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NICE PARABOLIC SUBALGEBRAS OF REDUCTIVE LIE ALGEBRAS

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Abstract. This paper gives a classification of parabolic subalgebras of simple Lie algebras over $\mathbb{C}$ that are complexifications of parabolic subalgebras of real forms for which Lynch's vanishing theorem for generalized Whittaker modules is non-vacuous. The paper also describes normal forms for the admissible characters in the sense of Lynch (at least in the quasi-split cases) and analyzes the important special case when the parabolic is defined by an even embedded TDS (three-dimensional simple Lie algebra).

Introduction

If $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$ and if $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, then there is a $\mathbb{Z}$-grade of $\mathfrak{g}$ as a Lie algebra, $\mathfrak{g} = \sum_j \mathfrak{g}_j$ such that $\mathfrak{p} = \sum_{j\geq 0} \mathfrak{g}_j$ and if $\mathfrak{n} = \sum_{j>0} \mathfrak{g}_j$, then $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{g}_1$ projects bijectively onto the abelianization $(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])$ of $\mathfrak{n}$. The purpose of this article is to give a classification of those parabolic subalgebras such that there is a Richardson element $x \in \mathfrak{n}$ such that $x \in \mathfrak{g}_1$. That is, there exists $x \in \mathfrak{g}_1$ such that $[\mathfrak{p}, x] = \mathfrak{n}$. A parabolic subalgebra will be called *nice* if it satisfies this condition. These parabolic subalgebras are exactly the complexifications of the real parabolic subalgebras whose nilradicals support admissible Lie algebra homomorphisms to $i\mathbb{R}$ in the sense of Lynch's thesis [L]. In that thesis Lynch proved a generalization of Kostant's vanishing theorem for Whittaker modules [K] (valid for generic Lie algebra homomorphisms to $i\mathbb{R}$ of nilradicals of Borel subalgebras of quasisplit real forms). Of course, Lynch's theorem is vacuous if the nilradical of the parabolic subalgebra admits no such homomorphisms. He introduced the term admissible for the parabolic sublagebras whose nilradicals admit admissible homomorphisms. Thus an admissible parabolic in the sense of Lynch is a real form of a nice parabolic in our sense.

It is clear that a parabolic subalgebra is nice if and only if its intersection with each ideal is nice. Thus it is enough to do the classification for simple Lie algebras over $\mathbb{C}$. The complete classification is given in section 1 and the rest of the paper is devoted to the proof of the correctness of the list (sections 3 and 4), and to a description of the corresponding Richardson elements (admissible elements in the sense of Lynch). The proofs of the assertions in section 5 will appear in [B].

In [W] the second named author used his extension of the Lynch results to prove a holomorphic continuation of generalized Jacquet integrals for degenerate principal

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series under the real analogue of the condition of niceness. These results contain all known cases of continuation, holomorphy and (essentially) uniqueness for Jaquet integrals and Whittaker models. Thus the results of this paper explain both the range of applicability of those results and their limitations.

In [L] Lynch studies the classification of his admissible parabolic subalgebras. His results are correct for types A_n, F_4 and G_2.

The paper [EK] studies the notion of a *good grade* of a semi-simple algebra. This is a $\mathbb{Z}$-grade of $\mathfrak{g} = \sum_j \mathfrak{g}_j$ as a Lie algebra such that there is an element $x \in \mathfrak{g}_2$ such that $\mathrm{ad}(x)$ is injective on $\sum_{j<0} \mathfrak{g}_j$. Theorem 2.1 implies that a good grade with all odd components equal to 0 defines a nice parabolic $\sum_{j\geq 0} \mathfrak{g}_j$. Thus a classification of good grades yields, as a special case, a classification of nice parabolic subalgebras. That said, we have decided that the special case was of sufficient importance to have an independent exposition even if the paper [EK] had no errors (see the remark before Theorem 1.4). Also, our description of the answer for the classical groups is (we hope) simpler than that given in [EK]. In addition, we have endeavored to give enough detail that a serious reader could with little additional effort check that the results are correct.

Acknowledgments. The authors would like to thank Elashvili for informing them of his work with Kac. For obvious reasons there is substantial overlap between our papers. Many of the general results in section 2 can be found in the second named author's manuscript [W] that has been freely available on his website (in various forms) for at least two years and in [EK].

1. Statement of the results

If not specified otherwise, $\mathfrak{g}$ will denote a simple Lie algebra over the complex numbers. Fix a Borel subalgebra $\mathfrak{b}$ in $\mathfrak{g}$, let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$. We will denote the set of simple roots relative to this choice by $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We always use the Bourbaki-numbering of simple roots.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ (where $\mathfrak{m}$ is a Levi factor and $\mathfrak{u}$ the corresponding nilpotent radical of $\mathfrak{p}$). After conjugation we can assume that $\mathfrak{p}$ contains the chosen Borel subalgebra and $\mathfrak{m} \supset \mathfrak{h}$. If $\mathfrak{b}$ is fixed, then we will say that $\mathfrak{p}$ is standard if $\mathfrak{p} \supset \mathfrak{b}$ from now on. In particular, if $\mathfrak{p}$ is standard, then it is given by a subset of Δ, namely the simple roots such that both root spaces $\mathfrak{g}_{\pm\alpha}$ belong to the Levi factor of $\mathfrak{p}$.

Thus such a parabolic subalgebra is described by an n-tuple, $(u_1, \dots, u_n)$ in $\{0,1\}^n$: ones correspond to simple roots with root spaces not in $\mathfrak{m}$. Equivalently, a parabolic subalgebra is given by a coloring of the Dynkin diagram of the Lie algebra: a black (colored) node corresponds to a simple root whose root space belongs to $\mathfrak{m}$. Here, one has to be very careful since there exist different notations. Our choice was motivated by the coloring for Satake diagrams. Let $(u_1, \dots, u_n)$ define the parabolic subalgebra $\mathfrak{p}$ and and $H \in \mathfrak{h}$ be defined by $\alpha_i(H) = u_i$. If we set $\mathfrak{g}_i = \{x \in \mathfrak{g} | [H,x] = ix\}$, then $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}_i$.

1.1. Results in the classical cases.

As is usual, we will refer to the simple Lie algebras of type $\mathrm{A}_n, \mathrm{B}_n, \mathrm{C}_n, \mathrm{D}_n$ as the classical Lie algebras and the remaining five simple Lie algebras will be called exceptional. We realize the classical Lie algebras as subalgebras of $\mathfrak{gl}_N$ for $N = n+1, 2n+1, 2n, 2n$ respectively. With A_n the trace zero matrices, $\mathrm{B}_n, \mathrm{D}_n$ the orthogonal Lie algebra of the symmetric form with matrix with all entries 0 except for those on the skew diagonal which are 1 and C_n the symplectic Lie algebra for the symplectic form with matrix whose only nonzero entries are skew diagonal and the first n are 1 and the last n are -1. With this realization we take as our choice of Borel subalgebra the intersection of the corresponding Lie algebra with the upper triangular matrices in $\mathfrak{gl}_N$. We will call a parabolic subalgebra that contains this Borel subalgebra standard. If $\mathfrak{p}$ is a standard parabolic subalgebra, then we refer to the Levi factor that contains the diagonal Cartan subalgebra, $\mathfrak{h}$, by $\mathfrak{m}$ and call it the standard Levi factor.

Thus for all classical Lie algebras the standard Levi factor is then in diagonal block form given by a sequence of square matrices on the diagonal. For the orthogonal and symplectic Lie algebras, these sequences are palindromic. If $\mathfrak{p}$ is a parabolic subalgebra for one of these Lie algebras and if $\mathfrak{m}$ is the standard Levi factor of the parabolic subalgebra to which it is conjugate, then we will say that $\mathfrak{m}$ is the standard Levi factor.

Theorem 1.1. *For type A_n a parabolic subalgebra is nice if and only if the sequence of the block lengths of its standard Levi factor is unimodal.*

Theorem 1.2. *For type B_n a parabolic subalgebra $\mathfrak{p}$ is nice if and only if its standard Levi factor either has unimodal block lengths or the block lengths are of the form*

$$a_1 \leq a_2 \leq \cdots \leq a_r > b_1 = b_2 = \cdots = b_s < a_r \geq \cdots \geq a_1$$

with $b_1 = a_r - 1$ and if $r > 1$, then $a_{r-1} < a_r$.

Theorem 1.3. *For type C_n a parabolic subalgebra is nice if and only if the sequence of block lengths of its standard Levi factor is unimodal and if the number of blocks is odd, then each odd block length occurs exactly twice.*

Before we state our result for D_n a word should be said about the ambiguity in describing parabolic subalgebras in the case of D_n. We note that the intersection of the standard parabolic subalgebra with block sizes $[a_1, a_2, \ldots, a_r, 1, 1, a_r, \ldots, a_2, a_1]$ for $\mathrm{GL}(d)$ ($d = 2(a_1 + \cdots + a_r + 1)$) with $\mathfrak{so}(d)$ is the same as the intersection with the standard parabolic with block lengths $[a_1, a_2, \ldots, a_r, 2, a_r, \ldots, a_2, a_1]$. In the following theorem we we will only look at the second version of the parabolic subgroup. The first two types in Theorem 6.4 (i),(ii) in [EK] do not appear in our result directly because of this choice.

We should also point out that there is a difference between our statement and that of [EK] involving the last part of this result and Theorem 6.4 (i) and (ii) in [EK]. Their list is missing all cases with s even and $a_r > 3$. In addition to having a (perhaps too) detailed proof of the following result in this paper the authors have done extensive computer computations which agree with our formulation for $n \leq 20$.

Theorem 1.4. *For type D_n a parabolic subalgebra is nice if and only if its standard Levi factor (taking into account the choice made above) has one of the following forms:*

1) It is unimodal with an odd number of blocks.

2) It is unimodal with an even number of block lengths and the odd block lengths occur exactly twice.

3) The block lengths are of the form

$$a_1 \leq a_2 \leq \cdots \leq a_r > b_1 = b_2 = \cdots = b_s < a_r \geq \cdots \geq a_1$$

with $b_1 = a_r - 1$, a_r is odd and and if s is even, then the odd block lengths occur exactly twice.

1.2. **Results in the exceptional cases.** In this subsection we will state the classification of nice parabolic subalgebras for the exceptional simple Lie algebras. The parabolic subalgebra will be given by an n-tuple where n is the rank and the entries are $\alpha_i(H)$ where H is the element that gives the grade corresponding to the parabolic subalgebra and the α_i are the simple roots in the Bourbaki order.

The only nice parabolic subalgebras in type G_2, F_4 are those that are given by an even $\mathfrak{sl}_2$-triple (see the next section for the definition). They are listed below.

G_2	F_4
$(1,1)$	$(1,1,1,1)$
$(1,0)$	$(1,1,0,1)$
$(0,0)$	$(1,1,0,0)$
	$(1,0,0,0)$
	$(0,1,0,1)$
	$(0,1,0,0)$
	$(0,0,0,1)$
	$(0,0,0,0)$

The nice parabolic subalgebras of type E are given in the following table:

	E_6	E_7	E_8
1	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,1,1)$
2	$(1,1,1,0,1,1)$	$(1,1,1,0,1,1,1)$	$(1,1,1,0,1,1,1,1)$
3	$(1,1,1,0,1,0)$	$(1,1,1,0,1,0,1)$	$(1,1,1,0,1,0,1,1)$
4	$(1,1,0,1,0,1)$	$(1,1,0,0,1,0,1)$	$(1,0,1,0,0,0,0,1)$
5	$(1,1,0,0,1,0)$	$(1,1,0,0,0,0,1)$	$(1,0,0,1,0,1,1,1)$
6	$(1,1,0,0,0,1)$	$(1,0,1,1,0,1,0)$	$(1,0,0,1,0,1,0,1)$
7	$(1,1,0,0,0,0)$	$(1,0,1,0,0,1,0)$	$(1,0,0,1,0,0,1,1)$
8	$(1,0,1,1,0,1)$	$(1,0,1,0,0,0,0)$	$(1,0,0,1,0,0,1,0)$
9	$(1,0,1,0,0,1)$	$(1,0,0,1,0,1,1)$	$(1,0,0,0,1,0,0,1)$
10	$(1,0,1,0,0,0)$	$(1,0,0,1,0,1,0)$	$(1,0,0,0,0,1,1,1)$
11	$(1,0,0,1,1,1)$	$(1,0,0,1,0,0,1)$	$(1,0,0,0,0,1,0,1)$
12	$(1,0,0,1,0,1)$	$(1,0,0,0,1,0,0)$	$(1,0,0,0,0,1,0,0)$
13	$(1,0,0,1,0,0)$	$(1,0,0,0,0,1,1)$	$(1,0,0,0,0,0,1,1)$
14	$(1,0,0,0,1,1)$	$(1,0,0,0,0,1,0)$	$(1,0,0,0,0,0,1,0)$
15	$(1,0,0,0,1,0)$	$(1,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,1)$
16	$(1,0,0,0,0,1)$	$(1,0,0,0,0,0,0)$	$(1,0,0,0,0,0,0,0)$
17	$(1,0,0,0,0,0)$	$(0,1,1,0,0,1,1)$	$(0,1,0,0,0,0,0,1)$
18	$(0,1,1,0,1,1)$	$(0,1,0,0,0,0,0)$	$(0,1,0,0,0,0,0,0)$
19	$(0,1,1,0,0,1)$	$(0,0,1,0,0,1,0)$	$(0,0,1,0,0,0,1,0)$
20	$(0,1,0,1,0,0)$	$(0,0,1,0,0,0,1)$	$(0,0,0,1,0,0,1,1)$
21	$(0,1,0,0,0,1)$	$(0,0,1,0,0,0,0)$	$(0,0,0,1,0,0,1,0)$
22	$(0,1,0,0,0,0)$	$(0,0,0,1,0,1,0)$	$(0,0,0,1,0,0,0,1)$
23	$(0,0,1,0,0,1)$	$(0,0,0,1,0,0,1)$	$(0,0,0,0,1,0,0,1)$
24	$(0,0,1,0,0,0)$	$(0,0,0,1,0,0,0)$	$(0,0,0,0,1,0,0,0)$
25	$(0,0,0,1,0,1)$	$(0,0,0,0,1,0,1)$	$(0,0,0,0,0,1,0,0)$
26	$(0,0,0,1,0,0)$	$(0,0,0,0,1,0,0)$	$(0,0,0,0,0,0,1,1)$
27	$(0,0,0,0,1,1)$	$(0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,1,0)$
28	$(0,0,0,0,1,0)$	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,1)$
29	$(0,0,0,0,0,1)$	$(0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$
30	$(0,0,0,0,0,0)$		

2. Characterizations of niceness

In this section we study the notion of niceness and prove properties of nice parabolic subalgebras. Recall that a parabolic subalgebra induces a $\mathbb{Z}$-grading of $\mathfrak{g}$ via the map $\operatorname{ad} H$ where $H \in \mathfrak{h}$ is defined as in the beginning of section 1. We will also use the notation B for the Killing form of $\mathfrak{g}$.

Theorem 2.1. *Let $\mathfrak{p} \subset \mathfrak{g}$ with associated grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$. The following are equivalent:*

1. *$\mathfrak{p}$ is nice*
2. *There exists $X \in \mathfrak{g}_1$ such that $\operatorname{ad}(X) : \mathfrak{g}_j \to \mathfrak{g}_{j+1}$ is surjective for all $j \geq 0$.*
3. *There exists $X \in \mathfrak{g}_1$ such that $\operatorname{ad}(X) : \mathfrak{g}_j \to \mathfrak{g}_{j+1}$ is surjective for all $j > 0$.*
4. *There exists $X \in \mathfrak{g}_1$ such that $\operatorname{ad}(X) : \mathfrak{g}_{j-1} \to \mathfrak{g}_j$ is injective for all $j \leq 0$.*

Proof. Let $X \in \mathfrak{g}_1$ be such that $[\mathfrak{p}, X] = \mathfrak{n}$. Then since

$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

we must have $\operatorname{ad}(X)\mathfrak{g}_i = \mathfrak{g}_{i+1}$ for $i \geq 0$. This proves the necessity of condition (2). The sufficiency is equally clear. To prove (3) we will show that (3) implies (2). Let Ω be the set of all $X \in \mathfrak{g}_1$ satisfying the condition in (3). Then Ω is Zariski open and dense in $\mathfrak{g}_1$. We also note that if Λ is the set of all $X \in \mathfrak{g}_1$ such that $[X, \mathfrak{p}] = \mathfrak{g}_1$, then it is also Zariski open and dense. Hence $\Omega \cap \Lambda \neq \emptyset$. We have proved that (3) implies (2).

We now prove (4). For this we observe that if $\overline{\mathfrak{n}} = \sum_{i<0} \mathfrak{g}_i$ (resp. $\overline{\mathfrak{p}} = \sum_{i\leq 0} \mathfrak{g}_i$), then $y \in \overline{\mathfrak{n}}$ (resp. $y \in \overline{\mathfrak{p}}$) is 0 if and only if $B(y, \mathfrak{n}) = \{0\}$ (resp. $B(y, \mathfrak{p}) = \{0\}$). Let $X \in \mathfrak{g}_1$ be such that $[\mathfrak{p}, X] = \mathfrak{n}$. Then if $y \in \overline{\mathfrak{n}}$ and $[X, y] = 0$, then

$$\{0\} = B([X, y], \mathfrak{p}) = B(y, [X, \mathfrak{p}]) = B(y, \mathfrak{n}).$$

So $y = 0$. Suppose that $X \in \mathfrak{g}_1$ and $\operatorname{ad} X : \overline{\mathfrak{n}} \to \mathfrak{g}$ is injective, then we assert that $\operatorname{ad} X : \mathfrak{p} \to \mathfrak{n}$ is surjective. Indeed, if not, there would exist $y \in \overline{\mathfrak{n}}$ with $B(y, [\mathfrak{p}, X]) = \{0\}$ but $y \neq 0$. But $B(y, [\mathfrak{p}, X]) = B([X, y], \mathfrak{p})$ and since $[X, \overline{\mathfrak{n}}] \subset \overline{\mathfrak{p}}$, we would have the contradiction $y \neq 0$ but $[X, y] = 0$. □

As a consequence we have the following useful criterion for the exceptional Lie algebras.

Corollary 2.2. *If $\mathfrak{p} \subset \mathfrak{g}$ is nice, then*
(i) $\dim \mathfrak{g}_j \geq \dim \mathfrak{g}_{j+1}$ *for all $j > 1$;*
(ii) $\dim \mathfrak{g}_1 > \dim \mathfrak{g}_2$.

Proof. Part (2) of Theorem 2.1 gives (i). Note that the line through X is in the kernel of the map $\operatorname{ad} X : \mathfrak{g}_1 \to \mathfrak{g}_2$. This gives part (ii). □

Any nonzero nilpotent element x of a complex semisimple Lie algebra is the nilpositive element of an $\mathfrak{sl}_2$-triple $\{x, h, y\}$ (Jacobson-Morozov Theorem, cf. Section 3.3 in [CM]). The next result describes a large class of examples of nice parabolic subalgebras (most of them in many simple Lie algebras) .

Lemma 2.3. *Let $\mathfrak{p} \subset \mathfrak{g}$ with grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$. Assume that there is a nonzero $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}$ such that the $\mathfrak{sl}_2$-triple of x is $\{x, 2H, y\}$. Then $\mathfrak{p}$ is nice.*

Proof. This follows from part (4) of Theorem 2.1 since the kernel of $\operatorname{ad} x$ is contained in the sum of the eigenspaces for the eigenvalues of $2H$ that are nonnegative integers. □

If $\mathfrak{p} \subset \mathfrak{g}$ is a subalgebra satisfying the assumptions of Lemma 2.3, we will say that $\mathfrak{p}$ is given by an even TDS (by a three-dimensional simple subalgebra).

We recall a helpful result of [W]: Let $\mathfrak{g}$ be a simple Lie algebra with Dynkin diagram $\triangle$. Let $\triangle'$ be a connected subdiagram of $\triangle$ and $\mathfrak{g}'$ the corresponding simple Lie algebra. If $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra we denote the parabolic subalgebra $\mathfrak{p} \cap \mathfrak{g}'$ by $\mathfrak{p}'$.

Lemma 2.4. *If $\mathfrak{p} \subset \mathfrak{g}$ is nice, then for every $\mathfrak{g}'$ obtained as above, $\mathfrak{p}'$ is nice.*

Proof. Let $\mathfrak{n}'$ be the nilradical of $\mathfrak{p}'$ and let $\mathfrak{n}$ be the nilradical of $\mathfrak{p}$. Let $u \in \mathfrak{h}$ be such that if $\alpha \notin \Delta'$, then $\alpha(u) = 1$. Let $\mathfrak{v}$ be the direct sum of the eigenspaces for strictly positive eigenvalues of $\operatorname{ad} u$. Then $\mathfrak{v}$ is an ideal in $\mathfrak{n}$ and since $\mathfrak{n} = \mathfrak{n}' \oplus \mathfrak{v}$, $\mathfrak{n}' \cong \mathfrak{n}/\mathfrak{v}$. Let p denote the projection from $\mathfrak{n}$ onto $\mathfrak{n}/\mathfrak{v}$. Let $X \in \mathfrak{g}_1$ be as in (2) of Theorem 2.1. Let X' be the unique element in $\mathfrak{n}'$ such that $p(X') = p(X)$. Then it is easy to see that $X' \in \mathfrak{g}_1'$ and satisfies (3) of Theorem 2.1. □

Lemma 2.4 has two consequences: on one hand, any subdiagram of a nice parabolic subalgebra is again nice. On the other hand, if a diagram contains a subdiagram that is not nice, then the "big" diagram is not nice.

The following result will be one of our main tools in our proofs of the theorems in the previous section.

Theorem 2.5. *Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ where $\mathfrak{m}$ is a Levi factor of $\mathfrak{p}$ and $\mathfrak{u}$ the corresponding nilpotent radical. Let $\mathfrak{g} = \sum \mathfrak{g}_j$ be the corresponding grade. Then $x \in \mathfrak{g}_1$ is a Richardson element of $\mathfrak{p}$ if and only if* $\dim \mathfrak{g}^x = \dim \mathfrak{m}$.

Proof. We denote the opposite nilradical, $\sum_{j<0} \mathfrak{g}_j$, by $\overline{\mathfrak{u}}$. If $x \in \mathfrak{u}$, then $\operatorname{ad}(x)\mathfrak{g} = \operatorname{ad}(x)\overline{\mathfrak{u}} + \operatorname{ad}(x)\mathfrak{p}$. Now $\operatorname{ad}(x)\mathfrak{p} \subset \mathfrak{u}$ and $\dim \operatorname{ad}(x)\overline{\mathfrak{u}} \le \dim \overline{\mathfrak{u}}$. Thus

$$\dim \operatorname{ad}(x)\mathfrak{g} \le \; 2 \dim \mathfrak{u}.$$

This implies for $x \in \mathfrak{u}$ that $\dim \mathfrak{g}^x \ge \dim \mathfrak{m}$ and equality implies that $\dim \operatorname{ad}(x)\mathfrak{p} = \dim \mathfrak{u}$. Thus equality implies that x is a Richardson element. If $x \in \mathfrak{g}_1$, then $\operatorname{ad}(x)\overline{\mathfrak{u}} \subset \sum_{j \le 0} \mathfrak{g}_j$. If x is a Richardson element, then since $\operatorname{ad}(x)$ is injective on $\overline{\mathfrak{u}}$ the kernel of $\operatorname{ad}(x)$ is contained in $\mathfrak{p}$ (see Theorem 2.1 (4)). Since $\operatorname{ad}(x)\mathfrak{p} = \mathfrak{u}$, we have proved that $\dim \mathfrak{g}^x = \dim \mathfrak{m}$. □

We recall that *distinguished* nilpotent elements of $\mathfrak{g}$ are nilpotent elements that are not contained in any Levi factor $\mathfrak{m} \subsetneq \mathfrak{g}$. Let $x \in \mathfrak{g}_1$ be nilpotent, let $x, 2H, y$ be a TDS and $\mathfrak{g} = \sum \mathfrak{g}_i$ the grading given by H. Then x is distinguished if and only if $\dim \mathfrak{m} = \dim \mathfrak{g}_1$ (cf. Section 8.2 of [CM]).

Corollary 2.6. *Let $\mathfrak{p} \subsetneq \mathfrak{g}$ and $x \in \mathfrak{g}_1$ a Richardson element with* $\dim \mathfrak{m} > \dim \mathfrak{g}_1$. *Then x is not a distinguished nilpotent element for $\mathfrak{g}$.*

Proof. Follows directly from Theorem 2.5. □

3. The determination of Jordan forms

We use the notation $M_{p,q}(\mathbb{C})$ for the $p \times q$ matrices over $\mathbb{C}$. Let $n_1, \dots, n_r > 0$ with $r \ge 2$ be given. Then we define a map $\phi : M_{n_1,n_2}(\mathbb{C}) \times M_{n_2,n_3}(\mathbb{C}) \times \cdots \times M_{n_{r-1},n_r}(\mathbb{C}) \to M_{n_1,n_r}(\mathbb{C})$ by $\phi(X_1, \dots, X_{r-1}) = X_1 \cdots X_{r-1}$. The following should be well known. Since it will be used several times we include a proof.

Lemma 3.1. *If $r \geq 2$, then ϕ is a surjection onto the variety $Y_{n_1,n_r,m} = \{X \in M_{n_1,n_r} | rank(X) \leq m\}$ with $m = \min\{n_1, \ldots, n_r\}$.*

Proof. If V is a vector space over $\mathbb{C}$, then we will use the notation $\mathbb{P}V$ for the projective space of V. We observe that ϕ induces a regular map of $\mathbb{P}M_{n_1,n_2}(\mathbb{C}) \times \mathbb{P}M_{n_2,n_2}(\mathbb{C}) \times \cdots \times \mathbb{P}M_{n_{r-1},n_r}(\mathbb{C})$ into $\mathbb{P}M_{n_1,n_r}(\mathbb{C})$. We will prove that the image of this map is the projective variety $Z = \{[X] \in \mathbb{P}M_{n_1,n_r}(\mathbb{C}) | \Delta(X) = 0$ for all $m+1 \times m+1$ minors of $X\}$. This will clearly prove the lemma. We first observe that $Y = Y_{n_1,n_r,m}$ is an irreducible affine variety. Indeed, define the $n_1 \times n_r$ matrix J by

$$J = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

with I the $m \times m$ identity matrix. Then the set of matrices in $M_{n_1,n_r}(\mathbb{C})$ of rank exactly m is $U = \{gJh^{-1} | g \in GL(n_1, \mathbb{C}), h \in GL(n_r, \mathbb{C})\}$. Since U is irreducible as a quasi-projective variety (indeed it is affine) and Y is the Zariski closure of U the irreducibility follows.

We now prove the result by induction on r. If $r = 2$, then ϕ is the identity map and the result is obvious. Assume this for r and we will now prove it for $r+1$. By the above and the inductive hypothesis it is enough tp show that the set $H = \{WX | X \in M_{n_r,n_{r+1}}(\mathbb{C}), W \in Y_{n_1,n_r,m}$ is Zariski dense in $Y_{n_1,n_{r+1},m'}$ with $m' = \min\{n_{r+1}, m\}$. To see this we observe that if J is as above and if K is $\begin{bmatrix} I & 0 \end{bmatrix}$ if $n_r \leq n_{r+1}$ or $\begin{bmatrix} I \\ 0 \end{bmatrix}$ if $n_r > n_{r+1}$, then the set $\{gJKh^{-1} | g \in GL(n_1, \mathbb{C}), h \in GL(n_{r+1}, \mathbb{C})\}$ is contained in H and JK is the "J" for $Y_{n_1,n_{r+1},m'}$. □

We now consider a standard parabolic subalgebra $\mathfrak{p}$ of A_n, $n \geq 1$. Then $\mathfrak{p}$ is given by integers $n_1, \ldots, n_r > 0$ with $n_1 + \cdots + n_r = n+1$. We think of the matrices as given in $r \times r$ block form with the i-th diagonal block of size $n_i \times n_i$. Then the grade corresponding to $\mathfrak{p}$ is given by bracket with

$$H = \begin{bmatrix} \frac{r-1}{2} I_{n_1} & 0 & 0 & 0 \\ 0 & \frac{r-3}{2} I_{n_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\frac{r-1}{2} I_{n_r} \end{bmatrix}.$$

Thus $x \in \mathfrak{g}_1$ is of the block form

$$x = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 \\ 0 & 0 & X_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & X_{r-1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with X_i an $n_i \times n_{i+1}$ matrix. If we consider the j-th power of x, then it has all of its (possibly) nonzero entries in the $j+1$-st super-diagonal and those entries are $X_1 X_2 \cdots X_j, X_2 X_3 \cdots X_{j+1}, \ldots, X_{r-j} X_{r+1-j} \cdots X_{r-1}$. We have

Lemma 3.2. *If $x \in \mathfrak{g}_1$ is generic, then x^j has rank equal to*

$$\sum_{i=1}^{r-j+1} \min\{n_i, n_{i+1}, \ldots, n_{i+j-1}\}.$$

This is a simple application of the above observation, and Lemma 3.1.

We now recall an algorithm for calculating the Jordan canonical form of a nilpotent matrix. Let x be a nilpotent $n+1 \times n+1$ matrix. We assume that its Jordan canonical form has a_s blocks of size s for $s = 1, \dots, m$ and $x^m = 0$. If J is a nilpotent Jordan block of size k, then the dimension of the kernel of J^s is s for $s = 1, \dots, k-1$ and k for $s \geq k$. Hence $a_1, \dots, a_r$ satsify the system of linear equations

$$\begin{aligned}
a_1 + a_2 + a_3 + \cdots + a_{r-1} + a_r &= \dim\ker x \\
a_1 + 2a_2 + 2a_3 + \cdots + 2a_{r-1} + 2a_r &= \dim\ker x^2 \\
&\vdots \\
a_1 + 2a_2 + 3a_3 + \cdots + (r-1)a_{r-1} + (r-1)a_r &= \dim\ker x^{r-1} \\
a_1 + 2a_2 + 3a_3 + \cdots + (r-1)a_{r-1} + ra_r &= \dim\ker x^r = n+1.
\end{aligned}$$

This system has a unique solution

$$(*) \qquad a_j = -\dim\ker x^{j+1} + 2\dim\ker x^j - \dim\ker x^{j-1}$$

for $j = 1, 2, \dots, r$.

If x is a generic element in $\mathfrak{g}_1$, then the previous lemma implies that

$$\dim\ker x^j = n + 1 - \sum_{i=1}^{r-j} \min\{n_i, n_{i+1}, \dots, n_{i+j}\}.$$

We have proved

Proposition 3.3. *Let $\mathfrak{p}$ be the parabolic subalgebra of A_n, $n \geq 1$ given by integers $n_1, \dots, n_r > 0$ with $n_1 + \cdots + n_r = n + 1$. Let x be a generic element of $\mathfrak{g}_1$ (i.e. $[\mathfrak{p}, x] = \mathfrak{g}_1$), then the Jordan canonical form of x has a_j blocks of size j for $j = 1, \dots, r$ with*

$$\begin{aligned}
a_j = &\sum_{i=1}^{r-j-1} \min\{n_i, n_{i+1}, \dots, n_{i+j+1}\} - 2\sum_{i=1}^{r-j} \min\{n_i, n_{i+1}, \dots, n_{i+j}\} \\
&+ \sum_{i=1}^{r-j+1} \min\{n_i, n_{i+1}, \dots, n_{i+j-1}\}.
\end{aligned}$$

We will now study the Jordan forms for the other classical Lie algebras. We will use the versions of these Lie algebras as they are given in [GW]. We will now give the analogue of Proposition 3.3 for these cases. We start with the case of C_n.

Proposition 3.4. *Let $\mathfrak{g}$ be of type C_n and let $\mathfrak{p}$ is a standard parabolic subalgebra. Let $x \in \mathfrak{g}_1$ be a generic element. If $\mathfrak{p}$ is given by an even number of blocks, then the element x is generic for the corresponding parabolic subgroup of $\mathfrak{sl}_{2n}$ and thus its Jordan form is calculated as in Proposition 3.3. If there are an odd number of blocks the Jordan form is calculated as follows: write the block sizes as $n_1, \dots, n_r, n_{r+1}, n_{r+2}, \dots, n_{2r+1}$ and denote this $2r+1$ tuple by $\mathbf{n}$. We define a nonnegative integer $r_{ij}(\mathbf{n})$ as follows. If $i < 1$, $j < 1$ or $i + j > 2r + 1$, then $r_{ij}(\mathbf{n}) = 0$. We now assume $i, j \geq 1$ and $i + j \leq 2r + 1$. If $i + j \leq r + 1$ or if $i \geq r + 1$, then $r_{ij}(\mathbf{n}) = \min\{n_i, n_{i+1}, \dots, n_{i+j}\}$. If $i \leq r$ and $i + j > r + 1$, then we set $s = \min\{n_i, n_{i+1}, \dots, n_{i+j}\}$, $u = \min\{r + 1 - i, i + j - r - 1\}$ and*

$v = \min\{n_{r+1-u}, n_{r+2-u}, \ldots, n_{r+1+u}\}$. If v is odd set $t = v - 1$ and if v is even set $t = v$; then $r_{ij}(\mathbf{n}) = \min\{s, t\}$. Then the rank of x^j is

$$r_j(\mathbf{n}) = \sum_i r_{ij}(\mathbf{n}), j > 0, r_0(\mathbf{n}) = 2n.$$

For each $1 \le j \le 2r+1$ there are $r_{j-1}(\mathbf{n}) - 2r_j(\mathbf{n}) + r_{j+1}(\mathbf{n})$ blocks of size j in the Jordan form of x.

The result for B_n and D_n is similarly complicated. Also we will follow the convention introduced in the discussion before Theorem 1.4 to resolve the ambiguity in the cases where there are an even number of block sizes and the central pair are both of size 1. These cases are ignored and we use the description of the same parabolic now with one less block size and all sizes the same except that the 1,1 is replaced by a 2.

Proposition 3.5. *Let $\mathfrak{g}$ be of type B_n or D_n and let $\mathfrak{p}$ is a standard parabolic subalgebra and let $d = 2n+1$ in the case of B_n and $d = 2n$ for D_n. Let $x \in \mathfrak{g}_1$ be a generic element. If $\mathfrak{p}$ is given by an odd number of blocks, then the element x is generic for the corresponding parabolic subgroup of $\mathfrak{sl}_d$ and thus its Jordan form is calculated as in Proposition 3.3. If there is an even number of blocks, the Jordan form is calculated as follows: write the block sizes as $n_1, \ldots, n_r, n_{r+2}, \ldots, n_{2r+1}$ and denote this $2r$ tuple by $\mathbf{n}$. We define a nonnegative integer $r_{ij}(\mathbf{n})$ as follows. If $i < 1$, $j < 1$ or $i + j > 2r$, then $r_{ij}(\mathbf{n}) = 0$. We now assume $i, j \ge 1$ and $i + j \le 2r$. If $i + j \le r$ or if $i \ge r$, then $r_{ij}(\mathbf{n}) = \min\{n_i, n_{i+1}, \ldots, n_{i+j}\}$. If $i \le r$ and $i+j \ge r+1$, then we set $s = \min\{n_i, n_{i+1}, \ldots, n_{i+j}\}$, $u = \min\{r+1-i, i+j-r\}$ and $v = \min\{n_{r+1-u}, n_{r+2-u}, \ldots, n_{r+u}\}$. If v is odd set $t = v - 1$ and if v is even set $t = v$; then $r_{ij}(\mathbf{n}) = \min\{s, t\}$. Then the rank of x^j is*

$$r_j(\mathbf{n}) = \sum_i r_{ij}(\mathbf{n}), j > 0, r_0(\mathbf{n}) = 2n.$$

For each $1 \le j \le 2r+1$ there are $r_{j-1}(\mathbf{n}) - 2r_j(\mathbf{n}) + r_{j+1}(\mathbf{n})$ blocks of size j in the Jordan form of x.

The intricacy of these statements is evidenced in the complicated proof. We warn the reader that it will be painful. We will devote the rest of this section to the proofs of these propositions. For B_n and D_n we will use the form with matrix having all entries 0 except for ones on the skew diagonal

$$L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

For C_n we use the form given by

$$M = \begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix}$$

with L as above. We will use the notation $A^{\#}$ for the adjoint of the matrix A with respect to the indicated form. Thus for $\mathrm{B}_n, \mathrm{D}_n$ if A has ij entry a_{ij}, then $A^{\#}$ has ij entry $a_{m-j,m-i}$ with $m = 2n+2$, $2n+1$ respectively. In the case C_n we write $\eta_i = 1$ if $i \le n$ and $\eta_i = -1$ if $i > n$. Then if A has ij entry a_{ij}, then $A^{\#}$ has ij entry $-\eta_i \eta_j a_{2n+1-j,2n+1-i}$. In all cases, if $e_i, i = 1, \ldots, q$ is the standard basis with $q = 2n+1, 2n, 2n$ for $\mathrm{B}_n, \mathrm{C}_n$ and D_n respectively, and if $(\dots,\dots)$ is the corresponding invariant form on $\mathbb{C}^q$, then $(e_i, e_{q+1-j}) = \pm\delta_{ij}$. This implies the following observations:

(a) If $i + p \le n+1$, then $(\dots,\dots)$ induces a perfect pairing between
$$V_{i,p} = \mathrm{span}_{\mathbb{C}}\{e_i, e_{i+1}, \dots, e_{i+p-1}\}$$
and
$$\check{V}_{i,p} = \mathrm{span}_{\mathbb{C}}\{e_{q+2-i-p}, e_{q+1-i-p}, \dots, e_{q+1-i}\}.$$

(b) In the case of B_n the form $(\dots,\dots)$ induces a nondegenerate form on $\mathrm{span}_{\mathbb{C}}\{e_i | n+1-p \le i \le n+1+p\}$ for $p = 0, \dots, n$.

(c) In the cases C_n and D_n the form $(\dots,\dots)$ induces a nondegenerate form on $\mathrm{span}_{\mathbb{C}}\{e_{n-i}, e_{n-i+1}, \dots, e_n, e_{n+1}, \dots, e_{n+i+1}\}$.

If $\mathfrak{p}$ is a standard parabolic subalgebra of one of these Lie algebras, then it is given by a palindromic sequence of the form (A) $n_1, \dots, n_{r-1}, n_r, n_r, n_{r-1}, \dots, n_1$ or (B) $n_1, \dots, n_r, m, n_r, \dots, n_1$ with $n_i > 0$ and $2(n_1 + \dots + n_r) = q$ in the former case and in the latter in addition $m > 0$ and $2(n_1 + \dots + n_r) + m = q$. We will use the corresponding grade on the Lie algebra. Each of these sequences delineates a block form for the $q \times q$ matrices. The elements of $\mathfrak{g}_1$ have block forms
$$x = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & X_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_{r-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
for case (A) and
$$x = \begin{bmatrix} 0 & X_1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & X_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
in case (B). Furthermore, for $i = 1, \dots, r-1$ we see X_i as a map of $V_{n_1+\dots+n_{i-1}+1, n_i}$ to $\check{V}_{n_1+\dots+n_{i-1}+1, n_i}$ and $Y_i = -X_i^{\#}$ with upper # indicating the adjoint map relative to the indicated perfect pairing. In case (A) $V_{n_1+\dots+n_{r-1}+1, n_r} = \check{V}_{n_1+\dots+n_{r-1}+1, n_r}$, so relative to the nondegenerate forms indicated in (b) and (c) above we have $X_r^{\#} = -X_r$. In case (B) we see X_r as a map from $V_{n_1+\dots+n_r+1, m}$ to $\check{V}_{n_1+\dots+n_r+1, m}$ and then $Y_r = -X_r^{\#}$.

We will now begin a calculation of the Jordan canonical form of a generic element x in $\mathfrak{g}_1$ as given above. As in the case A_n we must calculate the rank of x^j for $j = 1, \dots, 2r-1$ in case (A) and $j = 1, \dots, 2r$ in case (B). We need the following lemma to handle the case (B) for C_n. Let $V_1, \dots, V_r, W$ be finite dimensional nonzero vector spaces and assume that we have a symplectic (i.e. nondegenerate skew symmetric bilinear form), $\langle \ldots, \ldots \rangle$ on W. If $T : V_{i+1} \to V_i$ linear, then define $T^* : V_i^* \to V_{i+1}^*$ (as usual) by $T^*\lambda = \lambda \circ T$. If $T : W \to V_r$, then we define $T^\# : V_r^* \to W$ by $\langle T^\# \lambda, w \rangle = \lambda(Tw)$. We leave the following as an exercise whose proof is similar to that of Lemma 3.1 and will be left to the reader.

Lemma 3.6. *Let $T_i : V_{i+1} \to V_i$ be linear for $i = 1, \ldots, r-1$ and let $T_r : W \to V_r$ be linear, then the form on V_1^* defined by*

$$(\lambda, \mu)_T = \lambda(T\mu)$$

with $T = T_1 \cdots T_{r-1} T_r T_r^\# T_{r-1}^* \cdots T_1^*$ is skew symmetric. Let X_k be the space of all skew symmetric forms of rank k on V_1^*. Let s be the minimum of $\{\dim V_1, \ldots, \dim V_r, \dim W\}$, then if s is even set $t = s$ and if s is odd set $t = s - 1$, then the map

$$Hom(V_2, V_1) \times Hom(V_3, V_2) \times \cdots \times Hom(V_r, V_{r-1}) \times Hom(W, V_r) \to X_k$$

given by $(T_1, \ldots, T_r) \mapsto (\ldots, \ldots)_T$ with $T = T_1 \cdots T_{r-1} T_r T_r^\# T_{r-1}^* \cdots T_1^*$ is onto.

In case (A) the nonzero entries of x^j are in the blocks in the j-th diagonal (i.e. has blocks $i, i+j$, so the diagonal is the 0-th diagonal). The entries in that diagonal are of three possible types:

1-A. $X_i X_{i+1} \cdots X_{i+j-1}$ if $i \geq 1$ and $j + i \leq r + 1$ (the i-th entry).
2-A. $X_i X_{i+1} \cdots X_r Y_{r-1} \cdots Y_{2r-i-j+1}$ if $i \geq 1$, $2r > i + j > r + 1$ and $i \leq r$ (the i-th entry).
3-A. $Y_{r-i} Y_{r-i-1} \cdots Y_{r-i-j+1}$ if $i \geq 1, i + j \leq r$ the ($r + i$-th entry).

In case (B) there are also three possibilities:

1-B. $X_i X_{i+1} \cdots X_{i+j-1}$ if $i \geq 1$ and $j + i \leq r + 1$ (the i-th entry).
2-B. $X_i X_{i+1} \cdots X_r Y_r \cdots Y_{2r-i-j+1}$ if $i \geq 1$ and $2r > i + j > r + 1$ and $i \leq r$ (the i-th entry).
3-B. $Y_{r-i} Y_{r-i-1} \cdots Y_{r-i-j+1}$ if $i \geq 0$ and $i + j \leq r$ the ($r + i + 1$-th entry).

We will now compute the generic rank of each of the six types of matrices.

Proposition 3.7. *(i) In cases 1-A, and 1-B and $i + j \leq r$ the generic rank is* $\min\{n_i, \ldots, n_{i+j}\}$.

(ii) In case 1-A with $i + j = r + 1$ the generic rank is $\min\{n_i, \ldots, n_{r-1}, s\}$ with $s = n_r$ if n_r is even or the algebra is symplectic and $s = n_r - 1$ if n_r is odd and the algebra is orthogonal.

(iii) In case 1-B and $i + j = r + 1$ the generic rank is $\min\{n_i, \ldots, n_r, m\}$. (Here recall that m is the block size of the middle block.)

(iv) In case 2-A if $i = 2r - i - j$ set $s = \min\{n_i, \ldots, n_{2r-i-j}\}$. The generic rank is s if s is even and $s - 1$ if s is odd. If $i < 2r - i - j$ set $s = \min\{n_r, \ldots, n_{2r-i-j+1}\}$ and $t = \min\{n_i, n_{i+1}, \ldots, n_{2r-i-j}\}$, if $i > 2r - i - j$, then set $s = \min\{n_i, \ldots, n_r\}$ and $t = \min\{n_{2r-i-j}, \ldots, n_i\}$, then there are two possibilities.

a) If s is even or if we are in the case C_n, then the generic rank is $\min\{s, t\}$.

b) If s is odd, then the generic rank is $\min\{s - 1, t\}$.

(v) In case 3-A it is $\min\{n_{r-i+1}, n_{r-i}, \ldots, n_{r-i-j+1}\}$.

(vi) In case 3-B it is $\min\{n_{r-i+1}, n_{r-i}, \ldots, n_{r-i-j+1}\}$ if $i > 0$ and it is $\min\{m, n_r, n_{r-i-1}, \ldots, n_{r-i-j+1}\}$ *if $i = 0$.*

(vii) In case 2-B it is $\min\{n_i, \ldots, n_r, m, n_r, n_{r-1}, \ldots, n_{2r-i-j}\}$ for orthogonal algebras.

(viii) In case 2-B for symplectic algebras let k be the minimum of i and $2r-i-j$. Let $s = \min\{n_k, \ldots, n_r, m\}$ if s is even, then set $t = s$ and if s is odd set $t = s-1$. Then the generic rank is $\min\{n_i, \ldots, n_r, m, n_r, n_{r-1}, \ldots, n_{2r-i-j}, t\}$.

Proof. The cases 1-A for $i+j \leq r$, 3-A, 1-B, 3-B have already been observed in the study of A_n. In the case of 1-A with $i+j = r+1$ then in the symplectic case the generic rank of X_r is n_r in the orthogonal case it is n_r if n_r is even and $n_r - 1$ if it is odd. We look at the case 2-B. This case for orthogonal algebras follows from Lemma 3.1 above, the similar fact that (in the notation of Lemma 3.1) the set

$$Z = \{XX^T | X \in Y_{p,q,r}\} = \{Z | Z^T = Z, Z \in M_p(\mathbb{C}), rk(Z) = r\}$$

is irreducible and the last part of the proof of Lemma 3.1. For symplectic algebras we use the previous lemma and a similar argument. We are left with 2-A. In this case we must look at matrices of the form XZX^T with the matrix Z, $l \times l$ and $Z^T = -Z$ for the orthogonal cases and $MZ^TM = Z$ (with M as above) in the symplectic case. In the orthogonal cases if X is generic of rank a and Z is generic of rank b (note b is l if l is even and $l-1$ if l is odd), then the rank of XZX^T is $c = \min\{a, b\}$ if c is even and it is $c-1$ if c is odd. In the symplectic case the generic rank is the minimum of the generic ranks of X and Z (which must be l). $\square$

We will use the result above to calculate the Jordan forms of specific generic elements in the next few sections using formula $(*)$ above.

Corollary 3.8. *Let x be a generic element in $\mathfrak{g}_1$ for the standard parabolic subalgebra $\mathfrak{p}$. If $\mathfrak{g}$ is of type C_n and the corresponing sequence of block sizes is as in case (A) or if it is of type B_n or D_n, then the Jordan form of x is the same as that of x', a generic element in level 1 of the grade for the sequence of block sizes for $\mathfrak{sl}_q$ with $q = 2n$ for types C_n and D_n and $2n+1$ for type B_n.*

We will conclude this section with some observations that will be used in the determination of the nice parabolic subalgebras that are given in terms of even nilpotent elements as in Lemma 2.3.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{k}$ be a reductive subalgebra that is reductively embedded (that is the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{k}$ is completely reducible). Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{k}$ contained in $\mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$. Then we say that $\mathfrak{k}$ is evenly embedded in $\mathfrak{g}$ if the restriction to $\mathfrak{a}$ of a root of $\mathfrak{g}$ on $\mathfrak{h}$ is in the root lattice of $\mathfrak{k}$ in $\mathfrak{a}^*$. We note that this condition is independent of any choices made. We note that if $\mathfrak{k}$ is an embedded $\mathfrak{sl}_2$ with standard basis $\{e, f, h\}$, then $\mathfrak{k}$ is evenly embedded if and only if e is an even nilpotent element of $\mathfrak{g}$.

The following lemma is an obvious consequence of the definitions.

Lemma 3.9. *Let $\mathfrak{k}$ be evenly embedded in the semisimple Lie algebra $\mathfrak{g}$. Then $x \in \mathfrak{k}$ is an even nilpotent element of $\mathfrak{k}$ if and only if it is even as a nilpotent element of $\mathfrak{g}$.*

We apply this result to the classical Lie algebras with the following easy lemma that we will leave to the reader.

Lemma 3.10. *Standard embeddings of Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ into $\mathfrak{sl}_d$ (with $d = n$ and $d = 2n$ respectively) are even.*

4. The proofs of the main theorems

In this section we will prove the theorems of section 1. For the exceptional types the results were proved using the computer program GAP. Our proofs in the classical cases involve the computation of Jordan forms for generic elements as described in Proposition 3.3 and Proposition 3.7.

4.1. **The case of A_n.**

In this subsection we will prove Theorem 1.1. We will first prove the necessity of the condition. We will use the following result.

Theorem 4.1. *Let $n_1 \geq n_2 \geq \cdots \geq n_r > 0$ define the Jordan form of a nilpotent matrix in $M_n(\mathbb{C})$, then the dimension of its centralizer is $\sum_i m_i^2$ with $m_1 \geq m_2 \geq \cdots \geq m_s$ the dual partition.*

Suppose that there exist $i < j < k$ with $n_i > n_j < n_k$. We assert that this implies that there exist l and m such that $n_l < n_{l+1} = \cdots = n_{i+m} > n_{l+m+1}$. Indeed, let $k - i$ be minimal subject to the condition that there exists $i < j < k$ such that $n_i > n_j < n_k$. If there exists s with $i < s < j$ and $n_s \geq n_i$, then $n_s > n_j < n_k$ and $k - s < k - i$. Thus if $i < s < j$, then $n_i > n_s$. If $i < s < j$ and $n_s < n_j$, then $n_i > n_s < n_j$ and $j - i < k - i$. So if $i < s < j$, then $n_i > n_s \geq n_j$. If for such an s we have $n_s > n_j$, then we have $n_s > n_j < n_k$. Thus we have $n_s = n_j$, for $i < s < j$. We can use the same argument to complete the proof for the interval between j and k.

Lemma 4.2. *If $\mathfrak{g}$ is of type A_n and if $\mathfrak{p}$ is the standard parabolic subalgebra of A_n, then $n \geq 1$ given by integers $n_1, \ldots, n_r > 0$ with $n_1 + \cdots + n_r = n + 1$. Then if there exist $i < j < k \leq r$ with $n_i > n_j < n_k$, then $\mathfrak{p}$ is not nice.*

Proof. The above remarks imply that we may assume that we have

$$n_1 > n_2 = \cdots = n_m < n_{m+1}.$$

Then the above proposition implies that if the Jordan canonical form of a generic element of $\mathfrak{g}_1$ has Jordan canonical form with a_j blocks of size j for $j = 1, \ldots, m+1$, then $a_{m+1} = n_2$ and $a_i = 0$ for $1 < i < m+1$ and $a_1 = n_1 + n_{m+1} - 2n_2$. The corresponding partition is

$$m + 1 = \cdots = m + 1 > 1 = 1 = \cdots = 1$$

with n_2, $m+1$ and $n_1 + n_{m+1} - 2n_2$ one. The dual partition has one row of length $n_1 + n_{m+1} - n_2$ and m rows of length n_2. Thus the dimension of the centralizer of a generic element of $\mathfrak{g}_1$ in $M_{n+1}(\mathbb{C})$ is

$$\begin{aligned}(n_1 + n_{m+1} - n_2)^2 + mn_2^2 &= 2(n_1 - n_2)(n_{m+1} - n_2) + n_1^2 + n_{m+1}^2 + (m-1)n_2^2 \\ &> n_1^2 + n_{m+1}^2 + (m-1)n_2^2 = \dim \mathfrak{l}\end{aligned}$$

with $\mathfrak{l}$ the standard Levi factor of the corresponding parabolic subalgebra in $M_{n+1}(\mathbb{C})$. Thus Theorem 4.1 implies that $\mathfrak{p}$ is not nice. ☐

In light of this lemma the condition of the main theorem for A_n is necessary. We now prove that it is sufficient. We will prove the result by induction on r. If $r = 1$, there is nothing to prove. The above lemma implies that we may assume that the n_i satisfy one of the following systems of inequalities

(i) $$n_1 \le \cdots \le n_r > 0,$$

(ii) $$n_1 \ge \cdots \ge n_r > 0,$$

(iii) $$0 < n_1 \le \cdots \le n_{r_1} > n_{r_1+1} \ge \cdots \ge n_r > 0.$$

Using the Chevalley involution we see that case (i) is nice if and only if case (ii) is nice. But Proposition 3.3 above implies that in case (ii) the partition corresponding to the Jordan canonical form of a generic element of $\mathfrak{g}_1$ is the dual partition to $n_1 \ge \cdots \ge n_r$. Thus the dimension of the centralizer of a generic element in $\mathfrak{g}_1$ in $M_{n+1}(\mathbb{C})$ is $\sum n_i^2$. Thus the parabolic subalgebra is indeed nice in these cases. We now complete the proof by induction. If $r \le 2$, then we are in case (i) or (ii). We assume the result for $r-1 \ge 2$. We may assume that we are in case (iii). Applying the Chevalley automorphism we may assume that $n_1 \ge n_r$. The formula for the Jordan form implies that if $i > r-1$ the contribution to the a_i of n_r cancel out. Let a_i' be the number of blocks of size i for

$$n_1 \le \cdots \le n_{r_1} > n_{r_1+1} \ge \cdots \ge n_{r-1}.$$

Then $a_i' = a_i$ for $i = 2, \ldots, r-2$ and $a_{r-1}' = \min\{n_1.n_{r-1}\}$. We also note that $a_{r-1} = \min\{n_1, n_{r-1}\} - n_r$ and $a_r = n_r$. This implies that $a_1' = a_1$. Hence the dual partition for the a_i with $i = 1, \ldots, r$ differs from the dual partition for the a_i' with $i = 1, \ldots, r-1$ by having one additional row which is of length n_r. The inductive hypothesis implies that the sum of the squares of the entries in the dual partition for the a_i' is $n_1^2 + \cdots + n_{r-1}^2$ and so the sum of the squares for the dual partition for the a_i is $n_1^2 + \cdots + n_r^2$. This completes the proof.

Actually we have proved that if we have a parabolic subalgebra in the case A_n that corresponds to a unimodal sequence $n_1, \ldots, n_r$ and if p is the partition corresponding to the Jordan form of a generic element of $\mathfrak{g}_1$, then the dual partition to p is the n_i put in decreasing order.

The following result follows from the fact that a nilpotent element in $\mathfrak{sl}_n$ is even if and only if all of the block lengths in its Jordan canonical form have the same parity.

Theorem 4.3. *Let $\mathfrak{p}$ be a nice standard parabolic subalgebra of $\mathfrak{sl}_n$, then it corresponds to an even nilpotent element if and only if its sequence of block lengths is palindromic.*

This result implies that the even nilpotent orbits on $\mathfrak{sl}_n$ are in one-to-one correspondence with the unimodal, palindromic, compositions of n (recall that a composition of n is a sequence $m_1, \ldots, m_r$ of positive integers such that $m_1 + \cdots + m_r = n$). Let a_n denote the number of even nilpotent orbits (0 is considered to be an even nilpotent orbit) in $\mathfrak{sl}_n$. It is not hard to show that this description implies

Lemma 4.4. *The generating function for the a_n, $\sum_{n \ge 1} a_n q^n$, is*

$$\sum_{j \ge 1} \frac{q^j (1 + q^j)}{\prod_{i=1}^{j} (1 - q^{2i})}.$$

We note that this combined with the usual classification of even nilpotent orbits yields

$$\sum_{j\geq 1}\frac{q^j(1+q^j)}{\prod_{i=1}^{j}(1-q^{2i})}=\frac{1}{\prod_{i=1}^{\infty}(1-q^{2i+1})}+\frac{1}{\prod_{i=1}^{\infty}(1-q^{2i})}.$$

This sequence can be found in the Online Encyclopedia of Integer Sequences as number A096441. We note that Euler's identity (cf. [A] (2.2.5)) says that

$$1+\sum_{n=1}\frac{t^n}{\prod_{i=1}^{n}(1-q^i)}=\frac{1}{\prod_{i=0}^{\infty}(1-tq^i)}.$$

This implies the above equation also. However, the above method is purely combinatorial and implies the special cases $t=q$ and $q\to q^2$ and $t=q^2$ and $q\to q^2$.

4.2. **The case of C_n.** In this subsection we will prove Theorem 1.3. We will use the following result (cf. [CM]).

Theorem 4.5. *Let $\mathfrak{g}$ be of type* C. *Let $n_1\geq n_2\geq\cdots\geq n_r>0$ be a partition corresponding the Jordan form of an element $x\in\mathfrak{g}$, then if $m_1\geq\cdots\geq m_s$ is the dual partition, then the dimension of $\mathfrak{g}^x$ is*

$$\sum_i \tfrac{m_i^2}{2}+\tfrac{1}{2}|\{i|n_i\ odd\}|.$$

Before we get down to the business at hand we will prove a useful result.

Lemma 4.6. *Let the standard parabolic subalgebra, $\mathfrak{p}$, for C_n have block sizes r of size l, s of size m, and r of size l with s and l odd, $l<m$ and $n=rl+\frac{sm}{2}$ viz.*

$$\begin{bmatrix} l\times l & * & * \\ 0 & m\times m & * \\ 0 & 0 & l\times l\end{bmatrix}.$$

Then $\mathfrak{p}$ is nice if and only if $r=0,1$.

Proof. We will use the main result of the previous section. Let x be a generic element in $\mathfrak{g}_1$. Let r_i denote the rank of x^i. Our result implies that $r_i=2rl+(s-i)m$ for $i=1,\ldots,s$, $r_{s+i}=(2r-2i)l+i(l-1)$ for $i=1,\ldots,r$ and $r_{s+r+i}=(r-i)(l-1)$ for $i=1,\ldots,r$. Thus using the method of section one to calculate Jordan forms we see that the Jordan form of a generic x has $l-1$ blocks of size $2r+s$, 2 blocks of size $r+s$ and $m-l-1$ blocks of size s.

The dual partition has s blocks of size m, r blocks of size $l+1$ and r blocks of size $l-1$. We therefore see that the dimension of the centralizer of x is

$$\frac{sm^2}{2}+(l^2+1)r+\frac{m}{2}-1+\xi$$

with $\xi=1$ if r is even and 0 if r is odd. This number is $\frac{s-1}{2}m^2+\frac{m(m+1)}{2}+l^2r+r-1+\xi$. If r is even, this number is r and if r is odd, it is $r-1$. The lemma follows. ∎

We will first show that a standard parabolic subalgebra is nice only if the corresponing sequence defining the A_{2n-1} parabolic is unimodal. So suppose that we have a standard parabolic subalgebra $\mathfrak{p}$ corresponding to $n_1,\ldots,n_r,n_r,\ldots,n_1$ in case (A) or $n_1,\ldots,n_r,m,n_r,\ldots,n_1$ in case (B). We assume that the sequence is not unimodal and we derive a contradiction. We first consider case (A). Then Corollary 3.8 implies that the Jordan form in question is the same as the one for A_q with $q = 2(n_1 + \cdots + n_r)$. Let $m_1 \geq \cdots \geq m_s$ be the corresponding dual partition. If the sequence $n_1, \ldots, n_r, n_r, \ldots, n_1$ is not unimodal, then we have

$$2(n_1^2 + \cdots + n_r^2) < m_1^2 + \cdots + m_s^2$$

by the main theorem for A_n. This implies that

$$n_1^2 + \cdots + n_r^2 < \frac{(m_1^2 + \cdots + m_s^2)}{2} \leq \dim \mathfrak{g}^x$$

so the parabolic is not nice.

We now look at case (B). We assume that the squence $n_1, \ldots, n_r, m, n_r, \ldots, n_1$ is not unimodal but the corresponding parabolic subalgebra is nice. Then there must be a subsequence of the form

$$n_s > n_{s+1}, \ldots, m, \ldots, n_{s+1} < n_s$$

or of the form

$$n_r > m < n_r.$$

These correspond to subdiagrams of the diagram for the standard parabolic subalgebra. We assert that the corresponding parabolic subalgebra cannot be nice for the corresponding subalgebra of type $\mathrm{C}_{n_r + \frac{m}{2}}$. Let x be generic in $\mathfrak{g}_1$. Then according to Proposition 3.4,

$$\begin{aligned} rk(x) &= 2m, \\ rk(x^2) &= m, \\ rk(x^3) &= 0. \end{aligned}$$

Thus the Jordan canonical form of x has $2n_r - 2m$ blocks of size 1 and m blocks of size 3. Thus the dual partition is $2n_r - m \geq m = m$. So the dimension of $\mathfrak{g}^x$ is

$$\begin{aligned} &\frac{(2n_r - m)^2}{2} + m^2 + n_r + \frac{m}{2} \\ &= (n_r - m)^2 + n_r + n_r^2 + \frac{m(m+1)}{2}. \end{aligned}$$

Since $n_r^2 + \frac{m(m+1)}{2}$ is the dimension of the corresponding Levi factor we see that the corresponding parabolic is not nice. We can thus assume that we have

$$n_s > n_{s+1}, \ldots, m, \ldots, n_{s+1} < n_s$$

as a subsequence. We take s to be maximal with respect to this condition. Then we must have

$$n_s > n_{s+1} \leq \cdots \leq n_r, m, n_r \geq \cdots \geq n_{s+1} < n_s$$

if one of the inequalities $n_t \leq n_{t+1}$ were strict, then the type A result combined with the subdiagram theorem would imply that the parabolic is not nice. Thus we may assume that we have

$$n_s > n_{s+1} = \cdots = n_r, m, n_r = \cdots = n_{s+1} < n_s,$$

the relationship between n_r and m cannot be $n_r > m$ since this yields a subdiagram that we have just seen cannot be nice. We are thus left with 2 cases,

$$a > b = \cdots = b < a$$

or

$$a > b = \cdots = b < c > b = \cdots = b < a$$

where $a = n_s$ and in the former case $b = m$ in the latter case $n = n_{s+1}$ and $c = m$. We look at the latter case first. We assume first of all that b is even. Our usual method of calculation (assuming that there are r b's and that $n = 2rb + 2a + m$) yields

$$\dim \mathfrak{g}^x - l = (a-b)(a+2c+1-3b).$$

If b is odd, then the previous lemma implies that $r = 1$. Thus we are looking at

$$a > b < c > b < a.$$

In this case one finds that the dual partition is $2a+c-2b \geq b+1 = b+1 > b-1 = b-1$. We leave the algebra that shows that this case is not nice to the reader.

At this point we have shown that unimodality is a necessity for niceness. The lemma above implies that it is not sufficient. We will devote the rest of the section to the proof of Theorem 1.3. Let $\mathfrak{p}$ be a standard parabolic of $\mathfrak{g}$ with a unimodal sequence of block lengths. Then if the number of blocks is even or all of the block lengths are even then Corollary 3.8 implies that the Jordan canonical form of a generic element in $\mathfrak{g}_1$ is the same as the Jordan form for the corresponding parabolic in A_{2n-1}. Since the sequence is unimodal and palindromic, we see that the element corresponds to an even nilpotent element for A_{2n-1} and hence for $\mathfrak{g}_1$. Thus in this case the parabolic is nice. We may thus assume that the number of blocks is odd and the sequence of block sizes is of the form

$$0 < n_1 \leq \cdots \leq n_r \leq m \geq n_r \geq \cdots \geq n_1 > 0 \tag{1}$$

with at least one n_i odd. Lemma 4.6 implies that the result is true for $r = 1$. We will prove the theorem in this case by induction on r. We will prove that if we have a sequence as above that corresponds to a nice parabolic and if we extend it by adding blocks at either end of size n_0 yielding

$$0 < n_0 \leq n_1 \leq \cdots \leq n_r \leq m \geq n_r \geq \cdots \geq n_1 \geq n_0 > 0,$$

then the new parabolic subalgebra is nice if n_0 is even or if n_0 is odd and $n_0 < n_1$. We will show that it is not nice if n_0 is odd and $n_0 = n_1$. We will first prove the positive statements.

Assume that n_0 is even. Let x' be a generic element of the first level of the grade for the parabolic subalgebra corresponding to the block sequence (1) and let x be an element for the block sequence above. Let R'_j be the rank of $(x')^j$ and let R_j be the rank of x^j. Then since n_0 is even, Proposition 3.7 implies that

$$R_j = R'_j + 2n_0, 1 \leq j \leq 2r+1,$$
$$R_{2r+2} = R'_{2r+2} + n_0, R_{2r+j} = R'_{2r+j}, j > 2.$$

Thus if N_j and N'_j are the respective dimensions of the kernels of x^j and $(x')^j$, we have

$$N_j = N'_j, 1 \leq j \leq 2r+1,$$
$$N_{2r+2} = N'_{2r+2} + n_0, N_{2r+j} = N'_{2r+j} + 2n_0, j > 2.$$

So if the multiplicities in the Jordan forms of x and x' are respectively M_j and M_j', then we have

$$M_j = M_j', 1 \leq j \leq 2r,$$
$$M_{2r+1} = M_{2r+1}' - n_0, M_{2r+2} = M_{2r+2}' = 0, M_{2r+3} = n_0.$$

This implies that the number of odd blocks in the Jordan form is unchanged and that the only change in the dual partition is the addition of 2 blocks of size n_0. Thus as in the case of type A the parabolic subalgebra is nice in this case.

We now consider the case when n_0 is odd and $n_0 < n_1$. The argument is similar. We will use the same notation. We have

$$R_j = R_j' + 2n_0, 1 \leq j \leq 2r+1,$$
$$R_{2r+2} = R_{2r+2}' + n_0 - 1, R_{2r+j} = R_{2r+j}', j > 2.$$

Doing the obvious subtraction we have

$$N_j = N_j', 1 \leq j \leq 2r+1,$$
$$N_{2r+2} = N_{2r+2}' + n_0 + 1, N_{2r+j} = N_{2r+j}' + 2n_0, j > 2.$$

This yields

$$M_j = M_j', 1 \leq j \leq 2r,$$
$$M_{2r+1} = M_{2r+1}' - n_0 - 1, M_{2r+2} = M_{2r+2}' + 2, M_{2r+3} = n_0 - 1.$$

We therefore see that we have reuced the number of odd blocks by 2 and added 2 blocks to the dual partition of sizes $n_0 + 1$ and $n_0 - 1$. This completes the proof in this case.

Finally, we assume that n_0 is odd and $n_0 = n_1$. We may assume that $n_1 < n_2$ by the inductive hypothesis. We will show that in this case the parabolic subalgebra is not nice and the proof will be complete. This time we have

$$R_j = R_j' + 2n_0, 1 \leq j \leq 2r, R_{2r+1} = R_{2r+1}' + 2(n_0 - 1),$$
$$R_{2r+2} = R_{2r+2}' + n_0 - 1, R_{2r+j} = R_{2r+j}', j > 2.$$

So

$$N_j = N_j', 1 \leq j \leq 2r, N_{2r+1} = N_{2r+1}' + 2,$$
$$N_{2r+2} = N_{2r+2}' + n_0 + 1, N_{2r+j} = N_{2r+j}' + 2n_0, j > 2.$$

Hence

$$M_j = M_j', 1 \leq j \leq 2r-2, M_{2r-1} = M_{2r-1}', M_{2r} = M_{2r}' - 2 = 0,$$
$$M_{2r+1} = M_{2r+1}' - n_0 + 1 = 0, M_{2r+2} = M_{2r+2}' = 0, M_{2r+3} = n_0 - 1.$$

Here we see that there is a net change of two more odd blocks in the Jordan form. In the dual partition we remove one row of size $n_0 - 1$ and replace it with one of size $n_0 + 1$, we then add 2 rows of length $n_0 - 1$ to complete it. Thus the dimension of the centralizer for x is that of the one for x' plus

$$\frac{(n_0+1)^2}{2} - \frac{(n_0-1)^2}{2} + (n_0 - 1)^2 + 1 = n_0^2 + 2.$$

The inductive hypothesis now implies that the dimension of the centralizer of x is the dimension of a Levi factor of $\mathfrak{p}$ plus 2 and the result follows.

Using Theorem 4.3 we note that the methods of this section allow us to describe the standard nice parabolic subalgebras coming from even nilpotent elements in the case of C_n and thereby give a parametrization of the even nilpotent orbits.

Lemma 4.7. *The even nilpotent orbits of a Lie algebra of type* C_n *are in one-to-one correspondence with the unimodal, palindromic compositions of* $2n$ *which have the property that if the number of parts is odd, then all the parts are even.*

4.3. **The case of** B_n **and** D_n**.** In this subsection we will describe the standard nice parabolic subalgebras in the case when $\mathfrak{g}$ is an orthogonal Lie algebra that is of type B or D and thereby prove Theorems 1.2 and 1.4. We will use the following result (cf. [CM]).

Theorem 4.8. *Let* $\mathfrak{g}$ *be of type* B *or* D*. Let* $n_1 \geq n_2 \geq \cdots \geq n_r > 0$ *be a partition corresponding to the Jordan form of an element* $x \in \mathfrak{g}$*. Then if* $m_1 \geq \cdots \geq m_s$ *is the dual partition, then the dimension of* $\mathfrak{g}^x$ *is*

$$\sum_i \frac{m_i^2}{2} - \frac{1}{2}|\{i|n_i \text{ odd}\}|.$$

We will prove Theorems 1.2 and 1.4 simultaneously in the following form.

Theorem 4.9. *Let* $\mathfrak{p}$ *be a standard parabolic subgroup of* $\mathfrak{g}$*. We consider 4 conditions on the sequence of its of block lengths:*

1) It is a unimodal sequence with an odd number of elements.

2) It is a unimodal sequence of even length and each odd block length occurs at most twice.

3) The sequence is of the form

$$n_1 \leq \cdots \leq n_r > n_r - 1 = \cdots = n_r - 1 < n_r \geq \cdots \geq n_1$$

with the total number of blocks odd and $r = 1$ *or* $n_{r-1} < n_r$*.*

4) The sequence is of the form above with the total number of blocks even, the odd ones occurring at most twice and n_r *must be odd and at least* 3*.*

Then $\mathfrak{p}$ *is nice if and only if the sequence satisfies one of the above conditions.*

The proof of this result will take up the rest of the section. We will first prove that the shapes of the sequences are necessary conditions for niceness. So assume that we have a nice $\mathfrak{p}$ with block lengths $m_1, \ldots, m_s$ that is not unimodal. Since $m_j = m_{s+1-j}$, we see that if $m_i \leq m_{i+1}$ for all $i \leq \frac{s}{2} - 1$ if s is odd and $i \leq \frac{s-1}{2}$ if s is odd, then the sequence is unimodal. Thus we may assume that there is j such that $m_j > m_{j+1}$ and j is in the indicated range. If we have $m_j > m_k < m_l$ with $j < k < l < u$ and $u = \frac{s}{2}$ if s is even, $u = \frac{s-1}{2}$ if s is odd, then we would have a non-nice type A subdiagram of the diagram of $\mathfrak{p}$. Taking the j with $m_j > m_{j+1}$ in the indicted range to be as big as possible we see that there must be central subsequences of one of the following forms (even and odd correspond to the number of blocks, thus (1) in the even cases means that there are an even number of b's):

Even Cases:

(1) $a > b = \cdots = b < a$.
(2) $a > b = \cdots = b < c = c > b = \cdots = b < a$.

Odd Cases:

(1) $a > b = \cdots = b < a$.
(2) $a > b = \cdots = b < c > b = \cdots = b < a$.

We will show that the diagrams of type (2) are never nice and those of type (1) are nice if and only if $a = b+1$, and if there are an even number of blocks, then b is even. We first look at the case of b even, we are in the even type (1) case, so there are $2l$, b's. Then applying Proposition 3.5 we see that if x is a generic element in the degree 1 part of the grade and if R_i is the rank of x^i, then

$$R_1 = (2l+1)b, R_2 = 2lb, R_3 = (2l-1)b, \ldots, R_{2l+1} = b, R_j = 0, j > 2l+1.$$

The dimension of $\ker x^i$ will be denoted N_i. We find that

$$N_1 = 2a-b, N_2 = 2a, N_3 = 2a+b, \ldots, N_{2l+1} = 2a+(2l-1)b, N_j = 2a+2lb, j > 2l+1.$$

From this we find that the Jordan canonical form has $2(a-b)$ blocks of size 1 and b blocks of size $2l+2$ This implies that the Jordan form of x is $2a-b, b, \ldots, b$ with $2l+1$ b's. The sequence of block lengths has $2(a-b)$ odd parts thus

$$\dim \mathfrak{g}^x = \frac{(2a-b)^2}{2} + \frac{2l+1}{2}b^2 - a + b.$$

If $\mathfrak{m}$ is the standard Levi factor of $\mathfrak{p}$, then $\dim \mathfrak{m} = a^2 + lb^2$. A direct calculation yields that

$$\dim \mathfrak{g}^x - \dim \mathfrak{m} = (a-b)(a-b-1).$$

This says that in this case the parabolic subalgebra is nice if and only if $a = b+1$. We next look at type 1 with an odd number of blocks, then the argument is identical (since in the odd case the parity of b plays no role). We conclude the analysis of the type one cases by looking at the even case with with b odd and showing that this cannot correspond to a nice parabolic subalgebra. To simplify the calculation we first show that the case when we have blocks b, b, b, b with b odd is not nice. Indeed, in this case we have

$$R_1 = 3b-1, R_2 = 2b-2, R_3 = b-1, R_4 = 0.$$

Thus

$$M_1 = 0, M_2 = 2, M_3 = 0, M_4 = b-1.$$

If $b = 1$, then the dual partition is $2, 2$ so $\dim \mathfrak{g}^x - \dim \mathfrak{m} = 2$. If $b > 1$, then the dual partition is $b+1, b+1, b-1, b-1$ and we have $\dim \mathfrak{g}^x - \dim \mathfrak{m} = 2$. This proves our assertion. If we use Lemma 2.4, then we are reduced to proving that the parabolic with block lengths $a > b = b < a$ is not nice if b is odd. In this case we have

$$R_1 = 3b-1, R_2 = 2b-2, R_3 = b-1, R_4 = 0.$$

This yields

$$M_1 = 2a-2b, M_2 = 2, M_3 = 0, M_4 = b-1.$$

If $b = 1$, then the dual partition is

$$2a-b+1, b+1, b-1, b-1.$$

Hence we have

$$\dim \mathfrak{g}^x - \dim \mathfrak{m} = (a-b)^2 + a - b + 2 > 2.$$

This proves the result in this case.

We are thus left with analyzing the type (2) cases. When there are an even number of parts, then there are 4 cases: (i) b, c even, (ii) c even, b odd, (iii) c odd, b even and (iv) b, c odd. We will not bore the reader by going through all of the 4 type (2) even cases and the one type (2) odd case. We will do enough so that the reader should have no trouble checking the missing cases. Recall that we are trying to prove that all of these cases are not nice. We look at the odd case (2) first. In this case we have

$$R_j = (2l + 3 - j)b.$$

From this we see that the corresponding Jordan form has $2a - 3b + c$ blocks of size 1 and b blocks of size $2l + 3$. The dual partition is one block of size $2a - 2b + c$ and $2l + 2$ blocks of size b. Thus

$$\dim \mathfrak{g}^x = \frac{(2a - 2b + c)^2}{2} + (l + 1)b^2 - a + b - \frac{c}{2}.$$

The dimension of the corresponding Levi factor is $a^2 + lb^2 + \frac{c(c-1)}{2}$. One checks easily that the dimension of $\mathfrak{g}^x$ is strictly larger than the dimension of the Levi factor.

We now look at the even cases. We first look at the case when b and c are even and the number of b's is $2l$ (thus the total length is $2l + 4$). Then

$$R_1 = (2l + 2)b + c,\ R_i = (2l + 4 - i)b,\ 2 \leq i \leq 2l + 4.$$

This leads to a Jordan form with $2a - 2b$ ones, $c - b$ twos and b blocks of size $2l + 4$. The dual partition has one block of size $2a - 2b + c$, one block of size c and $2l + 2$ of size b. The dimension of a Levi factor is $a^2 + lb^2 + c^2$ and the dimension of $\mathfrak{g}^x$ is

$$\frac{(2a - 2b + c)}{2} + \frac{(c - b)}{2} + (l + 1)b^2 - a + b$$

the difference is $(a-b)(a+2c-3b-1) \geq 2(a-b) > 0$. Thus the parabolic subalgebra is not nice as asserted. If c is odd and b is even, then the only change in the ranks is $R_1 = (2l + 2)b + c - 1$. We leave it to the reader to show that the parabolic subalgebra is not nice in this case. We will now look at b, c odd. Then we have

$$\begin{aligned}
&R_1 = (2l + 2)b + c - 1, \quad R_2 = (2l + 2)b,\\
&R_j = (2l - 2(j - 3))b + (j - 2)(b - 1),\ 2 < j \leq l + 3,\\
&R_j = (2l + 4 - j)(b - 1),\ l + 4 \leq j \leq 2l + 4.
\end{aligned}$$

From this we find that the corresponding Jordan form has $2a - 2b + 2$ blocks of size 1, $c - b - 2$ blocks of size 2, 2 blocks of size $l + 3$ and $b - 1$ blocks of size $2l + 4$. The number of odd blocks is $2a - 2b + 2 + 2\xi$ with $\xi = 1$ if l is even and $\xi = 0$ if l is odd. The dual partition has one block of size $2a + c - 2b + 1$, one block of size $c - 1$, $l + 1$ blocks of size $b + 1$ and $l + 1$ blocks of size $b - 1$. We therefore have

$$\begin{aligned}
\dim \mathfrak{g}^x &= \frac{(2a - 2b + c + 1)^2}{2} + \frac{(c - 1)^2}{2} + (l + 1)(b^2 + 1) - a + b - \xi\\
&= 2a^2 - 4ab + 2b^2 + 2(a - b)(c + 1) + c^2 + 1 + lb^2 + b^2 + l - a + b - \xi\\
&= a^2 - 4ab + 3b^2 + 2(a - b)c + (1 - \xi) + l + a^2 + lb^2 + c^2\\
&= (a - b)^2 + 2(a - b)(c - b) + (1 - \xi) + l + a^2 + 2lb + c^2.
\end{aligned}$$

Since the dimension of the corresponding Levi factor is $a^2 + 2lb + c^2$, the corresponding parabolic subalgebra is not nice. If c is even and b is odd the only change in the ranks is $R_1 = (2l + 2)b + c$. We leave this case for the reader.

We conclude that if a parabolic subalgebra is nice and the corresponding sequence of block sizes not unimodal, then it must contain the following in the middle

$$b + 1 > b = \cdots = b < b + 1$$

with b even if the number of terms is even. We now prove that under these conditions on b and the number of blocks, the parabolic corresponding to

$$a \geq b+1 > b = \cdots = b < b+1 \leq a$$

is not nice. Let s be the number of b's. One checks that

$$R_1 = 2(b+1) + (s+1)b, R_j = (s+4-j)a, 2 \leq j \leq s+4.$$

This implies that the corresponding Jordan form one block of size $2(a-b-1)$ (unless this number is 0) has 2 blocks of size 2 and b blocks of size $s+4$. Thus the dual partition is one block of size $2a-b$ one of size $b+2$ (notice that these are the same if $a = b+1$) and b with multiplicity $s+2$. Thus the dimension of $\mathfrak{g}^x$ is $\frac{(2a-b)^2}{2} + \frac{(b+2)^2}{2} + \frac{s+2}{2}b^2 - (a-b-1) - \frac{b}{2}\xi$ with $\xi = 0$ if s is even and $\xi = 1$ if s is odd. The dimension of the corresponding Levi factor is $2(b+1)^2 + \frac{s}{2}b^2 - \frac{b}{2}\xi$. The difference is $2 + (a-b)(a-b-1)$ so the parabolic subalgebra is not nice as asserted. We leave it to the reader to check that if a is as above and $a < b+1$, then the parabolic subalgebra is nice.

We will prove the theorem by induction on the number of elements in the sequence of block lengths. We will first consider the unimodal case and we will show that if the number of blocks is odd, then the corresponding parabolic subalgebra is nice. If the number of blocks is even, then the corresponding parabolic subalgebra is nice if and only if each odd block occurs at most twice. We will prove this by induction on the number of blocks. If the number of blocks is one, then the corresponding parabolic subalgebra is the whole Lie algebra and the result is obvious. If the number of blocks is 2, then the corresponding parabolic subalgebra has commutative nilradical, so it is nice. Assume that we have proved the result for $r \geq 1$ blocks. If we take such a unimodal nice parabolic subalgebra with sequence

$$0 < n_1 \leq \cdots \geq n_1 > 0$$

and we adjoin a block at both ends to a block of size $n_0 \leq n_1$, then we will show that if r is odd, then the corresponding parabolic subalgebra is nice and if r is even, then it is nice if n_0 is even or if it is odd it is nice if and only if $n_0 < n_1$. We first assume that r is odd or n_0 is even. We will use arguments similar to those for the case of type C. We will use the primed notation in the same way as in the previous section. We find that

$$R_i = R_i' + 2n_0,\ 1 \leq i \leq r,$$
$$R_{r+1} = R_{r+1}^{'} + n_0,\ R_{r+2} = R_{r+2}'.$$

Thus the multiplicities in the Jordan form are

$$M_i = M_i' \text{ for } i \leq r-1,$$
$$M_r = M_r' - n_0,\ M_{r+1} = M_{r+1}' = 0,\ M_{r+2} = n_0.$$

This implies that the corresponding dual partition is obtained by adjoining two rows of length n_0 to the original partition. Thus we have added n_0^2 to the dimension of the centralizer and the same to the dimension of the Levi factor so the parabolic subalgebra is nice in this case. We now assume that n_0 is odd and $n_0 < n_1$. In this case we have

$$R_i = R_i' + 2n_0,\ 1 \leq i \leq r,$$
$$R_{r+1} = R_{r+1}^{'} + n_0 - 1,\ R_{r+2} = R_{r+2}' = 0.$$

The multiplicities for the Jordan form are given by

$$M_i = M_i' \text{ for } i \leq r-1,$$
$$M_r = M_r' - n_0 - 1,\ M_{r+1} = M_{r+1}' + 2 = 2,\ M_{r+2} = n_0 - 1.$$

This time we change the dual partition by adjoining a row of length $n_0 - 1$ and one of length $n_0 + 1$. In the Jordan form we have added to rows of odd length. So we find that the parabolic subalgebra is nice since the dimension of the Levi factor is the same as the centralizer.

To complete the unimodal case we must show that if r is even n_0 is odd and $n_0 = n_1$, then the parabolic subalgebra is not nice. We note that by the inductive hypothesis we must have $n_1 < n_2$. As above we have

$$R_i = R_i' + 2n_0,\ 1 \leq i \leq r-1,\ R_r = R_r' + 2(n_0 - 1),$$
$$R_{r+1} = R_{r+1}' + n_0 - 1,\ R_{r+2} = R_{r+2}' = 0.$$

This yields multiplicities in the Jordan form:

$$M_i = M_i' \text{ for } i \leq r-2,\ M_{r-1} = M_{r-1}' - 2 = 0,$$
$$M_r = M_r' - n_0 + 3 = 2,\ M_{r+1} = 0,\ M_{r+2} = n_0 - 1.$$

The upshot is two rows of length $r-1$ are lost and the original last row in the dual partition is replaced by a row of length $n_0 + 1$ and one row is added of length $n_0 - 1$. The upshot is that the dimension of the centralizer is two more than that of the Levi factor. We have completed the argument in this case.

We now look at the nonunimodal case. Here we have seen that the sequence must contain

$$b + 1 > b = \cdots = b < b + 1$$

with b even if the number of terms is even or

$$a < b + 1 > b = \cdots = b < b + 1 > a$$

with b even if the number of terms is even. Now using the results for type A we see that if we have a general nonunimodal sequence and not just of the above two forms it must be of the form

$$n_1 \leq \cdots \leq n_r \leq a < b + 1 > b = \cdots = b < b + 1 > a \geq n_r \geq \cdots \geq n_1$$

with b even if the number of blocks is even. Now exactly the same argument as in the unimodal case proves that the conditions (3) and (4) of the theorem are necessary and sufficient.

Using Theorem 4.3 we note that the methods of this section allow us to describe the standard nice parabolic subalgebras coming from even nilpotent elements in the case of $\mathfrak{so}(n)$ and thereby give a parametrization of the even nilpotent orbits.

Lemma 4.10. *The even nilpotent orbits of $\mathfrak{so}(n)$ are in one-to-one correspondence with the unimodal, palindromic compositions of n which have the property that if the number of parts is even, then all the block sizes are even.*

4.4. **The exceptional cases.** In this subsection we will explain how to derive the tables for the exceptional Lie algebras. In the case of G_2, Lemma 2.3 and Corollary 2.2 already give the classification of nice parabolic subalgebras. For F_4 there are eight parabolic subalgebras that are given by a TDS and seven parabolic subalgebras that are not nice by the dimension criterion in Corollary 2.2. The only remaining case is $(1,0,0,1)$. One uses Corollary 2.6 to show that this parabolic subalgebra is not nice: The corresponding Levi factor has dimension 12. The only nilpotent element in the Bala-Carter table (cf. [C], p. 401) that has a 12-dimensional centralizer satisfies $\dim\mathfrak{m} = \dim\mathfrak{g}_1$, i.e. is a distinguished nilpotent. But the grading associated to $(1,0,0,1)$ gives a much smaller dimension for $\mathfrak{g}_1$. Hence, this parabolic subalgebra is not nice.

Remark 4.11. Observe that for G_2 and F_4, the nice parabolic subalgebras are exactly those given by an even TDS.

The most complicated case is E_n. We describe the steps that led to the classification (cf. list in Section 1).

First step: Use the dimension criterion (Corollary 2.2). There remain 37, 46 resp. 40 parabolic subalgebras of E_6, E_7, resp. of E_8. We used Mathematica to check the dimensions of the graded parts.

In a second step, one identifies all parabolic subalgebras that are given by a TDS (10 for E_6, 24 for E_7 and 46 for E_8). They can be found using the tables of nilpotent orbits in [C], pp. 402-407.

Thirdly, the subdiagram result (Lemma 2.4) gives sixteen more nice parabolic subalgebras for E_6. It also excludes a couple of parabolics for all E-types that have a bad subdiagram (of type D_n, $n = 4,5,6,7$).

After that, there remain 30 parabolic subalgebras of E-types. For all these parabolic subalgebras, one calculates the dimension of the Levi factor. By Theorem 2.5, this is the same as the dimension of the centralizer for a Richardson element. In a case-by-case study, one checks whether there is a nilpotent element in $\mathfrak{g}_1$ whose stabilizer has the required dimension. This can be done using the Computer Algebra Program GAP:

Remark 4.12. The GAP-program uses the function 'random' to generate a generic element $x \in \mathfrak{g}_1$, namely it produces an element $x = \sum_{\alpha\in\mathfrak{g}_1} c_\alpha X_\alpha$ where all c_α are nonzero. Generic means that the dimension $\dim\mathfrak{m}^x$ of the centralizer of x in the Levi factor $\mathfrak{m}$ equals $\dim\mathfrak{m} - \dim\mathfrak{g}_1$.

The program tests whether x is generic and then calculates the dimension $\dim\mathfrak{g}^x$ of the centralizer of x in $\mathfrak{g}$. By Theorem 2.5, $\mathfrak{p}$ is nice if and only if this dimension equals $\mathfrak{m}$.

Remark 4.13. We came to the same list by hand. This involved studying the roots of $\mathfrak{g}_1$ using a variety of different methods.

The complete list of nice parabolic subalgebras in type E consists of 30 parabolic subalgebras of E_6, 29 of E_7 and 29 of E_8, most of those come from an even TDS. The next table lists all nice parabolic subalgebras that are not given by an even TDS nor by a subdiagram of a parabolic given by an even TDS. We use the Bourbaki numbering of simple roots:

$$\begin{array}{lll}
(1,1,0,0,1,0) & (1,1,0,0,1,0,1) & (1,0,1,0,0,0,0,1) \\
(1,1,0,0,0,0) & (1,1,0,0,0,0,1) & (1,0,0,0,0,0,1,0) \\
(0,1,1,0,0,1) & (0,1,1,0,0,1,1) & (0,0,1,0,0,0,1,0) \\
(0,1,0,0,0,1) & (0,0,1,0,0,0,1) & \\
 & (0,0,0,0,1,0,1) &
\end{array}$$

5. Nilpotent elements, a normal form

In this section, we introduce a normal form for Richardson elements in $\mathfrak{g}_1$. We will not give any proofs, given that the paper is already long. In a separate paper ([B]), the first named author discusses this normal form for the nilpotent elements in more detail and explains the weight structure of the representation $\mathfrak{g}_1$ of $\mathfrak{m}$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be a nice parabolic subalgebra of a classical Lie algebra. We will show how to choose a Richardson element in $\mathfrak{g}_1$ that projects into only the root spaces for the roots in a small set S_1. For each type of a parabolic subalgebra $\mathfrak{p}$ in type $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ we give a recipe that picks this set of $\mathfrak{g}_1$.

We view the classical Lie algebras in the corresponding $\mathfrak{gl}_N$ ($N = n+1, 2n+1, 2n, 2n$ for types $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$). If α_{ij}, $i<j$, is a positive root for $\mathfrak{gl}_N$, then the standard matrix with a 1 in the i,j position and zeros everywhere else, E_{ij}, is a basis for the root space $\mathfrak{g}_{\alpha_{ij}}$.

Recall that the sequence of the block lengths of the standard Levi factor are of the form $(a_1,\dots,a_{r+1})$ for $\mathfrak{sl}_{n+1}$. For the other classical Lie algebras, they are of the form $(a_1,\dots,a_r,a_r,\dots,a_1)$ (even number of blocks, (A)-case) or of the form $(a_1,\dots,a_r,a_{r+1},a_r,\dots,a_1)$ (odd number of blocks, (B)-case) respectively. These sequences give rise to a block decomposition of the first super-diagonal (cf. discussion above Lemma 3.6). We thus obtain r subsets of the positive roots. More precisely, the roots of $\mathfrak{g}_1$ are divided into r subsets.

We can think of the roots of such a subset as filling out a rectangle of size $a_i \times a_{i+1}$, $1 \le i \le r-1$ (resp. $i \le r$ in the (B)-case) and of size $a_r \times a_r$ in the (A)-case. We denote these rectangles by $R_{i,i+1}$ ($i \le r$). Note that the central one in the (A)-case is a square matrix of size $a_r \times a_r$ that is symmetric if the Lie algebra is symplectic and skew-symmetric if $\mathfrak{g}$ is an orthogonal Lie algebra. The entries of a rectangle $R_{i,i+1}$ will be labeled by (k,l), $1 \le k \le a_{i+1}$, $1 \le l \le a_i$, starting with $(1,1)$ in the lower left corner and ending with (a_{i+1},a_i) in the upper right corner. Every such entry (k,l) corresponds uniquely to a positive root α_{i_k,j_l} of $\mathfrak{gl}_N$ since every entry (k,l) of such a rectangle corresponds uniquely to a matrix entry E_{i_k,j_l}.

Remark 5.1. Our choice of the subset S_1 of the roots of $\mathfrak{g}_1$ has to do with the weight structure of the representation of the Levi factor on the space $\mathfrak{g}_1$. The subsets S_1 are related to our initial proofs of the theorems. Especially, they play a role in our first theoretical approach to the exceptional case.

Remark 5.2. Using Proposition 3.7 one checks that the constructed nilpotent element is generic (that is the X_R^j have generic rank). So the constructed matrix X_R is a Richardson element for $\mathfrak{p}$ in $\mathfrak{g}_1$

5.1. **Case** A_n**.** Let $\mathfrak{p}$ be a nice parabolic subalgebra of $\mathfrak{sl}_{n+1}$. Let $(a_1,\dots,a_{r+1})$ be the sequence of the block lengths of the standard Levi factor of $\mathfrak{p}$. By Theorem 1.1 this sequence is unimodal, so $a_1 \le \dots \le a_s \ge \dots \ge a_{r+1}$. We will now give the subset S_1 of the sets of roots of $\mathfrak{g}_1$ by explicitly describing the needed roots for each rectangle $R_{i,i+1}$.

Recipe 5.3. (1a) If i is odd, we choose the entries $(1,1)$, $(2,2)$, $\dots,(a_i,a_i)$ of the rectangle $R_{i,i+1}$.

(1b) If i is even, we choose entries (a_{i+1},a_i), $(a_{i+1}-1,a_i-1)$, $\dots,(a_{i+1}-a_i+1,1)$ of $R_{i,i+1}$.

(2) The subset S_1 of the roots of $\mathfrak{g}_1$ is then the union of all roots corresponding to the chosen entries.

(3) We define the nilpotent element X_R to be the matrix $\sum_{\alpha\in S_1} E_\alpha$.

The result is a nilpotent matrix that has r small boxes, one in each of the rectangles $R_{i,i+1}$. The small boxes consist of skew-diagonal matrices as shown in the following example.

Example 5.4. The parabolic subalgebra $(1,0,1,0,0,1)$ of A_6, i.e. $\mathfrak{p}$ is given by the sequence $(1,2,3,1)$ of block length of the standard Levi factor. The normal form of the Richardson element is

$$X_R := \begin{pmatrix} * & 1 & & & & & \\ & * & * & & & 1 & \\ & * & * & & 1 & & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & * & * & * & 1 \\ & & & & & & * \end{pmatrix}.$$

The roots that are chosen are $\alpha_1,\alpha_2+\alpha_3+\alpha_4,\alpha_6,\ \alpha_3+\alpha_4$.

Remark 5.5. In Example 4.6 of [GR] Röhrle and Goodwin describe Richardson elements of $\mathfrak{g}_1$ for parabolic subalgebras of $\mathfrak{gl}_N$. They follow a construction given by [BHRR]. The method amounts to choose an identity matrix of size $\min\{a_i,a_{i+1}\}$ in each rectangle $R_{i,i+1}$, starting with entry $(1,a_i)$. The two methods are equivalent. Our approach is motivated by the construction for the other classical Lie algebras as we will see.

5.2. **Case C_n.** Here, the pattern is slightly different. Either there is an even number of blocks in the standard Levi factor ((A)-case) or an odd number ((B)-case). In the first case the recipe is essentially the same as for $\mathfrak{sl}_{n+1}$. In the second case we modify Recipe 5.3. We divide the chosen entries into two sets, choosing half of them starting with entry $(1,1)$ and the other half starting with entry (a_{i+1},a_i). Let $b_i := \lfloor\frac{a_i}{2}\rfloor$ and $B_i := \lceil\frac{a_i}{2}\rceil$.

Recipe 5.6. (1A) For $i\le r$ we do the same as in Recipe 5.3. In particular, in rectangle $R_{r,r+1}$ we choose all entries on the skew-diagonal, i.e. $(1,1),\dots,(a_r,a_r)$.

(1B) If i is odd, we choose the entries $(1,1),\dots,(B_i,B_i)$ and the entries (a_{i+1},a_i), $\dots,(a_{i+1}-b_i+1,a_i-b_i+1)$ of the rectangle $R_{i,i+1}$. If i is even, we choose the entries $(1,1),\dots,(b_i,b_i)$ and the entries $(a_{i+1},a_i),\dots,(a_{i+1}-b_i,a_i-b_i)$.

(2) The subset S_1 of the roots of $\mathfrak{g}_1$ is the union of all roots that correspond to the chosen entries.

(3) We define the nilpotent element X_R to be the following matrix: If α_{ij} (with $i\le n$) is an element of S_1, then the matrix X_R has entry 1 at position (i,j) and the correspond entry at its adjoint position: It has entry -1 at position $(2n-j+1,2n-i+1)$ if $j>n$ and entry 1 at position $(2n-j+1,2n-i+1)$ if $j\le n$.

It is clear that this is a matrix of $\mathfrak{sp}_{2n}$.

Example 5.7. (1) The parabolic subalgebra $(0, 0, 1, 0, 0)$ of C_5. The standard Levi factor has block lengths $(3, 4, 3)$. There is an odd number of blocks:

$$X_R = \begin{pmatrix} * & * & * & & & & 1 & & & \\ * & * & * & & 1 & & & & & \\ * & * & * & 1 & & & & & & \\ & & & * & * & * & * & & & 1 \\ & & & * & * & * & * & & & \\ & & & * & * & * & * & & -1 & \\ & & & * & * & * & * & -1 & & \\ & & & & & & & * & * & * \\ & & & & & & & * & * & * \\ & & & & & & & * & * & * \end{pmatrix}.$$

We have $\dim\ker X_R = 4$, $\dim\ker X_R^2 = 8$. Thus the partition of X_R is $(2, 2, 3, 3)$, with one pair of odd entries. The dual partition is $(4, 4, 2)$. This gives $\dim \mathfrak{g}^{X_R} = \frac{1}{2}(16 + 16 + 4) + 1 = 19$ (cf. Theorem 4.5 which is equal to $\dim \mathfrak{m}$.

(2) The parabolic subalgebra $(0, 1, 0, 0, 0, 1)$ of C_6. The standard Levi factor has block lengths $(2, 4, 4, 2)$, i.e., an even number of blocks:

$$X_R = \begin{pmatrix} * & * & & 1 & & & & & & & & \\ * & * & 1 & & & & & & & & & \\ & & * & * & * & * & & & & 1 & & \\ & & * & * & * & * & & & 1 & & & \\ & & * & * & * & * & & 1 & & & & \\ & & * & * & * & * & 1 & & & & & \\ & & & & & & * & * & * & * & & \\ & & & & & & * & * & * & * & & \\ & & & & & & * & * & * & * & & -1 \\ & & & & & & * & * & * & * & -1 & \\ & & & & & & & & & & * & * \\ & & & & & & & & & & * & * \end{pmatrix}.$$

We compute the dimension of the centralizer of X_R in $\mathfrak{g}$. Note that $\dim\ker X_R = 12 - 8 = 4$, $\dim\ker X_R^2 = 8$, $\dim\ker X_R^3 = 10$ and $\dim\ker X_R^4 = 12$. Thus the partition is $(2, 2, 4, 4)$, which is equal to its dual partition. So $\dim \mathfrak{g}^{X_R} = \frac{1}{2}(16 + 16 + 4 + 4) = 20$. The Levi factor also has dimension 20.

5.3. **Cases** B_n**,** D_n**.** The choice in the case of the orthogonal Lie algebras is similar to the symplectic case. There are some differences:

Remark 5.8. In the construction for the special orthogonal case, there are some adaptions of the recipe: If there is an even number of blocks, i.e., in the (A)-case, and $a_i > a_{i+1}$ for some i, we pick a_{i+1} roots for $R_{i,i+1}$. In the (B)-case we choose twice $B_i := \lceil \frac{a_i}{2} \rceil$ entries in each rectangle $R_{i,i+1}$ if the parities of a_i and of a_{i+1} are different.

Recall that $b_i := \lfloor \frac{a_i}{2} \rfloor$ and $B_i := \lceil \frac{a_i}{2} \rceil$. To deal with nonunimodal sequences we introduce $s_i := \min\{a_i, a_{i+1}\}$.

Recipe 5.9. (1A) Let the number of blocks in the standard Levi factor be even.

Let $i \leq r + 1$. For odd i we choose entries $(1, 1)$, …, (s_i, s_i) in rectangle $R_{i,i+1}$. For even i we choose entries (a_{i+1}, a_i), …, $(a_{i+1} - s_i + 1, a_i - s_i + 1)$ of the rectangle $R_{i,i+1}$.

In rectangle $R_{r,r+1}$ we choose b_r two-by-two matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on the skew-diagonal, starting at $(1, 1)$ if r is odd or at (a_r, a_r) if r is even; cf. Example 5.11.

(1B) For each rectangle $R_{i,i+1}$ we pick the entries $(1, 1), \ldots, (B_i, B_i)$ and the entries (a_{i+1}, a_i), …, $(a_{i+1} - b_i, a_i - b_i)$. As an illustration, see Example 5.10 below and Example 5.12.

(2) The set S_1 is the union of all roots a_{ij} that correspond to the chosen entries.

(3) The nilpotent element X_R is the following matrix: If $\alpha_{ij} \in S_1$, then X_R has entry 1 at position (i,j) and entry -1 at the position adjoint to (i,j): at position $(2n-j+2, 2n-i+2)$ in the case of $\mathfrak{so}_{2n+1}$ resp. at $(2n-j+1, 2n-i+1)$ in the case of $\mathfrak{so}_{2n}$.

Example 5.12 treats nonunimodal cases.

Example 5.10. We present the constructed nilpotent element for parabolic subalgra of B_4 where the sequence of block lengths is unimodal and consists of an odd number of blocks. The parabolic subalgebra of B_4 is $(0,1,0,0)$, with block lengths $(2,5,2)$:

$$X_R = \begin{pmatrix}
* & * & * & & & & & 1 & & & \\
* & * & * & & 1 & & 1 & & & & \\
* & * & * & 1 & & & & & & & \\
 & & & * & * & * & * & * & & & -1 \\
 & & & * & * & * & * & * & & -1 & \\
 & & & * & * & * & * & * & & & \\
 & & & * & * & * & * & * & & -1 & \\
 & & & * & * & * & * & * & -1 & & \\
 & & & & & & & & * & * & * \\
 & & & & & & & & * & * & * \\
 & & & & & & & & * & * & *
\end{pmatrix}.$$

Here, $\dim\ker X_R = 5$, $\dim\ker X_R^2 = 8$ and $\dim\ker X_R^3 = 11$, giving the partition $(3,3,3,1,1)$ with five odd parts. The dual partition is $(5,3,3)$, and so $\dim \mathfrak{g}^{X_R} = \frac{1}{2}(25+9+9) - \frac{5}{2} = 19$ which is the dimension of the Levi factor.

Example 5.11. This matrix is the constructed nilpotent element for the parabolic subalgebra of D_5 that has α_2 and α_5 in the nilpotent radical. The sequence of the block lengths is $(2,3,3,2)$ with an even number of blocks:

$$X_R = \begin{pmatrix}
* & * & & 1 & & & & & & \\
* & * & 1 & & & & & & & \\
 & & * & * & * & & 1 & 0 & & \\
 & & * & * & * & & 0 & -1 & & \\
 & & * & * & * & & & & & \\
 & & & & & * & * & * & & \\
 & & & & & * & * & * & & -1 \\
 & & & & & * & * & * & -1 & \\
 & & & & & & & & * & * \\
 & & & & & & & & * & *
\end{pmatrix}.$$

We have $\dim\ker X_R = 4$, $\dim\ker X_R^2 = 6$ and $\dim\ker X_R^3 = 8$, $\dim\ker X_R^4 = 10$. The Jordan form is given by the partition $(1,1,4,4)$ with dual partition $(4,2,2,2)$. So the centralizer of X_R has dimension 13 which is equal to $\dim \mathfrak{m}$.

Example 5.12. The parabolic subalgebra is $(0,0,0,1,0)$ of B_5. The sequence of block length in the standard Levi factor is $(5,3,5)$:

$$X_R = \begin{pmatrix}
* & * & * & * & & & 1 & & & & \\
* & * & * & * & & 1 & & & & & \\
* & * & * & * & & 1 & & & & & \\
* & * & * & * & 1 & & & & & & \\
 & & & & * & * & * & & & & -1 \\
 & & & & * & * & * & & -1 & -1 & \\
 & & & & * & * & * & -1 & & & \\
 & & & & & & & * & * & * & * \\
 & & & & & & & * & * & * & * \\
 & & & & & & & * & * & * & * \\
 & & & & & & & * & * & * & *
\end{pmatrix}.$$

We leave the calculation of the dimension of the Levi factor to the reader. The second example is the parabolic subalgebra of D_6 where α_2, α_3 and α_6 are the simple roots in the Levi factor. The sequence of the block lengths is $(1,3,2,2,3,1)$, with an even number of blocks:

$$X_R = \begin{pmatrix}
* & 1 & & & & & & & & & & \\
 & * & * & * & & 1 & & & & & & \\
 & * & * & * & 1 & & & & & & & \\
 & * & * & * & & & & & & & & \\
 & & & & * & * & 1 & & & & & \\
 & & & & * & * & & -1 & & & & \\
 & & & & & & * & * & & & -1 & \\
 & & & & & & * & * & & -1 & & \\
 & & & & & & & & * & * & * & \\
 & & & & & & & & * & * & * & \\
 & & & & & & & & * & * & * & -1 \\
 & & & & & & & & & & & *
\end{pmatrix}.$$

Remark 5.13. In the (A)-cases, type B, D Recipe 5.9 chooses more roots than we do in the other case. We know that there are examples where this is necessary, e.g., the parabolic subalgebra of D_{11} with block lengths $(1,3,5,4,5,3,1)$. Here, one needs the "extra" roots chosen in $R_{1,2}$ and in $R_{2,3}$. If we choose only the commuting roots according to the recipe for type C (Recipe 5.6) we obtain a nilpotent X_R' with $\dim \mathfrak{g}^{X_R'} > 41$ while $\dim \mathfrak{m} = 41$. On the other hand, if there are only odd sized blocks, as in Example 5.10, there is a way to choose one entry less per rectangle $R_{i,i+1}$. In Example 5.10 one can alternatively choose $(2,1)$, $(3,2)$ and $(4,3)$ in $R_{1,2}$. Also, if $r = 2$, we can use part (1B) of Recipe 5.6 instead of (1B) in Recipe 5.9. These phenomena are discussed in detail in [B] where the first named author describes the properties of the roots involved in the construction of Richardson elements.

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