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CHARACTER SHEAVES ON DISCONNECTED GROUPS, VI

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Abstract. We define the character sheaves on a connected component of a reductive group and we show that the restriction functor takes a character sheaf to a direct sum of character sheaves.

Introduction

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G.

In Section 28 we define the character sheaves on a connected component of G generalizing the definition in [L3, I, §2]. In Section 29 we prove a semisimplicity property of the restriction functor, generalizing one in [L3, I, §3]. In Section 30 we show that any character sheaf is admissible, generalizing a result in [L3, I, §4]. In Section 31 we show that the restriction functor takes a character sheaf to a direct sum of character sheaves, generalizing a result in [L3, I, §6].

We adhere to the notation of [L9] and [BBD]. Here is some additional notation. If $K \in \mathcal{D}(X)$ and A is a simple perverse sheaf on X we write $A \dashv K$ instead of "A is a subquotient of ${}^pH^i(K)$ for some $i \in \mathbf{Z}$." Let $\mathcal{M}(X)$ be the subcategory of $\mathcal{D}(X)$ whose objects are the perverse sheaves on X.

Contents

28. Definition of character sheaves

28.1. Let T be a torus. For any $n \in \mathbf{N}^*_{\mathbf{k}}$, let $\mathfrak{s}_n(T)$ be the category whose objects are the local systems of rank 1 on T that are equivariant for the transitive T-action $z : t \mapsto z^n t$ on T; let $\mathfrak{s}(T)$ be the category whose objects are the local systems on T that are in $\mathfrak{s}_n(T)$ for some n as above.

If $f : T \to T'$ is a morphism of tori and $\mathcal{L}' \in \mathfrak{s}(T')$, then $f^*\mathcal{L}' \in \mathfrak{s}(T)$. The set $\underline{\mathfrak{s}}(T)$ of isomorphism classes of objects in $\mathfrak{s}(T)$ is an abelian group for tensor product

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of local systems. Let $\mathcal{X} = \mathrm{Hom}(T, \mathbf{k}^*)$ (homomorphisms of algebraic groups). From the definitions we see that

(a) *$\kappa \otimes \mathcal{E} \mapsto \kappa^*\mathcal{E}$ defines a group isomorphism $\mathcal{X} \otimes \underline{\mathfrak{s}}(\mathbf{k}^*) \xrightarrow{\sim} \underline{\mathfrak{s}}(T)$.*

We show that

(b) *for $\mathcal{L} \in \mathfrak{s}(T)$ there exists $\kappa \in \mathcal{X}$ and $\mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$ such that $\mathcal{L} \cong \kappa^*\mathcal{E}$.*

Indeed by (a) there exist $\kappa_i \in \mathcal{X}, \mathcal{E}_i \in \mathfrak{s}(\mathbf{k}^*), (i \in [1, m])$ such that $\mathcal{L} \cong \bigotimes_{i=1}^m \kappa_i^*\mathcal{E}_i$. By 5.3 we have $\underline{\mathfrak{s}}(\mathbf{k}^*) = \mathrm{Hom}(\mu_\infty(\mathbf{k}^*), \bar{\mathbf{Q}}_l^*) \cong \mathbf{Q}'/\mathbf{Z}$ where $\mathbf{Q}' = \bigcup_{n \in \mathbf{N}_\mathbf{k}^*} \frac{1}{n}\mathbf{Z} \subset \mathbf{Q}$. Hence we can find $\mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$, $n_i \in \mathbf{N}_\mathbf{k}^*$ such that $\mathcal{E}_i \cong \mathcal{E}^{\otimes n_i}$ for $i \in [1, m]$. Then $\mathcal{L} \cong \bigotimes_{i=1}^m \kappa_i^*\mathcal{E}^{\otimes n_i} \cong \kappa^*\mathcal{E}$ where $\kappa = \prod_{i=1}^m \kappa_i^{n_i}$ and (b) follows.

For any $\tau \in T$ define $h_\tau : T \to T$ by $h_\tau(t) = \tau t$. We show that

(c) *if $\tau \in T, \mathcal{L} \in \mathfrak{s}(T)$, then $h_\tau^*\mathcal{L} \cong \mathcal{L}$.*

Let n be such that $\mathcal{L} \in \mathfrak{s}_n(T)$. Then for any $z \in T$ we have $h_{z^n}\mathcal{L} \cong \mathcal{L}$. We can find $z \in T$ such that $z^n = \tau$. This proves (c).

28.2. Let $\mathcal{L} \in \mathfrak{s}(T)$, let $\kappa, \mathcal{E}$ be as in 28.1(b) and let n be the order of $\mathcal{E}$ in $\underline{\mathfrak{s}}(\mathbf{k}^*)$. Then $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$. We show that the following two conditions for a morphism $f : T \to T$ of tori are equivalent:

(i) *$f^*\mathcal{L} \cong \mathcal{L}$;*

(ii) *there exists $\kappa_1 \in \mathcal{X}$ such that $\kappa \circ f = \kappa\kappa_1^n$.*

Condition (i) is equivalent to $f^*\kappa^*\mathcal{E} \cong \kappa^*\mathcal{E}$, that is, $(\kappa \circ f)^*\mathcal{E} \cong \kappa^*\mathcal{E}$. Using the injectivity of the map 28.1(a) we see that this is equivalent to $(\kappa \circ f) \otimes (n'/n) = \kappa \otimes (n'/n)$ in $\mathcal{X} \otimes \mathbf{Q}'/\mathbf{Z}$ (here $n' \in \mathbf{Z}, 0 < n' \le n$ and n'/n is irreducible) which is clearly equivalent to condition (ii).

Assuming that (i) and (ii) hold, we show that

(a) *$\mathcal{L}$ is T-equivariant for the T-action $t_0 : t \mapsto f(t_0)tt_0^{-1}$ on T.*

The map $\kappa : T \to \mathbf{k}^*$ is compatible with the T-action (a) on T and the T-action $t_0 : z \mapsto \kappa_1(t_0)^n z$ on $\mathbf{k}^*$. Hence to show that $\mathcal{L} = \kappa^*\mathcal{E}$ is T-equivariant it suffices to show that $\mathcal{E}$ is T-equivariant. Since the T-action on $\mathbf{k}^*$ comes via κ_1 from the $\mathbf{k}^*$-action $z_0 : z \mapsto z_0^n z$ on $\mathbf{k}^*$, it suffices to show that $\mathcal{E}$ is $\mathbf{k}^*$-equivariant. This holds since $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$.

28.3. G acts on $\prod_{B \in \mathcal{B}} B/U_B$ by

$$x : (g_B U_B)_{B \in \mathcal{B}} \mapsto (g'_B U_B)_{B \in \mathcal{B}}$$

where $g'_{xBx^{-1}} U_{xBx^{-1}} = x g_B x^{-1} U_{xBx^{-1}}$. Let

$$\mathbf{T} = (\prod_{B \in \mathcal{B}} B/U_B)^{G^0}$$

(fixed point set of G^0). For any $B' \in \mathcal{B}$ we define $f_{B'} : \mathbf{T} \xrightarrow{\sim} B'/U_{B'}$ by $f_{B'}((g_B U_B)_{B \in \mathcal{B}}) = g_{B'} U_{B'}$. We use $f_{B'}$ to transport the algebraic group structure of $B'/U_{B'}$ to an algebraic group structure of $\mathbf{T}$. This structure is independent of the choice of B'. Thus $\mathbf{T}$ is naturally a torus over $\mathbf{k}$. The G action on $\prod_{B \in \mathcal{B}} B/U_B$ induces a G/G^0-action

$$D : t \mapsto \underline{D}(t)$$

on $\mathbf{T}$, respecting the algebraic group structure of $\mathbf{T}$. We say that $\mathbf{T}$ is the *canonical torus* of G^0.

For $w \in \mathbf{W}$ (see 26.1) there is a unique isomorphism $\mathbf{T} \xrightarrow{\sim} \mathbf{T}$ (denoted again by w) such that for any $(B, B') \in \mathcal{B} \times \mathcal{B}$ with $\mathrm{pos}(B, B') = w$ we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{T} & & \xrightarrow{\quad w \quad} & & \mathbf{T} \\ {\scriptstyle f_{B'}}\downarrow & & & & \downarrow{\scriptstyle f_B} \\ B'/U_{B'} & \xleftarrow{\sim} & (B \cap B')/(U_B \cap U_{B'}) & \xrightarrow{\sim} & B/U_B \end{array}$$

where the isomorphisms in the bottom row are induced by the obvious inclusions. We use this to identify $\mathbf{W}$ with a subgroup of the group $\mathrm{Aut}(\mathbf{T})$ of automorphisms of the torus $\mathbf{T}$. Let

$$\mathbf{W}^\bullet = \{w\underline{D}; w \in \mathbf{W}, D \in G/G^0\} \subset \mathrm{Aut}(\mathbf{T}).$$

This is a subgroup of $\mathrm{Aut}(\mathbf{T})$ normalizing $\mathbf{W}$ since $\underline{D}w = \epsilon_D(w)\underline{D} : \mathbf{T} \to \mathbf{T}$ for any $D \in G/G^0, w \in \mathbf{W}$; here ϵ_D is as in 26.2.

Let $\langle , \rangle : \mathrm{Hom}(\mathbf{k}^*, \mathbf{T}) \times \mathrm{Hom}(\mathbf{T}, \mathbf{k}^*) \to \mathbf{Z}$ be the standard pairing. Define subsets R, R^+ of $\mathrm{Hom}(\mathbf{T}, \mathbf{k}^*)$ as follows. Let $B \in \mathcal{B}$ and let T be a maximal torus of B. Consider the isomorphism $\mathbf{T} \xrightarrow{\sim} T$ (composition of $f_B : \mathbf{T} \xrightarrow{\sim} B/U_B$ with the obvious isomorphism $B/U_B \xrightarrow{\sim} T$). We require that the subset of $\mathrm{Hom}(T, \mathbf{k}^*)$ corresponding to R (resp. R^+) under this isomorphism is the set of roots of G^0 with respect to T (resp. the set of roots of G^0 with respect to T such that the corresponding root subgroup is contained in B). Let $R^- = R - R^+$. For any $\alpha \in R$ there is a unique $\check{\alpha} \in \mathrm{Hom}(\mathbf{k}^*, \mathbf{T})$ and a unique $s_\alpha \in \mathbf{W}$ such that $\langle \check{\alpha}, \alpha \rangle = 2$ and $t = s_\alpha(t)\check{\alpha}(\alpha(t))$ for all $t \in \mathbf{T}$. Then $s_\alpha^2 = 1$ and for $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ we have

(a) $$\mathcal{L} \cong s_\alpha^*\mathcal{L} \otimes \alpha^*(\check{\alpha}^*\mathcal{L}).$$

For $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ let

$$R_\mathcal{L} = \{\alpha \in R; \check{\alpha}^*\mathcal{L} \cong \bar{\mathbf{Q}}_l\}.$$

Pick $\kappa \in \mathrm{Hom}(\mathbf{T}, \mathbf{k}^*), \mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$ such that $\mathcal{L} \cong \kappa^*\mathcal{E}$; see 28.1(b). Let $n \in \mathbf{N}_\mathbf{k}^*$ be the order of $\mathcal{E}$ in $\underline{\mathfrak{s}}(\mathbf{k}^*)$. We show that

$$R_\mathcal{L} = \{\alpha \in R; \langle \check{\alpha}, \kappa \rangle \in n\mathbf{Z}\}.$$

Indeed, for $\alpha \in R$ we have $\check{\alpha}^*\mathcal{L} = \check{\alpha}^*\kappa^*\mathcal{E} = (\kappa \circ \check{\alpha})^*\mathcal{E} = f^*\mathcal{E}$ where $f : \mathbf{k}^* \to \mathbf{k}^*$ is $z \mapsto z^{\langle \check{\alpha}, \kappa \rangle}$. We now use the fact that, for $s \in \mathbf{Z}$, the inverse image of $\mathcal{E}$ under $\mathbf{k}^* \to \mathbf{k}^*, z \to z^s$ is $\bar{\mathbf{Q}}_l$ if and only if $s \in n\mathbf{Z}$.

Let

$$\mathbf{W}_\mathcal{L}^\bullet = \{a \in \mathbf{W}^\bullet; (a^{-1})^*\mathcal{L} \cong \mathcal{L}\}.$$

Let $\mathbf{W}_\mathcal{L}$ be the subgroup of $\mathbf{W}$ generated by $\{s_\alpha; \alpha \in R_\mathcal{L}\}$. From (a) we see that

(b) $$\mathbf{W}_\mathcal{L} \subset \mathbf{W}_\mathcal{L}^\bullet.$$

Moreover, $\mathbf{W}_\mathcal{L}$ is a normal subgroup of $\mathbf{W}_\mathcal{L}^\bullet$.

28.4. In the remainder of this section we fix a connected component D of G. Let $w \in \mathbf{W}$. Let

$$\mathfrak{Z}_D^w = \{(B, \xi); B \in \mathcal{B}, \xi \in U_B\backslash D/U_B, \mathrm{pos}(B, gBg^{-1}) = w \text{ for some/any } g \in \xi\}.$$

28.5. Let $B^* \in \mathcal{B}$ and let T be a maximal torus of B^*. We set $U^* = U_{B^*}$, $W_T = N_{G^0}T/T$.

We identify $W_T = \mathbf{W}$ *by* $w \leftrightarrow G^0$*-orbit of* $(B^*, \dot{w}B^*\dot{w}^{-1})$ *and* $T = \mathbf{T}$ *by*

(a) $$t_1 \leftrightarrow t, t_1U^* = f_{B^*}(t).$$

For any $w \in W_T = \mathbf{W}$ we fix a representative $\dot{w}$ of w in $N_{G^0}T$; we assume that for $s \in \mathbf{I}$, $\dot{s}$ belongs to the subgroup of G^0 generated by the two root subgroups of G^0 with respect to T corresponding to s. We also assume that $\dot{1} = 1$.

Any $\alpha \in R$ becomes a root α of G^0 with respect to T and $\check{\alpha}$ becomes the corresponding coroot $\mathbf{k}^* \to T$. We fix $d \in N_DB^* \cap N_DT$. For $w \in \mathbf{W}$ we have a diagram

$$T \xleftarrow{\phi} \hat{\mathfrak{Z}}_D^w \xrightarrow{\rho} \mathfrak{Z}_D^w$$

where

$$\hat{\mathfrak{Z}}_D^w = \{(hU^*, g); hU^* \in G^0/U^*, g \in \dot{w}dT\},$$

$$\rho(hU^*, g) = (hB^*h^{-1}, hU^*gU^*h^{-1}), \phi(hU^*, g) = d^{-1}\dot{w}^{-1}g.$$

Now ϕ is T-equivariant with respect to the T-action

$$t_0 : (hU^*, g) \mapsto (ht_0^{-1}U^*, t_0gt_0^{-1})$$

on $\hat{\mathfrak{Z}}_D^w$ and the T-action $t_0 : \mathrm{Ad}(d^{-1}\dot{w}^{-1})(t_0)tt_0^{-1}$ on T. Hence if $\mathcal{L} \in \mathfrak{s}(T)$ satisfies $\mathrm{Ad}((\dot{w}d)^{-1})^*\mathcal{L} \cong \mathcal{L}$, then $\phi^*\mathcal{L}$ is a T-equivariant local system on $\hat{\mathfrak{Z}}_D^w$. (See 28.2(a).) Since ρ is a principal T-bundle, there is a well-defined local system $\tilde{\mathcal{L}}$ on $\mathfrak{Z}_D^w$ such that $\rho^*\tilde{\mathcal{L}} = \phi^*\mathcal{L}$.

28.6. Let $w \in \mathbf{W}$. Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ be such that $w\underline{D} \in \mathbf{W}_{\mathcal{L}}^\bullet$. We associate to $\mathcal{L}$ a local system $\tilde{\mathcal{L}}$ of rank 1 on $\mathfrak{Z}_D^w$ as follows. Let $B^*, U^*, T, d, \dot{w}$ be as in 28.5. Using the identification $T = \mathbf{T}$ in 28.5, we regard $\mathcal{L}$ as a local system in $\mathfrak{s}(T)$. Then $\tilde{\mathcal{L}}$ is defined as in 28.5. We show that

(a) *the isomorphism class of* $\tilde{\mathcal{L}}$ *is independent of the choice of* $B^*, T, \dot{w}, d$.

Let us replace $B^*, T, \dot{w}, d$ by $xB^*x^{-1}, xTx^{-1}, x\dot{w}x^{-1}, xdx^{-1}$ where $x \in G^0$. Define ${}'\hat{\mathfrak{Z}}_D^w, {}'\phi, {}'\rho, {}'\mathcal{L}, {}'\tilde{\mathcal{L}}$ in terms of this new choice in the same way as $\hat{\mathfrak{Z}}_D^w, \phi, \rho, \mathcal{L}, \tilde{\mathcal{L}}$ were defined in terms of $B^*, T, \dot{w}, d$. We have a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{T} & \xrightarrow{a} & T & \xleftarrow{\phi} & \hat{\mathfrak{Z}}_D^w & \xrightarrow{\rho} & \mathfrak{Z}_D^w \\
{\scriptstyle =}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle =}\downarrow \\
\mathbf{T} & \xrightarrow{a'} & xTx^{-1} & \xleftarrow{{}'\phi} & {}'\hat{\mathfrak{Z}}_D^w & \xrightarrow{{}'\rho} & \mathfrak{Z}_D^w
\end{array}$$

where $b(t) = xtx^{-1}, c(hU^*, g) = (hx^{-1}xU^*x^{-1}, xgx^{-1})$, a is given by 28.5(a) and a' is the analogous isomorphism defined in terms of xB^*x^{-1}, xTx^{-1} instead of B^*, T. Then ${}'\mathcal{L} = b^*\mathcal{L}$, ${}'\phi^*{}'\mathcal{L} = c^*\phi^*\mathcal{L}$ and ${}'\tilde{\mathcal{L}} = \tilde{\mathcal{L}}$. Hence to prove (a) it suffices to show that if $B^*, T, \dot{w}, d$ are replaced by $B^*, T, \dot{w}t_1, dt_2$ where $t_1, t_2 \in T$, then the isomorphism class of $\tilde{\mathcal{L}}$ does not change. Note that $\hat{\mathfrak{Z}}_D^w, \rho$ remain unchanged under the replacement above. However, the map ϕ defined in terms of $B^*, T, \dot{w}, d$ is replaced by the composition of ϕ with a left translation on T. It remains to use that the inverse image of $\mathcal{L}$ under a left translation of T is isomorphic to $\mathcal{L}$; see 28.1(c).

28.7. Let J be a subset of $\mathbf{I}$. For any $B \in \mathcal{B}$ we denote by $Q_{J,B}$ the unique parabolic in $\mathcal{P}_J$ that contains B; we write $U_{J,B}$ instead of $U_{Q_{J,B}}$. Let $w \in \mathbf{W}$. Let

$$Z^w_{\emptyset,J,D} = \{(B, B', gU_{J,B}); B \in \mathcal{B}, B' \in \mathcal{B}, g \in D, gBg^{-1} = B', \mathrm{pos}(B, B') = w\}.$$

The map

$$\zeta : Z^w_{\emptyset,J,D} \to \mathfrak{Z}^w_D, \quad (B, B', gU_{J,B}) \mapsto (B, U_B g U_B)$$

is an affine space bundle.

Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ be such that $w\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Then $\zeta^*\tilde{\mathcal{L}}$ is a local system on $Z^w_{\emptyset,J,D}$ denoted again by $\tilde{\mathcal{L}}$.

28.8. Let $\mathbf{w} = (w_1, w_2, \dots, w_r)$ be a sequence in $\mathbf{W}$, let $[\mathbf{w}] = w_1 w_2 \dots w_r$ and let

$$\begin{aligned} Z^{\mathbf{w}}_{\emptyset,J,D} =& \{(B_0, B_1, \dots, B_r, gU_{J,B_0}); B_i \in \mathcal{B}(i \in [0, r]), g \in D, gB_0g^{-1} = B_r, \\ & \mathrm{pos}(B_{i-1}, B_i) = w_i (i \in [1, r])\}. \end{aligned}$$

We define a morphism

$$\zeta : Z^{\mathbf{w}}_{\emptyset,J,D} \to \mathfrak{Z}^{[\mathbf{w}]}_D, (B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_0, U_{B_0} n_1 n_2 \dots n_r n U_{B_0})$$

where $h_i \in G^0 (i \in [1, r])$ are such that $B_i = h_i B_0 h_i^{-1}$, $h_0 = 1$, T_0 is a maximal torus of B_0, $n_i \in N_{G^0} T_0 (i \in [1, r])$ are given by $h_{i-1}^{-1} h_i \in U_{B_0} n_i U_{B_0}$ and $n \in N_D B_0 \cap N_D T_0$ is given by $h_r^{-1} g \in U_{B_0} n$.

This is independent of the choices. (Another choice for h_i, g, T_0 must be of the form $h'_i = h_i u_i t_i, g' = gu', T' = uT_0u^{-1}$ where $u_i \in U_{B_0} (i \in [1, r])$, $t_i \in T_0 (i \in [1, r])$, $u \in U_{B_0}$, $u' \in U_{J,B_0}$. Define n'_i, n' in terms of this new choice in the same way as n_i, n were defined in terms of the original choice. We have $n'_i = ut_{i-1}^{-1} n_i t_i u^{-1}$ where $t_0 = 1$ and $n' = ut_r^{-1} n u^{-1}$. Hence $n'_1 n'_2 \dots n'_r n' = un_1 n_2 \dots n_r n u^{-1}$ and

$$U_{B_0} n'_1 n'_2 \dots n'_r n' U_{B_0} = U_{B_0} n_1 n_2 \dots n_r n U_{B_0},$$

as required.) One checks that ζ is an affine space bundle. Hence $Z^{\mathbf{w}}_{\emptyset,J,D}$ is smooth, connected.

Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ be such that $[\mathbf{w}]\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. The inverse image under ζ of the local system $\tilde{\mathcal{L}}$ on $\mathfrak{Z}^{[\mathbf{w}]}_D$ is a local system on $Z^{\mathbf{w}}_{\emptyset,J,D}$ denoted again by $\tilde{\mathcal{L}}$.

When $\mathbf{w}$ has a single term w, we have $Z^{\mathbf{w}}_{\emptyset,J,D} = Z^w_{\emptyset,J,D}$ and $\tilde{\mathcal{L}}$ defined above is the same as $\tilde{\mathcal{L}}$ defined in 28.7.

28.9. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I} \cup \{1\}$ and let

$$\begin{aligned} \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} =& \{(B_0, B_1, \dots, B_r, gU_{J,B_0}); B_i \in \mathcal{B}(i \in [0, r]), g \in D, gB_0g^{-1} = B_r, \\ & \mathrm{pos}(B_{i-1}, B_i) = 1 \text{ or } s_i (i \in [1, r])\}. \end{aligned}$$

Let

$$\mathcal{J}^0 = \{j \in [1, r]; s_j \in \mathbf{I}\}.$$

For any subset $\mathcal{J} \subset \mathcal{J}^0$ we consider the sequence $\mathbf{s}_{\mathcal{J}} = (s'_1, s'_2, \dots, s'_r)$ in $\mathbf{I} \cup \{1\}$ given by $s'_i = s_i$ if $i \notin \mathcal{J}$ and $s'_i = 1$ if $i \in \mathcal{J}$; let $[\mathbf{s}_{\mathcal{J}}] = s'_1 s'_2 \dots s'_r$. Then $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$ (see 28.8) is the locally closed subvariety of $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ defined by the conditions

$$B_{i-1} = B_i \text{ if } i \in \mathcal{J}, \mathrm{pos}(B_{i-1}, B_i) = s_i \text{ if } i \notin \mathcal{J}.$$

The sets $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}(\mathcal{J}\subset\mathcal{J}^0)$ form a partition of $\bar Z^{\mathbf{s}}_{\emptyset,J,D}$. We have $\mathbf{s}_\emptyset=\mathbf{s}$ and the corresponding piece $Z^{\mathbf{s}_\emptyset}_{\emptyset,J,D}=Z^{\mathbf{s}}_{\emptyset,J,D}$ is open dense in $\bar Z^{\mathbf{s}}_{\emptyset,J,D}$. Let $\mathcal{L}\in\mathfrak{s}(\mathbf{T})$ be such that $[\mathbf{s}]\underline{D}\in\mathbf{W}^\bullet_{\mathcal{L}}$. Let $\tilde{\mathcal{L}}$ be the local system on $Z^{\mathbf{s}}_{\emptyset,J,D}$ defined as in 28.8. Let

$$\bar{\mathcal{L}}=IC(\bar Z^{\mathbf{s}}_{\emptyset,J,D},\tilde{\mathcal{L}})\in\mathcal{D}(\bar Z^{\mathbf{s}}_{\emptyset,J,D}),$$

$$\begin{aligned}\mathcal{J}_{\mathbf{s}}&=\{j\in\mathcal{J}^0;s_rs_{r-1}\dots s_j\dots s_{r-1}s_r\in\epsilon_D(\mathbf{W}_{\mathcal{L}})\}\\&=\{j\in\mathcal{J}^0;s_1s_2\dots s_j\dots s_2s_1\in\mathbf{W}_{\mathcal{L}}\}.\end{aligned}$$

Lemma 28.10. *$\bar{\mathcal{L}}$ is a constructible sheaf on $\bar Z^{\mathbf{s}}_{\emptyset,J,D}$, which is a local system on the open subset $\bigcup_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}}Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$ of $\bar Z^{\mathbf{s}}_{\emptyset,J,D}$ and is 0 on its complement. For any $\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}$, we have $\bar{\mathcal{L}}|_{Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}}\cong\tilde{\mathcal{L}}$ (defined as in 28.8 in terms of $\mathbf{s}_{\mathcal{J}}$).*

For this to make sense, we must verify that, if $\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}$, then $[\mathbf{s}_{\mathcal{J}}]\underline{D}\in\mathbf{W}^\bullet_{\mathcal{L}}$. This follows from $[\mathbf{s}]\underline{D}\in\mathbf{W}^\bullet_{\mathcal{L}}$ and $s_1s_2\dots s_j\dots s_2s_1\in\mathbf{W}^\bullet_{\mathcal{L}}$ for all $j\in\mathcal{J}$. (See 28.3(b).)

Let $B^*,U^*,T,\dot s_j,d$ be as in 28.5. Let

$$\begin{aligned}\tilde Z^{\mathbf{s}}=&\{(h_0,h_1,\dots,h_r,g)\in G^0\times\dots\times G^0\times D;\\&h_{i-1}^{-1}h_i\in B^*\dot s_iB^*\cup B^*(i\in[1,r]),h_r^{-1}gh_0\in N_GB^*\}.\end{aligned}$$

The map $\tilde Z^{\mathbf{s}}\to\bar Z^{\mathbf{s}}_{\emptyset,J,D}$,

(a) $\quad(h_0,h_1,\dots,h_r,g)\mapsto(h_0B^*h_0^{-1},h_1B^*h_1^{-1},\dots,h_rB^*h_r^{-1},gU_{J,h_0B^*h_0^{-1}})$

is a locally trivial fibration with connected, smooth fibres. For $\mathcal{J}\subset\mathcal{J}^0$, the inverse image under (a) of the subvariety $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$ of $\bar Z^{\mathbf{s}}_{\emptyset,J,D}$ is the subvariety $Z'^{\mathbf{s}_{\mathcal{J}}}$ of $\tilde Z^{\mathbf{s}}$ defined by the conditions

$$h_{i-1}^{-1}h_i\in B^*\dot s_iB^*(i\in[1,r]-\mathcal{J}),h_{i-1}^{-1}h_i\in B^*(i\in\mathcal{J}).$$

It suffices to prove the statement analogous to that in the lemma for the inverse image under (a) of $\bar{\mathcal{L}}$. For $\mathcal{J}\subset\mathcal{J}^0$, define $\psi_{\mathcal{J}}:Z'^{\mathbf{s}_{\mathcal{J}}}\to T$ by

$$(h_0,h_1,\dots,h_r,g)\mapsto d^{-1}\dot{\mathbf{s}}_{\mathcal{J}}^{-1}n_1n_2\dots n_rn$$

where $n_i\in N_{G^0}T$ are given by $h_{i-1}^{-1}h_i\in U^*n_iU^*$ and $n\in N_GB^*\cap N_GT$ is given by $h_r^{-1}gh_0\in U^*n$. (We write $\dot{\mathbf{s}}_{\mathcal{J}}=\dot s'_1\dot s'_2\dots\dot s'_r$ for $\mathbf{s}_{\mathcal{J}}=(s'_1,s'_2,\dots,s'_r)$.) It suffices to prove the following statements:

$IC(\tilde Z^{\mathbf{s}},\psi_\emptyset^*\mathcal{L})$ is a local system on the open subset $\bigcup_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}}Z'^{\mathbf{s}_{\mathcal{J}}}$ of $\tilde Z^{\mathbf{s}}$ and is 0 on its complement.

For any $\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}$, we have $IC(\tilde Z^{\mathbf{s}},\psi_\emptyset^*\mathcal{L})|_{Z'^{\mathbf{s}_{\mathcal{J}}}}\cong\psi_{\mathcal{J}}^*\mathcal{L}$.

By the change of variables

$$utd=h_r^{-1}gh_0,y_i=h_{i-1}^{-1}h_i(i\in[1,r-1]),y_r=h_{r-1}^{-1}h_rt,$$

$\tilde Z^{\mathbf{s}}$ becomes

$$\{(h_0,y_1,\dots,y_r,u,t)\in G^0\times\dots\times G^0\times U^*\times T;y_i\in B^*\dot s_iB^*\cup B^*(i\in[1,r])\}$$

and for $\mathcal{J}\subset\mathcal{J}^0$, $Z'^{\mathbf{s}_{\mathcal{J}}}$ becomes the subset of $\tilde Z^{\mathbf{s}}$ defined by the conditions

$$y_i\in B^*\dot s_iB^*(i\in[1,r]-\mathcal{J}),y_i\in B^*(i\in\mathcal{J}).$$

Moreover, $\psi_{\mathcal{J}}$ becomes

$$(h_0,y_1,\dots,y_r,u,t)\mapsto d^{-1}\dot{\mathbf{s}}_{\mathcal{J}}^{-1}n_1n_2\dots n_rd$$

where $n_i \in N_{G^0}T$ are given by $y_i \in U^*n_iU^*$. Since h_0, u, t now play passive roles, we can omit them. Thus, we set

$$'\bar{Z}^{\mathbf{s}} = \{(y_1, \dots, y_r) \in (G^0)^r; y_i \in B^*\dot{s}_iB^* \cup B^*(i \in [1, r])\}$$

and, for $\mathcal{J} \subset \mathcal{J}^0$, we set

$$'Z^{\mathbf{s}_\mathcal{J}} = \{(y_1, \dots, y_r) \in (G^0)^r; y_i \in B^*\dot{s}_iB^*(i \in [1, r] - \mathcal{J}), y_i \in B^*(i \in \mathcal{J}).$$

Let

$$'\psi_\mathcal{J} : 'Z^{\mathbf{s}_\mathcal{J}} \to T, (y_1, \dots, y_r) \mapsto \dot{\mathbf{s}}_\mathcal{J}^{-1} n_1 n_2 \dots n_r$$

where $n_i \in N_{G^0}T$ are given by $y_i \in U^*n_iU^*$. Let $\mathcal{L}' = (\underline{D}^{-1})^*\mathcal{L} \in \mathfrak{s}(\mathbf{T}) = \mathfrak{s}(T)$. It suffices to prove the following statements:

(b) *$IC('\bar{Z}^{\mathbf{s}}, '\psi_\emptyset^*\mathcal{L}')$ is a local system on the open subset $\bigcup_{\mathcal{J} \subset \mathcal{J}_\mathbf{s}} 'Z^{\mathbf{s}_\mathcal{J}}$ of $'\bar{Z}^{\mathbf{s}}$ and is 0 on its complement.*

(c) *for any $\mathcal{J} \subset \mathcal{J}_\mathbf{s}$, we have $IC('\bar{Z}^{\mathbf{s}}, '\psi_\emptyset^*\mathcal{L}')|_{'Z^{\mathbf{s}_\mathcal{J}}} \cong '\psi_\mathcal{J}^*\mathcal{L}'$.*

For $j \in \mathcal{J}^0$ let Δ_j be the closure of the subvariety $\Delta_j^0 = 'Z^{\mathbf{s}_{\{j\}}}$ of $'\bar{Z}^{\mathbf{s}}$. Clearly, $\{\Delta_j, j \in \mathcal{J}^0\}$ are smooth divisors with normal crossing in the smooth variety $'\bar{Z}^{\mathbf{s}}$. Using [L3, I, 1.6], we see that to prove (b) it suffices to prove the following statement.

(b′) *For $j \in \mathcal{J}^0$, the monodromy of $'\psi_\emptyset^*\mathcal{L}'$ around the divisor Δ_j is trivial if and only if $j \in \mathcal{J}_\mathbf{s}$.*

Let U_j be the root subgroup of U^* with respect to T such that $\dot{s}_jU_j\dot{s}_j^{-1} \cap U^* = \{1\}$ and let $x : \mathbf{k} \xrightarrow{\sim} U_j$ be an isomorphism. Let α be the root of G^0 with respect to T such that $tx(a)t^{-1} = x(\alpha(t)a)$ for all $t \in T$, $a \in \mathbf{k}$. For $a \in \mathbf{k}$ we set $x'(a) = (\dot{s}_1, \dots, \dot{s}_{j-1}, \dot{s}_jx(a)\dot{s}_j^{-1}, \dot{s}_{j+1}, \dots, \dot{s}_r) \in Z$. Then $x' : \mathbf{k}^* \to Z$ is a cross section to Δ_j^0 in Z; we have $x'(0) \in \Delta_j^0$, $x'(a) \in Z - \Delta_j^0$ for $a \neq 0$. For $a \neq 0$ we have $\dot{s}_jx(a)\dot{s}_j^{-1} \in U^*\dot{s}_j\check{\alpha}(a)U^*$. Hence

$$'\psi_\emptyset(x'(a)) = \dot{\mathbf{s}}^{-1}\dot{s}_1 \dots \dot{s}_{j-1}\dot{s}_j\check{\alpha}(a)\dot{s}_{j+1} \dots \dot{s}_r = \dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1}\check{\alpha}(a)\dot{s}_{j+1} \dots \dot{s}_r = t_0\check{\beta}(a)$$

where β is the root of G^0 with respect to T that corresponds to the reflection $s_r \dots s_{j+1}s_js_{j+1} \dots s_r$ and t_0 is a fixed element of T. We see that $x'^*{}'\psi_\emptyset^*\mathcal{L}' \cong \check{\beta}^*\mathcal{L}'$. Thus, $x'^*{}'\psi_\emptyset^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$ if and only if $\check{\beta}^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$, that is, if $s_r \dots s_{j+1}s_js_{j+1} \dots s_r \in \epsilon_D(\mathbf{W}_\mathcal{L})$, that is, if $j \in \mathcal{J}_\mathbf{s}$. This proves (b′).

We prove (c). Let $\underline{\mathcal{L}'}$ be the local system $IC('\bar{Z}^{\mathbf{s}}, '\psi_\emptyset^*\mathcal{L}')|_{\bigcup_{\mathcal{J}' \subset \mathcal{J}_\mathbf{s}} 'Z^{\mathbf{s}_{\mathcal{J}'}}}$. Assume that (c) is known in the case where $\mathcal{J}$ consists of one element. We now consider a general $\mathcal{J} \subset \mathcal{J}_\mathbf{s}$. We argue by induction on the number of elements of $\mathcal{J}$. If $\mathcal{J} = \emptyset$, (c) is obvious. Assume now that $\mathcal{J} \neq \emptyset$. We pick $j \in \mathcal{J}$. Let $\mathcal{J}' = \mathcal{J} - \{j\}$. Since $'Z^{\mathbf{s}_{\mathcal{J}'}}$ is open dense in the smooth irreducible variety $'Z := 'Z^{\mathbf{s}_{\mathcal{J}'}} \cup 'Z^{\mathbf{s}_\mathcal{J}}$, there is (up to isomorphism) at most one local system on $'Z$ whose restriction to $'Z^{\mathbf{s}_{\mathcal{J}'}}$ is isomorphic to $'\psi_{\mathcal{J}'}^*\mathcal{L}'$. By our assumption (applied to $\mathbf{s}_{\mathcal{J}'}$ instead of $\mathbf{s}$), a local system as in the previous sentence exists (we denote it by $\mathcal{E}$) and its restriction to $'Z^{\mathbf{s}_\mathcal{J}}$ is isomorphic to $'\psi_\mathcal{J}^*\mathcal{L}'$.

By the induction hypothesis we have $\underline{\mathcal{L}'}|_{'Z^{\mathbf{s}_{\mathcal{J}'}}} \cong '\psi_{\mathcal{J}'}^*\mathcal{L}'$. Thus $\underline{\mathcal{L}'}|_{'Z}$ is a local system whose restriction to $'Z^{\mathbf{s}_{\mathcal{J}'}}$ is isomorphic to $'\psi_{\mathcal{J}'}^*\mathcal{L}'$. Hence $\underline{\mathcal{L}'}|_{'Z} \cong \mathcal{E}$. It follows that the restriction of $\underline{\mathcal{L}'}|_{'Z}$ to $'Z^{\mathbf{s}_\mathcal{J}}$ is isomorphic to $'\psi_\mathcal{J}^*\mathcal{L}'$. Thus, $\underline{\mathcal{L}'}|_{'Z^{\mathbf{s}_\mathcal{J}}} \cong '\psi_\mathcal{J}^*\mathcal{L}'$. Thus (c) holds for $\mathcal{J}$.

We see that it is enough to prove (c) in the case where $\mathcal{J}$ has exactly one element. It suffices to show that

(c′) *if $\mathcal{L}' \in \mathfrak{s}(\mathbf{T}) = \mathfrak{s}(T)$ and $j \in [1, r]$ satisfies $s_rs_{r-1} \dots s_j \dots s_{r-1}s_r \in \mathbf{W}_{\mathcal{L}'}$, then there exists a local system $\mathcal{F}$ (necessarily unique up to isomorphism) on $'Z =$*

${}'Z^{\mathbf{s}_\emptyset} \cup {}'Z^{\mathbf{s}_{\{j\}}}$ *such that* $\mathcal{F}|_{'Z^{\mathbf{s}_\emptyset}} \cong {}'\psi_\emptyset^* \mathcal{L}'$ *and* $\mathcal{F}|_{'Z^{\mathbf{s}_{\{j\}}}} \cong {}'\psi_{\{j\}}^* \mathcal{L}'$.

This statement involves only the component G^0 of G. Hence to prove it, we may assume that $G = G^0$. Let $\tilde{G} \to G$ be a surjective homomorphism of connected reductive groups whose kernel is a central torus in $\tilde{G}$ and such that $\tilde{G}$ has simply connected derived group. The desired statement for G follows from the analogous statement for $\tilde{G}$. Thus, we may assume that $G = G^0$ has simply connected derived group. We may assume that $\mathcal{L}' = \kappa^* \mathcal{E}$ where $\kappa \in \mathrm{Hom}(T, \mathbf{k}^*)$, $\mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$. Let m be the order of $\mathcal{E}$ in $\underline{\mathfrak{s}}(\mathbf{k}^*)$. By assumption, we have $\check{\beta}^* \mathcal{L}' \cong \bar{\mathbf{Q}}_l$ hence $\langle \check{\beta}, \kappa \rangle = mm_1$ with $m_1 \in \mathbf{Z}$. Since G has simply connected derived group, we can find $\kappa_1 \in \mathrm{Hom}(T, \mathbf{k}^*)$ such that $\langle \check{\beta}, \kappa_1 \rangle = m_1$. Then $\langle \check{\beta}, \kappa\kappa_1^{-m} \rangle = 0$. We have $(\kappa\kappa_1^{-m})^* \mathcal{E} \cong \kappa^* \mathcal{E}$. Hence replacing κ by $\kappa\kappa_1^{-m}$, we may assume that $\langle \check{\beta}, \kappa \rangle = 0$. Then there is a unique homomorphism of algebraic groups $\chi : B^* \dot{s}_j B^* \cup B^* \to \mathbf{k}^*$ such that

$$\chi(t) = \kappa(\dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1} t \dot{s}_{j+1} \dots \dot{s}_r) \text{ for all } t \in T.$$

Since $\dot{s}_j$ is in the derived subgroup of $B^* \dot{s}_j B^* \cup B^*$, we have $\chi(\dot{s}_j) = 1$. Define a morphism $\tilde{f} : {}'Z \to \mathbf{k}^*$ by

$$\begin{aligned}\tilde{f}(y_1, \dots, y_r) &= \chi(\dot{s}_j^{-1} \dot{s}_{j-1}^{-1} \dots \dot{s}_1^{-1} n_1 n_2 \dots n_{j-1} y_j n_{j+1} \dots n_r \dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1}) \\ &= \chi(\dot{s}_j^{-1} \dot{s}_{j-1}^{-1} \dots \dot{s}_1^{-1} n_1 n_2 \dots n_{j-1} n_j n_{j+1} \dots n_r \dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1}),\end{aligned}$$

where $n_i \in N_G T$ are given by $y_i \in U^* n_i U^*$. If $y_j \in B^* \dot{s}_j B^*$, we have

$$\begin{aligned}\tilde{f}(y_1, \dots, y_r) &= \kappa(\dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1} \dot{s}_j^{-1} \dot{s}_{j-1}^{-1} \dots \dot{s}_1^{-1} n_1 n_2 \dots n_{j-1} n_j n_{j+1} \dots n_r) \\ &= \kappa({}'\psi_\emptyset(y_1, \dots, y_r)).\end{aligned}$$

If $y_j \in B^*$, we have

$$\begin{aligned}\tilde{f}(y_1, \dots, y_r) &= \chi(\dot{s}_{j-1}^{-1} \dots \dot{s}_1^{-1} n_1 n_2 \dots n_{j-1} n_j n_{j+1} \dots n_r \dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1}) \\ &= \kappa(\dot{s}_r^{-1} \dots \dot{s}_{j+1}^{-1} \dot{s}_{j-1}^{-1} \dots \dot{s}_1^{-1} n_1 n_2 \dots n_{j-1} n_j n_{j+1} \dots n_r) = \kappa({}'\psi_{\{j\}}(y_1, \dots, y_r)).\end{aligned}$$

Hence the local system $\mathcal{F} = \tilde{f}^*(\mathcal{E})$ on ${}'Z$ has the required properties. The lemma is proved.

Lemma 28.11. *In the setup of 28.9 assume that* $r \geq 2$, *that* $j \in [2, r]$ *and* $s_{j-1} = s_j \in \mathbf{I}$. *Let* Z_1 *be the open subset of* $Z_{\emptyset,J,D}^{\mathbf{s}}$ *defined by* $\mathrm{pos}(B_{j-2}, B_j) = s_j$. *Let* $\mathbf{s}' = (s_1, s_2, \dots, s_{j-1}, s_{j+1}, \dots, s_r)$. *Define* $\delta : Z_1 \to Z_{\emptyset,J,D}^{\mathbf{s}'}$ *by*

$$(B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_{j-2}, B_j, B_{j+1}, \dots, B_r, gU_{J,B_0}).$$

Let $\tilde{\mathcal{L}}$ *be the local system on* $Z_{\emptyset,J,D}^{\mathbf{s}}$ *associated to* $\mathcal{L}$ *as in 28.8; in the case where* $j \in \mathcal{J}_{\mathbf{s}}$, *let* $\tilde{\mathcal{L}}'$ *be the analogous local system on* $Z_{\emptyset,J,D}^{\mathbf{s}'}$ *associated to* $\mathcal{L}$. *Let* $\tilde{\mathcal{L}}_1$ *be the restriction of* $\tilde{\mathcal{L}}$ *to* Z_1. *If* $j \in \mathcal{J}_{\mathbf{s}}$, *then* $\tilde{\mathcal{L}}_1 \cong \delta^* \tilde{\mathcal{L}}'$. *If* $j \notin \mathcal{J}_{\mathbf{s}}$, *then* $\delta_! \tilde{\mathcal{L}}_1 = 0$.

Consider the union $Z_{\emptyset,J,D}^{\mathbf{s}_\emptyset} \cup Z_{\emptyset,J,D}^{\mathbf{s}_{\{j\}}}$ inside $\bar{Z}_{\emptyset,J,D}^{\mathbf{s}}$. Recall that we may identify $Z_{\emptyset,J,D}^{\mathbf{s}_\emptyset} = Z_{\emptyset,J,D}^{\mathbf{s}}$, $Z_{\emptyset,J,D}^{\mathbf{s}_{\{j\}}} = Z_{\emptyset,J,D}^{\mathbf{s}'}$. For

$$(B_0, B_1, \dots, B_{j-2}, B_j, B_{j+1}, \dots, B_r, gU_{J,B_0}) \in Z_{\emptyset,J,D}^{\mathbf{s}'},$$

$$\begin{aligned}F = \{&(B_0, B_1, \dots, B_{j-2}, B, B_j, B_{j+1}, \dots, B_r, gU_{J,B_0}); \mathrm{pos}(B_{j-2}, B) = s_j, \\ &\mathrm{pos}(B, B_j) = 1 \text{ or } s_j\}\end{aligned}$$

is a cross section to $Z_{\emptyset,J,D}^{\mathbf{s}_{\{j\}}}$ in $\bar{Z}_{\emptyset,J,D}^{\mathbf{s}}$ which intersects $Z_{\emptyset,J,D}^{\mathbf{s}_{\{j\}}}$ in the point π defined by $B = B_j$. If $j \notin \mathcal{J}_{\mathbf{s}}$, then the proof of Lemma 28.10 shows that the restriction of $\tilde{\mathcal{L}}$ to $F - \{\pi\} \cong \mathbf{k}^*$ is a local system in $\mathfrak{s}(\mathbf{k}^*)$ not isomorphic to $\bar{\mathbf{Q}}_l$. Hence $H^i_c(F - \{\pi\}, \tilde{\mathcal{L}}) = 0$ for all i. Now $F - \{\pi\}$ is the fibre of δ at $(B_0, B_1, \dots, B_{j-2}, B_j, B_{j+1}, \dots, B_r, gU_{J,B_0})$. We see that the cohomology with compact support of any fibre of δ with coefficients in $\tilde{\mathcal{L}}$ is 0. Thus, $\delta_!\tilde{\mathcal{L}} = 0$ as required.

Assuming that $j \in \mathcal{J}_{\mathbf{s}}$, the same argument shows that the restriction of $\tilde{\mathcal{L}}_1$ to any fibre of δ is $\bar{\mathbf{Q}}_l$. Hence $\tilde{\mathcal{L}} = \delta^*\mathcal{E}$ where $\mathcal{E}$ is a local system of rank 1 on $Z^{\mathbf{s}'}_{\emptyset,J,D}$. The proof of Lemma 28.10 shows that there exists a local system $\mathcal{F}$ on $Z^{\mathbf{s}}_{\emptyset,J,D} \cup Z^{\mathbf{s}'}_{\emptyset,J,D}$ such that $\mathcal{F}|_{Z^{\mathbf{s}}_{\emptyset,J,D}} = \tilde{\mathcal{L}}$ and $\mathcal{F}|_{Z^{\mathbf{s}'}_{\emptyset,J,D}} = \tilde{\mathcal{L}}'$. Let V be the open subset of $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ defined by $\mathrm{pos}(B_{i-1}, B_i) = s_i$ if $i \in [1, r], i \neq j$ and $\mathrm{pos}(B_{j-2}, B_j) = s_j$. We have $V = Z_1 \cup Z^{\mathbf{s}'}_{\emptyset,J,D}$. Then $\mathcal{F}^1 := \mathcal{F}|_V$ is a local system on V such that $\mathcal{F}^1|_{Z_1} = \tilde{\mathcal{L}}_1$ and $\mathcal{F}^1|_{Z^{\mathbf{s}'}_{\emptyset,J,D}} = \tilde{\mathcal{L}}'$. Define $\tilde{\delta} : V \to Z^{\mathbf{s}'}_{\emptyset,J,D}$ by the same formula as δ. Then $\mathcal{F}^1, \tilde{\delta}^*\mathcal{E}$ are local systems on V with the same restriction $\tilde{\mathcal{L}}_1$ on the open dense subset Z_1 of V. Hence $\mathcal{F}^1 \cong \tilde{\delta}^*\mathcal{E}$. Since $\mathcal{F}^1|_{Z^{\mathbf{s}'}_{\emptyset,J,D}} = \tilde{\mathcal{L}}'$, $\tilde{\delta}^*\mathcal{E}|_{Z^{\mathbf{s}'}_{\emptyset,J,D}} = \mathcal{E}$, we see that $\tilde{\mathcal{L}}' \cong \mathcal{E}$. Thus, $\tilde{\mathcal{L}} \cong \delta^*\tilde{\mathcal{L}}'$. The lemma is proved.

28.12. Let $\epsilon_D, Z_{J,D}$ be as in 26.2. Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$. For $w \in \mathbf{W}$ we define

$$\pi : Z^w_{\emptyset,J,D} \to Z_{J,D}, (B, B', gU_{J,B}) \mapsto (Q_{J,B}, Q_{\epsilon_D(J),B'}, gU_{J,B}).$$

If w satisfies $w\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, we set $K^{w,\mathcal{L}}_{J,D} = \pi_!\tilde{\mathcal{L}} \in \mathcal{D}(Z_{J,D})$ where $\tilde{\mathcal{L}}$ is the local system on $Z^w_{\emptyset,J,D}$ defined in 28.7.

For a sequence $\mathbf{w} = (w_1, w_2, \dots, w_r)$ in $\mathbf{W}$ we define $\pi_{\mathbf{w}} : Z^{\mathbf{w}}_{\emptyset,J,D} \to Z_{J,D}$ by

(a) $$(B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (Q_{J,B_0}, Q_{\epsilon_D(J),B_r}, gU_{J,B_0}).$$

If $\mathbf{w}$ satisfies $w_1w_2\dots w_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, we set $K^{\mathbf{w},\mathcal{L}}_{J,D} = \pi_{\mathbf{w}!}\tilde{\mathcal{L}} \in \mathcal{D}(Z_{J,D})$ where $\tilde{\mathcal{L}}$ is the local system on $Z^{\mathbf{w}}_{\emptyset,J,D}$ defined in 28.8.

For a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I} \cup \{1\}$ we define $\bar{\pi}_{\mathbf{s}} : \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \to Z_{J,D}$ by (a). If $\mathbf{s}$ satisfies $s_1s_2\dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, we set $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} = \bar{\pi}_{\mathbf{s}!}\bar{\mathcal{L}} \in \mathcal{D}(Z_{J,D})$ where $\bar{\mathcal{L}}$ is as in 28.9. Then

(b) $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} \in \mathcal{D}(Z_{J,D})$ *is a semisimple complex.*

This follows by applying the decomposition theorem [BBD, 5.4.5, 5.3.8] to the proper map $\bar{\pi}$.

Proposition 28.13. *Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ and let A be a simple perverse sheaf on $Z_{J,D}$. The following conditions on A are equivalent:*

(i) $A \dashv K^{w,\mathcal{L}}_{J,D}$ for some $w \in \mathbf{W}$ such that $w\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$;

(ii) $A \dashv K^{\mathbf{w},\mathcal{L}}_{J,D}$ for some $\mathbf{w} = (w_1, w_2, \dots, w_r)$ with $w_i \in \mathbf{W}$, $w_1w_2\dots w_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$;

(iii) $A \dashv K^{\mathbf{s},\mathcal{L}}_{J,D}$ for some $\mathbf{s} = (s_1, s_2, \dots, s_r)$ with $s_i \in \mathbf{I} \cup \{1\}$, $s_1s_2\dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$;

(iv) $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ for some $\mathbf{s} = (s_1, s_2, \dots, s_r)$ with $s_i \in \mathbf{I} \cup \{1\}$, $s_1s_2\dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$;

(v) $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ for some $\mathbf{s} = (s_1, s_2, \dots, s_r)$ with $s_i \in \mathbf{I}$, $s_1s_2\dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$.

If in (ii), $\mathbf{w}$ reduces to a single element w, then $K^{\mathbf{w},\mathcal{L}}_{J,D} = K^{w,\mathcal{L}}_{J,D}$. Thus, (i) $\implies$ (ii). The implication (iii) $\implies$ (ii) is trivial. We now prove that (ii) $\implies$ (iii). Let $\mathbf{w} = (w_1, w_2, \dots, w_r)$ be a sequence in $\mathbf{W}$ such that $w_1w_2\dots w_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, and for some

$i \in [1,r]$, let w'_i, w''_i be elements of $\mathbf{W}$ such that $w_i = w'_i w''_i$, $l(w_i) = l(w'_i) + l(w''_i)$. Let $\tilde{\mathbf{w}} = (w_1, \dots, w_{i-1}, w'_i, w''_i, w_{i+1}, \dots, w_r)$. The map

$$(B_0, B_1, \dots, B_{r+1}, gU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_{r+1}, gU_{J,B_0})$$

defines an isomorphism $Z^{\tilde{\mathbf{w}}}_{J,D} \xrightarrow{\sim} Z^{\mathbf{w}}_{J,D}$ compatible with the maps $\pi_{\tilde{\mathbf{w}}}, \pi_{\mathbf{w}}$ and with the local systems $\tilde{\mathcal{L}}$ defined on $Z^{\tilde{\mathbf{w}}}_{J,D}, Z^{\mathbf{w}}_{J,D}$ in terms of $\mathcal{L}$ as in 28.8. Hence

(a) $$K^{\tilde{\mathbf{w}},\mathcal{L}}_{J,D} = K^{\mathbf{w},\mathcal{L}}_{J,D}.$$

Applying (a) repeatedly we see that $K^{\mathbf{w},\mathcal{L}}_{J,D}$ is equal to $K^{\mathbf{w}',\mathcal{L}}_{J,D}$ for some sequence $\mathbf{w}'$ in $\mathbf{I}$. Thus, (ii) $\implies$ (iii).

We prove the equivalence of (iii) and (iv). Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I} \cup \{1\}$ such that $s_1 s_2 \dots s_r \underline{D} \in \mathbf{W}^{\bullet}_{\mathcal{L}}$. Let $\bar{\mathcal{L}}$ be as in 28.9.

Define a sequence ${}^0Z \supset {}^1Z \supset \dots$ of closed subsets of $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ by

$${}^iZ = \bigcup_{\mathcal{J} \subset \mathcal{J}^0; |\mathcal{J}| \ge i} Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}.$$

Let $f^i : {}^iZ \to \bar{Z}^{\mathbf{w}}_{\emptyset,J,D}, f'^i : {}^iZ - {}^{i+1}Z \to \bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ be the inclusions. The natural distinguished triangle in $\mathcal{D}(Z_{J,D})$

$$(\bar{\pi}_{\mathbf{s}!} f'^i_!(f'^i)^* \bar{\mathcal{L}}, \bar{\pi}_{\mathbf{s}!} f^i_!(f^i)^* \bar{\mathcal{L}}, \bar{\pi}_{\mathbf{s}!} f^{i+1}_!(f^{i+1})^* \bar{\mathcal{L}})$$

gives rise for any $i \ge 0$ to a long exact sequence in $\mathcal{M}(Z_{J,D})$:

$$\dots \to {}^pH^{j-1}(\bar{\pi}_{\mathbf{s}!} f^{i+1}_!(f^{i+1})^* \bar{\mathcal{L}}) \to \bigoplus_{\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}; |\mathcal{J}| = i} {}^pH^j(K^{\mathbf{s}_{\mathcal{J}},\mathcal{L}}_{J,D})$$

(b)

$$\to {}^pH^j(\bar{\pi}_{\mathbf{s}!} f^i_!(f^i)^* \bar{\mathcal{L}}) \to {}^pH^j(\bar{\pi}_{\mathbf{s}!} f^{i+1}_!(f^{i+1})^* \bar{\mathcal{L}}) \to \bigoplus_{\substack{\mathcal{J} \subset \mathcal{J}_{\mathbf{s}} \\ |\mathcal{J}| = i}} {}^pH^{j+1}(K^{\mathbf{s}_{\mathcal{J}},\mathcal{L}}_{J,D}) \to \dots.$$

Here we have used the isomorphism $\bar{\pi}_{\mathbf{s}!} f'^i_!(f'^i)^* \bar{\mathcal{L}} = \bigoplus_{\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}; |\mathcal{J}| = i} K^{\mathbf{s}_{\mathcal{J}},\mathcal{L}}_{J,D}$ which follows from Lemma 28.10. Note that

($*$) $\bar{\pi}_{\mathbf{s}!} f^0_!(f^0)^* \bar{\mathcal{L}} = \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}, \bar{\pi}_{\mathbf{s}!} f^i_!(f^i)^* \bar{\mathcal{L}} = 0$ for i large.

We set $m(\mathbf{s}) = \sharp(i \in [1,r]; s_i \in \mathbf{I})$. If $m(\mathbf{s}) = 0$, then $Z^{\mathbf{s}}_{\emptyset,J,D} = \bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ and $K^{\mathbf{s},\mathcal{L}}_{J,D} = \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$. Hence in this case we have $A \dashv K^{\mathbf{s},\mathcal{L}}_{J,D}$ if and only if $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$. It suffices to verify the following statement.

(c) *Assume that* $\mathbf{s}$ *satisfies* $m(\mathbf{s}) = m \ge 1$ *and that for any sequence* $\mathbf{s}' = (s'_1, s'_2, \dots, s'_r)$ *in* $\mathbf{I} \cup \{1\}$ *with* $s'_1 s'_2 \dots s'_r \underline{D} \in \mathbf{W}^{\bullet}_{\mathcal{L}}$ *and with* $m(\mathbf{s}') < m$ *we have* $A \not\dashv K^{\mathbf{s}',\mathcal{L}}_{J,D}$. *Then* $A \dashv K^{\mathbf{s},\mathcal{L}}_{J,D}$ *if and only if* $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$.

Using (b) and our hypothesis we see that for any $i > 0$ we have $A \dashv \bar{\pi}_{\mathbf{s}!} f^i_!(f^i)^* \bar{\mathcal{L}}$ if and only if $A \dashv \bar{\pi}_{\mathbf{s}!} f^{i+1}_!(f^{i+1})^* \bar{\mathcal{L}}$. Applying this repeatedly for $i = N, N-1, \dots, 1$ (with N large) we see that $A \not\dashv \bar{\pi}_{\mathbf{s}!} f^1_!(f^1)^* \bar{\mathcal{L}}$. Using this, together with (b) we see that $A \dashv K^{\mathbf{s},\mathcal{L}}_{J,D}$ if and only if $A \dashv \bar{\pi}_{\mathbf{s}!} f^0_!(f^0)^* \bar{\mathcal{L}}$, that is, $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ (see ($*$)). This proves (c). The equivalence of (iii) and (iv) is established.

The equivalence of (iv) and (v) is obvious.

Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D} \in \mathbf{W}^{\bullet}_{\mathcal{L}}$. Assume that $r \ge 2$ and that, for some $j \in [2,r]$ we have $s_{j-1} = s_j$. We have a partition $Z^{\mathbf{s}}_{\emptyset,J,D} = Z_1 \cup Z_2$ where Z_1 (resp. Z_2) is the open (resp. closed) subset of $Z^{\mathbf{s}}_{\emptyset,J,D}$ defined by $\mathrm{pos}(B_{j-2}, B_j) = s_j$ (resp. by $B_{j-2} = B_j$). Let π_1, π_2 be the restrictions of $\pi_{\mathbf{s}}$ to Z_1, Z_2. The natural distinguished triangle $(\pi_{1!}\tilde{\mathcal{L}}, K^{\mathbf{s},\mathcal{L}}_{J,D}, \pi_{2!}\tilde{\mathcal{L}})$ in $\mathcal{D}(Z_{J,D})$ (where the restrictions of $\tilde{\mathcal{L}}$ from $Z^{\mathbf{s}}_{\emptyset,J,D}$ to Z_1, Z_2 are denoted again by $\tilde{\mathcal{L}}$) gives rise to a long exact sequence in $\mathcal{M}(Z_{J,D})$:

$$\dots \to {}^pH^i(\pi_{1!}\tilde{\mathcal{L}}) \to {}^pH^i(K^{\mathbf{s},\mathcal{L}}_{J,D}) \to {}^pH^i(\pi_{2!}\tilde{\mathcal{L}}) \to {}^pH^{i+1}(\pi_{1!}\tilde{\mathcal{L}}) \to \dots.$$

Let $\mathbf{s}' = (s_1, s_2, \dots, s_{j-1}, s_{j+1}, \dots, s_r)$, $\mathbf{s}'' = (s_1, s_2, \dots, s_{j-2}, s_{j+1}, \dots, s_r)$. Then

$$\delta : (B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_{j-2}, B_j, B_{j+1}, \dots, B_r, gU_{J,B_0})$$

makes Z_1 into a locally trivial $\mathbf{k}^*$-bundle over $Z^{\mathbf{s}'}_{\emptyset,J,D}$ and

$$(B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_{j-2}, B_{j+1}, \dots, B_r, gU_{J,B_0})$$

makes Z_2 into a locally trivial affine line bundle over $Z^{\mathbf{s}''}_{\emptyset,J,D}$. The local system $\tilde{\mathcal{L}}$ on Z_2 is the inverse image of the local system $\tilde{\mathcal{L}}$ on $Z^{\mathbf{s}''}_{\emptyset,J,D}$ defined as in 28.8 in terms of $\mathcal{L}$. By 28.11, if $j \in \mathcal{J}_{\mathbf{s}}$, the local system $\tilde{\mathcal{L}}$ on Z_1 is the inverse image under δ of the local system $\tilde{\mathcal{L}}$ on $Z^{\mathbf{s}'}_{\emptyset,J,D}$ defined as in 28.8 in terms of $\mathcal{L}$; if $j \notin \mathcal{J}_{\mathbf{s}}$, then $\delta_!\tilde{\mathcal{L}} = 0$. It follows that

$$\pi_{2!}\tilde{\mathcal{L}} = K^{\mathbf{s}'',\mathcal{L}}_{J,D}[[-1]]$$

and, if $j \notin \mathcal{J}_{\mathbf{s}}$, we have $\pi_{1!}\tilde{\mathcal{L}} = 0$. If $j \in \mathcal{J}_{\mathbf{s}}$, we have a natural distinguished triangle $(\pi_{1!}\tilde{\mathcal{L}}, K^{\mathbf{s}',\mathcal{L}}_{J,D}[[-1]], K^{\mathbf{s}',\mathcal{L}}_{J,D})$ in $\mathcal{D}(Z_{J,D})$. Hence we have long exact sequences in $\mathcal{M}(Z_{J,D})$:

(d) $\dots \to {}^pH^i(\pi_{1!}\tilde{\mathcal{L}}) \to {}^pH^i(K^{\mathbf{s},\mathcal{L}}_{J,D}) \to {}^pH^{i-2}(K^{\mathbf{s}'',\mathcal{L}}_{J,D})(-1) \to {}^pH^{i+1}(\pi_{1!}\tilde{\mathcal{L}}) \to \dots,$

(e) $\dots \to {}^pH^i(\pi_{1!}\tilde{\mathcal{L}}) \to {}^pH^{i-2}(K^{\mathbf{s}',\mathcal{L}}_{J,D})(-1) \to {}^pH^i(K^{\mathbf{s}',\mathcal{L}}_{J,D}) \to {}^pH^{i+1}(\pi_{1!}\tilde{\mathcal{L}}) \to \dots,$

if $j \in \mathcal{J}_{\mathbf{s}}$, and isomorphisms

(f) $${}^pH^i(K^{\mathbf{s},\mathcal{L}}_{J,D}) \xrightarrow{\sim} {}^pH^{i-2}(K^{\mathbf{s}'',\mathcal{L}}_{J,D})(-1)$$

if $j \notin \mathcal{J}_{\mathbf{s}}$.

We prove that (v) $\implies$ (i). Assume that $A \dashv K^{\mathbf{s},\mathcal{L}}_{J,D}$ where $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is as in (v). We may assume that r is minimum possible. We want to show that (i) holds. Assume first that $l(s_1s_2\dots s_r) < r$. We show that this contradicts the minimality of r. We can find $j \in [2, r]$ such that $l(s_js_{j+1}\dots s_r) = r - j + 1$ and $l(s_{j-1}s_j\dots s_r) < r-j+2$. We can find $s'_j, s'_{j+1}, \dots, s'_r \in \mathbf{I}$ such that $s'_js'_{j+1}\dots s'_r = s_js_{j+1}\dots s_r = y$ and $s'_j = s_{j-1}$. Let

$$\mathbf{u}' = (s_1, s_2, \dots, s_{j-1}, s'_j, s'_{j+1}, \dots, s'_r), \quad \mathbf{u}'' = (s_1, s_2, \dots, s_{j-1}, y).$$

From (a) we have $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{\mathbf{u}',\mathcal{L}}_{J,D} = K^{\mathbf{u}'',\mathcal{L}}_{J,D}$. Hence we may assume that $s_{j-1} = s_j$. If $j \notin \mathcal{J}_{\mathbf{s}}$, then (f) shows that $A \dashv K^{\mathbf{s}'',\mathcal{L}}_{J,D}$; since the sequence $\mathbf{s}''$ has $r-2$ terms, this contradicts the minimality of r. Assume now that $j \in \mathcal{J}_{\mathbf{s}}$. By the minimality of r we have $A \not\dashv K^{\mathbf{s}',\mathcal{L}}_{J,D}$. From (e) it follows that $A \not\dashv \pi_{1!}\tilde{\mathcal{L}}$. This, together with (d) shows that $A \dashv K^{\mathbf{s}'',\mathcal{L}}_{J,D}$. This again contradicts the minimality of r. We see that $l(s_1s_2\dots s_r) = r$. By (a), we have $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{w,\mathcal{L}}_{J,D}$ where $w = s_1s_2\dots s_r$ and the desired conclusion follows. Thus, we have (v) $\implies$ (i). The proposition is proved.

28.14. Let A be a simple perverse sheaf on $Z_{J,D}$ and let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$. We write $A \in \hat{Z}^{\mathcal{L}}_{J,D}$ if A satisfies the equivalent conditions (i)–(v) in 28.13. We write $A \in \hat{Z}_{J,D}$ if $A \in \hat{Z}^{\mathcal{L}}_{J,D}$ for some $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$; we then say that A is a *parabolic character sheaf* on $Z_{J,D}$ (see [L10]).

In the case where $J = \mathbf{I}$ we identify $D = Z_{J,D}$ by $g \mapsto (G^0, G^0, g)$; we write $A \in \hat{D}^{\mathcal{L}}, A \in \hat{D}$ instead of $A \in \hat{Z}^{\mathcal{L}}_{J,D}, A \in \hat{Z}_{J,D}$. We say that A is a *character sheaf* on D if $A \in \hat{D}$.

In the case where $J = \mathbf{I}$ we write $K^{w,\mathcal{L}}_D, K^{\mathbf{s},\mathcal{L}}_D, \bar{K}^{\mathbf{s},\mathcal{L}}_D$ instead of $K^{w,\mathcal{L}}_{J,D}$, $K^{\mathbf{s},\mathcal{L}}_{J,D}$, $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$.

28.15. Let $A \in \hat{Z}_{J,D}$. We can find $n \in \mathbf{N}^*_{\mathbf{k}}$ and $\mathcal{L} \in \mathfrak{s}_n(\mathbf{T})$ such that $A \in \hat{Z}^{\mathcal{L}}_{J,D}$. We show that

(a) *A is equivariant for the action*

$$(z, x) : (P, P', gU_P) \mapsto (xPx^{-1}, xP'x^{-1}, xz^n gx^{-1}U_{xPx^{-1}})$$

of $H = {}^D\mathcal{Z}^0_{G^0} \times G^0$ on $Z_{J,D}$.

We can find $w \in \mathbf{W}$ such that $w\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$ and $A \dashv \pi_!\tilde{\mathcal{L}}$ where $\tilde{\mathcal{L}}$ is the local system on $Z^w_{\emptyset,J,D}$ defined in 28.7 and $\pi : Z^w_{\emptyset,J,D} \to Z_{J,D}$ is as in 28.12. Now H acts on $Z^w_{\emptyset,J,D}$ and on $\mathfrak{Z}^w_D$ by

$$(z, x) : (B, B', gU_{J,B}) \mapsto (xBx^{-1}, xB'x^{-1}, xz^n gx^{-1}U_{J,xBx^{-1}}),$$

$$(z, x) : (B, U_B gU_B) \mapsto (xBx^{-1}, U_{xBx^{-1}}xz^n gx^{-1}U_{xBx^{-1}})$$

and π and $\zeta : Z^w_{\emptyset,J,D} \to \mathfrak{Z}^w_D$ (see 28.7) are compatible with the H-actions. It suffices to show that the local system $\tilde{\mathcal{L}}$ on $\mathfrak{Z}^w_D$ (see 28.5) is H-equivariant. Let $T \xleftarrow{\phi} \hat{\mathfrak{Z}}^w_D \xrightarrow{\rho} \mathfrak{Z}^w_D$ be as in 28.5. Now H acts on T by $(z, x) : t \mapsto z^n t$ and on $\hat{\mathfrak{Z}}^w_D$ by $(z, x) : (hU^*, g) \mapsto (xhU^*, z^n g)$; note that ϕ, ρ are compatible with the H-actions. Using the definitions we see that it suffices to show that $\mathcal{L}$ is H-equivariant. This follows from the fact that $\mathcal{L} \in \mathfrak{s}_n(T)$. This proves (a).

28.16. Consider a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I} \cup \{1\}$ with $s_1 s_2 \dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, $s_1 \in J \cup \{1\}$. Let $\mathbf{s}' = (s_2, s_3, \dots, s_r, \epsilon_D(s_1))$. We have $s_2 s_3 \dots s_r \epsilon_D(s_1)\underline{D} \in s_1\mathbf{W}^\bullet_{\mathcal{L}} s_1 = \mathbf{W}^\bullet_{\mathcal{L}'}$ where $\mathcal{L}' = s_1^*\mathcal{L} \in \mathfrak{s}(\mathbf{T})$. Now

$$(B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_1, B_2, \dots, B_r, gB_1g^{-1}, gU_{J,B_1})$$

is a well-defined isomorphism $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \xrightarrow{\sim} \bar{Z}^{\mathbf{s}'}_{\emptyset,J,D}$ which restricts to an isomorphism $Z^{\mathbf{s}}_{\emptyset,J,D} \xrightarrow{\sim} Z^{\mathbf{s}'}_{\emptyset,J,D}$. Under these isomorphisms, the local system $\tilde{\mathcal{L}}$ on $Z^{\mathbf{s}}_{\emptyset,J,D}$ defined in 28.8 in terms of $\mathcal{L}$ corresponds to the analogous local system $\tilde{\mathcal{L}}'$ on $Z^{\mathbf{s}'}_{\emptyset,J,D}$ defined in terms of $\mathcal{L}'$. Similarly, the constructible sheaf $\bar{\mathcal{L}} = IC(\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}, \tilde{\mathcal{L}})$ corresponds to the constructible sheaf $\bar{\mathcal{L}}' = IC(\bar{Z}^{\mathbf{s}'}_{\emptyset,J,D}, \tilde{\mathcal{L}}')$. It follows that

(a) $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{\mathbf{s}',\mathcal{L}'}_{J,D}$,

(b) $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} = \bar{K}^{\mathbf{s}',\mathcal{L}'}_{J,D}$.

28.17. Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ and let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I} \cup \{1\}$ such that $s_1 s_2 \dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Let $m = \sharp(i \in [1, r]; s_i \neq 1) + \dim G/U_Q$ where $Q \in \mathcal{P}_J$. For any $j \in \mathbf{Z}$ we have:

(a) $${}^pH^j(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}) \cong {}^pH^{2m-j}(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}),$$

(b) $$\mathfrak{D}({}^pH^j(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D})) \cong {}^pH^{2m-j}(\bar{K}^{\mathbf{s},\check{\mathcal{L}}}_{J,D})$$

in $\mathcal{M}(Z_{J,D})$. Now (a) is a special case of the "relative hard Lefschetz theorem" [BBD, 6.2.10] applied to the projective morphism $\bar{\pi}_{\mathbf{s}} : \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \to Z_{J,D}$ and to the perverse sheaf $\bar{\mathcal{L}}[m]$ on $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$.

We prove (b). Define $\tilde{\check{\mathcal{L}}}, \bar{\check{\mathcal{L}}}$ in terms of $\check{\mathcal{L}}$ in the same way as $\tilde{\mathcal{L}}, \bar{\mathcal{L}}$ are defined in 28.8, 28.9 in terms of $\mathcal{L}$. Note that Verdier duality commutes with $\bar{\pi}_{\mathbf{s}}$; it is then enough to show that $\mathfrak{D}(\bar{\mathcal{L}}[m]) = \bar{\check{\mathcal{L}}}[m]$. By the definition of $\bar{\mathcal{L}}$ it is enough to show that $\check{\tilde{\mathcal{L}}} = \tilde{\check{\mathcal{L}}}$. This follows from the definition of $\tilde{\mathcal{L}}$.

28.18. Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$. Using 28.17(b), we see that

(a) *if* $A \in \hat{Z}^{\mathcal{L}}_{J,D}$, *then its Verdier dual* $\mathfrak{D}(A)$ *is in* $\hat{Z}^{\check{\mathcal{L}}}_{J,D}$.

28.19. Define an isomorphism $\partial : Z_{J,D} \xrightarrow{\sim} Z_{\epsilon_D(J),D^{-1}}$ by

$$\partial(Q, Q', gU_Q) = (Q', Q, g^{-1}U_{Q'}).$$

Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ and let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Let $\tilde{\mathbf{s}} = (s_r, s_{r-1}, \dots, s_1), \mathcal{L}' = (\underline{D}^{-1})^*\mathcal{L}$. We have $s_r s_{r-1} \dots s_1 \underline{D}^{-1} \in \mathbf{W}^\bullet_{\mathcal{L}'}$. We have a commutative diagram

$$\begin{array}{ccc} \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} & \xrightarrow{\tilde{\partial}} & \bar{Z}^{\tilde{\mathbf{s}}}_{\emptyset,\epsilon_D(J),D^{-1}} \\ \downarrow & & \downarrow \\ Z_{J,D} & \xrightarrow{\partial} & Z_{\epsilon_D(J),D^{-1}} \end{array}$$

where $\tilde{\partial}$ is the isomorphism

$$(B_0, B_1, \dots, B_r, gU_{J,B_0}) \mapsto (B_r, B_{r-1}, \dots, B_0, g^{-1}U_{\epsilon_D(J),B_r})$$

and the verical maps are of type 28.12(a). Under the isomorphism $\tilde{\partial}$, the constructible sheaf $\bar{\mathcal{L}}$ on $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ defined in 28.9 in terms of $\mathcal{L}$ corresponds to the analogous constructible sheaf $\bar{\mathcal{L}}'$ on $\bar{Z}^{\tilde{\mathbf{s}}}_{\emptyset,\epsilon_D(J),D^{-1}}$ defined in terms of $\mathcal{L}'$. It follows that

$$\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} = \partial^* \bar{K}^{\tilde{\mathbf{s}},\mathcal{L}'}_{\epsilon_D(J),D^{-1}}.$$

We see also that

(a) $A \in \hat{Z}^{\mathcal{L}}_{J,D} \implies \partial_! A \in \hat{Z}^{\mathcal{L}'}_{\emptyset,\epsilon_D(J),D^{-1}}$; $A' \in \hat{Z}^{\mathcal{L}'}_{\emptyset,\epsilon_D(J),D^{-1}} \implies \partial^* A' \in \hat{Z}^{\mathcal{L}}_{J,D}$.

29. Restriction functor for character sheaves

29.1. Let D be a connected component of G and let P be a parabolic of G^0 such that $N_D P \neq \emptyset$. Let L be a Levi of P. Let $G' = N_G P \cap N_G L$, a reductive group with $G'^0 = L$. Let $D' = G' \cap D$, a connected component of G'. Let $\mathrm{res}_D^{D'} : \mathcal{D}(D) \to \mathcal{D}(D')$ be as in 23.3. Let $\alpha = \dim U_P$. We write $\epsilon : \mathbf{W} \to \mathbf{W}$ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$ (see 26.2).

In this section we begin the study of $\mathrm{res}_D^{D'}(A)$ where A is a character sheaf on D. One of the main results of this section is that $\mathrm{res}_D^{D'}(A)$ is a direct sum of shifts of character sheaves on D'. (Here the words "shifts of" can be omitted but this will only come after further work in Section 31.) The results in this section extend results in the connected case that appeared in [L3, I, §3]. An obscure point in the proof in [L3, I, 3.5] (pointed out to me by J.G.M. Mars in 1985) is here removed following in part [L11].

Let $\mathcal{B}^\dagger$ be the variety of Borel subgroups of L. We show that the canonical torus $\mathbf{T}$ of G^0 (see 28.3) is canonically isomorphic to the analogously defined canonical torus $\mathbf{T}^\dagger$ of L. We define a map $\prod_{B\in\mathcal{B}} B/U_B \to \prod_{\beta\in\mathcal{B}^\dagger}\beta/U_\beta$ by $(g_BU_B)_{B\in\mathcal{B}} \mapsto (h_\beta U_\beta)_{\beta\in\mathcal{B}^\dagger}$ where, for $\beta\in\mathcal{B}^\dagger$, h_β is the image of $g_{\beta U_P}$ under the obvious homomorphism $\beta U_P\to\beta$. This map restricts to a map $\mathbf{T}\to\mathbf{T}^\dagger$ which is an isomorphism of tori. We use this isomorphism to identify $\mathbf{T}^\dagger=\mathbf{T}$.

Similarly, the Weyl group $\mathbf{W}$ of G^0 (see 26.1) contains the analogously defined Weyl group $\mathbf{W}^\dagger$ of L as a subgroup. The imbedding $\mathbf{W}^\dagger\to\mathbf{W}$ is obtained by associating to the L-orbit of $(\beta,\beta')\in\mathcal{B}^\dagger\times\mathcal{B}^\dagger$ the G^0-orbit of $(\beta U_P,\beta' U_P)\in\mathcal{B}\times\mathcal{B}$. If J is the subset of $\mathbf{W}^\dagger$ analogous to the subset $\mathbf{I}$ of $\mathbf{W}$, then the imbedding $\mathbf{W}^\dagger\to\mathbf{W}$ restricts to an imbedding $J\subset\mathbf{I}$. The length function of $\mathbf{W}^\dagger$ is just the restriction of the length function of $\mathbf{W}$. With the notation of 26.1, we have $\mathbf{W}^\dagger=\mathbf{W}_J$. Define $\mathrm{pos}^\dagger:\mathcal{B}^\dagger\times\mathcal{B}^\dagger\to\mathbf{W}_J$ in terms of L in the same way as $\mathrm{pos}:\mathcal{B}\times\mathcal{B}\to\mathbf{W}$ was defined in terms of G^0. For any Borel B of G^0 we set

$$P^B=(P\cap B)U_P,$$

a Borel of P.

Let ${}^J\mathbf{W}$ be as in 26.1. Then $y\mapsto\mathbf{W}_Jy$ is a bijection ${}^J\mathbf{W}\xrightarrow{\sim}\mathbf{W}_J\backslash\mathbf{W}$. We also have a bijection from the set of P-orbits on $\mathcal{B}$ (for the conjugation action) to $\mathbf{W}_J\backslash\mathbf{W}$: the P-orbit of $B\in\mathcal{B}$ corresponds to the $\mathbf{W}_J$ coset of $\mathrm{pos}(P^B,B)\in\mathbf{W}$. Let $v(y)$ be the P-orbit on $\mathcal{B}$ corresponding to $\mathbf{W}_Jy, y\in\mathbf{W}$.

If $y\in{}^J\mathbf{W}$ and $s\in\mathbf{I}$, there are three possibilities for ys:

(i) $ys\in{}^J\mathbf{W}$ and $l(ys)>l(y)$; then $v(y)\subset\overline{v(ys)}-v(ys)$.

(ii) $ys\in{}^J\mathbf{W}$ and $l(ys)<l(y)$; then $v(ys)\subset\overline{v(y)}-v(y)$.

(iii) $ys\notin{}^J\mathbf{W}$; then $ysy^{-1}\in J$ and $v(ys)=v(y)$.

For any $y\in\mathbf{W},g\in D$ we have $gv(y)g^{-1}=v(\epsilon(y))$.

Define a homomorphism $\pi:N_GP\to G'$ by $\pi(z\omega)=z$ where $z\in G',\omega\in U_P$ (see 1.26).

29.2. Until the end of 29.9 we fix a sequence $\mathbf{s}=(s_1,s_2,\dots,s_r)$ in $\mathbf{I}$ and $\mathcal{L}\in\mathfrak{s}(\mathbf{T})$ such that $s_1s_2\dots s_r\underline{D}\in\mathbf{W}^\bullet_{\mathcal{L}}$. Let $\mathcal{J}_{\mathbf{s}}\subset[1,r]$ be as in 28.9.

We write $\bar Z^{\mathbf{s}}$ instead of $\bar Z^{\mathbf{s}}_{\emptyset,\mathbf{I},D}$ (see 28.9). For $\mathcal{T}\subset[1,r]$ let

$$\begin{aligned}Z^{\mathbf{s}\mathcal{T}}=&\{(B_0,B_1,\dots,B_r,g)\in\bar Z^{\mathbf{s}};B_{i-1}=B_i\text{ for }i\in\mathcal{T},\\&\mathrm{pos}(B_{i-1},B_i)=s_i\text{ for }i\in[1,r]-\mathcal{T}\}.\end{aligned}$$

We have a partition $\bar Z^{\mathbf{s}}=\bigsqcup_{\mathcal{T}\subset[1,r]}Z^{\mathbf{s}\mathcal{T}}$. Let

$$\bar Z'=\{(B_0,B_1,\dots,B_r,g)\in\bar Z^{\mathbf{s}};g\in N_DP\}.$$

Define $\bar\pi':\bar Z'\to D'$ by $(B_0,B_1,\dots,B_r,g)\mapsto\pi(g)$. Any sequence $\mathbf{y}=(y_0,y_1,\dots,y_r)$ in ${}^J\mathbf{W}$ defines a locally closed subvariety

$$\bar Z'_{\mathbf{y}}=\{(B_0,B_1,\dots,B_r,g)\in\bar Z^{\mathbf{s}};g\in N_DP,B_i\in v(y_i)(i\in[0,r])\}$$

of $\bar Z'$. Clearly, $\bar Z'_{\mathbf{y}}=\emptyset$ unless $\mathbf{y}$ satisfies

(a) $$y_i=y_{i-1}\text{ or }y_i=y_{i-1}s_i\text{ for all }i\in[1,r],\epsilon(y_0)=y_r.$$

Let $\mathfrak{i}_{\mathbf{y}}:\bar Z'_{\mathbf{y}}\to\bar Z^{\mathbf{s}}$ be the inclusion. Let $\bar\pi'_{\mathbf{y}}:\bar Z'_{\mathbf{y}}\to D'$ be the restriction of $\bar\pi'$.

29.3. Until the end of 29.11 we fix a sequence $\mathbf{y}$ satisfying 29.2(a). We set

$$\begin{aligned}\mathbf{d}(\mathbf{y}) &= \alpha + \sharp(i \in [1,r]; y_{i-1}s_i \in {}^J\mathbf{W}, l(y_{i-1}s_i) < l(y_{i-1}) \\ &= \alpha + \sharp(i \in [1,r]; y_i s_i \in {}^J\mathbf{W}, l(y_i s_i) < l(y_i)).\end{aligned}$$

We show that these two definitions of $\mathbf{d}(\mathbf{y})$ are equivalent. Let
$c = \sharp(i \in [1,r]; y_{i-1} = y_i, y_i s_i \in {}^J\mathbf{W}, l(y_{i-1}s_i) < l(y_{i-1}))$,
$c' = \sharp(i \in [1,r]; l(y_i) - l(y_{i-1}) = -1)$,
$c'' = \sharp(i \in [1,r]; l(y_i) - l(y_{i-1}) = 1)$.
The two definitions of $\mathbf{d}(\mathbf{y})$ are $\alpha+c+c'$, $\alpha+c+c''$. Hence it suffices to show that $c' = c''$. Clearly, $l(y_r) - l(y_0) = c'' - c'$. Since $y_r = \epsilon(y_0)$, we have $l(y_r) = l(y_0)$ hence $c' = c''$, as required. An equivalent definition of $\mathbf{d}(\mathbf{y})$ is

$$\begin{aligned}\mathbf{d}(\mathbf{y}) &= \alpha + \sharp(i \in [1,r]; v(y_i s_i) \subset \overline{v(y_i)} - v(y_i)) \\ &= \alpha + \sharp(i \in [1,r]; v(y_{i-1}s_i) \subset \overline{v(y_{i-1})} - v(y_{i-1})).\end{aligned}$$

We define a sequence $(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_r)$ in $\mathbf{I} \cup \{1\}$ by

$$\tilde{s}_i = s_i \text{ if } y_{i-1}s_i \in \mathbf{W}_J y_i, \quad \tilde{s}_i = 1 \text{ if } y_{i-1}s_i \notin \mathbf{W}_J y_i.$$

Define $\mathbf{t} = (t_1, t_2, \dots, t_r)$ by $t_i = y_{i-1}\tilde{s}_i y_i^{-1}$. Then $t_i \in J \cup \{1\}$ and

$$t_i = y_{i-1}s_i y_{i-1}^{-1} \text{ if } y_{i-1}s_i \in \mathbf{W}_J y_{i-1}, t_i = 1 \text{ if } y_{i-1}s_i \notin \mathbf{W}_J y_{i-1}.$$

Let

$$\begin{aligned}\bar{Z}^{\mathbf{t}\dagger} = \{&(\beta_0, \beta_1, \dots, \beta_r, h) \in \mathcal{B}^\dagger \times \dots \times \mathcal{B}^\dagger \times D'; \mathrm{pos}^\dagger(\beta_{i-1}, \beta_i) = 1 \text{ or } t_i, \\ &\beta_r = h\beta_0 h^{-1}\}.\end{aligned}$$

This is a variety like $\bar{Z}^{\mathbf{s}}_{\emptyset,\mathbf{I},D}$ in 28.9 with $G, D, \mathbf{s}, \mathbf{I}$ replaced by $G', D', \mathbf{t}, J$. Define

$$\rho : \bar{Z}'_{\mathbf{y}} \to \bar{Z}^{\mathbf{t}\dagger}, (B_0, B_1, \dots, B_r, g) \mapsto (\pi(P^{B_0}), \pi(P^{B_1}), \dots, \pi(P^{B_r}), \pi(g)).$$

Lemma 29.4. *ρ is an iterated affine space bundle with fibres of dimension* $\mathbf{d}(\mathbf{y})$.

Let F be the fibre of ρ over $(\beta_0, \beta_1, \dots, \beta_r, h) \in \bar{Z}^{\mathbf{t}\dagger}$. We show only that
(a) *F is an iterated affine space bundle over a point and* $\dim F = \mathbf{d}(\mathbf{y})$.
For any $k \in [0,r]$ let F_k be the set of all sequences $(B_0, B_1, \dots, B_k)$ in $\mathcal{B}$ such that

$$\mathrm{pos}(B_{i-1}, B_i) = 1 \text{ or } s_i (i \in [1,k]), B_i \in v(y_i) (i \in [0,k]), \pi(P^{B_i}) = \beta_i (i \in [0,k]).$$

Let $F_{r+1} = F$. We have obvious maps

$$F = F_{r+1} \xrightarrow{\xi_{r+1}} F_r \xrightarrow{\xi_r} F_{r-1} \xrightarrow{\xi_{r-1}} \dots \xrightarrow{\xi_1} F_0.$$

It is easy to see that $F_0 \cong \mathbf{k}^{l(y_0)}$ and that $\xi_{r+1} : F_{r+1} \to F_r$ is an affine space bundle with fibres of dimension $\alpha - l(y_r) = \alpha - l(y_0)$. Moreover, for $i \in [1,r]$,
(b) $\xi_i : F_i \to F_{i-1}$ *is an affine space bundle with fibres of dimension* 1 *if* $v(y_i s_i) \subset \overline{v(y_i)} - v(y_i)$ *and of dimension* 0*, otherwise.*
Now (a) follows from (b). This completes the proof.

29.5. Let $\mathcal{J}=\{i\in[1,r];\tilde{s}_i=1\}$. We have $\mathcal{J}=\mathcal{J}_1\sqcup\mathcal{J}_2$ where

$$\mathcal{J}_1=\{i\in\mathcal{J};v(y_{i-1})\subset\overline{v(y_{i-1}s_i)}-v(y_{i-1}s_i)\},$$
$$\mathcal{J}_2=\{i\in\mathcal{J};v(y_{i-1}s_i)\subset\overline{v(y_{i-1})}-v(y_{i-1})\}.$$

Let $\mathcal{K}^0=\{i\in[1,r];t_i\ne 1\}$. We have

$$\mathcal{J}\cap\mathcal{K}^0=\emptyset.$$

Indeed, if $i\in\mathcal{J}\cap\mathcal{K}^0$, then $\tilde{s}_i=1$ hence $v(y_i)\ne v(y_{i-1}s_i), v(y_i)=v(y_{i-1})$, $v(y_{i-1})\ne v(y_{i-1}s_i)$ and $t_i\in J,\tilde{s}_i=1$ hence $v(y_{i-1}s_i)=v(y_{i-1})$, a contradiction.

We show:

(a) *If $i\in\mathcal{K}^0$, then $v(y_is_i)=v(y_i)$. If $i\in\mathcal{J}_1$, then $v(y_i)\subset\overline{v(y_is_i)}-v(y_is_i)$. If $i\in\mathcal{J}_2$, then $v(y_is_i)\subset\overline{v(y_i)}-v(y_i)$.*

Assume first that $i\in\mathcal{K}^0$. Then $t_i\ne 1$, $v(y_{i-1}s_i)=v(y_{i-1})$, $s_i\ne 1$. If $v(y_i)=v(y_{i-1})$ we get $v(y_is_i)=v(y_i)$; if $v(y_i)=v(y_{i-1}s_i)$ we get again $v(y_is_i)=v(y_i)$.

Assume next that $i\in\mathcal{J}_1$. Then $v(y_{i-1})\subset\overline{v(y_{i-1}s_i)}-v(y_{i-1}s_i)$, $\tilde{s}_i=1$, $v(y_i)\ne v(y_{i-1}s_i)$ hence $v(y_i)=v(y_{i-1})$ and $v(y_i)\subset\overline{v(y_is_i)}-v(y_is_i)$.

Finally, assume that $i\in\mathcal{J}_2$. Then $v(y_{i-1}s_i)\subset\overline{v(y_{i-1})}-v(y_{i-1})$, $\tilde{s}_i=1$, $v(y_i)\ne v(y_{i-1}s_i)$ hence $v(y_i)=v(y_{i-1})$ and $v(y_is_i)\subset\overline{v(y_i)}-v(y_i)$.

29.6. For any subset $\mathcal{K}\subset\mathcal{K}^0$ let

$$Z^{\mathbf{t}_\mathcal{K}\dagger}=\{(\beta_0,\beta_1,\dots,\beta_r,h)\in\bar{Z}^{\mathbf{t}\dagger};\mathrm{pos}^\dagger(\beta_{i-1},\beta_i)=t_i(i\in[1,r]-\mathcal{K}),$$
$$\beta_{i-1}=\beta_i(i\in\mathcal{K})\}.$$

We shall write $Z^{\mathbf{t}_\dagger}$ instead of $Z^{\mathbf{t}_\emptyset\dagger}$. We have $\bar{Z}^{\mathbf{t}\dagger}=\bigsqcup_{\mathcal{K}\subset\mathcal{K}^0}Z^{\mathbf{t}_\mathcal{K}\dagger}$. Hence $\bar{Z}'_{\mathbf{y}}=\bigsqcup_{\mathcal{K}\subset\mathcal{K}^0}Z'_{\mathbf{y},\mathcal{K}}$ where $Z'_{\mathbf{y},\mathcal{K}}=\rho^{-1}(Z^{\mathbf{t}_\mathcal{K}\dagger})$. We show that for $(B_0,B_1,\dots,B_r,h)\in Z'_{\mathbf{y},\mathcal{K}}$, conditions (i) and (ii) below are equivalent:

(i) $\mathcal{K}\cup\mathcal{J}_1\subset\{i\in[1,r];B_{i-1}=B_i\}\subset\mathcal{K}\cup\mathcal{J}$;

(ii) $\{i\in\mathcal{K}^0;\pi(P^{B_{i-1}})=\pi(P^{B_i})\}=\mathcal{K}$.

Assume that (i) holds. If $i\in\mathcal{K}$, then by (i) we have $B_{i-1}=B_i$, hence $\pi(P^{B_{i-1}})=\pi(P^{B_i})$. Conversely, let $i\in\mathcal{K}^0$ be such that $\pi(P^{B_{i-1}})=\pi(P^{B_i})$. Using 29.4(b) we see that $B_{i-1}=B_i$ (since $v(y_is_i)\not\subset\overline{v(y_i)}-v(y_i)$, by 29.5(a)). Using (i) we see that $i\in\mathcal{K}\cup\mathcal{J}$. Since $\mathcal{K}^0\cap\mathcal{J}=\emptyset$ we deduce that $i\in\mathcal{K}$. We see that (i) $\implies$ (ii).

Assume that (ii) holds. If $i\in\mathcal{K}$, then, by (ii), we have $\pi(P^{B_{i-1}})=\pi(P^{B_i})$; using 29.4(b) we see that $B_{i-1}=B_i$ (since $v(y_is_i)\not\subset\overline{v(y_i)}-v(y_i)$, by 29.5(a)). If $i\in\mathcal{J}_1$, then $\tilde{s}_i=1$, $v(y_i)\ne v(y_{i-1}s_i)$, hence $v(y_i)=v(y_{i-1})$ and $v(y_{i-1})\ne v(y_{i-1}s_i)$. Then $t_i=1$ hence $\pi(P^{B_{i-1}})=\pi(P^{B_i})$. Using 29.4(b) we see that $B_{i-1}=B_i$ (since $v(y_is_i)\not\subset\overline{v(y_i)}-v(y_i)$, by 29.5(a)). Thus, the first inclusion in (i) holds. Conversely, if $i\in\mathcal{K}^0,B_{i-1}=B_i$, then $\pi(P^{B_{i-1}})=\pi(P^{B_i})$ and using (ii) we see that $i\in\mathcal{K}$. If $i\in[1,r]-\mathcal{K}^0,B_{i-1}=B_i$, then $t_i=1$, $v(y_{i-1}s_i)\ne v(y_{i-1})$, $v(y_i)=v(y_{i-1})$ hence $v(y_i)\ne v(y_{i-1}s_i)$, $\tilde{s}_i=1$ hence $i\in\mathcal{J}$. Thus the second inclusion in (i) holds. We see that (ii) $\implies$ (i).

The equivalence of (i) and (ii) can be also formulated as follows:

(a) $$Z'_{\mathbf{y},\mathcal{K}}=\bigsqcup_{\mathcal{J}';\mathcal{J}'\subset\mathcal{J}_2}Z^{\mathbf{s}_{\mathcal{K}\cup\mathcal{J}_1\cup\mathcal{J}'}}\cap\bar{Z}'_{\mathbf{y}}.$$

29.7. Let $(\beta_0, \beta_1, \dots, \beta_r, h) \in \bar{Z}^{\mathbf{t}_{\mathcal{K}}\dagger}$ and let F, F_k, ξ_k be as in 29.4. From 29.6(a) we see that

$$F = \bigsqcup_{\mathcal{J}';\mathcal{J}' \subset \mathcal{J}_2} (F \cap Z^{\mathbf{s}_{\mathcal{K}\cup\mathcal{J}_1\cup\mathcal{J}'}}).$$

For $i \in [1, r]$, let $F^i = \{(B_0, B_1, \dots, B_r, g) \in F; B_{i-1} = B_i\}$. We show:

(a) *For $i \in \mathcal{J}_2$, F^i is a smooth hypersurface in F. For $i \in \mathcal{K} \cup \mathcal{J}_1$ we have $F^i = F$. For $i \in [1, r] - (\mathcal{K} \cup \mathcal{J})$ we have $F^i = \emptyset$.*

If $F^i \neq \emptyset$, then, using $F \subset \bigcup_{\mathcal{J}' \subset \mathcal{J}_2} Z^{\mathbf{s}_{\mathcal{K}\cup\mathcal{J}_1\cup\mathcal{J}'}}$, we see that $i \in \mathcal{K} \cup \mathcal{J}_1 \cup \mathcal{J}'$ for some $\mathcal{J}' \subset \mathcal{J}_2$; thus, $i \in \mathcal{K} \cup \mathcal{J}$. In the rest of the proof we assume that $i \in \mathcal{K} \cup \mathcal{J}$.

For each $k \in [i, r]$ let F^i_k be the set of all $(B_0, B_1, \dots, B_k) \in F_k$ such that $B_{i-1} = B_i$. Let $F^i_{r+1} = F^i$. From the definitions we see that for $k \in [i+1, r+1]$ we have a cartesian diagram

$$\begin{array}{ccc} F^i_k & \longrightarrow & F_k \\ \big\downarrow & & \big\downarrow{\scriptstyle \xi_k} \\ F^i_{k-1} & \longrightarrow & F_{k-1} \end{array}$$

where the horizontal maps are inclusions.

Assume first that $i \in \mathcal{J}_2$. Using the cartesian diagram above, it suffices to show that F^i_i is a smooth hypersurface in F_i. From 29.5(a) we see that $v(y_i s_i) \subset \overline{v(y_i)} - v(y_i)$; hence $\xi_i : F_i \to F_{i-1}$ is an affine line bundle (see 29.4(b)). It suffices to show that ξ_i restricts to an isomorphism $F^i_i \xrightarrow{\sim} F_{i-1}$. Let $(B_0, B_1, \dots, B_{i-1}) \in F_{i-1}$. It suffices to show that $(B_0, B_1, \dots, B_{i-1}, B_{i-1}) \in F_i$. Hence it suffices to show that $v(y_{i-1}) = v(y_i)$ and $\beta_{i-1} = \beta_i$. Since $i \in \mathcal{J}$, we have $\tilde{s}_i = 1$, hence $v(y_i) \neq v(y_{i-1} s_i)$ hence $v(y_i) = v(y_{i-1})$ and $t_i = 1$. Since $\mathrm{pos}^\dagger(\beta_{i-1}, \beta_i) = 1$ or t_i, we see that $\beta_{i-1} = \beta_i$, as required.

Assume next that that $i \in \mathcal{K} \cup \mathcal{J}_1$. Using the cartesian diagram above, it suffices to show that $F^i_i = F_i$. From 29.5(a) we see that $v(y_i s_i) \not\subset \overline{v(y_i)} - v(y_i)$; hence $\xi_i : F_i \to F_{i-1}$ is an isomorphism (see 29.4(b)). It suffices to show that ξ_i restricts to an isomorphism $F^i_i \xrightarrow{\sim} F_{i-1}$. If $i \in \mathcal{J}_1$, this is shown exactly as in the first part of the proof. Assume now that $i \in \mathcal{K}$. We have $t_i \neq 1$, hence $v(y_{i-1} s_i) = v(y_{i-1}) = v(y_i)$. From the definitions we have $\beta_{i-1} = \beta_i$. Hence $F^i_i \xrightarrow{\sim} F_{i-1}$ as in the first part of the proof. This proves (a).

Lemma 29.8. *The map $\rho_1 : \bar{Z}'_{\mathbf{y}} \cap Z^{\mathbf{s}_{\mathcal{J}}} \to Z^{\mathbf{t}\dagger}$ (restriction of ρ) is an iterated affine space bundle.*

Let $(\beta_0, \beta_1, \dots, \beta_r, h) \in Z^{\mathbf{t}\dagger}$. We show only that the fibre $\bar{F}$ of ρ_1 at $(\beta_0, \beta_1, \dots, \beta_r, h)$ is an iterated affine space bundle over a point and

$$\dim \bar{F} = \alpha + \sharp(i \in [1, r] - \mathcal{J}; v(y_i s_i) \subset \overline{v(y_i)} - v(y_i)).$$

For any $k \in [0, r]$ let

$$\begin{aligned} \bar{F}_k =& \{(B_0, B_1, \dots, B_k) \in \mathcal{B}^{k+1}; \mathrm{pos}(B_{i-1}, B_i) = s_i (i \in [1, k], i \notin \mathcal{J}), \\ & B_{i-1} = B_i (i \in [1, k] \cap \mathcal{J}), B_i \in v(y_i)(i \in [0, k]), \pi(P^{B_i}) = \beta_i (i \in [0, k])\}. \end{aligned}$$

Let $\bar{F}_{r+1} = \bar{F}$. We have obvious maps

$$\bar{F} = \bar{F}_{r+1} \to \bar{F}_r \to \bar{F}_{r-1} \to \dots \to \bar{F}_0.$$

It is easy to see that $\bar{F}_0 \cong \mathbf{k}^{l(y_0)}$ and that $\bar{F}_{r+1} \to \bar{F}_r$ is an affine space bundle with fibres of dimension $\alpha - l(y_r) = \alpha - l(y_0)$. Moreover, for $i \in [1, r]$, $\bar{F}_i \to \bar{F}_{i-1}$ is an affine space bundle with fibres of dimension 1 if $v(y_i s_i) \subset \overline{v(y_i)} - v(y_i)$, $i \notin \mathcal{J}$ and of dimension 0, otherwise. This completes the proof.

29.9. For $k \in [1, r]$ we set

$$\underline{s}_{r,k} = s_r s_{r-1} \dots s_k \dots s_{r-1} s_r, \quad \underline{\tilde{s}}_{r,k} = \tilde{s}_r \tilde{s}_{r-1} \dots \tilde{s}_{k+1} s_k \tilde{s}_{k+1} \dots \tilde{s}_{r-1} \tilde{s}_r.$$

Let Γ be a subgroup of W such that

$$k \in [1, r], \tilde{s}_k = 1 \implies \underline{s}_{r,k} \in \Gamma.$$

We show:

(a) *For $i \in [1, r]$ we have $\underline{s}_{r,i} \in \Gamma$ if and only if $\underline{\tilde{s}}_{r,i} \in \Gamma$.*

We argue by induction on $r - i$. If $r - i = 0$, we have $\underline{s}_{r,i} = \underline{\tilde{s}}_{r,i}$ so the result is obvious. Assume now that $r - i \geq 1$. We have

$$k \in [1, r-1], \tilde{s}_k = 1 \implies \underline{s}_{r-1,k} \in s_r \Gamma s_r.$$

By the induction hypothesis we have $\underline{s}_{r-1,i} \in s_r \Gamma s_r$ if and only if $\underline{\tilde{s}}_{r-1,i} \in s_r \Gamma s_r$. If $\tilde{s}_r = s_r$, then $\underline{s}_{r-1,i} = s_r \underline{s}_{r,i} s_r$, $\underline{\tilde{s}}_{r-1,i} = s_r \underline{\tilde{s}}_{r,i} s_r$. Hence we have $\underline{s}_{r,i} \in \Gamma$ if and only if $\underline{\tilde{s}}_{r,i} \in \Gamma$. If $\tilde{s}_r = 1$, then $\underline{s}_{r-1,i} = s_r \underline{s}_{r,i} s_r$, $\underline{\tilde{s}}_{r-1,i} = \underline{\tilde{s}}_{r,i}$. Hence we have $\underline{s}_{r,i} \in \Gamma$ if and only if $\underline{\tilde{s}}_{r,i} \in \Gamma$. (We use that $s_r \in \Gamma$.) This proves (a).

We show:

(b) *If $i \in [1, r]$ and $t_i \neq 1$, then $t_r t_{r-1} \dots t_i \dots t_{r-1} t_r = y_r \underline{\tilde{s}}_{r,i} y_r^{-1}$.*

We argue by induction on $r - i$. If $r - i = 0$, we have $t_r \neq 1$, hence $t_r = y_{r-1} \tilde{s}_r y_r^{-1} = y_{r-1} s_r y_{r-1}^{-1}$. We see that $y_{r-1} = y_r \tilde{s}_r s_r = y_r s_r \tilde{s}_r$ and $t_r = y_r s_r y_r^{-1}$ as required.

Assume now that $r - i \geq 1$. By the induction hypothesis the left-hand side of the equality in (b) is $t_r y_{r-1} \underline{\tilde{s}}_{r-1,i} y_{r-1}^{-1} t_r$ and the right-hand side is $y_r \tilde{s}_r \underline{\tilde{s}}_{r-1,i} \tilde{s}_r y_r^{-1}$. It then suffices to show that $t_r y_{r-1} = y_r \tilde{s}_r$; this follows from the definitions since $t_r = t_r^{-1}$. This proves (b).

29.10. We set $y = y_0$. In the case where $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$ we set ${}^y\mathcal{L} = \mathrm{Ad}(y^{-1})^* \mathcal{L} \in \mathfrak{s}(\mathbf{T})$ and we show that

$$\text{(a)} \qquad t_1 t_2 \dots t_r \underline{D} \in \mathbf{W}^\bullet_{{}^y\mathcal{L}}.$$

For $i \in [1, r]$ define $u_i \in \mathbf{W}$ by $u_i = \epsilon^{-1}(s_r s_{r-1} \dots s_i \dots s_{r-1} s_r)$ if $i \in \mathcal{J}$ and by $u_i = 1$ if $i \notin \mathcal{J}$. If $i \in \mathcal{J}$, we have $i \in \mathcal{J}_{\mathbf{s}}$, hence $u_i \in \mathbf{W}^\bullet_{\mathcal{L}}$. Then

$$\tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_r \underline{D} = s_1 s_2 \dots s_r \underline{D} u_1 u_2 \dots u_r \in \mathbf{W}^\bullet_{\mathcal{L}}.$$

We have $\tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_r = y^{-1} t_1 t_2 \dots t_r y_r$, hence $y^{-1} t_1 t_2 \dots t_r y_r \underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Since $y_r = \epsilon(y)$ we have $y_r \underline{D} = \underline{D} y$, hence $y^{-1} t_1 t_2 \dots t_r \underline{D} y \in \mathbf{W}^\bullet_{\mathcal{L}}$ and (a) follows.

Using (a) we can define a constructible sheaf ${}^y\bar{\mathcal{L}}$ on $\bar{Z}^{\mathbf{t}\dagger}$ and a complex $\bar{K}_{D'}^{\mathbf{t}, {}^y\mathcal{L}} \in \mathcal{D}(D')$ in terms of $\mathbf{t}, {}^y\mathcal{L}, G'$ in the same way as $\bar{\mathcal{L}}$ on $\bar{Z}^{\mathbf{s}}$ and $\bar{K}_D^{\mathbf{s}, \mathcal{L}} \in \mathcal{D}(D)$ are defined in 28.12 in terms of $\mathbf{s}, \mathcal{L}, G$.

Lemma 29.11. *(a) If $\mathcal{J} \not\subset \mathcal{J}_{\mathbf{s}}$, then $\bar{\pi}'_{\mathbf{y}!} \mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}} = 0$.*

(b) If $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$, then $\bar{\pi}'_{\mathbf{y}!} \mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}} = \bar{K}_{D'}^{\mathbf{t}, {}^y\mathcal{L}}[[-\mathbf{d}(\mathbf{y})]]$.

Let $\bar{\pi}_{\mathbf{t}} : \bar{Z}^{\mathbf{t}\dagger} \to D'$ be the obvious projection. We have $\bar{\pi}'_{\mathbf{y}!} = \bar{\pi}_{\mathbf{t}!} \rho_!$ and $\rho_! \rho^* ({}^y\bar{\mathcal{L}}) = {}^y\bar{\mathcal{L}}[[-\mathbf{d}(\mathbf{y})]]$ (we use 29.4). Hence it suffices to prove:

(a′) *If $\mathcal{J} \not\subset \mathcal{J}_{\mathbf{s}}$, then $\rho_!(\mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}}) = 0$.*

(b′) *If* $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$, *then* $\mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}} \cong \rho^*({}^y\bar{\mathcal{L}})$.

We prove (a′). Let F be the fibre of ρ over a point of $Z^{\mathbf{t}_{\mathcal{K}}\dagger}(\mathcal{K} \subset \mathcal{K}^0)$. We must show that $H_c^*(F, \bar{\mathcal{L}}|_F) = 0$. (We write $\bar{\mathcal{L}}$ instead of $\mathfrak{i}_{\mathbf{y}}^*\bar{\mathcal{L}}$.) If $\mathcal{K} \cup \mathcal{J}_2 \not\subset \mathcal{J}_{\mathbf{s}}$ then $\bar{\mathcal{L}}|_F = 0$ (we use 29.6(a) and 28.10) and the desired vanishing follows. Assume now that $\mathcal{K} \cup \mathcal{J}_2 \subset \mathcal{J}_{\mathbf{s}}$ but $\mathcal{J} \not\subset \mathcal{J}_{\mathbf{s}}$. Using 29.6(a) and 28.10 we see that $\bar{\mathcal{L}}|_F$ is a local system on

$$\bigsqcup_{\mathcal{J}';\mathcal{J}' \subset \mathcal{J}_2; \mathcal{J}' \subset \mathcal{J}_{\mathbf{s}}} (F \cap Z^{\mathbf{s}_{\mathcal{K}\cup\mathcal{J}_1\cup\mathcal{J}'}})$$

and is zero elsewhere. Hence $\bar{\mathcal{L}}|_F$ is a local system on $F - \bigcup_{j \in \mathcal{J}_2 - \mathcal{J}_{\mathbf{s}}} F^j$ (F^j as in 29.7) and is zero on $\bigcup_{j \in \mathcal{J}_2 - \mathcal{J}_{\mathbf{s}}} F^j$. Let i be the largest number in $\mathcal{J}_2 - \mathcal{J}_{\mathbf{s}}$. It suffices to show that for any $(B_0, B_1, \dots, B_{i-1}) \in F_{i-1}$ (see 29.4) we have $H_c^*(F', \bar{\mathcal{L}}|_{F'}) = 0$ where F' is the fibre of the obvious map $F \to F_{i-1}$ at $(B_0, B_1, \dots, B_{i-1})$. Let $F'^i = F' \cap F^i$. If $B_{j-1} = B_j$ for some $j < i, j \in \mathcal{J}_2 - \mathcal{J}_{\mathbf{s}}$, then $F' \subset F^j$ and $\bar{\mathcal{L}}|_{F'} = 0$; the desired vanishing follows. Thus we may assume that $\mathrm{pos}(B_{j-1}, B_j) = s_j$ for all $j < i, j \in \mathcal{J}_2 - \mathcal{J}_{\mathbf{s}}$. Then $\bar{\mathcal{L}}|_{F' - F'^i}$ is a local system and $\bar{\mathcal{L}}|_{F'^i} = 0$. Let F'' be the fibre of $\xi_i : F_i \to F_{i-1}$ (see 29.4) at $(B_0, B_1, \dots, B_{i-1})$. Let $F''^i = \{(B_0, B_1, \dots, B_{i-1}, B_{i-1})\}$, a point on the affine line F''. Let $u : F' \to F''$ be the restriction of the obvious map $F \to F_i$. Then u is an iterated affine space bundle (see 29.4(a)), $F'^i = u^{-1}(F''^i)$, and there is a well-defined local system $\mathcal{E}$ on $F'' - F''^i \cong \mathbf{k}^*$ such that $\mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$, $\bar{\mathcal{L}}|_{F' - F'^i} = u^*(\mathcal{E})$. Then $H_c^*(F' - F'^i, \bar{\mathcal{L}}) \cong H_c^*(\mathbf{k}^*, \mathcal{E})$ and it suffices to show that $\mathcal{E} \not\cong \bar{\mathbf{Q}}_l$. It also suffices to show that $\bar{\mathcal{L}}|_{F' - F'^i}$ has non-trivial monodromy around the smooth hypersurface F'^i of F'. This is the same as the monodromy of $\bar{\mathcal{L}}|_{Z^{\mathbf{s}}}$ around the hypersurface $Z^{\mathbf{s}_{\{i\}}}$. This monodromy is non-trivial by 28.10(b′). This proves (a′).

We prove (b′). We define $\mathbf{W}_{J, {}^y\mathcal{L}}$ in terms of $G', \mathbf{W}_J, {}^y\mathcal{L}$ in the same way as $\mathbf{W}_{\mathcal{L}}$ was defined in terms of $G^0, \mathbf{W}, \mathcal{L}$. Let

$$\begin{aligned} \mathcal{J}_{\mathbf{t}} &= \{i \in [1, r]; t_i \in J, t_r t_{r-1} \dots t_i \dots t_{r-1} t_r \in \epsilon(\mathbf{W}_{J, {}^y\mathcal{L}})\} \\ &= \{i \in [1, r]; t_i \in J, t_r t_{r-1} \dots t_i \dots t_{r-1} t_r \in \epsilon(\mathbf{W}_{{}^y\mathcal{L}})\}. \end{aligned}$$

(The two definitions coincide since $t_r t_{r-1} \dots t_i \dots t_{r-1} t_r$ is a reflection in $\mathbf{W}_J$.) We show that

(c) $$\mathcal{J}_{\mathbf{t}} = \mathcal{J}_{\mathbf{s}} \cap \mathcal{K}^0.$$

Using 29.9(b) it suffices to show that for $i \in [1, r]$ such that $t_i \neq 1$, we have

$$y_r \tilde{s}_r \dots \tilde{s}_{i+1} s_i \tilde{s}_{i+1} \dots \tilde{s}_r y_r^{-1} \in \epsilon(\mathbf{W}_{{}^y\mathcal{L}}) \leftrightarrow s_r s_{r-1} \dots s_i \dots s_{r-1} s_r \in \epsilon(\mathbf{W}_{\mathcal{L}}).$$

Using 29.9(a), we see that it suffices to show that $\epsilon^{-1}(y_r^{-1}) \mathbf{W}_{{}^y\mathcal{L}} \epsilon^{-1}(y_r) = \mathbf{W}_{\mathcal{L}}$ or that ${}^y\mathcal{L} = \mathrm{Ad}(\epsilon^{-1}(y_r^{-1}))^* \mathcal{L}$. This follows from the definitions using $y_r = \epsilon(y)$.

Using 28.10 for G' instead of G, we see that ${}^y\bar{\mathcal{L}}$ is a local system on $Z^1 = \bigcup_{\mathcal{K} \subset \mathcal{J}_{\mathbf{t}}} Z^{\mathbf{t}_{\mathcal{K}}\dagger}$ and is zero on its complement in $\bar{Z}^{\mathbf{t}\dagger}$. Using 28.10 and 29.6(a) we see that $\mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}}$ is a local system on

$$Z^2 = \bigcup_{\mathcal{J}' \subset \mathcal{J}_2; \mathcal{K} \subset \mathcal{J}_{\mathbf{s}}} Z^{\mathbf{s}_{\mathcal{K}\cup\mathcal{J}_1\cup\mathcal{J}'}} \cap \bar{Z}'_{\mathbf{y}} = \bigcup_{\mathcal{K} \subset \mathcal{J}_{\mathbf{t}}} Z'_{\mathbf{y}, \mathcal{K}}$$

and is zero on its complement in $\bar{Z}'_{\mathbf{y}}$. (We have used (c).) Since Z^2 is an iterated affine space bundle over Z^1 (via ρ) and the restriction of $\mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}}$ to any fibre of $\rho : Z^2 \to Z^1$ is a local system of rank 1 with finite monodromy of order invertible in $\mathbf{k}$ (hence it is $\bar{\mathbf{Q}}_l$) we see that $\mathfrak{i}_{\mathbf{y}}^* \bar{\mathcal{L}}|_{Z^2} = \rho^* \mathcal{E}$ for a well-defined local system $\mathcal{E}$ of rank 1 on Z^1.

It suffices to show that $\mathcal{E} \cong {}^y\bar{\mathcal{L}}|_{Z^1}$. Since Z^1 is smooth and $Z^{\mathbf{t}\dagger}$ is open dense in Z^1, it suffices to show that $\mathcal{E}|_{Z^{\mathbf{t}\dagger}} \cong {}^y\bar{\mathcal{L}}|_{Z^{\mathbf{t}\dagger}}$. Let $Z^3 = Z^{\mathbf{s}_\mathcal{J}} \cap \bar{Z}'_{\mathbf{y}}$. This is a closed subset of the open subset $\rho^{-1}(Z^{\mathbf{t}\dagger})$ of Z_2. Since the restriction of ρ is an iterated affine space bundle $\rho_1 : Z^3 \to Z^{\mathbf{t}\dagger}$ (see 29.8), it suffices to show that $\rho_1^*(\mathcal{E}|_{Z^{\mathbf{t}\dagger}}) \cong \rho_1^*({}^y\bar{\mathcal{L}}|_{Z^{\mathbf{t}\dagger}})$. Since $\rho_1^*(\mathcal{E}|_{Z^{\mathbf{t}\dagger}}) = \bar{\mathcal{L}}|_{Z^3}$, it suffices to show that $\bar{\mathcal{L}}|_{Z^3} \cong \rho_1^*({}^y\bar{\mathcal{L}}|_{Z^{\mathbf{t}\dagger}})$. Using 28.10, once for G and once for G', we see that $\bar{\mathcal{L}}|_{Z^3} = \tilde{\mathcal{L}}|_{Z^3}$, ${}^y\bar{\mathcal{L}}|_{Z^{\mathbf{t}\dagger}} = {}^y\tilde{\mathcal{L}}$ where $\tilde{\mathcal{L}}$ (on $Z^{\mathbf{s}_\mathcal{J}}$) is defined as in 28.8 in terms of $G, \mathbf{s}_\mathcal{J}, \mathcal{L}$ and ${}^y\tilde{\mathcal{L}}$ (on $Z^{\mathbf{t}\dagger}$) is defined analogously in terms of $G', \mathbf{t}, {}^y\mathcal{L}$. Thus it suffices to show that

(d) $$\tilde{\mathcal{L}}|_{Z^3} \cong \rho_1^*({}^y\tilde{\mathcal{L}}).$$

To prove (d), we choose B^*, T, d and $\dot{w}$ (for $w \in \mathbf{W}$) as in 28.5, in such a way that $B^* \subset P, T \subset L$. We have necessarily $d \in D'$. Let $\beta^\dagger = \pi(B^*) \in \mathcal{B}^\dagger$. Let $U^* = U_{B^*}, U^\dagger = U_{\beta^\dagger}$. Let

$$\mathcal{Z} =\{(h_0U^*, h_1U^*, \dots, h_rU^*, g) \in (G^0/U^*)^{r+1} \times D; h_{i-1}^{-1}h_i \in B^*\dot{\tilde{s}}_iB^*$$
$$\text{for } i \in [1,r], h_r^{-1}gh_0 \in N_GB^*, g \in N_GP, h_i \in P\dot{y}_iU^* \text{ for } i \in [0,r]\},$$

$$\mathcal{Z}' = \{(h'_0U^\dagger, h'_1U^\dagger, \dots, h'_rU^\dagger, g') \in (L/U^\dagger)^{r+1} \times D';$$
$${h'_{i-1}}^{-1}h'_i \in \beta^\dagger\dot{t}_i\beta^\dagger \text{ for } i \in [1,r], {h'_r}^{-1}g'h'_0 \in N_G\beta^\dagger\}.$$

Define $\zeta : \mathcal{Z} \to \mathcal{Z}'$ by

$$(h_0U^*, h_1U^*, \dots, h_rU^*, g) \mapsto (h'_0U^\dagger, h'_1U^\dagger, \dots, h'_rU^\dagger, g')$$

where $h_i \in p_i\dot{y}_iU^*, p_i \in P, h'_i = \pi(p_i), g' = \pi(g)$. (We show that $h'_iU^\dagger$ is well defined or equivalently that p_iU^* is well defined. It suffices to show that $p\dot{y}_iU^* = p'\dot{y}_iU^*, p, p' \in P \implies p'U^* = pU^*$. It also suffices to show that $P \cap \dot{y}_iU^*\dot{y}_i^{-1} \subset U^*$. Since $y_i \in {}^J\mathbf{W}$ and $B^* \subset P$ we have $P^{\dot{y}_iB^*\dot{y}_i^{-1}} = B^*$. Hence $P \cap \dot{y}_iB^*\dot{y}_i^{-1} \subset B^*$. Thus $P \cap \dot{y}_iU^*\dot{y}_i^{-1}$ is contained in the set of unipotent elements of B^*, that is, in U^*.) We have a commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\zeta} & \mathcal{Z}' \\ a\downarrow & & a'\downarrow \\ Z^3 & \xrightarrow{\rho_1} & Z^{\mathbf{t}\dagger} \end{array}$$

where $a : (h_0U^*, h_1U^*, \dots, h_rU^*, g) \mapsto (h_0B^*h_0^{-1}, h_1B^*h_1^{-1}, \dots, h_rB^*h_r^{-1}, g)$,

$$a' : (h'_0U^\dagger, h'_1U^\dagger, \dots, h'_rU^\dagger, g') \mapsto (h'_0\beta^\dagger{h'_0}^{-1}, h'_1\beta^\dagger{h'_1}^{-1}, \dots, h'_r\beta^\dagger{h'_r}^{-1}, g').$$

Since a is a locally trivial fibration with smooth connected fibres, to prove (d) it suffices to prove that $a^*(\tilde{\mathcal{L}}|_{Z^3}) \cong a^*\rho_1^*({}^y\tilde{\mathcal{L}})$ or that $a^*(\tilde{\mathcal{L}}|_{Z^3}) \cong \zeta^*a'^*({}^y\tilde{\mathcal{L}})$. Define $\xi : \mathcal{Z} \to T$ by

$$(h_0U^*, h_1U^*, \dots, h_rU^*, g) \mapsto d^{-1}(\dot{s}'_1\dot{s}'_2 \dots \dot{s}'_r)^{-1}n_1n_2 \dots n_rn$$

where $n_i \in N_{G^0}T$ are given by $h_{i-1}^{-1}h_i \in U^*n_iU^*$ and $n \in N_GB^* \cap N_GT$ is given by $h_r^{-1}gh_0 \in U^*n$. Define $\xi_1 : \mathcal{Z}' \to T$ by

$$(h'_0U^\dagger, h'_1U^\dagger, \dots, h'_rU^\dagger, g') \mapsto d^{-1}(\dot{t}_1\dot{t}_2 \dots \dot{t}_r)^{-1}\bar{n}_1\bar{n}_2 \dots \bar{n}_r\bar{n}$$

where $\bar{n}_i \in N_LT$ are given by ${h'_{i-1}}^{-1}h'_i \in U^\dagger\bar{n}_iU^\dagger$ and $\bar{n} \in N_G\beta^\dagger \cap N_GT$ is given by ${h'_r}^{-1}g'h'_0 \in U^\dagger\bar{n}$. From the definitions we have

$$a^*(\tilde{\mathcal{L}}|_{Z^3}) = \xi^*\mathcal{L}, a'^*({}^y\tilde{\mathcal{L}}) = \xi_1^*\mathrm{Ad}(\dot{y}^{-1})^*\mathcal{L}$$

where $\mathcal{L} \in \mathfrak{s}(T)$ is as in 28.6. Therefore it suffices to show that

$$\xi^*\mathcal{L} \cong \zeta^*\xi_1^*\mathrm{Ad}(\dot{y}^{-1})^*\mathcal{L}.$$

Define $\xi' : \mathcal{Z} \to T$ by

$$(h_0U^*, h_1U^*, \dots, h_rU^*, g) \mapsto d^{-1}(\dot{t}_1\dot{t}_2\dots\dot{t}_r)^{-1}\bar{n}_1\bar{n}_2\dots\bar{n}_r\bar{n}$$

where $h_i \in p_i\dot{y}_iU^*$, $p_i \in P$, $p_{i-1}^{-1}p_i \in U^*\bar{n}_iU^*$, $p_r^{-1}gp_0 \in U^*\bar{n}$, $\bar{n}_i \in N_LT$, $\bar{n} \in N_GB^* \cap N_GT$. Then $\xi' = \xi_1\zeta$ and it suffices to show that

$$\xi^*\mathcal{L} \cong \xi'^*\mathrm{Ad}(\dot{y}^{-1})^*\mathcal{L}.$$

Using 28.1(c) it suffices to show that there exists $t \in T$ such that

$$t\xi(z) = \mathrm{Ad}(\dot{y}^{-1})\xi'(z) \text{ for all } z \in \mathcal{Z}.$$

Let $(h_0, h_1, \dots, h_r, g) \in (G^0)^{r+1} \times D$ be such that $z = (h_0U^*, h_1U^*, \dots, h_rU^*, g) \in \mathcal{Z}$. We define $p_i, n_i, n, \bar{n}_i, \bar{n}$ in terms of h_i as in the definition of ξ, ξ'. From $h_r^{-1}gh_0 \in U^*n$, $p_r^{-1}gp_0 \in U^*\bar{n}$, we deduce $\dot{y}_r^{-1}p_r^{-1}gp_0\dot{y} \in U^*n$, $U^*\bar{n}\dot{y}U^* = U^*\dot{y}_rnU^*$, hence $\bar{n}\dot{y} = \dot{y}_rn$. We show that

$$\bar{n}_i = \dot{y}_{i-1}n_i\dot{y}_i^{-1} \text{ for any } i \in [1, r].$$

From $h_{i-1}^{-1}h_i \in U^*n_iU^*$, $p_{i-1}^{-1}p_i \in U^*\bar{n}_iU^*$, we deduce $\dot{y}_{i-1}^{-1}p_{i-1}^{-1}p_i\dot{y}_i \in U^*n_iU^*$, hence $\dot{y}_{i-1}un_iu'\dot{y}_i^{-1} \in U^*\bar{n}_iU^*$ for some $u, u' \in U^*$. Assume first that $t_i \ne 1$. Then

$$y_i = y_{i-1}, y_{i-1}s_i = t_iy_i, l(y_{i-1}s_i) = l(t_iy_i) = l(y_{i-1}) + 1 = l(y_i) + 1,$$

hence

$$\dot{y}_{i-1}un_iu' \in U^*\dot{y}_{i-1}n_iU^*, U^*\bar{n}_iU^*\dot{y}_i \subset U^*\bar{n}_i\dot{y}_iU^*.$$

Thus, $U^*\dot{y}_{i-1}n_iU^* = U^*\bar{n}_i\dot{y}_iU^*$ and $\dot{y}_{i-1}n_i = \bar{n}_i\dot{y}_i$, as required. Next, assume that $t_i = 1, \tilde{s}_i \ne 1$. Then $y_i = y_{i-1}s_i \ne y_{i-1}$, $\bar{n}_i \in T$. If $l(y_{i-1}s_i) = l(y_{i-1}) + 1$, then $\dot{y}_{i-1}un_iu' \in U^*\dot{y}_{i-1}n_iU^*$ and $U^*\bar{n}_iU^*\dot{y}_i \subset U^*\bar{n}_i\dot{y}_iU^*$ so that $U^*\dot{y}_{i-1}n_iU^* = U^*\bar{n}_i\dot{y}_iU^*$ and $\dot{y}_{i-1}n_i = \bar{n}_i\dot{y}_i$, as required. If $l(y_{i-1}s_i) = l(y_{i-1}) - 1$, then $l(s_iy_i^{-1}) = l(y_i^{-1}) + 1$. We have $un_iu'\dot{y}_i^{-1} \in U^*n_i\dot{y}_i^{-1}U^*$ and $\dot{y}_{i-1}^{-1}U^*\bar{n}_iU^* \subset U^*\dot{y}_{i-1}^{-1}\bar{n}_iU^*$ so that $U^*n_i\dot{y}_i^{-1}U^* = U^*\dot{y}_{i-1}^{-1}\bar{n}_iU^*$ and $n_i\dot{y}_i^{-1} = \dot{y}_{i-1}^{-1}\bar{n}_i$, as required. Finally, assume that $\tilde{s}_i = 1$. Then $t_i = 1, y_{i-1}s_i \ne y_{i-1} = y_i$, $n_i \in T$, $\bar{n}_i \in T$. We have $\dot{y}_{i-1}un_iu' \in U^*\dot{y}_{i-1}n_iU^*$ and $U^*\bar{n}_iU^*\dot{y}_i \subset U^*\bar{n}_i\dot{y}_iU^*$ so that $U^*\dot{y}_{i-1}n_iU^* = U^*\bar{n}_i\dot{y}_iU^*$ and $\dot{y}_{i-1}n_i = \bar{n}_i\dot{y}_i$, as required.

We have

$$\begin{aligned}
&\mathrm{Ad}(\dot{y}^{-1})\xi'(z) = \dot{y}^{-1}d^{-1}(\dot{t}_1\dot{t}_2\dots\dot{t}_r)^{-1}\bar{n}_1\bar{n}_2\dots\bar{n}_r\bar{n}\dot{y}\\
&= \dot{y}^{-1}d^{-1}(\dot{t}_1\dot{t}_2\dots\dot{t}_r)^{-1}(\dot{y}n_1\dot{y}_1^{-1})(\dot{y}_1n_2\dot{y}_2^{-1})\dots(\dot{y}_{r-1}n_r\dot{y}_r^{-1})\dot{y}_rn\\
&= \dot{y}^{-1}d^{-1}(\dot{t}_1\dot{t}_2\dots\dot{t}_r)^{-1}\dot{y}n_1n_2\dots n_rn\\
&= td^{-1}(\dot{\tilde{s}}_1\dot{\tilde{s}}_2\dots\dot{\tilde{s}}_r)^{-1}n_1n_2\dots n_rn = t\xi(z)
\end{aligned}$$

where

$$t = \dot{y}^{-1}d^{-1}(\dot{t}_1\dot{t}_2\dots\dot{t}_r)^{-1}\dot{y}(\dot{\tilde{s}}_1\dot{\tilde{s}}_2\dots\dot{\tilde{s}}_r)d.$$

We have $t \in T$. (Equivalently, $y\tilde{s}_1\tilde{s}_2\dots\tilde{s}_r = t_1\dots t_ry_r$, which is clear from the definitions.) This completes the proof of (d), hence that of (b′). The lemma is proved.

29.12. We consider the sequence $Z_0 \subset Z_1 \subset \dots$ of closed subsets of $\bar{Z}'$ defined by $Z_i = \bigcup_{\mathbf{y};c(\mathbf{y})\le i} \bar{Z}'_{\mathbf{y}}$ where $\mathbf{y}$ is a sequence $(y_0, y_1, \dots, y_r)$ of elements in ${}^J\mathbf{W}$ satisfying 29.2(a) and $c(\mathbf{y}) = \sum_{i\in[0,r]} \dim v(y_i)$. Let $k_i : Z_i \to \bar{Z}'(i \ge 0)$ and $k'_i : Z_i - Z_{i-1} \to \bar{Z}'(i \ge 1)$ be the inclusions. For any $i \ge 1$, the natural distinguished triangle

$$(\bar{\pi}'_! k'_{i!} k'^*_i \bar{\mathcal{L}}, \bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}}, \bar{\pi}'_! (k_{i-1})_! k^*_{i-1} \bar{\mathcal{L}})$$

in $\mathcal{D}(D')$ gives rise to a long exact sequence in $\mathcal{M}(D')$:

$$\dots \to {}^pH^{j-1}(\bar{\pi}'_!(k_{i-1})_! k^*_{i-1}\bar{\mathcal{L}}) \xrightarrow{\delta} \bigoplus_{\mathbf{y};c(\mathbf{y})=i} {}^pH^j(\bar{\pi}'_{\mathbf{y}!} \mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}}) \to$$

(a) $${}^pH^j(\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}}) \to {}^pH^j(\bar{\pi}'_!(k_{i-1})_! k^*_{i-1}\bar{\mathcal{L}}) \xrightarrow{\delta} \dots .$$

We now prove the following result.

Lemma 29.13. *(a) The maps δ in 29.12(a) are zero.*

(b) For $i \ge 0$, $\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}} \in \mathcal{D}(D')$ is a semisimple complex; it is isomorphic to $\bigoplus_{\mathbf{y};c(\mathbf{y})\le i} \bar{\pi}'_{\mathbf{y}!} \mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}}$.

(c) $\bar{\pi}'_! \bar{\mathcal{L}} \in \mathcal{D}(D')$ is a semisimple complex; it is isomorphic to $\bigoplus_{\mathbf{y}} \bar{\pi}'_{\mathbf{y}!} \mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}}$.

(c) is a special case of (b), for large i. Assuming that (a) and the first assertion of (b) are proved, we prove the second assertion of (b) as follows. Since both complexes in question are semisimple (see 29.11 and 28.12(b)), it suffices to show that they have the same ${}^pH^j$ for any j. Using (a) we see that 29.12(a) decomposes into short exact sequences of semisimple objects in $\mathcal{M}(D')$. Hence

$${}^pH^j(\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}}) \cong {}^pH^j(\bar{\pi}'_!(k_{i-1})_! k^*_{i-1}\bar{\mathcal{L}}) \oplus \bigoplus_{\mathbf{y};c(\mathbf{y})=i} {}^pH^j(\bar{\pi}'_{\mathbf{y}!} \mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}}).$$

This proves the desired equality for ${}^pH^j$ by induction on i. (The case where $i = 0$ is trivial.)

It remains to prove (a) and the first assertion of (b). By general principles, we may assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$, that G, P, D are defined over $\mathbf{F}_q$ and that G^0 is split over $\mathbf{F}_q$. By taking $\mathbf{F}_q$ large enough, we may assume that 29.12(a) and the isomorphisms in 29.11(a),(b) are realized in the category of mixed complexes with $\mathcal{L}$ pure of weight 0. Now $\bar{K}^{\mathbf{t}, {}^{y_0}\mathcal{L}}_{D'}$ in 29.11(b) is pure of weight 0 (by Deligne's theorem [D, 6.2.6]) since it is a direct image under a proper map of ${}^{y_0}\bar{\mathcal{L}}$ which is pure of weight 0; after applying to it $[[-\mathbf{d}(\mathbf{y})]]$, it remains pure of weight 0; see [BBD, 6.1.4]. Hence by 29.11, $\bar{\pi}'_{\mathbf{y}!}(\mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}})$ is pure of weight 0; it follows that

(d) $\bigoplus_{\mathbf{y};c(\mathbf{y})=i} {}^pH^j(\bar{\pi}'_{\mathbf{y}!} \mathfrak{i}^*_{\mathbf{y}} \bar{\mathcal{L}})$ is pure of weight j.

We now show by induction on i that ${}^pH^j(\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}})$ is pure of weight j for any i. For $i = 0$ this follows from (d). If we assume that this holds for $i - 1$ where $i \ge 1$ then the statement for i follows from the statement for $i - 1$ and 29.12(a) (using (d)); we also use the following fact: if $K_1 \to K_2 \to K_3$ is an exact sequence of mixed perverse sheaves with K_1, K_3 pure of weight j, then K_2 is pure of weight j. Using [BBD, 5.4.4] it follows that $\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}}$ is pure of weight 0. Using the decomposition theorem [BBD, 5.4.5, 5.3.8] it follows that $\bar{\pi}'_! k_{i!} k^*_i \bar{\mathcal{L}}$ is a semisimple complex. The vanishing of δ in 29.12(a) follows from the fact that δ is a morphism between two pure perverse sheaves of different weights. The lemma is proved.

Proposition 29.14. *In $\mathcal{D}(D')$ we have $\mathrm{res}_D^{D'}(\bar{K}_D^{\mathbf{s},\mathcal{L}}) \cong \bigoplus_{\mathbf{y}} \bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}}[[-\mathbf{d}(\mathbf{y})]]$ where $\mathbf{y}$ runs over all sequences satisfying 29.2(a) such that $\{i \in [1,r], \tilde{s}_i = 1\} \subset \mathcal{J}_{\mathbf{s}}$ and $\tilde{s}_i, \mathbf{t}$ are defined in terms of $\mathbf{y}$ as in 29.3. In particular, $\mathrm{res}_D^{D'}(\bar{K}_D^{\mathbf{s},\mathcal{L}})$ is a direct sum of shifts of character sheaves on D'.*

From the definitions we have $\mathrm{res}_D^{D'}(\bar{K}_D^{\mathbf{s},\mathcal{L}}) \cong \bar{\pi}'_! \bar{\mathcal{L}}$. (We ignore the Tate twist (α).) The result follows from 29.13(c) and 29.11.

Proposition 29.15. *Let A be a character sheaf on D. Then $\mathrm{res}_D^{D'} A \in \mathcal{D}(D')$ is a direct sum of shifts of character sheaves on D'.*

We can find $\mathbf{s}, \mathcal{L}$ as in 29.2 such that $A \dashv \bar{K}_D^{\mathbf{s},\mathcal{L}}$. Using 28.12(b) we see that for some $j \in \mathbf{Z}$, $A[-j]$ is a direct summand of $\bar{K}_D^{\mathbf{s},\mathcal{L}}$. Hence $\mathrm{res}_D^{D'} A[-j]$ is a direct summand of $\mathrm{res}_D^{D'}(\bar{K}_D^{\mathbf{s},\mathcal{L}})$, which is a semisimple complex by 29.14. It follows that $\mathrm{res}_D^{D'} A$ is a semisimple complex. Now ${}^pH^i(\mathrm{res}_D^{D'} A)$ is a direct summand of ${}^pH^{i+j}(\mathrm{res}_D^{D'}(\bar{K}_D^{\mathbf{s},\mathcal{L}}))$ which, by 29.14, is a direct sum of character sheaves on D'. Hence ${}^pH^i(\mathrm{res}_D^{D'} A)$ is a direct sum of character sheaves on D'. This completes the proof.

30. Admissibility of character sheaves

30.1. In this section we fix a connected component D of G. We write $\epsilon : \mathbf{W} \to \mathbf{W}$ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$ (see 26.2).

Lemma 30.2. *Let $H = {}^D\mathcal{Z}_{G^0}^0 \times G^0$. Let A be a simple perverse sheaf on D which is cuspidal (see 23.3). Assume that there exists $n \in \mathbf{N}_{\mathbf{k}}^*$ such that A is equivariant for the H-action $(z,x) : g \mapsto xz^n gx^{-1}$ on D. Let $Z = \mathrm{supp} A$, $m = \dim Z$. There exists a unique pair $(S, \mathcal{E})$ where S is an isolated stratum S of D and $\mathcal{E}$ is an irreducible cuspidal local system $\mathcal{E} \in \mathcal{S}(S)$ (up to isomorphism) such that $A[-m] = IC(\bar{S}, \mathcal{E})$ extended by 0 on $D - \bar{S}$.*

The intersections of Z with the various strata of D form a finite partition of Z into locally closed subsets. Since Z is irreducible, one of these intersections is open dense in Z. Thus there exists $(L, S) \in \mathbf{A}$ such that $S \subset D$ and $Y_{L,S} \cap Z$ is open dense in Z. Let P be a parabolic of G^0 with Levi L such that $S \subset N_G P$. Let $a = \dim U_P$. We can find an open dense smooth subset V of Z and an irreducible local system $\mathcal{E}$ on V such that $A = IC(Z, \mathcal{E})[m]$ extended by 0 on $D - Z$. Replacing if necessary $V, \mathcal{E}$ by $V \cap Y_{L,S}, \mathcal{E}|_{V \cap Y_{L,S}}$, we may assume that $V \subset Y_{L,S}$. For any $h \in H$, the h-translate ${}^hV$ of V is an open dense smooth subset of Z. Hence $V' = \bigcup_h {}^hV$ is an open dense smooth subset of Z. Since $V \subset Y_{L,S}$ and $Y_{L,S}$ is H-stable, we have ${}^hV \subset Y_{L,S}$ for $h \in H$ hence $V' \subset Y_{L,S}$. Now $A' = A[-m]|_V$ is an H-equivariant intersection cohomology complex on V' such that $A'|_V$ is a local system and $A'|_{{}^hV}$ is automatically a local system for any $h \in H$. Since $\bigcup_h {}^hV$ is an open covering of V', we see that A' is a local system on V'. Replacing $V, \mathcal{E}$ by V', A', we see that we may assume in addition that V is H-stable and $\mathcal{E}$ is an H-equivariant local system on V. Define $f : G^0 \times (V \cap S^*) \to V$ by $(y, g) \mapsto ygy^{-1}$. Then f is surjective since $V \subset Y_{L,S}$. Moreover, f is a principal bundle with group $\Gamma = \{x \in N_{G^0} L; xSx^{-1} = S\}$ which acts on $G^0 \times (V \cap S^*)$ by $x : (y, g) \mapsto (yx^{-1}, xgx^{-1})$. (We show this only at the level of sets. It suffices to show that, if $(y, g), (y', g')$ are elements of $V \cap S^*$ such that $ygy^{-1} = y'g'y'^{-1}$, then the element $x = y'^{-1}y \in G^0$ satisfies $xLx^{-1} = L, xSx^{-1} = S$. We have $xgx^{-1} = g'$.

Since $g \in S^*, g' \in S^*$, we have $L = L(g) = L(g')$ (see 3.9) and $L(g') = xL(g)x^{-1}$, hence $xLx^{-1} = L$. Since xSx^{-1}, S are strata of $N_G L$ with a common element g, we must have $xSx^{-1} = S$, as required.) Since V is irreducible and Γ is of pure dimension $\dim L$, it follows that $V \cap S^*$ is non-empty, of pure dimension $m - 2a$: we have

$$\dim(V \cap S^*) + \dim G^0 = \dim V + \dim \Gamma = \dim V + \dim L = m + \dim G^0 - 2a.$$

Let $g \in V \cap S^*$. Let U' be the orbit of g under U_P-conjugation. Since U' is an orbit of an action of a unipotent group on an affine variety, U' is closed in D. We have $U' \subset gU_P$. (Indeed if $x \in U_P$, then $xgx^{-1} = g(g^{-1}xg)x^{-1}$ and $g^{-1}xg \in U_P$ since $g \in N_G P$.) The isotropy group $U_{P,g}$ of g in U_P is contained in

$$U_P \cap Z_G(g) \subset U_P \cap Z_G(g_s) \subset U_P \cap Z_G(g_s)^0$$

(the last inclusion follows from 1.11). Since $g \in S^*$, we have $Z_G(g_s)^0 \subset L$ hence $U_{P,g} \subset U_P \cap L = \{1\}$. Thus, $U_{P,g} = \{1\}$. We see that $\dim U' = \dim U_P$. Since U' is closed in gU_P, we have $U' = gU_P$. Since V is stable under U_P-conjugation and U' is the U_P-orbit of $g \in V$, it follows that $U' \subset V$. Thus, $gU_P \subset V$. Now $\mathcal{E}|_{gU_P}$ is a U_P-equivariant local system (for the conjugation action of U_P which has trivial isotropy group). It follows that $\mathcal{E}|_{gU_P} \cong \bar{\mathbf{Q}}_l^c$ for some $c \geq 1$. Hence $H_c^{2a}(gU_P, \mathcal{E}) \neq 0$. Equivalently, $H_c^{2a-m}(gU_P, A) \neq 0$.

For any $i \in \mathbf{Z}$, we denote by $\mathcal{X}^i$ the set of all U_P-cosets R in $N_D P$ such that $H_c^i(R, A) \neq 0$. Then, for any $g \in V \cap S^*$, we have $gU_P \in \mathcal{X}^{2a-m}$. The map

$$V \cap S^* \to N_D P/U_P, g \mapsto gU_P$$

is injective: if $g, g' \in V \cap S^*$ and $gU_P = g'U_P$, then

$$g^{-1}g' \in (N_G P \cap N_G L) \cap U_P = \{1\}$$

(see 1.26), hence $g = g'$. We see that $\dim \mathcal{X}^{2a-m} \geq \dim(V \cap S^*)$ hence $\dim \mathcal{X}^{2a-m} \geq m - 2a$. Thus,

$$\dim(\text{supp}\mathcal{H}^{2a-m}(\text{res}_D^{D'} A)) \geq m - 2a$$

where $D' = N_D P \cap N_D L$. If $P \neq G^0$, then our assumption that A is cuspidal gives

$$\dim(\text{supp}\mathcal{H}^{2a-m}(\text{res}_D^{D'} A)) < m - 2a,$$

a contradiction. Thus, $P = G^0, L = G^0$ and S must be an isolated stratum of D, so that $Y_{L,S} = S$. Since V is H-stable, contained in S and S is a single H-orbit, it follows that $V = S$ and $\mathcal{E}$ is an H-equivariant local system on S, that is, $\mathcal{E} \in \mathcal{S}(S)$. Using 23.3(a), we see that $\mathcal{E}$ is a cuspidal local system. The lemma is proved.

30.3. For $J \subset \mathbf{I}$ such that $\epsilon(J) = J$, let

$$V_{J,D} = \{(P, gU_P); P \in \mathcal{P}_J, gU_P \in N_D P/U_P\}.$$

Let $P_0 \in \mathcal{P}_J$. Since $\epsilon(J) = J$, $N_D P_0$ is a connected component of $N_G P_0$ and $D_0 = N_D P_0/U_{P_0}$ is a connected component of $N_G P_0/U_{P_0}$. Consider the diagram

$$D_0 \xleftarrow{a} G^0 \times D_0 \xrightarrow{b} V_{J,D}$$

where $a(x, gU_{P_0}) = gU_{P_0}, b(x, gU_{P_0}) = (xPx^{-1}, xgx^{-1}U_{xPx^{-1}})$ with $x \in G^0$, $g \in N_D P_0$. Then a, b are smooth morphisms with connected fibres; more precisely, b is a principal P_0-bundle where P_0 acts on $G^0 \times D_0$ by $p : (x, gU_{P_0}) \mapsto (xp^{-1}, pgp^{-1}U_{P_0})$. Let A_0 be a perverse sheaf on D_0 equivariant for the conjugation action of P_0/U_{P_0}. Then $a^\bigstar A_0 = b^\bigstar A_0^\flat$ for a well-defined perverse sheaf $A_0^\flat$ on $V_{J,D}$.

30.4. For any $J \subset J' \subset \mathbf{I}$ such that $\epsilon(J) = J, \epsilon(J') = J'$ let $V_{J,J',D}$ be the variety consisting of all pairs (P, gU_Q) where $P \in \mathcal{P}_J, gU_Q \in N_D P/U_Q$ and Q is the unique parabolic in $\mathcal{P}_{J'}$ such that $P \subset Q$. We have a diagram

$$V_{J,D} \xleftarrow{c} V_{J,J',D} \xrightarrow{d} V_{J',D}$$

where $c(P, gU_Q) = (P, gU_P), d(P, gU_Q) = (Q, gU_Q)$. Define

$$\tilde{f}_{J,J'} : \mathcal{D}(V_{J,D}) \to \mathcal{D}(V_{J',D}), \tilde{e}_{J,J'} : \mathcal{D}(V_{J',D}) \to \mathcal{D}(V_{J,D})$$

by $\tilde{f}_{J,J'}A = d_! c^* A, \tilde{e}_{J,J'}A' = c_! d^* A'$. Define

$$f_{J,J'} : \mathcal{D}(V_{J,D}) \to \mathcal{D}(V_{J',D}), e_{J,J'} : \mathcal{D}(V_{J',D}) \to \mathcal{D}(V_{J,D})$$

by $f_{J,J'}A = \tilde{f}_{J,J'}A[\alpha] = d_! c^\bigstar A, e_{J,J'}A' = \tilde{e}_{J,J'}A'[\alpha](\alpha)$ where $\alpha = \dim \mathcal{P}_J - \dim \mathcal{P}_{J'}$.

Let $P_0 \in \mathcal{P}_J, P'_0 \in \mathcal{P}_{J'}$ be such that $P_0 \subset P'_0$. Let $D_0 = N_D P_0/U_{P_0}, D'_0 = N_D P'_0/U_{P'_0}$. We have $\alpha = \dim U_{P_0} - \dim U_{P'_0}$. We show:

(a) *If $A \in \mathcal{M}(V_{J,D})$ is of the form $A = A_0^\flat$ where A_0 is a direct sum of admissible simple perverse sheaves on D_0, then $A' := f_{J,J'}A$ is of the form $A_0'^\flat$ where $A'_0 = \text{ind}_{D_0}^{D'_0} A_0$ is a direct sum of admissible simple perverse sheaves on D'_0. In particular, $f_{J,J'}A \in \mathcal{M}(V_{J',D})$.*

We have a commutative diagram

$$\begin{array}{ccccccc}
D_0 & \xleftarrow{r} & V_1 & \xrightarrow{s} & V_2 & \xrightarrow{t} & D'_0 \\
\uparrow a & & \uparrow j & & \uparrow h & & \uparrow a' \\
G^0 \times D_0 & \xleftarrow{1\times r} & G^0 \times V_1 & \xrightarrow{1\times s} & G^0 \times V_2 & \xrightarrow{1\times t} & G^0 \times D'_0 \\
\downarrow b & & & & \downarrow k & & \downarrow b' \\
V_{J,D} & & \xleftarrow{\quad c \quad} & & V_{J,J',D} & \xrightarrow{d} & V_{J',D}
\end{array}$$

Here
$V_1 = P'_0/U_{P'_0} \times N_D P_0/U_{P'_0}$,
$V_2 = \{(P, gU_{P'_0}); P \in \mathcal{P}_J, P \subset P'_0, g \in N_D P\}$,
b is as in 30.3, b' is the analogous map (with P_0 replaced by P'_0),
a, j, h, a', r, t are given by the second projection,
$s(p'U_{P'_0}, gU_{P'_0}) = (p'P_0 p'^{-1}, p'gp'^{-1}U_{P'_0})$ where $p' \in P'_0, g \in N_D P_0$,
$k(x, P, gU_{P'}) = (xPx^{-1}, xgx^{-1}U_{xP'_0x^{-1}})$ where $x \in G^0$, $(P, gU_{P'}) \in V_2$.

All morphisms in this diagram (except $t, 1 \times t, d$) are smooth with connected fibres. Moreover, s, b, k, b' are principal bundles with group $P_0/U_{P'_0}, P_0, P'_0, P'_0$. We may assume that A_0, A are simple. There is a well-defined simple perverse sheaf A_1 on V_2 such that $r^\bigstar A_0 = s^\bigstar A_1$. We have $A'_0 = t_! A_1$. Using the commutativity of the diagram above we see that $h^\bigstar A_1 = k^\bigstar(c^\bigstar A)$. Since the squares $(h, t, 1 \times t, a')$ and $(1 \times t, b', k, d)$ are cartesian, we have $(1 \times t)_! h^\bigstar A_1 = a'^\bigstar A_0$, $(1 \times t)_! k^\bigstar(c^\bigstar A) = b'^\bigstar d_!(c^\bigstar A) = b'^\bigstar A'$. It follows that $a'^\bigstar A_0 = b'^\bigstar A'$. From 27.2(d) we see that $A'_0 = \text{ind}_{D_0}^{D'_0} A_0$ is a direct sum of admissible simple perverse sheaves on D'_0. Hence $a'^\bigstar A_0 = b'^\bigstar A'$ is a direct sum of simple perverse sheaves on $G^0 \times D'_0$. Hence A' is a direct sum of simple perverse sheaves on $V_{J',D}$. This proves (a).

We show:

(b) *Let C_0 be a $P'_0/U_{P'_0}$-equivariant simple perverse sheaf on D'_0 and let $C = C_0^\flat$ be the corresponding simple perverse sheaf on $V_{J',D}$. Then for any $i \in \mathbf{Z}$ we have ${}^pH^i(e_{J,J'}C) = ({}^pH^i(\mathrm{res}_{D'_0}^{D_0} C_0))^\flat$ (equality of perverse sheaves on $V_{J,D}$).*

We have a commutative diagram

$$\begin{array}{ccccc} D_0 & \xleftarrow{f'} & V_3 & \xrightarrow{f} & D'_0 \\ {\scriptstyle a}\uparrow & & {\scriptstyle v}\uparrow & & {\scriptstyle a'}\uparrow \\ G^0 \times D_0 & \xleftarrow{1\times f'} & G^0 \times V_3 & \xrightarrow{1\times f} & G^0 \times D'_0 \\ {\scriptstyle b}\downarrow & & {\scriptstyle m}\downarrow & & {\scriptstyle b'}\downarrow \\ V_{J,D} & \xleftarrow{c} & V_{J,J',D} & \xrightarrow{d} & V_{J',D} \end{array}$$

where a, b, a', b', c, d are as above, $V_3 = N_D P_0/U_{P'_0}$, f, f' are the obvious maps, v is the second projection and

$$m(x, pU_{P'_0}) = (xP_0x^{-1}, xpx^{-1}U_{xP'_0x^{-1}}).$$

From this commutative diagram we see that $v^*f^*C_0[\dim U_{P'_0}] = m^*d^*C$ (we use that $b'^*C = a'^*C_0[\dim U_{P'_0}]$). Since the squares $(f', a, 1\times f', v)$ and $(v, f, 1\times f, a')$ are cartesian, we have

$$a^*f'_!(f^*C_0) = (1\times f')_!v^*(f^*C_0), b^*c_!d^*C = (1\times f')_!m^*(d^*C),$$

hence $a^*f'_!(f^*C_0)[\dim U_{P'_0}] = b^*c_!d^*C$. Thus,

$$a^*(\mathrm{res}_{D'_0}^{D_0} C_0)(-\alpha)[\dim U_{P'_0}] = b^*(e_{J,J'}C)[-\alpha](-\alpha).$$

Hence $a^*(\mathrm{res}_{D'_0}^{D_0} C_0)[\dim U_{P_0}] = b^*(e_{J,J'}C)$ and $b^\bigstar(e_{J,J'}C) = a^\bigstar(\mathrm{res}_{D'_0}^{D_0} C_0)$,

$${}^pH^i(b^\bigstar(e_{J,J'}C)) = {}^pH^i(a^\bigstar(\mathrm{res}_{D'_0}^{D_0} C_0)).$$

Using this and [L3, I, (1.8.1)], we have

$$b^\bigstar({}^pH^i(e_{J,J'}C)) = a^\bigstar({}^pH^i(\mathrm{res}_{D'_0}^{D_0} C_0)) = b^\bigstar(({}^pH^i(\mathrm{res}_{D'_0}^{D_0} C_0))^\flat).$$

Since $\beta^\bigstar$ is fully faithful [L3, I, (1.8.3)], we deduce the required equality ${}^pH^i(e_{J,J'}C) = ({}^pH^i(\mathrm{res}_{D'_0}^{D_0} C_0))^\flat$.

Lemma 30.5. *Let $A \in \mathcal{D}(V_{J,D})$, $A' \in \mathcal{D}(V_{J',D})$. We have*

$$\mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_{J,J'}A', A) = \mathrm{Hom}_{\mathcal{D}(V_{J',D})}(A', f_{J,J'}A).$$

Using the fact that d is proper (hence $d_* = d_!$) and that c is an affine space bundle with fibres of dimension α (hence $c^!A = c^*A[2\alpha](\alpha)$), we have

$$\mathrm{Hom}(c_!d^*A'[\alpha](\alpha), A) = \mathrm{Hom}(d^*A'[\alpha](\alpha), c^!A) = \mathrm{Hom}(d^*A'[\alpha](\alpha), c^*A[2\alpha](\alpha))$$
$$= \mathrm{Hom}(d^*A', c^*A[\alpha]) = \mathrm{Hom}(A', d_*c^*A[\alpha]) = \mathrm{Hom}(A', d_!c^*A[\alpha]).$$

The lemma is proved.

Theorem 30.6. *Let A be a character sheaf on D.*

(a) Let P_0 be a parabolic of G^0 such that $N_D P_0 \neq \emptyset$ and let $D_0 = N_D P_0/U_{P_0}$, a connected component of $N_G P_0/U_{P_0}$. Let A_1 be a character sheaf on D_0. Then $\mathrm{ind}_{D_0}^D A_1 \in \mathcal{M}(D)$.

(b) Let P_0, D_0 be as in (a). Then $\mathrm{res}_D^{D_0} A \in \mathcal{D}(D_0)^{\leq 0}$.

(c) Let P_0, D_0 be as in (a). Let A_1 be a character sheaf on D_0. Then $\mathrm{Hom}_{\mathcal{D}(D_0)}(\mathrm{res}_D^{D_0} A, A_1) = \mathrm{Hom}_{\mathcal{D}(D)}(A, \mathrm{ind}_{D_0}^D A_1)$.

(d) There exist P_0, D_0 as in (a) and a cuspidal character sheaf A_1 on D_0 such that A is a direct summand of $\mathrm{ind}_{D_0}^D A_1$.

(e) A is admissible.

If $G^0 = \{1\}$, the theorem is obvious. Assume now that $\dim G > 0$ and that the theorem is true when G is replaced by a reductive group of dimension $< \dim G$. The proof of the theorem for G assuming this inductive assumption is given in 30.7–30.11.

30.7. We show that 30.6(a) holds for G. If $P_0 = G^0$, we have $D = D_0$, $\mathrm{ind}_{D_0}^D A_1 = A_1$ and the result is obvious. Assume now that $P_0 \neq G^0$. By 30.6(e) for $N_G P_0/U_{P_0}$, A_1 is admissible on D_0. Using 27.2(d) we see that $\mathrm{ind}_{D_0}^D A_1 \in \mathcal{M}(D)$, as required.

30.8. We show that 30.6(b) holds for G. If $P_0 = G^0$, we have $\mathrm{res}_D^{D_0} A = A \in \mathcal{M}(D_0)$. Assume now that $P_0 \neq G^0$. Let J be such that $P_0 \in \mathcal{P}_J$. We identify $V_{\mathbf{I},D} = D$ in the obvious way. We show:

(a) ${}^pH^i(e_{J,\mathbf{I}}A) = 0$ for $i > 0$.

Assume that this is not so; let i be the largest integer such that ${}^pH^i(e_{J,\mathbf{I}}A) \neq 0$. Then $i > 0$ and there exists a nonzero morphism $e_{J,\mathbf{I}}A \to {}^pH^i(e_{J,\mathbf{I}}A)[-i]$. Using 30.5 we deduce that

$$\mathrm{Hom}_{\mathcal{D}(D)}(A, f_{J,\mathbf{I}}({}^pH^i(e_{J,\mathbf{I}}A)[-i])) \neq 0.$$

Using 30.4(b) we have

$$f_{J,\mathbf{I}}({}^pH^i(e_{J,\mathbf{I}}A)) = f_{J,\mathbf{I}}(({}^pH^i(\mathrm{res}_D^{D_0} A))^\flat).$$

By 29.15, ${}^pH^i(\mathrm{res}_D^{D_0} A)$ is a finite direct sum of character sheaves on D_0 hence, by 30.6(e) for $N_G P_0/U_{P_0}$ it is a finite direct sum of admissible complexes on D_0. Using 30.4(a), we see that $C := f_{J,\mathbf{I}}(({}^pH^i(\mathrm{res}_D^{D_0} A))^\flat) \in \mathcal{M}(D)$. Thus we have $\mathrm{Hom}_{\mathcal{D}(D)}(A, C[-i]) \neq 0$ with $A, C \in \mathcal{M}(D), i > 0$. This contradicts [L3, II, 7.4]. Thus, (a) holds.

Using (a) and 30.4(b) we see that for $i > 0$ we have $({}^pH^i(\mathrm{res}_D^{D_0} A))^\flat = 0$, hence ${}^pH^i(\mathrm{res}_D^{D_0} A) = 0$. It follows that $\mathrm{res}_D^{D_0} A \in \mathcal{D}(D_0)^{\leq 0}$. Thus, 30.6(b) holds for G.

30.9. We show that 30.6(c) holds for G. If $P_0 = G^0$, the result is obvious. Assume now that $P_0 \neq G^0$. Let J be as in 30.8. By 30.6(e) for $N_G P_0/U_{P_0}$, A_1 is admissible on D_0. Let $A_1^\flat$ be the simple perverse sheaf on $V_{J,D}$ corresponding to A_1 as in 30.3. From 30.4 we have

(a) $$\mathrm{ind}_{D_0}^D A_1 = f_{J,\mathbf{I}} A_1^\flat, a^\bigstar e_{J,\mathbf{I}} A = b^\bigstar \mathrm{res}_D^{D_0} A,$$

where a, b are as in 30.3. We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(D_0)}(\mathrm{res}_D^{D_0} A, A_1) &= \mathrm{Hom}_{\mathcal{D}(G^0 \times D_0)}(b^\bigstar \mathrm{res}_D^{D_0} A, b^\bigstar A_1) \\ &= \mathrm{Hom}_{\mathcal{D}(G^0 \times D_0)}(a^\bigstar e_{J,\mathbf{I}} A, a^\bigstar A_1^\flat) = \mathrm{Hom}_{\mathcal{D}(V_{J,D})}(e_{J,\mathbf{I}} A, A_1^\flat) \\ &= \mathrm{Hom}_{\mathcal{D}(D)}(A, f_{J,\mathbf{I}} A_1^\flat) = \mathrm{Hom}_{\mathcal{D}(D)}(A, \mathrm{ind}_{D_0}^D A_1);\end{aligned}$$

the first equality comes from [L3, I, (1.8.2)], and 30.6(b); the second equality comes from (a); the third equality comes from [L3, I, (1.8.2)], and 30.8(a); the fourth equality comes from 30.5; the fifth equality comes from (a). We see that 30.6(c) holds for G.

30.10. We show that 30.6(d) holds for A. If A is cuspidal, we can take $P = G^0, A_1 = A$ and the desired result holds. Thus, we may assume that A is not cuspidal. Then there exist P_0, D_0 as in 30.6(a) such that $P \neq G^0$ and $\mathrm{res}_D^{D_0} A[-1] \notin \mathcal{D}(D_0)^{\le 0}$. Then ${}^pH^i(\mathrm{res}_D^{D_0} A) \neq 0$ for some $i \ge 0$. By 30.6(b) we have ${}^pH^j(\mathrm{res}_D^{D_0} A) = 0$ for all $j > 0$. It follows that ${}^pH^0(\mathrm{res}_D^{D_0} A) \neq 0$ and there exists a non-zero morphism $\mathrm{res}_D^{D_0} A \to {}^pH^0(\mathrm{res}_D^{D_0} A)$ in $\mathcal{D}(D_0)$. Since ${}^pH^0(\mathrm{res}_D^{D_0} A)$ is a direct sum of character sheaves on D_0 (see 29.15) it follows that there exists a character sheaf A_2 on D_0 and a non-zero morphism $\mathrm{res}_D^{D_0} A \to A_2$ in $\mathcal{D}(D_0)$. Using 30.6(c) it follows that there exists a non-zero morphism $A \to \mathrm{ind}_{D_0}^D A_2$ in $\mathcal{D}(D)$. By 30.6(a) this is a non-zero morphism in $\mathcal{M}(D)$. This must be injective since A is simple. By the induction hypothesis, A_2 is a direct summand of a complex of the form $\mathrm{ind}_{D_1}^{D_0} A_3$ where $D_1 = N_{D_0}Q/U_Q$, Q is a parabolic of $N_G P/U_P$ such that $N_{D_0}Q \neq \emptyset$ and A_3 is a cuspidal character sheaf on D_1. By the induction hypothesis, A_3 is admissible. By the transitivity property 27.3(a) we have $\mathrm{ind}_{D_0}^D(\mathrm{ind}_{D_1}^{D_0} A_3) = \mathrm{ind}_{D_1}^D A_3$. Since $\mathrm{ind}_{D_0}^D$ commutes with direct sums, we see that $\mathrm{ind}_{D_0}^D A_2$ is a direct summand of $\mathrm{ind}_{D_1}^D A_3$. Hence A is isomorphic to a subobject of $\mathrm{ind}_{D_1}^D A_3$. From 27.2(d) we see that $\mathrm{ind}_{D_1}^D A_3$ is a semisimple perverse sheaf hence A is a direct summand of it. Thus, 30.6(d) holds for G.

30.11. We show that 30.6(e) holds for A. Assume first that A is cuspidal. Then A is admissible by 30.2, which is applicable in view of 28.15(a) with $J = \mathbf{I}$. Next assume that A is not cuspidal. Then, by 30.6(d) and its proof, we see that there exist P_0, D_0 as in 30.6(a) and a cuspidal character sheaf A_1 on D_0 such that $P_0 \neq G^0$ and A is a direct summand of $\mathrm{ind}_{D_0}^D A_1$. By the induction hypothesis, A_1 is admissible. Using 27.2(d) we see that A is admissible. Thus 30.6(e) holds. Theorem 30.6 is proved.

Corollary 30.12. *Let $J \subset \mathbf{I}$ and let X be a parabolic character sheaf on $Z_{J,D}$ (see §26). Then X is admissible in the sense of 26.3.*

By [L10, 4.13] we have $X = \hat{A}$ where $\hat{A}$ is obtained from some $\mathbf{t}, C, A$ as in 26.3 except that A is a character sheaf on C instead of being an admissible complex on C. However, by 30.6(e), A is automatically admissible on C hence $\hat{A}$ is admissible on $Z_{J,D}$, by the definition in 26.3.

31. Character sheaves and Hecke algebras

31.1. In this section we show that the restriction functor studied in §29 takes a character sheaf to a direct sum of character sheaves (Theorem 31.14). In the connected case this result appeared in [L3, I, §6] with a proof based on a connection of character sheaves with Hecke algebras. The present proof in the general case is an extension of the proof in [L3, I, §6], taking also into account the approach given later by Mars and Springer [MS, §9].

31.2. Until the end of 31.13 we fix $n \in \mathbf{N}_{\mathbf{k}}^*$. We write $\mathfrak{s}_n$ instead of $\mathfrak{s}_n(\mathbf{T})$. Let $\underline{\mathfrak{s}}_n$ be the set of isomorphism classes of objects in $\mathfrak{s}_n$. We have canonically

$$\underline{\mathfrak{s}}_n = \mathrm{Hom}(\mu_n(\mathbf{T}), \bar{\mathbf{Q}}_l^*)$$

(notation of 5.3). Thus $|\underline{\mathfrak{s}}_n| = n^{\dim \mathbf{T}} < \infty$. Now $\mathbf{W}^\bullet$ acts on $\underline{\mathfrak{s}}_n$ by $a : \lambda \mapsto a\lambda$ where λ is the isomorphism class of $\mathcal{L} \in \mathfrak{s}_n$ and $a\lambda$ is the isomorphism class of

$(a^{-1})^*\mathcal{L} \in \mathfrak{s}_n$. For $\lambda \in \underline{\mathfrak{s}}_n$ we set $\mathbf{W}_\lambda = \mathbf{W}_\mathcal{L}$ (see 28.3) where λ is the isomorphism class of $\mathcal{L} \in \mathfrak{s}_n$ (this is independent of the choice of $\mathcal{L}$).

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. We now introduce an associative $\mathcal{A}$-algebra H_n. (In Part VII it will be shown that H_n is closely related to the algebra of double cosets of a finite Chevalley group with respect to a maximal unipotent subgroup, studied in [Y].) We define H_n by the generators $T_w (w \in \mathbf{W}), 1_\lambda (\lambda \in \underline{\mathfrak{s}}_n)$ and the relations

$1_\lambda 1_\lambda = 1_\lambda$ for $\lambda \in \underline{\mathfrak{s}}_n$, $1_\lambda 1_{\lambda'} = 0$ for $\lambda \neq \lambda'$ in $\underline{\mathfrak{s}}_n$,
$T_w T_{w'} = T_{ww'}$ for $w, w' \in \mathbf{W}$ with $l(ww') = l(w) + l(w')$,
$T_w 1_\lambda = 1_{w\lambda} T_w$ for $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$,
$T_s^2 = v^2 T_1 + (v^2 - 1) \sum_{\lambda; s \in \mathbf{W}_\lambda} T_s 1_\lambda$ for $s \in \mathbf{I}$,
$T_1 = \sum_\lambda 1_\lambda$.

Note that $T_1 = \sum_\lambda 1_\lambda$ is the unit element of H_n and that

(a) $\{T_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n\}$ *is an* $\mathcal{A}$*-basis of* H_n.

In the case where $n = 1$, H_n is just the Iwahori-Hecke algebra attached to $\mathbf{W}$ and the proof of (a) is standard (see for example [L12, 3.3]). The proof in the general case is quite similar. Consider the free $\mathcal{A}$-module M with basis

$$\{[\lambda', w, \lambda]; \lambda, \lambda' \in \underline{\mathfrak{s}}_n, w \in \mathbf{W}, w\lambda = \lambda'\}.$$

If (a) is true, then M may be identified with H_n so that $[\lambda', w, \lambda] \in M$ corresponds to $T_w 1_\lambda = 1_{\lambda'} T_w \in H_n$; hence M is naturally an (H_n, H_n) bimodule. Conversely, if we can make M naturally into an (H_n, H_n) bimodule, then (a) can be easily deduced. For any $s \in \mathbf{I}$ we define $\mathcal{A}$-linear maps $m \mapsto T_s m$ and $m \mapsto m T_s$ of M into itself by

$T_s[\lambda', w, \lambda] = [s\lambda', sw, \lambda]$ if $l(sw) = l(w) + 1$,
$T_s[\lambda', w, \lambda] = v^2[s\lambda', sw, \lambda] + (v^2 - 1)[s\lambda', w, \lambda]$ if $l(sw) = l(w) - 1, s \in \mathbf{W}_{\lambda'}$,
$T_s[\lambda', w, \lambda] = v^2[s\lambda', sw, \lambda]$ if $l(sw) = l(w) - 1, s \notin \mathbf{W}_{\lambda'}$,
$[\lambda', w, \lambda]T_s = [\lambda', ws, s\lambda]$ if $l(ws) = l(w) + 1$,
$[\lambda', w, \lambda]T_s = v^2[\lambda', ws, s\lambda] + (v^2 - 1)[\lambda', w, s\lambda]$ if $l(ws) = l(w) - 1, s \in \mathbf{W}_\lambda$,
$[\lambda', w, \lambda]T_s = v^2[\lambda', ws, s\lambda]$ if $l(ws) = l(w) - 1, s \notin \mathbf{W}_\lambda$.

For any $\tilde{\lambda} \in \underline{\mathfrak{s}}_n$ we define $\mathcal{A}$-linear maps $m \mapsto 1_{\tilde{\lambda}} m, m \mapsto m 1_{\tilde{\lambda}}$ of M into itself by

$1_{\tilde{\lambda}}[\lambda', w, \lambda] = \delta_{\tilde{\lambda}, \lambda'}[\lambda', w, \lambda], [\lambda', w, \lambda]1_{\tilde{\lambda}} = \delta_{\tilde{\lambda}, \lambda}[\lambda', w, \lambda]$.

One shows that this defines an (H_n, H_n) bimodule structure on M. We omit further details.

Remark. In [MS, 3.3.1] an algebra structure on M (with λ restricted in a fixed W-orbit) is considered which is similar to the one above, coming from H_n, but differs from it in the following way: for $s \in \mathbf{I}$ and λ such that $s \notin \mathbf{W}_\lambda$, $[s\lambda, s, \lambda]^2$ is equal, in our definition, to $v^2[s\lambda, 1, \lambda]$, while in the definition of [MS] it equals $[s\lambda, 1, \lambda]$.

31.3. We return to the general case. For $s \in \mathbf{I}$, T_s is invertible in H_n; we have

$$T_s^{-1} = v^{-2} T_s + (v^{-2} - 1) \sum_{\lambda; s \in \mathbf{W}_\lambda} 1_\lambda.$$

Moreover,

$$T_s^{-1} T_s^{-1} = v^{-2} + (v^{-2} - 1) \sum_{\lambda; s \in \mathbf{W}_\lambda} T_s^{-1} 1_\lambda.$$

It follows that T_w is invertible in H_n for any $w \in \mathbf{W}$ and that

$$T_{y^{-1}}^{-1} T_{w^{-1}}^{-1} = T_{(yw)^{-1}}^{-1} \text{ if } y, w \in \mathbf{W}, l(yw) = l(y) + l(w).$$

For any $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$ we have $T_{w^{-1}}^{-1} 1_\lambda = 1_{w\lambda} T_{w^{-1}}^{-1}$. Hence there is a unique ring homomorphism $^-: H_n \to H_n$ such that $\overline{T_w} = T_{w^{-1}}^{-1}$ for all $w \in \mathbf{W}$, $\overline{v^m 1_\lambda} = v^{-m} 1_\lambda$ for all λ and all $m \in \mathbf{Z}$. Note that:

(a) *The square of* $^-: H_n \to H_n$ *is* 1. *In particular,* $^-$ *is an isomorphism of rings.* Indeed, the generators $v, v^{-1}, T_w (w \in \mathbf{W})$ and $1_\lambda (\lambda \in \underline{\mathfrak{s}}_n)$ are mapped to themselves by the square of $^-$.

31.4. For any $D \in G/G^0$, the assignment $1_\lambda \mapsto 1_{\underline{D}\lambda}, T_w \mapsto T_{\epsilon_D(w)}$ defines an automorphism of the algebra H_n denoted by $h \mapsto \mathfrak{a}_D(h)$. Let $\tilde{H}_n$ be the free left H_n-module with basis $\{[D]; D \in G/G^0\}$. We regard $\tilde{H}_n$ as an associative $\mathcal{A}$-algebra with unit $1 = [G^0]$ such that $(h[D])(h'[D']) = h\mathfrak{a}_D^i(h')[DD']$ for $h, h' \in H_n, D, D' \in G/G^0$. We have $\mathfrak{a}_D(h) = [D]h[D]^{-1}$ in $\tilde{H}_n$. Define a group involution $^-: \tilde{H}_n \to \tilde{H}_n$ by $\overline{h[D]} = \bar{h}[D]$ where $h \in H_n$, $D \in G/G^0$ and $^-: H_n \to H_n$ is as in 31.3. Then $^-: \tilde{H}_n \to \tilde{H}_n$ is a ring involution (we use the fact that $^-: H_n \to H_n$ commutes with $\mathfrak{a}_D$).

31.5. For $s \in \mathbf{I} \cup \{1\}, \lambda \in \underline{\mathfrak{s}}_n$, we set

$$C_\lambda^s = (T_s + u)1_\lambda = 1_{s\lambda}(T_s + u) \in H_n,$$

where $u = 1$ if $s \in \mathbf{W}_\lambda \cap \mathbf{I}$ and $u = 0$ if $s \notin \mathbf{W}_\lambda \cap \mathbf{I}$. We have

(a) $$\overline{C_\lambda^s} = v^{-2} C_\lambda^s \text{ if } s \in \mathbf{I}, \quad \overline{C_\lambda^1} = C_\lambda^1.$$

If $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I} \cup \{1\}$ and $\lambda \in \underline{\mathfrak{s}}_n$, we set

$$C_\lambda^{\mathbf{s}} = (C_{s_2 \dots s_r \lambda}^{s_1})(C_{s_3 \dots s_r \lambda}^{s_2}) \dots (C_\lambda^{s_r}) \in H_n.$$

One checks that

(b) *the* $\mathcal{A}$*-module of* H_n *is generated by the elements* $C_\lambda^{\mathbf{s}}$ *with* $\mathbf{s}, \lambda$ *as above.*

31.6. If A is a simple perverse sheaf on an algebraic variety V and K is a perverse sheaf on V, we write $(A : K)$ for the multiplicity of A in a Jordan-Hölder series of K.

If $\mathbf{k}$ is an algebraic closure of $\mathbf{F}_q$, V has a fixed $\mathbf{F}_q$-rational structure and K is a mixed complex on V, we denote by ${}^pH_j^i(K)$ the j-th subquotient of the weight filtration of ${}^pH^i(K)$ so that ${}^pH_j^i(K)$ is a pure perverse sheaf of weight j; for A as above, we set

$$\chi_v^A(K) = \sum_{i,j} (-1)^i (A : {}^pH_j^i(K)) v^j \in \mathcal{A}.$$

We return to the general case. In the remainder of this section we fix a connected component D of G. We write $\epsilon : \mathbf{W} \to \mathbf{W}$ instead of $\epsilon_D : \mathbf{W} \to \mathbf{W}$. Until the end of 31.9, J denotes a subset of $\mathbf{I}$ and A denotes a fixed parabolic character sheaf on $Z_{J,D}$. For

(a) $\lambda \in \underline{\mathfrak{s}}_n$ and a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I} \cup \{1\}$ with $s_1 s_2 \dots s_r \underline{D} \lambda = \lambda$, we set

$$\gamma_\lambda^A(\mathbf{s}) = \sum_{j \in \mathbf{Z}} (A : {}^pH^j(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}}))(-v)^j \in \mathcal{A}$$

where $\mathcal{L} \in \mathfrak{s}_n$ is in the isomorphism class λ.

Proposition 31.7. *Let $c_0 = \dim G/U_Q$ where $Q \in \mathcal{P}_J$. There is a unique $\mathcal{A}$-linear map $\zeta^A : H_n[D] \to \mathcal{A}$ such that for any $\lambda \in \underline{\mathfrak{s}}_n$ and any sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I} \cup \{1\}$ we have*

$$\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) = v^{-c_0}\gamma^A_\lambda(\mathbf{s}) \text{ if } s_1 s_2 \dots s_r \underline{D}\lambda = \lambda,$$

$$\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) = 0 \text{ if } s_1 s_2 \dots s_r \underline{D}\lambda \neq \lambda.$$

Let X be the free $\mathcal{A}$-module with basis $[\lambda', \mathbf{s}, \lambda]$ where $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I} \cup \{1\}$ and $\lambda, \lambda' \in \underline{\mathfrak{s}}_n$ are such that $s_1 s_2 \dots s_r \underline{D}\lambda = \lambda'$. Define an $\mathcal{A}$-linear map $b : X \to H_n[D]$ by $b[\lambda', \mathbf{s}, \lambda] = C^{\mathbf{s}}_{\underline{D}\lambda}[D]$. Define an $\mathcal{A}$-linear map $b' : X \to \mathcal{A}$ by $b'[\lambda', \mathbf{s}, \lambda] = v^{-c_0}\gamma^A_\lambda(\mathbf{s})$ if $\lambda' = \lambda$, $b'[\lambda', \mathbf{s}, \lambda] = 0$ if $\lambda' \neq \lambda$. Since b is surjective (see 31.5(b)) we see that it suffices to prove the following statement.

(a) *If $\xi \in \ker(b)$, then $\xi \in \ker(b')$.*

By a standard argument [BBD, §6], to prove (a), we may assume that

(b) $\mathbf{k}$ *is an algebraic closure of a finite field $\mathbf{F}_q$ with $q - 1 \in n\mathbf{Z}$, G has a fixed $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$ which induces the identity map on $\mathbf{W}$ and on G/G^0 and the map $t \mapsto t^q$ on $\mathbf{T}$.*

Then each $\mathcal{L} \in \mathfrak{s}_n$ may be regarded as pure of weight 0 and each $K^{\mathcal{L},w}_{J,D}, K^{\mathcal{L},\mathbf{w}}_{J,D}, \bar{K}^{\mathcal{L},\mathbf{s}}_{J,D}$ (as in 28.13) may be regarded as a mixed complex on $Z_{J,D}$. Define an $\mathcal{A}$-linear map $\tilde{\zeta}^A : H_n[D] \to \mathcal{A}$ by

$$\tilde{\zeta}^A(T_w 1_{\underline{D}\lambda}[D]) = v^{-c_0}\chi^A_v(K^{w,\mathcal{L}}_{J,D}), \text{ if } w\underline{D}\lambda = \lambda,$$

$$\tilde{\zeta}^A(T_w 1_{\underline{D}\lambda}[D]) = 0 \text{ if } w\underline{D}\lambda \neq \lambda$$

where $\mathcal{L} \in \mathfrak{s}_n$ is in the isomorphism class λ.

Let $\lambda, \mathcal{L}, \mathbf{s}, r$ be as in 31.6(a); we will show that

(c) $$\chi^A_v(K^{\mathbf{s},\mathcal{L}}_{J,D}) = v^{c_0}\tilde{\zeta}^A(T_{s_1}T_{s_2}\dots T_{s_r}1_{\underline{D}\lambda}[D]),$$

(d) $$\chi^A_v(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}) = v^{c_0}\tilde{\zeta}^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]).$$

We prove (c) by induction on r. We may assume that $s_i \in \mathbf{I}$ for all i. If $r = 0$, we have $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{1,\mathcal{L}}_{J,D}$ and the result is clear; we have

$$\chi^A_v(K^{1,\mathcal{L}}_{J,D}) = v^{c_0}\tilde{\zeta}^A(1_{\underline{D}\lambda}[D]).$$

Assume that $r \geq 1$. Assume first that $l(s_1 s_2 \dots s_r) = r$. Using 28.13(a) repeatedly we have $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{w,\mathcal{L}}_{J,D}$ where $w = s_1 s_2 \dots s_r$. Then the right-hand side of (c) equals $v^{c_0}\tilde{\zeta}^A(T_w 1_{\underline{D}\lambda})$; the result follows. Assume next that $l(s_1 s_2 \dots s_r) < r$. We can find $j \in [2, r]$ such that $s_j \dots s_{r-1}s_r$ is a reduced expression in $\mathbf{W}$ and $s_{j-1}s_j \dots s_{r-1}s_r$ is not a reduced expression. We can find $s'_j, \dots, s'_{r-1}, s'_r$ in $\mathbf{I}$ such that $s'_j \dots s'_{r-1}s'_r = s_j \dots s_{r-1}s_r$ and $s'_j = s_{j-1}$. Let

$$\mathbf{u} = (s_1, s_2, \dots, s_{j-1}, s'_j, \dots, s'_{r-1}, s'_r).$$

As in the proof of the implication (v) $\implies$ (i) in 28.13 we see that $K^{\mathbf{s},\mathcal{L}}_{J,D} = K^{\mathbf{u},\mathcal{L}}_{J,D}$. Hence

$$\chi^A_v(K^{\mathbf{s},\mathcal{L}}_{J,D}) = \chi^A_v(K^{\mathbf{u},\mathcal{L}}_{J,D}).$$

Moreover, both the right-hand side of (c) and the analogous expression obtained by replacing $\mathbf{s}$ by $\mathbf{u}$ are equal to

$$v^{c_0}\tilde{\zeta}^A(T_{s_1}\dots T_{s_{j-1}}T_{s_j s_{j+1}\dots s_r}1_{\underline{D}\lambda}[D]).$$

Hence to prove the lemma for $\mathbf{s}$ it suffices to prove it for $\mathbf{u}$. Thus we are reduced to the case where $s_j = s_{j-1}$. In this case we will use the notation in 28.13(d),(e),(f).

If $j \notin \mathcal{J}_{\mathbf{s}}$, then from 28.13(f) we have $\chi_v^A(K_{J,D}^{\mathbf{s},\mathcal{L}}) = v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}})$. In this case we have

$$T_{s_{j-1}}T_{s_j}1_{s_{j+1}\dots s_r\underline{D}\lambda} = v^2 1_{s_{j+1}\dots s_r\underline{D}\lambda}$$

since $s_j \notin \mathbf{W}_{s_{j+1}\dots s_r\underline{D}\lambda}$. Hence the right-hand side of (c) is equal to

$$v^2 v^{c_0}\tilde{\zeta}^A(T_{s_1}\dots T_{s_{j-2}}T_{s_{j+1}}T_{s_{j+2}}T_{s_r}1_{\underline{D}\lambda}[D])$$

which by the induction hypothesis is equal to $v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}})$. Thus, (c) holds in this case.

If $j \in \mathcal{J}_{\mathbf{s}}$, then from 28.13(d),(e) we have

$$\chi_v^A(K_{J,D}^{\mathbf{s},\mathcal{L}}) = \chi_v^A(\pi_{1!}\tilde{\mathcal{L}}) + v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}}), \quad v^2\chi_v^A(K_{J,D}^{\mathbf{s}',\mathcal{L}}) = \chi_v^A(\pi_{1!}\tilde{\mathcal{L}}) + \chi_v^A(K_{J,D}^{\mathbf{s}',\mathcal{L}}).$$

(Indeed since the weight filtrations are strictly compatible with morphisms [BBD, 5.3.5], the exact sequences 28.13(d),(e) remain exact when each ${}^pH^i$ is replaced by its pure subquotient of a fixed weight.) It follows that

$$\chi_v^A(K_{J,D}^{\mathbf{s},\mathcal{L}}) = v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}}) + (v^2-1)\chi_v^A(K_{J,D}^{\mathbf{s}',\mathcal{L}}).$$

Using the induction hypothesis for $\mathbf{s}'', \mathbf{s}'$ we see that

$$v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}}) + (v^2-1)\chi_v^A(K_{J,D}^{\mathbf{s}',\mathcal{L}}) = v^2 v^{c_0}\tilde{\zeta}^A(T_{s_1}\dots T_{s_{j-2}}T_{s_{j+1}}T_{s_r}1_{\underline{D}\lambda}[D])$$
$$+ (v^2-1)v^{c_0}\tilde{\zeta}^A(T_{s_1}\dots T_{s_{j-1}}T_{s_{j+1}}\dots T_{s_r}1_{\underline{D}\lambda}[D]).$$

Substituting here

$$v^2 1_{s_{j+1}\dots s_r\underline{D}\lambda} + (v^2-1)T_{s_{j-1}}1_{s_{j+1}\dots s_r\underline{D}\lambda} = T_{s_{j-1}}T_{s_j}1_{s_{j+1}\dots s_r\underline{D}\lambda}$$

which holds since $s_{j-1} = s_j \in \mathbf{W}_{s_{j+1}\dots s_r\underline{D}\lambda}$, we see that $v^2\chi_v^A(K_{J,D}^{\mathbf{s}'',\mathcal{L}}) + (v^2 - 1)\chi_v^A(K_{J,D}^{\mathbf{s}',\mathcal{L}})$ is equal to the right-hand side of (c). Thus (c) holds.

We prove (d). We will use the notation in 28.13(b). Using 28.13(b) we get

$$\chi_v^A(\bar{\pi}_{\mathbf{s}!}f_!^i(f^i)^*\bar{\mathcal{L}}) = \chi_v^A(\bar{\pi}_{\mathbf{s}!}f_!^{i+1}(f^{i+1})^*\bar{\mathcal{L}}) + \sum_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}};|\mathcal{J}|=i}\chi_v^A(K_{J,D}^{\mathbf{s}_{\mathcal{J}},\mathcal{L}})$$

for any i. Summing these equalities over all $i \ge 0$ we find

$$\chi_v^A(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}}) = \sum_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}}\chi_v^A(K_{J,D}^{\mathbf{s}_{\mathcal{J}},\mathcal{L}}).$$

We now use (c) for each $\mathbf{s}_{\mathcal{J}}$ in the last sum. We see that $\chi_v^A(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}})$ is a sum of 2^k terms (each term is v^{c_0} times a product of basis elements of H_n times $[D]$) where $k = |\mathcal{J}_{\mathbf{s}}|$. Clearly, the right-hand side of (d) is the sum of the same 2^k terms. This proves (d).

As in the proof of 29.14, $\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}}$ is pure of weight 0, hence ${}^pH_j^i(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}})$ equals ${}^pH^i(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}})$ if $i = j$ and equals 0 if $i \ne j$. It follows that $\chi_v^A(\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}}) = \gamma_\lambda^A(\mathbf{s})$, hence (d) implies $b'[\lambda, \mathbf{s}, \lambda] = \tilde{\zeta}^A(C_{\underline{D}\lambda}^{\mathbf{s}})[D]) = \tilde{\zeta}^A(c[\lambda, \mathbf{s}, \lambda])$. Clearly, $b'[\lambda', \mathbf{s}, \lambda] = \tilde{\zeta}^A(b[\lambda', \mathbf{s}, \lambda])$ if $\lambda \ne \lambda'$ (both sides are 0). We see that $b' = \tilde{\zeta}^A \circ b$. Hence (a) holds. The proposition is proved. The previous proof shows also that

(e) $$\zeta^A = \tilde{\zeta}^A : H_n[D] \to \mathcal{A}.$$

Lemma 31.8. *Let $H_{J,n}$ be the subalgebra of H_n generated as an $\mathcal{A}$-submodule by $\{T_w 1_\lambda; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n\}$. For any $h \in H_{J,n}, h' \in H_n$ we have $\zeta^A(hh'[D]) = \zeta^A(h'[D]h)$.*

By 31.5(b) and its analogue for $H_{J,n}$, we may assume that $h = a_1 \dots a_{p-1}$, $h' = a_p a_{p+1} \dots a_r$ where

$$a_i = C^{s_i}_{s_{i+1}\dots s_{p-1}\underline{D}\tilde{\lambda}} \text{ for } i \in [1, p-1],$$
$$a_i = C^{s_i}_{s_{i+1}\dots s_r\underline{D}\lambda} \text{ for } i \in [p, r];$$

here $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I}$, $1 \le p \le r$, $s_i \in J$ for $i \in [1, p-1]$ and $\lambda, \tilde{\lambda} \in \underline{\mathfrak{s}}_n$. If $s_p \dots s_r\underline{D}\lambda \ne \underline{D}\tilde{\lambda}$ or $s_1 \dots s_{p-1}\underline{D}\tilde{\lambda} \ne \lambda$, then $\zeta^A(hh'[D]) = 0$ and $\zeta^A(h'[D]h) = 0$. Thus we may assume that $\underline{D}\tilde{\lambda} = s_p \dots s_r\underline{D}\lambda$ and $s_1 \dots s_{p-1}\underline{D}\tilde{\lambda} = \lambda$. Then we have

$$a_i = C^{s_i}_{s_{i+1}\dots s_r\underline{D}\lambda} \text{ for } i \in [1, r]$$

and $s_1 s_2 \dots s_r\underline{D}\lambda = \lambda$. Hence $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ is defined (with $\mathcal{L} \in \mathfrak{s}_n$ in the isomorphism class λ). Let

$$\mathbf{s}' = (s_p, s_{p+1}, \dots, s_r, \epsilon(s_1), \dots, \epsilon(s_{p-1})),$$

let $\lambda' = s_{p-1} \dots s_1\lambda$ and let $\mathcal{L}' \in \mathfrak{s}_n$ be in the isomorphism class λ'. Then

$$s_p s_{p+1} \dots s_r \epsilon(s_1) \dots \epsilon(s_{p-1})\underline{D}\lambda' = \lambda',$$

hence $\bar{K}^{\mathbf{s}',\mathcal{L}'}_{J,D}$ is defined. Using 28.16(b) $p-1$ times, we have $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} = \bar{K}^{\mathbf{s}',\mathcal{L}'}_{J,D}$. Hence $\gamma^A_\lambda(\mathbf{s}) = \gamma^A_{\lambda'}(\mathbf{s}')$. By 31.7, we have $\zeta^A(a_1 a_2 \dots a_r[D]) = \zeta^A(a'_1 a'_2 \dots a'_r[D])$ with a_i as above and

$$a'_i = C^{s_{p+i-1}}_{s_{p+i}\dots s_r\epsilon(s_1\dots s_{p-1})\underline{D}\lambda'} = C^{s_{p+i-1}}_{s_{p+i}\dots s_r\underline{D}\lambda} = a_{p+i-1}$$

for $i \in [1, r-p+1]$,

$$a'_i = C^{\epsilon(s_{i-r+p-1})}_{\epsilon(s_{i-r+p}\dots s_{p-1})\underline{D}\lambda'} = C^{\epsilon(s_{i-r+p-1})}_{\underline{D}s_{i-r+p}\dots s_r\underline{D}\lambda} = [D]a_{i-r+p-1}[D]^{-1}$$

for $i \in [r-p+2, r]$. We have therefore $\zeta^A(hh'[D]) = \zeta^A(h'[D]h)$. The lemma is proved.

Lemma 31.9. *Let $^-: \mathcal{A} \to \mathcal{A}$ be the ring homomorphism such that $\overline{v^t} = v^{-t}$ for all $t \in \mathbf{Z}$. For any $h \in H_n$ we have $\zeta^A(\overline{h}[D]) = \overline{\zeta^A(h[D])}$.*

Using 31.5(b) we may assume that $h = C^{\mathbf{s}}_\lambda$ where $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I}$ and $\lambda \in \underline{\mathfrak{s}}_n$. Using 31.5(a) we see that $\overline{h} = v^{-2r}h$. Hence it suffices to show that $\overline{\zeta^A(h[D])} = v^{-2r}\zeta^A(h[D])$. If $s_1 s_2 \dots s_r\underline{D}\lambda \ne \lambda$, then $\zeta^A(h[D]) = 0$ and the result is obvious. Thus we may assume that $s_1 s_2 \dots s_r\underline{D}\lambda = \lambda$ so that $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ is defined (with $\mathcal{L} \in \mathfrak{s}_n$ in the isomorphism class λ). By 31.7 we have $\zeta^A(h[D]) = v^{-c_0}\gamma^A_\lambda(\mathbf{s})$. Hence it suffices to show that

$$\overline{\gamma^A_\lambda(\mathbf{s})} = v^{-2m}\gamma^A_\lambda(\mathbf{s})$$

where $m = r + c_0$. Using 28.17(a) we have

$$\overline{\gamma^A_\lambda(\mathbf{s})} = \sum_j (-v)^{-j}(A : {}^pH^j(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D})) = \sum_j (-v)^{-j}(A : {}^pH^{2m-j}(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}))$$
$$= \sum_j (-v)^{j-2m}(A : {}^pH^j(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D})) = v^{-2m}\gamma^A_\lambda(\mathbf{s}).$$

The lemma is proved.

Lemma 31.10. *Let $I \subset \mathbf{I}, y \in {}^I\mathbf{W}, s \in \mathbf{I}, \lambda' \in \underline{\mathfrak{s}}_n$. We have*

(a) $$T_y C^s_{\lambda'} = v^{2\delta(y)} \sum_{y_1} C^{t_1}_{y_1\lambda'} T_{y_1};$$

here $\delta(y)$ is 1 if $ys < y, ys \in {}^I\mathbf{W}$ and is 0 otherwise; the sum is taken over all $y_1 \in {}^I\mathbf{W} \cap \{y, ys\}$ such that

$$ys \notin \mathbf{W}_I y, y_1 = y \implies s \in \mathbf{W}_{\lambda'};$$

$t_1 \in I \cup \{1\}$ is defined by $ys = t_1 y$ if $ys \in \mathbf{W}_I y$ and $t_1 = 1$ if $ys \notin \mathbf{W}_I y$.

If $s \notin \mathbf{W}_{\lambda'}, ys > y$, then both sides of (a) are equal to $T_{ys}1_{\lambda'}$. If $s \notin \mathbf{W}_{\lambda'}, ys < y$, then both sides of (a) are equal to $v^2 T_{ys}1_{\lambda'}$. If $s \in \mathbf{W}_{\lambda'}, ys > y$, then both sides of (a) are equal to $(T_{ys} + T_y)1_{\lambda'}$. If $s \in \mathbf{W}_{\lambda'}, ys < y$, then both sides of (a) are equal to $v^2(T_{ys} + T_y)1_{\lambda'}$. The lemma is proved.

Lemma 31.11. *Let $I \subset \mathbf{I}, y \in {}^I\mathbf{W}, \lambda' \in \underline{\mathfrak{s}}_n$ and let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$. We have*

$$T_y C^{\mathbf{s}}_{\lambda'} = \sum_{\mathbf{y}} v^{2\delta(\mathbf{y})} C^{\mathbf{t}}_{y_r\lambda'} T_{y_r};$$

here the sum is taken over all sequences $\mathbf{y} = (y_0, y_1, \dots, y_r)$ in ${}^I\mathbf{W}$ such that $y = y_0$ and $y_i \in \{y_{i-1}, y_{i-1}s_i\}$ for $i \in [1, r]$, $\mathbf{t} = (t_1, t_2, \dots, t_r)$ is the sequence in $I \cup \{1\}$ defined by $y_{i-1}s_i = t_i y_{i-1}$ if $y_{i-1}s_i \in \mathbf{W}_I y_{i-1}$ and $t_i = 1$ if $y_{i-1}s_i \notin \mathbf{W}_I y_{i-1}$; these are subject to the requirement

$$i \in [1, r], t_i = 1, y_{i-1} = y_i \implies s_i \in \mathbf{W}_{s_{i+1}\dots s_r\lambda'};$$

moreover,

(a) $$\delta(\mathbf{y}) = \sharp(i \in [1, r]; y_{i-1}s_i < y_{i-1}, y_{i-1}s_i \in {}^I\mathbf{W}).$$

This follows by applying r times Lemma 31.10.

31.12. Until the end of 31.14 we fix D, P, L, G', D' as in 29.1. Let $I \subset \mathbf{I}$ be such that $P \in \mathcal{P}_I$. Since the Weyl group of L is naturally the subgroup $\mathbf{W}_I$ of $\mathbf{W}$ and the canonical torus of L may be identified with $\mathbf{T}$ as in 29.1, we may identify $H_{I,n}$ (as in 31.8 with I instead of J) with the $\mathcal{A}$-algebra defined in terms of L in the same way as H_n was defined in 31.2 in terms of G^0. Note that H_n is naturally a left $H_{I,n}$-module (using left multiplication). This $H_{I,n}$-module is free with basis $\{T_y; y \in {}^I\mathbf{W}\}$. (The elements βT_y where β runs through $\{T_w 1_\lambda; w \in \mathbf{W}_I, \lambda \in \underline{\mathfrak{s}}_n\}$ and $y \in {}^I\mathbf{W}$, form the $\mathcal{A}$-basis 31.2(a) of H_n.) Applying to this basis the ring involution$^-$: $H_n \to H_n$ which restricts to an involution of $H_{I,n}$ we see that $\{T^{-1}_{y^{-1}}; y \in {}^I\mathbf{W}\}$ is again a basis of the left $H_{I,n}$-module H_n.

Lemma 31.13. *Let $\lambda \in \underline{\mathfrak{s}}_n$ and let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D}\lambda = \lambda$. Let $\mathcal{L} \in \mathfrak{s}_n$ be in the isomorphism class λ. Let $m = r + \dim G$. For any $j \in \mathbf{Z}$ we have*

$${}^pH^j(\mathrm{res}^{D'}_D(\bar{K}^{\mathbf{s},\mathcal{L}}_D)) \cong {}^pH^{2m-j}(\mathrm{res}^{D'}_D(\bar{K}^{\mathbf{s},\mathcal{L}}_D)).$$

Let Ξ be the set of all pairs $(\mathbf{y}, \mathbf{t})$ where $\mathbf{y} = (y_0, y_1, \dots, y_r)$ is a sequence in ${}^I\mathbf{W}$ such that $y_r = \epsilon(y_0)$ and $y_i \in \{y_{i-1}, y_{i-1}s_i\}$ for $i \in [1, r]$, $\mathbf{t} = (t_1, t_2, \dots, t_r)$ is the sequence in $I \cup \{1\}$ defined by $y_{i-1}s_i = t_i y_{i-1}$ if $y_{i-1}s_i \in \mathbf{W}_I y_{i-1}$ and $t_i = 1$ if $y_{i-1}s_i \notin \mathbf{W}_I y_{i-1}$; these are subject to the requirement

$$i \in [1, r], t_i = 1, y_{i-1} = y_i \implies s_i \in \mathbf{W}_{s_{i+1}\dots s_r\underline{D}\lambda}.$$

Using 29.14 we see that it suffices to show that for any j we have

$$\bigoplus_{(\mathbf{y},\mathbf{t})\in\Xi} {}^pH^{j-2\mathbf{d}(\mathbf{y})}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}}) \cong \bigoplus_{(\mathbf{y},\mathbf{t})\in\Xi} {}^pH^{2m-j-2\mathbf{d}(\mathbf{y})}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}}).$$

Here both sides are semisimple complexes (see 28.12); hence it suffices to show that for any character sheaf A' on D' we have

$$\sum_{(\mathbf{y},\mathbf{t})\in\Xi;j}(-v)^j(A' : {}^pH^{j-2\mathbf{d}(\mathbf{y})}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}}))$$
$$= \sum_{(\mathbf{y},\mathbf{t})\in\Xi;j}(-v)^j(A' : {}^pH^{2m-j-2\mathbf{d}(\mathbf{y})}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}})).$$

or equivalently

$$v^{2\mathbf{d}(\mathbf{y})}\sum_{(\mathbf{y},\mathbf{t})\in\Xi;j}(-v)^j(A' : {}^pH^{j}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}}))$$
$$= v^{2m-2\mathbf{d}(\mathbf{y})}\sum_{(\mathbf{y},\mathbf{t})\in\Xi;j}(-v)^{-j}(A' : {}^pH^{j}(\bar{K}_{D'}^{\mathbf{t},{}^{y_0}\mathcal{L}})).$$

Using 31.7 for $G', A', \mathbf{t}$ instead of $G, A, \mathbf{s}$, we see that it suffices to show

$$\sum_{(\mathbf{y},\mathbf{t})\in\Xi}\zeta^{A'}(C^{\mathbf{t}}_{\underline{D}y_0\lambda}[D])v^{\dim L}v^{2\mathbf{d}(\mathbf{y})} = \sum_{(\mathbf{y},\mathbf{t})\in\Xi}\overline{\zeta^{A'}(C^{\mathbf{t}}_{\underline{D}y_0\lambda}[D])v^{\dim L}v^{2\mathbf{d}(\mathbf{y})}}v^{2m}.$$

Here $\zeta^{A'} : H_{I,n}[D] \to \mathcal{A}$ is defined as in 3.7 for G', A' instead of G, A. We substitute $\mathbf{d}(\mathbf{y}) = \delta(y) + \dim U_P$ with $\delta(\mathbf{y})$ as in 31.11(a), and use $\underline{D}y_0\lambda = y_r\underline{D}\lambda$ and $\dim L + 2\dim U_P = \dim G$; we see that it suffices to show

$$\zeta^{A'}(\Psi[D]) = \overline{\zeta^{A'}(\Psi[D])}v^{2r}$$

where $\Psi = \sum_{(\mathbf{y},\mathbf{t})\in\Xi} C^{\mathbf{t}}_{y_r\underline{D}\lambda}v^{2\delta(\mathbf{y})} \in H_{I,n}$. We write the matrix of right multiplication by $C^{\mathbf{s}}_{\underline{D}\lambda}$ in H_n (an $H_{I,n}$-linear map) in the $H_{I,n}$-basis $\{T_y; y \in {}^I\mathbf{W}\}$:

(a)
$$T_yC^{\mathbf{s}}_{\underline{D}\lambda} = \sum_{y'\in{}^I\mathbf{W}} a_{y,y'}T_{y'}$$

where $y \in {}^I\mathbf{W}$ and $a_{y,y'} \in H_{I,n}$. Using 31.11 (with $\lambda' = \underline{D}\lambda$) we see that $\sum_y a_{y,\epsilon(y)} = \Psi$ where y runs over ${}^I\mathbf{W}$. Hence it suffices to show that

$$\zeta^{A'}(\sum_y a_{y,\epsilon(y)}[D]) = \overline{\zeta^{A'}(\sum_y a_{y,\epsilon(y)}[D])}v^{2r}.$$

Using 31.9 (for G', A' instead of G, A) we see that

$$\overline{\zeta^{A'}(\sum_y a_{y,\epsilon(y)}[D])} = \zeta^{A'}(\sum_y \overline{a_{y,\epsilon(y)}}[D]).$$

Hence it suffices to show that

$$\sum_y \zeta^{A'}(a_{y,\epsilon(y)}[D]) = \sum_y \zeta^{A'}(\overline{a_{y,\epsilon(y)}}[D])v^{2r}.$$

Applying $^- : H_n \to H_n$ to the equality (a) and using $\overline{C^{\mathbf{s}}_{\underline{D}\lambda}} = v^{-2r}C^{\mathbf{s}}_{\underline{D}\lambda}$ (see the proof of Lemma 31.9) we obtain

$$v^{-2r}T^{-1}_{y^{-1}}C^{\mathbf{s}}_{\underline{D}\lambda} = \sum_{y'\in{}^I\mathbf{W}} \overline{a_{y,y'}}T^{-1}_{y'^{-1}}.$$

Since $\{T_y; y \in {}^I\mathbf{W}\}, \{T_{y^{-1}}^{-1}; y \in {}^I\mathbf{W}\}$ are two $H_{I,n}$-bases of H_n, we have

$$T_y = \sum_{y'} c_{y,y'} T_{y'^{-1}}^{-1}, \quad T_{y^{-1}}^{-1} = \sum_{y'} d_{y,y'} T_{y'}$$

where y, y' run over ${}^I\mathbf{W}$ and $c_{y,y'}, d_{y,y'} \in H_{I,n}$. For any y we have

$$\sum_{y'} \overline{a_{y,y'}} T_{y'^{-1}}^{-1} = v^{-2r} T_{y^{-1}}^{-1} C_{\underline{D}\lambda}^{\mathbf{s}} = v^{-2r} \sum_{y''} d_{y,y''} T_{y''} C_{\underline{D}\lambda}^{\mathbf{s}}$$
$$= v^{-2r} \sum_{y''} d_{y,y''} \sum_{y_1} a_{y'',y_1} T_{y_1} = v^{-2r} \sum_{y''} d_{y,y''} \sum_{y_1} a_{y'',y_1} \sum_{y'} c_{y_1,y'} T_{y'^{-1}}^{-1},$$

hence

$$\overline{a_{y,y'}} = v^{-2r} \sum_{y'',y_1} d_{y,y''} a_{y'',y_1} c_{y_1,y'}$$

for any y, y'. Hence it suffices to show that

$$\sum_y \zeta^{A'}(a_{y,\epsilon(y)}[D]) = \sum_y \zeta^{A'}(\sum_{y'',y_1} d_{y,y''} a_{y'',y_1} c_{y_1,\epsilon(y)}[D]).$$

By 31.8 (for G', A' instead of G, A) we have

$$\zeta^{A'}(d_{y,y''} a_{y'',y_1} c_{y_1,\epsilon(y)}[D]) = \zeta^{A'}(a_{y'',y_1} c_{y_1,\epsilon(y)}[D] d_{y,y''}).$$

Hence it suffices to show that

$$\sum_y \zeta^{A'}(a_{y,\epsilon(y)}[D]) = \zeta^{A'}(\sum_{y,y'',y_1} a_{y'',y_1} c_{y_1,\epsilon(y)}[D] d_{y,y''}).$$

We have

$$\sum_{y'} d_{\epsilon(y),\epsilon(y')} T_{\epsilon(y')} = T_{\epsilon(y)^{-1}}^{-1} = [D] T_{y^{-1}}^{-1} [D]^{-1}$$
$$= \sum_{y'} [D] d_{y,y'} [D]^{-1} [D] T_{y'} [D]^{-1} = \sum_{y'} [D] d_{y,y'} [D]^{-1} T_{\epsilon(y')},$$

hence $d_{\epsilon(y),\epsilon(y')} = [D] d_{y,y'} [D]^{-1}$. Hence it suffices to show that

(b) $$\sum_y \zeta^{A'}(a_{y,\epsilon(y)}[D]) = \zeta^{A'}(\sum_{y,y'',y_1} a_{y'',y_1} c_{y_1,\epsilon(y)} d_{\epsilon(y),\epsilon(y'')}[D]).$$

from the definitions, $\sum_y c_{y_1,\epsilon(y)} d_{\epsilon(y),\epsilon(y'')}$ is 1 if $y_1 = \epsilon(y'')$ and is 0, otherwise. Hence (b) holds. The lemma is proved.

Theorem 31.14. *Let D, P, L, G', D' be as in 29.1. Let A be a character sheaf on D. Then $\mathrm{res}_D^{D'} A$ is a direct sum of character sheaves on D'.*

We can find $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ and a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D} \in \mathbf{W}_{\mathcal{L}}^{\bullet}$ and such that A is a direct summand of ${}^pH^i(K)$ for some $i \in \mathbf{Z}$, where $K = \bar{K}_D^{\mathbf{s},\mathcal{L}}[m]$, $m = r + \dim G$. Let $K' = \mathrm{res}_D^{D'}(K)$. For any i, let

$$K_i = {}^pH^i(K), K'_i = \mathrm{res}_D^{D'}(K_i).$$

For any character sheaf A' on D', let $b_{i,j} = (A' : {}^pH^j(K'_i))$, $b_j = (A' : {}^pH^j(K'))$. From 28.12(b) we have

$${}^pH^j(K') = {}^pH^j(\bigoplus_i \mathrm{res}_D^{D'}(K_i)[-i]) = \bigoplus_i {}^pH^{j-i}(K'_i),$$

hence $b_j = \sum_i b_{i,j-i}$. Using 31.13, which is applicable since $\mathcal{L} \in \mathfrak{s}_n$ for some $n \in \mathbf{N}^*_{\mathbf{k}}$, we get $b_j = b_{-j}$ for all j, hence

(a)
$$0 = \sum_j j b_j = \sum_{i,j} j b_{i,j-i} = \sum_{i,j} (i+j) b_{i,j}.$$

From 28.17(a) we have $K_i = K_{-i}$. It follows that $b_{i,j} = b_{-i,j}$ so that $\sum_{i,j} i b_{i,j} = 0$. Introducing this into (a) we find $\sum_{i,j} j b_{i,j} = 0$. From 28.12(b) and 30.6(b) we see that $b_{i,j} = 0$ for all $j > 0$. Therefore, we have $\sum_{i,j;j \le 0} j b_{i,j} = 0$. Since $j b_{i,j} \le 0$ for all terms of the previous sum, we must have $j b_{i,j} = 0$ for all i, j. It follows that $b_{i,j} = 0$ for $j \neq 0$. Since, by 29.15, ${}^p H^j(K'_i)$ is a direct sum of character sheaves, it follows that ${}^p H^j(K'_i) = 0$ for $j \neq 0$. In other words, for any i, K'_i is a perverse sheaf on D' which is a direct sum of character sheaves. Since A is a direct summand of K_i for some i, we see that $\text{res}_D^{D'} A$ is a direct summand of $\text{res}_D^{D'}(K_i) = K'_i$, hence $\text{res}_D^{D'} A$ is a perverse sheaf on D' which is a direct sum of character sheaves. The theorem is proved.

Corollary 31.15. *Let A be a character sheaf on D. Then A is cuspidal (see 23.3) if and only if it is strongly cuspidal (see 23.3).*

We may assume that A is cuspidal. Let P, L, D' be as in 31.14 such that $P \neq G^0$. Since A is cuspidal we have ${}^p H^i(\text{res}_D^{D'} A) = 0$ for all $i \geq 0$. By 31.14 we have ${}^p H^i(\text{res}_D^{D'} A) = 0$ for all $i \neq 0$. Hence ${}^p H^i(\text{res}_D^{D'} A) = 0$ for all i. It follows that $\text{res}_D^{D'} A = 0$. Thus, A is strongly cuspidal. The corollary is proved.

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