ let $X_{S'}$ be as in 5.6. Let $\mathfrak{X} = \psi^{-1}(\mathfrak{T})$ and let $\psi' : \mathfrak{X} \to \mathfrak{T}$ be the restriction of ψ. We show:

(a) $\mathfrak{X} \subset X_S$;

(b) ψ' *has finite fibres;*
(c) $\mathfrak{X}$ *is smooth.*
Let $g \in \mathfrak{T}$. We must show that $\psi^{-1}(g) \cap X_{S'}$ is empty if $S' \neq S$ and is finite if $S' = S$. By 4.4(b), $\dim(\psi^{-1}(g) \cap X_{S'}) \le (\dim G^0/L - \dim \mathbf{c}_1 + \dim C')/2$ where $\mathbf{c}_1$ is the G^0-conjugacy class of g and C' is any L-conjugacy class in S'. Let C be an L-conjugacy class in S. We have $\dim C' \le \dim C$ with strict inequality if $S' \neq S$. Hence $\dim(\psi^{-1}(g) \cap X_{S'}) \le (\dim G^0/L - \dim \mathbf{c}_1 + \dim C)/2$ with strict inequality if $S' \neq S$. As in the proof of 25.10 we have $\dim G^0/L + \dim C = \dim \mathbf{c}$ where $\mathbf{c}$ is a G^0-conjugacy class in $Y_{L,S}$. Also $\dim \mathbf{c} = \dim \mathbf{c}_1$ by the choice of g. Thus $\dim(\psi^{-1}(g) \cap X_{S'}) \le 0$ with strict inequality if $S' \neq S$. This completes the proof of (a),(b).

Using (a) and Lemma 25.12 we see that $\mathfrak{X}$ is an open subset of X_S which is smooth. Hence (c) holds.

Proposition 25.14. *Let* $(L, S) \in \mathbf{A}$*, let* $\mathcal{E} \in \mathcal{S}(S)$ *and let* $\mathfrak{T} = \mathfrak{T}_{L,S}$*. Then* $IC(\bar{Y}_{L,S}, \pi_!\tilde{\mathcal{E}})|_{\mathfrak{T}}$ *is a constructible sheaf.*

Let $\bar{\mathcal{E}}$ be the local system on X_S defined in 5.6. Using 5.7 it is enough to show that $\psi_!(IC(X, \bar{\mathcal{E}}))|_{\mathfrak{T}}$ is a constructible sheaf or equivalently that $\psi'_!(IC(X, \bar{\mathcal{E}})|_{\mathfrak{X}})$ is a constructible sheaf. Since $\mathfrak{X} \subset X_S$ (see 25.13(a)) we have $IC(X, \bar{\mathcal{E}})|_{\mathfrak{X}} = \bar{\mathcal{E}}|_{\mathfrak{X}}$ and it is enough to show that $\psi'_!(\bar{\mathcal{E}}|_{\mathfrak{X}})$ is a constructible sheaf. This is clear from 25.13(b).

26. The variety $Z_{J,D}$

26.1. Let $\mathcal{B}$ be the variety of Borel subgroups of G^0. Let $\mathbf{W}$ be the set of G^0-orbits on $\mathcal{B} \times \mathcal{B}$ (G^0 acts by conjugation on both factors). For $B, B' \in \mathcal{B}$ we write $\mathrm{pos}(B, B') = w$ if the G^0-orbit of (B, B') is w. There is a unique group structure on $\mathbf{W}$ such that whenever $B, B', B'' \in \mathcal{B}$ have a common maximal torus, we have $\mathrm{pos}(B, B')\mathrm{pos}(B', B'') = \mathrm{pos}(B, B'')$. Then $\mathbf{W}$ is a finite Coxeter group (called *the Weyl group*) with length function $l : \mathbf{W} \to \mathbf{N}$ which attaches to a G^0-orbit its dimension minus $\dim \mathcal{B}$. Let $\le$ be the standard partial order of the Coxeter group $\mathbf{W}$. Let $\mathbf{I} = \{w \in \mathbf{W}; l(w) = 1\}$. For $J \subset \mathbf{I}$ let $\mathbf{W}_J$ be the subset of $\mathbf{W}$ generated by J; let $\mathbf{W}^J$ (resp. ${}^J\mathbf{W}$) be the set of all $w \in \mathbf{W}$ such that $l(ws) > l(w)$ (resp. $l(sw) > l(w)$) for all $s \in J$. Let w_J^0 be the unique element of maximal length in $\mathbf{W}_J$. For $J, J' \subset \mathbf{I}$ let ${}^J\mathbf{W}^{J'} = {}^J\mathbf{W} \cap \mathbf{W}^{J'}$.

If P is a parabolic of G^0, the set of all $w \in \mathbf{W}$ such that $w = \mathrm{pos}(B, B')$ for some $B, B' \in \mathcal{B}, B \subset P, B' \subset P$ is of the form $\mathbf{W}_J$ for a well-defined $J \subset \mathbf{I}$; we then say that P has type J. For $J \subset \mathbf{I}$ let $\mathcal{P}_J$ be the set of all parabolics of type J of G^0. For $P \in \mathcal{P}_J, Q \in \mathcal{P}_K$ there is a well-defined element $u = \mathrm{pos}(P, Q) \in \mathbf{W}$ such that $u \le \mathrm{pos}(B, B')$ for any $B, B' \in \mathcal{B}, B \subset P$, $B' \subset Q$ and $u = \mathrm{pos}(B_1, B_1')$ for some $B_1, B_1' \in \mathcal{B}, B_1 \subset P, B_1' \subset Q$.

26.2. In the remainder of this section we fix a connected component D of G. There is a unique isomorphism $\epsilon_D : \mathbf{W} \to \mathbf{W}$ such that $\epsilon_D(\mathbf{I}) = \mathbf{I}$ and such that

$$g \in D, P \in \mathcal{P}_J \implies gPg^{-1} \in \mathcal{P}_{\epsilon_D(J)}.$$

Let $J \subset \mathbf{I}$. Following [B], let $\mathcal{T}(J, \epsilon_D)$ be the set of all sequences $(J_n, w_n)_{n\ge 0}$ where $J_n \subset \mathbf{I}$ and $w_n \in \mathbf{W}$ are such that

$$J = J_0 \supset J_1 \supset J_2 \supset \dots,$$
$$J_n = J_{n-1} \cap \epsilon_D^{-1}(w_{n-1}J_{n-1}w_{n-1}^{-1}) \text{ for } n \ge 1,$$
$$w_n \in {}^{\epsilon_D(J_n)}\mathbf{W}^{J_n} \text{ for } n \ge 0,$$
$$w_n \in \mathbf{W}_{\epsilon_D(J_n)} w_{n-1} \mathbf{W}_{J_{n-1}} \text{ for } n \ge 1.$$

Then $\mathcal{T}(J, \epsilon_D)$ is a finite set.

For $(P, P') \in \mathcal{P}_J \times \mathcal{P}_{\epsilon_D(J)}$ let $A_D(P, P') = \{g \in D; gPg^{-1} = P'\}$. Let $\bar{A}_D(P, P') = U_{P'} \backslash A_D(P, P') = A_D(P, P')/U_P$. Let

$$Z_{J,D} = \{(P, P', \gamma); P \in \mathcal{P}_J, P' \in \mathcal{P}_{\epsilon_D(J)}, \gamma \in \bar{A}_D(P, P')\}.$$

Following [L10, 3.11], to any $(P, P', \gamma) \in Z_{J,D}$ we associate an element $(J_n, w_n)_{n\ge 0} \in \mathcal{T}(J, \epsilon_D)$ and two sequences of parabolics $P^n, P'^n, (n \ge 0)$ by the requirements:

$$P'^0 = P', P^0 = P, P'^n = (P^{n-1} \cap P'^{n-1})U_{P'^{n-1}} \in \mathcal{P}_{\epsilon_D(J_n)},$$
$$P^n = g^{-1}P'^n g \in \mathcal{P}_{J_n}, g \in \gamma, w_n = \mathrm{pos}(P'^n, P^n).$$

We write $(J_n, w_n)_{n\ge 0} = \beta'(P, P', \gamma)$. For $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)$ let

$${}^{\mathbf{t}}\, Z_{J,D} = \{(P, P', \gamma) \in Z_{J,D}; \beta'(P, P', \gamma) = \mathbf{t}\}.$$

Then $({}^{\mathbf{t}}\, Z_{J,D})_{\mathbf{t}\in\mathcal{T}(J,\epsilon_D)}$ is a partition of $Z_{J,D}$ into locally closed subvarieties. Now G^0 acts on $Z_{J,D}$ by $h : (P, P', \gamma) \mapsto (hPh^{-1}, hP'h^{-1}, h\gamma h^{-1})$. This action preserves each of the pieces ${}^{\mathbf{t}}Z_{J,D}$.

26.3. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}(J, \epsilon_D)$. For $r \gg 0$, J_r, w_r are independent of r; we denote them by J_∞, w. Then

$$wJ_\infty w^{-1} = \epsilon_D(J_\infty), w \in {}^{\epsilon_D(J_\infty)}\mathbf{W}^{J_\infty}.$$

Let $R_{\mathbf{t}} = \{(\tilde{Q}, \tilde{Q}', \gamma') \in Z_{J_\infty,D}, \mathrm{pos}(\tilde{Q}', \tilde{Q}) = w\}$. We choose $Q \in \mathcal{P}_{J_\infty}, Q' \in \mathcal{P}_{\epsilon_D(J_\infty)}$ such that $\mathrm{pos}(Q', Q) = w$. We can find a common Levi L for Q and Q'. Let

$$C = \{g \in D; gLg^{-1} = L, gQg^{-1} = Q'\} = \{g \in D; gLg^{-1} = L, \mathrm{pos}(gQg^{-1}, Q) = w\}.$$

Let A be a simple perverse sheaf on C which is admissible in the sense of 6.7 (this makes sense since C is a connected component of the reductive group $N_G L$). Then A is L-equivariant for the conjugation action of L hence it is also $(Q\cap Q')$-equivariant where $Q\cap Q'$ acts via its quotient $(Q\cap Q')/U_{Q\cap Q'} = L$. Hence there is a well-defined simple perverse sheaf A' on $G^0 \times_{Q\cap Q'} C$ (here $Q\cap Q'$ acts on G^0 by right translation) such that $j^\bigstar A' = pr_2^\bigstar A$ in the obvious diagram $G^0 \times_{Q\cap Q'} C \xleftarrow{j} G^0 \times C \xrightarrow{pr_2} C$. We may regard A' as a simple perverse sheaf on $R_{\mathbf{t}}$ via the isomorphism

(a) $$G^0 \times_{Q\cap Q'} C \xrightarrow{\sim} R_{\mathbf{t}}, (g, c) \mapsto (gQg^{-1}, gQ'g^{-1}, gU_{Q'}cU_Q g^{-1}).$$

Define $\vartheta_{\mathbf{t}} : {}^{\mathbf{t}}Z_{J,D} \to R_{\mathbf{t}}$ by $(P, P', \gamma) \mapsto (P^r, P'^r, \gamma U_{P^r})$ where $r \gg 0$ and P^r, P'^r are attached to (P, P', γ) as in 26.2. Now G^0 acts on $R_{\mathbf{t}}$ by $h : (\tilde{Q}, \tilde{Q}', \gamma') \mapsto (h\tilde{Q}h^{-1}, h\tilde{Q}'h^{-1}, h\gamma' h^{-1})$ and $\vartheta_{\mathbf{t}}$ is G^0-equivariant. By [L10, 3.12], $\vartheta_{\mathbf{t}}$ is an iterated affine space bundle. Let $\tilde{A} = \vartheta^\bigstar A'$, a simple perverse sheaf on ${}^{\mathbf{t}}Z_{J,D}$. Let $\hat{A}$ be the simple perverse sheaf on $Z_{J,D}$ whose support is the closure in $Z_{J,D}$ of $\mathrm{supp}(\tilde{A})$ and whose restriction to ${}^{\mathbf{t}}Z_{J,D}$ is $\tilde{A}$. A simple perverse sheaf on ${}^{\mathbf{t}}Z_{J,D}$ is said to be *admissible* if it is of the form $\tilde{A}$ for some A as above. This concept does not depend on the choice of Q, Q', L since any two such triples are G^0-conjugate. Note that $A \mapsto \tilde{A}$ is a bijection between the set of isomorphism classes of simple perverse sheaves on C that are admissible and the set of isomorphism classes of simple perverse sheaves on ${}^{\mathbf{t}}Z_{J,D}$ that are admissible. A simple perverse sheaf on $Z_{J,D}$ is said to be *admissible* if it is of the form $\hat{A}$ for some $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)$ and some A as above. Note that $(\mathbf{t}, A) \mapsto \hat{A}$ is a bijection between the set of pairs consisting of an element of $\mathcal{T}(J, \epsilon_D)$ and an isomorphism class of a simple perverse sheaf on ${}^{\mathbf{t}}Z_{J,D}$ that is admissible and the set of isomorphism classes of simple perverse sheaves on $Z_{J,D}$ that are admissible.

When $J = \mathbf{I}$, then $\mathcal{T}(J, \epsilon_D)$ consists of a single element $\mathbf{t} = (J_n, w_n)_{n \geq 0}$ where $J_n = \mathbf{I}, w_n = 1$ for all n. Now $Z_{\mathbf{I},D}$ consists of all triples (G^0, G^0, g) with $g \in D$. We identify $Z_{\mathbf{I},D} = D$ in the obvious way. A simple perverse sheaf on $Z_{\mathbf{I},D}$ is admissible in the sense just defined if and only if it is admissible on D in the sense of 6.7.

26.4. In this and the next subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure with Frobenius map $F : G \to G$. There are induced maps $F : \mathcal{B} \to \mathcal{B}, F : \mathbf{W} \to \mathbf{W}$; the last map restricts to a bijection $F : \mathbf{I} \to \mathbf{I}$. Let $J \subset \mathbf{I}$ be such that $F(J) = J$. Then $F : G \to G$ induces a map $F : \mathcal{P}_J \to \mathcal{P}_J$. We assume that $F(D) = D$. Then $\epsilon_D : \mathbf{I} \to \mathbf{I}$ commutes with F hence $F(\epsilon_D(J)) = \epsilon_D(J)$. Hence $F : G \to G$ induces a map $F : \mathcal{P}_{\epsilon_D(J)} \to \mathcal{P}_{\epsilon_D(J)}$.

For $(P, P') \in \mathcal{P}_J \times \mathcal{P}_{\epsilon_D(J)}$ we have $g \in A_D(P, P') \implies F(g) \in A_D(F(P), F(P'))$ and $g \mapsto F(g)$ induces a map $F : \bar{A}_D(P, P') \to \bar{A}_D(F(P), F(P'))$. Define $F : Z_{J,D} \to Z_{J,D}$ by $F(P, P', \gamma) = (F(P), F(P'), F(\gamma))$; this is the Frobenius map for an $\mathbf{F}_q$-rational structure on $Z_{J,D}$. The G^0-action on $Z_{J,D}$ restricts to a G^{0F}-action on $Z^F_{J,D}$. Let $\mathbf{U}$ be the vector space of functions $Z^F_{J,D} \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-orbits.

Theorem 26.5. *Let $\mathcal{A}_J$ be a set of representatives for the isomorphism classes of admissible simple perverse sheaves K on $Z_{J,D}$ such that $F^*K \cong K$. For each $K \in \mathcal{A}_J$ we choose an isomorphism $\alpha : F^*K \xrightarrow{\sim} K$. The characteristic functions $\chi_{K,\alpha}$ (one for each $K \in \mathcal{A}_J$) form a $\bar{\mathbf{Q}}_l$-basis of $\mathbf{U}$.*

In this proof we write $Z, {}^{\mathbf{t}}Z$ instead of $Z_{J,D}, {}^{\mathbf{t}}Z_{J,D}$. Define $F : \mathcal{T}(J, \epsilon_D) \xrightarrow{\sim} \mathcal{T}(J, \epsilon_D)$ by $(J_n, w_n)_{n \geq 0} \mapsto (F(J_n), F(w_n))_{n \geq 0}$. For any $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)$ we have $F({}^{\mathbf{t}}Z) = {}^{F(\mathbf{t})}Z$. In particular, we have $Z^F = \bigsqcup_{\mathbf{t} \in \mathcal{T}(J,\epsilon_D); F(\mathbf{t}) = \mathbf{t}} {}^{\mathbf{t}}Z^F$ where ${}^{\mathbf{t}}Z^F = {}^{\mathbf{t}}Z \cap Z^F$. It follows that $\mathbf{U} = \bigoplus_{\mathbf{t} \in \mathcal{T}(J,\epsilon_D); F(\mathbf{t}) = \mathbf{t}} {}^{\mathbf{t}}\mathbf{U}$ where ${}^{\mathbf{t}}\mathbf{U}$ is the vector space of functions ${}^{\mathbf{t}}Z^F \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-orbits. (We identify any such function with a function $Z^F \to \bar{\mathbf{Q}}_l$ which is zero on the complement of ${}^{\mathbf{t}}Z^F$.)

For any integer t let $Z_{\leq t} = \bigcup_{\mathbf{t} \in \mathcal{T}(J,\epsilon_D); |\mathbf{t}| \leq t} {}^{\mathbf{t}}Z$ where $|\mathbf{t}| = \dim {}^{\mathbf{t}}Z$. This is a closed subvariety of Z since $\bigcup_{\mathbf{t}} {}^{\mathbf{t}}Z$ is a partition of Z into finitely many locally closed subvarieties. Moreover, if $|\mathbf{t}| = t$, then ${}^{\mathbf{t}}Z \cup Z_{\leq t-1}$ is a closed subvariety of Z.

Let $\mathbf{U}_{\leq t}$ be the vector space of functions $Z^F_{\leq t} \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-orbits. (We identify any such function with a function $Z^F \to \bar{\mathbf{Q}}_l$ which is zero on the complement of ${}^{\mathbf{t}}Z^F$.) We have $\mathbf{U}_{\leq 0} \subset \mathbf{U}_{\leq 1} \subset \dots$ and $\mathbf{U}_{\leq t} = \bigoplus_{\mathbf{t} \in \mathcal{T}(J,\epsilon_D); F(\mathbf{t}) = \mathbf{t}, |\mathbf{t}| \leq t} {}^{\mathbf{t}}\mathbf{U}$.

Now let $K \in \mathcal{A}_J$. Since the sets ${}^{\mathbf{t}}Z \cap \mathrm{supp}(K)$ form a partition of $\mathrm{supp}(K)$ into finitely many locally closed subsets, there is a unique $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)$ such that ${}^{\mathbf{t}}Z \cap \mathrm{supp}(K)$ is open dense in $\mathrm{supp}(K)$. Since $F^*K \cong K$ we have necessarily $F(\mathbf{t}) = \mathbf{t}$. Let $t = |\mathbf{t}|$. Since ${}^{\mathbf{t}}Z \cap \mathrm{supp}(K) \subset {}^{\mathbf{t}}Z$ and ${}^{\mathbf{t}}Z \cup Z_{\le t-1}$ is closed, we see that the closure of ${}^{\mathbf{t}}Z \cap \mathrm{supp}(K)$ is contained in ${}^{\mathbf{t}}Z \cup Z_{\le t-1}$. Thus, $\mathrm{supp}(K) \subset {}^{\mathbf{t}}Z \cup Z_{\le t-1}$. Hence we can write uniquely $\chi_{K,\alpha} = \chi'_{K,\alpha} + \chi''_{K,\alpha}$ where $\chi'_{K,\alpha}$ is the restriction of $\chi_{K,\alpha}$ to ${}^{\mathbf{t}}Z$ (extended by 0 on $Z - {}^{\mathbf{t}}Z$) and $\chi''_{K,\alpha} \in \mathbf{U}_{\le t-1}$. Note that $\chi_{K,\alpha}, \chi'_{K,\alpha}$ are contained in $\mathbf{U}_{\le t}$ and they are equal modulo $\mathbf{U}_{\le t-1}$. Thus, in order to prove that the functions $\chi_{K,\alpha}$ form a basis of $\mathbf{U}$ it is enough to prove that the functions $\chi'_{K,\alpha}$ form a basis of $\mathbf{U}$. More precisely, we will show that, for any $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)^F$, the functions $\chi'_{K,\alpha}$ with $K \in \mathcal{A}_J$ such that ${}^{\mathbf{t}}Z \cap \mathrm{supp}(K)$ is open dense in $\mathrm{supp}(K)$, form a basis of ${}^{\mathbf{t}}\mathbf{U}$. An equivalent statement is:

(a) *For any $\mathbf{t} \in \mathcal{T}(J, \epsilon_D)^F$ let ${}^{\mathbf{t}}\mathcal{A}_J$ be a set of representatives for the isomorphism classes of admissible simple perverse sheaves K' on ${}^{\mathbf{t}}Z$ such that $F^*K' \cong K'$. For each $K' \in {}^{\mathbf{t}}\mathcal{A}_J$ we choose an isomorphism $\alpha' : F^*K' \xrightarrow{\sim} K'$. Then the characteristic functions $\chi_{K',\alpha'}$ (one for each $K' \in {}^{\mathbf{t}}\mathcal{A}_J$) form a $\bar{\mathbf{Q}}_l$-basis of ${}^{\mathbf{t}}\mathbf{U}$.*

Let $R = R_{\mathbf{t}}, Q, Q', L, C$ be as in 26.3. Then R is defined over $\mathbf{F}_q$, with Frobenius map $F : R_{\mathbf{t}} \to R_{\mathbf{t}}, (\tilde{Q}, \tilde{Q}', \gamma') \mapsto (F(\tilde{Q}), F(\tilde{Q}'), F(\gamma'))$. We may assume that $F(Q) = Q, F(Q') = Q', F(L) = L'$. Then $F(C) = C$. The map $\vartheta_{\mathbf{t}} : {}^{\mathbf{t}}Z \to R$ in 26.3 is G^0-equivariant and commutes with F.

We show that $\vartheta_{\mathbf{t}}$ induces a bijection between the set of G^{0F}-orbits on ${}^{\mathbf{t}}Z$ and the set of G^{0F}-orbits on R. Now $\vartheta_{\mathbf{t}}$ is a composition

$$ {}^{\mathbf{t}}Z_{J,D} \xrightarrow{{}^{1}\vartheta} {}^{\mathbf{t}_1}Z_{J_1,D} \xrightarrow{{}^{2}\vartheta} \dots \xrightarrow{{}^{r}\vartheta} {}^{\mathbf{t}_r}Z_{J_r,D} = R $$

where $\mathbf{t}_i = (J_n, w_n)_{n \ge i}$, ${}^1\vartheta(P, P', \gamma) = (P^1, P'^1, \gamma U_{P^1})$ (notation of 26.2) and ${}^i\vartheta$ is defined for $i \ge 2$ just like ${}^1\vartheta$ with $\mathbf{t}_{i-1}$ instead of $\mathbf{t}$. Each ${}^i\vartheta$ is G^0-equivariant and commutes with F. Hence it is enough to show that ${}^i\vartheta$ induces a bijection between the set of G^{0F}-orbits on ${}^{\mathbf{t}_{i-1}}Z$ and the set of G^{0F}-orbits on ${}^{\mathbf{t}_i}Z$. We may assume that $i = 1$. Let Φ be the fibre of ${}^1\vartheta$ at some $\mathbf{F}_q$-rational point $(\tilde{P}, \tilde{P}', \tilde{\gamma}) \in {}^{\mathbf{t}_1}Z_{J_1,D}$. It is enough to show that $\Phi^F \ne \emptyset$ and that Φ^F is contained in a single G^{0F}-orbit. Since Φ is an affine space (see [L10, 3.12(b)]) defined over $\mathbf{F}_q$, it must contain some F-fixed point (P, P', γ). By [L10, 3.8], Φ is a homogeneous $U_P \cap P'$ space (the action being the restriction of the G^0-action on ${}^{\mathbf{t}}Z$) and the isotropy group of (P, P', γ) is $U_P \cap U_{P'}$ (see [L10, 3.9]). Since $U_P \cap P', U_P \cap U_{P'}$ are connected, it follows that $(U_P \cap P')^F$ acts transitively on Φ^F; thus, Φ^F is contained in a single G^{0F}-orbit, as required.

Using this and the definitions, we see that (a) would be a consequence of the following statement:

Let $\mathcal{A}'$ be a set of representatives for the isomorphism classes of admissible simple perverse sheaves A on C such that $F^*A \cong A$. For each $A \in \mathcal{A}'$ we choose an isomorphism $\alpha : F^*A \xrightarrow{\sim} A$; we define A' (a simple perverse sheaf on R) as in 26.3 and let $\alpha' : F^*A' \xrightarrow{\sim} A'$ be the isomorphism induced by α. Then the characteristic functions $\chi_{A',\alpha'}$ (one for each $A \in \mathcal{A}'$) form a $\bar{\mathbf{Q}}_l$-basis of the vector space of functions $R^F \to \bar{\mathbf{Q}}_l$ that are constant on G^{0F}-orbits.

This is an immediate consequence of 21.21 applied to $N_G L$ instead of G. The theorem is proved.

27. Induction

27.1. Let D be a connected component of G and let P be a parabolic of G^0 such that $N_D P \neq \emptyset$. Let $\pi' : N_D P \to N_D P/U_P$ be the obvious map. Note that $N_D P/U_P$ is a connected component of $N_G P/U_P$. Consider the diagram

$$N_D P/U_P \xleftarrow{\underline{a}} V_1 \xrightarrow{a} V_2 \xrightarrow{a'} D$$

where
$V_1 = \{(g, x) \in G \times G^0; x^{-1}gx \in N_D P\}$,
$V_2 = \{(g, xP) \in G \times G^0/P; x^{-1}gx \in N_D P\}$,
$\underline{a}(g, x) = \pi'(x^{-1}gx), a'(g, x) = (g, xP), a''(g, xP) = g$.
Then $\underline{a}$ (resp. a') is a smooth morphism with connected fibres of dimension $\dim P + 2\dim U_P$ (resp. $\dim P$). To any $A \in \mathcal{M}(N_D P/U_P)$ which is P/U_P-equivariant for the conjugation action of P/U_P we associate a complex $\mathrm{ind}_{N_D P/U_P}^D A \in \mathcal{D}(D)$ as follows. The complex $\underline{a}^\bigstar A \in \mathcal{M}(V_1)$ is P-equivariant for the action $p : (g, x) \mapsto (g, xp^{-1})$ of P on V_1. Since a' is a principal P-bundle, there is a well-defined complex $A_1 \in \mathcal{M}(V_2)$ such that $\underline{a}^\bigstar A = a'^\bigstar A_1$ and a well-defined complex $A'_1 \in \mathcal{D}(V_2)$ such that $a^* A = a'^* A'_1$. Then $A'_1 = A_1[-2\alpha]$ where $\alpha = \dim U_P$. We set

$$\mathrm{ind}_{N_D P/U_P}^D (A) = a''_! A_1 = a''_! A'_1[2\alpha].$$

Let L be a Levi of P. Let $G' = N_G P \cap N_G L$, a reductive group with $G'^0 = L$. Let $D' = G' \cap D$, a connected component of G'. Define a homomorphism $\xi : N_G P \to G'$ by $\xi(z\omega) = z$ where $z \in G', \omega \in U_P$ (see 1.26). We identify $N_G P/U_P = G', N_D P/U_P = D'$ via ξ. Then $\mathrm{ind}_{N_D P/U_P}^D$ may be viewed as a procedure which associates to any $A \in \mathcal{M}(D')$ which is L-equivariant for the conjugation action of L, a complex $\mathrm{ind}_{D'}^D A = a''_! A_1 = a''_! A'_1[2\alpha] \in \mathcal{D}(D)$ where $a : V_1 \to D'$ is $a(g, x) = \xi(x^{-1}gx)$ and $A_1 \in \mathcal{M}(V_2), A'_1 \in \mathcal{D}(V_2)$ are given by $a^\bigstar A = a'^\bigstar A_1$, $a^* A = a'^* A'_1$.

27.2. Let L' be a Levi of a parabolic of L. Let S be an isolated stratum of $N_{G'} L'$ such that $S \subset D$ and S normalizes some parabolic of L with Levi L'. Then S is also an isolated stratum of $N_G L'$ such that S normalizes some parabolic of G^0 with Levi L'. We have a commutative diagram in which all squares except the top two are cartesian:

$$\begin{tikzcd}
S'^* & S'^* \arrow[l, "="'] \arrow[r] & S'^* & \\
\hat{Y}' \arrow[u, "r_0"] & \hat{Z}_1 \arrow[l, "c"'] \arrow[r, "c"] \arrow[u, "r_1"] & \hat{Z}_2 \arrow[u, "r_2"] & \\
\tilde{Y}' \arrow[u, leftarrow, "q_0"] & Z_1 \arrow[l, "b"'] \arrow[r, "b"] \arrow[u, leftarrow, "q_1"] & Z_2 \arrow[u, leftarrow, "q_2"] & \tilde{Y} \arrow[l, "k"'] \\
D' \arrow[u, leftarrow, "p_0"] & V_1 \arrow[l, "a"'] \arrow[r, "a"] \arrow[u, leftarrow, "p_1"] & V_2 \arrow[r, "a'"] \arrow[u, leftarrow, "p_2"] & D
\end{tikzcd}$$

Here
$S'^* = \{g \in S; Z_{G'}(g_s)^0 \subset L'\}$,
$\tilde{Y}' = \{(g, lL') \in G' \times L/L'; l^{-1}gl \in S'^*\}$,
$S^* = \{g \in S; Z_G(g_s)^0 \subset L'\}$,

$\tilde{Y} = \{(g, xL') \in G \times G^0/L'; x^{-1}gx \in S^*\}$,
$Z_1 = \{(g, lL', x) \in G \times \times L/L' \times G^0; l^{-1}x^{-1}gxl \in S'^*U_P\}$,
$Z_2 = \{(g, xL'U_P) \in G \times G^0/(L'U_P); x^{-1}gx \in S'^*U_P\}$,
$\hat{Y}' = \{(g, l) \in G' \times L; l^{-1}gl \in S'^*\}$,
$\hat{Z}_1 = \{(g, l, x) \in G \times \times L \times G^0; l^{-1}x^{-1}gxl \in S'^*U_P\}$,
$\hat{Z}_2 = \{(g, xU_P) \in G \times G^0/U_P; x^{-1}gx \in S'^*U_P\}$,
$c(g, l, x) = (\xi(x^{-1}gx), l), c'(g, l, x) = (g, xlU_P)$,
$b(g, lL', x) = (\xi(x^{-1}gx), lL'), b'(g, lL', x) = (g, xlL'U_P)$,
$k(g, xL') = (g, xL'U_P), q_0(g, l) = (g, lL'), q_1(g, l, x) = (g, lL', x)$,
$q_2(g, vU_P) = (g, vL'U_P), p_0(g, lL') = g, p_1(g, lL', x) = (g, x)$,
$p_2(g, vL'U_P) = (g, vP), r_0(g, l) = l^{-1}gl$,
and $r_1(g, l, x) = s_1 \in S'^*, r_2(g, vU_P) = s_2 \in S'^*$ are defined by

$$l^{-1}x^{-1}gxl \in s_1U_P, v^{-1}gv \in s_2U_P.$$

Let $Y' = \bigcup_{l \in L} lS'^*l^{-1}$, a locally closed smooth irreducible subvariety of D'; see 3.16, 3.17. Let $Y'_1 = a^{-1}(Y')$, a locally closed smooth irreducible subvariety of V_1 which is P-stable since Y' is stable under L-conjugacy (we use that a is smooth with connected fibres). Then $Y'_1 = a'^{-1}(Y'_2)$ where Y'_2 is a well-defined locally closed smooth irreducible subvariety of V_2. Let $Y = \bigcup_{x \in G^0} xS^*x^{-1}$, a locally closed smooth irreducible subvariety of D. Let $\bar{Y}', \bar{Y}'_1, \bar{Y}'_2, \bar{Y}$ be the closure of Y', Y'_1, Y'_2, Y in D', V_1, V_2, D. Then $a'^{-1}(\bar{Y}_2) = \bar{Y}_1 = a^{-1}(\bar{Y}')$. Let

$$\mathcal{W}' = \{n \in N_L L'; nSn^{-1} = n\}/L', \ \mathcal{W} = \{n \in N_{G^0} L'; nSn^{-1} = n\}/L'.$$

Then $\mathcal{W}' \subset \mathcal{W}$ are finite groups. Now $\mathcal{W}'$ acts freely on $\tilde{Y}', Z_1, Z_2$ by
$n : (g, lL') \mapsto (g, ln^{-1}L'), n : (g, lL', x) \mapsto (g, ln^{-1}L', x)$,
$n : (g, xL'U_P) \mapsto (g, xn^{-1}L'U_P)$.
These actions are compatible with $\tilde{Y}' \xleftarrow{b} Z_1 \xrightarrow{b} Z_2$. By 3.13, $p_0 : \tilde{Y}' \to Y'$ is a principal $\mathcal{W}'$-bundle. It follows that $p_1 : Z_1 \to Y'_1$ and $p_2 : Z_2 \to Y'_2$ are principal $\mathcal{W}'$-bundles.

Now $\mathcal{W}$ acts freely on $\tilde{Y}$ by $n : (g, xL') \mapsto (g, xn^{-1}L')$. From 3.13 we see that $p : \tilde{Y} \to Y, (g, xL') \mapsto g$ is a principal $\mathcal{W}$-bundle. From the definitions we see that

(a) *the restriction of the $\mathcal{W}$-action on $\tilde{Y}$ to the subgroup $\mathcal{W}'$ is compatible via $k : \tilde{Y} \to Z_2$ with the $\mathcal{W}'$-action on Z_2.*

Let $\mathcal{E} \in \mathcal{S}(S)$. Let $\tilde{\mathcal{E}}$ be the local system on $\tilde{Y}$ defined as in 5.6 (with G, L', S instead of G, L, S). Define local system $\tilde{\mathcal{E}}', \tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2$ on $\tilde{Y}', Z_1, Z_2$ (respectively) by $q_0^*\tilde{\mathcal{E}}' = r_0^*\mathcal{E}, q_1^*\tilde{\mathcal{E}}_1 = r_1^*\mathcal{E}, q_2^*\tilde{\mathcal{E}}_2 = r_2^*\mathcal{E}$. We have $b^*\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}_1 = b'^*\tilde{\mathcal{E}}_2$. Hence $(p_1)_!b^*\tilde{\mathcal{E}}' = (p_1)_!b'^*\tilde{\mathcal{E}}_2 = (p_1)_!\tilde{\mathcal{E}}_1$. Now $(p_0)_!\tilde{\mathcal{E}}', (p_2)_!\tilde{\mathcal{E}}_2$ are local systems on Y', Y'_2 and $a^*(p_0)_!\tilde{\mathcal{E}}' = a'^*(p_2)_!\tilde{\mathcal{E}}_2 = (p_1)_!\tilde{\mathcal{E}}_1$ as local systems on Y'_1. We have $\tilde{\mathcal{E}} = k^*\tilde{\mathcal{E}}_2$. Let

$K' = IC(\bar{Y}', (p_0)_!\tilde{\mathcal{E}}')[\dim \tilde{Y}'], K_1 = IC(\bar{Y}'_1, (p_1)_!\tilde{\mathcal{E}}_1)[\dim Z_1]$,
$K_2 = IC(\bar{Y}'_2, (p_2)_!\tilde{\mathcal{E}}_2)[\dim Z_2], K = IC(\bar{Y}, p_!\tilde{\mathcal{E}})[\dim \tilde{Y}]$,
regarded as perverse sheaves on D', V_1, V_2, D, zero on $D' - \bar{Y}', V_1 - \bar{Y}'_1, V_2 - \bar{Y}'_2, D - \bar{Y}$ respectively. Since a, a' are smooth morphisms with connected fibres, we see that $a^\bigstar K' = K_1 = a'^\bigstar K_2$ in $\mathcal{M}(V_1)$. Hence

(b) $\mathrm{ind}_{D'}^D(K') = a''_! K_2$.

We show that

(c) $a''_! K_2 = K$ *canonically.*

Let P' be a parabolic of L with Levi L' such that $S \subset N_{G'}P'$. Then $P'_1 = P'U_P$ is a parabolic of G^0 with Levi L' such that $S \subset N_G P'_1$. We have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \tilde{Y}' & \xleftarrow{b} & Z_1 & \xrightarrow{b} & Z_2 \\ {\scriptstyle j_0}\downarrow & & {\scriptstyle j_1}\downarrow & & {\scriptstyle j_2}\downarrow \\ X' & \xleftarrow{e} & Z & \xrightarrow{e} & X \\ {\scriptstyle t_0}\downarrow & & {\scriptstyle t_1}\downarrow & & {\scriptstyle t_2}\downarrow \\ D' & \xleftarrow{a} & V_1 & \xrightarrow{a'} & V_2 \end{array}$$

Here
$X' = \{(g, lP') \in G' \times L/P'; l^{-1}gl \in \bar{S}U_{P'}\}$,
$X = \{(g, vP'_1) \in G \times G^0/P'_1; v^{-1}gv \in \bar{S}U_{P'_1}\}$,
$Z = \{(g, lP', x) \in G \times \times L/P' \times G^0; l^{-1}x^{-1}gxl \in \bar{S}U_{P'_1}\}$,
$e(g, lP', x) = (\xi(x^{-1}gx), lP'), e'(g, lP', x) = (g, xlP'_1)$,
$t_0(g, lP') = g, t_1(g, lP', x) = (g, x), t_2(g, vP'_1) = (g, vP)$,
$j_0(g, lL') = (g, lP'), j_1(g, lL', x) = (g, lP', x), j_2(g, xL'U_P) = (g, xP'_1)$.
Since j_0 is an open imbedding (by 5.5) we see that j_1, j_2 are open imbeddings. We identify $\tilde{Y}', Z_1, Z_2$ with open subsets of X', Z, X via j_0, j_1, j_2. The composition $\tilde{Y} \to Z_2 \xrightarrow{j_2} X$ is the map $(g, xL') \mapsto (g, xP'_1)$ which is an open imbedding by 5.5. Since Z_2 is an open subset of X via j_2, we see that $\tilde{Y}$ may be identified with an open subset of Z_2 via k. Since e, e' are smooth morphisms with connected fibres, we see that

$$e^{\star} IC(X', \tilde{\mathcal{E}}')[\dim X'] = IC(Z, \tilde{\mathcal{E}}_1)[\dim Z] = e'^{\star} IC(X, \tilde{\mathcal{E}}_2)[\dim X] \text{ in } \mathcal{M}(Z).$$

From 5.7 we see that $K' = (t_0)_! IC(X', \tilde{\mathcal{E}}')[\dim X']$. Hence

$$K_1 = (t_1)_! IC(Z, \tilde{\mathcal{E}}_1)[\dim Z], K_2 = (t_2)_! IC(X, \tilde{\mathcal{E}}_2)[\dim X].$$

Since $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_2|_{\tilde{Y}}$, we have $IC(X, \tilde{\mathcal{E}}_2) = IC(X, \tilde{\mathcal{E}})$ and $K_2 = (t_2)_! IC(X, \tilde{\mathcal{E}})[\dim X]$. The composition $a''t_2 : X \to D$ is $(g, vP'_1) \mapsto g$. Using 5.7 we have $a''_! K_2 = (a''t_2)_! IC(X, \tilde{\mathcal{E}})[\dim X] = K$. This proves (c).

Define $\mathbf{E} = \bigoplus_{w \in \mathcal{W}} \mathbf{E}_w$ as in 7.10 (for $G, L', S, \mathcal{E}$ instead of $G, L, S, \mathcal{E}$). Define $\mathbf{E}' = \bigoplus_{w \in \mathcal{W}'} \mathbf{E}'_w$ in the same way (for $G', L', S, \mathcal{E}$ instead of $G, L, S, \mathcal{E}$). Then $\mathbf{E}$ is naturally an algebra and $\mathbf{E}'$ is a subalgebra of $\mathbf{E}$. Since a, a' are smooth morphisms with connected fibres, we see that

$$\mathbf{E}' = \mathrm{End}((p_0)_! \tilde{\mathcal{E}}') = \mathrm{End}(K_0) = \mathrm{End}(K_1) = \mathrm{End}(K_2).$$

Now $a''_!$ defines a ring homomorphism $\mathrm{End}(K_2) \to \mathrm{End}(a''_! K_2)$. Thus $a''_! K_2$ becomes an $\mathbf{E}'$-module. On the other hand, $\mathbf{E} = \mathrm{End}(K)$. Using (a) and the definitions we see that the restriction of the $\mathbf{E}$-module structure of K to $\mathbf{E}'$ corresponds under (c) to the $\mathbf{E}'$-module structure on $a''_! K_2$.

Let Γ be a subgroup of $\mathcal{W}'$ and let $\mathbf{E}'_\Gamma = \bigoplus_{w \in \Gamma} \mathbf{E}'_w$, a subalgebra of $\mathbf{E}'$, hence of $\mathbf{E}$. Let ρ be a $\mathbf{E}'_\Gamma$-module of finite dimension over $\bar{\mathbf{Q}}_l$. Let
$K'(\rho) = \mathrm{Hom}_{\mathbf{E}'_\Gamma}(\rho, K') \in \mathcal{M}(D'), K_1(\rho) = \mathrm{Hom}_{\mathbf{E}'_\Gamma}(\rho, K_1) \in \mathcal{M}(V_1)$,
$K_2(\rho) = \mathrm{Hom}_{\mathbf{E}'_\Gamma}(\rho, K_2) \in \mathcal{M}(V_2), K(\rho) = \mathrm{Hom}_{\mathbf{E}'_\Gamma}(\rho, K) \in \mathcal{M}(D)$.

Then $a^\bigstar K'(\rho)=K_1(\rho)=a'^\bigstar K_2(\rho)$ and

(d) $$\mathrm{ind}_{D'}^{D}(K'(\rho))=a''_!K_2(\rho)=K(\rho)\in\mathcal{M}(D).$$

27.3. Let P,L,G',D,D' be as in 27.1. Let Q be a parabolic of L with Levi M such that $N_{D'}Q\ne\emptyset$. Let $G''=N_{G'}Q\cap N_{G'}M$. Then $G''^0=M$, $N_{D'}Q$ is a connected component of $N_{G'}Q$ and $D''=G''\cap D'$ is a connected component of G''. Let $R=QU_P$, a parabolic of G^0 with Levi M. We have $R\subset P$ and $N_DR\ne\emptyset$. (Indeed, $D''\subset N_DR$; more precisely, $N_DR\cap N_DM=D''$.) Note that $N_GR\cap N_GM$ contains G'' as a subgroup of finite index and both have D'' as a connected component.

Let M' be a Levi of a parabolic of M. Let S be an isolated stratum of $N_{G''}M'$ such that $S\subset D$ and S normalizes some parabolic of M with Levi M'. Then S is also an isolated stratum of $N_{G'}M'$ such that S normalizes some parabolic of L with Levi M'. Moreover, S is an isolated stratum of N_GM' such that S normalizes some parabolic of G^0 with Levi M'. Let

$S''^*=\{g\in S;Z_{G''}(g_s)^0\subset M'\}$,
$\tilde Y''=\{(g,mM')\in G''\times M/M';m^{-1}gn\in S''^*\}$,
$S'^*=\{g\in S;Z_{G'}(g_s)^0\subset M'\}$,
$\tilde Y'=\{(g,lL')\in G'\times L/M';l^{-1}gl\in S'^*\}$,
$S^*=\{g\in S;Z_G(g_s)^0\subset M'\}$,
$\tilde Y=\{(g,xM')\in G\times G^0/M';x^{-1}gx\in S^*\}$.

Let

$Y''=\bigcup_{m\in M}mS''^*m^{-1},Y'=\bigcup_{l\in L}lS'^*l^{-1},Y=\bigcup_{x\in G^0}xS^*x^{-1}$.

Define

$p'':\tilde Y''\to Y'',p':\tilde Y'\to Y',p:\tilde Y\to Y$

by the first projection. Let

$\mathcal{W}''=\{n\in N_MM';nSn^{-1}=n\}/M'$,
$\mathcal{W}'=\{n\in N_LM';nSn^{-1}=n\}/M'$,
$\mathcal{W}=\{n\in N_{G^0}M';nSn^{-1}=n\}/M'$.

Then $\mathcal{W}''\subset\mathcal{W}'\subset\mathcal{W}$ are finite groups. As in 3.13, p'' is a principal $\mathcal{W}''$-bundle, p' is a principal $\mathcal{W}'$-bundle, p is a principal $\mathcal{W}$-bundle. Let $\mathcal{E}\in\mathcal{S}(S)$. Define a local system $\tilde{\mathcal{E}}$ on $\tilde Y$ as 5.6 (with G,M',S instead of G,L,S). Define a local system $\tilde{\mathcal{E}}'$ on $\tilde Y'$ as 5.6 (with G',M',S instead of G,L,S). Define a local system $\tilde{\mathcal{E}}''$ on $\tilde Y''$ as 5.6 (with G'',M',S instead of G,L,S). Then $p''_!\tilde{\mathcal{E}}'',p'_!\tilde{\mathcal{E}}',p_!\tilde{\mathcal{E}}$ is a local system on Y'',Y',Y respectively. We regard

$K''=IC(\bar Y'',p''_!\tilde{\mathcal{E}}'')[\dim\tilde Y'']$,
$K'=IC(\bar Y',p'_!\tilde{\mathcal{E}}')[\dim\tilde Y']$,
$K=IC(\bar Y,p_!\tilde{\mathcal{E}})[\dim\tilde Y]$,

as perverse sheaves on D'',D',D respectively, zero on $D''-\bar Y'',D'-\bar Y',D-\bar Y$. Define $\mathbf{E}=\bigoplus_{w\in\mathcal{W}}\mathbf{E}_w$ as in 7.10 (for $G,M',S,\mathcal{E}$ instead of $G,L,S,\mathcal{E}$). Define in the same way $\mathbf{E}'=\bigoplus_{w\in\mathcal{W}'}\mathbf{E}'_w$ (for $G',M',S,\mathcal{E}$ instead of $G,L,S,\mathcal{E}$) and $\mathbf{E}''=\bigoplus_{w\in\mathcal{W}''}\mathbf{E}''_w$ (for $G'',M',S,\mathcal{E}$ instead of $G,L,S,\mathcal{E}$). Then $\mathbf{E}$ is naturally an algebra, $\mathbf{E}'$ is a subalgebra of $\mathbf{E}$ and $\mathbf{E}''$ is a subalgebra of $\mathbf{E}'$. We have naturally

$$\mathbf{E}''=\mathrm{End}(K''),\mathbf{E}'=\mathrm{End}(K'),\mathbf{E}=\mathrm{End}(K).$$

Thus, K'',K',K are naturally $\mathbf{E}''$-modules. Let ρ be a finite dimensional $\mathbf{E}''$-module over $\bar{\mathbf{Q}}_l$. Let

$K''(\rho)=\mathrm{Hom}_{\mathbf{E}''}(\rho,K'')\in\mathcal{M}(D'')$,
$K'(\rho)=\mathrm{Hom}_{\mathbf{E}''}(\rho,K')\in\mathcal{M}(D')$,

$K(\rho) = \mathrm{Hom}_{\mathbf{E}''}(\rho, K) \in \mathcal{M}(D)$.
Applying 27.2(d) with G', Q, M, M' instead of G, P, L, L' and $\Gamma = \mathcal{W}''$ we have

$$\mathrm{ind}_{D''}^{D'}(K''(\rho)) = K'(\rho) \in \mathcal{M}(D').$$

Applying 27.2(d) with G, P, L, M' instead of G, P, L, L' and $\Gamma = \mathcal{W}''$ we have

$$\mathrm{ind}_{D'}^{D}(K'(\rho)) = K(\rho) \in \mathcal{M}(D).$$

Applying 27.2(d) with G, R, M, M' instead of G, P, L, L' and $\Gamma = \mathcal{W}''$ we have

$$\mathrm{ind}_{D''}^{D}(K''(\rho)) = K(\rho) \in \mathcal{M}(D').$$

Hence we have the following *transitivity formula*:

(a) $$\mathrm{ind}_{D'}^{D}(\mathrm{ind}_{D''}^{D'}(K''(\rho))) = \mathrm{ind}_{D''}^{D}(K''(\rho)).$$

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