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ON MINIMAL REPRESENTATIONS DEFINITIONS AND PROPERTIES

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Abstract. This paper gives a self-contained exposition of minimal representations. We introduce a notion of weakly minimal representations and prove a global rigidity result for them. We address issues of uniqueness and existence and prove many key properties of minimal representations needed for global applications.

1. Introduction

Let F be a local field of characteristic zero and residue characteristic p, and let G be a simple, simply connected linear algebraic group over F with Lie algebra $\mathfrak{g}$. There is a notion of "minimal representation" of the locally compact group $G(F)$ (or some covering group of $G(F)$) which we briefly recall. Suppose first that F is non-archimedean and π is an irreducible smooth representation of $G(F)$. The character distribution χ_π of π can be regarded as a distribution on a neighbourhood of $0 \in \mathfrak{g}$ (via the exponential map) and a result of Howe and Harish-Chandra says that

$$\chi_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$$

where the sum ranges over the nilpotent $G(F)$-orbits $\mathcal{O}$ in $\mathfrak{g}^*$, $\mu_{\mathcal{O}}$ is the distribution given by a suitably normalized $G(F)$-invariant measure on $\mathcal{O}$ and $\hat{\mu}_{\mathcal{O}}$ denotes the Fourier transform of the distribution $\mu_{\mathcal{O}}$. Now there is a unique nilpotent G-orbit $\mathbf{O}_{\min}$ in $\mathfrak{g}(\bar{F})^*$ which has the smallest non-zero dimension; this is simply the orbit of a highest weight vector of the coadjoint representation. Assume that $\mathbf{O}_{\min} \cap \mathfrak{g}$ is non-empty, and that it consists of a unique $G(F)$-orbit $\mathcal{O}_{\min}$. One can then make the following definition.

Definition. If F is non-archimedean, an irreducible smooth representation π of $G(F)$ is called a minimal representation of $G(F)$ if

$$\chi_\pi = \widehat{\mu}_{\mathcal{O}_{min}} + c_0.$$

Although this definition is aesthetically pleasing, practice has shown that a different definition, in terms of a certain *local Fourier-Jacobi model*, is better especially if certain applications are to hold without any restrictions on the residual characteristic. Details of this definition are given in Section 3, and the equivalence of the two definitions is proved, assuming that $p \neq 2$.

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For archimedean local fields, a representation π is minimal if its annihilator $\mathcal{J}$ in the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ is equal to the Joseph ideal (see Section 4). In particular, $\mathcal{J}$ is a completely prime ideal whose associated variety is the closure of $\mathbf{O}_{\min}$, so π has the smallest possible Gelfand-Kirillov dimension amongst the infinite dimensional representations of $G(F)$.

The purpose of this paper is to give a self-contained treatment of minimal representations. Besides addressing the issues of existence and uniqueness, we prove many key properties of minimal representations which are crucial for global applications, such as constructing the Arthur packets on G_2 using the minimal representations of (quasi-split) D_4 [GGJ], or constructing analogues of theta series on G_2 using the minimal representation of quaternionic E_8 [G].

Our approach to the construction of minimal representations uses a global result. For this, we relax somewhat the definition of minimality, and introduce a notion of weakly minimal representations (cf. Sections 3 and 4). Using an idea of Kazhdan, we prove a (global) rigidity result for weakly minimal representations in Section 5. This says that if one local component of an irreducible automorphic subrepresentation of an adelic group $G(\mathbb{A})$ is weakly minimal, then all its local components are weakly minimal. To exploit this result for the construction of minimal representations, we shall give sufficient conditions for a weakly minimal representation to be minimal. A feature of our approach is that one addresses the automorphic realization of a global minimal representation concurrently.

At this point, we should mention that we restrict ourselves to groups for which one expects the minimal representation to be unique. These are the groups for which $\mathcal{O}_{\min}$ exists and is unique, and which satisfy some other conditions (which assures uniqueness of the admissible data attached to $\mathcal{O}_{\min}$; see Torasso [To]). The list of these groups is given in Section 6. The view point taken here is that if the minimal representation exists and is unique, then it must correspond to a remarkable parameter. Thus, in Section 7, we give the parameter of the candidate minimal representation for each (p-adic) group under our consideration. For example, if G splits over an unramified extension of F, then the minimal representation has Iwahori-fixed vectors and we write down the corresponding Hecke algebra representation. In general, we compute the exponents of the candidate representation; these calculations are given in Section 8.

From Section 9 onwards, we restrict our groups to those of absolute type D_4 and E_n. This restriction to exceptional groups is natural for two reasons. First, the minimal representations of groups that have a maximal parabolic subgroup with abelian unipotent radical (which includes all classical groups) can be obtained and studied via a Fourier-Jacobi functor. This relatively simple and elegant approach has been used in a recent paper of Weissman [W]. Second, we can give a fairly uniform treatment of the structure of exceptional groups (Section 10) to derive important properties of their minimal representations (Section 11). This paper culminates with Section 12, where the minimality of our candidate representations is proved using the rigidity theorem.

Finally, we should mention that minimal representations for various groups have been constructed by various people: Vogan [V1], Brylinski and Kostant [BK], Gross and Wallach [GW], Kazhdan [K], Kazhdan and Savin [KS], Savin [S2], Rumelhart [R], Torasso [To], as well as in the paper of Weissman [W]. It has been our intention to gather at one place all the key properties of minimal representations needed for applications and to address certain inaccuracies and gaps in the existing literature (cf. Remarks 4.5 and 5.9). Thus, our treatment in this paper is largely self-contained and our approach somewhat different from these other works.

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2. Heisenberg and Weil Representations

In this section, we introduce the Heisenberg group and the Weil representation in some detail.

2.1. **Heisenberg group.** Let F be any field of characteristic zero (for simplicity) and let (V,ω) be a symplectic vector space over F so that ω is a nondegenerate alternating form on V. The Heisenberg group N_V associated to (V,ω) is the algebraic group whose underlying variety is $V\oplus F$ and whose group structure is given by

$$(x,z)\cdot(y,z')=(x+y,z+z'+\frac{1}{2}\omega(x,y)).$$

It is a 2-step nilpotent group with 1-dimensional center $Z=\{(0,t):t\in F\}$. The Lie algebra $Lie(N_V)$ can be naturally identified with N_V and we shall write e for the element $(0,1)\in Lie(Z)$.

A polarization of V is a decomposition $V=V^+\oplus V^-$ of V into the direct sum of two Lagrangian subspaces. Given a polarization, let $N_V^-=V^-\oplus F$. Then N_V^- is a maximal (normal) abelian subgroup of N_V.

2.2. **Heisenberg representation.** Now assume that F is a local field. Let ψ be a non-trivial unitary character of $Z(F)$. The set of extensions of ψ to a character of $N_V^-(F)$ is a principal homogeneous space for $N_V(F)/N_V^-(F)\cong V^+$. To fix ideas, for any $x\in V^+$, we define the extension ψ_x of ψ by

$$\psi_x(n)=\psi(\omega(x,n)),\quad \text{for } n\in V^-.$$

For any extension ψ_x of ψ, let ρ_ψ be the (smooth) induced representation $ind_{N_V^-}^{N_V}\psi_x$ of $N_V(F)$. This representation is independent of the choice of the extension, up to equivalence. For the extension ψ_0, an explicit model for ρ_ψ is given on the space $\mathcal{S}(V^+)$ of

$$\begin{cases}\text{locally constant, compactly supported functions on } V^+ \text{ if } F \text{ is non-archimedean;}\\ \text{Schwarz functions on } V^+ \text{ if } F \text{ is archimedean.}\end{cases}$$

The action of N_V is given as follows:

$$\begin{cases}(\rho_\psi(x')f)(x)=f(x+x') \text{ if } x'\in V^+,\\ (\rho_\psi(y)f)(x)=\psi_x(y)f(x) \text{ if } y\in V^-,\\ (\rho_\psi(z)f)(x)=\psi(z)f(x) \text{ if } z\in Z(F).\end{cases}$$

As explained in [We, §11], the space $\mathcal{S}(V^+)$ has a natural topology (trivial when F is non-archimedean) for which ρ_ψ is a smooth representation. Further, one sees easily that the natural inner product

$$\{f_1, f_2\} = \int_{V^+} f_1(x) \cdot \overline{f_2(x)} dx$$

on $\mathcal{S}(V^+)$ is $N_V(F)$-invariant. The completion of $\mathcal{S}(V^+)$ is thus a unitary representation of $N_V(F)$. In fact, by the Stone-von Neumann theorem, it is the unique irreducible unitary representation of $N_V(F)$ with central character ψ.

Now we have:

2.3. **Proposition.** *Assume that F is non-archimedean. The representation ρ_ψ is an irreducible admissible representation of $N_V(F)$ which is independent of the choice of the extension ψ_x. Moreover, for any extension ψ_x, $dim(\rho_\psi)_{N_V^-,\psi_x} = 1$.*

Proof. See [MVW, Ch. 2, §I.2, pp. 28–30] for the first statement and [MVW, Ch. 6, Lemma 6.1, pp. 133–134] for the second. □

2.4. **Weil representation.** Let $Sp(V)$ denote the symplectic group associated to (V, ω). Then $Sp(V)$ act on N_V by

$$g : (x, z) \mapsto (gx, z)$$

with $x \in V$ and $z \in Z$. This gives a semi-direct product $Sp(V) \ltimes N_V$. The following is a well-known result of Shale and Weil [We]:

2.5. **Proposition.** *There exists a two-fold covering $Mp(V)(F) \to Sp(V)(F)$ (trivial when $F = \mathbb{C}$) and a smooth representation ρ_ψ of $Mp(V)(F)$ on $\mathcal{S}(V^+)$ such that*

$$\rho_\psi(g)\rho_\psi(n)\rho_\psi(g)^{-1} = \rho_\psi(gng^{-1})$$

for all $g \in Mp(V)(F)$ and $n \in N_V(F)$. Further, the natural inner product $\{-,-\}$ on $\mathcal{S}(V^+)$ is preserved by the action of $Mp(V)(F)$.

2.6. **Archimedean case.** We conclude this section by taking a closer look at the representation ρ_ψ when F is archimedean. Since the representation ρ_ψ is smooth, we obtain an associated representation of the *real* Lie algebra $\mathfrak{sp}(V) \oplus \mathfrak{n}$, and thus a representation of the complexified Lie algebra $\mathfrak{sp}(V) \otimes_{\mathbb{R}} \mathbb{C}$. On the level of Lie algebras, this extension of ρ_ψ to $Mp(V)(F)$ can be more easily described. This observation reflects the fact that $\mathfrak{sp}(V)$ is isomorphic to $\mathrm{Sym}^2(V)$ as a representation of $Sp(V)$.

To be precise, we have a map

$$\mathfrak{sp}(V) \longrightarrow (V^* \otimes V^*)^{S_2},$$

where S_2 is the symmetric group on 2 letters, which is defined by sending an element $A \in \mathfrak{sp}(V)$ to the symmetric bilinear form

$$\omega_A(x, y) = \omega(Ax, y).$$

Using ω, we identify V with V^* by sending $x \in V$ to the linear form $y \mapsto \omega(x, y)$. Thus we have a map

$$\mathfrak{sp}(V) \to (V \otimes V)^{S_2}.$$

This is in fact a $Sp(V)$-equivariant isomorphism; its inverse is given by sending $x \otimes y + y \otimes x$ to the linear map

$$A : z \mapsto \omega(y, z)x + \omega(x, z)y$$

which is an element of $\mathfrak{sp}(V)$. Since there is a natural map $(V \otimes V)^{S_2} \to U(\mathfrak{n})$, we have a map $A \mapsto n_A$ from $\mathfrak{sp}(V)$ to $U(\mathfrak{n})$ and we set

$$p_A = e \cdot A - \frac{1}{2} n_A.$$

It is an element in the universal enveloping algebra of $\mathfrak{sp}(V) \oplus \mathfrak{n}$ and satisfies the properties

$$\begin{cases} [p_A, x] = 0, \text{ for all } x \in U(\mathfrak{n}); \\ [p_A, B] = p_{[A,B]}, \text{ for all } B \in \mathfrak{sp}(V). \end{cases}$$

Of course, the discussion in this paragraph remains valid over any field F of characteristic zero.

Before stating the result on the action of $\mathfrak{sp}(V)$ on $\mathcal{S}(V^+)$, we need to introduce some notation for the case when $F = \mathbb{C}$. In this case, $\mathfrak{sp}(V)$ is itself a complex Lie algebra and

$$\mathfrak{sp}(V) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sp}(V)_L \times \mathfrak{sp}(V)_R.$$

The isomorphism is defined by

$$A \otimes \lambda \mapsto (\lambda \cdot A, \overline{\lambda} \cdot A),$$

so that $\mathfrak{sp}(V)_L$ is isomorphic to $\mathfrak{sp}(V)$ (as complex Lie algebras) and $\mathfrak{sp}(V)_R$ is isomorphic to the conjugate of $\mathfrak{sp}(V)$. Moreover, the subset $\mathfrak{sp}(V) \hookrightarrow \mathfrak{sp}(V) \otimes \mathbb{C}$ is the set of diagonal elements in $\mathfrak{sp}(V)_L \times \mathfrak{sp}(V)_R$.

2.7. **Lemma.** *(i) When $F = \mathbb{R}$, for any $A \in \mathfrak{sp}(V) \otimes \mathbb{C}$, we have*

$$\rho_\psi(e) \cdot \rho_\psi(A) = \frac{1}{2} \cdot \rho_\psi(n_A).$$

In other words, the element p_A is in the annihilator of $(\rho_\psi, \mathcal{S}(V^+))$ for any non-trivial character ψ.

(ii) When $F = \mathbb{C}$, for any $A \in \mathfrak{sp}(V)$, the elements $p_A \otimes 1$ and $1 \otimes p_A$ of $\mathfrak{sp}(V)_L \otimes \mathfrak{sp}(V)_R$ are in the annihilator of $(\rho_\psi, \mathcal{S}(V^+))$ for any non-trivial character ψ.

Proof. (i) This was shown in [GV, §5, Thm. 5.19].

(ii) This can be deduced from the result of (i) as follows. For concreteness, let us choose the unitary character ψ of $Z(\mathbb{C})$ to be given by $z \mapsto exp(2\pi i Re(z))$. The real group $Res_{\mathbb{C}/\mathbb{R}} N$ is not a Heisenberg group over $\mathbb{R}$, since its center $Res_{\mathbb{C}/\mathbb{R}} Z$ has dimension 2 over $\mathbb{R}$. However, the 1-dimensional subgroup $Im(Z) = \{(0, it) : t \in \mathbb{R}\}$ acts trivially on $\mathcal{S}(V^+)$ and hence ρ_ψ is a representation of $N' = N(\mathbb{C})/Im(Z)$. The quotient group N' is the Heisenberg group associated to the real symplectic space (V', ω') with

$$V' = Res_{\mathbb{C}/\mathbb{R}}(V) \quad \text{and} \quad \omega'(x,y) = Re(\omega(x,y))$$

and the action of N' on $\mathcal{S}(V^+)$ via ρ_ψ is precisely the Heisenberg representation ρ'_ψ associated to N'. This representation extends to one for the group $Mp(V', \omega')$, which contains $Res_{\mathbb{C}/\mathbb{R}} Sp(V, \omega)$ as a subgroup. Further, it is well known that the restriction of ρ'_ψ to $Res_{\mathbb{C}/\mathbb{R}} Sp(V, \omega)$ is precisely the representation ρ_ψ. But we already know by (i) how to describe the action of $\mathfrak{sp}(V', \omega')$ on $\mathcal{S}(V^+)$. The assertion of (ii) now follows by a careful computation. □

3. Minimality (p-adic case)

In this section, we will introduce several notions of minimal and weakly minimal representations for p-adic groups, and discuss the relation between different definitions.

3.1. **Groups.** Let G be a simply-connected simple linear algebraic group over F. Fix a maximal split torus A of G, and H a maximal torus containing A. Let Φ be the relative root system associated to A, and Φ_a the absolute root system associated to H. Let $\mathfrak{h} = Lie(H)$. For every γ in Φ_a define h_γ in $\mathfrak{h}(\bar{F})$ so that

$$\kappa(h_\gamma, h) = \frac{2\gamma(h)}{\kappa(h_\gamma, h_\gamma)}$$

for every element h in $\mathfrak{h}(\bar{F})$. A root γ in Φ_a is said to be defined over F, if h_γ is in fact an element of $\mathfrak{a} = Lie(A)$. Next, fix a choice of positive roots Φ_a^+ in Φ_a. Let α be the corresponding highest root. The following is a result of Torasso [To, Prop. 4.3]:

3.2. **Proposition.** *Let $\mathbf{O}_{\min}$ be the minimal nilpotent orbit in $\mathfrak{g}(\bar{F})$. The following two conditions are equivalent:*
i) $\mathbf{O}_{\min}$ intersects $\mathfrak{g}$.
ii) The highest root α is defined over F.

Therefore, from now on, we shall assume that the group G satisfies the condition

- (H0) The highest root α is defined over F.

The adjoint action of h_α on $\mathfrak{g}$ defines a $\mathbb{Z}$-gradation by

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} : [h_\alpha, x] = i \cdot x\}.$$

Then $\mathfrak{g}(i) = 0$ for $|i| > 2$ and $\bigoplus_{i \geq 0} \mathfrak{g}(i)$ is a parabolic subalgebra whose unipotent radical $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is a Heisenberg Lie algebra with center $\mathfrak{z} = \mathfrak{g}(2)$. Let e_α and $e_{-\alpha}$ be a non-zero element in $\mathfrak{g}(2)$, and a non-zero element in $\mathfrak{g}(-2)$. We can pick these elements so that

$$h_\alpha = [e_\alpha, e_{-\alpha}].$$

We let $P = MN$ be the corresponding parabolic subgroup and call it the Heisenberg parabolic. We shall assume that P satisfies the following working hypotheses:

- (H1) M acts transitively on non-trivial elements in Z, with stabilizer $M_{ss} = [M, M]$.

We note that for split groups, the condition (H1) is satisfied if G is not of type A_n or C_n. Since $\mathbf{O}_{\min}$ is the $G(\bar{F})$-orbit of e_α, the condition (H1) assures that $\mathcal{O}_{\min} = \mathbf{O}_{\min} \cap \mathfrak{g}$ is one G-orbit when F is p-adic. Indeed, the stabilizer of e_α in G is $M_{ss}N$ and M_{ss} is simply-connected since G is also. Thus the claim follows from the fact that the first Galois cohomology group of semi-simple, simply connected p-adic groups is trivial.

Set $V = \mathfrak{g}(1)$ for ease of notation and choose an isomorphism of $\mathfrak{z}$ with F by fixing the basis element e (such as e_α). Then V possesses a natural nondegenerate alternating form ω. More precisely, for $x, y \in V$, we set

$$[x, y] = \omega(x, y) \cdot e \in \mathfrak{z}.$$

We give $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{z}$ a group structure given by

$$(x, z) \cdot (y, z') = (x + y, z + z' + \frac{1}{2}\omega(x, y)).$$

With this group structure, $\mathfrak{n}$ is naturally isomorphic to the Heisenberg group N_V. Since F has characteristic zero, the exponential map defines an isomorphism of varieties

$$exp : \mathfrak{n} = N_V \to N.$$

By the Campbell-Hausdorff formula, we see that exp is an isomorphism of unipotent algebraic groups. We may thus apply the construction of the previous section to obtain a smooth representation ρ_ψ of $Mp(V)(F) \ltimes N(F)$.

3.3. **The representation W_ψ.** Let $P_{ss} = M_{ss} \cdot N$, where M_{ss} is the derived group of M. Note that M_{ss} is simply-connected since G is also. Since M_{ss} acts trivially on Z, it preserves the symplectic form ω on V and thus we have a map $M_{ss} \to Sp(V)$. The two-fold covering $Mp(V)(F) \to Sp(V)(F)$ is not necessarily split over the image of $M_{ss}(F)$ in general. In order not to introduce more notations, we make an additional working hypothesis:

- (H2) Suppose that there is a splitting $M_{ss}(F) \to Mp(V)(F)$.

The hypothesis (H2) is not needed if we are willing to work with a covering group of $G(F)$. Together with Proposition 2.5, it gives rise to an extension of ρ_ψ to $P_{ss}(F) = M_{ss}(F) \ltimes N(F)$. This extension will be unique if $M_{ss}(F)$ is perfect. Since M_{ss} is simply-connected, this will be the case if we require (as the last condition):

- (H3) M_{ss} has no anisotropic factors.

The simple groups satisfying the conditions (H0) through (H3) can be read off from [To, p. 346], and will be listed in Section 6.

3.4. **Definition.** Let (ρ_ψ, W_ψ) be the unique extension to $P_{ss}(F)$ of the representation ρ_ψ of $N(F)$ on $\mathcal{S}(V^+)$.

The representation W_ψ will play a significant role in the rest of the paper.

3.5. **Local Fourier-Jacobi Model.** Now assume that F is p-adic. For any smooth representation π of $G(F)$, define

$$\pi_{Z,\psi} = \pi / \langle \pi(z)v - \psi(z)v | z \in Z, v \in \pi \rangle.$$

Note that this definition makes perfect sense even when ψ is replaced by the trivial character. In that case the corresponding quotient will be denoted by π_Z.

As it is shown in [W, Cor. 2.4], $\pi_{Z,\psi}$, as an $N(F)$-module, is a direct sum of W_ψ:

$$\pi_{Z,\psi} = Hom_N(W_\psi, \pi_{Z,\psi}) \otimes W_\psi.$$

The space $Hom_N(W_\psi, \pi_{Z,\psi})$ is naturally an $M_{ss}(F)$-module; the action of $m \in M_{ss}(F)$ being given by the formula

$$T \mapsto \pi_{Z,\psi}(m) T \rho_\psi(m^{-1})$$

for any $T \in Hom_N(W_\psi, \pi_{Z,\psi})$. Here is the definition of minimality in the non-archimedean case.

3.6. **Definition** (Minimality in p-adic case)**.** Suppose that F is non-archimedean. Let π be an irreducible smooth representation of $G(F)$ and let ψ be any non-trivial character of $Z(F)$.

(i) Say that π is *weakly minimal* if $Hom_N(W_\psi, \pi_{Z,\psi})$ is a trivial $M_{ss}(F)$-module (possibly infinitely many copies of $\mathbb{C}$ as a representation of $M_{ss}(F)$).
(ii) Say that π is *minimal* if $Hom_N(W_\psi, \pi_{Z,\psi}) \cong \mathbb{C}$, as an $M_{ss}(F)$-module.

There is of course no reason why a minimal representation should exist. Nevertheless, we can derive certain properties of a minimal representation from its definition. The following proposition gives an alternative characterization of minimality, in terms of the Howe-Harish-Chandra local character expansion.

3.7. **Proposition.** *Assume that $p \neq 2$. An irreducible unitarizable representation π is minimal if and only if*

$$\chi_\pi = \hat{\mu}_{\mathcal{O}_{min}} + c_0.$$

Proof. The *if* part is an immediate consequence of a result of Moeglin and Waldspurger [MW]. To prove that a minimal representation has the desired character expansion, we need to consider a representation of $P(F)$ defined by

$$W = ind_{P_0}^{P} W_\psi \quad \text{(smooth compact induction)}.$$

The main properties of W are given by the following lemma.

3.8. **Lemma.** *The representation W is the unique smooth representation of $P(F)$ such that for any character ψ' of $Z(F)$, we have*

$$(W)_{Z,\psi'} \cong \begin{cases} W_{\psi'} \text{ if } \psi' \neq 1, \\ 0 \text{ if } \psi' = 1. \end{cases}$$

In particular, W is irreducible and independent of ψ.

Proof. This follows from the property (H1), by a version of the Mackey Lemma. See [MVW, Ch. 6, Lemma 6.1, pp. 133–134]. □

Let π be a minimal representation, and $\pi(Z)$ the $P(F)$-submodule of π such that $\pi_Z = \pi/\pi(Z)$. Since $P(F)$ acts transitively on the set of non-trivial characters of Z, the minimality of π implies

$$\pi(Z)_{Z,\psi'} \cong \begin{cases} W_{\psi'} \text{ if } \psi' \neq 1, \\ 0 \text{ if } \psi' = 1. \end{cases}$$

In particular, it follows from the previous lemma that $\pi(Z) \cong W$.

Next, we shall define a projection operator on π. Fix a (suitably chosen) special maximal compact subgroup G_0 of $G(F)$ and let $\{G_n\}$ be the associated principal congruence subgroups. Let $\mathfrak{g}_n$ be a filtration of $\mathfrak{g}$ by lattices such that $G_n = \exp(\mathfrak{g}_n)$. Assume that the element e_α is a basis element for $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$ and the additive character ψ has conductor zero. For $n \geq 0$, define a function χ_n supported on $\mathfrak{g}_n$ by

$$\chi_n(x) = \psi(\kappa(x, \varpi^{-n-1} e_{-\alpha})),$$

where κ is the Killing form. If $n \geq 1$, then $\chi_n([x, y]) = 1$, and the function χ_n can be viewed as a character of G_n via the exponential map. Let π^{G_n,χ_n} be the subspace of all vectors in π on which G_n acts as the character χ_n. Note that

$$\int_{\mathfrak{g}} \chi_n(x)\pi(\exp(x))\, dx$$

defines a projection operator on π^{G_n,χ_n}. For all sufficiently large integers n, the rank of this projector can be computed in terms of the character expansion of π:

$$dim\ \pi^{G_n,\chi_n} = \sum_{\mathcal{O}} c_{\mathcal{O}} \mu_{\mathcal{O}}(\hat{\chi}_n),$$

where $\hat{\chi}_n$ denotes the Fourier transform of χ_n. Note that $\hat{\chi}_n$ is equal to the characteristic function of $\varpi^{-n-1}e_{-\alpha} + \mathfrak{g}_n^*$, where $\mathfrak{g}_n^*$ is the dual of $\mathfrak{g}_n$ with respect to κ. Since any non-zero nilpotent orbit $\mathcal{O}$ contains $\mathcal{O}_{\min}$ in its closure, it follows that $\mu_{\mathcal{O}}(\hat{\chi}_n) > 0$. Since $\chi_n(x) = \chi_0(\varpi^{-n} \cdot x)$, the homogeneity property of the distributions $\mu_{\mathcal{O}}$ implies that

$$dim\ \pi^{G_n,\chi_n} = \sum_{\mathcal{O}} c_{\mathcal{O}} q^{n \cdot dim(\mathcal{O})/2} \mu_{\mathcal{O}}(\hat{\chi}_0).$$

On the other hand, we have an obvious inclusion $\pi^{G_n,\chi_n} \subseteq \pi^{P_n,\chi_n}$, where $P_n = P \cap G_n$. We shall now compute π^{P_n,χ_n}. Since $\pi_Z^{P_n,\chi_n} = 0$ and $\pi(Z) \cong W$, we need to compute W^{P_n,χ_n}. Let λ be the character of P defined by the conjugation action of P on $\mathfrak{g}(2)$. Let $P(n)$ be the subset of all p in P such that $|\lambda(p)| = q^n$. Note that $P(0)$ is a subgroup of P containing P_{ss}. The P-module W, when restricted to $P(0)$, breaks up as a sum

$$W = \bigoplus_{-\infty<n<\infty} W_n$$

where W_n consists of functions supported on $P(n)$. Next, note that $W^{P_n,\chi_n} = W_{n+1}^{P_n,\chi_n}$. Let $\psi_{n+1}(x) = \psi(\varpi^{-n-1}x)$. Since the dimension of $W_{\psi_{n+1}}^{N_n,\chi_n}$ is $q^{(n-1)\cdot dim(V)/2}$, and the number of $P_{ss}P_n$ cosets in $P(n+1)$ is $q^n - q^{n-1}$, it follows that

$$dim\ \pi^{P_n,\chi_n} = dim\ W_{n+1}^{P_n,\chi_n} \leq (q^n - q^{n-1}) q^{(n-1)\cdot dim(V)/2}.$$

Finally, since $dim\ \pi^{G_n,\chi_n} \leq dim\ \pi^{P_n,\chi_n}$, the dimension of any leading orbit in the character expansion must be less then or equal to $dim(V)+2$, which is the dimension of the unique minimal orbit $\mathcal{O}_{\min}$. This shows that the character expansion of π has the form

$$\chi_\pi = c_1 \hat{\mu}_{\mathcal{O}_{min}} + c_0.$$

Using the minimality of π and the results of Moeglin and Waldspurger again, we deduce that $c_1 = 1$. The proposition is proved. □

Next, suppose we have a unitarizable minimal representation π; we shall describe the unitary completion of π. Note that W has a natural $P(F)$-invariant inner product defined by

$$(f_1, f_2) = \int_{P_{ss}(F)\backslash P(F)} \{f_1(p), f_2(p)\} dp$$

where $\{-,-\}$ is the natural inner product on the space $\mathcal{S}(\mathfrak{g}(1)^+)$ of W_ψ.

3.9. **Proposition.** *The product $(-,-)$ defined above is the unique (up to a scalar in $\mathbb{R}^\times$) non-trivial $P(F)$-invariant hermitian form on W.*

Proof. Let $\bar{\psi}$ be the complex conjugate of the character ψ. Any non-trivial $P(F)$-invariant hermitian form on $ind_{P_{ss}}^P(W_\psi)$, combined with the (conjugate linear) bijection

$$f \mapsto \bar{f}$$

from $ind_{P_{ss}}^P(W_{\bar\psi})$ to $ind_{P_{ss}}^P(W_\psi)$, defines a nondegenerate $P(F)$-invariant bilinear pairing between $ind_{P_{ss}}^P(W_{\bar\psi})$ and $ind_{P_{ss}}^P(W_\psi)$. Since the smooth dual of $ind_{P_{ss}}^P(W_{\bar\psi})$ is $\tilde W = Ind_{P_{ss}}^P(W_\psi)$, any non-zero $P(F)$-invariant hermitian form on W defines a non-zero element of $Hom_P(W, \tilde W)$. By Frobenius reciprocity, we have

$$\begin{aligned} Hom_P(W,\tilde W) = Hom_{P_{ss}}(W, W_\psi) &= Hom_{P_{ss}}((W)_{Z,\psi}, W_\psi) \\ &= Hom_{P_{ss}}(W_\psi, W_\psi) \cong \mathbb{C}. \end{aligned}$$

The proposition follows. □

3.10. **Proposition.** *Let π be a unitarizable minimal representation. Then its unitary completion $\hat\pi$ is isomorphic to the unitary completion of W, as $P(F)$-modules.*

Proof. Let $\pi(Z)$ be the $P(F)$-submodule of π such that $\pi_Z = \pi/\pi(Z)$. We have seen in the proof of Proposition 3.7 that $\pi(Z) \cong W$, and we may identify $\pi(Z)$ with W. By Proposition 3.9, we can assume that the restriction to W of the inner product on π is given by the product $(-,-)$. To finish the proof of the proposition, we need to show that the space W is dense in $\hat\pi$. To see this, suppose on the contrary that $\hat W^\perp \subset \hat\pi$ is non-zero. Since π is dense in $\hat\pi$, we have a $P(F)$-equivariant projection map $pr : \pi \to \hat W^\perp$ whose image $pr(\pi)$ is a dense subspace of $\hat W^\perp$ consisting of $P(F)$-smooth vectors, thus affording a smooth representation of $P(F)$. Since W is in the kernel of pr, we see that $pr(\pi)_{Z,\psi} = 0$ for any non-trivial character of $Z(F)$ and consequently $Z(F)$ acts trivially on $pr(\pi)$ and thus on $\hat W^\perp$. But by a well-known result of Howe and Moore [HM], $\hat\pi$ does not possess a non-zero $Z(F)$-invariant vector. This shows that $\hat W = \hat\pi$ as claimed. □

3.11. **Corollary.** *Let G be a simply connected group satisfying the conditions (H0) through (H3). Then G admits at most one unitarizable minimal representation.*

Proof. The corollary follows from the fact that the representation of P on $\hat W$ can be extended to a representation of G in at most one way; this was shown in [To] and [S1]. (The groups in question have a unique set of admissible data attached to the unique minimal orbit $\mathcal{O}_{\min}$, see [To]). □

4. Minimality (archimedean case)

In this section, assume that F is an archimedean local field. Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ and let $K \subset G(F)$ be a maximal compact subgroup. The goal is to introduce the notion of minimality and weak minimality for real and complex groups.

4.1. **Associated variety.** We first assume that $F = \mathbb{R}$, so that $\mathfrak{g}_\mathbb{C}$ is a simple complex Lie algebra. By the PBW theorem, the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ has a filtration $U_{n-1}(\mathfrak{g}_\mathbb{C}) \subset U_n(\mathfrak{g}_\mathbb{C})$ so that $U_n(\mathfrak{g}_\mathbb{C})/U_{n-1}(\mathfrak{g}_\mathbb{C})$ is naturally isomorphic to the n-th symmetric power $S^n(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$. The isomorphism preserves the adjoint action of $G(\mathbb{C})$. For any two-sided ideal $\mathcal{J}$ of $U(\mathfrak{g}_\mathbb{C})$, we can associate an ideal J in the symmetric algebra $S(\mathfrak{g}_\mathbb{C}) = \bigoplus_{n=0}^\infty S^n(\mathfrak{g}_\mathbb{C})$ defined by

$$J = \bigoplus_{n=0}^\infty \mathcal{J}_n/\mathcal{J}_{n-1}$$

where $\mathcal{J}_n = \mathcal{J} \cap U_n(\mathfrak{g}_\mathbb{C})$. The zero set defined by J in $\mathfrak{g}_\mathbb{C} \cong \mathfrak{g}_\mathbb{C}^*$ is the associated variety of $\mathcal{J}$ and will be denoted by $Ass(\mathcal{J})$.

4.2. **Minimal orbit.** We shall now describe some results of Kostant and Garfinkle, which will play an important role in the rigidity theorem of the next section. Let J_0 be the prime ideal in the symmetric algebra $S(\mathfrak{g}_{\mathbb{C}})$ corresponding to the closure of the minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}^*$. Recall that α is the highest root. Kostant has shown that, as a $\mathfrak{g}_{\mathbb{C}}$-module,

$$S(\mathfrak{g}_{\mathbb{C}})/J_0 = \sum_{m=0}^{\infty} V(m\alpha),$$

and that J_0 is generated by a $\mathfrak{g}_{\mathbb{C}}$-submodule W in $S^2(\mathfrak{g}_{\mathbb{C}})$, such that

$$S^2(\mathfrak{g}_{\mathbb{C}}) = V(2\alpha) + W.$$

The structure of W was determined by Garfinkle as follows. We shall assume, as we do throughout this paper, that $\mathfrak{g}_{\mathbb{C}}$ is not of type A_n or C_n (otherwise a small modification to what follows is needed). Write $[\mathfrak{m},\mathfrak{m}] = \bigoplus \mathfrak{m}_i$ as a direct sum of simple summands. Let α_i be the highest root of the factor $\mathfrak{m}_i$. Then

$$W = \bigoplus_i V(\alpha+\alpha_i) + V(0).$$

Next, note that each irreducible $\mathfrak{g}_{\mathbb{C}}$-module $V(\alpha+\alpha_i)$ can be considered, in a canonical fashion, as a submodule of $U_2(\mathfrak{g}_{\mathbb{C}})$, as it appears with multiplicity one there.

4.3. **Proposition.** *Let $\mathcal{J}$ be a primitive ideal in $U(\mathfrak{g}_{\mathbb{C}})$. Assume also that $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}$ is not finite dimensional. Then the following are equivalent:*

(i) $\mathcal{J}$ contains $\bigoplus_i V(\alpha+\alpha_i)$.

(ii) $\mathcal{J}$ contains the elements

$$p_x = x\cdot e - \frac{1}{2}n_x \in U_2(\mathfrak{g}_{\mathbb{C}}), \quad x \in [\mathfrak{m},\mathfrak{m}]$$

introduced in 2.6.

(iii) The graded ideal J of $S(\mathfrak{g}_{\mathbb{C}})$ associated to $\mathcal{J}$ is equal to J_0.

(iv) $\mathcal{J}$ is completely prime and $Ass(\mathcal{J}) = \bar{\mathbf{O}}_{\min}$.

Proof. Let x_i be a non-zero vector in the root space for α_i. Using the commutation relations of the elements p_x discussed in 2.6, an easy check shows that p_{x_i} is in fact a highest weight vector in $V(\alpha+\alpha_i)$. From this, the equivalence of (i) and (ii) follows immediately.

Now we show that (i) implies (iii). Let Ω be the Casimir element in $U(\mathfrak{g}_{\mathbb{C}})$. Since $\Omega - c$ is in $\mathcal{J}$ for some constant c, it follows that $V(0)$ is in J. Thus, if $\mathcal{J}$ contains $\bigoplus_i V(\alpha+\alpha_i)$, we have $J_0 \subset J$. Since J_0 is prime and the associated variety of J is non-trivial (and therefore contains $\mathcal{O}_{\min}$), we have the opposite inclusion as well.

The implication (iii) $\Longrightarrow$ (iv) is immediate, since $J = J_0$ is prime. Finally, we need to show that (iv) implies (i). If $Ass(\mathcal{J}) = \bar{\mathbf{O}}_{\min}$, then J contains a power of J_0. Since J_0^k is generated by the symmetric power $S^k(W)$ over $S(\mathfrak{g}_{\mathbb{C}})$, it follows that the $\mathfrak{g}_{\mathbb{C}}$-types of J_0^k/J_0^{k+1} are contained in

$$S^k(W) \otimes (\bigoplus_{m=0}^{\infty} V(m\alpha)).$$

Thus, the highest weights of $\mathfrak{g}_{\mathbb{C}}$-types in $S(\mathfrak{g}_{\mathbb{C}})/J$ are located on finitely many lines parallel to α. In particular, for all sufficiently large integers n, $V(n\alpha + n\alpha_i)$ does not appear as a submodule of $S(\mathfrak{g}_{\mathbb{C}})/J$. Since $S(\mathfrak{g}_{\mathbb{C}})/J$ is isomorphic to $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}$ as a $\mathfrak{g}_{\mathbb{C}}$-module, $V(n\alpha+n\alpha_i)$ does not appear as a submodule of $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}$ for large n. Thus, for such n, $a_i^n = 0$ (modulo $\mathcal{J}$), where a_i is a highest weight vector in $V(\alpha + \alpha_i) \subset U_2(\mathfrak{g}_{\mathbb{C}})$. Since $\mathcal{J}$ is completely prime, a_i must be in $\mathcal{J}$ so that (i) holds. The proposition is proved. $\square$

4.4. **Joseph Ideal.** In [J], Joseph constructed a completely prime 2-sided ideal $\mathcal{J}_0$ whose associated variety is $\bar{\mathbf{O}}_{min}$. He also derived a number of properties of $\mathcal{J}_0$; for example, he computed its infinitesimal character. An alternative construction was given by Garfinkle. She defined $\mathcal{J}_0$ as the ideal generated by $\bigoplus_i V(\alpha + \alpha_i)$ and $\Omega - c_0$, where Ω is the Casimir operator, and c_0 the eigenvalue of Ω for the infinitesimal character that Joseph obtained. We shall refer to $\mathcal{J}_0$ as the Joseph ideal.

4.5. *Remarks.* In [J], Joseph proved that $\mathcal{J}_0$ is the unique completely prime two-sided ideal whose associated variety is the closure of the minimal orbit. However, it was noticed by Savin [S] that there is a gap in the proof of [J, Lemma 8.8]. The above proposition shows, however, that this uniqueness holds if the eigenvalue of the Casimir operator is fixed.

4.6. **Definition** (Minimality in the archimedean case)**.** Let π be a non-trivial irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$ module and let $\mathcal{J}$ be the annihilator of π in $U(\mathfrak{g}_{\mathbb{C}})$. Assume first that $F = \mathbb{R}$.

(i) We say that π is *weakly minimal* if the ideal $\mathcal{J}$ is completely prime and its associated variety is the closure of the minimal orbit.

(ii) We say that π is *minimal* if $\mathcal{J}$ is the Joseph ideal $\mathcal{J}_0$.

When $F = \mathbb{C}$, $\mathfrak{g}$ is itself a simple complex Lie algebra and $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_L \times \mathfrak{g}_R$, where the notations $\mathfrak{g}_L$ and $\mathfrak{g}_R$ are as in 2.6. In this case, a primitive ideal has the form [V2, Prop. 7.11],

$$\mathcal{J} = \mathcal{J}_L \otimes U(\mathfrak{g}_R) + U(\mathfrak{g}_L) \otimes \mathcal{J}_R \subset U(\mathfrak{g}_L) \otimes U(\mathfrak{g}_R)$$

for two primitive ideals $\mathcal{J}_L$ and $\mathcal{J}_R$. One then makes analogous definitions for "weakly minimal" and "minimal", using the corresponding pair of ideals $(\mathcal{J}_L, \mathcal{J}_R)$.

By Remark 4.5, we have the following:

4.7. **Proposition.** *If π is weakly minimal and has the same infinitesimal character as $\mathcal{J}_0$, then π is minimal.*

4.8. **Smooth representations.** An irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-module π has a canonical smooth Frechet globalization π^∞ introduced by Casselman and Wallach. We can make analogous definitions for π^∞ to be minimal or weakly minimal. Since the elements of $\mathfrak{g}_{\mathbb{C}}$ act continuously on π^∞, the annihilator of π^∞ in $U(\mathfrak{g}_{\mathbb{C}})$ is the same as that of π, and thus π^∞ is minimal (or weakly minimal) iff π is also.

4.9. **Uniqueness of minimal representation.** Here we shall show that two minimal $(\mathfrak{g}_{\mathbb{C}}, K)$-modules are isomorphic if they have a common K-type. The following beautiful argument is due to Kostant and is communicated to us by Nolan Wallach. To keep the exposition uniform, we shall assume here that K has finite center. In other words, we are not in a hermitian symmetric case.

4.10. **Proposition.** *Assume that K has finite center. Let π be a minimal $(\mathfrak{g}_{\mathbb{C}}, K)$-module. Then the multiplicities of K-types are 1. Moreover, two minimal $(\mathfrak{g}_{\mathbb{C}}, K)$-modules are isomorphic if and only if they have a common K-type.*

Proof. Let $U(\mathfrak{g}_{\mathbb{C}})^K$ be the subalgebra of $U(\mathfrak{g}_{\mathbb{C}})$ consisting of K-invariants under the adjoint action of K. Now, if π is a $(\mathfrak{g}_{\mathbb{C}}, K)$-module, then $U(\mathfrak{g}_{\mathbb{C}})^K$ acts on the space $Hom_K(\tau, \pi)$ where τ is (any) K-type of π. If the module π is irreducible, then the corresponding $U(\mathfrak{g}_{\mathbb{C}})^K$-module is also irreducible. Moreover, two irreducible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules are isomorphic if and only if they have a common K-type, and the corresponding $U(\mathfrak{g}_{\mathbb{C}})^K$-modules are isomorphic. □

4.11. **Lemma.** *Let Ω_K be the Casimir operator of K. Then $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K$ is isomorphic to the ring of polynomials in Ω_K.*

Proof. First note that

$$(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K = \sum_{k=0}^{\infty} V(k\alpha)^K.$$

By the Cartan-Helgason theorem, the space of K-invariants in $V(k\alpha)$ is at most one-dimensional. Moreover, since we assume that K has finite center, $V(k\alpha)^K$ is one-dimensional if and only if k is even. Next, note that $S^2(\mathfrak{g}_{\mathbb{C}})^K$ is spanned by Ω (Casimir) and Ω_K. In particular, Ω_K projects non-trivially on $V(2\alpha)$ since Ω does not. Finally, since $\mathcal{J}_0$ is completely prime, the powers of Ω_K are linearly independent, and therefore span the space of K-invariants. The lemma is proved. □

We can now finish the proof of Proposition 4.10 easily. The multiplicity one for K-types follows from the fact that $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K$ is commutative. Moreover, since $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K$ is generated by the Casimir operator of K, two minimal representations with a common K-type will clearly give rise to isomorphic $(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K$-modules, so they are isomorphic. □

5. **Rigidity**

The main result of this section is the rigidity result stated in Theorem 5.4. Let F be a number field. For any place v of F, F_v will denote the completion of F at v, and $\mathbb{A}$ will denote the adele ring of F. For a fixed place v, we shall write $\mathbb{A} = F_v \times \mathbb{A}'$. Let G be a simple linear algebraic group over F and suppose that G possesses a maximal parabolic subgroup P so that the working hypotheses (H0), (H1), (H2) and (H3) are satisfied at every place v of F.

5.1. **Global Heisenberg representation.** Let $\psi = \prod_v \psi_v$ be a non-trivial unitary character of $F\backslash\mathbb{A}$. As in 2.2, one can define a Heisenberg representation (ρ_ψ, W_ψ) of $N(\mathbb{A})$. This representation is realized on the space of Schwarz-Bruhat functions $\mathcal{S}(V_{\mathbb{A}}^+)$ on the adelic vector space $V_{\mathbb{A}}^+$ and the action of $N(\mathbb{A})$ is given by the formulas in 2.2. Moreover, it extends to a smooth representation of the semi-direct product $Mp(V_{\mathbb{A}}) \ltimes N(\mathbb{A})$ and thus we obtain a smooth representation ρ_ψ of $P_{ss}(\mathbb{A})$ under our working hypotheses (H0), (H1), (H2) and (H3).

This smooth representation of $P_{ss}(\mathbb{A})$ is almost the restricted tensor product of the local representations (ρ_v, W_{ψ_v}). The reason for "almost" is essentially topological. To be more precise, if S_∞ denotes the set of archimedean places of F, then

$$\mathcal{S}(V_{\mathbb{A}}^{+}) = \mathcal{S}(V_{\infty}^{+}) \bigotimes (\bigotimes_{v \notin S_\infty} \mathcal{S}(V_v^{+}))$$

as representations of $P_{ss}(\mathbb{A})$. The space $\mathcal{S}(V_\infty^+)$ is, however, not the algebraic tensor product of $\mathcal{S}(V_v^+)$ for $v \in S_\infty$. Nevertheless, the algebraic tensor product has a natural inclusion into $\mathcal{S}(V_\infty^+)$ (given by the product of functions) with dense image. Because of the density, this subtlety will not be important.

On the other hand, let

$$\mathcal{C}_\psi^\infty(N(F)\backslash N(\mathbb{A})) = \{h \in \mathcal{C}^\infty(N(F)\backslash N(\mathbb{A})) : h(zu) = \psi(z)h(u) \quad \text{for all } z \in Z(\mathbb{A}) \text{ and } u \in N(\mathbb{A})\}$$

on which $N(\mathbb{A})$ acts by right translation. Moreover, there is an action of $M_0(F)$ given by

$$(m \cdot h)(u) = h(m^{-1}um) \quad \text{for } m \in M_{ss}(F) \text{ and } u \in N(\mathbb{A}).$$

The following is a well-known result of Weil [We]:

5.2. **Proposition.** *There is a topological isomorphism $W_\psi \to \mathcal{C}_\psi^\infty(N(F)\backslash N(\mathbb{A}))$ which is equivariant with respect to the action of $M_{ss}(F) \ltimes N(\mathbb{A})$. This isomorphism is given by sending $f \in W_\psi$ to the function on $N(\mathbb{A})$ given by*

$$u \mapsto \sum_{x \in V_F^+} (\rho_\psi(u)f)(x).$$

Via this isomorphism, we obtain an action of $M_{ss}(\mathbb{A})$ on $\mathcal{C}_\psi^\infty(N(F)\backslash N(\mathbb{A}))$ by transport of structure. Henceforth, we shall denote this representation of $P_{ss}(\mathbb{A})$ on $\mathcal{C}_\psi^\infty(N(F)\backslash N(\mathbb{A}))$ by (ρ_ψ, W_ψ) as well.

5.3. **Rigidity.** Let $\pi = \bigotimes_v \pi_v$ be an irreducible admissible representation of $G(\mathbb{A})$. In particular, when v is archimedean, π_v is the Casselman-Wallach globalization of the underlying Harish-Chandra module $(\pi_v)_{K_v}$. Let R be the right regular representation of $G(\mathbb{A})$ on the space of automorphic forms $\mathcal{A}(G)$ (no K_v-finiteness assumption for archimedean v). The purpose of this section is to prove the following result.

5.4. **Theorem.** *Suppose that π is an irreducible submodule of $\mathcal{A}(G)$. If π_{v_0} is weakly minimal for some place v_0, then π_v is weakly minimal for all places v.*

The rest of the section is devoted to the proof of the theorem. Define a continuous map $T : \pi \to W_\psi$ by

$$T(f)(u) = f_{Z,\psi}(u) = \int_{Z(F)\backslash Z(\mathbb{A})} f(zu) \cdot \overline{\psi(z)} dz.$$

Clearly, this map is $N(\mathbb{A})$-equivariant. In addition, T intertwines the two actions of $M_{ss}(F)$. However, there is no prior reason why it is $M_{ss}(\mathbb{A})$-equivariant. We now note the following lemma.

5.5. **Lemma.** *(i) If T is $M_{ss}(F_v)$-equivariant for some place v, then T is $M_{ss}(\mathbb{A})$-equivariant.*

(ii) If π_v is weakly minimal for some place v, then T is $M_{ss}(F_v)$-equivariant.

Proof. (i) Not surprisingly, this will follow from the strong approximation theorem and here is how it is done. For every f in π, define the Fourier-Jacobi coefficient of f to be the smooth function on $M_{ss}(\mathbb{A})$ given by

$$f_{FJ}(m) = (\rho_\psi(m^{-1})T(R_m \cdot f))(1).$$

Now, if T does not intertwine the two actions of $M_{ss}(\mathbb{A})$, then f_{FJ} is non-constant for at least one function f in π. On the other hand, T intertwines the two actions of $M_{ss}(F)$ so that f_{FJ} is left $M_{ss}(F)$-invariant. By the assumption that T intertwines the action of $M_{ss}(F_v)$, one sees that f_{FJ} is constant when restricted to $M_{ss}(F_v)$. In other words, for any $f \in \pi$, the smooth function f_{FJ} is constant when restricted to $M_{ss}(F) \cdot M_{ss}(F_v)$. Since M_{ss} is simply-connected and $M_{ss}(F_v)$ has no compact simple factors, it follows from the strong approximation theorem that f_{FJ} is constant and (i) is proved.

(ii) To prove the second part, assume first that π_v is weakly minimal at a real place v. To show that T is $M_{ss}(F_v)$-equivariant, it is necessary and sufficient to show that the restriction of f_{FJ} to $M_{ss}(F_v)$ is a constant function for any $f \in \pi$. For this, it suffices to show that $l_X f_{FJ} = 0$ as a function on $M_{ss}(F_v)$, where l_X denotes the action of $X \in \mathfrak{m}_{ss,v}$ by infinitesimal left translation. But

$$l_X f_{FJ}(m) = (\rho_\psi(m^{-1})\rho_\psi(X)T(R_m \cdot f))(1) - (\rho_\psi(m^{-1})T(R_X R_m \cdot f))(1).$$

Hence, it suffices to show that

$$\rho_\psi(X) \circ T = T \circ R_X;$$

in other words, that T is $\mathfrak{m}_{ss,v}$-equivariant.

Since $\rho_\psi(e)$ acts by a non-zero scalar and T intertwines the two actions of e, we are reduced to showing that T intertwines the two actions of $e \cdot X$. But by Lemma 2.7, we have

$$\rho_\psi(e) \cdot \rho_\psi(X) = \frac{1}{2} \cdot \rho_\psi(n_X).$$

On the other hand, since π_v is weakly minimal,

$$R_e \cdot R_X = \frac{1}{2} \cdot R_{n_X}.$$

Now the desired result follows from the fact that T intertwines the two actions of $n_X \in U(\mathfrak{n}_v)$.

When v is complex, the element $X \in \mathfrak{m}_{ss,v}$ is equal to $1 \otimes X + X \otimes 1$ when regarded as an element of $U(\mathfrak{g}_L) \otimes U(\mathfrak{g}_R) \cong U(\mathfrak{g}_v \otimes_{\mathbb{R}} \mathbb{C})$. Thus it suffices to show that

$$\begin{cases} \rho_\psi(1 \otimes X) \circ T = T \circ R_{1\otimes X}, \\ \rho_\psi(X \otimes 1) \circ T = T \circ R_{X\otimes 1}. \end{cases}$$

Now the element $\rho_\psi(1 \otimes e)$ (resp. $\rho_\psi(e \otimes 1)$) acts by a non-zero scalar and T intertwines the two actions of $1 \otimes e$ (resp. $e \otimes 1$). Thus we are reduced to showing that T intertwines the two actions of $1 \otimes (e \cdot X)$ (resp. $(e \cdot X) \otimes 1$). As before, this follows from the assumption that π_v is weakly minimal and Lemma 2.7 which ensures that the elements $1 \otimes p_X$ and $p_X \otimes 1$ are both in $Ann(W_\psi)$.

Now assume that v is finite. Then the map $T : \pi \to W_\psi$ factors through the quotient $\pi \to \pi_{Z_v,\psi_v}$. By the assumption, as a $P_{ss}(F_v)$-module, π_{Z_v,ψ_v} is a direct sum of copies of W_{ψ_v}. Let $U \cong W_{\psi_v}$ be an irreducible summand. Then T induces a map from U into W_ψ. But

$$Hom_{N(F_v)}(W_{\psi_v}, W_\psi) = \bigotimes_{v' \neq v} W_{\psi_{v'}}$$

as $M_{ss}(F_v)$-modules. Hence, the map from U to W_{ψ_v} must intertwine the two actions of $P_{ss}(F_v)$ since $\bigotimes_{v' \neq v} W_{\psi_{v'}}$ is a trivial $M_{ss}(F_v)$-module. The lemma is proved. □

Under the hypothesis of the theorem, the lemma implies that the map $T : \pi \to W_\psi$ is $P_{ss}(\mathbb{A})$-equivariant. By Frobenius reciprocity, we obtain a $P(\mathbb{A})$-equivariant map

$$\tilde{T} : \pi \to W := Ind_{P_{ss}(\mathbb{A})}^{P(\mathbb{A})} W_\psi.$$

We shall denote the action of $P(\mathbb{A})$ on W by ρ. Note that the representation (ρ, W) is *not* the restricted tensor product of the analogous local representations $W_v = Ind_{P_{ss}(F_v)}^{P(F_v)} W_{\psi_v}$. This is because the quotient $P(\mathbb{A})/P_{ss}(\mathbb{A})$ is non-compact and thus a smooth function on $P(\mathbb{A})$ cannot in general be written as a finite sum of products of functions on $P(F_v)$.

5.6. **Lemma.** *The map $\tilde{T}$ is injective.*

Proof. Let f be in the kernel of $\tilde{T}$. Since for every non-trivial character ψ' of $Z(F)\backslash Z(\mathbb{A})$, there exists an element p' in $P(F)$ such that

$$f_{Z,\psi'}(p) = f_{Z,\psi}(p'p),$$

we conclude that $f_{Z,\psi'}(p) = 0$ for all $p \in P(\mathbb{A})$ and for all non-trivial characters ψ'. This implies that the restriction of f to $P(\mathbb{A})$ is left $Z(\mathbb{A})$-invariant, and hence right $Z(\mathbb{A})$-invariant, since P normalizes Z.

Let z be in $Z(\mathbb{A})$ and consider $g = f - R_z \cdot f$. We know that g is left-invariant under $G(F)$ and is zero when restricted to $P(\mathbb{A})$. Furthermore, since g is right K_v-invariant for at least one place v (almost all in fact), the function g is zero when restricted to $G(F_v)$. By the strong approximation theorem, g is zero on $G(\mathbb{A})$. It follows that $R_z f = f$ for all z.

Now if $f \neq 0$, then the above shows that for almost all finite places v, the K_v-spherical vector lies in the non-zero space $(\pi_v)^{Z(F_v)}$. Since $P(F_v)$ normalizes $Z(F_v)$ and the K_v-spherical vector generates π_v over $P(F_v)$, we deduce that $Z(F_v)$ is in the kernel of the representation π_v. This is a contradiction, since π_v is not the trivial representation and $G(F_v)$ does not contain any infinite normal proper subgroup (since G is simply connected). □

We now proceed with the proof of the rigidity theorem. Assume first that v is finite. Let U be the kernel of the natural map $\pi \to \pi_{Z_v}$ so that $\pi_{Z_v,\psi_v} = U_{Z_v,\psi_v}$. Thus it suffices to show that as a representation of $P_{ss}(F_v)$, U_{Z_v,ψ_v} is the isotypic direct sum of copies of W_{ψ_v}. For this, we shall show the following lemma.

5.7. **Lemma.** *The image of U under $\tilde{T}$ is contained in the algebraic tensor product*

$$ind_{P_{ss}(F_v)}^{P(F_v)} W_{\psi_v} \otimes Ind_{P_{ss}(\mathbb{A}')}^{P(\mathbb{A}')} W_{\psi'}$$

where $\psi' = \prod_{v' \neq v} \psi_{v'}$.

Proof. The proof of this lemma is not difficult. Every element in U is a finite linear combination of $f = f_v \otimes f'$, where f_v is in $\pi_v(Z_v)$, and f' is in $\bigotimes_{v' \neq v} \pi_{v'}$. Let $\tilde{f} = \tilde{T}(f)$. Recall, from the proof of Proposition 3.7, the decomposition $P(F_v) = \bigcup_{-\infty < n < \infty} P(n)$. Now we have:

Claim: There exist integers n_1 and n_2 such that, if $p \notin \bigcup_{-n_1 \leq n \leq n_2} P(n)$, then

$$\tilde{f}(pq) = 0 \quad \text{for any } q \in P(\mathbb{A}').$$

To prove the claim, we shall explain how to pick the integers. The vector f_v is smooth, so it is invariant under a compact subgroup $Z_{n_2} = Z(F_v) \cap G_{n_2}$ of $Z(F_v)$. This gives n_2. On the other hand, f_v is in $\pi_v(Z_v)$, hence averaging f_v over a compact subgroup Z_{n_1} of $Z(F_v)$, for a sufficiently large n_1, will give 0. This gives n_1, and the claim is proved.

Choose $m > 0$ such that f_v is invariant under the congruence subgroup $P_m = P(F_v) \cap G_m$ of $P(F_v)$. Then there exists a finite set $\{t_i\}$ of elements of $P(F_v)$ such that

$$\bigcup_{-n_1 \leq n \leq n_2} P(n) = \bigcup_{t_i} P_{ss}(F_v) t_i P_m.$$

Now let

$$J = P_{ss}(F_v) \cap (\bigcap_i t_i P_m t_i^{-1}).$$

Then J is an open compact subgroup of P_{ss}. Further, observe that $\tilde{f}(t_i q) \in W_\psi$ is fixed by J. Thus we have

$$\tilde{f}(t_i q) \in W_{\psi_v}^J \otimes W_{\psi'} \quad \text{for any } i \text{ and any } q \in P(\mathbb{A}').$$

Since W_{ψ_v} is an admissible representation of $P_{ss}(F_v)$, $W_{\psi_v}^J$ is finite-dimensional. If $\{w_j\}$ is a basis of $W_{\psi_v}^J$, then we may write

$$\tilde{f}(t_i q) = \sum_j w_j \otimes \tilde{f}_{ij}(q)$$

for some $\tilde{f}_{ij}(q) \in W_{\psi'}$. Further, the function $\tilde{f}_{ij} : P(\mathbb{A}') \to W_{\psi'}$ is an element of W'. Write $\tilde{f}$ as a sum of $\tilde{f}_i$ where $\tilde{f}_i$ is supported on $C_i = P_{ss}(F_v) t_i P_m$. For any $p \in C_i$ and any $q \in P(\mathbb{A}')$, we have

$$\tilde{f}_i(pq) = \sum_j \rho_\psi(p_{ss})(w_j) \otimes \tilde{f}_{ij}(q)$$

where p_{ss} is in P_{ss} such that $p = p_{ss} t_i p_m$ for some p_m in P_m. Note, however, that the choice of p_{ss} in this decomposition is unique only up to a (right) factor in $P_{ss}(F_v) \cap t_i P_m t^{-1} \subset J$. Therefore, $\rho_\psi(p_{ss}) w_j$ is well defined, since w_j is fixed by J. Since

$$\phi_{ij}(p) = char(C_i)(p) \cdot \rho_\psi(p_{ss})(w_j)$$

is an element of W_v, we see that

$$\tilde{f} = \sum_{i,j} \phi_{ij} \otimes \tilde{f}_{ij}$$

lies in $W_v \otimes W'$, as desired. The lemma is proved. □

The lemma shows that we have a $P(\mathbb{A})$-equivariant embedding $\tilde{T}: U \hookrightarrow W_v \otimes W'$. By the exactness of Jacquet functor and the fact that $(W_v)_{Z_v,\psi_v} \cong W_{\psi_v}$, it follows that U_{Z_v,ψ_v} is the isotypic direct sum of copies of W_{ψ_v} and thus π_v is weakly minimal. This proves the theorem for finite v.

Finally, assume that v is archimedean. When v is real, we need to show that any $\tilde{f} \in W$ is annihilated by the element p_X for $X \in \mathfrak{m}_{ss,v}$. For this, it suffices to show that for any $\tilde{f} \in W$, the restriction of $\tilde{f}$ to $P(F_v)$ (which is an element of W_v) is annihilated by p_X. In other words, for any $p \in P(F_v)$, we need to show that

$$\rho(p_X)(\tilde{f})(p) = 0.$$

But

$$\rho(p_X)(\tilde{f})(p) = (\rho(Ad(p)(p_X))(\rho(p)\tilde{f}))(1) = \rho_{Ad(p)^{-1}\psi_v}(p_X)(\tilde{f}(p))$$

and the result follows by the fact that p_X is in the annihilator of $(\rho_{\psi'_v}, W_{\psi'_v})$ for any non-trivial ψ'_v. This proves the proposition when v is real. Since we do not need it for our applications, we shall omit the details when v is complex. Theorem 5.4 is proved completely.

5.8. **Corollary.** *Let $\pi = \bigotimes \pi_v$ be an infinite dimensional automorphic representation. Assume that the infinitesimal character of π_∞ is the same as the infinitesimal character of the Joseph ideal. If, at a finite place v_0, the local component π_{v_0} is weakly minimal, then the annihilator of π_∞ in $U(\mathfrak{g}_\mathbb{C})$ is the Joseph ideal.*

5.9. *Remarks.* An L^2-version of the rigidity theorem for minimal representations was shown in [K]. However, an important analytic detail was claimed but not established there. Namely, it was claimed that the image of the map $\tilde{T}: \pi \hookrightarrow W$ is contained in the space of functions on $P(\mathbb{A})$ (with values in W_ψ) which are square-integrable over $P_{ss}(\mathbb{A})\backslash P(\mathbb{A}) \cong \mathbb{A}^\times$. We do not know how to justify this. In any case, we prefer to work in the smooth category rather than the unitary one.

6. Preliminaries on p-adic groups

Let G be a simple, simply connected group defined over a p-adic field F, satisfying the conditions (H0) through (H3) introduced in Section 3. These conditions should assure the uniqueness of the minimal representation. Indeed, under the additional assumption of unitarity, we have shown this to be true. In the following sections, we shall address the issue of existence. More precisely, we shall specify a candidate minimal representation, derive a number of its important properties and show ultimately (using the rigidity theorem) that it is minimal.

6.1. **The p-adic groups.** The simple p-adic groups satisfying the conditions (H0) through (H3) can be read off from [To, p. 346]. In addition to all split groups of type G_2, D_n and E_n, we have quasi-split groups D_4^3, E_6^2, and D_{n+1}^2 (of relative type G_2, F_4, and B_n, respectively), and three inner forms:

- an inner form of absolute type E_6 and relative type G_2;
- an inner form of absolute type E_7 and relative type F_4;
- an inner form of absolute type D_{n+2} and relative type B_n ($n > 2$).

Each of these inner forms can be constructed by means of a division algebra D over F, of rank 9 in the first case, and rank 4 in the other two cases. Details of one such construction (for the relative types G_2 and F_4) will be given in Section 10. For our purposes here, however, it suffices to say that the relative root system is reduced and the rank-one subgroup corresponding to any short root is isomorphic to $SL_2(D)$. Note also that the relative types of these inner forms are the same as the relative types of the quasi-split groups.

In the next few sections, we shall assume that the group G is not G_2, even though G_2 is in the list above.

6.2. **Affine root system.** Assume first that G is unramified, which means that it splits over an unramified extension of F. In this case, the minimal representation will have Iwahori-fixed vectors. The purpose of this section is to describe the Iwahori-Hecke algebra, and write down some relevant facts about it. We follow the exposition of Casselman [Cs] closely.

Let B be a minimal parabolic subgroup of G, and A a maximal split torus contained in B. Let T be the centralizer of A, and U the unipotent radical of B. Let Φ be the set of roots with respect to A. Then $\Phi \subset Hom(A/A_0, \mathbb{R})$, where A_0 is the maximal compact subgroup of A. This is the relative root system of G. Assume further that this root system is reduced, as it is true for our groups. This will simplify the discussion in [Cs] considerably. For any root α in Φ, let N_α be the subgroup of G with Lie algebra $\mathfrak{g}_\alpha$. A root α is said to be positive if N_α is contained in N. Let Δ be the set of simple roots corresponding to this choice of positive roots.

Let $\mathcal{B}$ be the Bruhat-Tits building of $G(F)$, and let $\mathcal{A}$ be the unique apartment stabilized by A. Let $\tilde{\Phi}$ be the affine root system on $\mathcal{A}$, and $\tilde{W}$ the corresponding affine Weyl group. Let x_0 be a special vertex in $\mathcal{A}$ and let Φ_0 be the roots of $\tilde{\Phi}$ vanishing on x_0. Since A/A_0 can be identified with a lattice in $\mathcal{A}$, Φ_0 is yet another root system in $Hom(A/A_0, \mathbb{R})$. The two root systems are the same if G is quasi-split. Otherwise every root in Φ is a positive multiple of a unique root in Φ_0. This bijection, denoted by λ, exchanges the long and short roots. Set

$$\begin{cases} \Delta_0 = \lambda(\Delta), \\ \Phi_0^+ = \lambda(\Phi^+). \end{cases}$$

Also, let $\tilde{\alpha}$ be the highest root in Φ_0^+, and let

$$\tilde{\Delta} = \Delta_0 \cup \{1 - \tilde{\alpha}\}.$$

Then let $\mathcal{C}$ be the unique chamber in $\mathcal{A}$, which has x_0 as a vertex, given by $\alpha(x) > 0$ for all α in $\tilde{\Delta}$.

Let $\mathcal{N}$ be the set of nilpotent elements in $G(F)$. For each affine root α, define a subgroup by

$$U(\alpha) = \{x \in \mathcal{N} | ux = x \text{ for all } x \in \mathcal{A} \text{ with } \alpha(x) \geq 0\}.$$

Then $U(\alpha + 1)$ is strictly contained in $U(\alpha)$, and its index will be denoted by q_α. If α is a root in Φ, then U_α is the union of $U(\lambda(\alpha) + k)$ for all integers k. Let I be the Iwahori subgroup fixing the chamber $\mathcal{C}$ pointwise. Then we have the Iwahori factorization $I = U_1^- T_0 U_0$ where T_0 is the intersection of T with K (the special maximal compact subgroup of $G(F)$ which fixes x_0), and

$$\begin{cases} U_0 = \prod_{\alpha \in \Phi_0^+} U(\alpha), \\ U_1^- = \prod_{\alpha \in \Phi_0^+} U(1 - \alpha). \end{cases}$$

6.3. **Hecke algebras of split groups.** Let $\tilde{H}$ be the Hecke algebra of I-biinvariant compactly supported functions on $G(F)$. If G is split (and simply-connected), then the Hecke algebra $\tilde{H}$ can be described as follows. Put $\alpha_0 = 1 - \tilde{\alpha}$ and let $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. Then $\tilde{H}$ is generated by T_α, $\alpha \in \tilde{\Delta}$ modulo the usual braid relations (cf. [Bo]) and

$$(T_\alpha - q)(T_\alpha + 1) = 0.$$

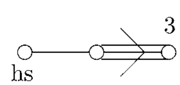
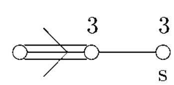
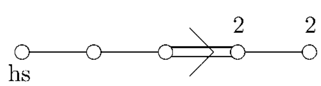
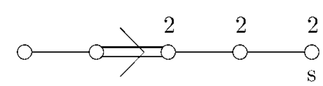
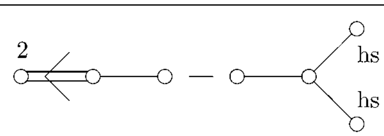
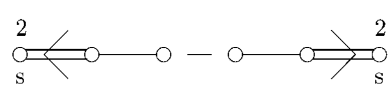
6.4. **Hecke algebras of non-split unramified groups.** Next, we shall describe the Hecke algebra of the non-split unramified groups satisfying the hypotheses (H0)–(H3); these were listed above. Recall that such a group has a relative root system of type G_2, F_4 and B_n which is obtained as fixed points of an automorphism of the root system D_4, E_6 and D_{n+1} respectively. Let i be the order of that automorphism. Thus $i = 3$, 2 and 2 in the three respective cases. Let Δ be a set of simple roots in G_2, F_4 or B_n. We can extend Δ in two ways

$$\begin{cases} \tilde{\Delta}_l = \Delta \cup \{1 - \alpha_l\}, \\ \tilde{\Delta}_s = \Delta \cup \{1 - \alpha_s\} \end{cases}$$

by adding either the (negative of the) highest long root or the highest short root. Let $\tilde{H}_l$ and $\tilde{H}_s$ be the affine Hecke algebra generated by T_α for each α in $\tilde{\Delta}_l$ and $\tilde{\Delta}_s$ respectively, modulo the usual relations, except that

$$(T_\alpha - q^i)(T_\alpha + 1) = 0$$

for every short root α. As we run through the root systems G_2, F_4 and B_n, the algebra $\tilde{H}_l$ is the Hecke algebra of the quasi-split groups D_4^3, E_6^2 and D_{n+1}^2. The algebra $\tilde{H}_s$, on the other hand, is the Hecke algebra of non-trivial inner forms of E_6, E_7 and D_{n+2}.

Δ	Δ_l	Δ_s
G_2		
F_4		
B_n		

6.5. **Exponents.** Now let G be an arbitrary unramified group under our consideration. Let Ω^+ be the cone in T/T_0 corresponding to elements t in T such that $tU^+t^{-1} \subseteq U^+$. Then every element ω in the lattice T/T_0 can be written as $\omega = \omega_1 - \omega_2$ for some two elements ω_1 and ω_2 in Ω^+. For any ω, define an element $\hat{T}_\omega$ in $\tilde{H}$ by

$$\hat{T}_\omega = q^{-\rho_U(\omega)} T_{\omega_1} \cdot T_{\omega_2}^{-1}$$

where ρ_U is equal to $1/2$ the sum of all positive roots, each taken with multiplicity equal to the dimension of the corresponding root space.

If V is a smooth $G(F)$-module, then V^I (the space of I-fixed vectors in V) is naturally a $\tilde{H}$-module. It is a well-known result of Borel [Bo] that this correspondence defines an equivalence between the category of representations of G generated by its I-fixed vectors and the category of representations of $\tilde{H}$.

Assume that V is generated by its I-fixed vectors. If V is in addition admissible, then V_U is a finite dimensional $T(F)$-module on which T_0 acts trivially. For every unramified character μ of T, define

$$V_U^\mu = \{v \in V_U | (t - \rho_U \mu(t))^\infty v = 0,\, t \in T(F)\}.$$

The characters μ such that V_U^μ is non-trivial are called the exponents of V, and the dimension of V_U^μ is called the multiplicity of μ. On the other hand, define

$$V_\mu^I = \{v \in V^I | (\hat{T}_\omega - \mu(\omega))^\infty v = 0,\, \omega \in T/T_0\}.$$

The following is a well-known result of Borel (cf. [Bo, Lemma 4.7] and [Cs, Props. 2.4 and 2.5]):

6.6. **Proposition.** *The natural projection form V^I on V_U induces an isomorphism of V_μ^I and V_U^μ.*

In particular, the exponents of representations generated by I-fixed vectors can be calculated in terms of the corresponding $\tilde{H}$-modules.

6.7. **Principal Series.** Here we generalize a result of Rodier on regular principal series representations of split groups [Ro] to the groups considered here. Thus our group is either quasi-split with the splitting field E, or an inner form constructed by means of a division algebra D.

Let Φ be the set of roots corresponding to the maximal split torus A as above. Let T be the centralizer of A as before. For every short root α, the coroot homomorphism from $F^\times$ to A can be extended to a homomorphism from $E^\times$ to T (in the quasi-split case) and from $D^\times$ to T in the case of inner forms. It will be convenient for us to consider the coroots in this extended sense.

Let $B = TU$ be a minimal parabolic subgroup of G, and let χ be a character of $T(F)$. Define the absolute value $|-|_{\mathbb{N}}$ on E, respectively on D, by

$$|x|_{\mathbb{N}} = \begin{cases} |\mathbb{N}(x)| \text{ if } x \in E; \\ |\mathbb{N}(x)|^{deg(D)} \text{ if } x \in D. \end{cases}$$

Here $\mathbb{N}$ is the norm on E or D and $deg(D)$ is such that $dim(D) = deg(D)^2$. Note that if α is a short simple root, then

$$\rho_U \circ \alpha^\vee = |-|_{\mathbb{N}}$$

in either case.

We are now ready to describe irreducible subquotients of $Ind_B^G(\chi)$ for a regular character χ. Let Φ_χ be the set of all roots such that

$$\chi \circ \alpha^\vee = \begin{cases} |\cdot|^{\pm 1}, & \text{if } \alpha \text{ is long;} \\ |\cdot|_{\mathbb{N}}^{\pm 1}, & \text{if } \alpha \text{ is short.} \end{cases}$$

Let $\mathcal{V}$ be the real vector space spanned by Φ (our relative root system) and put

$$\mathcal{V}_\chi = \mathcal{V} \setminus (\bigcup_{\alpha \in \Phi_\chi} \{x \in \mathcal{V} | \alpha^\vee(x) = 0\}).$$

Let $\mathcal{V}^+$ be the positive Weyl chamber in $\mathcal{V}$ and define W_χ to be the set of all Weyl group elements w such that $w(\mathcal{V}^+)$ is in the connected component of $\mathcal{V}_\chi$ containing $\mathcal{V}^+$. The following is a well-known result of Rodier [Ro, p. 418] (It is a formal consequence of rank-one reducibilities, so it applies to our groups since the reducibilities for $SL_2(D)$ are known. For example, they can be determined by methods of [MuS].)

6.8. **Proposition.** *Let χ be a regular character. Let V be the unique irreducible submodule of $Ind_B^G(\chi)$ (normalized induction). Then*

$$V_U = \bigoplus_{w \in W_\chi} \rho_U \cdot \chi^w$$

where χ^w is the character $t \mapsto \chi(wtw^{-1})$.

7. Candidates for Minimal Representations

We are finally ready to specify a candidate for the minimal representation, calculate its Jacquet module with respect to the minimal parabolic subgroup, and describe the minimal K-type(s) for the (hyper)special maximal compact subgroup. The minimal representation for unramified groups (i.e. those that split over unramified extensions) will turn out to possess Iwahori-fixed vectors, and in this case we specify the candidate by writing down the corresponding Hecke algebra representation. If the group is ramified, the minimal representation will turn out to be a submodule of a principal series representation induced from, fortunately, a regular character. Thus, its Jacquet module can be calculated using the result of Rodier (Proposition 6.8).

7.1. **Unramified Groups.** Assume first that G is unramified and has splitting field E (unramified over F). Thus, in addition to the split groups, we shall consider here two groups with relative root system G_2 (a quasi split form of D_4, and an inner form of E_6), two groups with relative root system F_4 (a quasi-split form of E_6 and an inner form of E_7) and two groups with relative root system B_n (a quasi-split form of D_{n+1} and an inner form of D_{n+2}).

We shall describe our candidate for the minimal representation in terms of the corresponding representation of the Iwahori-Hecke algebra. Let $\tilde{H}$ be the Hecke algebra of any of these unramified groups and let $\tilde{\Delta}$ be the corresponding set of simple affine roots. Let Λ be the subset of long roots in $\tilde{\Delta}$. We are now ready to define a representation of $\tilde{H}$. Let

$$V = \bigoplus_{\alpha \in \Lambda} \mathbb{C} e_\alpha.$$

The action of T_α for every α in Λ is given by

$$T_\alpha e_\beta = \begin{cases} -e_\beta & \text{if } \alpha = \beta, \\ qe_\beta + q^{\frac{1}{2}} e_\alpha & \text{if } n_{\alpha\beta} = 120^o, \\ qe_\beta & \text{if } n_{\alpha\beta} = 90^o, \end{cases}$$

where $n_{\alpha\beta}$ is the angle between α and β. If G is split, this completes the construction, since all roots are long. Otherwise, for every short root α, we simply let T_α act as the scalar q^i, where i is the degree of the splitting field E.

Let π be the irreducible representation of $G(F)$ such that $\pi^I = V$. This is our candidate minimal representation.

7.2. **Minimal K-types.** Fix a chamber $\mathcal{C}$ in $\mathcal{B}(G,F)$, and a chamber $\mathcal{C}_E$ in $\mathcal{B}(G,E)$ containing $\mathcal{C}$. A maximal parahoric subgroup K_x in G is called absolutely maximal if x is a vertex of $\mathcal{C}_E$. A remarkable fact, worth emphasizing, is that V is spanned by lines fixed under absolutely maximal parahoric subgroups:

7.3. **Proposition.** *Let $\mathcal{C}_{abs}$ be the set of absolutely maximal vertices of the chamber $\mathcal{C}$. For every x in $\mathcal{C}_{abs}$, the maximal parahoric subgroup K_x fixes a unique line V_x in V, and*

$$V = \bigoplus_{x \in \mathcal{C}_{abs}} V_x.$$

Proof. Recall that vertices of $\mathcal{C}$ can be identified with $\tilde{\Delta}$. Under this identification, K_x is absolutely maximal if and only if the vector part of the corresponding affine root α_x is long. Note that the Hecke algebra $H_x \subset \tilde{H}$ of I-bi-invariant functions supported on K_x is generated by T_α with $\alpha \neq \alpha_x$. Let

$$V^x = \bigoplus_{\alpha \in \Lambda \setminus \{\alpha_x\}} \mathbb{C} e_\alpha.$$

Then V^x is an H_x-submodule, and V/V^x is the one-dimensional character of H_x corresponding to the trivial representation of K_x. The proposition follows. □

Let K_{x_0} be a special maximal compact subgroup fixing the vertex x_0. Then, by inspection, K_{x_0} is hyperspecial if and only if K_{x_0} is absolutely maximal, which is the case for split and quasi-split groups.

7.4. **Corollary.** *Let G be split or quasi-split (and unramified). Then the representation π is unramified.*

Let $K_{x_0,1}$ be the first principal congruence subgroup, which is the set of all elements in G fixing pointwise all chambers containing the vertex x_0. We shall now describe the $K_{x_0}/K_{x_0,1}$ module $\pi^{K_{x_0,1}}$ in *all* cases (i.e. even for G not quasi-split).

Consider first the split case. Then $K_{x_0}/K_{x_0,1}$ is simply $G(\mathbb{F}_q)$. Under the action of the finite Hecke algebra H_{x_0}, the $\tilde{H}$-module V decomposes as

$$V = V_{x_0} \oplus R_{x_0}$$

where V_{x_0} is the one-dimensional character corresponding to the trivial representation of K_0, and R_{x_0} is the reflection representation of H_{x_0}. (Observe that specializing $q = 1$ makes H_{x_0} the group algebra of the Weyl group W_{x_0}, and R_{x_0} its reflection representation.) The corresponding representation U_{x_0} of $G(\mathbb{F}_q)$ has been studied by Kilmoyer and others [CIK]. Its dimension is

$$dim\ U_{x_0} = q^{m_1} + \ldots + q^{m_r}$$

where r is the dimension of R_{x_0}, and m_i are the exponents of the Weyl group W_{x_0}. The exponents can be defined in terms of the W_{x_0} invariant homogeneous polynomials on R_{x_0}. Indeed, $m_i + 1 = deg(I_i)$, where I_i are homogeneous polynomials such that

$$\mathbb{C}[R_{x_0}]^{W_{x_0}} = \mathbb{C}[I_1, \ldots I_r].$$

If G is not split, then the finite group $K_{x_0}/K_{x_0,1}$ is quasi-split, and its type depends on the relative type of G. If the relative root system of G is G_2, F_4 and B_n, then $K_{x_0}/K_{x_0,1}$ is $D_4^3(q)$, $E_6^2(q)$, and $D_{n+1}^2(q)$ respectively. Let W'_{x_0} be the quotient of W_{x_0} by the normal subgroup generated by short root reflections. In this case

$$V = V_{x_0} \oplus R'_{x_0} \quad \text{ or } \quad R'_{x_0}$$

depending on whether G is quasi-split or not. Here R'_{x_0} is an irreducible H_{x_0} module, which, after specializing $q = 1$, becomes the reflection representation of W'_{x_0}. Let U_{x_0} be the corresponding representation of $K_{x_0}/K_{x_0,1}$. Its dimension can be found in Carter's book, and can be expressed in terms of the exponents as follows. Let σ be the outer automorphism of the root system of type D_4, E_6, and D_{n+1}, so that the σ-fixed roots form a system of type G_2, F_4, and B_n, respectively. The invariant homogeneous polynomials I_i (for the simply laced root systems) can be picked so that

$$\sigma(I_i) = \epsilon_i I_i$$

for some roots ϵ_i. We have

$$dim\ U_{x_0} = \epsilon_1 q^{m_1} + \ldots + \epsilon_r q^{m_r}.$$

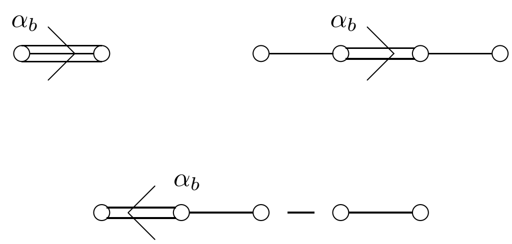
7.5. **Split Groups.** We shall now compute the exponents of the candidate representation π. This can be done on the level of $\tilde{H}$-module V, by Proposition 6.6. When G is split, B is a Borel subgroup of G and T is a maximal torus. Let χ be an unramified character of $T(F)$. If χ is real, define a number n_α by $\chi \circ \check{\alpha} = |\cdot|^{-n_\alpha}$ for every simple coroot $\check{\alpha} : F^\times \to T(F)$.

Recall that G is of type D_n or E_n. Let χ be the real, unramified character such that the family $\{2n_\alpha\}$ gives the marking of the Dynkin diagram corresponding to the subregular nilpotent orbit. This means that $2n_\alpha = 2$ for all simple roots α different from α_b and $2n_{\alpha_b} = 0$ where α_b corresponds to the unique vertex which is adjacent to three vertices.

The exponents of π are given by [Lu, Theorem 4.7]:

7.6. **Proposition.** *For any $\alpha_k \in \Delta$, let $\alpha_b, \alpha_l \ldots, \alpha_k = \alpha$ be the geodesic path on the Dynkin diagram for G from α_b to α_k. Let s_i be the reflection corresponding to α_i, and put $w = s_l \ldots s_k$. Then $\chi_k : t \mapsto \chi(wtw^{-1})$ is an exponent. All exponents are obtained in this way and have multiplicity one, except the multiplicity of $\chi_b = \chi$ is 2.*

7.7. **Inner forms.** We now consider the inner forms. Recall that the diagrams G_2, F_4 and B_n are obtained from those of D_4, E_6 and D_{n+1} by identifying some vertices. In particular, one can identify the long roots in G_2, F_4 and B_n with some long roots in D_4, E_6 and D_n. Let α_b be the long root in G_2, F_4 and D_{n+1} corresponding to the branching vertex in D_4, E_6 and D_{n+1}:



We define a character χ_b of $T(F)$ by

$$\chi_b \circ \alpha^\vee = \begin{cases} |\cdot|_{\mathbb{N}}^{-1} & \text{for every short root } \alpha, \\ |\cdot| & \text{if } \alpha = \alpha_b, \\ |\cdot|^{-2} & \text{if } \alpha \text{ is long and adjacent to } \alpha_b, \\ |\cdot|^{-1} & \text{for other long roots.} \end{cases}$$

7.8. **Proposition.** *Let V be the Hecke module defined above. For each long root α_k on the relative Dynkin diagram of G, let χ_k be defined as in Proposition 7.6. Then the exponents of π are the characters χ_k, for every long root α_k. Moreover, each exponent has multiplicity one.*

The proof of this statement will be given in the next section on a case-by-case basis. Note however, that the character χ_b is regular, so we have this consequence:

7.9. **Corollary.** *In the inner forms case, the candidate representation π is the unique irreducible submodule of $Ind_B^G \chi_b$.*

7.10. **Quasi-split groups: Galois case.** We now consider the quasi-split (but non-split) groups with Galois splitting fields. In particular, we allow the group to be ramified here. Hence, we need to specify the candidate representation for the ramified groups, and to compute the exponents in all cases.

Let α_b be the long root in G_2, F_4 and B_n as before. We define a character $\bar{\chi}_b$ of $T(F)$ by

$$\bar{\chi}_b \circ \alpha^\vee = \begin{cases} |\cdot|_{\mathbb{N}}^{-1} & \text{for every short root } \alpha, \\ |\cdot|^{-1} & \text{for every long root } \alpha \neq \alpha_b, \\ \chi_E & \text{if } \alpha = \alpha_b. \end{cases}$$

Here χ_E is the character of $F^\times$ corresponding to E via the local class field theory.

The character $\bar{\chi}_b$ defined above is a regular character of $T(F)$ and we let π be the unique irreducible submodule of $Ind_B^G \bar{\chi}_b$. *Then π is the candidate minimal representation in this case.* Of course, if E is unramified, we need to show that π corresponds to the Hecke module V. To do so, we need to calculate the exponents.

The answer turns out to be analogous to that in the split case, except here we take only paths over the long roots. More precisely, define

$$\chi_b(t) = \bar{\chi}_b(s_b \cdot t \cdot s_b^{-1})$$

where s_b is the reflection corresponding to the root α_b. Then we have:

7.11. **Proposition.** *Assume that G is unramified. For each long root α_k on the relative Dynkin diagram of G, let χ_k be obtained from χ_b as in Proposition 7.6. Then the exponents of π are the characters $\bar{\chi}_b$ and χ_k for every long root α_k. Moreover, each exponent has multiplicity one.*

The proof of this statement on a case-by-case basis will be given in the next section. We note the following consequence.

7.12. **Corollary.** *(i) If E is unramified, then the Hecke module V corresponds to π.*

(ii) If E is ramified, the exponents of π are the ones given by the proposition above.

Proof. The proposition implies that V corresponds to a submodule of $Ind_B^G(\bar{\chi}_b)$. Since π is the unique irreducible submodule here, the first statement follows. In view of Proposition 6.8, the exponents of π clearly do not depend on E. Hence the result of the proposition holds regardless of whether E is unramified, saving us a combinatorial calculation. □

7.13. **Quasi-split groups: non-Galois case.** Finally, suppose that G is the quasi-split group of absolute type D_4 which is associated to a non-cyclic cubic extension E. The non-Galois extension E determines a surjective homomorphism

$$\rho_E : Gal(\overline{F}/F) \to S_3 \quad \text{(the symmetric group on 3 letters)}.$$

If r denotes the 2-dimensional irreducible representation of S_3, then $r \circ \rho_E$ is an irreducible 2-dimensional representation of $Gal(\overline{F}/F)$ which determines an irreducible self-contragredient supercuspidal representation τ of $GL_2(F)$. Let χ_τ denote the quadratic central character of τ.

If G^{ad} denotes the adjoint group of G, then the non-Heisenberg parabolic Q^{ad} has Levi factor

$$L^{ad}(F) = (GL_2(F) \times E^\times)/\Delta F^\times.$$

Define a representation σ of $L^{ad}(F)$ by

$$\sigma = \tau \otimes (\chi_\tau \circ \mathbb{N})$$

and pull it back to a representation of $L(F)$ using the natural map $L(F) \to L^{ad}(F)$. This representation σ of $L(F)$ is still irreducible. Indeed, as we shall see later, G contains a subgroup of type G_2 such that $L(F) \cap G_2(F) \cong GL_2(F)$. Moreover, this $GL_2(F)$ maps isomorphically to the $GL_2(F)$-factor in $L^{ad}(F)$ and the representation σ is irreducible when restricted to this subgroup. Now consider the induced representation $I(\sigma) = Ind_Q^G \delta_Q^{-1/3} \sigma$ (normalized induction). This is reducible and has a unique irreducible submodule π, which is our candidate minimal representation.

8. **Calculations of Exponents**

In this section, on a case-by-case basis, we calculate the exponents of π for the non-split unramified groups, thus giving the proofs of Propositions. 7.8 and 7.11. We rely heavily on calculations of Lusztig.

8.1. **Unramified groups with relative root system G_2.** In this subsection, we prove Propositions. 7.8 and 7.11 when G has relative root system G_2.

Let $\Delta = \{\alpha_1, \alpha_2\}$ be a set of simple roots with α_1 short, and α_2 long root. Then

$$\begin{cases} \alpha_l = 3\alpha_1 + 2\alpha_2, \\ \alpha_s = 2\alpha_1 + \alpha_2. \end{cases}$$

Consider first the quasi-split case. The affine root system is the same as the system for split G_2. Let ω_1 and ω_2 be the fundamental coweights. They are the coroots attached to α_s and α_l respectively and they span the lattice $T(F)/T_0$. Then the formulas of Lusztig in the split G_2 case apply here, and

$$\begin{cases} T_{\omega_2} = T_0T_2T_1T_2T_1T_2, \\ T_{\omega_1} = T_0T_2T_1T_2T_1T_0T_2T_1T_2T_1, \end{cases}$$

where T_0 is the generator corresponding to $\alpha_0 = 1 - \alpha_l$.

The representation V is spanned by vectors e_0 and e_2. Define

$$\begin{cases} v_\rho = e_0 + (q^{1/2} + \rho q^{3/2})e_2, \\ v_{\bar\rho} = e_0 + (q^{1/2} + \bar\rho q^{3/2})e_2 \end{cases}$$

where ρ is a primitive cube root of 1. Then the vectors v_ρ and $v_{\bar\rho}$ form a basis of eigenvectors:

$$\begin{cases} \hat{T}_{\omega_1}(v_\rho) = q^6 v_\rho, \\ \hat{T}_{\omega_1}(v_{\bar\rho}) = q^6 v_{\bar\rho} \end{cases}$$

and

$$\begin{cases} \hat{T}_{\omega_2}(v_\rho) = \bar\rho q^3 v_\rho, \\ \hat{T}_{\omega_2}(v_{\bar\rho}) = \rho q^3 v_{\bar\rho}. \end{cases}$$

From this, one can figure out the exponents of π using Proposition 6.6.

Consider now the inner form case. Then the short root space has a structure of a rank nine division algebra D. In this case the lattice $T(F)/T_0$ contains the lattice $A(F)/A_0$ with index 3. This comes from the fact that the range of valuations of norms of elements in $D^\times$ is $\mathbb{Z}$, whereas the valuations of elements in $F^\times \subset D^\times$ are divisible by 3. In particular, $T(F)/T_0$ contains the element $\omega_1/3$ as well. The corresponding affine Weyl group is denoted by G_2^I in [Ti]. The formulas in this case are dual to the one obtained above, with respect to the involution of the Coxeter diagram of G_2 switching the long and short roots. More precisely, if we set

$$\omega_1' = \omega_1/3 \text{ and } \omega_2' = \omega_2,$$

then

$$\begin{cases} T_{\omega_1'} = T_0T_1T_2T_1T_2T_1, \\ T_{\omega_2'} = T_0T_1T_2T_1T_2T_0T_1T_2T_1T_2, \end{cases}$$

where T_0 is the generator corresponding to $\alpha_0 = 1 - \alpha_s$. The representation V in this case is one-dimensional and it is trivial to check that the unique exponent is χ_2.

We summarize the above discussion by the following proposition.

8.2. **Proposition.** *When G is quasi-split, the exponents of π, when restricted to A, are given by*

$$\begin{cases} \chi_2 = -6\alpha_1 - (3 + \log_q \rho)\alpha_2; \\ \bar{\chi_2} = -6\alpha_1 - (3 + \log_q \bar{\rho})\alpha_2. \end{cases}$$

When G is an inner form of the split group, the exponent of π, when restricted to A, is equal to $\chi_2 = -15\alpha_1 - 7\alpha_2$.

8.3. **Unramified groups with relative root system F_4.** In this subsection, we prove Propositions. 7.8 and 7.11 when G has relative root system F_4.

Let $\Delta = \{\alpha_1, \alpha_2\, \alpha_3, \alpha_4\}$ be the set of simple roots as in Lusztig's paper [Lu, p. 640]. Consider first the quasi-split case. Then the affine root system is the same as the affine root system of the split F_4. Again the formulas of Lusztig can be copied here. In particular,

$$T_{\omega_2} = T_0T_2T_4T_3T_1T_4T_2T_3T_4T_3T_2T_4T_1T_3T_4T_2,$$

and the formulas for the other T_{ω_i}'s can be found in [Lu, p. 646]. The representation V is spanned by vectors e_0, e_2 and e_4. Here $b = 4$. The eigenvectors corresponding to the exponents χ_2, χ_4 and $\bar{\chi}_4$ are

$$\begin{cases} v_2 = e_0 + (q^{1/2} + q^{7/2})e_2 + (q - q^2 + q^3)e_4, \\ v_4 = e_0 + (q^{1/2} - q^{5/2})e_2 + (q - q^2 + q^3)e_4, \\ \bar{v}_4 = e_0 + (q^{1/2} - q^{5/2})e_2 + (q - q^2 - q^3)e_4. \end{cases}$$

Consider next the inner form case. Then the short root space has the structure of a quaternion divison algebra D. In this case the lattice T/T_0 contains the lattice A/A_0 with index $(2, 2)$. This comes from the fact that the range of valuations of norms of elements in $D^\times$ is $\mathbb{Z}$, whereas the valuations of elements in $F^\times \subset D^\times$ are divisible by 2. In particular, T/T_0 contains the elements $\omega_1/2$ and $\omega_3/2$ as well. The corresponding affine Weyl group is denoted F_4^I in [Ti]. Again, the formulas in this case are dual to the one obtained above, with respect to the involution of the Coxeter diagram of F_4 switching the long and short roots, which corresponds to the permutation $(2, 1)(4, 3)$ of indices. More precisely, if we set

$$\begin{cases} \omega_i' = \omega_i/2, \text{ if } i = 1, 3; \\ \omega_i' = \omega_i, \text{ if } i = 2, 4, \end{cases}$$

then

$$T_{\omega_1'} = T_0T_1T_3T_4T_2T_3T_1T_4T_3T_4T_1T_3T_2T_4T_3T_1.$$

The formulas for the other $T_{\omega_i'}$ are obtained in a similar fashion from the formulas for T_{ω_i} in the quasi-split case. The representation V is spanned by the vectors e_2 and e_4. Here $b = 4$, and the the eigenvectors corresponding to the eigenvalues χ_4 and χ_2 are $v_4 = e_4$ and $v_2 = e_4 + q^{1/2}e_2$.

We summarize the above discussion with the following proposition.

8.4. **Proposition.** *When G is quasi-split, the exponents of π, when restricted to A, are given by*

$$\begin{cases} \bar{\chi}_4 = -12\alpha_1 - (8+\log_q(-1))\alpha_2 - 22\alpha_3 - 15\alpha_4, \\ \chi_4 = -12\alpha_1 - (8+\log_q(-1))\alpha_2 - 22\alpha_3 - (15+\log_q(-1))\alpha_4, \\ \chi_2 = -12\alpha_1 - 7\alpha_2 - 22\alpha_3 - (15+\log_q(-1))\alpha_4. \end{cases}$$

When G is an inner form of the split group, the exponents of π, when restricted to A, are

$$\begin{cases} \chi_4 = -20\alpha_1 - 13\alpha_2 - 36\alpha_3 - 24\alpha_4, \\ \chi_2 = -20\alpha_1 - 11\alpha_2 - 36\alpha_3 - 24\alpha_4. \end{cases}$$

8.5. **Unramified groups with relative root system B_n.** In this subsection, we prove Propositions 7.11 and 7.8 when G has relative root system B_n. This is similar to the previous two subsections. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the set of simple roots as in [Lu, p. 637]. In particular, α_n is short and the other roots are long.

Consider first the quasi-split case. Let $\{\omega_1, \omega_2, \ldots, \omega_n\}$ be the fundamental coweights. Because our group is simply-connected, the lattice $T(F)/T_0 = A(F)/A_0$ is not equal to the coweight lattice. Rather, it is equal to the coroot lattice which is of index 2 in the coweight lattice. Hence, we cannot express T_{ω_i} in terms of T_i in general.

Consider the Iwahori-Hecke algebra $\tilde{\mathcal{H}}_{ad}$ of the adjoint group of G (in the sense of Lusztig [Lu]). It contains $\tilde{\mathcal{H}}$ as a subalgebra, and is generated by $\tilde{\mathcal{H}}$ and the extra element τ satisfying

$$\begin{cases} \tau^2 = 1, \\ \tau T_0 \tau = T_1, \\ \tau T_i \tau = T_i, \text{ for } i \neq 0, 1. \end{cases}$$

Then

$$\begin{cases} T_{\omega_1} = \tau T_1 ... T_{n-1} T_n T_{n-1} ... T_1, \\ \hat{T}_{\omega_{i+1}} \hat{T}_{\omega_i}^{-1} = \hat{T}_i^{-1} \hat{T}_{\omega_i} \hat{T}_{\omega_{i-1}}^{-1} \hat{T}_i^{-1}, (1 \leq 1 \leq n-1) \end{cases}$$

where we have set $\omega_0 = 0$.

We extend the $\tilde{\mathcal{H}}$-module V to a module over $\tilde{\mathcal{H}}_{ad}$ by setting

$$\begin{cases} \tau(e_{\alpha_0}) = e_{\alpha_1}, \\ \tau(e_{\alpha_1}) = e_{\alpha_0}, \\ \tau(e_\alpha) = e_\alpha \text{ if } \alpha \neq \alpha_0, \alpha_1. \end{cases}$$

Then the above formulas allow us to compute the action of T_{ω_i} on V and hence figure out the exponents of V. In this case $b = n-1$, and the eigenvector with eigenvalue $\bar{\chi}_{n-1}$ is

$$\bar{v}_{n-1} = e_0 + qe_1 + \sum_{i=2}^{n-1} (q^{(i-1)/2} + q^{(i+1)/2}) e_i.$$

Since $\bar{\chi}_{n-1}(\alpha_{n-1}^\vee) \neq |t|^{\pm 1}$, the exponent χ_{n-1} appears as well. Continuing in this fashion, we get the exponents χ_{n-1} through χ_1. Since the dimension of V is n, this is a complete list.

Now consider the inner form case. In this case, the lattice $T(F)/T_0$ contains $A(F)/A_0$ with index 2:

$$\begin{cases} A/A_0 = \langle \omega_1, \dots, \omega_{n-1}, 2\omega_n \rangle, \\ T/T_0 = \langle \omega_1, \dots, \omega_{n-1}, \omega_n \rangle. \end{cases}$$

The Hecke algebra $\tilde{H}$ is generated by $\langle T_0, ..., T_n \rangle$ (here T_0 corresponds to $\alpha_0 = 1 - \alpha_s$) and contains the elements T_{ω_i}. We deduce from [Lu, p. 644, Type $\tilde{C}_n$] that

$$\begin{cases} T_{\omega_1} = T_0 T_1 ... T_{n-1} T_n T_{n-1} ... T_1, \\ \hat{T}_{\omega_{i+1}} \hat{T}_{\omega_i}^{-1} = \hat{T}_i^{-1} \hat{T}_{\omega_i} \hat{T}_{\omega_{i-1}}^{-1} \hat{T}_i^{-1}, (1 \leq 1 \leq n-2), \\ \hat{T}_{\omega_n} \hat{T}_{\omega_{n-1}}^{-1} = \hat{T}_{n-1}^{-1} \hat{T}_{\omega_{n-1}} \hat{T}_{\omega_{n-2}}^{-1} \hat{T}_{n-1}^{-1}, \end{cases}$$

where we have set $\omega_0 = 0$.

In this case, $b = n-1$ and the eigenvector with eigenvalue χ_{n-1} is

$$v_{n-1} = e_{n-1}.$$

Since $\chi_{n-1}(\alpha_{n-2}^\vee) \neq |t|^{\pm 1}$, the exponent χ_{n-2} appears as well. Continuing in this fashion, we get the exponents χ_{n-2} through χ_1. Since the dimension of V is $n-1$, this is a complete list.

9. **Split Exceptional Lie Algebras**

Let Φ be a root system of type D_4 or E_n, $n = 6, 7, 8$. In this section we shall give a description of the corresponding simple split Lie algebra $\mathfrak{g}$, and derive some structural features which will play an important role in the construction of the non-split forms of $\mathfrak{g}$, given in the next section.

Pick Φ^+ a choice of positive roots, and

$$\Delta = \{\alpha_1, ..., \alpha_l\}$$

the corresponding set of simple roots. Let $\langle \alpha, \beta \rangle$ be the Killing form on Φ, and let B be the integer valued bilinear form defined by

$$B(\alpha_i, \alpha_j) = \begin{cases} 0 & \text{if } i < j, \\ \frac{1}{2}\langle \alpha_i, \alpha_j \rangle & \text{if } i = j, \\ \langle \alpha_i, \alpha_j \rangle & \text{if } i > j. \end{cases}$$

Note that $\langle \alpha, \beta \rangle = B(\alpha, \beta) + B(\beta, \alpha)$. Let $c_{\alpha,\beta} = (-1)^{B(\alpha,\beta)}$. The split Lie algebra $\mathfrak{g}$ is spanned by h_α, $\alpha \in \Delta$ and e_β, $\beta \in \Phi$ satisfying the relations

$$\begin{cases} [e_\alpha, e_{-\alpha}] = h_\alpha, \\ [h_\alpha, e_\beta] = \langle \alpha, \beta \rangle e_\beta, \\ [e_\beta, e_\gamma] = \begin{cases} c_{\beta,\gamma} \cdot e_{\beta+\gamma} & \text{if } \beta + \gamma \text{ is a root}, \\ 0 & \text{otherwise}. \end{cases} \end{cases}$$

9.1. **$\mathbb{Z}/3\mathbb{Z}$-filtration of $\mathfrak{g}$.** Let $\alpha_{\max}$ be the highest root in Φ^+. Let α be the unique simple root not perpendicular to $\alpha_{\max}$. Let $\mathfrak{m} \subset \mathfrak{g}$ be the simple algebra whose Dynkin diagram is the Dynkin diagram for $\mathfrak{g}$ with α removed. Extend the Dynkin diagram for $\mathfrak{g}$ by adding $-\alpha_{\max}$. Assume first that the root system is not D_4. Let δ be the unique simple root of $\mathfrak{m}$ not perpendicular to α. Remove the vertex corresponding to the simple root δ. The diagram breaks into two pieces, one of which is the A_2 diagram corresponding to $\{\alpha, -\alpha_{\max}\}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the simple algebra corresponding to the other piece of the diagram. Consider the adjoint action of $sl(3) + \mathfrak{h}$ on $\mathfrak{g}$. Clearly, $-\delta$ is a highest weight for that representation. Let $V \otimes J \subset \mathfrak{g}$ be the corresponding irreducible representation. Since $\langle -\delta, \alpha \rangle = 1$ and $\langle -\delta, -\alpha_{\max} \rangle = 0$, it follows that V is a fundamental 3-dimensional irreducible representation V_3 of $sl(3)$. A similar computation determines J; see the table below. Since the adjoint representation is self dual, we have

$$\mathfrak{g} \supseteq sl(3) + \mathfrak{h} + V \otimes J + V^* \otimes J^*.$$

Checking the dimensions of both sides implies that we in fact have an equality. In the D_4 case we have a similar decomposition, except that here $\mathfrak{h}$ is the orthogonal complement of $\mathfrak{t}$ in the maximal Cartan algebra of $\mathfrak{g}$, where $\mathfrak{t}$ is the diagonal subalgebra of $\mathfrak{sl}_3$. In any case, this gives a $\mathbb{Z}/3\mathbb{Z}$-filtration of $\mathfrak{g}$, with

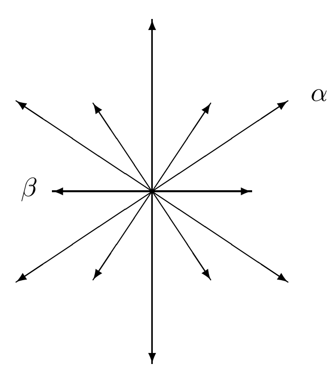
$$\begin{cases} \mathfrak{g}_0 = \mathfrak{sl}_3 \oplus \mathfrak{h}, \\ \mathfrak{g}_1 = V \otimes J, \\ \mathfrak{g}_{-1} = V^* \otimes J^*. \end{cases}$$

$\mathfrak{g}$	$\mathfrak{m}$	$\mathfrak{h}$	J
D_4	A_1^3	F^2	F^2
E_6	A_5	$A_2 \times A_2$	$V_3 \oplus V_3$
E_7	D_6	A_5	$\wedge^2 V_6$
E_8	E_7	E_6	27 dim.

9.2. **Restricted root system of type G_2.** Let $\mathfrak{t}$ be the diagonal subalgebra of $sl(3)$. Decompose $\mathfrak{g}$ under the action of $\mathfrak{t}$

$$\mathfrak{g} = \bigoplus_{\gamma \in \mathfrak{t}^*} \mathfrak{g}_\gamma$$

Let Ψ be the set of all γ with $\mathfrak{g}_\gamma \neq 0$. It follows from the $\mathbb{Z}/3\mathbb{Z}$-filtration that Ψ consists of the roots of $sl(3)$ plus the weights of V and V^*. Thus, Ψ is the root system of type G_2:



We shall fix β to be the weight in V such that $\alpha_{\max} = 2\alpha + 3\beta$. In particular, we have a choice of positive roots given by

$$\Psi^+ = \{\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, \alpha_{\max}\}.$$

The structure of each $\mathfrak{g}_\gamma$ can be described using the $\mathbb{Z}/3\mathbb{Z}$-filtration. For $\gamma = 0$, or a long root in Ψ,

$$\begin{cases} \mathfrak{g}_0 = \mathfrak{k} + \mathfrak{h}, \\ dim\ \mathfrak{g}_\gamma = 1, \quad \gamma \text{ a long root.} \end{cases}$$

In fact, when γ is long, it can be considered a root in Φ, so $\mathfrak{g}_\gamma$ is spanned by e_γ. The structure of short root spaces is more interesting, and we have

$$\begin{cases} \mathfrak{g}_\gamma \cong J \text{ if } \gamma \text{ is a weight of } V, \\ \mathfrak{g}_\gamma \cong J^* \text{ if } \gamma \text{ is a weight of } V^*. \end{cases}$$

9.3. **Trilinear Form.** A simple consequence of this G_2 root system is the existence of an $\mathfrak{h}$-invariant trilinear form on J. More precisely, if x, y and z are in $g_\beta \cong J$, the commutator

$$[x, [y, [z, e_\alpha]]]$$

is a scalar multiple of $e_{\alpha_{\max}-\alpha}$ (note that $\alpha_{\max} - \alpha = \alpha + 3\beta$ in Ψ).

9.4. **Proposition.** *The trilinear form (x, y, z) on $g_\beta \cong J$ by*

$$[x, [y, [z, e_\alpha]]] = (x, y, z) \cdot e_{\alpha_{\max}-\alpha}$$

is $\mathfrak{h}$-invariant

Proof: Obvious, since $\mathfrak{h}$ centralizes both e_α and $e_{\alpha_{\max}-\alpha}$.

9.5. **Restricted root system of type F_4.** Assume again that the type is not D_4. We shall now construct a subalgebra of $\mathfrak{g}$ of type D_4. The adjoint action of its split Cartan subalgebra on $\mathfrak{g}$ will generate a root system of type F_4.

Note that the short root β in Ψ is determined by $\langle \alpha, \beta \rangle = -1$ and $\langle \alpha_{\max}, \beta \rangle = 0$. It follows that $g_\beta \cong J$ has a basis consisting of e_γ with γ in Π where

$$\Pi = \{\gamma \in \Phi | \langle \alpha, \gamma \rangle = -1 \text{ and } \langle \alpha_{\max}, \gamma \rangle = 0\}.$$

9.6. **Proposition.** *Let $\{\eta_1, \eta_2, \eta_3\}$ be a subset of Π. Then the following two are equivalent:*

- *$\{\eta_1, \eta_2, \eta_3\}$ is a maximal subset of orthogonal roots in Π.*
- *$\alpha + \eta_1 + \eta_2 + \eta_3 = \alpha_{\max} - \alpha$.*

Proof. Let $\eta_1, \eta_2, \ldots, \eta_k$ be a sequence of orthogonal roots in Π. Here we do not assume that $k = 3$. Recall that in a simply laced root system, the sum of two roots γ and γ' is a root if and only if $\langle \gamma, \gamma' \rangle = -1$. Thus, it is easy to check that

$$\alpha + \eta_1,\ \alpha + \eta_1 + \eta_2, \ldots, \alpha + \eta_1 + \ldots + \eta_k$$

are all roots. Next, note that $\langle \alpha+\eta_1+\ldots+\eta_k, \alpha_{\max}-\alpha \rangle = k-1$. Since $\alpha_{\max}-\alpha$ is a root in Φ, $\langle \gamma, \alpha_{\max}-\alpha \rangle \leq 2$ for any root γ, and it is equal to 2 only if $\gamma = \alpha_{\max}-\alpha$. It follows that $k \leq 3$, and if $k = 3$, then $\alpha + \eta_1 + \eta_2 + \eta_3 = \alpha_{\max} - \alpha$. In particular, we have shown that the first statement implies the second.

Next, note that $[\mathfrak{g}_\beta, \mathfrak{g}_\beta] = 0$ since 2β is not a root in Ψ. In particular, it follows that $\eta_i + \eta_j$ is not a root in Φ, and $\langle \eta_i, \eta_j \rangle$ has to be non-negative (0 or 1). Since $\eta_1 + \eta_2 + \eta_3 = \alpha_{\max} - 2\alpha$, one easily checks that

$$\langle \eta_1 + \eta_2 + \eta_3, \eta_1 + \eta_2 + \eta_3 \rangle = 6.$$

Writing out the left side, and using $\langle \eta_i, \eta_i \rangle = 2$, we obtain

$$\sum_{i \neq j} \langle \eta_i, \eta_j \rangle = 0.$$

Since each term on the left side is non-negative, they all have to be 0, and the proposition is proved. □

Fix a set $\{\gamma_1, \gamma_2, \gamma_3\}$ of perpendicular roots in Π. Then

$$\{\alpha, \gamma_1, \gamma_2, \gamma_3\}$$

is a set of simple roots for the root system of type D_4. Consider the adjoint action of the corresponding split Cartan subalgebra of rank four on $\mathfrak{g}$. The generated restricted root system is of type F_4. To explain this, note that the root system F_4 contains the root system D_4, (precisely the long roots), and short roots are $1/2(\alpha+\beta)$ for every pair of orthogonal long roots. For example, let $\delta_k = 1/2(\gamma_i+\gamma_j)$, where $\{i, j, k\} = \{1, 2, 3\}$. Then $\mathfrak{g}_\beta \cong J$ breaks up as

$$F \cdot e_{\gamma_1} + F \cdot e_{\gamma_2} + F \cdot e_{\gamma_3} + \mathfrak{g}_{\delta_1} + \mathfrak{g}_{\delta_2} + \mathfrak{g}_{\delta_3}.$$

The dimension of each short root space $\mathfrak{g}_{\delta_i}$ is equal to 2,4 and 8 for the three exceptional Lie algebras E_6, E_7 and E_8, respectively. Moreover, $x \mapsto (x, x, e_{\gamma_i})$ defines a non-degenerate quadratic form on $\mathfrak{g}_{\delta_i}$.

This decomposition of $\mathfrak{g}_\lambda \cong J$, reveals that J is a rank 3 Jordan algebra, with coefficients in a composition algebra of dimension 2, 4 and 8 respectively. This observation will be exploited in the next section to give a construction of the non-split forms of $\mathfrak{g}$.

10. Non-split forms of Exceptional Lie algebras

In this section we shall describe a construction of exceptional Lie algebras which gives rise (by inspection) to all non-split forms over a p-adic field F.

10.1. $\mathbb{Z}/3\mathbb{Z}$-gradation of simple Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra over a field F, together with a $\mathbb{Z}/3\mathbb{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

The Killing form $\langle -, - \rangle$ on $\mathfrak{g}$, restricts to the Killing form $\langle -, - \rangle_0$ on $\mathfrak{g}_0$. This shows that $\langle -, - \rangle_0$ is nondegenerate, in particular, the algebra $\mathfrak{g}_0$ is semisimple. Next, the Killing form gives an $\mathfrak{g}_0$-invariant pairing

$$\langle -, - \rangle_{00} : \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to F.$$

In particular, $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$ as $\mathfrak{g}_0$-modules. Finally, the Lie bracket induces $\mathfrak{g}_0$-invariant maps

$$\begin{cases} [-, -] : \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_{-1}, \\ [-, -] : \wedge^2 \mathfrak{g}_1 \to \mathfrak{g}_{-1}. \end{cases}$$

10.2. Proposition. *The Lie bracket on $\mathfrak{g}$ is completely determined by the $\mathfrak{g}_0$-invariant pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, and the two maps $[-, -]$ as above.*

Proof. To prove this proposition, we need only to show how to define the bracket between an element x in $\mathfrak{g}_1$ and an element y in $\mathfrak{g}_{-1}$. Since the bracket $[x, y]$ should lie in $\mathfrak{g}_0$, it suffices to specify the functional $z \mapsto \langle [x, y], z \rangle_0$ on $\mathfrak{g}_0$. This is done by

$$\langle [x, y], z \rangle_0 = \langle x, [y, z] \rangle_{00}$$

which is also equal to $\langle [z, x], y \rangle_{00}$ by invariance of the form $\langle -, - \rangle_{00}$. □

We shall use this proposition to construct exceptional Lie algebras. However, to do so, we first need to give some basic facts on certain cubic algebras.

10.3. **Cubic algebra J..** In this section J will denote one of the following:

(i) A cubic algebra of the *first* type: the field F, an étale cubic F-algebra E, or a 9-dimensional central simple F-algebra D. Let $\mathbb{N}(x)$ be its reduced norm (if $J = F$, then take $\mathbb{N}(x) = x^3$). Define a symmetric trilinear form on J by

$$(x, x, x) = 6\mathbb{N}(x).$$

(ii) A cubic algebra of the *second* type: a Jordan algebra J_C of 3×3 Hermitian matrices with entries in a composition algebra C. An element of J_C has the form:

$$x = \begin{pmatrix} x_1 & \bar{z}_3 & z_2 \\ z_3 & x_2 & \bar{z}_1 \\ \bar{z}_2 & z_1 & x_3 \end{pmatrix} \quad \text{with } x_i \in F \text{ and } z_i \in C.$$

Let $\mathbb{N}'$ be the reduced norm on C. Define a symmetric trilinear form on J by $(x, x, x) = 6\mathbb{N}(x)$ where

$$\mathbb{N}(x) = x_1x_2x_3 - x_1\mathbb{N}'(z_1) - x_2\mathbb{N}'(z_2) - x_3\mathbb{N}'(z_3) + Tr(z_1z_2z_3).$$

Let x and y be any two elements in J. By $x \times y$, we shall mean the element of J^* defined by the linear functional $z \mapsto (x, y, z)$. As a special case, we introduce:

$$x^{\#} = \frac{1}{2} x \times x.$$

We will identify J and J^* using the (reduced) trace form Tr, normalized so that $Tr(1) = 3$, where 1 is the identity element of J. Under this identification,

$$x \cdot x^{\#} = \mathbb{N}(x) \cdot 1.$$

In particular, if $\mathbb{N}(x) \neq 0$, then $x^{\#}/\mathbb{N}(x)$ is the inverse of x in J. Thus, if J is a division algebra (and thus of the first kind), then $x^{\#} = 0$ implies $x = 0$.

10.4. **Exceptional Lie algebras.** Motivated by the discussion of the previous section and following the ideas of Proposition 10.2, we can now construct a simple Lie algebra $\mathfrak{g}$ over F with a $\mathbb{Z}/3\mathbb{Z}$-grading using J. Let V denote the standard representation of $\mathfrak{sl}_3$ with the standard basis $\{e_1, e_2, e_3\}$ and the dual basis $\{e_1^*, e_2^*, e_3^*\}$. Let $\mathfrak{h}$ be the Lie algebra consisting of linear maps A on J such that

$$(Ax, y, z) + (x, Ay, z) + (x, y, Az) = 0$$

for all $x, y, z \in J$. Now set

$$\begin{cases} \mathfrak{g}_0 = \mathfrak{sl}_3 \oplus \mathfrak{h}, \\ \mathfrak{g}_1 = V \otimes J, \\ \mathfrak{g}_{-1} = V^* \otimes J^*, \end{cases}$$

and

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Since $\mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$, we have a $\mathfrak{g}_0$-invariant pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$. The Lie bracket on $\mathfrak{g}_1$ is given by

$$[e_i \otimes x, e_j \otimes y] = \begin{cases} \pm e_k^* \otimes (x \times y) & \text{if } (i, j, k) \text{ is a permutation of } (1, 2, 3); \\ 0 & \text{otherwise,} \end{cases}$$

where $\pm$ is the sign of the permutation (i, j, k). The Lie bracket on $\mathfrak{g}_{-1}$ is defined similarly. In view of Proposition 10.2, this uniquely determines the Lie bracket on

$\mathfrak{g}$. Of course, we have not shown here that the bracket satisfies the Jacobi identity. This (hard) work was done by Rumelhart [R]. Here we tabulate his results.

10.5. **Relative system G_2.** Suppose that J is of the first type. If J is the split algebra, the Lie algebra $\mathfrak{g}$ is also split of type shown in the table below. If J is a division algebra, the above construction results in a simple Lie algebra of F-rank 2. Indeed, the maximal split Cartan subalgebra of $sl(3) \subset \mathfrak{g}$ is a maximal split Cartan subalgebra of $\mathfrak{g}$ as well. In particular, the relative root system is of type G_2:

J	F	E	D
Absolute Root System	G_2	D_4	E_6

10.6. **Relative system F_4.** Suppose that J is of second type. Again, if the composition algebra C is split, then $\mathfrak{g}$ is split too, of type given in the table below. If C is a division algebra, then $\mathfrak{g}$ has relative root system F_4. If $F = \mathbb{R}$, then the series of so-called quaternionic real Lie algebras is obtained:

Q	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
Absolute Root System	F_4	E_6	E_7	E_8

The set of long roots forms a subsystem of type D_4. A set of simple roots

$$\{\alpha, \gamma_1, \gamma_2, \gamma_3\}$$

can be picked as follows. As in the previous section, let Ψ be the restricted system of type G_2 obtained by the adjoint action of the maximal Cartan subalgebra of $sl_3 \subset \mathfrak{g}$. Then α restricts to the long simple root in Ψ (which was also denoted by α). If β is the short simple root in Ψ, then the root space $\mathfrak{g}_\beta$ is isomorphic to J. Under this identification, the root spaces of γ_i correspond to the diagonal entries in J.

10.7. **Quasi split D_4.** If J is an etalé cubic algebra over F, we get a quasi-split Lie algebra of type D_4:

J	$F \oplus F \oplus F$	$F \oplus K$	E
Relative Root System	D_4	B_3	G_2

10.8. **Heisenberg parabolic.** We can now describe a maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ of $\mathfrak{g}$ using the $\mathbb{Z}/3\mathbb{Z}$-grading.

Let $\mathfrak{t}$ be the maximal Cartan subalgebra in $sl(3)$ consisting of diagonal matrices. Again, the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ generates a restricted root system of type G_2. We shall identify it with the root system Ψ introduced in the last section so that $h_{\alpha_{\max}} = diag(1, 0, -1)$. Define a $\mathbb{Z}$-grading on $\mathfrak{g}$ by

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} : [h_{\alpha_{\max}}, x] = i \cdot x\}.$$

Then we have

$$\begin{cases} \mathfrak{g}(0) = J_\beta \oplus \mathfrak{t} \oplus \mathfrak{h} \oplus J^*_{-\beta}, \\ \mathfrak{g}(1) = Fe_\alpha \oplus J_{\alpha+\beta} \oplus J^*_{\alpha+2\beta} \oplus Fe_{\alpha+3\beta}, \\ \mathfrak{g}(2) = Fe_{\alpha_{\max}}. \end{cases}$$

The parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ is given by $\mathfrak{m} = \mathfrak{g}(0)$ and $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Its nilradical $\mathfrak{n}$ is a Heisenberg Lie algebra with 1-dimensional center $\mathfrak{z} = \mathfrak{g}(2)$.

10.9. **The split subalgebra of type G_2.** It is tempting to ask when the above construction is functorial for the inclusion of cubic algebras. In one case, $F \subseteq J$ (the inclusion of scalars), this is clear, as the Lie algebra $\mathfrak{h}$ for F is trivial. In particular, the inclusion $F \subseteq J$ gives rise to an inclusion of $\mathfrak{g}_2$, the Lie algebra of type G_2 into any $\mathfrak{g}$ considered here. Morally speaking, $\mathfrak{g}$ is the *Lie algebra of type G_2 with coefficients in J.*

10.10. **Groups.** Let G be the simply-connected linear algebraic group over F with Lie algebra $\mathfrak{g}$. Then the groups arising in this way satisfy the conditions (H0) through (H3); indeed they exhaust the list of such groups which are of absolute type D_4 and E_n. The existence of the split subalgebra of type G_2 gives rise to a split subgroup of type G_2 in G. This subgroup has already been used in 7.13.

11. N/Z-spectrum

In this section we shall assume that the field F is non-archimedean, and that the group is of absolute type D_4 or E_n. Let N be the unipotent radical of the Heisenberg parabolic P and Z the center of N. Given a smooth representation π of G, the N/Z-spectrum of π is defined to be the set of non-trivial characters ψ of $N(F)$ such that $\pi_{N,\psi} \neq 0$. Clearly, the N/Z-spectrum of π is invariant under the action of $M(F)$. Thus it is a union of $M(F)$-orbits of characters of N. As the first main result of this section, we describe the minimal M-orbit Ω of non-trivial characters, and show that the N/Z-spectrum of a weakly minimal representation is supported on Ω. Moreover, if π is minimal we show further that

$$dim(\pi_{N,\psi}) = 1$$

for every ψ in the minimal orbit Ω. Of course, this result follows at once from [MW], provided that $p \neq 2$. Our proof is elementary, and removes this restriction on the residual characteristic.

The second main result gives a condition under which a weakly minimal representation is minimal. We show that the condition is satisfied by our candidate representations π introduced in Section 7. This allows us to describe π_Z as a smooth $P(F)$-module. This knowledge is the basis of many applications, as it allows one to compute the Jacquet modules of π with respect to various dual pairs in $G(F)$.

11.1. **Minimal orbit Ω.** Fix a character ψ of F (which is necessarily unitary). Then the character group $Hom(N(F), \mathbb{C}^\times)$ can be identified with $\mathfrak{g}(-1)$ using the exponential map $exp : \mathfrak{n} \to N$, the character ψ and the Killing form $\langle -, - \rangle$. More precisely, for any $x \in \mathfrak{g}(-1)$, one sets

$$\psi_x(exp(n)) = \psi(\langle x, n \rangle).$$

Recall that

$$\mathfrak{g}(-1) = Fe_{-\alpha-3\beta} \oplus J_{-\alpha-2\beta} \oplus J^*_{-\alpha-\beta} \oplus Fe_{-\alpha}.$$

We shall thus denote an element of $\mathfrak{g}(-1)$ by (a, x, y, d), with $a, d \in F$, $x \in J$, and $y \in J^*$. Then Ω is defined as the M-orbit of the vector $(0, 0, 0, 1)$ in $\mathfrak{g}(-1)$. The description of $\mathfrak{m}$ in 10.8 exhibits a parabolic subalgebra

$$\mathfrak{s} = (\mathfrak{t} \oplus \mathfrak{h}) \oplus J_\beta \subset \mathfrak{m}.$$

From the definition of the Lie brackets, it is not difficult to check that for z in J_β, the nilpotent radical of $\mathfrak{s}$, we have

$$z : (a, x, y, d) \mapsto (0, az, x \times z, \langle y, z \rangle).$$

It follows that $(0, 0, 0, 1)$ is a highest weight vector for the action of $\mathfrak{m}$ on $\mathfrak{g}(-1)$. Let S be the corresponding parabolic subgroup of M. Since F has characteristic 0, one may use the exponential map to obtain the action of a group element in the unipotent radical of S.

11.2. **Proposition.** *A non-zero element $(a, x, y, d) \in \mathfrak{g}(-1)$ lies in Ω if and only if it satisfies the following quadratic relations:*

$$\begin{cases} x \cdot y = ad, \\ x^\# = ay, \\ y^\# = dx. \end{cases}$$

Here, the product $x \cdot y$ refers to multiplication in the algebra J.

Proof. To prove any element in Ω satisfies the quadratic relations, it suffices to check that the relations hold on a dense (in the Zariski topology) subset Ω' of Ω. This will be accomplished by means of the Bruhat decomposition of M with respect to the parabolic subgroup S. To do so, we need to describe the double cosets

$$W_S \backslash W_M / W_S$$

where W_M and W_S are relative Weyl groups of M, and the Levi factor of S. Since W_M acts transitively on long roots in $\mathfrak{g}(-1)$, and the stabilizer of the root corresponding to $(0, 0, 0, 1)$ is W_S, the set W_M/W_S can be identified with the set of long roots in $\mathfrak{g}(-1)$. Thus $W_S \backslash W_M / W_S$ can be identified with the W_S-orbits of long roots in $\mathfrak{g}(-1)$. In particular,

$$\# W_S \backslash W_M / W_S = \begin{cases} 8 & \text{in the split } D_4 \text{ case;} \\ 2 & \text{in the relative } G_2 \text{ case;} \\ 4 & \text{in all other cases.} \end{cases}$$

In any case, we shall define Ω' to be the S-orbit of $(1, 0, 0, 0)$, which consists of the elements

$$a(1, x, x^\#, \mathbb{N}(x)), \quad x \in J.$$

One easily checks that the relations are satisfied by this type of elements and one implication of the proposition is proved. We omit the proof of the converse, since we do not need it in the sequel. □

11.3. **Corollary.** *Any non-zero M-orbit in $\mathfrak{g}(-1)$ has a representative of the form $(1, 0, y, *)$. Moreover, y can be arranged to be non-zero if and only if the orbit is not Ω.*

Proof. Indeed, in every non-zero orbit there is an element such that $a \neq 0$. This can be arranged by means of the unipotent group opposite to S. Then, rescaling by an element in T, we can achieve $a = 1$. Finally, acting by the unipotent radical of S, we can arrange $x = 0$.

The only if part of the second statement follows form the proposition. The if part is a bit tricky. We need to show that $(1, 0, 0, d)$ with $d \neq 0$ can be conjugated into the desired form. Recall that $\mathfrak{g}_2$, the exceptional Lie algebra of type G_2, sits in $\mathfrak{g}$, and this embedding corresponds to the embedding of F into J as scalars. Since $(1, 0, 0, d)$ sits in $\mathfrak{g}_2(-1)$, it suffices to check this statement for the exceptional group G_2. In this case $\mathfrak{g}_2(-1)$ can be identified with the space of binary cubic forms $p(u, v) = au^3 + bu^2v + cuv^2 + dv^3$, such that

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M \cong GL_2(F)$$

acts by

$$g \circ p(u, v) = \frac{1}{det(g)} \cdot p(Au + Bv, Cu + Dv).$$

It follows that the diagonal matrix $diag(-1, 1)$ changes the sign of d, and we can assume that $p(u, v) = u^3 - dv^3$, with $d \neq 1$. Let

$$g = \begin{pmatrix} 1 & d \\ 1 & 1 \end{pmatrix}.$$

Then $p'(u, v) = g \circ p(u, v) = u^3 + 3duv^2 + (d + d^2)v^3$, and the proposition follows. □

11.4. N/Z**-spectrum.** The following proposition gives us good control on the N/Z-spectrum of a minimal representation.

11.5. **Proposition.** *Let π be a weakly minimal smooth representation of $G(F)$.*

(i) The N/Z-spectrum of π is supported on Ω.

(ii) π is minimal if and only if

$$dim(\pi_{N,\psi}) = 1 \textit{ for any } \psi \in \Omega.$$

Proof. (i) Let ψ be a non-trivial character of N. By Corollary 11.3, we may thus assume that ψ corresponds to an element in $\mathfrak{g}(-1)$ of the form $(1, 0, y, *)$.

Consider the subgroup R of codimension one in N such that $N/R \cong N_\alpha$, where N_α is the one-parameter subgroup corresponding to α. Clearly, to prove the first part, it suffices to show that if $y \neq 0$, then

$$\pi_{R,\psi} = 0.$$

Let $P' = M'N'$ be the Heisenberg parabolic subgroup of G such that the center Z' of N' is the one-parameter subgroup corresponding to $\alpha + 3\beta$. Hence, N' is generated by the root subgroups corresponding to $\alpha + 2\beta$, $\alpha + 3\beta$, $2\alpha + 3\beta$, β and $-\alpha$. On the level of Lie algebras, we have

$$\mathfrak{n} \cap \mathfrak{n}' = J_{\alpha+2\beta} \oplus Fe_{\alpha+3\beta} \oplus Fe_{2\alpha+3\beta}.$$

It follows that $N \cap N'$ is a maximal abelian (polarizing) subgroup of N'. We shall now restrict the character ψ further down to $N \cap N'$ and Z'. Note that this restriction to Z' is non-trivial. Since π is weakly minimal, we have

$$\pi_{Z',\psi} \cong E \otimes W_\psi$$

for some trivial M'_{ss}-module E. Since $N \cap N'$ is a polarization of N', Proposition 2.3 implies that

$$\pi_{N\cap N',\psi} \cong E,$$

as $R \cap M'_{ss}$-modules. All claims now follow at once, including the multiplicity one result if π is minimal. □

11.6. **The stabilizer M_ψ.** Let π be a minimal representation, so that $\pi_{N,\psi}$ is 1-dimensional. Let $M_\psi \subset M$ be the stabilizer of the character ψ. We want to determine $\pi_{N,\psi}$ as a representation of $M_\psi(F)$.

Without loss of generality, we may assume that ψ is the character corresponding to $e_{-\alpha-3\beta} \in \mathfrak{g}(-1)$. Let $P' = M'N'$ be again the Heisenberg subgroup such that the center Z' of N' is the one-parameter subgroup corresponding to $\alpha + 3\beta$. Note that $M_\psi \subseteq M'_{ss}$ and, as we have seen in the previous section,

$$\pi_{N,\psi} \cong \pi_{N\cap N',\psi}.$$

Since $\pi_{N\cap N',\psi}$ is a quotient of $\pi_{Z',\psi}$, the character of M_ψ on $\pi_{N\cap N',\psi}$ is obtained by restricting to M_ψ the action of M'_{ss} on $\pi_{Z',\psi} \cong W_\psi$. In order to describe this character, note that

$$M_\psi/[M_\psi, M_\psi] \cong M/[M,M].$$

If $\alpha_0 = 2\alpha + 3\beta$ is the highest root, then the basis of $X^*(M)$, the F-rational character group of M, can be chosen so that its restriction to the maximal split torus is equal to α_0. With this notation in hand, it is not too difficult to calculate the character.

11.7. **Proposition.** *Let π be a minimal representation. Let χ be the character of $F^\times$ such that the action of M_ψ on the 1-dimensional space $\pi_{N,\psi}$ is given by $\chi \circ \alpha_0$.*

(i) Suppose that G is an inner form of a split group, then $\chi = |-|^s$ where the value of s is given by the following table:

G	D_4	E_6	E_7	E_8
s	*1*	*2*	*3*	*5*

(ii) If G is the quasi-split group of type E_6 associated to a quadratic field extension K, then $\chi = \chi_K \cdot |-|^2$, where χ_K is the quadratic character attached to K.

(iii) If G is the quasi-split group of type D_4 attached to an étale cubic algebra E, let K_E be the étale quadratic algebra associated to the discriminant of E. Then $\chi = \chi_{K_E} \cdot |-|$.

11.8. **Conditions for Minimality.** We give some sufficient conditions for a weakly minimal representation to be minimal. This will be important later for showing that the representations constructed in Section 7 are minimal.

Recall that our groups admit a relative root system of type G_2. This assures that two parabolic subgroups are defined over F. One is the Heisenberg maximal parabolic P. The other has a 3-step nilpotent subgroup U as its unipotent radical, and it will be denoted by $Q = LU$. On the level of Lie algebras, the Levi factor is $\mathfrak{gl}_2 \oplus \mathfrak{h}$ where

$$\mathfrak{gl}_2 = Fe_{-\alpha} \oplus \mathfrak{t} \oplus Fe_\alpha.$$

This suggests that L has a normal subgroup isomorphic to GL_2. This is in fact the case: we have observed in Section 10 that G contains a subgroup G_2 and it is easy to see that $Q \cap G_2$ is the non-Heisenberg maximal parabolic of G_2. The $\mathfrak{gl}_2$-factor in the Lie algebra of L is precisely the Lie algebra of $L \cap G_2 \cong GL_2$. The Jacquet module π_U is a representation of L and thus can be restricted to the subgroup GL_2. Now we have the following proposition.

11.9. **Proposition.** *If π is weakly minimal and π_U has a 1-dimensional space of Whittaker functionals for $GL_2 = G_2 \cap L$, then π is minimal.*

Proof. Let ψ be the character of N which corresponds to $(0,0,0,1) \in \mathfrak{g}(-1)$. By the previous proposition, we need to show that

$$dim(\pi_{N,\psi}) = 1.$$

Let R denote the unipotent radical of the parabolic subgroup $P \cap Q$ so that $R/N \cong N_\beta$ where N_β is the unipotent group corresponding to the root β. Then R normalizes N and fixes the character ψ. Hence R acts naturally on $\pi_{N,\psi}$. If we can show that N_β acts trivially on it, then it would follow that

$$\pi_{N,\psi} = \pi_{R,\psi_0}$$

where ψ_0 is the unique extension of ϕ to R which restricts to the trivial character on N_β. But the space π_{R,ψ_0} is precisely the space of Whittaker functionals of π_U for GL_2. Thus the hypothesis of the proposition would imply that

$$dim(\pi_{N,\psi}) = dim(\pi_{R,\psi_0}) = 1,$$

as desired. It remains to prove that N_β acts trivially on $\pi_{N,\psi}$. This is similar to the proof of Proposition 11.5 except that it is more complicated since one needs to use the restricted F_4 root system.

To begin, identify $Hom(N_\beta, \mathbb{C}^\times)$ with $J_{-\beta}$, as usual. In particular, each extension of ψ to R corresponds to an element x in $J_{-\beta}$, and will be denoted by ψ_x. Let x be such that

$$\pi_{R,\psi_x} \neq 0.$$

We need to show that $x = 0$.

Let $P' = M'N'$ be the Heisenberg parabolic such that the center of N' is the one-parameter subgroup corresponding to $\alpha+3\beta$. Then $R \cap N'$ is generated by the root subgroups corresponding to $\alpha+2\beta$, $\alpha+3\beta$, $2\alpha+3\beta$ and β. Since $\pi_{R\cap N',\psi_x} \neq 0$, we see that the N'/Z' spectrum of π contains an element of the form $(*, x, 0, 0)$. Since the spectrum is concentrated on the minimal orbit (by Proposition 11.2), we see that this is only possible if $x^\# = 0$.

At this point, the argument divides into different cases, according to the type of J we are considering. If the relative root system of G is of type G_2, then J is a division algebra and $x^\# = 0$ implies $x = 0$, so that we are done.

Assume now that J is of the second type. We shall make use of the F_4 root system which exists in this case. Recall that the long roots form a subsystem of type D_4, and a there is a choice of the set of simple roots

$$\{\alpha, \gamma_1, \gamma_2, \gamma_3\}$$

where α is the simple root in Ψ as above. Moreover, if we identify J_β with the Jordan algebra of 3×3 hermitian symmetric matrices with coefficients in the composition algebra C. The diagonal entries belong to the root spaces of γ_1, γ_2, and γ_3.

The relation $x^\# = 0$ implies that x is of rank one. Without any loss of generality, we may assume that $x = d \cdot e_{-\gamma_1}$. Next, note that $\alpha + \gamma_1$ is a long root, and let $P'' = M''N''$ be the Heisenberg parabolic such that the center Z'' of N'' is the one-parameter subgroup corresponding to $\alpha + \gamma_1$. A short computation shows that

(pictorially)

$$\mathfrak{r} \cap \mathfrak{g}''(1) = Fe_\alpha \oplus \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \oplus Fe_{\gamma_1}.$$

From this, one sees easily that any extension of $\psi_x|_{R\cap N''}$ to a character of N'' is represented by an element in $\mathfrak{g}''(-1)$ of the form

$$\left(1, \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}, d\right).$$

Now since $\pi_{R,\psi_x} \neq 0$ by assumption, we see that $\pi_{R\cap N'',\psi_x} \neq 0$, and therefore $\pi_{N'',\psi''} \neq 0$ for some character ψ'' of N" of the above form. Now Proposition 11.5 implies that such ψ'' lies in in the minimal orbit, and is therefore of the form $(1, z, z^\#, \mathbb{N}(z))$ by Proposition 11.2. Since z has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

then $\mathbb{N}(z) = 0$. Thus $d = 0$, as desired.

The remaining cases correspond to the case when $J = E$ is an étale cubic algebra which is not a field. The argument here is similar and in fact simpler; one just works directly with the root system of the group and we omit the details. The proposition is proved. □

11.10. **Corollary.** *Let G be of absolute type D_4 or E_n. If the candidate representation π is weakly minimal, then it is minimal.*

Proof. Using the information on exponents provided in Section 7, one can determine π_U precisely as a representation of $L(F) = (GL_2 \times_{\mu_3} H)(F)$; we leave this somewhat tedious computation to the reader. Indeed, as a representation of $SL_2(F) \times H(F)$, π_U has the form

$$\pi_U = (\sigma \otimes 1) \oplus (1 \otimes \sigma_H)$$

where σ is an irreducible infinite-dimensional $GL_2(F)$-module and σ_H is some smooth representation of $H(F)$. Hence, π_U has a 1-dimensional space of Whittaker functionals for GL_2, and the corollary follows from the proposition. □

12. Minimality of Candidate Representations

In this section, we shall derive some important consequences. The proofs are based on the rigidity theorem, and on the following simple observation regarding the complex groups.

12.1. **The Minimal Representation of Complex Groups.** If $F = \mathbb{C}$, then we have the isomorphism

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{g}_L \times \mathfrak{g}_R$$

defined by $a \otimes \lambda \mapsto (\lambda a, \bar{\lambda} a)$. Note that $\mathfrak{g}_L = \mathfrak{g}$ and $\mathfrak{g}_R = \bar{\mathfrak{g}}$. Fix a Cartan involution θ on $\mathfrak{g}$. Then θ is a $\mathbb{C}$-linear isomorphism

$$\theta : \bar{\mathfrak{g}} \longrightarrow \mathfrak{g}.$$

By composition, we obtain a $\mathbb{C}$-linear isomorphism

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{g}_L \times \mathfrak{g}_R \xrightarrow{id \times \theta} \mathfrak{g} \times \mathfrak{g}.$$

If $\mathfrak{k} = \{x \in \mathfrak{g} : \theta(x) = x\}$, then $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of $G(\mathbb{C})$. Via the above isomorphism, the subalgebra $\mathfrak{k} \otimes \mathbb{C}$ is identified with the diagonal $\Delta\mathfrak{g}$.

Let $\mathcal{J}_0$ be the Joseph ideal of $U(\mathfrak{g})$. Then $(a, b) \in \mathfrak{g} \times \mathfrak{g}$ acts on $U(\mathfrak{g})/\mathcal{J}_0$ by

$$x \mapsto ax - xb.$$

This defines a $(\mathfrak{g} \times \mathfrak{g}, \Delta\mathfrak{g})$-module π_0. It is clearly spherical and its annihilator is equal to $\mathcal{J}_0 \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes \mathcal{J}_0$. Further, it is irreducible and self-contragredient. We call π_0 the minimal representation of $G(\mathbb{C})$.

12.2. **Split groups.** Let k be a number field and assume that G is a split, simply connected group over k of type D_n or E_n. According to a conjecture of Arthur, the everywhere unramified irreducible representations $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$ which occur as submodules in the part of the discrete spectrum of $L^2(G(k)\backslash G(\mathbb{A}))$ with cuspidal support along the Borel subgroup are parameterized by unipotent orbits $\mathcal{O}$ in the dual group $\hat{G}(\mathbb{C})$ which do not intersect any proper Levi subgroup. More precisely, recall that unipotent orbits correspond to the conjugacy classes of homomorphisms

$$\varphi : SL_2(\mathbb{C}) \to \hat{G}(\mathbb{C}).$$

For a given orbit $\mathcal{O}$, with associated homomorphism φ, the components of the corresponding representation π can be specified as follows. At each finite place v of k, the Satake parameter of the unramified representation π_v is

$$\varphi \begin{pmatrix} |\varpi|^{\frac{1}{2}} & 0 \\ 0 & |\varpi|^{-\frac{1}{2}} \end{pmatrix}$$

where ϖ is a uniformizing element in k_v. At each real place of k, the infinitesimal character of the local component is given by

$$d\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, this is also the infinitesimal character for the two Lie algebra actions at each complex place.

For the groups of type D_n and E_n one such orbit is the subregular orbit. The occurrence of the corresponding globally unramified representation π in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$ was shown by Ginzburg, Rallis and Soudry in [GRS]. At every finite place v, the local component π_v is precisely the representation introduced in Section 7, and at every real place we have a spherical representation whose infinitesimal character is equal to the infinitesimal character of the Joseph ideal $\mathcal{J}_0$. Let F be a local field of characteristic 0 different from $\mathbb{C}$. Then one can pick a number field k so that

$$\begin{cases} k_v \cong F, & \text{for some place } v; \\ k_\infty \cong \mathbb{C}, & \text{for some place } \infty. \end{cases}$$

Since π_∞ is spherical and its infinitesimal character is known, it is isomorphic to π_0, the minimal representation of $G(\mathbb{C})$. The rigidity theorem Theorem 5.4, Corollary 5.8, and Corollary 11.10 thus imply:

12.3. **Theorem.** *Let G be the split simply connected group of type D_n or E_n defined over a local field F of characteristic 0.*

- *If $F = \mathbb{R}$, let π be the unique spherical representation of $G(\mathbb{R})$ whose infinitesimal character is equal to the infinitesimal character of the Joseph ideal $\mathcal{J}_0$. Then π is a unitarizable minimal representation.*
- *If F is non-archimedean, let π be the unramified representation corresponding to the reflection representation of the affine Hecke algebra. Then π is the unique unitarizable minimal representation (in the sense of Definition 3.6).*

Of course, the first part of this theorem goes back to an old work of Vogan [V1]. We feature it here as a simple consequence of the rigidity theorem. In the rest of the paper, we shall show how to use the rigidity theorem to show the minimality of the candidate π for non-split groups.

12.4. **Degenerate Principal Series.** In order to discuss a construction of the global minimal representation for non-split groups, we note that our candidate representation π occurs in certain degenerate principal series representations.

12.5. **Proposition.** *Let F be a non-archimedean local field. Assume that G is of absolute type D_4, E_6, E_7 or E_8, but not the quasi-split form of D_4 associated to a non-Galois cubic extension of F. Then the candidate representation π is the unique irreducible submodule of $I(-s_0) = Ind_P^G \chi \cdot \delta_P^{-s_0}$ (normalized induction), where*

$$\chi = \begin{cases} \text{the trivial character if } G \text{ is split or is an inner form;} \\ \text{the character } \chi_E \text{ if } G \text{ is quasi-split and has splitting field } E, \end{cases}$$

and s_0 is given by the following table:

Absolute type of G	D_4	E_6	E_7	E_8
s_0	3/10	7/22	11/34	19/58

Proof. The exponents of π are known, so one can easily check that π embeds into $I(-s_0)$. If G is split, then it is shown in [S1] that $I(-s_0)$ has unique submodule. If G is not split, the parameter is regular, and the uniqueness is trivial. $\square$

12.6. **Real Groups.** When $F = \mathbb{R}$, one may consider the same degenerate principal series as in the above proposition. If G is quasi-split of type D_4, then it is still true that $I(-s_0)$ has a unique irreducible submodule. This is because the representation $I(s_0)$ is a quotient of an induced representation of Langlands type (cf. [GGJ, Prop. 4.3]). When G is of type E_n, we don't know if the same is true. However, if G is split, we shall insist on working with the spherical minimal representation given in Theorem 12.3 and $I(s_0)$ clearly contains this as an irreducible subquotient with multiplicity one. When G is not split, then it belongs to the quaternionic series (cf. 10.6) and we shall see later that $I(s_0)$ has at most one irreducible subquotient which can be minimal.

12.7. **Eisenstein Series.** We recall the basic theory of Eisenstein series. Let $\mathbb{G}$ be a simply connected group defined over a global field k of absolute type D_4 or E_n. We shall assume that $\mathbb{G}$ is defined by means of a cubic algebra $\mathbb{J}$ over k, as in Section 10. Let $\mathbb{P}$ be the Heisenberg parabolic subgroup of $\mathbb{G}$. Consider the degenerate principal series representation of $\mathbb{G}(\mathbb{A})$:

$$I(s) = Ind_{\mathbb{P}}^{\mathbb{G}} \chi \cdot \delta_{\mathbb{P}}^{s}$$

where χ is a unitary character. For a flat section $f_s \in I(s)$, one can form the Eisenstein series

$$E(f_s, g) = \sum_{\gamma \in \mathbb{P}(k)\backslash \mathbb{G}(k)} f_s(\gamma g)$$

which converges absolutely and locally uniformly in g when the real part of s is sufficiently large, and defines an element in the space $\mathcal{A}(\mathbb{G})$ of automorphic forms on $\mathbb{G}$.

At a point of convergence, one thus has an equivariant map

$$E_s : I(s) \longrightarrow \mathcal{A}(\mathbb{G}).$$

Moreover, for each K-finite flat section f_s, it is known that the $\mathcal{A}(\mathbb{G})$-valued function $s \mapsto E(f_s, -)$ has a meromorphic continuation to the whole complex plane. Suppose that, at $s = s_0$, these functions have a pole of order $\leq k$ and the pole is attained by some f_s. Then we may consider the leading term map

$$Res(s_0) : I(s_0) \longrightarrow \mathcal{A}(\mathbb{G})$$
$$f \mapsto (s - s_0)^k E(f_s, -)|_{s=s_0}.$$

The map $Res(s_0)$ is $\mathbb{G}(\mathbb{A})$-equivariant.

Now suppose we have a non-split group G over a local field $F \neq \mathbb{C}$ which is constructed from a cubic algebra J (which is non-split). We choose a number field k such that $k_v \cong F$ for some place v. Then choose a cubic algebra $\mathbb{J}$ over k such that $\mathbb{J}_v \cong J$. The associated group $\mathbb{G}$ over k then satisfies $\mathbb{G}_v \cong G$. Now consider the family of degenerate principal series $I(s)$ of Proposition 12.5. If v is archimedean and $\mathbb{G}_v$ is split, we let $I'_v(s) \subset I_v(s)$ be the submodule generated by the spherical vector. Then $I'_v(s_0)$ has a unique irreducible quotient which is spherical. Let

$$I'(s) = (\bigotimes_{v \in S_{\mathbb{G}}} I'_v(s)) \bigotimes (\bigotimes_{v \notin S_{\mathbb{G}}} I_v(s))$$

where $S_{\mathbb{G}}$ is the set of archimedean places v such that $\mathbb{G}_v$ is split. Then we have:

12.8. **Proposition.**

- *For any flat section $f_s \in I(s)$, the Eisenstein series $E(f_s, -)$ has a pole of order $\leq k$ at $s = s_0$, where*

$$k = \begin{cases} 2 \text{ if } \mathbb{G} \text{ is split of type } D_4; \\ 1 \text{ otherwise.} \end{cases}$$

- *The pole of order k at $s = s_0$ is attained by an element of $I'(s_0)$.*
- *The image of the map $Res(s_0)$ is a square-integrable automorphic representation Π. When $\mathbb{G}$ is quasi-split of type D_4, then Π is irreducible.*

Proof. This requires a case-by-case analysis. More precisely, the proposition was established by Gan, Gurevich and Jiang [GGJ] when $\mathbb{G}$ is of type D_4, by Rumelhart (unpublished) if $\mathbb{G}$ is of type E_6 and constructed by means of a cubic divison algebra, and by Gan [G2] if $\mathbb{G}$ is constructed by means of a Jordan algebra over a quaternion or octonion division algebra (so $\mathbb{G}$ of type E_7 or E_8).

The only case not covered is the case of quasi-split E_6, which is constructed by means of a Jordan algebra over a quadratic field extension of k. For this, we have verified the proposition but the details are too boring and involved to include here.

We note that the irreducibility of Π for the D_4 case is a consequence of the fact that $I(s_0)$ has a unique irreducible quotient, and as we remarked before, this is not known when G is of type E_n. However, we shall shortly be able to show the irreducibility of Π in all cases. □

As a consequence of the proposition, we have:

12.9. **Theorem.** *Let π be an irreducible summand of Π.*

- *For each finite place v, the local component π_v is the unique unitarizable minimal representation of $G(F)$, and is isomorphic to the candidate representation defined in Section 7.*
- *For each archimedean place v, π_v is minimal and unitarizable. It is spherical if $\mathbb{G}_v$ is split.*

Proof. For every place v, the local component π_v is obviously an irreducible quotient of $I_v(s_0)$. Thus, when v is finite, it must be isomorphic to the representation defined in Section 7, which is the unique irreducible quotient of $I_v(s_0)$ by Proposition 12.5. Since $\mathbb{G}$ splits at some finite place v, the representation π_v is unramified and already minimal by Theorem 12.3. The minimality of π_v at any other place, including archimedean, follows from the rigidity theorem, Corollary 5.8 and Corollary 11.10. The theorem is proved. □

Observe that when $F = \mathbb{R}$ and G is non-split (and thus belongs to the quaternionic series), we have constructed at least one minimal representation of $G(F)$. However, a priori we do not know what this representation is, except that it is a unitarizable quotient of $I(s_0)$. For example, the theorem does not tell us the K-types of π_v. This is a serious drawback in applications but we shall address this in a moment.

12.10. **Uniqueness of minimal representation.** We shall now consider the minimal representations of the quaternionic real groups of type E_n. For these groups, the maximal compact subgroup K is of the form $SU_2(\mathbb{C}) \times_{\mu_2} M_0$, where M_0 is the connected compact group with Lie algebra $\mathfrak{m}$ and center μ_2 (cf. 9.1). Gross and Wallach [GW] have constructed a minimal representation π_0 of $G(\mathbb{R})$, The K types of π_0 are as follows. Let λ be the fundamental weight of M_0, such that $\mathfrak{g}/\mathfrak{k}$ is isomorphic to $\mathbb{C}^2 \otimes V(\lambda)$ as a K-module. Then

$$\pi_0|_K = \bigoplus_{n=0}^{\infty} S^{k+n}(\mathbb{C}^2) \otimes V(n\lambda)$$

where $S^m(\mathbb{C}^2)$ denotes the m-th symmetric power of $\mathbb{C}^2$, and $k = 2, 4$ or 8 for groups of absolute type E_6,E_7 or E_8, respectively.

12.11. **Proposition.** *Let G be a quaternionic real group of type E_n. Assume that π is a minimal representation of $G(\mathbb{R})$. Then π is isomorphic to π_0, the minimal representation constructed by Gross and Wallach.*

Proof. We shall first show that π does not contain any K-type outside the K-types of π_0. Recall (Lemma 4.11) that the algebra of K-invariants in $U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0$ is generated by the Casimir operator Ω_K. In fact, since $\Omega_K = \Omega_{SU(2)} + \Omega_{M_0}$, we have

$$(U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K \cong \mathbb{C}[\Omega_{SU(2)}] \cong \mathbb{C}[\Omega_{M_0}].$$

Let $Z(\mathfrak{m}_{\mathbb{C}})$ be the center of the Lie algebra $\mathfrak{m}_{\mathbb{C}}$. Then $Z(\mathfrak{m}_{\mathbb{C}}) \subseteq (U(\mathfrak{g}_{\mathbb{C}})/\mathcal{J}_0)^K$. Thus, modulo the ideal $\mathcal{J}_0$, there is an identification of $Z(\mathfrak{m}_{\mathbb{C}})$ and $\mathbb{C}[\Omega_{SU(2)}]$. A precise form of this identification can be read off the K-types of the minimal representation π_0, and can be interpreted as a correspondence of infinitesimal characters (see [Li]). In particular, if

$$S^m(\mathbb{C}^2) \otimes V(\mu)$$

is a K-type of π, and $m \geq k$, then $\mu = (m-k)\lambda$, since it is so for π_0. Otherwise, if $m < k$, we claim that there is no such K-type in π. Indeed, modulo $\mathcal{J}_0$, we have

$$3(\Omega_{SU(2)} + 1/2) = \Omega_{M_0} + 3(k+1)^2/2.$$

Since $\Omega_{SU(2)} + 1/2$ acts on $S^m(\mathbb{C}^2)$ as the scalar $(m+1)^2/2$, we see that Ω_{M_0} is negative on $V(\mu)$, if $m < k$. However, Casimir is non-negative on every finite dimensional representation. Therefore, there are no K-types with $m < k$, as desired.

We have shown that the K-types of π must be among $S^{k+n}(\mathbb{C}^2) \otimes V(n\lambda)$ for $n \geq 0$. In particular, π must be isomorphic to π_0, by Proposition 4.10. □

12.12. **Corollary.** *In the setting of Proposition 12.8, the square-integrable representation* Π *is irreducible in all cases. In particular, if* $\mathbb{G}_v$ *is a quaternionic real group of type* E_n*, then*

- *The local component* Π_v *is minimal and is isomorphic to the representation* π_0 *constructed by Gross-Wallach.*
- *The restriction of* Π_v *to* $K = SU_2(\mathbb{C}) \times_{\mu_2} M_0$ *is equal to*

$$\bigoplus_{n \geq 0} S^{k+n}(\mathbb{C}^2) \otimes V(n\lambda).$$

Proof. It is easy to check that $I_v(s_0)$ contains the minimal K-type of π_0 with multiplicity one (cf. [G, §8]). Hence the proposition implies that $I_v(s_0)$ has at most one irreducible quotient which is minimal and the corollary follows. □

12.13. **Quasi-split** D_4**.** Finally, we consider the case when G is the quasi-split form of D_4 associated to a non-Galois cubic field extension K of k. The field K determines a cuspidal representation σ_K of $\mathbb{L}(\mathbb{A})$. We consider the principal series representation

$$I(\sigma_K, s) = Ind_{\mathbb{Q}}^{\mathbb{G}} \delta_{\mathbb{Q}}^s \sigma_K.$$

Let Π be the unique irreducible quotient $I(\sigma_K, 1/3)$. Note that uniqueness here is clear since Π is simply the restricted tensor product of local Langlands quotients. Also, at every finite place, the representation Π_v is isomorphic to the candidate representation defined in Section 7 (see [HMS] for the description in terms of Langlands parameters). As before, we have:

12.14. **Theorem.**
- *For any flat section* $f_s \in I(\sigma_K, s)$*, the Eisenstein series* $E(f_s, -)$ *has a pole of order at most 1 at* $s = 1/3$*. The image of the map* $Res(1/3)$ *is an irreducible square-integrable automorphic representation isomorphic to* Π.
- *Any local component* Π_v *is a unitarizable minimal representation.*

Proof. The first statement is stated in [K]. The second follows from the rigidity result as above. □

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