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THE UNITARY $\mathbb{I}$–SPHERICAL DUAL FOR SPLIT p–ADIC GROUPS OF TYPE F_4

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Abstract. It is known that the determination of the Iwahori-spherical unitary dual for p-adic groups can be reduced to the classification of unitary representations with real infinitesimal character for the associated Hecke algebras. In this setting, I determine the Iwahori–spherical unitary dual for split groups of type F_4.

1. Introduction

The purpose of this paper is to describe the unitary representations with nontrivial Iwahori fixed vectors for a split reductive p–adic group G of type F_4. From [BM1] and [BM2], this is equivalent to the determination of the unitary representations with real infinitesimal character of the corresponding graded Hecke algebra $\mathbb{H}$. Using the classification of simple Hecke algebra modules, the unitary dual is partitioned into subsets parametrized by nilpotent orbits in the dual Lie algebra. Most of the techniques here are the same as those used in [BM3] and [B2] for the classification of the spherical unitary spectrum of classical groups.

I present an outline of the paper. Section 2 has the ingredients needed in the description of the unitary dual. I recall the results of Barbasch and Moy mentioned in the first paragraph of the introduction and the basic definitions and facts about the classification of irreducible modules for the graded Hecke algebra, standard modules and intertwining operators that will be used throughout the paper. I also give a summary of the results in [B2] for classical groups.

In Section 3, I determine the spherical unitary dual. This is done entirely using the relevant K-types (in the sense of [B2]), which are a minimal set of Weyl group representations that are sufficient for determining the unitarity. The payoff is that one hopes to match these Weyl representations with K-types in the real split group F_4, so that the spherical real unitary dual would follow. The details about this correspondence, as applied for classical groups by Barbasch in [B2], are presented in Appendix C.

The spherical $\mathbb{H}$–modules can be parametrized in terms of semisimple elements s in the dual Cartan subalgebra $\check{\mathfrak{a}}$ (as mentioned before, one can assume s is real). Barbasch and Moy described how one can attach to such an element s a unique nilpotent orbit $\check{\mathcal{O}}$. Therefore, it is natural to partition the spherical unitary dual by nilpotent orbits $\check{\mathcal{O}}$ in the dual Lie algebra. To each $\check{\mathcal{O}}$, one attaches a set of unitary parameters, called *complementary series*.

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This description of the spherical dual has some beautiful consequences. In the case of classical groups, as in [BM3], the complementary series of $\check{\mathcal{O}}$ can *always* be identified with the complementary series associated to the trivial nilpotent in the Lie algebra of the centralizer of $\check{\mathcal{O}}$. For F_4, this does not hold when the nilpotent is $\check{\mathcal{O}} = A_1 + \widetilde{A}_1$. It is the only exception for type F_4. I note, however, that there are examples of similar exceptions when the group is of types E_7 and E_8 (although these examples appear very rarely).

A second feature of the description of the spherical unitary dual for the classical groups is that each parameter s in a complementary series can be deformed irreducibly to a parameter which is unitarily induced irreducible from some special unitary spherical parameter of a $\mathbb{H}_M$ ($M \subset G$ a Levi subgroup). This second feature is preserved for F_4.

Section 4 deals with the determination of the Iwahori–spherical unitary dual of the Hecke algebra of type F_4. I compare the part of the $\mathbb{I}$–spherical dual associated to each nilpotent $\check{\mathcal{O}}$ with the spherical unitary dual of the centralizer of $\check{\mathcal{O}}$. The main tools are computations of the intertwining operators introduced in [BM3] and [B2], restricted to some special K-types and the determination of the composition series of standard modules. The connection between the results in the two sections is provided by the Iwahori–Matsumoto involution. This is an involution of $\mathbb{H}$ which preserves unitarity when acting on $\mathbb{H}$–modules.

In Section 5, I give a table with the unitary representations ordered by infinitesimal characters and nilpotents.

In Appendix A, one can find the explicit description of irreducible Weyl representations (as in [L5]) used for constructing realizations of W–representations. In Appendix B, I reproduce the unitary spherical dual for G_2. This is well known of course, by the work of G. Muić ([M]) in the p–adic case and D. Vogan in the real case ([V1]). I just present it here in terms of the affine graded Hecke algebra and give the relevant K-types to justify the claimed connection (from Sections 3 and 4) between the unitary parameters associated to the nilpotent orbit $\widetilde{A}_2$ and its centralizer, which is of type G_2.

In Appendix C, I present the background and methods from [B2] needed to connect the determination of the spherical unitary dual of split p–adic groups with that for the split real groups.

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2. Preliminaries

2.1. **The Iwahori–Hecke algebra.** Let $\mathbb{F}$ denote a p–adic field with a discrete valuation $|\ |$. $\mathcal{R} = \{x \in \mathbb{F} : |x| \leq 1\}$ is its ring of integers and $\mathcal{P} = \{x \in \mathbb{F} : |x| < 1\}$ is the unique maximal ideal in $\mathcal{R}$. $\mathcal{R}/\mathcal{P}$, the residue field, is isomorphic to a finite field $\mathbb{F}_q$.

Let $G = \mathcal{G}(\mathbb{F})$ be the $\mathbb{F}$–points of a split reductive algebraic group defined over $\mathbb{F}$. $K = \mathcal{G}(\mathcal{R})$ is a maximal compact open subgroup in G. Let B be a Borel subgroup such that $G = KB$. $B = AN$, where A is a maximal split torus and N is the unipotent radical.

There is a short exact sequence

$$\{1\} \to K_1 \to K \to G(\mathbb{F}_q) \to \{1\},$$

where $K_1 = \{x \in G : \ x \equiv 1 \bmod \mathcal{P}\}$. Define the *Iwahori subgroup*, $\mathbb{I} \subset G$, to be the inverse image in K of a Borel subgroup in $G(\mathbb{F}_q)$.

The *unitary dual problem* for the group G refers to the determination of all irreducible unitary representations of G. By a representation of G, I will always mean a smooth admissible representation. As mentioned in the introduction, this paper determines the *Iwahori–spherical dual* of G, that is, the irreducible unitary representations (π, V) of G, such that $V^{\mathbb{I}} \neq \{0\}$. An important particular case is that of *spherical representations*, which are representations with nontrivial fixed K-vectors.

Define the *Iwahori–Hecke algebra*, $\mathcal{H} = \mathcal{H}(\mathbb{I}\backslash G/\mathbb{I})$, to be the set of compactly supported $\mathbb{I}$–biinvariant functions on G. This is an algebra under the convolution of functions. If (π, V) is an $\mathbb{I}$–spherical representation of G, then $\mathcal{H}$ acts on $V^{\mathbb{I}}$ via:

$$\pi(f)v := \int_G f(g)(\pi(g)v)\ dg, \text{ for } v \in V^{\mathbb{I}} \text{ and } f \in \mathcal{H}.$$

This action makes $V^{\mathbb{I}}$ a finite–dimensional $\mathcal{H}$–module.

Theorem 2.1 (Borel–Casselman)**.** *The functor $V \to V^{\mathbb{I}}$ is an equivalence of categories between the category of finite–length admissible representations of G with the property that each subquotient is generated by its $\mathbb{I}$–fixed vectors and the category of finite–dimensional modules of $\mathcal{H}$.*

Note that, in particular, the theorem implies that irreducible $\mathbb{I}$–spherical representations of G are in one–to–one correspondence with finite dimensional simple $\mathcal{H}$–modules.

The algebra $\mathcal{H}$ has a star operation defined as $f \to f^*$, $f^*(g) := \overline{f(g^{-1})}$ and therefore one can define Hermitian and unitary modules for $\mathcal{H}$. The following result gives the reduction of the unitarity problem for the group G to the similar problem for the Iwahori–Hecke algebra (see [BM1]).

Theorem 2.2 ([BM1], [BM2])**.** *An $\mathbb{I}$–spherical irreducible representation V of G is unitary if and only if $V^{\mathbb{I}}$ is a unitary simple $\mathcal{H}$–module.*

Moreover, in [BM2], Barbasch and Moy showed that the determination of the unitary dual of $\mathcal{H}$ can be reduced to the determination of the unitary dual of the associated *affine graded Hecke algebra* $\mathbb{H}$. Next, I give the description of $\mathbb{H}$ in terms of generators and relations and recall some basic definitions and results about the parametrization of simple $\mathbb{H}$–modules.

2.2. **The Affine Graded Hecke Algebra.** Let G_m be the multiplicative group of $\mathbb{F}$, $X = \mathrm{Hom}(G_m, A)$ the lattice of one–parameter subgroups of A and $\check{X} = \mathrm{Hom}(A, G_m)$. Let R, R^+ and let Π be the sets of roots of A in G, positive roots, simple roots and $\check{R}$, $\check{R}^+$ and let $\check{\Pi}$ be the sets of coroots, positive coroots and simple coroots. Let $\check{G}$ be the complex dual group of G and $\check{\mathfrak{g}}$ be its Lie algebra. $\mathfrak{a} = X \otimes_{\mathbb{Z}} \mathbb{C}$ and $\check{\mathfrak{a}} = \check{X} \otimes_{\mathbb{Z}} \mathbb{C}$. Let W denote the Weyl group and $\mathbb{C}[W]$ the group algebra of W.

As a vector space, $\mathbb{H}$ is $\mathbb{C}[W] \otimes \mathbb{A}$, where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{a}$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{a}$. The relations between the generators are:

$t_w t'_w = t_{ww'}$, for all $w, w' \in W$;

$t_s^2 = 1$, for any simple reflection $s \in W$;

$t_s \omega = s(\omega) t_s + \langle \omega, \check{\alpha} \rangle$, for simple reflections $s = s_\alpha$.

$\mathbb{H}$ is also a star algebra with the star operation given on generators as follows (as in [BM2]):

$t_w^* = t_{w^{-1}}$, $w \in W$;

$\omega^* = -\overline{\omega} + \sum_{\alpha \in R^+} \langle \overline{\omega}, \check{\alpha} \rangle t_\alpha$, $\omega \in \mathfrak{a}$.

2.3. **Simple $\mathbb{H}$–modules.** As I mentioned before, the problem of the $\mathbb{I}$–spherical unitary dual of G comes down to the determination of all the unitary simple modules for $\mathbb{H}$. To this end, I need to recall some of the basic results about the classification and parametrization of simple Hecke algebra modules as in [KL] and [L1] and present Langlands classification in the setting of the affine graded Hecke algebra. The presentation of these basic results is influenced by D. Barbasch's exposition in [B1].

Theorem 2.3 ([L1])**.** *The irreducible $\mathbb{H}$–modules are parametrized by $\check{G}$-conjugacy classes $(s, \check{\mathcal{O}}, \psi)$, where $s \in \check{\mathfrak{g}}$ is semisimple, $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ is a nilpotent orbit which has a standard Lie triple $\{\check{e}, \check{h}, \check{f}\}$ such that $[s, \check{e}] = \check{e}$ and $\psi \in \widehat{A(s, \check{e})}$ is an irreducible representation of $A(s, \check{e})$, the component group of the centralizer of s and $\check{e}$. The representations ψ that appear come from the Springer correspondence.*

More precisely, if $(s, \check{\mathcal{O}}, \psi)$ is a parameter as in the theorem and $\{\check{e}, \check{h}, \check{f}\}$ is a Lie triple for $\check{\mathcal{O}}$, the infinitesimal character s can be written as $s = \frac{1}{2}\check{h} + \nu$, with ν centralizing the triple $\{\check{e}, \check{h}, \check{f}\}$. To each pair $(s, \check{\mathcal{O}})$, one attaches a *standard module* $X(s, \check{\mathcal{O}})$. The standard module may be reducible and it decomposes into a direct sum:

$$X(s, \check{\mathcal{O}}) = \bigoplus_{\psi \in \widehat{A(s,\check{e})}} X(s, \check{\mathcal{O}}, \psi).$$

In this direct sum, not all $\psi \in \widehat{A(s, \check{e})}$ appear. Each standard module $X(s, \check{\mathcal{O}}, \psi)$ has a unique irreducible quotient $L(s, \check{\mathcal{O}}, \psi)$ and each irreducible $\mathbb{H}$–module is isomorphic to such a $L(s, \check{\mathcal{O}}, \psi)$.

To each nilpotent orbit, one attaches, by the Springer correspondence, some representations of the Weyl group, which will be refered to as the *lowest K-types* of the nilpotent orbit. Their construction and properties will be recalled at the beginning of Section 4, when I will make use of them in an essential way.

Finally, all the factors of $X(s, \check{\mathcal{O}}, \psi)$ have parameters $(s, \check{\mathcal{O}}', \psi')$ such that $\check{\mathcal{O}}' \neq \check{\mathcal{O}}$ and $\check{\mathcal{O}} \subset \overline{\check{\mathcal{O}}'}$. This fact is crucial for the method of determination of the spherical unitary dual used in this paper.

Definition 2.4. Let $(s, \check{\mathcal{O}}, \psi)$ be a parameter corresponding to a simple $\mathbb{H}$–module. If the semisimple element s has trivial elliptic part, the parameter is called *real*. A parameter is called *tempered* if $\nu = 0$. If, in addition, $\check{\mathcal{O}}$ is distinguished, i.e. it does not meet any proper Levi component, the parameter is called a *discrete series*.

The above definitions are justified by the Borel–Caselman correspondence with irreducible representations of G and the results in [KL]. An essential fact for us is that simple $\mathbb{H}$-modules parametrized by a tempered parameter (as in the above definition) are formed by the Iwahori–fixed vectors of tempered representations of the group G and, therefore, are unitary. They represent the starting point for building the unitary dual of $\mathbb{H}$.

The results in [BM2] show that it is sufficient to classify the unitary simple $\mathbb{H}$–modules with real parameters. Actually, [BM2] implies that the classification of the unitary dual for the Iwahori–Hecke algebra $\mathcal{H}$ is equivalent to the classification of the unitary simple modules with real parameter for the graded Hecke algebra $\mathbb{H}$ *and* for similar graded Hecke algebras corresponding to groups of lower ranks. Since the unitary dual for graded Hecke algebras of classical types is known (from [BM3]) and the unitary dual for groups of type G_2 was also determined ([M]), it remains to determine the unitary dual (for real parameters) when $\mathbb{H}$ is of type F_4.

From now on, all the parameters will be assumed real.

If $P = MN$ is a (standard) parabolic subgroup of G with the Levi component M and V is a module for the affine graded Hecke algebra $\mathbb{H}_M$ associated to M, one can form the induced module $I(M, V) = \mathbb{H} \otimes_{\mathbb{H}_M} V$. The Langlands classification in this setting (as in [BM3]) says that every irreducible module of $\mathbb{H}$ appears as the unique irreducible quotient $L(M, V, \nu)$ (called *Langlands quotient*) of an induced module $X(M, V, \nu) = I(M, V \otimes \nu)$, where:

1. M is a Levi component of a parabolic subgroup of G;
2. V is a tempered irreducible representation of $\mathbb{H}_M$;
3. $\nu \in \mathfrak{a}^*$, ν real, satisfying $\langle \nu, \alpha \rangle = 0$, for all $\alpha \in R_M^+$ and $\langle \nu, \alpha \rangle > 0$, for all $\alpha \in R^+ - R_M^+$ ($R_M \subset R$ denotes the root subsystem associated to the Levi component $M \subset G$).

Moreover, two Langlands quotients are isomorphic if and only if the data (M, V, ν) that characterize them are conjugate by an element in the group G.

Next, I will explain the connection between Kazhdan–Lusztig and Langlands classifications. Suppose $X_G(s, \check{\mathcal{O}})$ is a standard module for $\mathbb{H} = \mathbb{H}_G$ and that $s = \frac{1}{2}\check{h} + \nu$ and $\{\check{h}, \check{e}, \check{f}\} \subset \check{\mathcal{O}}$ are contained in a Levi component $\check{\mathfrak{m}}$. Let $M \subset G$ be the Levi subgroup whose Lie algebra has dual $\check{\mathfrak{m}}$. One can form the standard module $X_M(s, \check{\mathcal{O}}_M)$ ($\check{\mathcal{O}}_M$ is the $\check{M}$–orbit of e in $\check{\mathfrak{m}}$) . Then

$$X_G(s, \check{\mathcal{O}}) = I(M, X_M(s, \check{\mathcal{O}}_M)).$$

For $\phi \in \widehat{A_M(s, \check{e})}$, the induced module from $X_M(s, \check{\mathcal{O}}_M, \phi)$ breaks into a direct sum of standard modules of G corresponding to the representations of $A_G(s, \check{e})$ which contain ϕ in their restriction to $A_M(s, \check{e})$ (we view $A_M(s, \check{e})$ as a subgroup of $A_G(s, \check{e})$):

$$I(M, X_M(s, \check{\mathcal{O}}_M, \phi)) = \bigoplus_{\psi \in \widehat{A_G(s,\check{e})}} [\psi|_{A_M(s,\check{e})} : \phi] X_G(s, \check{\mathcal{O}}, \psi).$$

If $\check{M}$ denotes the centralizer in $\check{G}$ of ν and M the corresponding subgroup in G, the standard modules $X_G(s, \check{\mathcal{O}}, \psi)$ can also be seen as induced modules:

$$X_G(s, \check{\mathcal{O}}, \psi) = I(M, X_M(\frac{1}{2}\check{h}, \check{\mathcal{O}}_M, \phi) \otimes \mathbb{C}_\nu),$$

for some ϕ in the restriction of ψ to $A_M(s, \check{e})$. By Definition 2.4, $V = X_M(\frac{1}{2}\check{h}, \check{\mathcal{O}}_M, \phi)$ is a tempered module of $\mathbb{H}_M$. This shows the connection between the two classifications.

2.4. **Intertwining Operators and Hermitian Forms.** I recall the construction of Hermitian forms and intertwining operators from [BM3].

Let $w = s_1 \dots s_k$ be a reduced decomposition of w. For each simple root α, define $r_\alpha = (t_\alpha \check{\alpha} - 1)(\check{\alpha} - 1)^{-1}$. Then define $r_w := r_{\alpha_1} \dots r_{\alpha_k}$. A priori, r_w could depend on the reduced expression of w, but Lemma 1.6. in [BM3] shows that actually r_w is well defined. w_0 will denote the long Weyl element. Denote by $W(M)$ the Weyl group of W viewed as a subgroup of W.

I will use the following results from [BM3]:

Theorem 2.5 ([BM3])**.** *Let M be the Levi component of a parabolic subgroup P, V be a tempered module for $\mathbb{H}_M$ and ν a real character as before.*

(1) *The Langlands quotient $L(M, V, \nu)$ is Hermitian if and only if there exists a Weyl group element w which conjugates the triple (M, V, ν) to $(M, V, -\nu)$.*
(2) *Assume $L(M, V, \nu)$ is Hermitian with w as above. Let w_m be the shortest element in the double coset $W(M)wW(M)$. The operator*

$$I(w_m, \nu) : X(M, V, \nu) \to X(M, V, -\nu), \quad x \otimes (v \otimes 1_\nu) \to x r_{w_m} \otimes (v \otimes 1_{-\nu})$$

is an intertwining operator. Moreover, the image of $I(w, \nu)$ is the Langlands quotient $L(w, V, \nu)$ and the Hermitian form on $L(M, V, \nu)$ is given by:

$$\langle x \otimes (v \otimes 1_\nu), y \otimes (v' \otimes 1_\nu) \rangle = (x \otimes (v \otimes 1_\nu), y r_w \otimes (v' \otimes 1_{-\nu}))_h,$$

where $(\ ,\ )_h$ denotes the pairing with the Hermitian dual.

For practical calculations in F_4, w_m can be chosen to be the shortest element in the double coset $W(M)w_0W(M)$.

Of great importance for the actual classification is the $\mathbb{C}[W]$–structure of the standard modules. Recall the Peter–Weyl decomposition

$$\mathbb{C}[W] = \sum_{\sigma \in \widehat{W}} V_\sigma \otimes V_\sigma^*,$$

(σ, V_σ) denoting the irreducible representations of the Weyl group, which, by analogy with the real groups, are called K*-types.* The Weyl group representations for type F_4 are classified by Kondo in [K]. The K-structure of standard modules is given by the Green polynomials calculated in [K] and can also be read from the (unpublished) tables of Alvis (see [A]).

Consider the intertwining operators of the form $I(w, \nu)$: $X(M, V, \nu) \to X(M, V, -\nu)$. We assume here that $w\nu = -\nu$. As a $\mathbb{C}[W]$–module, $X(M, V, \nu) = \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V$. For any K-type (σ, V_σ), $I(w, \nu)$ induces an operator

$$r_\sigma(w, M, \nu) : \operatorname{Hom}_{\mathbb{C}[W]}(V_\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V) \to \operatorname{Hom}_{\mathbb{C}[W]}(V_\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V).$$

By Frobenius reciprocity,

$$\operatorname{Hom}_{\mathbb{C}[W]}(V_\sigma, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} V) \cong \operatorname{Hom}_{\mathbb{C}[W(M)]}(V_\sigma, V).$$

In conclusion, $I(w, \nu)$ gives rise to an operator

$$r_\sigma(w, M, \nu) : \operatorname{Hom}_{\mathbb{C}[W(M)]}(V_\sigma, V) \to \operatorname{Hom}_{\mathbb{C}[W(M)]}(V_\sigma, V),$$

or, equivalently,

$$r_\sigma(w, M, \nu) : (V_\sigma^*)^V \to (V_\sigma^*)^V.$$

Theorem 2.5 implies that if the Langlands quotient were unitary, all the operators $r_\sigma(w, M, \nu)$, obtained by the restriction to K-types, would be positive semidefinite.

As in [BM3] and [B2], one of the main tools for showing modules are *not* unitary is to compute the signature of these operators.

2.5. **Spherical $\mathbb{H}$–modules.** For the rest of the section, I will present the special case of spherical modules and the results for classical groups from [B2]. The general machinery presented so far can be described in considerably simpler terms for this case. The $\mathbb{H}$-modules which correspond to the spherical group representations are precisely those which viewed as $\mathbb{C}[W]$–modules contain the trivial Weyl group representation.

If a simple spherical $\mathbb{H}$–module is parametrized by a Kazhdan–Lusztig triple $(s, \check{\mathcal{O}}, \psi)$, the representation ψ must be the trivial representation. Moreover, the semisimple element s determines the nilpotent orbit uniquely. Fix a semisimple $s \in \check{\mathfrak{g}}$ (actually, one can assume $s \in \check{\mathfrak{a}}$). The characterization of the nilpotent orbit $\check{\mathcal{O}}$ as in [B2] is the following. Let $\check{\mathfrak{g}}_1$ and $\check{\mathfrak{g}}_0$ be the 1–eigenspace, respectively the 0–eigenspace of $\mathrm{ad}(s)$:

$$\check{\mathfrak{g}}_1 = \{x \in \check{\mathfrak{g}} : \ [s, x] = x\}, \quad \check{\mathfrak{g}}_0 = \{x \in \check{\mathfrak{g}} : \ [s, x] = 0\}.$$

Let $\check{G}_0 \subset \check{G}$ be the complex Lie group with Lie algebra $\check{\mathfrak{g}}_0$. $\check{G}_0$ acts on $\check{\mathfrak{g}}_1$ and it has a unique dense orbit in $\check{\mathfrak{g}}_1$. Call it $\check{\mathcal{O}}_1$. Then there is a unique nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$ which meets $\check{\mathfrak{g}}_1$ in $\check{\mathcal{O}}_1$.

The nilpotent orbit $\check{\mathcal{O}}$ admits a second, equivalent, description:

Proposition 2.6 ([BM1])**.** *Let $s \in \check{\mathfrak{a}}$ be a semisimple element and $\check{\mathcal{O}}$ the associated nilpotent orbit constructed before. Let $\{\check{e}, \check{h}, \check{f}\}$ be a Lie triple associated to the orbit $\check{\mathcal{O}}$. Then $\check{\mathcal{O}}$ has the property that it is unique subject to the following two conditions:*

1. *there exists $w \in W$ such that $ws = \frac{1}{2}\check{h} + \nu$, where ν is a semisimple element in the Lie algebra $\mathfrak{z}(\check{\mathcal{O}})$ of the centralizer of the Lie triple;*
2. *if s satisfies the first property for a different $\check{\mathcal{O}}'$, then $\check{\mathcal{O}}' \subset \overline{\check{\mathcal{O}}}$.*

For the spherical case, I consider the *principal series module* $X(s) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_s$, $s \in \check{\mathfrak{a}}$. Since $X(s)$ is isomorphic as a W–representation to $\mathbb{C}[W]$, it follows that the trivial W–representation appears with multiplicity one in $X(s)$ and therefore, there is a unique spherical subquotient $L(s)$. Consequently, I will refer to a semisimple element s to be unitary if the spherical module parametrized by it is unitary.

Consider the intertwining operator given by w_0, the long element in the Weyl group, called the *long intertwining operator.* I cite the following result from [B2].

Theorem 2.7 ([B2])**.** *If s is dominant (i.e., $\langle s, \check{\alpha}\rangle \geq 0$ for all positive roots $\alpha \in R^+$), then the image of $I(w_0, s)$ is $L(s)$.*

Moreover, $L(s)$ is Hermitian if and only if $w_0 s = -s$.

Note that $r_{w_0} = r_{\alpha_1} \cdots r_{\alpha_k}$ acts on the right and therefore, each α_j in the decomposition into r_{α_j}'s can be replaced by the scalar $\langle \check{\alpha}_j, s_{j+1}s_{j+2}\cdots s_k(\nu)\rangle$ in the intertwining operator $I(w_0, \nu)$. Consequently, we can think of r_{w_0} as an element in $\mathbb{C}[W]$.

The discussion about the intertwining operators and Hermitian forms in Section 2.4 implies the following remark in the spherical case.

Remark 2.8. The long intertwining operator gives rise to operators on the K-types (σ, V_σ): $r_\sigma(w_0, s) : (V_\sigma)^* \to (V_\sigma)^*$. As before, the Hermitian form on the module $L(s)$ is positive definite (and therefore $L(s)$ is unitary) if and only if $w_0 s = -s$ and all the operators $r_\sigma(w_0, s)$ are positive semidefinite.

Note that this fact suggests the following combinatorial description of the spherical unitary dual. One can consider (real) parameters s in the dominant Weyl chamber. They parametrize spherical $\mathbb{H}$–modules. Since in the Weyl group of type F_4, w_0 acts on any such s by -1, any parameter s is Hermitian. In order to determine if s is unitary, one would have to compute the operators $r_\sigma(w_0, s)$ on the K-type σ. An operator $r_\sigma(w_0, s)$ can only change its signature in the dominant Weyl chamber on a hyperplane where $\langle s, \check{\alpha}\rangle = 1$ for $\alpha \in R^+$ or $\langle s, \check{\alpha}\rangle = 0$ for $\alpha \in \Pi$. Therefore, the spherical unitary dual can be viewed as a (bounded) union of closed facets in this arrangement of hyperplanes. I will use this observation in the description of the spherical unitary dual in Section 3.

For the explicit description, the spherical unitary dual is partitioned into subsets, each subset being parametrized by a nilpotent orbit in $\check{\mathfrak{g}}$. To such a nilpotent orbit $\check{\mathcal{O}}$, one attaches the set of parameters corresponding to $\check{\mathcal{O}}$ which are unitary.

Definition 2.9. These sets of parameters $s = \frac{1}{2}\check{h} + \nu$ associated to a nilpotent orbit $\check{\mathcal{O}}$ which are unitary are called the *complementary series attached to* $\check{\mathcal{O}}$.

When G is of classical type, the explicit description of the spherical unitary dual of the associated affine graded Hecke algebra from [B2] can be summarized in the following theorem. I mention that for type A, the unitary dual for p–adic $GL(n, \mathbb{F})$ had already been classified by Tadić (see [T]).

Theorem 2.10 ([B2])**.** *Let $s \in \check{\mathfrak{a}}$ be a semisimple element and $\check{\mathcal{O}}$ the unique maximal nilpotent orbit such that $s = \frac{1}{2}\check{h} + \nu$, with ν a semisimple element in $\mathfrak{z}(\check{\mathcal{O}})$.*

(1) *s is in the complementary series of $\check{\mathcal{O}}$ if and only if ν is in the complementary series of the trivial nilpotent orbit of $\mathfrak{z}(\check{\mathcal{O}})$.*
(2) *The (real) parameters $s = (\nu_1, \nu_2, \dots, \nu_n)$, $n = \text{rank } G$, in the complementary series associated to the trivial nilpotent orbit can be described explicitly as follows:*
   (a) **A***: s has to be of the form $(\nu_1, \dots, \nu_k, -\nu_k, \dots, -\nu_1)$ if $n = 2k$ or $(\nu_1, \dots, \nu_k, 0, -\nu_k, \dots, -\nu_1)$ if $n = 2k+1$, with $0 \le \nu_1 \le \nu_2 \le \dots \le \nu_k < \frac{1}{2}$.*
   (b) **B***: $0 \le \nu_1 \le \nu_2 \le \dots \le \nu_n < \frac{1}{2}$.*
   (c) **C, D***: $0 \le \nu_1 \le \nu_2 \le \dots \le \nu_k \le \frac{1}{2} < \nu_{k+1} < \nu_{k+2} < \dots < \nu_n$, so that $\nu_i + \nu_j \ne 1$ if $i \ne j$ and there is an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \le \frac{1}{2}$ and an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.*

(Note that the types A, B, C, D in the theorem refer to *the group G*.)

Moreover, in view of Remark 2.8, the spherical unitary dual for classical groups is determined by the operators restricted to a small set of K-types, as follows from [B2].

Theorem 2.11 ([B2])**.** *For G of classical type, a spherical parameter s is unitary if and only if the operators $r_\sigma(w_0, s)$ are positive semidefinite for the following representations σ of W:*

(1) **A***: $(m, n-m)$, $0 \le m \le [\frac{n}{2}]$.*
(2) **B, C***: $(n-m) \times (m)$, $0 \le m \le n$ and $(m, n-m) \times (0)$, $m \le [\frac{n}{2}]$.*
(3) **D***: $(n-m) \times (m)$, $0 \le m \le [\frac{n}{2}]$ and $(m, n-m) \times (0)$, $m \le [\frac{n}{2}]$.*

The K-types appearing in Theorem 2.11 are called *relevant K-types.*

2.6. **Coordinates for F_4.** Throughout this paper, I will use the following realization of the root system of the group G of type F_4 (α_i are the simple roots and ω_i the corresponding simple weights):

$$\begin{array}{ll} \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) & \omega_1 = \epsilon_1 \\ \alpha_2 = \epsilon_4 & \omega_2 = \frac{3}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + \frac{1}{2}\epsilon_3 + \frac{1}{2}\epsilon_4 \\ \alpha_3 = \epsilon_3 - \epsilon_4 & \omega_3 = 2\epsilon_1 + \epsilon_2 + \epsilon_3 \\ \alpha_4 = \epsilon_2 - \epsilon_3 & \omega_4 = \epsilon_1 + \epsilon_2 \end{array}$$

α_1 and α_2 are the short roots and α_3 and α_4 the long roots. Note that all the calculations with the intertwining operators, being done in the dual group $\check{G}$, will use the coroots $\check{\alpha}_i$. All the parameters will be expressed in the coordinates $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$.

3. The unitary spherical dual

Recall that, in the spherical case, the Langlands quotients are uniquely determined by their infinitesimal character s: there is a unique maximal nilpotent orbit $\check{\mathcal{O}}$ such that $s = \frac{1}{2}\check{h} + \nu$, where $\check{h}$ denotes the middle element of a standard Lie triple corresponding to $\check{\mathcal{O}}$ and ν is a semisimple element centralizing the standard triple, which can be written as a vector with real entries of length $n = \text{rank } \check{\mathfrak{g}}$. In this way, each spherical parameter corresponds to a unique nilpotent orbit.

3.1. **The Iwahori–Matsumoto involution.** I recall that the graded Hecke algebra has an involution called the *Iwahori–Matsumoto involution*, IM, defined on the generators as follows:

$$IM(t_w) = (-1)^{l(w)} t_w, \qquad IM(\omega) = -\omega,\ \omega \in \mathfrak{a}.$$

IM acts therefore on the modules of $\mathbb{H}$.

The induced action of the Iwahori–Matsumoto involution on the K-types is tensoring with the sign representation of W. The use of the Iwahori–Matsumoto involution is justified by the following result from [BM1].

Theorem 3.1 ([BM1])**.** *Let V denote a module of $\mathbb{H}$ and IM the Iwahori-Matsumoto involution. Then V is unitary if and only if $IM(V)$ is unitary.*

In particular, if one considers a spherical module $L(s)$ parametrized by $s = \frac{1}{2}\check{h}$, where $\check{h}$ is the middle element of a nilpotent orbit $\check{\mathcal{O}}$, $IM(L(s))$ is a tempered $\mathbb{H}$–module (in the sense of Section 2). It follows that $IM(L(s))$ is unitary and therefore, $L(s)$ is spherical unitary.

Definition 3.2. A spherical parameter of the form $s = \frac{1}{2}\check{h}$ is called *anti-tempered* (or spherical unipotent).

The anti-tempered parameters are unitary and they will play an important role in the determination of the spherical unitary dual.

Note that the distinguished orbits parametrize spherical unitary representations which are the Iwahori–Matsumoto dual of discrete series. They are therefore unitary. I just record them here, each with its corresponding parameter:

$$\begin{array}{ll} F_4 & (\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2}); \\ F_4(a_1) & (\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}); \\ F_4(a_2) & (\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}); \\ F_4(a_3) & (\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}). \end{array}$$

Now assume an orbit $\check{\mathcal{O}}$ is not distinguished and let $s \in \check{\mathfrak{a}}$ be a parameter corresponding to $\check{\mathcal{O}}$. Then $\check{\mathcal{O}}$ meets $\check{\mathfrak{m}}$, the Levi component of a parabolic subalgebra, in a distinguished orbit $\check{\mathcal{O}}_M$. More precisely, $\check{\mathfrak{m}}$ is the centralizer in $\check{\mathfrak{g}}$ of a Cartan subalgebra of $\mathfrak{z}(\check{\mathcal{O}})$. Let $M \subset G$ be the Levi subgroup whose Lie algebra $\mathfrak{m}$ has $\check{\mathfrak{m}}$ as its dual. Let $L_M(s)$ be the irreducible spherical module of $\mathbb{H}_M$ parametrized by s and $\check{\mathcal{O}}_M$.

Theorem 3.3 ([BM1]). *The spherical Langlands quotient $L(s)$ parametrized by $\check{\mathcal{O}}$ is the unique spherical subquotient of*

$$X_M(s) := I(M, L_M(s)) = \mathbb{H} \otimes_{\mathbb{H}_M} L_M(s).$$

3.2. **Maximal Parabolics Cases.** As the starting case for the determination of the spherical unitary dual, I consider the modules which are Iwahori–Matsumoto duals induced from discrete series on the Levi component of some maximal parabolic tensored with a character ν. These modules are parametrized by nilpotent orbits which meet the Levi component of a maximal parabolic subalgebra in a distinguished nilpotent orbit. They will be refered to as *maximal parabolic cases.*

In the case when the Hecke algebra is of type F_4, the maximal parabolic cases correspond to the nilpotent orbits B_3, C_3, $A_2+\widetilde{A}_1$, $\widetilde{A}_2+A_1$ and $C_3(a_1)$. The notation is the same as in Bala–Carter's classification of nilpotent orbits in the exceptional Lie algebras (see [Ca]). The Levi components of the maximal parabolic subalgebras are parametrized by the root subsystems of type B_3, C_3 and A_2+A_1 and the $\widetilde{\ }$ stands for short roots. Each nilpotent meets the Levi component in the principal nilpotent, except $C_3(a_1)$, where the nilpotent orbit in the Lie algebra of type C_3 is (42).

The intertwining operator calculations are done exclusively with the K-types 1_1, 2_3, 4_2, 8_1 and 9_1. The notation is from [K] and the explicit description of each irreducible $W(F_4)$–representation is given in the Appendix. I just note here that in the notation d_k is for a Weyl representation, d is the degree of the representation and that 1_1 and 4_2 are the trivial, respectively the reflection representations of $W(F_4)$.

Following [B2], these K-types will be called *relevant.*

Lemma 3.4. *The K-types 1_1, 2_3, 4_2, 8_1 and 9_1 are a minimal set for determining the unitarity of the spherical parameters in the maximal parabolic cases.*

Proof. The proof consists of checking each maximal parabolic case separately. I construct explicit matrix realizations of the relevant K-types using the descriptions of the K-types as given in the Appendix. The long Weyl element of type F_4 has a reduced decomposition $w_0 = s_1 s_2 \cdots s_{24}$ and, as explained in Section 2, it gives rise to operators on each K-type.

For each nilpotent orbit $\check{\mathcal{O}}$ in the five maximal parabolic cases, the parameter is of the form $s = \frac{1}{2}\check{h}+\nu$, where ν is a real number, which can be assumed nonnegative.

Fix a K-type σ. For σ, the calculation comes down to a multiplication of 24 matrices of dimension $\dim(\sigma)$ with a single parameter $\nu \in \mathbb{R}^+$. Since in any standard module for the maximal parabolic cases, the multiplicity of a relevant K-type in the Langlands quotient is at most two, it is easy enough to determine explicitly the nonzero eigenvalues of these matrices. Recall that by Remark 2.8, it is sufficient to consider (in the spherical case) the long intertwining operator. For all explicit computations, I used the software "Mathematica".

We construct the induced modules $X_M(\nu) := I(M, V \otimes \nu)$, in each of the maximal cases. Explicitly, when M is of types B_3 and $A_2 + A_1$, V is the trivial module of the graded Hecke algebra $\mathbb{H}_M$. When M is of type C_3, there are two cases. If the nilpotent orbit is C_3, V is again trivial. If the nilpotent is $C_3(a_1)$, V is the spherical representation (IM dual of a discrete series) parametrized by the nilpotent orbit (42) in the Lie algebra of type C_3. As a $W(C_3)$–representation, V decomposes into $3 \times 0 + 2 \times 1$.

With these constructions, the spherical Langlands quotient $L(s)$, which is parametrized by $\check{\mathcal{O}}$ ($s = \frac{1}{2}\check{h} + \nu$), is actually equal to $X_M(\nu)$, for ν such that $X_M(\nu)$ is irreducible. At $\nu = 0$, $X_M(\nu)$ is irreducible and unitary, being anti-tempered. Therefore, it has to remain unitary until the first point of reducibility. Recall that from the Kazhdan–Lusztig classification, we know that at any reducibility point, the spherical parameter corresponds to a bigger nilpotent $\check{\mathcal{O}}'$, $\check{\mathcal{O}} \subset \overline{\check{\mathcal{O}}'}$.

The first point of reducibility is $\nu = \frac{1}{2}$ for C_3, $C_3(a_1)$, $\widetilde{A}_2 + A_1$ and $A_2 + \widetilde{A}_1$, and $\nu = 1$ for B_3.

The same method applies for all five nilpotents: beyond the first nonzero reducibility point, I show that the intertwining operator is not positive semidefinite on at least one of the relevant K-types. I include the tables of signatures for the nonzero eigenvalues of the operators induced by the long intertwining operator on the five relevant K-types.

$\mathbf{B_3}$:

ν	1_1	4_2	9_1	8_1
	$+$	$+$	$+$	$+$
1	$+$	$+$	$+$	0
	$+$	$+$	$+$	$-$
2	$+$	$+$	0	0
	$+$	$+$	$-$	$+$
4	$+$	0	0	0
	$+$	$-$	$+$	$-$

$\mathbf{C_3}$:

ν	1_1	4_2	9_1	2_3
	$+$	$+$	$+$	$+$
$\frac{1}{2}$	$+$	$+$	$+$	0
	$+$	$+$	$+$	$-$
$\frac{5}{2}$	$+$	$+$	0	$-$
	$+$	$+$	$-$	$-$
$\frac{11}{2}$	$+$	0	0	0
	$+$	$-$	$+$	$+$

$\mathbf{C_3(a_1)}$:

ν	1_1	4_2	4_2	2_3	9_1	9_1	8_1
	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\frac{1}{2}$	$+$	$+$	$+$	$+$	$+$	0	$+$
	$+$	$+$	$+$	$+$	$+$	$-$	$+$
$\frac{5}{2}$	$+$	$+$	0	0	$+$	0	0
	$+$	$+$	$+$	$-$	$+$	$-$	$-$
$\frac{7}{2}$	$+$	$+$	0	0	0	0	0
	$+$	$+$	$-$	$+$	$-$	$+$	$+$

$\mathbf{\widetilde{A}_1+A_2}$:

ν	1_1	4_2	9_1	9_1	8_1	8_1
	+	+	+	+	+	+
$\frac{1}{2}$	+	+	0	+	0	+
	+	+	−	+	+	+
1	+	+	0	+	0	+
	+	+	+	+	−	+
$\frac{3}{2}$	+	+	0	0	0	0
	+	+	−	−	+	+
$\frac{5}{2}$	+	0	0	0	0	0
	+	−	+	+	−	−

$\mathbf{A_1+\widetilde{A}_2}$:

ν	1_1	4_2	2_3	9_1	9_1	8_1
	+	+	+	+	+	+
$\frac{1}{2}$	+	+	+	+	0	+
	+	+	+	+	−	+
$\frac{3}{2}$	+	+	0	+	0	0
	+	+	−	+	−	−
$\frac{5}{2}$	+	+	0	0	−	0
	+	+	−	−	−	+
$\frac{7}{2}$	+	0	0	0	0	0
	+	−	+	+	+	−

In this way, I obtain the set of spherical unitary parameters (the complementary series) for each nilpotent orbit in the maximal parabolic cases:

B_3	$(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$	$0\le\nu<1$;
C_3	$(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	$0\le\nu<\frac{1}{2}$;
$C_3(a_1)$	$(\nu,\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$0\le\nu<\frac{1}{2}$;
$\widetilde{A}_1+A_2$	$(\frac{1}{2}+2\nu,\nu,-1+\nu,\frac{1}{2})$	$0\le\nu<\frac{1}{2}$;
$A_1+\widetilde{A}_2$	$(\frac{1}{4}+\frac{3\nu}{2},\frac{3}{4}+\frac{\nu}{2},-\frac{1}{4}+\frac{\nu}{2},-\frac{5}{4}+\frac{\nu}{2})$	$0\le\nu<\frac{1}{2}$.

□

3.3. **Main Results.** With the maximal parabolic cases done, one can determine the unitarity of the spherical parameters for each of the remaining nilpotent orbits. The main result follows. The explicit description of the complementary series for each $\check{\mathcal{O}}$ will be given in the proof and recorded again in Section 5. Recall that we restrict (as we may) to the case of *real* parameters.

Theorem 3.5. *Consider the graded Hecke algebra $\mathbb{H}$ of type F_4.*

(1) *Let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{\mathfrak{g}}$ and $s=\frac{1}{2}\check{h}+\nu$ a (real) spherical parameter associated to $\check{\mathcal{O}}$, where ν is a semisimple element in $\mathfrak{z}(\check{\mathcal{O}})$.*
   (a) *If $\check{\mathcal{O}}\neq A_1+\widetilde{A}_1$, s is in the complementary series of $\check{\mathcal{O}}$ if and only if ν is in the complementary series attached to the trivial nilpotent orbit in $\mathfrak{z}(\check{\mathcal{O}})$.*
   (b) *If $\check{\mathcal{O}}=A_1+\widetilde{A}_1$ and if s is in the complementary series of $\check{\mathcal{O}}$, then ν is in the complementary series attached to the trivial nilpotent orbit in $\mathfrak{z}(\check{\mathcal{O}})$, but the converse is false.*

(2) *The complementary series associated to the trivial nilpotent with dominant (real) infinitesimal character $(\nu_1,\nu_2,\nu_3,\nu_4)$, $\nu_1\ge\nu_2\ge\nu_3\ge\nu_4\ge0$, $\nu_1-\nu_2-\nu_3-\nu_4\ge0$ are:*
   (a) $\left\{\nu_1<\frac{1}{2}\right\}$;
   (b) $\{\nu_1+\nu_2+\nu_3+\nu_4>1,\ \nu_1+\nu_2+\nu_3-\nu_4<1\}$.

Proof. The proof is based on the following induction: for a fixed nilpotent orbit $\check{\mathcal{O}}$, one divides the parameter space into open regions determined by the hyperplanes where the standard module is reducible. Assume first that the nilpotent orbit has trivial component group $A(\check{e})$, and so, the standard module is irreducible at the origin (i.e, for $s=\frac{1}{2}\check{h}$). On any reducibility hyperplane, the spherical module corresponds to a bigger nilpotent in the closure ordering for which I have already found the unitary parameters. In this way one can rule out the regions which are bounded by hyperplanes with nonunitary parameters. In the end, there only remain parameters close to the origin, in regions that are bounded by unitary walls. For these, I show that they can be deformed irreducibly to parameters which are unitarily induced irreducible from unitary parameters of classical groups of rank less than four.

There is an extra difficulty for nilpotents $\check{\mathcal{O}}$ with nontrivial component group. In these cases, the standard module $I(M, triv \otimes \nu) = \mathrm{Ind}_{\mathbb{H}_M}^{\mathbb{H}}(triv \otimes \mathbb{C}_\nu)$ is reducible at the origin (see Section 2.3) and one finds reducibility hyperplanes through the origin where the spherical factor is still parametrized by $\check{\mathcal{O}}$. The method outlined above is not sufficient and one needs extra calculations with the long intertwining operator on the relevant K-types in order to rule out some nonunitary regions close to the origin or on the reducibility hyperplanes parametrized by $\check{\mathcal{O}}$.

For the nilpotent orbits in the maximal cases (all having the centralizer of type A_1), one can see from the previous calculations that the (one-dimensional) complementary series are the same as those for the centralizers. The method outlined above is best illustrated in the case of the nilpotent orbits which admit a two-dimensional parameter. For them, I present pictures with the reducibility lines. In these pictures, the red lines represent nonunitary spherical factors and the green lines the unitary spherical factors. Any open region bounded by some red line is necessarily nonunitary.

In the case of nilpotents A_1 and $\widetilde{A}_1$, the arguments are more involved, as these two nilpotents admit a three-dimensional parameter. $\widetilde{A}_1$ has only a two-dimensional complementary series. The proof of this fact is easy if one uses the signatures of the two lowest K-types associated to this nilpotent orbit (the definition and the argument will be presented in Section 4), but more difficult if we restrict to relevant K-types only.

For all the nilpotent orbits, I present the infinitesimal characters, reducibility hyperplanes with the nilpotent orbit parametrizing the spherical factor on each such hyperplane and the complementary series. I also show how the infinitesimal character in the complementary series can be deformed without reducibility to unitarily induced modules from smaller rank groups.

The cases of the parameters associated to the trivial nilpotent orbit and the orbits A_1 and $\widetilde{A}_1$ will be presented in more detail.

$\mathbf{B_2}$: infinitesimal character $(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$, $\nu_1 \geq \nu_2 \geq 0$. The reducibility lines are: $\nu_1 = \frac{1}{2}$ and $\nu_2 = \frac{1}{2}$ from $C_3(a_1)$, $\nu_1 + \nu_2 = 3$ and $\nu_1 - \nu_2 = 3$ from B_3, $\nu_1 = \frac{5}{2}$ and $\nu_2 = \frac{5}{2}$ from C_3, $\nu_1 = \nu_2$ where the spherical module is still parametrized by B_2. The complementary series is $\{0 \leq \nu_2 \leq \nu_1 < \frac{1}{2}\}$. On the line $\nu_2 = 0$ and $0 \leq \nu_1 < \frac{1}{2}$ the module is unitarily induced irreducible from a complementary series associated to the nilpotent (411) in C_3 (see Figure 1).

$\mathbf{A_2}$: infinitesimal character $(\frac{1}{2} + \nu_1 + \nu_2, -\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_2, \frac{1}{2})$, $\nu_1 \geq \nu_2 \geq 0$. Reducibility lines: $2\nu_1 + \nu_2 = 1$, $\nu_1 + 2\nu_2 = 1$ and $\nu_1 - \nu_2 = 1$ from $\widetilde{A}_1 + A_2$, $\nu_1 + \nu_2 = 1$, $\nu_1 = 1$ and $\nu_2 = 1$ from B_2, $\nu_1 + \nu_2 = 2$, $\nu_1 = 2$ and $\nu_2 = 2$ from B_3 and finally $\nu_2 = 0$, where the spherical module is parametrized by A_2. The complementary series is $\{\nu_2 = 0,\ 0 \leq \nu_1 < \frac{1}{2}\}$. At the origin, the spherical module is unitarily induced irreducible from the trivial in A_2 (see Figure 2).

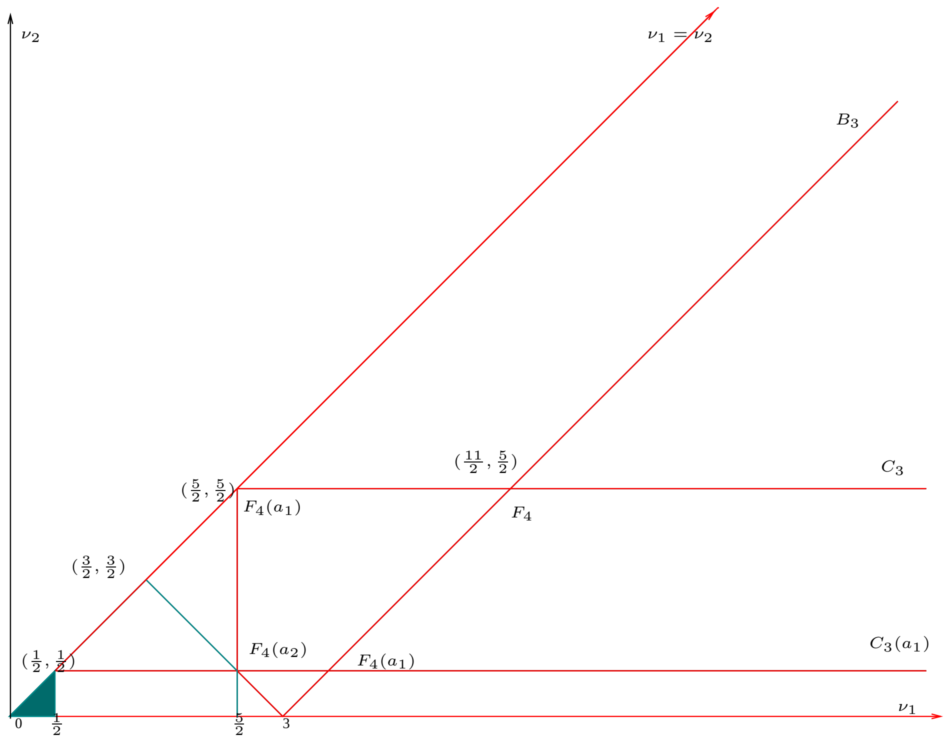

FIGURE 1. Spherical unitary dual attached to the orbit B_2

$\widetilde{\mathbf{A}}_{\mathbf{2}}$: infinitesimal character $(\nu_2 + \frac{3\nu_1}{2}, 1 + \frac{\nu_1}{2}, \frac{\nu_1}{2}, -1 + \frac{\nu_1}{2}),\ \nu_1 \geq 0,\ \nu_2 \geq 0$. Reducibility lines: $2\nu_2 + 3\nu_1 = 1$, $\nu_2 + 3\nu_1 = 1$, $\nu_2 = 1$ from $A_1 + \widetilde{A}_2$, $\nu_2 + 2\nu_1 = 1$, $\nu_2 + \nu_1 = 1$ and $\nu_1 = 1$ from $C_3(a_1)$, $\nu_2 + 2\nu_1 = 3$, $\nu_2 + \nu_1 = 3$ and $\nu_1 = 3$ from C_3. The complementary series is $\{2\nu_2 + 3\nu_1 < 1\}$ and $\{\nu_2 + 2\nu_1 < 1 < \nu_2 + 3\nu_1\}$. On the line $\nu_2 = 0$, for $0 \leq \nu_1 < \frac{1}{2}$, the parameter is unitarily induced irreducible from a complementary series associated to the nilpotent (33) in C_3 (see Figure 3).

$\mathbf{A_1} + \widetilde{\mathbf{A}}_{\mathbf{1}}$: infinitesimal character $(\nu_1, \frac{1}{2} + \nu_2, -\frac{1}{2} + \nu_2, \frac{1}{2})$, $\nu_1 \geq 0$, $\nu_2 \geq 0$. Reducibility lines: $\nu_1 = \frac{1}{2}$ from A_2, $\nu_1 = \frac{3}{2}$ from B_2, $\nu_1 = \frac{5}{2}$ from B_3, $\nu_2 = 1$ from $C_3(a_1)$, $\nu_2 = 2$ from C_3, $\nu_1 - 2\nu_2 = -\frac{3}{2}$, $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $\nu_1 - 2\nu_2 = \frac{3}{2}$ from $\widetilde{A}_1 + A_2$, $\nu_1 - \nu_2 = -\frac{3}{2}$, $\nu_1 + \nu_2 = \frac{3}{2}$ and $\nu_1 - \nu_2 = \frac{3}{2}$ from $A_1 + \widetilde{A}_2$. The complementary series is $\{\nu_1 + 2\nu_2 < \frac{3}{2},\ \nu_1 < \frac{1}{2}\}$ and $\{2\nu_2 - \nu_1 > \frac{3}{2},\ \nu_2 < 1\}$. On the line $\nu_1 = 0$, $0 \leq \nu_2 < \frac{3}{4}$ and $\frac{3}{4} < \nu_2 < 1$, the standard module is unitarily induced irreducible from a complementary series associated to the nilpotent (222) in C_3 (see Figure 4).

$\widetilde{\mathbf{A}}_{\mathbf{1}}$: infinitesimal character $(\nu_1, \nu_2, \frac{1}{2} + \nu_3, -\frac{1}{2} + \nu_3)$, $\nu_1 \geq \nu_2 \geq 0$, $\nu_3 \geq 0$. The planes of reducibility are:

$\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{1}{2}$ and $\nu_1 \pm \nu_2 \pm 2\nu_3 = \pm 1$	from $A_1 + \widetilde{A}_1$;
$\nu_3 = 1$ and $\nu_1 \pm \nu_2 = 2$	from B_2;
$\nu_1 \pm \nu_3 = \pm\frac{3}{2}$, $\nu_2 \pm \nu_3 = \pm\frac{3}{2}$	from $\widetilde{A}_2$.

Also, there is reducibility on the plane $\nu_3 = 0$ (and planes conjugate to it), but the spherical factor is still parametrized by $\widetilde{A}_1$.

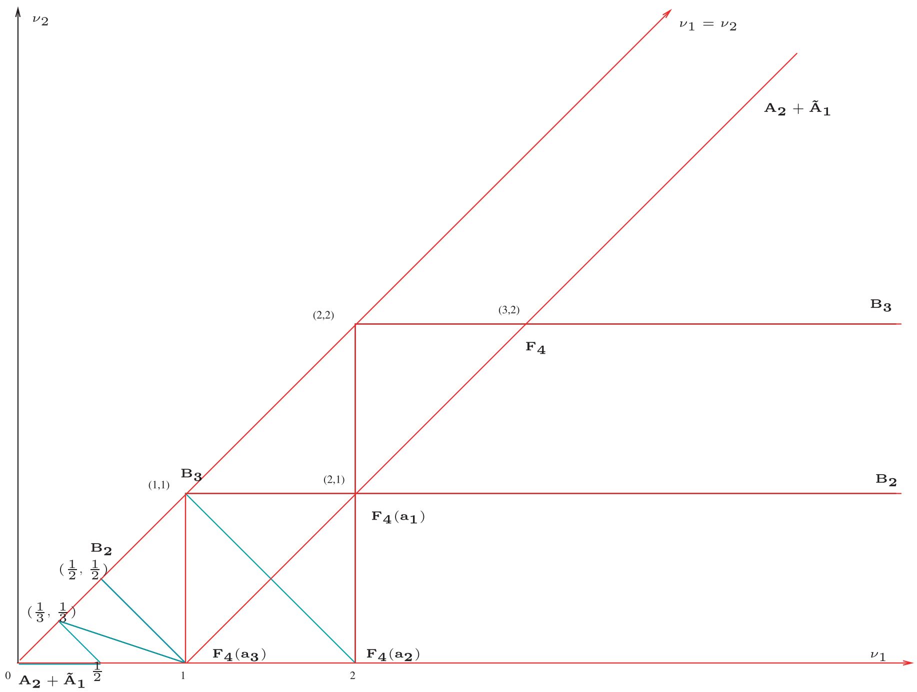

FIGURE 2. Spherical unitary dual attached to the orbit A_2

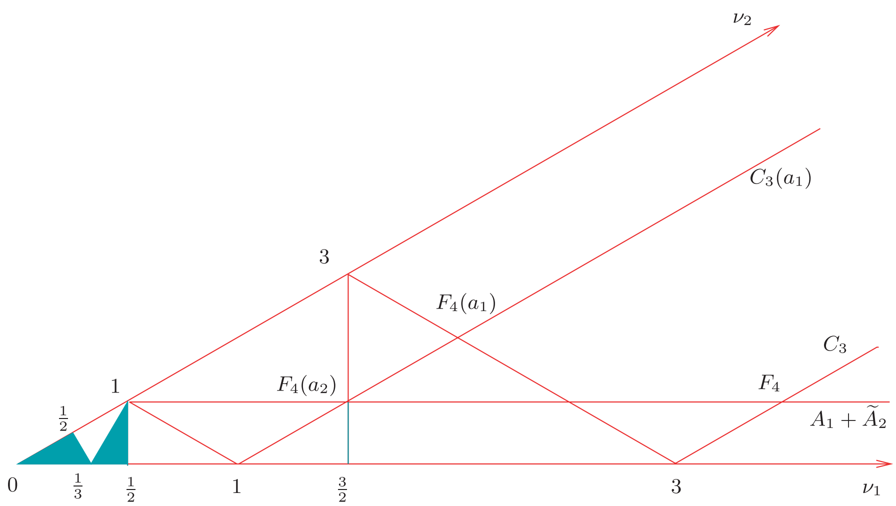

FIGURE 3. Spherical unitary dual attached to the orbit $\widetilde{A}_2$

I will show that the infinitesimal character can only be unitary on the plane $\nu_3 = 0$. This is done as follows.

Assume the parameter is unitary, with the corresponding standard module irreducible and $\nu_3 > 0$. Then one can deform the parameter to the closest reducibility hyperplane, but keeping $\nu_3 > 0$; the parameter has to remain unitary. On each

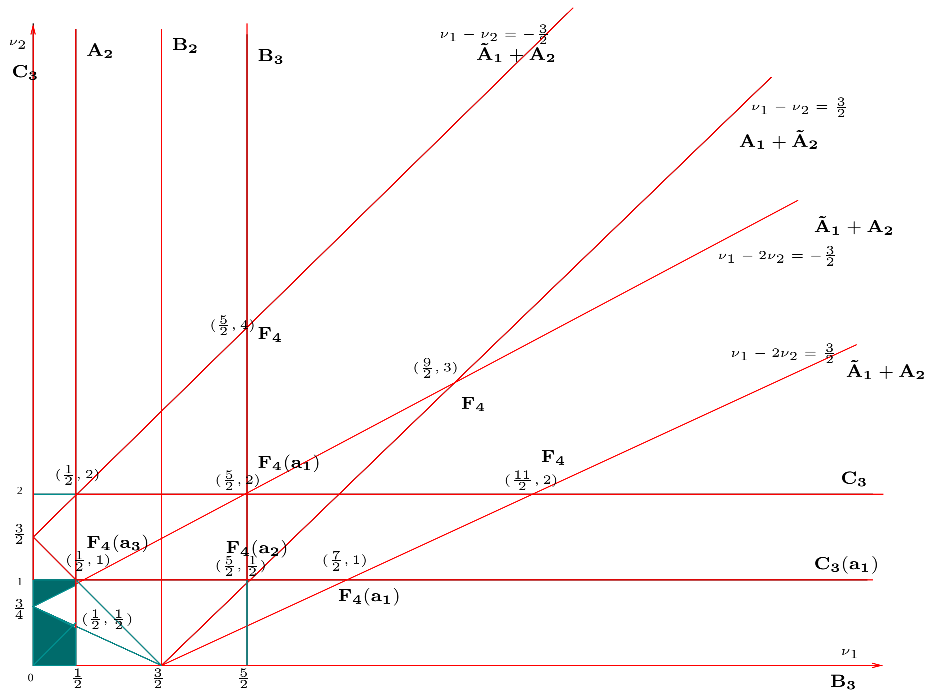

FIGURE 4. Spherical unitary dual attached to the orbit $A_1 + \widetilde{A}_1$

of the reducibility hyperplanes on which the spherical factor is parametrized by a strictly bigger nilpotent, one knows which parameters are unitary. Checking the parameters on each of these hyperplanes, it follows that the only hyperplanes that could bound a (three-dimensional) unitary region are (the parameters are listed by the nilpotent orbit to which the spherical factor belongs):

B_2	$\nu_3 = 1$	with $0 \leq \nu_1 < \frac{1}{2}$
	$\nu_1 + \nu_2 = 2$	with $1 < \nu_1 < \frac{3}{2} - \nu_3$
$A_1 + \widetilde{A}_1$	$\nu_1 = \frac{1}{2}$	with $0 < \nu_3 < 1$
	$\nu_1 + \nu_2 + 2\nu_3 = 1$	with $\nu_1 + \nu_2 < 1$
$\widetilde{A}_2$	$\nu_1 + \nu_3 = \frac{3}{2}$	with $\frac{3}{4} < \nu_1 < \frac{5}{4}$
	$\nu_2 + \nu_3 = \frac{3}{2}$	with $\frac{3}{4} < \nu_2 < \frac{5}{4}$

Assume again that the parameter $(\nu_1, \nu_2, \frac{1}{2} + \nu_3, -\frac{1}{2} + \nu_3)$ belongs to a (three-dimensional) unitary region. The claim is that $\nu_1 + \nu_3 < \frac{3}{2}$. Assume that $\nu_1 + \nu_3 > \frac{3}{2}$. Then one can deform the parameter ν_1 upward, leaving ν_2 and ν_3 fixed, and it cannot hit any of the unitary facets listed above. This is because the unitary facets involving ν_1 have the property that $\nu_1 + \nu_3 < \frac{3}{2}$.

Therefore in order for a parameter associated to $\widetilde{A}_1$ to be unitary, it is necessary that $\nu_1 + \nu_3 < \frac{3}{2}$.

Now, I use a direct calculation involving the relevant K-types 2_3 and 4_2. Both appear with multiplicity 2 in the standard module induced from $\widetilde{A}_1$. Denote by

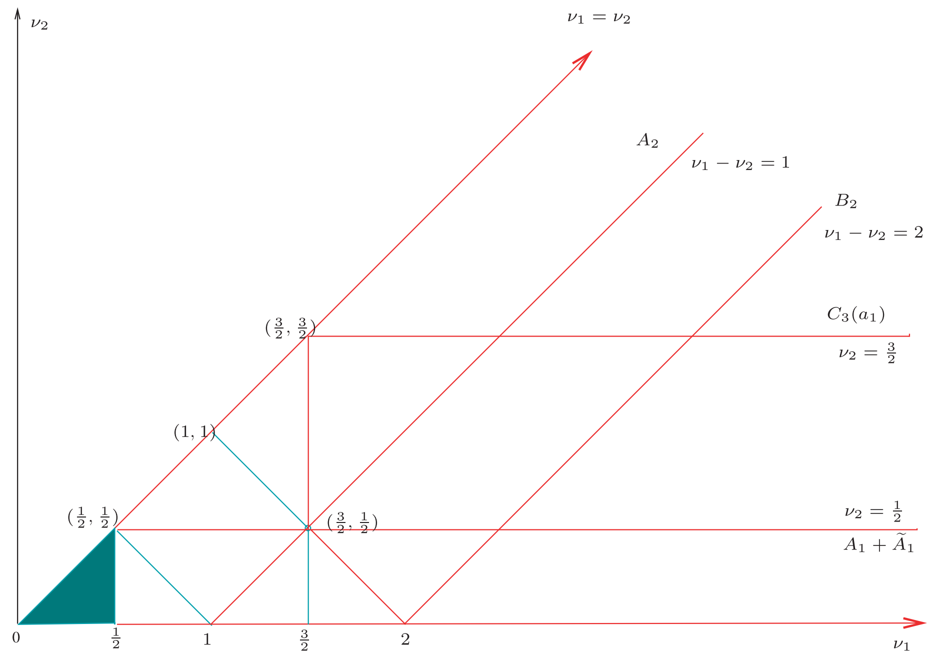

FIGURE 5. Spherical unitary dual attached to the orbit $\widetilde{A}_1$

$prod(\sigma)$, the product of the nonzero eigenvalues of the long intertwining operator on the K-type σ. The ratio $prod(4_2)/prod(2_3)$ is

$$\frac{prod(4_2)}{prod(2_3)} = -\frac{(\frac{3}{2} - \nu_1 - \nu_3)(\frac{3}{2} - \nu_1 + \nu_3)(\frac{3}{2} - \nu_2 - \nu_3)(\frac{3}{2} - \nu_2 + \nu_3)}{(\frac{3}{2} + \nu_1 + \nu_3)(\frac{3}{2} + \nu_1 - \nu_3)(\frac{3}{2} + \nu_2 + \nu_3)(\frac{3}{2} + \nu_2 - \nu_3)},$$

which shows that the region $\nu_1 + \nu_3 < \frac{3}{2}$ must be nonunitary.

This argument implies that the only possible unitarity is on the plane $\nu_3 = 0$.

The complementary series, $\{\nu_3 = 0,\ 0 \le \nu_2 \le \nu_1 < \frac{1}{2}\}$, is unitarily induced irreducible from a complementary series associated to the nilpotent (31^4) in B_3 (see Figure 5).

$\mathbf{A_1}$: infinitesimal character $(\nu_1, \nu_2, \nu_3, \frac{1}{2})$, $\nu_1 \ge \nu_2 \ge \nu_3 \ge 0$. The reducibility planes are

$$\begin{array}{ll} \nu_i = \frac{1}{2},\ i = 1,2,3 & \text{from } \widetilde{A}_1; \\ \nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2} & \text{from } A_2; \\ \nu_i = \frac{3}{2},\ i = 1,2,3 & \text{from } B_2; \\ \nu_i \pm \nu_j = 1,\ 1 \le i < j \le 3 & \text{from } A_1 + \widetilde{A}_1. \end{array}$$

I will show that the complementary series is $0 \le \nu_3 \le \nu_2 \le \nu_1 < \frac{1}{2}$. On the plane $\nu_3 = 0$, $0 \le \nu_2 \le \nu_1 < \frac{1}{2}$, the infinitesimal character is unitarily induced irreducible from a complementary series associated to the nilpotent (21^4) in C_3.

One immediate observation is that all the reducibility hyperplanes on which the spherical factor is parametrized by A_2 cannot bound any three-dimensional unitary region. This is because the complementary series associated to A_2 has dimension one only. On the rest of reducibility hyperplanes, the spherical factor is parametrized by one of nilpotent orbits $\widetilde{A}_1$, B_2 and $A_1+\widetilde{A}_1$. Using the complementary series for these nilpotent orbits it follows that the unitary three-dimensional regions could be bounded by the following hyperplanes (listed by the nilpotent orbit parametrizing the spherical factor):

$\widetilde{A}_1$	$\nu_1=\frac{1}{2}$	with $0\le\nu_3\le\nu_2<\frac{1}{2}$
B_2	$\nu_1=\frac{3}{2}$	with $0\le\nu_3\le\nu_2<\frac{1}{2}$
$A_1+\widetilde{A}_1$	$\nu_1-\nu_2=1$	with $2\nu_2+\nu_3<\frac{1}{2}$ or $\{2\nu_2-\nu_3>\frac{1}{2},\ 0\le\nu_2<\frac{1}{2}\}$
	$\nu_1+\nu_2=1$	with $0\le\nu_3\le\nu_2<\frac{1}{2}$
	$\nu_1-\nu_3=1$	with $0\le\nu_3\le\nu_2<\frac{1}{2}$
	$\nu_1+\nu_3=1$	with $0\le\nu_3\le\nu_2<\frac{1}{2}$

Assume that the parameter $(\nu_1,\nu_2,\nu_3,\frac{1}{2})$ is unitary and the corresponding standard module is irreducible. In particular, the walls of this region can only be among the 6 hyperplanes listed above.

The first step is to show that $\nu_2<\frac{1}{2}$. Assume $\nu_2>\frac{1}{2}$. If $\nu_1<\frac{3}{2}$, deform $\nu=\nu_1$ upward. The first reducibility wall that can be met is one of the following: $\nu_1=\frac{3}{2}$, $\nu_1-\nu_3=1$, $\nu_1+\nu_3=1$ and $\nu_1-\nu_2=1$. On each of these hyperplanes the corresponding parameter is nonunitary because $\nu_2>\frac{1}{2}$. Now, if the case is $\nu_1>\frac{3}{2}$, $\nu_2>\frac{1}{2}$, move $\nu=\nu_1$ downward. The first reducibility wall must be one of the following: $\nu_1=\frac{3}{2}$, $\nu_1-\nu_3=1$ or $\nu_1-\nu_2=1$, but on these the corresponding parameter is nonunitary as before.

From now on, I consider $0\le\nu_3\le\nu_2<\frac{1}{2}$.

If $\nu_1<\frac{1}{2}$, one can deform ν_3 to zero without any reducibility. The parameter becomes $(0,\nu_1,\nu_2,\frac{1}{2})$ which is unitarily induced irreducible from (21^4) in C_3 and as $0\le\nu_2\le\nu_1<\frac{1}{2}$, it is unitary.

I want to show that this is the only unitary region associated with A_1. Assume $0\le\nu_3\le\nu_2<\frac{1}{2}<\nu_1$. Move $\nu'=\nu_3$ toward zero. If it gets to zero without passing any reducibility point, the parameter is nonunitary being unitarily induced irreducible from some nonunitary parameter in C_3. Note that if there is reducibility, this cannot involve ν_2, as $0<\nu_3<\nu_2<\frac{1}{2}$. The only cases of reducibility involve ν_1 and they are

$\nu_1>1$ and $\nu_3>\nu_1-1$ which implies $0<\nu_1-1<\nu_3\le\nu_2<\frac{1}{2}<1<\nu_1$,

$\nu_1<1$ and $\nu_3>1-\nu_1$ which implies $0<1-\nu_1<\nu_3\le\nu_2<\frac{1}{2}<\nu_1<1$.

In both of these cases one can move $\nu'=\nu_3$ to ν_2 and no reducibility occurs. The resulting parameter $(\nu_2,\nu_2,\nu_1,\frac{1}{2})$ is conjugate to $(\nu_2,-\nu_2,\nu_1,\frac{1}{2})$ which is unitarily induced irreducible from B_3. B_3 is given by the roots α_1, α_2 and α_3 and if one changes the coordinates into the standard coordinates for type B_3, the parameter becomes $(2\nu_2,\frac{1}{2}+\nu_1,-\frac{1}{2}+\nu_1)$. This is nonunitary since $\nu_1>\frac{1}{2}$, which is a contradiction. This completes the analysis in this case.

1: Finally, the complementary series associated to the trivial is determined as follows. The full induced from the trivial has parameter $(\nu_1,\nu_2,\nu_3,\nu_4)$ with $\nu_1\ge\nu_2\ge\nu_3\ge\nu_4\ge0$. The most important observation is that any region bounded by a wall on which a short root is 1 is *not* unitary. On any such wall, there is a factor coming from $\widetilde{A}_1$ which can't be unitary at all points since the complementary series for $\widetilde{A}_1$ is two-dimensional.

Therefore, one needs to only look at regions bounded by long roots. Moreover, regions bounded by any of the following hyperplanes: $\nu_2 = \frac{1}{2}$, $\nu_3 = \frac{1}{2}$, $\nu_4 = \frac{1}{2}$ cannot be unitary because on these hyperplanes the factor corresponding to A_1 has nonunitary parameter ($\nu_1 > \frac{1}{2}$). So it remains to check $\nu_1 = \frac{1}{2}$ and the following hyperplanes:

$$
\begin{array}{lll}
(1) & \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 = 1, & \nu_1 - \nu_2 - \nu_3 - \nu_4 = 1,\\
(2) & \epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 = 1, & \nu_1 - \nu_2 - \nu_3 + \nu_4 = 1,\\
(3) & \epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4 = 1, & \nu_1 - \nu_2 + \nu_3 - \nu_4 = 1,\\
(4) & \epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4 = 1, & \nu_1 - \nu_2 + \nu_3 + \nu_4 = 1,\\
(5) & \epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4 = 1, & \nu_1 + \nu_2 - \nu_3 - \nu_4 = 1,\\
(6) & \epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4 = 1, & \nu_1 + \nu_2 - \nu_3 + \nu_4 = 1,\\
(7) & \epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4 = 1, & \nu_1 + \nu_2 + \nu_3 - \nu_4 = 1,\\
(8) & \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 1, & \nu_1 + \nu_2 + \nu_3 + \nu_4 = 1.
\end{array}
$$

The above hyperplanes are listed in the partial ordering of the roots: if α and β are two positive roots, then $\alpha > \beta$ if and only if $\alpha - \beta$ is a sum of positive roots.

One can show, in a case-by-case analysis, that none of the hyperplanes (1)–(6) can bound a unitary region, since on them the factor coming from A_1 has nonunitary parameter.

It follows that any unitary region can only have as reducibility walls the following hyperplanes: $\nu_1 = \frac{1}{2}$, $\nu_1 + \nu_2 + \nu_3 - \nu_4 = 1$ and $\nu_1 + \nu_2 + \nu_3 + \nu_4 = 1$. This implies right away that any unitary region has to satisfy $\nu_1 + \nu_2 + \nu_3 - \nu_4 < 1$.

a) Assume $\nu_1 + \nu_2 + \nu_3 + \nu_4 < 1$. Then $\nu_2 < \frac{1}{2}$. Move ν_4 toward zero, no reducibility can occur and at zero, the parameter is $(0, \nu_1, \nu_2, \nu_3)$ which is unitarily induced irreducible from C_3. It is unitary iff $\nu_1 < \frac{1}{2}$ (this comes from the complementary series of C_3). The first unitary region is then

$$\nu_1 < \frac{1}{2}.$$

b) $\nu_1 + \nu_2 + \nu_3 + \nu_4 > 1$, but $\nu_1 + \nu_2 + \nu_3 - \nu_4 < 1$. Again $\nu_2 < \frac{1}{2}$. Move $\nu = \nu_4$ up toward ν_3, no reducibility can appear (à priori the only reducibility could come from short roots involving ν_1, but both $\nu_1 - \nu_3 < 1 < \nu_1 - \nu_4$ and $\nu_1 + \nu_4 < 1 < \nu_1 + \nu_3$ are impossible). At $\nu = \nu_3$ the parameter, $(\nu_3, -\nu_3, \nu_1, \nu_2)$, is unitarily induced irreducible from B_3. Inside B_3 the parameter looks like $(2\nu_3, \nu_1 + \nu_2, \nu_1 - \nu_2)$ and because $\nu_1 + \nu_2 + 2\nu_3 > 1$, it is unitary iff $\nu_1 > \frac{1}{2}$ (note that this condition is automatically satisfied since the parameter is dominant).

The second unitary region is a complementary series from B_3:

$$\nu_1 + \nu_2 + \nu_3 + \nu_4 > 1,\ \nu_1 + \nu_2 + \nu_3 - \nu_4 < 1.$$

$\square$

From the proof, it follows immediately:

Corollary 3.6. *The K-types from Lemma* 3.4 *are sufficient for the determination of the spherical unitary dual of type F_4.*

Also, one can reformulate the classification of the spherical dual presented in the proof in an analogous way to the results for classical groups in [BM3]:

Theorem 3.7. *A spherical parameter s associated to a nilpotent orbit $\check{\mathcal{O}}$ in type F_4 is unitary if and only if it can be deformed irreducibly to a parameter which is induced irreducible from an antitempered module (on the affine graded Hecke algebra of a Levi subgroup) tensored with a parameter in a GL-complementary series.*

3.4. **Computer Calculations.** I conclude this section with some remarks about the calculation of the spherical dual of F_4 by computer. As mentioned in Section 2, Remark 2.8 reduces the determination of the spherical unitary dual to the computation of operators $r_\sigma(w_0, s)$, s an element in the dominant Weyl chamber and σ any irreducible W–representation. One can therefore use the following method:

(1) find rational matrix realizations for all Weyl representations σ;
(2) partition the dominant Weyl chamber into (a finite number of) cells coming from the arrangement of hyperplanes $\langle s, \alpha\rangle = 1$, or 0, for $\alpha \in R^+$ and choose in each cell a point s with rational entries;
(3) compute the operator $r_\sigma(w_0, s)$ on each sample point s from step (2) and find the signature of this operator;
(4) keep only the cells for which the corresponding sample points give positive semidefinite operators in *all* representations σ. This set of cells describes the spherical unitary dual inside the dominant Weyl chamber.

I mention that the results of the present paper were completed in September 2002. The computational method explained above was applied by J. Adams, J. Stembridge and J.-K. Yu in an effort to determine by computer the spherical unitary dual of split p–adic exceptional groups. As a result, they obtained, in 2003, a description of the spherical unitary dual for F_4. Their answer matches perfectly the description of the spherical unitary dual presented in this section (note, however, that, in practice, a “translation” between the two forms of the result is not completely straightforward).

4. Unitary $\mathbb{I}$–spherical dual

Theorem 3.7 gives an explanation of how the spherical unitary modules arise. The same kind of a result does not hold for the Iwahori–spherical unitary dual. There are unitary parameters which cannot be explained as deformations from unitarily induced modules coming from smaller groups.

In this section, I determine the full unitary dual of $\mathbb{H}$. The explicit description will be listed for convenience in Section 5. In view of the Kazhdan–Lusztig parametrization of simple $\mathbb{H}$–modules, the unitary dual will be partitioned again by nilpotent orbits $\check{\mathcal{O}}$. I try to match the unitary parameters associated of each nilpotent orbit $\check{\mathcal{O}}$ with the spherical unitary dual of its centralizer, $\mathfrak{z}(\check{\mathcal{O}})$. They will not always be the same and I will emphasize the unitary parameters which do not have a correspondent in the centralizer.

4.1. **Lowest K-types.** Let $X(s, \check{\mathcal{O}})$ be a standard module and $\check{e} \in \check{\mathcal{O}}$ a nilpotent element as in the Kazhdan–Lusztig classification. I recall some facts about the W–structure of standard modules as treated in [BM1].

Let $u = exp(\check{e})$ be the unipotent element in the group $\check{G}$. Consider $\mathcal{B}_u$, the complex variety of Borel subgrups of $\check{G}$ containing u and $H^*(\mathcal{B}_u)$, the cohomology groups of $\mathcal{B}_u$. The component group $A(\check{e})$ acts on $H^*(\mathcal{B}_u)$ and let $H^*(\mathcal{B}_u)^\phi = \mathrm{Hom}_{A(\check{e})}[\phi : H^*(\mathcal{B}_u)]$ be the ϕ–isotypic component of $H^*(\mathcal{B}_u)$, $\phi \in \widehat{A(\check{e})}$. There is an action of W on each $H^*(\mathcal{B}_u)^\phi$ (Springer). If d_u is the dimension of $\mathcal{B}_u$, then

$(H^{d_u}(\mathcal{B}_u))^\phi$ is either zero or it is irreducible as a representation of W. Denote this representation $\sigma(\check{\mathcal{O}},\phi)$. The resulting correspondence $\phi \to \sigma(\check{\mathcal{O}},\phi)$ is the *Springer correspondence*.

As W–representations $X(s,\check{\mathcal{O}}) \cong H^*(\mathcal{B}_u) \otimes sgn$ ([KL]). Then $\widehat{A(s,\check{e})}$ acts on the right-hand side via the inclusion $A(s,\check{e}) \subset A(\check{e})$. Fix $\psi \in \widehat{A(s,\check{e})}$. If $\phi \in \widehat{A(\check{e})}$ appears in the Springer correspondence and ϕ contains ψ in its restriction to $A(s,\check{e})$, then the W–representation $\sigma(\check{\mathcal{O}},\phi) \otimes sgn$ appears with multiplicity one in the standard module $X(s,\check{\mathcal{O}},\psi)$. Following [BM1], I will call these representations ϕ *lowest K-types* for $X(s,\check{\mathcal{O}},\psi)$. They have the property that $L(s,\check{\mathcal{O}},\psi)$ is the unique subquotient of $X(s,\check{\mathcal{O}},\psi)$ which contains the lowest K-types $\sigma(\check{\mathcal{O}},\phi)$.

Moreover, if the parameter is tempered ($s = \frac{1}{2}\check{h}$), then $A(s,\check{e}) = A(\check{e})$ and $X(s,\check{\mathcal{O}},\psi)$ has a unique lowest K-type $\sigma(\check{\mathcal{O}},\psi)$.

4.2. **Unitary dual of $\mathbb{H}$.** From the discussion in the previous section, it follows that it is natural to partition the unitary dual of $\mathbb{H}$ by nilpotent orbits and lowest K-types. I mention that the lowest K-types attached to nilpotent orbits are known, they can be read for example from [Ca]. For each nilpotent in F_4, I would like to determine the unitarity of the factors containing the lowest K-types.

In the action of the Iwahori–Matsumoto involution, modules containing the sign representation are taken into spherical modules. Since this involution preserves unitarity, the complementary series associated to a nilpotent $\check{\mathcal{O}}$ are transformed into unitary modules containing the sign W–representation which are parametrized by $\check{\mathcal{O}}$ in the Kazhdan–Lusztig classification. They give most of the unitary dual of $\mathbb{H}$ associated to $\check{\mathcal{O}}$. Note also that the set of unitary parameters associated to the trivial nilpotent are just the complementary series coming from the spherical case. This is because the lowest K-type of the trivial nilpotent is the trivial W–representation.

I explain the calculations with the intertwining operators in this case. Using the notation in Section 2, if a simple $\mathbb{H}$–module $L(M,V,\nu)$ is Hermitian with $w \in W$ such that $w \cdot (M,V,\nu) = (M,V,-\nu)$, the intertwining operator $I(w,\nu)$ gives rise to an operator $r_\sigma(w,M,\nu)$ on the space $\mathrm{Hom}_{\mathbb{C}[W(M)]}(V_\sigma,V)$, for each K-type (σ,V_σ). I would like to calculate the signature of this operator. Explicitly, the method is the following:

(1) using the description in Appendix A, construct an explicit (matrix) realization for σ;
(2) determine the vectors in V_σ which transform like V under the action of $W(M)$. For almost all cases, V as a $W(M)$–representation is just the sign representations, so one only needs to find the vectors that transform like the sign. The number of linearly independent such vectors is the same as the multiplicity of σ in $X(M,V,\nu)$.
(3) write a reduced decomposition for w and compute the matrix given by the action of $r_\sigma(w,M,\nu)$ on the vectors in (2). One obtains in this way a hermitian matrix of dimension equal to the dimension of σ.

A lowest K-type (abbreviated LKT) σ appears with multiplicity one in $L(M,V,\nu)$, so $r_\sigma(w,M,\nu)$ is a scalar. All the intertwining operators calculated are normalized so that this scalar is $+1$.

There are four distinguished orbits and the modules associated to them are discrete series and therefore unitary. They are

$\mathbf{F_4}$	parameter $(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	LKT 1_4
$\mathbf{F_4(a_1)}$	parameter $(\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	LKTs 4_5 and 2_4
$\mathbf{F_4(a_2)}$	parameter $(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	LKTs 9_4 and 2_2
$\mathbf{F_4(a_3)}$	parameter $(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	LKTs 12_1, 9_2, 6_2 and 1_2

Then, there is the special case of the nilpotents coming from maximal parabolics. For them, the same argument used in [BM3] applies: for a standard module parametrized by such a nilpotent orbit $\mathcal{O}$ and lowest K-type μ, the next bigger nilpotent $\mathcal{O}'$ has the property that a factor attached to $\mathcal{O}'$ and with lowest K-type μ' appears at the first point of reducibility. Moreover, μ' appears with multiplicity one in the standard module. Beyond this point, μ and μ' stay in the same factor and they have opposite signatures at ∞. By Proposition 2.4 in [BM3], two such K-types have opposite signatures at ∞ if and only if their respective lowest harmonic degrees have different parity.

Proposition 4.1. *Suppose the standard module $X(M,V,\nu)$, $\nu \geq 0$ is parametrized by a Levi component M of a maximal parabolic in F_4. If $\nu = \nu_0$ is its first point of reducibility on the half line $\nu > 0$, then $L(M,V,\nu)$ is unitary if and only if $0 \leq \nu \leq \nu_0$.*

Proof. There are five nilpotent orbits coming from maximal parabolics: B_3, C_3, $C_3(a_1)$, $A_1+\widetilde{A}_2$ and $\widetilde{A}_1+A_2$. For each of them, I use the argument outlined above, but also compute explicitly the intertwining operator (which is a scalar) on μ', normalized by μ. This scalar turns out to be in all cases $\frac{\nu_0-\nu}{\nu_0+\nu}$.

$\mathbf{B_3}$: The infinitesimal character is $(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$, the centralizer is A_1 and LKT 8_2. The standard module is $X(B_3,sgn,\nu)$. The first reducibility point is at $\nu=1$, where there are factors with LKT 9_4 and 2_2 coming from $F_4(a_2)$. For $\nu>1$, these K-types will stay in the same factor with 8_2. 8_2 and 9_4, or 8_2 and 2_2, have opposite signs at ∞, ruling out $\nu>1$.

The intertwining operators are

$$2_2 : \frac{1-\nu}{1+\nu} \text{ and } 9_4 : \frac{1-\nu}{1+\nu}$$

and this shows independently that the unitary parameter is $0\leq\nu\leq 1$. At the endpoint, corresponding to parameter $(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$, the factor is just 8_2.

$\mathbf{C_3}$: The infinitesimal character is $(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$, the centralizer is A_1 and LKT 8_4. The standard module is $X(C_3,sgn,\nu)$. The first reducibility point is at $\nu=\frac{1}{2}$, where there is a factor with LKT 9_4 coming from $F_4(a_2)$. For $\nu>\frac{1}{2}$, this K-type will stay in the same factor with 8_2. 8_2 and 9_4 have opposite signs at ∞, ruling out $\nu>\frac{1}{2}$.

The intertwining operator is

$$9_4 : \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}.$$

The unitary parameter is $0\leq\nu\leq\frac{1}{2}$. At the endpoint $\nu=\frac{1}{2}$, the factor is 8_4+2_4 with parameter $(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$.

$\mathbf{C_3(a_1)}$: The infinitesimal character is $(\nu,\frac{3}{2},\frac{1}{2},\frac{1}{2})$, the centralizer is A_1 and LKTs 16_1 and 4_3.

There are two lowest K-types 16_1 and 4_3. The corresponding two standard modules are $X(C_3, V_1, \nu)$ and $X(C_3, V_2, \nu)$. Here V_1 is the discrete series in $\mathbb{H}(C_3)$ with K-structure $1 \times 11 + 0 \times 1^3$, parametrized by the nilpotent orbit (42) and LKT 1×11, while V_2 is the discrete series with K-structure $1^3 \times 0$, parametrized by the nilpotent orbit (42) and LKT $1^3 \times 0$.

The first reducibility point for the 16_1 standard module is at $\nu = \frac{1}{2}$ corresponding to $F_4(a_3)$ and lowest K-type 12_1. 16_1 and 12_1 stay in the same factor except at $\nu = \frac{1}{2}$ and they have opposite signs at ∞, therefore the 16_1-factor is not unitary for $\nu > \frac{1}{2}$. The intertwining operator (normalized by 16_1) is

$$12_1 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}.$$

The unitary parameter is $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the parameter is $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the factor $16_1 + 9_4$.

For the standard module containing LKT 4_3, the first reducibility is again at $\nu = \frac{1}{2}$ corresponding to $F_4(a_3)$ and the factor with LKT 1_2. 4_3 and 1_2 stay in the same factor after that and they have opposite signs at ∞. The intertwining operator (normalized by 4_3) is

$$1_2 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}.$$

The unitary parameter is $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$ and parameter $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the lowest K-type 4_3 forms a factor by itself.

$\mathbf{A_1} + \widetilde{\mathbf{A}}_\mathbf{2}$: The infinitesimal character is $(\frac{1}{4} + \frac{3}{2}\nu, \frac{3}{4} + \frac{1}{2}\nu, -\frac{1}{4} + \frac{1}{2}\nu, -\frac{5}{4} + \frac{1}{2}\nu)$, centralizer is A_1 and LKT 6_1. The standard module is $X(A1 + \widetilde{A}_2, sgn, \nu)$. The first point of reducibility is $\nu = \frac{1}{2}$, where there is a factor coming from $C_3(a_1)$ with LKT 16_1. If 16_1 and 6_1 come apart again, there should be again a factor from $C_3(a_1)$, with LKT 16_1, but now this factor should also contain 6_2. Since 6_2 does not appear in the induced form $A_1 + \widetilde{A}_2$ in F_4, it follows that 16_1 and 6_1 stay in the same factor for $\nu > \frac{1}{2}$. As they have opposite signs at ∞, they rule out $\nu > \frac{1}{2}$. This argument also implies that 6_1, 16_1, 12_1 and 9_2 are in the same factor for $\nu > \frac{1}{2}$.

The intertwining operator is

$$16_1 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu},$$

which confirms that the unitary parameter is $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the parameter becomes $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the factor is 6_1 by itself.

$\widetilde{\mathbf{A}}_\mathbf{1} + \mathbf{A_2}$: The infinitesimal character is $(\frac{1}{2} + 2\nu, \nu, -1 + \nu, \frac{1}{2})$, centralizer is A_1 and the LKT 4_4. The standard module is $X(\widetilde{A}_1 + A_2, sgn, \nu)$. The first reducibility point is $\nu = \frac{1}{2}$, where there is a factor from $A_1 + \widetilde{A}_2$ with LKT 6_1. If 6_1 and 4_4 come apart again, there should be a factor with LKT 6_1 which would contain 9_2. But 9_2 does not appear in the induced from $\widetilde{A}_1 + A_2$. Therefore, 4_4 and 6_1 stay in the same factor for $\nu > \frac{1}{2}$ and have opposite signs at ∞.

The intertwining operator is

$$6_1 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}.$$

The unitary parameter is $0 \le \nu \le \frac{1}{2}$. At the endpoint $\nu = \frac{1}{2}$, the factor is 4_4 by itself corresponding to parameter $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. $\square$

The rest of the nilpotents are treated case-by-case as in the closure ordering. For each nilpotent, I give the infinitesimal character, centralizer and lowest K-type(s). The main idea is the following: assume we try to determine the unitarity of a standard module parametrized by an orbit $\check{\mathcal{O}}$ and containing a lowest K-type μ. The corresponding standard module is an $X(M, V, \nu)$. We look at lowest K-types σ of nilpotent orbits $\check{\mathcal{O}}'$ which are bigger than $\check{\mathcal{O}}$, but close to $\check{\mathcal{O}}$ in the closure ordering. We compute the operators $r_\sigma(w, M, \nu)$. We try to match these operators with (spherical) intertwining operators on the relevant K-types of the centralizer of $\check{\mathcal{O}}$.

I will say that two such operators *match* if they have the same characteristic polynomials (in particular they have the same signature).

However, one cannot always match in this way all the relevant K-types of the centralizer, and it is unclear at this point how one can predict which relevant K-types can be matched and what the (abstract) reason for this is.

There is a more delicate point concerning the nilpotent orbits with nontrivial component group. Let $\check{\mathcal{O}} \subset \check{\mathfrak{g}}$ be a non–distinguished nilpotent orbit and $\{\check{e}, \check{h}, \check{f}\}$ a standard Lie triple. Let the standard module attached to $\check{\mathcal{O}}$ be $X(M, sgn, \nu) = \mathrm{Ind}_{\mathbb{H}_M}^{\mathbb{H}}(sgn \otimes \mathbb{C}_\nu)$ and s a semisimple element with $s = \frac{1}{2}\check{h} + \nu$. If $A_G(\check{e}) \ne \{1\}$, then by Springer's correspondence, there are at least two lowest K-types attached to $\check{\mathcal{O}}$. In F_4, if $\check{\mathcal{O}}$ is non–distinguished and has nontrivial component group, there are *exactly* two LKTs for $\check{\mathcal{O}}$ ($\check{\mathcal{O}}$ is one of the following: $C_3(a_1)$, B_2, A_2 and $\widetilde{A}_1$).

When $\nu = 0$, the standard module breaks into a sum of two factors, each factor corresponding to one of the two LKTs. The question is how to determine when the two LKTs are again in separate factors for $\nu \ne 0$. This fact is controlled by the component group $A(s, \check{e}) \subset A(\check{e}) = S_2$ (S_2 is the group with two elements). We use now the Kazhdan–Lusztig classification and the connection with Langlands classification (see Section 2.3).

The two lowest K-types are in separate factors if and only if

(1) there exists a Levi subgroup M' with $M \subset M' \subset G$ such that
$$X_G(M, sgn, \nu) = \mathrm{Ind}_{\mathbb{H}_{M'}}^{\mathbb{H}}(X'_M(M, sgn, \frac{1}{2}\check{h}_{M'}) \otimes \mathbb{C}_\nu),$$
where by $\check{h}_{M'}$, I denote the middle element of the nilpotent orbit $\check{\mathcal{O}}_{M'}$ parametrized by M in the dual Lie algebra of M', $\check{\mathfrak{m}}'$;

(2) and the nilpotent orbit $\check{\mathcal{O}}_{M'}$ in $\check{\mathfrak{m}}'$ has nontrivial component group.

Concretely, for a parameter s we check condition (1) by verifying if there exists M' such that s (or rather a W–conjugate of s) is in $\check{\mathfrak{a}}_{M'} \subset \check{\mathfrak{a}}$ (the dual Cartan subalgebra corresponding to M').

Next, I begin the analysis of unitarity for the remaininig nilpotent orbits. As in the case of spherical parameters, I present pictures of the two-dimensional cases. In the pictures, "green" and "red" refer now to the unitarity of the lowest K-type factors.

$\mathbf{B_2}$: The infinitesimal character is $(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$ with $0 \le \nu_2 \le \nu_1$, the centralizer is $A_1 + A_1$. The standard module is $X(B_2, sgn, \nu)$.

There are two lowest K-types, 9_3 and 4_1 which have the same lowest harmonic degree and therefore the same sign at ∞. They stay in the same factor everywhere

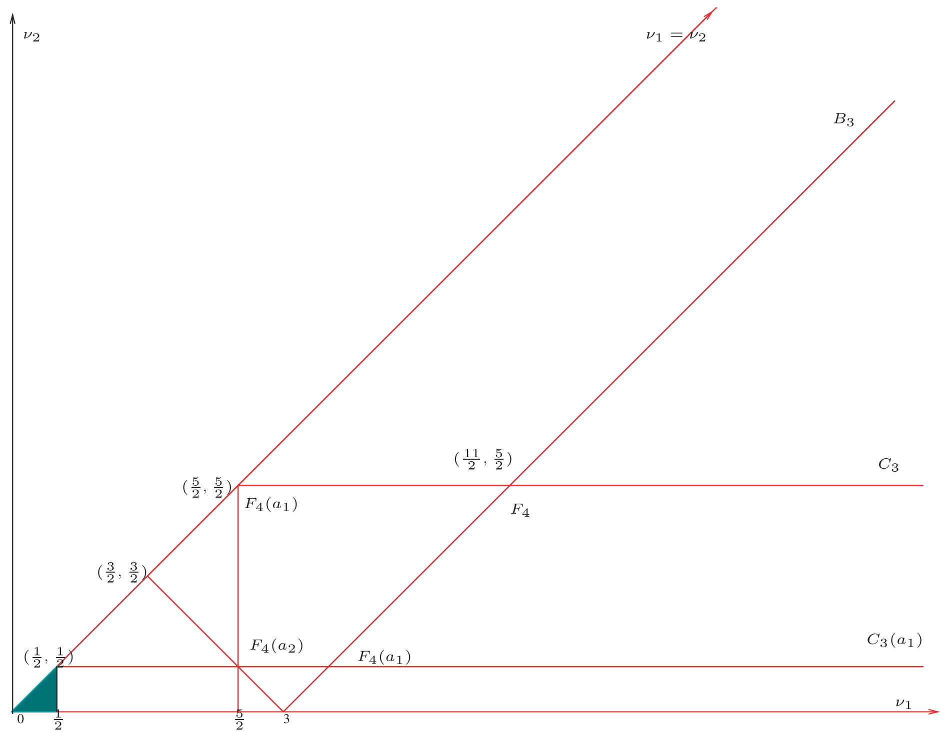

FIGURE 6. Unitary representations parametrized by B_2

except on the line $\nu_1 = \nu_2 = \nu$. On this line the parameter can be conjugated to $(\nu, -\nu, \frac{3}{2}, \frac{1}{2}) \in \check{\mathfrak{a}}_{C_3}$. The nilpotent corresponding to B_2 in the Lie algebra of type B_3 is (511) and it has two LKTs.

The intertwining operator is

$$16_1 : \begin{pmatrix} \frac{\frac{1}{2}-\nu_1}{\frac{1}{2}+\nu_1} & 0 \\ 0 & \frac{\frac{1}{2}-\nu_2}{\frac{1}{2}+\nu_2} \end{pmatrix}.$$

Since the first lines of reducibility are $\nu_1 = \frac{1}{2}$ and $\nu_2 = \frac{1}{2}$ (coming from $C_3(a_1)$), this implies the parameter is unitary if $0 \leq \nu_2 < \nu_1 \leq \frac{1}{2}$.

On the line $\nu_1 = \nu_2 = \nu$, the first reducibility occurs at $\nu = \frac{1}{2}$, corresponding to $F_4(a_3)$. Both factors have a copy of the K-type 16_1 and the value of the interwining operator is the same $\frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$ for both copies (when normalized by 9_3, respectively 4_1). This fact shows the parameter is unitary if $0 \leq \nu \leq \frac{1}{2}$ for both factors on the line $\nu_1 = \nu_2$.

$\mathbf{A_2}$: The infinitesimal character is $(\frac{1}{2} + \nu_1 + \nu_2, -\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_2, \frac{1}{2})$, with $0 \leq \nu_2 \leq \nu_1$, centralizer A_2, LKTs 8_3 and 1_3. The standard module is $X(A_2, sgn, \nu)$.

The two lowest K-types, 8_3 and 1_3 are separate only on the line $\nu_2 = 0$. On this line, the parameter can be conjugated to $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1)$, which is an element of $\check{\mathfrak{a}}_{C_3}$. The corresponding nilpotent orbit (331) in the Lie algebra of type B_3 has two lowest K-types.

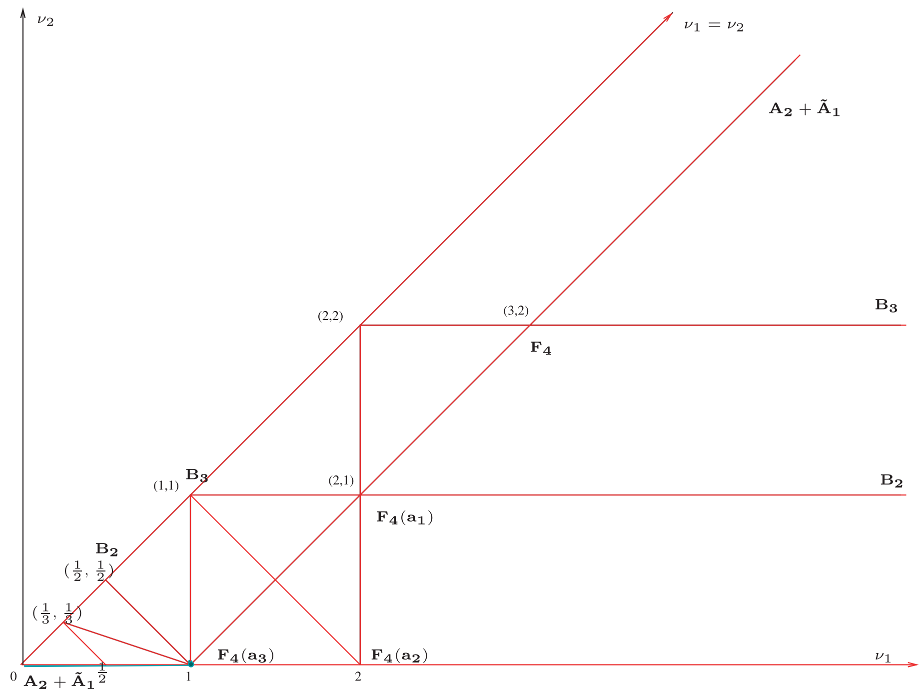

FIGURE 7. Unitary representations parametrized by A_2

The lowest K-types, 8_3 and 1_3 have opposite signs at ∞; therefore, the factor containing both 8_3 and 1_3 cannot be unitary. So one restricts to $\nu_2 = 0$, where the infinitesimal character becomes $(\frac{1}{2} + \nu, -\frac{1}{2} + \nu, \frac{1}{2}, \frac{1}{2})$, $\nu \geq 0$.

The factor containing 8_3 is unitary for $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, there is a first reducibility point corresponding to $\widetilde{A}_1 + A_2$. The intertwining operator is

$$4_4 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu} \cdot \frac{1 - \nu}{1 + \nu} \text{ and } 6_1 : \frac{(1-\nu)^2}{(1+\nu)^2} \cdot \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu},$$

which implies the factor is not unitary for $\nu \geq \frac{1}{2}$, except maybe at $\nu = 1$.

At $\nu = 1$, the parameter is $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the factor $(8_3 + 12_1 + 9_3 + 8_2)$ is unitary (this factor is also the IM dual of a unitary factor, endpoint of a complementary series in $\widetilde{A}_2$).

The factor with LKT=1_3 has first reducibility only at $\nu = 1$, corresponding to $F_4(a_3)$. The intertwining operator (normalized by 1_3) is

$$4_3 : \frac{1 - \nu}{1 + \nu}.$$

Therefore, the unitary parameters associated to 1_3 are $0 \leq \nu \leq 1$.

$\widetilde{\mathbf{A}}_{\mathbf{2}}$: The infinitesimal character is $(\nu_2 + \frac{3\nu_1}{2}, 1 + \frac{\nu_1}{2}, \frac{\nu_1}{2}, -1 + \frac{\nu_1}{2})$ with $\nu_1 \geq 0$, $\nu_2 \geq 0$, the centralizer is G_2 and the LKT 8_1. The standard module is $X(\widetilde{A}_2, sgn, \nu)$.

One can match the calculations with those for the spherical unitary dual for G_2 (see Appendix B).

Explicit calculations with the intertwining operator give:

$$\begin{array}{ll} 8_1 & \text{matches } 1_1 \text{ in } G_2, \\ 6_1 & \text{matches } 1_4 \text{ in } G_2, \\ 16_1 & \text{matches } 2_2 \text{ in } G_2, \\ 12_1 & \text{matches } 2_1 \text{ in } G_2, \\ 6_2 & \text{matches } 1_3 \text{ in } G_2. \end{array}$$

The hyperplanes of reducibility in $\widetilde{A}_2 \subset F_4$ are those from G_2 and $\nu_2 + 2\nu_1 = 3$, $\nu_2 + \nu_1 = 3$, $\nu_1 = 3$. However, these extra hyperplanes do not intersect the unitary dual of G_2 except at the point $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. Also, as seen above, the relevant K-types in G_2 are all matched, following that the unitary parameters for $\widetilde{A}_2$ are exactly those of the spherical G_2:

$$2\nu_2 + 3\nu_1 \leq 1,\ \nu_2 + 2\nu_1 \leq 1 \leq \nu_2 + 3\nu_1,$$

and the point $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, where the LKT factor is just (8_1).

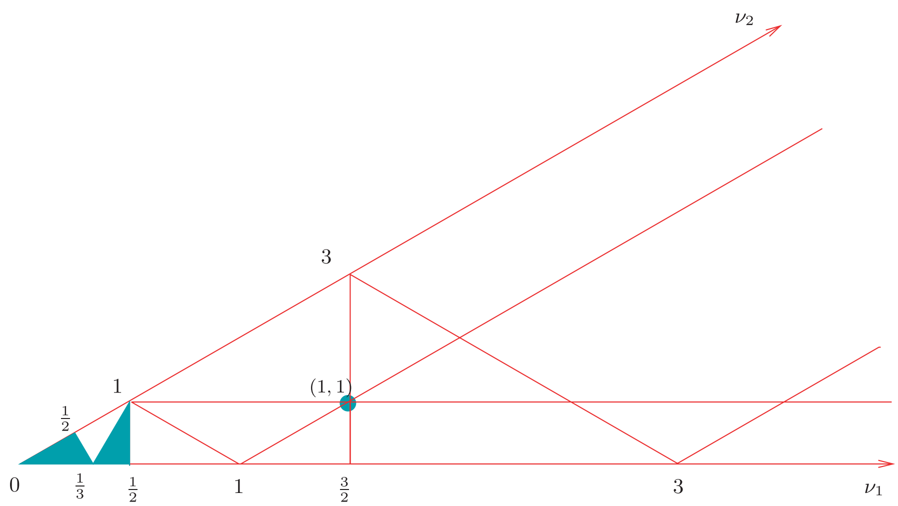

FIGURE 8. Unitary representations parametrized by $\widetilde{A}_2$

$\mathbf{A_1} + \widetilde{\mathbf{A}}_\mathbf{1}$: The infinitesimal character is $(\nu_1, \frac{1}{2} + \nu_2, -\frac{1}{2} + \nu_2, \frac{1}{2})$, $\nu_1 \geq 0$, $\nu_2 \geq 0$, the centralizer is $A_1 + A_1$ and LKT 9_1. The standard module is $X(A_1 + \widetilde{A}_1, sgn, \nu)$.

One tries to match the unitary parameters with the spherical unitary dual of $A_1 + A_1$. The intertwining operators are

$$\begin{array}{ll} 9_1: & +1, \\ 8_3: & \frac{\frac{1}{2}-\nu_1}{\frac{1}{2}+\nu_1}, \\ 8_1: & \frac{1-\nu_2}{1+\nu_2}. \end{array}$$

This implies that the unitary dual is included in $0 \leq \nu_2 \leq 1$, $0 \leq \nu_1 \leq \frac{1}{2}$. However, there are two lines that cut through this region: $\nu_1 + 2\nu_2 = \frac{3}{2}$ and $-\nu_1 + 2\nu_2 = \frac{3}{2}$. On these lines there is a factor from $\widetilde{A}_1 + A_2$, the parameter can be written as $(\frac{1}{2}+2\nu, \nu, -1+\nu, \frac{1}{2})$ and the K-structure of the 9_1-factor is

$$9_1 + 3\times 16_1 + 8_3 + 8_1 + 2\times 12_1 + 2\times 9_2 + 9_3 + 6_1 \\ + 4_1 + 2\times 8_4 + 8_2 + 4_3 + 2\times 9_4 + 6_2 + 4_5 + 2_4.$$

The factor parametrized by $\widetilde{A}_1 + A_2$ has LKT 4_4. By computing the intertwining operator on 4_4, one rules out the region $0 < \nu_1 < \frac{1}{2},\ \nu_1 + 2\nu_2 > \frac{3}{2},\ -\nu_1 + 2\nu_2 < \frac{3}{2}$. On the line $\nu_1 = \frac{1}{2}$, however, 4_4 is not in the same factor as 9_1.

Along the line $\nu_1 = \frac{1}{2}$, the parameter can be written as $(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2}, \frac{1}{2})$ and there is reducibility coming from A_2. The generic K-structure for the LKT factor is

$$9_1 + 2\times 16_1 + 8_1 + 12_1 + 2\times 9_2 + 6_1 + 4_1 + 2\times 8_4 + 4_3 + 9_4 + 2_4.$$

This factor is only reducible at $\nu = 1$, so it will be unitary for $0 \le \nu \le 1$.

In conclusion, the unitary dual attached to this nilpotent is as seen in Figure 9.

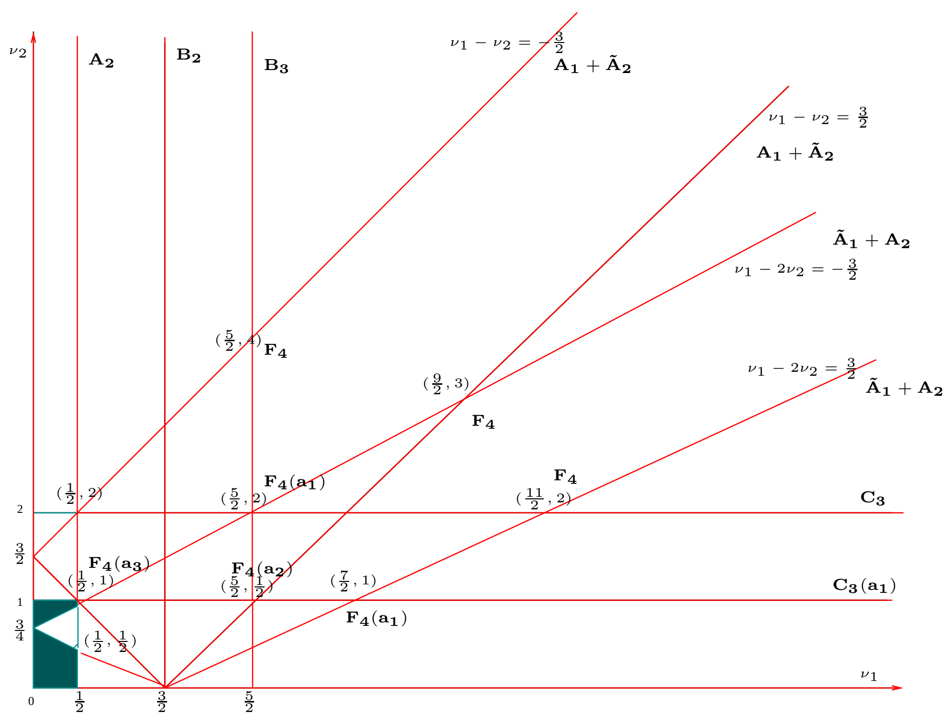

FIGURE 9. Unitary representations parametrized by $A_1 + \widetilde{A}_1$

$\widetilde{\mathbf{A}}_{\mathbf{1}}$: The infinitesimal character is $(\nu_1, \nu_2, \frac{1}{2}+\nu_3, -\frac{1}{2}+\nu_3)$, $\nu_1 \ge \nu_2 \ge 0$, $\nu_3 \ge 0$, the centralizer is A_3 and the LKTs 4_2 and 2_1. The standard module is $X(\widetilde{A}_1, sgn, \nu)$.

The two lowest K-types have opposite signs at ∞; therefore, if they are in the same factor, that factor will be nonunitary. They are separate if the parameter can be conjugated to an element in $\check{\mathfrak{a}}_{C_2}$ since the nilpotent orbit corresponding to $\widetilde{A}_1$ in the Lie algebra of type B_2 has two lowest K-types.

This happens when $\nu_3 = 0$ or $\nu_1 = \nu_2$. The parameters on the two planes are W-conjugate, and for the purpose of calculation, it is more convenient to consider $\nu_3 = 0$.

I restrict to this hyperplane and the parameter is $(\nu_1, \nu_2, \frac{1}{2}, \frac{1}{2})$, with $0 \leq \nu_2 \leq \nu_1$.

First, I look at the factor with LKT 4_2. The reducibility lines are $\nu_1 = \frac{1}{2}$ and $\nu_2 = \frac{1}{2}$ coming from $A_1 + \widetilde{A}_1$, $\nu_1 + \nu_2 = 1$ and $\nu_1 - \nu_2 = 1$ coming from A_2, $\nu_1 = \frac{3}{2}$ and $\nu_2 = \frac{3}{2}$ coming from $C_3(a_1)$ and $\nu_1 + \nu_2 = 2$ and $\nu_1 - \nu_2 = 2$ coming from B_2. The first reducibility line is $\nu_1 = \frac{1}{2}$, so in the region $0 \leq \nu_2 \leq \nu_1 \leq \frac{1}{2}$, the factor corresponding to 4_2 is unitary. Note also that on these reducibility lines, the 4_2-factor is self-dual, so I can't use the results for the other nilpotent orbits and the Iwahori-Matsumoto involution.

I am trying to match the unitary dual in this case with the spherical dual of C_2. The intertwining operator on the K-type 9_1 having multiplicity 2 in the 4_2-factor, normalized by the value on 4_2, matches the intertwining operator on the K-type 1×1 in C_2. Moreover, 4_1 matches 0×11 and 4_4 matches the product of the operators on 0×11 and 11×0. However, one also needs to use the intertwining operator on 8_3 (multiplicity 2). I list the parameters according to the nilpotent orbits in C_2.

(4): parameter $(\nu_1, \nu_2) = (\frac{3}{2}, \frac{1}{2})$. The 4_2-factor is $(4_2 + 6_2)$ and it is unitary (it is dual to a factor in $F_4(a_3)$).

(22): parameter $(\frac{1}{2} + \nu, -\frac{1}{2} + \nu)$. The 4_2-factor is self-dual:

$$4_2 + 9_1 + 8_1 + 8_3 + 9_2 + 9_3 + 2 \times 12_1 + 8_4 + 8_2 + 2 \times 16_1 + 2 \times 6_2 + 9_4 + 4_5.$$

The operator on 9_1 is $\frac{1-\nu}{1+\nu}$ and on 8_3 is $\frac{1-\nu}{1+\nu} \cdot \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$. The first point of reducibility is at $\nu = \frac{1}{2}$, where there is an extra factor coming from $\widetilde{A}_1 + A_2$, so this shows the factor above is unitary for $0 \leq \nu < \frac{1}{2}$. At $\nu = \frac{1}{2}$, the factor is just

$$4_2 + 6_2 + 9_1 + 8_1 + 16_1 + 9_2 + 12_1 + 8_4.$$

(211): parameter $(\nu, \frac{1}{2})$. The 4_2-factor is as in the case (22). The operator on 9_1 is $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$. On this line, the first reducibility point is at $\nu = \frac{3}{2}$, so the factor is unitary for $0 \leq \nu < \frac{3}{2}$.

(1^4): parameter is (ν_1, ν_2). The operator on 9_1 is positive definite only in the regions $0 \leq \nu_2 \leq \nu_1 < \frac{1}{2}$, which I know is unitary, and $1 - \nu_2 < \nu_1 < 1 + \nu_2$, $0 < \nu_2 < \frac{1}{2}$. The second region is ruled out by the operator on 4_4.

The answer is illustrated by Figure 10.

Next, I analyze the unitarity of the other lowest K-type factor, containing 2_1, on the same plane. The lines of reducibility are $\nu_1 + \nu_2 = 1$, $\nu_1 - \nu_2 = 1$, where there is a factor from $A_1 + \widetilde{A}_1$, $\nu_1 + \nu_2 = 2$, $\nu_1 - \nu_2 = 2$ with a factor from B_2 and $\nu_1 = \frac{3}{2}$, $\nu_2 = \frac{3}{2}$ where one finds a factor from $C_3(a_1)$ and also one from $\widetilde{A}_2$.

I will give the explicit expressions for the intertwining operators on some K-types of F_4 (normalized by the value on 2_1). I will only remark that there is a matching with the spherical unitary dual of a graded Hecke algebra of type B_2, but with parameter $c = \frac{3}{2}$ (I will denote it $B_2(\frac{3}{2})$). The operators on $B_2(\frac{3}{2})$ were also computed explicitly. 8_1 has multiplicity 2 and it matches 1×1 for $B_2(\frac{3}{2})$, 9_1 matches 11×0 and 6_1 matches 0×11. One also needs the operator on 4_3. Again I list the infinitesimal characters by their correspondents in B_2.

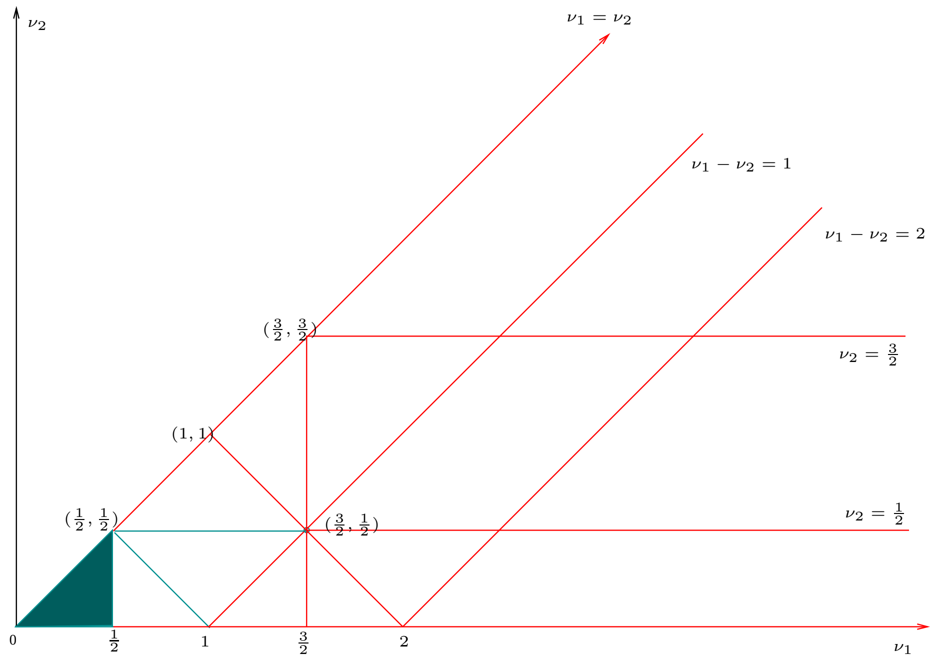

FIGURE 10. Unitary representations parametrized by $\widetilde{A}_1$ and 4_2

(5): parameter $(\frac{5}{2}, \frac{3}{2})$. 2_1 forms a factor by itself which is unitary (dual to a $F_4(a_2)$ factor).

(311): parameter $(\nu, \frac{3}{2})$. The 2_1-factor at generic points is dual to one in B_2 and has K-structure

$$2_1 + 9_1 + 16_1 + 8_1 + 4_1 + 9_2.$$

The intertwining operator on 8_1 is $\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$ and on 9_1 is $\frac{\frac{1}{2}+\nu}{\frac{1}{2}-\nu} \cdot \frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$ (note that the second one has a pole at $\nu = \frac{1}{2}$). The first reducibility point is at $\nu = \frac{1}{2}$, therefore the calculations imply the factor is unitary for $0 \le \nu < \frac{1}{2}$. At $\nu = \frac{1}{2}$, the factor is just

$$2_1 + 8_1 + 9_2.$$

(221): parameter $(\frac{1}{2} + \nu, -\frac{1}{2} + \nu)$. The 2_1-factor at generic points is dual to one in A_2 and has K-structure

$$2_1 + 8_1 + 9_2 + 4_3 + 1_2.$$

The intertwining operator on 8_1 is $\frac{2-\nu}{2+\nu}$ and on 4_3 is $\frac{1-\nu}{1+\nu} \cdot \frac{(2-\nu)^2}{(2+\nu)^2}$. Since the first reducibility is at $\nu = 1$, it follows that the factor is unitary for $0 \le \nu < 1$.

(1^5): parameter (ν_1, ν_2). From the previous calculations, it follows that the generic 2_1-factor is unitary for $0 \le \nu_2 \le \nu_1 < 1 - \nu_2$ and $1 + \nu_2 < \nu_1 < \frac{3}{2}$.

This is also seen in the Figure 11.

Note: On the line $\nu_2 = 0$, $\nu_1 = \nu$, the parameter is $(0, \nu, \frac{1}{2}, \frac{1}{2})$, so it is induced from C_3. The parameter $(\nu, \frac{1}{2}, \frac{1}{2})$ in C_3 comes from (2211) and there are two separate lowest K-type factors. The 2×1-factor is unitary for $0 \le \nu < \frac{1}{2}$ and

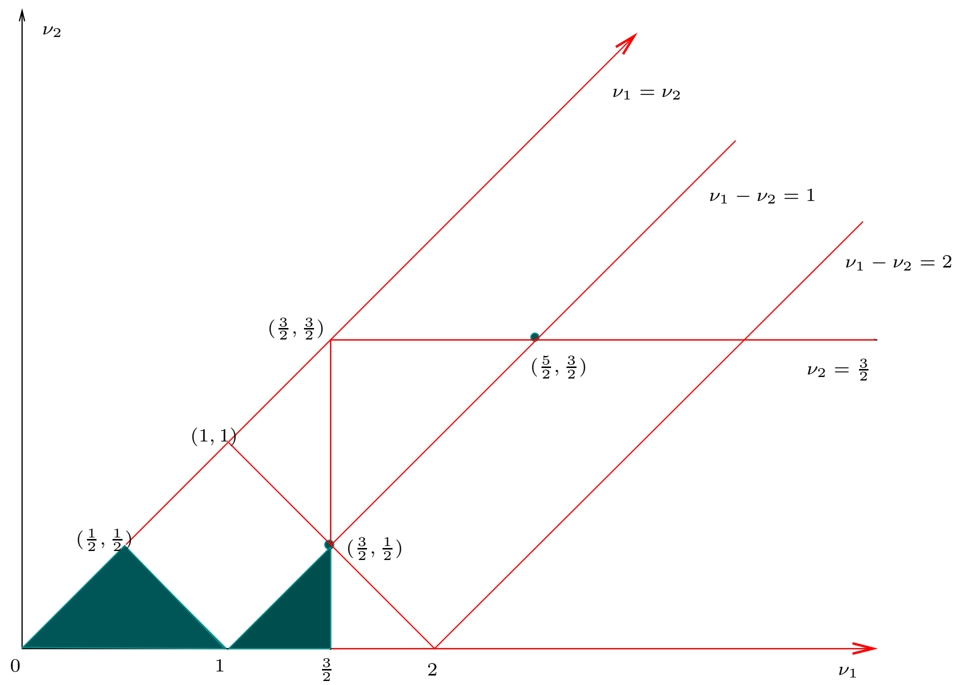

FIGURE 11. Unitary representations parametrized by $\widetilde{A}_1$ and 2_1

induces up to the 4_2-factor in F_4, while the 12×0-factor is unitary for $0 \leq \nu < \frac{3}{2}$ and induces to the 2_1-factor in F_4. This fact is consistent with the above calculations.

$\mathbf{A_1}$: The infinitesimal character is $(\nu_1, \nu_2, \nu_3, \frac{1}{2})$ with $0 \leq \nu_3 \leq \nu_2 \leq \nu_1$, the centralizer is C_3 and the LKT 2_3. The standard module is $X(A_1, sgn, \nu)$.

I compute the intertwining operator on K-types, normalized by the value on the LKT 2_3 and try to match the unitary dual with the spherical dual of C_3. The K-types that match intertwining operators in C_3 are

$$\begin{array}{ll} 2_3 & \text{with } 3\times 0, \\ 4_2 & \text{with } 0\times 3, \\ 8_1 & \text{with } 0\times 12, \\ 9_1 & \text{with } 1\times 2, \\ 4_3 & \text{with } 0\times 1^3. \end{array}$$

Since it is impossible to match all the relevant K-types for C_3, one cannot conclude if the unitary parameters of A_1 are identical or not with the spherical unitary dual of C_3. One also needs the intertwining operators calculated on 4_4, 8_3 and 1_3.

I list the infinitesimal characters as in C_3, ordered by the nilpotents in C_3.

(6): parameter $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$. The LKT factor is $2_3 + 8_3$ and it is unitary (dual of a factor in C_3).

(42): parameter $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. The LKT factor is $2_3 + 8_3 + 9_3$ and is unitary (dual of a factor in $F_4(a_3)$).

(411): parameter $(\nu, \frac{3}{2}, \frac{1}{2})$. The matched intertwining operators are all zero for this parameter. 8_3 gives $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}$, 1_3 gives $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu} \cdot \frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$ and 4_4 gives $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu} \cdot \frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu} \cdot \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$. The LKT factor is unitary for $0 \le \nu < \frac{1}{2}$ ($\nu = \frac{1}{2}$ is the first reducibility point) and at $\nu = \frac{7}{2}$ (this point does not appear in C_3). At $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2})$, the factor is 2_3 by itself (dual of a $F_4(a_1)$ factor).

(33): parameter $(1+\nu, \nu, -1+\nu)$. The first reducibility is for $\nu = \frac{1}{2}$. 4_2 gives $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$ and 9_1 gives $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu} \cdot \frac{\frac{1}{2}-\nu}{\frac{1}{2}+\nu}$. The LKT factor is unitary for $0 \le \nu < \frac{1}{2}$.

(222): parameter $(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{1}{2})$. 9_1 gives $\frac{(2-\nu)(1-\nu)}{(2+\nu)(1+\nu)}$. 8_3 has two nonzero eigenvalues with product $\frac{(2-\nu)(1-\nu)^2}{(2+\nu)(1+\nu)^2}$. The first reducibility is at $\nu = 1$, so the LKT factor is unitary for $0 \le \nu < 1$.

(2211): parameter $(\nu, \frac{1}{2}, \frac{1}{2})$. 9_1 gives $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$. 1_3 is $\frac{(\frac{1}{2}-\nu)(\frac{3}{2}-\nu)^2(\frac{5}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{3}{2}+\nu)^2(\frac{5}{2}+\nu)}$. The first reducibility point in F_4 is $\nu = \frac{1}{2}$, so the LKT factor is unitary for $0 \le \nu < \frac{1}{2}$.

(2211): parameter $(\frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, \nu_2)$. The matched operators on 9_1, 8_1 and 4_2 rule out this entire plane except the region $\frac{1}{2} < \nu_2 < \frac{3}{2} - \nu_1$. In $A_1 \subset F_4$, there is the line $2\nu_1 + \nu_2 = \frac{3}{2}$, $\frac{1}{2} < \nu_2 < \frac{3}{2}$ cutting through this region. The two resulting open regions are ruled out by 4_4 and 8_3. On this line the parameter can be written as $(\frac{1}{2}+2\nu, \nu, -1+\nu, \frac{1}{2})$, which is unitary for $0 \le \nu < \frac{1}{2}$ (it is dual of a unitary factor in $A_1 + \widetilde{A}_1$).

(21^4): parameter $(\nu_1, \nu_2, \frac{1}{2})$ with $0 \le \nu_2 \le \nu_1$. The only nonzero matched operator is on 9_1, which is $\frac{(\frac{3}{2}-\nu_1)(\frac{3}{2}-\nu_2)}{(\frac{3}{2}+\nu_1)(\frac{3}{2}+\nu_2)}$. The calculations with 4_4 and 8_3 give the following reducibility lines: $\nu_1 \pm \nu_2 = 1$, $\nu_1 \pm \nu_2 = 2$, $\nu_1 = \frac{3}{2}$ and $\nu_2 = \frac{3}{2}$. Checking each of the resulting regions and segments not ruled out by 9_1 already, it follows that the LKT factor is unitary in the regions $\nu_1 + \nu_2 < 1$ and $1 + \nu_2 < \nu_1 < \frac{3}{2}$ (dual of a unitary factor in $\widetilde{A}_1$) and on the segment $(1+\nu, 1-\nu, \frac{1}{2})$, $0 \le \nu \le \frac{1}{2}$ (the parameter is conjugate to $(\nu, \nu, \frac{3}{2}, \frac{1}{2})$ and the module is the IM-dual of a factor in B_2).

(1^6): parameter (ν_1, ν_2, ν_3). The matched operators rule out all the space except in the regions $\nu_1 < \frac{1}{2}$ which is unitary, the plane $\nu_1 = \frac{1}{2}$ being the first plane of reducibility, and $0 < 1 - \nu_1 < \nu_3 < 1 - \nu_2 < \frac{1}{2} < \nu_2 < 1 - \nu_3 < \nu_1 < 1 + \nu_3$. In F_4, the plane $\nu_1 + \nu_2 - \nu_3 = \frac{3}{2}$ divides this region into two parts. The K-type 8_3 rules out the two open subregions and the wall between them. It follows that the only unitary parameters here are in the first region.

5. Table of unitarity

The first list is the unitary spherical dual partitioned by complementary series associated to nilpotent orbits. All parameters are assumed real.

Orbit	**Parameter**	**Conditions**
F_4	$(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	
$F_4(a_1)$	$(\frac{7}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	
$F_4(a_2)$	$(\frac{5}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})$	
C_3	$(\nu,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	$0\le\nu<\frac{1}{2}$
B_3	$(\frac{3}{2}+\nu,-\frac{3}{2}+\nu,\frac{3}{2},\frac{1}{2})$	$0\le\nu<1$
$F_4(a_3)$	$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	
$C_3(a_1)$	$(\nu,\frac{3}{2},\frac{1}{2},\frac{1}{2})$	$0\le\nu<\frac{1}{2}$
$A_1+\widetilde{A}_2$	$(\frac{1}{4}+\frac{3\nu}{2},\frac{3}{4}+\frac{\nu}{2},-\frac{1}{4}+\frac{\nu}{2},-\frac{5}{4}+\frac{\nu}{2})$	$0\le\nu<\frac{1}{2}$
B_2	$(\nu_1,\nu_2,\frac{3}{2},\frac{1}{2})$	$0\le\nu_2\le\nu_1<\frac{1}{2}$
$\widetilde{A}_1+A_2$	$(\frac{1}{2}+2\nu,\nu,-1+\nu,\frac{1}{2})$	$0\le\nu<\frac{1}{2}$
$\widetilde{A}_2$	$(\nu_2+\frac{3\nu_1}{2},1+\frac{\nu_1}{2},\frac{\nu_1}{2},-1+\frac{\nu_1}{2})$	$3\nu_1+2\nu_2<1$
		$2\nu_1+\nu_2<1<3\nu_1+\nu_2$
A_2	$(\frac{1}{2}+\nu,-\frac{1}{2}+\nu,-\frac{1}{2},\frac{1}{2})$	$0\le\nu<\frac{1}{2}$
$A_1+\widetilde{A}_1$	$(\nu_1,\frac{1}{2}+\nu_2,-\frac{1}{2}+\nu_2,\frac{1}{2})$	$\nu_1+2\nu_2<\frac{3}{2},\ \nu_1<\frac{1}{2}$
$\widetilde{A}_1$	$(\nu_1,\nu_2,\frac{1}{2},\frac{1}{2})$	$0\le\nu_2\le\nu_1<\frac{1}{2}$
A_1	$(\nu_1,\nu_2,\nu_3,\frac{1}{2})$	$0\le\nu_3\le\nu_2\le\nu_1<\frac{1}{2}$
1	$(\nu_1,\nu_2,\nu_3,\nu_4)$	$\nu_1<\frac{1}{2}$
		$\nu_1+\nu_2+\nu_3-\nu_4<1$
		$<\nu_1+\nu_2+\nu_3+\nu_4$

Note that the parameter for the trivial nilpotent orbit should be dominant.

Next, I will give a list with all unitary representations organized by the nilpotent orbits, (real) infinitesimal characters and lowest K-type of the unitary factor. The unitary parameters for the trivial nilpotent coincide with the spherical dual and will not be listed for economy.

Orbit	**Parameter**	**Conditions**	**LKT**
F_4	$(\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$		1_4
$F_4(a_1)$	$(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		4_5
			2_4
$F_4(a_2)$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		9_4
			2_2
C_3	$(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	$0 \leq \nu < \frac{1}{2}$	8_4
	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		8_4
B_3	$(\frac{3}{2} + \nu, -\frac{3}{2} + \nu, \frac{3}{2}, \frac{1}{2})$	$0 \leq \nu < 1$	8_2
	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		8_2
$F_4(a_3)$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		12_1
			9_2
			6_2
			1_2
$C_3(a_1)$	$(\nu, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$0 \leq \nu < \frac{1}{2}$	16_1
			4_3
	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		16_1
			4_3
$A_1 + \widetilde{A}_2$	$(\frac{1}{4} + \frac{3\nu}{2}, \frac{3}{4} + \frac{\nu}{2}, -\frac{1}{4} + \frac{\nu}{2}, -\frac{5}{4} + \frac{\nu}{2})$	$0 \leq \nu < \frac{1}{2}$	6_1
	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		6_1
B_2	$(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$	$0 \leq \nu_2 < \nu_1 < \frac{1}{2}$	$9_3 + 4_1$
	$(\nu, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$0 \leq \nu < \frac{1}{2}$	$9_3 + 4_1$
	$(1 + \nu, 1 - \nu, \frac{1}{2}, \frac{1}{2})$	$0 \leq \nu < \frac{1}{2}$	9_3
			4_1
	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		9_3
			4_1

$\widetilde{A}_1+A_2$	$(\frac12+2\nu,\nu,-1+\nu,\frac12)$	$0\le\nu<\frac12$	4_4
	$(\frac32,\frac12,\frac12,\frac12)$		4_4
$\widetilde{A}_2$	$(\nu_2+\frac{3\nu_1}{2},1+\frac{\nu_1}{2},\frac{\nu_1}{2},-1+\frac{\nu_1}{2})$	$3\nu_1+2\nu_2<1$	8_1
		$2\nu_1+\nu_2<1<3\nu_1+\nu_2$	8_1
	$(\nu,\frac32,\frac12,\frac12)$	$0\le\nu<\frac12$	8_1
	$(1+\nu,\nu,-1+\nu,\frac12)$	$0\le\nu<\frac12$	8_1
	$(\frac32,\frac12,\frac12,\frac12)$		8_1
	$(\frac52,\frac32,\frac12,\frac12)$		8_1
A_2	$(\frac12+\nu,-\frac12+\nu,-\frac12,\frac12)$	$0\le\nu<\frac12$	8_3
		$0\le\nu<1$	1_3
	$(\frac32,\frac12,\frac12,\frac12)$		8_3
			1_3
	$(1,\frac12,\frac12,0)$		8_3
$A_1+\widetilde{A}_1$	$(\nu_1,\frac12+\nu_2,-\frac12+\nu_2,\frac12)$	$\nu_1+2\nu_2<\frac32,\ \nu_1<\frac12$	9_1
		$2\nu_2-\nu_1>\frac32,\ \nu_2<1$	
	$(\frac12+\nu,-\frac12+\nu,\frac12,\frac12)$	$0\le\nu<1$	9_1
	$(\frac12+2\nu,\nu,-1+\nu,\frac12)$	$0\le\nu<\frac12$	9_1
	$(\frac32,\frac12,\frac12,\frac12)$		9_1
$\widetilde{A}_1$	$(\nu_1,\nu_2,\frac12,\frac12)$	$0\le\nu_2\le\nu_1<\frac12$	4_2
		$0\le\nu_2<1-\nu_1$	2_1
		$1+\nu_2<\nu_1<\frac32$	2_1
	$(\frac12+\nu,-\frac12+\nu,\frac12,\frac12)$	$0\le\nu<\frac12$	4_2
		$0\le\nu<1$	2_1
	$(\nu,\frac12,\frac12,\frac12)$	$0\le\nu<\frac32$	4_2
	$(\nu,\frac32,\frac12,\frac12)$	$0\le\nu<\frac12$	2_1
	$(\frac32,\frac12,\frac12,\frac12)$		4_2
			2_1
	$(1,\frac12,\frac12,0)$		4_2
	$(\frac52,\frac32,\frac12,\frac12)$		2_1

A_1	$(\nu_1, \nu_2, \nu_3, \frac{1}{2})$	$0 \le \nu_3 \le \nu_2 \le \nu_1 < \frac{1}{2}$	2_3
	$(\nu_1, \nu_2, \frac{1}{2}, \frac{1}{2})$	$\nu_1 + \nu_2 < 1$	2_3
		$1 + \nu_2 < \nu_1 < \frac{3}{2}$	2_3
	$(\frac{1}{2} + 2\nu, \nu, -1 + \nu, \frac{1}{2})$	$0 \le \nu < \frac{1}{2}$	2_3
	$(\nu, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$0 \le \nu < \frac{1}{2}$	2_3
	$(1 + \nu, \nu, -1 + \nu, \frac{1}{2}$	$0 \le \nu < \frac{1}{2}$	2_3
	$(\frac{1}{2} + \nu, -\frac{1}{2} + \nu, \frac{1}{2}, \frac{1}{2})$	$0 \le \nu < 1$	2_3
	$(1 + \nu, 1 - \nu, \frac{1}{2}, \frac{1}{2})$	$0 \le \nu < \frac{1}{2}$	2_3
	$(\nu, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$0 \le \nu < \frac{1}{2}$	2_3
	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$		2_3
	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		2_3
	$(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$		2_3

Appendix A. Irreducible Weyl characters

Recall that the calculations with the intertwining operators are done in the dual complex group $\check{G}$. Let s_i be the reflection in the Weyl group corresponding to the simple coroot $\check{\alpha}_i$, $i = 1, \ldots, 4$. Recall that $\check{\alpha}_1$, $\check{\alpha}_2$ are the *long* simple coroots. The description of irreducible characters ordered by dimension follows.

1_1 unit representation;
1_4 sign representation;
1_2 s_1, s_2 act by $+1$, s_3, s_4 by -1;
1_3 $= 1_2 \otimes 1_4$.
2_1 s_1, s_2 act trivially,
s_3, s_4 act by reflections as in the representation (21) of $GL(3)$;
2_2 $= 2_1 \otimes 1_4$;
2_3 s_1, s_2 act by reflections as in the representation (21) of $GL(3)$,
s_3, s_4 act trivially;
2_4 $= 2_3 \otimes 1_4$.
4_1 $= 2_1 \otimes 2_3$;
4_2 the reflection representation;
4_3 $= 4_2 \otimes 1_2$;
4_4 $= 4_2 \otimes 1_3$;
4_5 $= 4_2 \otimes 1_4$.
6_2 second exterior power of 4_2;
6_1 $= 6_2 \otimes 1_2 = 6_2 \otimes 1_3$.
$8_1 = 4_2 \otimes 2_1$; $8_2 = 8_1 \otimes 1_4$; $8_3 = 4_2 \otimes 2_3$; $8_4 = 8_3 \otimes 1_4$.
9_1 second symmetric power of 4_2 from which one substracts 1_1;
9_2 $= 9_1 \otimes 1_2$;
9_3 $= 9_1 \otimes 1_3$;
9_4 $= 9_1 \otimes 1_4$.
$12_1 = 6_1 \otimes 2_1$; $16_1 = 4_1 \otimes 4_2$.

Appendix B. Unitary spherical dual for G_2

Let $\check{G}$ be the *complex dual* of a p–adic group of type G_2. I use the following parametrization for simple roots, coroots and coweights of $\check{G}$:

$$\begin{array}{lll} \alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) & \check{\alpha}_1 = (2,-1,-1) & \check{\omega}_1 = (1,1,-2) \\ \alpha_2 = (-1,1,0) & \check{\alpha}_2 = (-1,1,0) & \check{\omega}_2 = (0,1,-1) \end{array}$$

Note that α_1 is the short simple root.

The closure ordering of the nilpotents orbits is

$$G_2 - G_2(a_1) - \widetilde{A}_1 - A_1 - 1.$$

The following two sets of K-types are sufficient for the determination of the spherical unitary dual:

$$\{1_1, 1_3, 1_4, 2_2\} \quad \text{and} \quad \{1_1, 2_1, 2_2\}$$

where 1_1 is the trivial representation, 2_1 is the reflection representation, 1_2 is the sign representation, 1_3 is the one-dimensional on which t_1 acts by 1 and t_2 by -1, 1_4 is the one-dimensional on which t_1 acts by -1 and t_2 by 1 and $2_2 = 2_1 \otimes 1_3$ (see [A]).

The long intertwining operators, corresponding to the long Weyl element $w_0 = (s_1 s_2)^3$, for a parameter (associated to the trivial nilpotent orbit) $(\nu_1, \nu_1+\nu_2, -2\nu_1-\nu_2)$ with $\nu_1 \geq 0,\ \nu_2 \geq 0$ are

$$\begin{array}{ll} 1_1: & +1 \\ 1_3: & \frac{1-\nu_2}{1+\nu_2} \cdot \frac{1-(3\nu_1+2\nu_2)}{1+(3\nu_1+2\nu_2)} \cdot \frac{1-(3\nu_1+\nu_2)}{1+(3\nu_1+\nu_2)} \\ 1_4: & \frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)} \cdot \frac{1-\nu_1}{1+\nu_1} \cdot \frac{1-(2\nu_1+\nu_2)}{1+(2\nu_1+\nu_2)} \\ 2_1,\ 2_2: & \text{are } 2\times 2 \text{ matrices with determinant} \\ & \frac{1-\nu_2}{1+\nu_2} \cdot \frac{1-(3\nu_1+2\nu_2)}{1+(3\nu_1+2\nu_2)} \cdot \frac{1-(3\nu_1+\nu_2)}{1+(3\nu_1+\nu_2)} \cdot \frac{1-(\nu_1+\nu_2)}{1+(\nu_1+\nu_2)} \cdot \frac{1-\nu_1}{1+\nu_1} \cdot \frac{1-(2\nu_1+\nu_2)}{1+(2\nu_1+\nu_2)} \end{array}$$

The lines of reducibility are $3\nu_1+2\nu_2 = 1$, $3\nu_1+\nu_2 = 1$, $2\nu_1+\nu_2 = 1$, $\nu_1+\nu_2 = 1$, $\nu_1 = 1$ and $\nu_2 = 1$.

The spherical unitary parameters are (as seen in the picture):

$$\begin{array}{l} \{3\nu_1 + 2\nu_2 < 1\}; \\ \{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2\}; \\ (\nu_1, \nu_2) = (1,1) \end{array}$$

The spherical unitary dual in the picture is partitioned by nilpotent orbits as follows:

$\mathbf{G_2}$: parameter $\rho = (1,2,-3)$ corresponding in the picture to the point $(1,1)$. The standard module is just 1_1.

$\mathbf{G_2(a_1)}$: parameter $\omega_2 = (0,1,-1)$ corresponding to the point $(0,1)$. The K-structure of the standard module is $(1_1+2_1)+(1_3)$.

$\widetilde{\mathbf{A}}_\mathbf{1}$: parameter $\frac{1}{2}\alpha_1 + \nu\omega_2 = (1, -\frac{1}{2}+\nu, -\frac{1}{2}-\nu)$. The standard module has K-structure $1_1+2_1+1_3+2_2$. $\nu = 0$ corresponds to $(\frac{1}{2}, 0)$ in the picture. The standard module decomposes as follows:

ν	Decomposition	Spherical factor orbit
$\frac{1}{2}$	$(1_1+2_1)+(1_3)+(2_2)$	$G_2(a_1)$
$\frac{5}{2}$	$(1_1)+(2_1+2_2+1_3)$	G_2

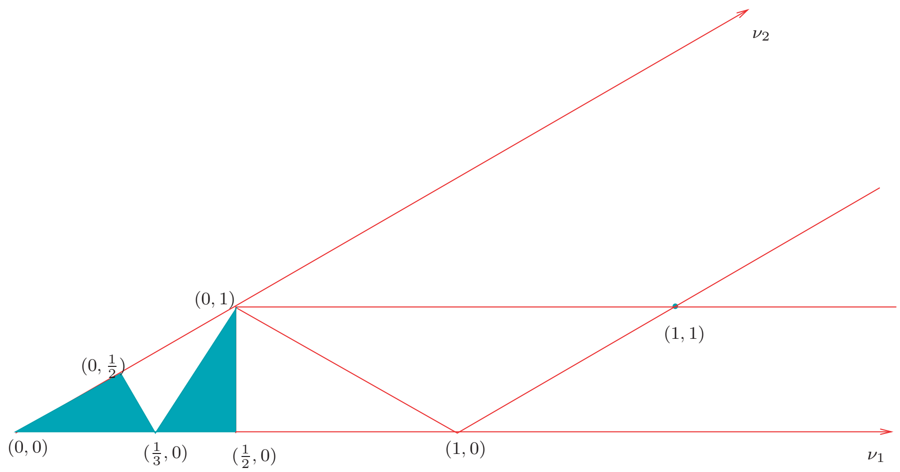

FIGURE 12. Unitary spherical dual for G_2

The complementary series is $0 \leq \nu < \frac{1}{2}$. The calculations with K-types are (1_4 does not appear):

1_1 : $+1$;

1_3 : $\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$ for $\nu \neq 0$, and 0 at $\nu = \frac{1}{2}$

2_2 : $\frac{(\frac{1}{2}-\nu)(\frac{5}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{5}{2}+\nu)}$;

2_1 : $\frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}$.

ν	1_1	1_3	2_2	2_1
	$+$	$+$	$+$	$+$
$\frac{1}{2}$	$+$	$+$	0	$+$
	$+$	$+$	$-$	$+$
$\frac{5}{2}$	$+$	0	0	0
	$+$	$-$	$+$	$-$

$\mathbf{A_1}$: parameter $\frac{1}{2}\alpha_2 + \nu\omega_1 = (-\frac{1}{2}+\nu, \frac{1}{2}+\nu, -2\nu)$. The standard module has K-structure $1_1+2_1+2_2+1_4$. $\nu = 0$ corresponds to $(0, \frac{1}{2})$ in the picture. The standard module decomposes as follows:

ν	Decomposition	Spherical factor orbit
$\frac{1}{2}$	$(1_1 + 2_1) + (2_2) + (1_4)$	$G_2(a_1)$
$\frac{3}{2}$	$(1_1) + (2_1 + 2_2 + 1_4)$	G_2

The complementary series is $0 \leq \nu < \frac{1}{2}$. The calculations with K-types are (1_3 does not appear):

1_1 : $+1$;

1_4 : $\frac{(\frac{1}{2}-\nu)^2(\frac{3}{2}-\nu)}{(\frac{1}{2}+\nu)^2(\frac{3}{2}+\nu)}$;

2_2 : $\frac{(\frac{1}{2}-\nu)(\frac{3}{2}-\nu)}{(\frac{1}{2}+\nu)(\frac{3}{2}+\nu)}$;

2_1 : $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$.

ν	1_1	1_4	2_2	2_1
	$+$	$+$	$+$	$+$
$\frac{1}{2}$	$+$	0	0	$+$
	$+$	$+$	$-$	$+$
$\frac{3}{2}$	$+$	0	0	0
	$+$	$-$	$+$	$-$

$\mathbf{1}$: parameter $(\nu_1, \nu_1+\nu_2, -2\nu_1-\nu_2)$; the complementary series is formed by the two 2–dimensional unitary regions in the picture.

Appendix C. Connections with the real split case

In this appendix, I plan to present the connection between the calculation of the spherical unitary dual in the split p–adic case, which is the subject of Section 3, and the spherical unitary dual for *real* split groups. I will review basic definitions and results from the theory of unitary representations of real groups and summarize the results of D. Barbasch from [B2]. The following exposition is fundamentally influenced by [B2] and the notes of D. Vogan in [V2].

Let us fix the notation. $G = \mathcal{G}(\mathbb{R})$ will denote the $\mathbb{R}$–points of a reductive algebraic group defined over $\mathbb{R}$. In section C.3 we will restrict to split groups G. K is a maximal compact subgroup (the fixed points of a Cartan involution θ). P will denote a parabolic subgroup with the (Langlands) decomposition $P = MAN$ and $G = PK$. In C.3, we will consider P to be a Borel subgroup, $P = B$, which contains a largest split torus.

C.1. $(\mathfrak{g}, K)$–**modules and unitarity.** The problem is to determine the irreducible representations (π, V) of G which are *spherical*, that is, $V^K \neq \{0\}$.

Definition C.1. (π, V) is called a $(\mathfrak{g}, K)$–module if V is a complex vector space, which is a $\mathcal{U}(\mathfrak{g})$–module ($\mathcal{U}(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$) and a semisimple K-representation such that the two actions are compatible:

(1) $\pi(k) \cdot \pi(X)v = \pi(\mathrm{Ad}k(X)) \cdot \pi(k)v$, $v \in V$, $k \in K$, $X \in \mathcal{U}(\mathfrak{g})$;
(2) if F is a K-stable finite dimensional subspace of V, then the representation of K on F is differentiable and its differential is $\pi|_{\mathfrak{k}}$ ($\mathfrak{k}$ is the Lie algebra of K).

A $(\mathfrak{g}, K)$–module (π, V) decomposes into a sum of K-isotypic components. We will always consider that the modules are *admissible*, which means that all the K-isotypic components are finite dimensional.

If (π, V) is a representation of G, $v \in V$ is called *smooth vector* if the orbit map $c_v : G \to V$, $c_v(g) = \pi(g)v$ is C^∞. $v \in V$ is called K-finite if the subspace generated by $\{\pi(k)v : k \in K\}$ is finite dimensional.

Let V_0 be the space of smooth K-finite vectors of V. To a representation (π, V) of G, one attaches the *Harish–Chandra module* (π, V_0), which is a $(\mathfrak{g}, K)$–module. The unitarity question can be translated to the category of $(\mathfrak{g}, K)$–modules.

Theorem C.2 (Harish–Chandra)**.** *$V \to V_0$ defines a bijection from the set of equivalence classes of irreducible unitary representations of G onto the set of equivalence classes of irreducible $(\mathfrak{g}, K)$–modules admitting a positive definite invariant Hermitian form.*

From now on, by an irreducible (unitary) admissible representation of G we will actually mean an irreducible (unitary) admissible $(\mathfrak{g}, K)$–module. Moreover, the term "equivalent" for representations of G will actually mean "infinitesimally equivalent" (the associated $(\mathfrak{g}, K)$–modules are equivalent).

C.2. **Langlands Classification.** I will present the construction of Langlands representations and associated intertwining operators following [KZ].

Definition C.3. An admissible representation (π, V) of G is called a *discrete series* if its matrix coefficients are in $L^2(G)$. (π, V) is called *tempered* if its matrix coefficients are in $L^{2+\epsilon}(G)$ for all $\epsilon > 0$.

Consider the following parameters:

(1) $P = MAN$ a parabolic subgroup.
(2) π an irreducible tempered representation of M.
(3) ν a character of $\mathfrak{a}$, the Lie algebra of A such that $\operatorname{Re}\nu$ is in the open dominant Weyl chamber given by the roots of A in P.

Note that ν gives rise to a one-dimensional character of A, which will be denoted e^ν.

Let $I(P, \pi, \nu)$ be the induced module

$$I(P, \pi, \nu) = \operatorname{Ind}_P^G(\pi \otimes e^\nu \otimes 1).$$

If $\overline{P} = MA\overline{N}$ denotes the opposite parabolic subgroup, define the (integral) intertwining operator

$$A_P(\pi, \nu) : I(P, \pi, \nu) \to I(\overline{P}, \pi, \nu),$$

by

$$(A_P(\pi, \nu)f)(x) = \int_{\overline{N}} f(x\bar{n})\, d\bar{n},\ \ f \in I(P, \pi, \nu),\ \ x \in G.$$

Define $L(P, \pi, \nu)$ to be the image of $A_P(\pi, \nu)$. This is the *Langlands quotient*.

Theorem C.4 (Langlands)**.** *$L(P, \pi, \nu)$ is irreducible admissible and every irreducible admissible representation of G is equivalent to a Langlands quotient. Two sets of parameters, (P, π, ν) and (P', π', ν'), parametrize the same representation if and only if they are conjugate under G.*

The unitarity question then amounts to classifying which Langlands quotients $L(P, \pi, \nu)$ are unitary. The following theorem gives the necessary and sufficient conditions for the Langlands quotients to be Hermitian.

Theorem C.5 (Knapp–Zuckerman)**.** *Let $L(P, \pi, \nu)$ be as in Theorem* C.4*. Then $L(P, \pi, \nu)$ admits an invariant Hermitian form if and only if there exists $w \in W(G, A)$* ($W(G, A) = N_G(A)/Z_G(A)$) *conjugating the triple (P, π, ν) to the triple* $(\overline{P}, \pi, -\bar{\nu})$.

In this case, the Hermitian form is positive definite if and only if the intertwining operator $A = R(w)A(\pi, \nu)$, where $R(w)$ denotes the right translation by w, is either positive semidefinite or negative semidefinite.

Three remarks regarding Theorem C.4:

a) If one required the parabolic subgroups in the parametrization to be standard, the sets (P, π, ν) would always parametrize inequivalent representations.
b) The Langlands classification can be reformulated so that π is a discrete series. This is because every irreducible tempered representation is equivalent to a summand of an induced representation from a discrete series (Langlands). In this formulation though, ν would be required to be in the closed Weyl chamber and the Langlands quotient as defined above would be reducible and Langlands classification would state that every irreducible admissible representation of G appears as a summand of a Langlands quotient (see, for example, Theorem 14.92 in [Kn]).
c) For unitary representations, one can reduce the classification to the case of real infinitesimal characters, i.e., one can assume $\nu \in \mathfrak{a}^*$ is real (as in Theorem 16.10 in [Kn]).

In the next subsection, when we will restrict to the case of spherical representations, the classification will become simpler and we will discuss the intertwining operator in more detail. Furthermore, the intertwining operator will be normalized so that the condition for unitarity as in the Theorem C.5 will require the operator to be positive semidefinite.

C.3. **The Spherical Split Case.** Let $B = MAN$ be a Borel subgroup. A is the identity component of T, a largest split torus in G and $M = T \cap K$.

Consider the induced representations (principal series)

$$X_B(\delta,\nu) := \mathrm{Ind}_B^G(\delta \otimes e^{\nu} \otimes 1),$$

where δ is a unitary character of T, trivial on M and ν is a real character of $\mathfrak{a}$.

Langlands classification for spherical representations says that any spherical representation of G is equivalent to the Langlands quotient of an induced representation $X_B(\delta,\nu)$ with ν dominant. Moreover, it is possible to reduce the study of unitary spherical representations to the case when δ *is trivial.* We will assume from now on that this is the case, so that the spherical representations will be parametrized only by a dominant character ν. Consequently, they will be denoted $L(\nu)$ and we will view them as the irreducible quotients of $X_B(\nu)$.

From Theorem C.5, we know that there is an intertwining operator $A(\nu)$: $X_B(\nu) \to X_{\overline{B}}(\nu)$ and let $A(\nu)$ be normalized so that it is $+1$ on the K-fixed vector. $L(\nu)$ is the image of this operator and it is Hermitian if and only if there exist $w \in W$, such that $w\nu = -\nu$ (recall that ν is real). This is equivalent in this case to $w_0\nu = -\nu$.

Let (μ, V_μ) be a K*-type* of G (a representation of K). The following construction is due to Barbasch and Vogan. The intertwining operator $A(\nu)$ induces a map

$$A_\mu(\nu): \ \mathrm{Hom}_K(V_\mu, X_B(\nu)) \to \mathrm{Hom}_K(V_\mu, X_{\overline{B}}(\nu)).$$

By Frobenius reciprocity

$$\mathrm{Hom}_K(V_\mu, X_B(\nu)) \cong \mathrm{Hom}_M(V_\mu, \mathbb{C}) \cong \mathrm{Hom}_K(V_\mu, X_{\overline{B}}(\nu)).$$

Since $\mathrm{Hom}_M(V_\mu, \mathbb{C}) \cong (V_\mu^*)^M$, we obtain an operator

$$A_\mu(\nu): \ (V_\mu^*)^M \to (V_\mu^*)^M.$$

The normalization of the intertwining operator implies that $A_{triv}(\nu) = +1$. To summarize the discussion, we have:

Proposition C.6. *A spherical representation $L(\nu)$, $\nu \in \mathfrak{a}^*$, real and dominant, is unitary if and only if $w_0\nu = -\nu$ and all the operators $A_\mu(\nu)$ induced on the spaces $(V_\mu^*)^M$ are positive semidefinite.*

Since $W \cong N_K(T)/M$, there is an action of the Weyl group W on $(V_\mu^*)^M$. Denote this W–representation by $\tau(\mu)$. In general $\tau(\mu)$ may be reducible. Clearly, the dimension of $\tau(\mu)$ is the same as the multiplicity of μ in $X_B(\nu)$.

The operator $A(\nu)$ has a factorization corresponding to a reduced decomposition of w_0, so each operator $A_\mu(\nu)$ will have such a factorization. For a given μ, the factors of $A_\mu(s_\alpha, \nu)$ corresponding to the simple roots $\alpha \in \Pi$ can be described explicitly.

For each simple root α of T in G, there is a homomorphism $\Psi_\alpha : SL(2,\mathbb{R}) \to G$, coming from the Lie algebra homomorphism which takes the Lie triple of $\mathfrak{sl}(2,\mathbb{R})$ to the Lie triple corresponding to α. Via Ψ_α, $SO(2)$ is embedded into K. Therefore, the K-representation (μ, V_μ) has a grading coming from the action of $SO(2)$:

$$V_\mu = \bigoplus_{j\in\mathbb{Z}} V_\mu(j), \quad V_\mu(j) = \{v \in V_\mu :\ \mu(\Psi_\alpha(x))v = \chi_j(x)v\}.$$

Recall that the irreducible representations of $SO(2)$ are parametrized by integers:

$$\chi_j(e^{i\theta}) = e^{ij\theta}, \quad \text{for } e^{i\theta} \in SO(2) \cong S^1.$$

The action of $M \subset K$ preserves $V_\mu(j) + V_\mu(-j)$ and it could have fixed vectors only if j is even. Denote $(V_\mu(2j))^M := (V_\mu(2j) + V_\mu(-2j))^M$. Then we have a grading on $(V_\mu^*)^M$:

$$(V_\mu^*)^M = \bigoplus_{j\in\mathbb{Z}_+} (V_\mu(2j)^*)^M.$$

The following well-known result gives the action of $A_\mu(s_\alpha, \nu)$ on each $(V_\mu(2j)^*)^M$.

Theorem C.7. *On $(V_\mu(2j)^*)^M$,*

$$A_\mu(s_\alpha,\nu) = \begin{cases} Id & \text{if } j = 0, \\ \displaystyle\prod_{0<l\le j} \frac{(2l-1) - \langle \nu, \check{\alpha}\rangle}{(2l-1) + \langle \nu, \check{\alpha}\rangle} Id & \text{if } j \neq 0. \end{cases}$$

In order to match these operators to those from the p–adic case, we need to restrict to a special class of K-types.

Definition C.8. A representation $(\mu, V_\mu) \in \widehat{K}$ is called *petite* if, for every simple root α, the representation of $SO(2)$ (via Ψ_α) on V_μ contains only the characters χ_j with $|j| \le 3$.

For petite K-types, Theorem C.7 can be reformulated as follows.

Corollary C.9. *If (μ, V_μ) is a petite K-type, $A_\mu(s_\alpha, V)$ acts on $(V_\mu^*)^M$ as:*

$$A_\mu(s_\alpha,\nu) = \begin{cases} 1 & \textit{on the } (+1)\textit{–eigenspace of } s_\alpha, \\ \dfrac{1 - \langle \nu, \check{\alpha}\rangle}{1 + \langle \nu, \check{\alpha}\rangle} & \textit{on the } (-1)\textit{–eigenspace of } s_\alpha. \end{cases}$$

The corollary implies that on petite K-types, $A_\mu(\nu)$ is a product of $A_\mu(s_\alpha, \nu)$ corresponding to the reduced decomposition of w_0 and *it depends only on the W–structure of $(V^*)^M$*.

In the p–adic case, from the explicit description of the long intertwining operator on a Weyl representation (σ, V_σ), we can see that $r_\sigma(w_0, \nu) : (V_\sigma)^* \to (V_\sigma)^*$ decomposes into a product corresponding to the reduced decomposition of w_0 and the action of each factor $r_\sigma(s_\alpha, \nu)$ is given explicitly by

$$r_\sigma(s_\alpha,\nu) = \begin{cases} 1 & \text{on the } (+1)\text{–eigenspace of } t_\alpha, \\ \dfrac{1 - \langle \nu, \check{\alpha}\rangle}{1 + \langle \nu, \check{\alpha}\rangle} & \text{on the } (-1)\text{–eigenspace of } t_\alpha. \end{cases}$$

Note that $s_\alpha \in W$ and $t_\alpha \in \mathbb{C}[W]$ have the same action on V_σ. This shows the connection between the real and p–adic case.

Theorem C.10 (Barbasch, Vogan)**.** *If (μ, V_μ) is a petite real K-type, the real operator $A_\mu(\nu)$ coincides with the p–adic operator $r_{\tau(\mu)}(w_0, \nu)$.*

One can use this theorem to rule out nonunitary parameters in the real case, based on the calculations for the p–adic case. In the p–adic case, we have a list of relevant K-types, which are enough for the determination of unitarity. The conjecture is that the relevant K-types match some petite real K-types.

Conjecture C.11 (Barbasch)**.** *For each relevant K-type, $\sigma \in \widehat{W}$, for the p–adic group, one can find a petite K-type μ for the real group such that $\tau(\mu) = \sigma$.*

In the classical cases, this conjecture was proven by Barbasch (see [B2]). He calculates explicitly the petite K-type corresponding to each relevant K-type in the list from Theorem 2.11 in Section 2. I reproduce here his correspondence for the split noncompact classical groups:

Type	**Petite K-type**	**Relevant K-type**
A	$(\underbrace{2,\dots,2}_{m},\underbrace{0,\dots,0}_{n-m};+)$	$(m, n-m)$
B, D	$(\underbrace{1,\dots,1}_{m},\underbrace{0,\dots,0}_{n-m};+)\otimes(\underbrace{1,\dots,1}_{m},\underbrace{0,\dots,0}_{n-m};+)$	$(n-m)\times(m)$
	$(\underbrace{0,\dots,0}_{n};+)\otimes(\underbrace{2,\dots,2}_{m},\underbrace{0,\dots,0}_{n-m};+)$	$(m, n-m)\times(0)$
C	$(\underbrace{2,\dots,2}_{m},\underbrace{0,\dots,0}_{n-m})$	$(n-m)\times(m)$
	$(\underbrace{1,\dots,1}_{m},\underbrace{0,\dots,0}_{n-2m},\underbrace{-1,\dots,-1}_{m})$	$(m, n-m)\times(0)$

In the table, the real groups are:

G	**K**	**Type**
$GL(n,\mathbb{R})$	$O(n)$	A
$O(n,n)$	$O(n)\times O(n)$	B
$Sp(n,\mathbb{R})$	$U(n)$	C
$O(n+1,n)$	$O(n+1)\times O(n)$	D

and the notation for the representations of K is the classical one.

One hopes that the same machinery works for the exceptional groups and that a similar correspondence between petite (real) K-types and relevant (p–adic) K-types exists.

Finally, I should stress the point that Theorem C.10 gives a criterion for nonunitarity. After ruling out the parameters which are not positive semidefinite on the relevant K-types, one needs to show that the remaining parameters are unitary. The method in the p–adic case relies on the fact that the parameters corresponding to $\frac{1}{2}\check{h}$, where $\check{h}$ is the middle element of a nilpotent orbit $\check{\mathcal{O}}$, are unitary (these are the so–called anti-tempered parameters). In the real case, the same parameters do not come from tempered representations, but it is known that they are unitary (Barbasch). For a proof of this fact, see [B3] (also section 9 in [B2]).

References

[A] D. Alvis *Induce/Restrict matrices for exceptional Weyl groups*, preprint.

[B1] D. Barbasch *Spherical dual for p-adic groups*, Progress in Mathematics, vol. 158, 1998, Birkhäuser Boston, pp. 1-19. MR1486132 (98k:22068)

[B2] ——— *Unitary spherical spectrum for split classical groups* (to appear).

[B3] ——— *Unipotent representations for real reductive groups*, Proceedings of ICM, Kyoto 1990, Springer–Verlag, The Mathematical Society of Japan, 1990, pp. 769–777. MR1159263 (93m:22012)

[BM1] D. Barbasch, A. Moy *A unitarity criterion for p-adic groups*, Invent. Math. 98, 1989, pp. 19-38. MR1010153 (90m:22038)

[BM2] ——— *Reduction to real infinitesimal character in affine Hecke algebras*, J. Amer. Math. Soc. 6(3), 1993, pp. 611-635. MR1186959 (93k:22015)

[BM3] ——— *Unitary spherical spectrum for p-adic classical groups*, Acta Appl. Math. 44, 1996, pp. 3-37. MR1407038 (98k:22067)

[BW] A. Borel, N. Wallach *Continuous cohomology, discrete subgroups and representations of reductive groups*, volume 94 of Annals of Mathematics Studies, Princeton University Press, Princeton, 1980. MR0554917 (83c:22018)

[Ca] R. Carter *Finite groups of Lie type*, Wiley-Interscience, New York, 1985. MR0794307 (87d:20060)

[K] T. Kondo *The characters of the Weyl group of type F_4*, J. Fac. Sci. Univ. Tokyo, 11, 1965, pp. 145-153. MR0185018 (32 #2488)

[KZ] A. Knapp, G. Zuckerman *Classification theorems for representations of semisimple Lie groups*, in *Non–commutative harmonic analysis (Actes Colloq., Marseille–Luminy, 1976)*, Lecture Notes in Math., Vol. 587, Springer, 1977, pp. 138–159. MR476923 (57 #16474)

[Kn] A. Knapp *Representation theory of semisimple groups: an overview based on examples*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1986. MR0855239 (87j:22022)

[KL] D. Kazhdan, G. Lusztig *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. 87, 1987, pp. 153-215. MR0862716 (88d:11121)

[L1] G. Lusztig *Affine Hecke algebras and their graded version*, Jour. AMS 2, 1989, pp. 599-635. MR0991016 (90e:16049)

[L2] ——— *Intersection cohomology complexes on a reductive group*, Invent. Math. 75, 1984, pp. 205-272. MR0732546 (86d:20050)

[L3] ——— *Cuspidal local systems and graded algebras I*, Publ. Math. IHES 67, 1988, pp. 145-202. MR0972345 (90e:22029)

[L4] ——— *Cuspidal local systems and graded algebras II*, Representations of groups (Banff, AB, 1994), Amer. Math. Soc., Providence, 1995, pp. 217-275. MR1357201 (96m:22038)

[L5] ——— *Characters of reductive groups over finite fields*, Annals of Math. Studies, vol. 107, Princeton University Press. MR0742472 (86j:20038)

[M] G. Muić *The unitary dual of p–adic G_2*, Duke Math. J., 90(3), 1997, pp. 465–493. MR1480543 (98k:22073)

[T] M. Tadić *Classification of unitary representations in irreducible representations of the general linear group (nonarchimedean case)*, Ann. Scient. Ec. Norm. Sup. 19 (1986), pp. 335–382. MR0870688 (88b:22021)

[V1] D. A. Vogan Jr. *The unitary dual of G_2*, Invent. Math. 116, 1994, no. 1-3, pp. 677-791. MR1253210 (95b:22037)

[V2] ——— *Computing the unitary dual*, notes at atlas.math.umd.edu/papers

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