REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 9, Pages 209–266 (March 25, 2005)
S 1088-4165(05)00278-5

CHARACTER SHEAVES ON DISCONNECTED GROUPS, VII

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Abstract. We define and study convolution of parabolic character sheaves. As an application, we attach to any parabolic character sheaf the orbit of a tame local system on the maximal torus under a subgroup of the Weyl group.

Introduction

Throughout this paper, G denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field $\mathbf{k}$. This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on G.

The usual convolution of class functions on a connected reductive group over a finite field makes sense also for complexes in $\mathcal{D}(G^0)$ and then it preserves (see [Gi]) in the derived sense the class of character sheaves on G^0. In §32 we define, more generally, a natural convolution operation for parabolic character sheaves (see 32.21(a)). A key role in our study of convolution is played by Theorem 32.6 which describes explicitly the convolution of two basic complexes of the form $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$ in terms of multiplication in some Hecke algebra. Using this we define a map which to each parabolic character sheaf associates an orbit of a subgroup of the Weyl group on the set of isomorphism classes of "tame" local systems of rank 1 on the torus $\mathbf{T}$ (see 32.25(b)); in fact, we define a refinement of this map in 32.25(a). The main result of §33 is Proposition 33.3 (a generalization of [L3, III,14.2(b)]). It asserts that (under a cleanness assumption), the cohomology sheaves of a character sheaf restricted to an open subset of the support of a different character sheaf are disjoint from the local system given by the second character sheaf on that open subset. (This plays a key role in the argument in Lemma 35.21.) In §34 we study the algebra H_n of 31.2 (or rather an extension H_n^D of it) in the spirit of our earlier study [L12] of a usual Iwahori-Hecke algebra by means of the asymptotic Hecke algebra. This allows us to construct representations of H_n^D starting from representations of $H_n^{D,1}$, the specialization of H_n^D at $v = 1$. In 34.19 we define some invariants $b^v_{A,u}$ of a character sheaf A which depend also on an irreducible representation E_u of $H_n^{D,1}$. These generalize the invariants $c_{A,E}$ of [L3, III, 12.10]. From the definition, $b^v_{A,u}$ is a rational function in the indeterminate v and one of our goals is to show that $b^v_{A,u}$ is in fact a constant. This goal is achieved in §35 under a cleanness assumption and a quasi-rationality assumption on E_u. (See Theorem 35.22 which is a generalization of [L3, III, 14.9].) In §35 we prove an orthogonality formula (Proposition 35.15) for the characteristic functions of complexes of the form $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$

Received by the editors November 4, 2004.
2000 *Mathematics Subject Classification.* Primary 20G99.
Supported in part by the National Science Foundation.

(over a finite field) in the spirit of [L3, 13.5, III]. A variant of this formula (see Corollary 32.23) can be obtained in an entirely different way as an application of the results on convolution in §32. As an application we associate a sign ± 1 to any character sheaf on a connected component of G (see 35.17), under a cleanness assumption. This generalizes [L3, III, 13.10].

Erratum to Part V. In 25.6 replace $R_1^* = R^* \cap R$ by $R_1^* = R^* \cap R_1$.

Erratum to Part VI. In 28.19 replace $\mathcal{L}' = (\underline{D}^{-1})^* \mathcal{L}$ by $\mathcal{L}' = (\underline{D}^{-1})^* \check{\mathcal{L}}$. In 31.4 replace ι_D by $\underline{D}$.

Contents

32. Convolution

32.1. In this section we define and study the convolution of parabolic character sheaves.

32.2. Let Δ be a connected component of G. Let $J \subset \mathbf{I}$. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$, $\mathbf{s}' = (s'_1, s'_2, \dots, s'_{r'})$ be two sequences in $\mathbf{I}$. Let $a, a' \in \mathbf{W}$. Let

$$\mathbf{w} = (s_1, s_2, \dots, s_r, a, s'_1, s'_2, \dots, s'_{r'}, a'), \quad [\mathbf{w}] = s_1 s_2 \dots s_r a s'_1 s'_2 \dots s'_{r'} a'.$$

Let $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ be such that $[\mathbf{w}]\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Let

$$\mathcal{T} = \{i \in [1, r]; s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1 \in \mathbf{W}_{\mathcal{L}}\},$$
$$\mathcal{T}' = \{j \in [1, r']; a'^{-1} s'_r \dots s'_{j+1} s'_j s'_{j+1} \dots s'_{r'} a' \in \epsilon_\Delta(\mathbf{W}_{\mathcal{L}})\},$$

$$\begin{aligned}\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta} = \{&(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}); B_i \in \mathcal{B}(i \in [0, r]),\\ &B'_j \in \mathcal{B}(j \in [0, r']), B \in \mathcal{B}, x \in \Delta, \mathrm{pos}(B_{i-1}, B_i) \in \{s_i, 1\}(i \in \mathcal{T}),\\ &\mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1, r] - \mathcal{T}), \mathrm{pos}(B'_{j-1}, B'_j) \in \{s'_j, 1\}(j \in \mathcal{T}'),\\ &\mathrm{pos}(B'_{j-1}, B'_j) = s'_j(j \in [1, r'] - \mathcal{T}'), \mathrm{pos}(B_r, B'_0) = a, \mathrm{pos}(B'_{r'}, B) = a',\\ &xB_0x^{-1} = B\}.\end{aligned}$$

Then $Z^{\mathbf{w}}_{\emptyset,J,\Delta}$ (see 28.8) is naturally an open subset of $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta}$. By 28.8, $\mathcal{L}$ gives rise to a local system $\tilde{\mathcal{L}}$ on $Z^{\mathbf{w}}_{\emptyset,J,\Delta}$.

(a) *$\tilde{\mathcal{L}}$ extends uniquely to a local system $\bar{\mathcal{L}}$ on $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta}$.*

Indeed, let $a = t_1 t_2 \dots t_m, a' = t'_1 t'_2 \dots t'_{m'}$ be reduced expressions for a, a' in $\mathbf{W}$ and let

$$\mathbf{t} = (s_1, s_2, \dots, s_r, t_1, t_2, \dots, t_m, s'_1, s'_2, \dots, s'_{r'}, t'_1, t'_2, \dots, t'_{m'}).$$

We identify in an obvious way $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta}$ with an open subset of $\bar{Z}^{\mathbf{t}}_{\emptyset,J,\Delta}$ (see 28.9) contained in $\cup_{\mathcal{J} \subset \mathcal{J}_{\mathbf{t}}} Z^{\mathbf{t}_{\mathcal{J}}}_{\emptyset,J,\Delta}$ (notation of 28.9) and we use the fact that $\tilde{\mathcal{L}}$, regarded as a local system on $Z^{\mathbf{w}}_{\emptyset,J,\Delta} = Z^{\mathbf{t}}_{\emptyset,J,\Delta}$ extends to a local system on $\cup_{\mathcal{J} \subset \mathcal{J}_{\mathbf{t}}} Z^{\mathbf{t}_{\mathcal{J}}}_{\emptyset,J,\Delta}$; see 28.10. This extension is unique up to isomorphism since $Z^{\mathbf{w}}_{\emptyset,J,\Delta}$ is open dense in the smooth irreducible variety $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta}$ (which is itself open dense in the smooth irreducible variety $\bar{Z}^{\mathbf{t}}_{\emptyset,J,\Delta}$).

32.3. For any $\mathbf{a} = (a_0, a_1, \dots, a_{r+r'}) \in \mathbf{W}^{r+r'+1}$ let

$$\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} = \{(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}) \in \underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta};$$
$$\mathrm{pos}(B_k, B'_{r'}) = a_k (k \in [0,r]), \mathrm{pos}(B_r, B'_{r+r'-k}) = a_k (k \in [r, r+r'])\}.$$

Define $\pi_{\mathbf{w},\mathbf{a}} : \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} \to Z_{J,\Delta}$ by

$$(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'r', B, xU_{J,B_0}) \mapsto (Q_{J,B_0}, Q_{\epsilon_\Delta(J),B}, xU_{J,B_0})$$

(notation of 28.7). Now $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ is empty unless

(i) $a_{r+r'} = a$,
(ii) $a_k \in \{a_{k-1}, s_k a_{k-1}\}$ for $k \in [1,r]$,
(iii) $a_k \in \{a_{k-1}, a_{k-1}s'_{r+r'+1-k}\}$ for $k \in [r+1, r+r']$.

Indeed, let $(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}) \in \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. Clearly, (i) holds. Let $k \in [1,r]$. From $\mathrm{pos}(B_k, B_{k-1}) \in \{1, s_k\}$, $\mathrm{pos}(B_{k-1}, B'_{r'}) = a_{k-1}$, we deduce $\mathrm{pos}(B_k, B'_{r'}) \in \{a_{k-1}, s_k a_{k-1}\}$ and (ii) holds. Let $k \in [r+1, r+r']$. From

$$\mathrm{pos}(B_r, B'_{r+r'+1-k}) = a_{k-1}, \mathrm{pos}(B'_{r+r'+1-k}, B'_{r+r'-k}) \in \{1, s'_{r+r'+1-k}\},$$

we deduce $\mathrm{pos}(B_r, B'_{r+r'-k}) \in \{a_{k-1}, a_{k-1}s'_{r+r'+1-k}\}$ and (iii) holds.

Let $\bar{\mathcal{L}}_{\mathbf{a}}$ be the restriction of the local system $\bar{\mathcal{L}}$ from $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta}$ to $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. We have a partition $\underline{Z}^{\mathbf{w}}_{\emptyset,J,\Delta} = \sqcup_{\mathbf{a}} \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ with $\mathbf{a} \in \mathbf{W}^{r+r'+1}$ subject to (i),(ii),(iii). We set

$$N_{\mathbf{a}} = |\{k \in [1,r], a_k > s_k a_k\}| + |\{k \in [r+1, r+r'], a_k > a_k s'_{r+r'+1-k}\}|,$$
$$\mathcal{T}_{\mathbf{a}} = \{i \in \mathcal{T}; a_{i-1} = a_i < s_i a_i\},$$
$$\mathcal{T}'_{\mathbf{a}} = \{j \in \mathcal{T}'; a_{r+r'-j} = a_{r+r'-j+1} < a_{r+r'-j+1}s'_j\}.$$

Lemma 32.4. *Assume that* $\mathbf{a}$ *satisfies 32.3(i),(ii),(iii).*

(a) $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ *is non-empty if and only if* $i \in [1,r], a_{i-1} = a_i < s_i a_i \implies i \in \mathcal{T}$, *and* $j \in [1,r'], a_{-j+r+r'} = a_{-j+1+r+r'} < a_{-j+1+r+r'}s'_j \implies j \in \mathcal{T}'$.

(b) If $(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}) \in \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$, *then* $B_{i-1} = B_i$ *for any* $i \in \mathcal{T}_{\mathbf{a}}$ *and* $B'_{j-1} = B'_j$ *for any* $j \in \mathcal{T}'_{\mathbf{a}}$.

(c) Let ${}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ *be the subset of* $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ *defined by the following conditions: for* $i \in [1,r]$ *we have* $B_{i-1} = B_i$ *if and only if* $i \in \mathcal{T}_{\mathbf{a}}$*; for* $j \in [1,r']$ *we have* $B'_{j-1} = B'_j$ *if and only if* $j \in \mathcal{T}'_{\mathbf{a}}$. *If* $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} \neq \emptyset$, *then it is smooth, irreducible and* ${}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ *is open dense in* $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$.

(d) If $a_k = a_{k-1}$ *for some* $k \in [1,r]$ *with* $k \notin \mathcal{T}$ *or for some* $k \in [r+1, r+r']$ *with* $r+r'+1-k \notin \mathcal{T}'$, *then* $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = 0$.

(e) If $a_k \neq a_{k-1}$ *for any* $k \in [1,r]$ *with* $k \notin \mathcal{T}$ *and for any* $k \in [r+1, r+r']$ *with* $r+r'+1-k \notin \mathcal{T}'$, *then* $a_0 a' \underline{\Delta} \in \mathbf{W}^{\bullet}_{\mathcal{L}}$*; moreover,* $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ *is an iterated affine space bundle over* $Z^{(a_0,b)}_{\emptyset,J,\Delta}$ *with fibres of dimension* $N_{\mathbf{a}}$ *and* $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = K^{(a_0,a'),\mathcal{L}}_{J,\Delta}[[-N_{\mathbf{a}}]]$.

We prove (d),(e) by induction on $r+r'$. If $r+r' = 0$, then $\mathbf{w} = (a_0, a'), \mathbf{a} = \{a_0\}$ and

$$\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} = Z^{\mathbf{w}}_{\emptyset,J,\Delta} = \{(B_0, B'_0, B, xU_{J,B_0}); B_0 \in \mathcal{B}, B'_0 \in \mathcal{B}, B \in \mathcal{B}, x \in \Delta,$$
$$\mathrm{pos}(B_0, B'_0) = a_0, \mathrm{pos}(B'_0, B) = a', xB_0x^{-1} = B\}.$$

Hence $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = K^{\mathbf{w},\mathcal{L}}_{J,\Delta}$. Now assume that $r + r' \ge 1$. Assume first that $r' \ge 1$. Let

$$\mathbf{w}' = (s_1, s_2, \dots, s_r, a_{r+r'-1}, s'_{r'-1}, s'_{r'-2}, \dots, s'_1, a'),$$
$$[\mathbf{w}'] = s_1 s_2 \dots s_r a_{r+r'-1} s'_{r'-1} s'_{r'-2} \dots s'_1 a',$$
$$\mathbf{a}' = (a_0, a_1, \dots, a_r, a_{r+1}, \dots, a_{r+r'-1}),$$

$$\begin{aligned}
&Y = \{(B_0, B_1, \dots, B_r, B'_1, B'_2, \dots, B'_{r'}, B, xU_{J,B_0}); B_i \in \mathcal{B}(i \in [0, r]),\\
&B'_j \in \mathcal{B}(j \in [1, r']), B \in \mathcal{B}, x \in \Delta, \mathrm{pos}(B_{i-1}, B_i) \in \{s_i, 1\}(i \in \mathcal{T}),\\
&\mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1, r] - \mathcal{T}), \mathrm{pos}(B'_{j-1}, B'_j) \in \{s'_j, 1\}(j \in \mathcal{T}' \cap [2, r']),\\
&\mathrm{pos}(B'_{j-1}, B'_j) = s'_j(j \in [2, r'] - \mathcal{T}'), \mathrm{pos}(B_k, B'_{r'}) = a_k(k \in [0, r]),\\
&\mathrm{pos}(B_r, B'_{r+r'-k}) = a_k(k \in [r, r + r' - 1]), \mathrm{pos}(B'_{r'}, B) = a', xB_0x^{-1} = B'\}.
\end{aligned}$$

Define $\pi^Y : Y \to Z_{J,\Delta}$, $f : \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} \to Y$ by

$$\pi^Y : (B_0, B_1, \dots, B_r, B'_1, B'_2, \dots, B'_{r'}, B, xU_{J,B_0}) \mapsto (Q_{J,B_0}, Q_{\epsilon_\Delta(J),B}, xU_{J,B_0}),$$
$$f : (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_r, B'_1, B'_2, \dots, B'_{r'}, xU_{J,B_0}).$$

The fibre of f at $(B_0, B_1, \dots, B_r, B'_1, B'_2, \dots, B'_{r'}, xU_{J,B_0}) \in Y$ may be identified with

$$\{B'_0 \in \mathcal{B}; \mathrm{pos}(B'_0, B'_1) \in \{s'_1, 1\} \text{ if } 1 \in \mathcal{T}', \mathrm{pos}(B'_0, B'_1) = s'_1 \text{ if } 1 \notin \mathcal{T}', \mathrm{pos}(B_r, B'_0) = a_{r+r'}\}.$$

If $a_{r+r'}s'_1 = a_{r+r'-1}$, then $[\mathbf{w}']\underline{\Delta} = [\mathbf{w}]\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}$ and the sets analogous to $\mathcal{T}, \mathcal{T}'$ (for $\mathbf{w}'$ instead of $\mathbf{w}$) are $\mathcal{T}, \mathcal{T}' \cap [2, r']$. Moreover, $\underline{Z}^{\mathbf{w}',\mathbf{a}'}_{\emptyset,J,\Delta} = Y$ and f is an isomorphism if $a_{r+r'} < a_{r+r'}s'_1$ and an affine line bundle if $a_{r+r'} > a_{r+r'}s'_1$. In both cases, $\bar{\mathcal{L}}_{\mathbf{a}} = f^*(\bar{\mathcal{L}}_{\mathbf{a}'})$. Hence

$$f_!\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}'}, \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}f_!\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}\bar{\mathcal{L}}_{\mathbf{a}'} \text{ if } a_{r+r'} < a_{r+r'}s'_1,$$
$$f_!\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}'}[[-1]], \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}f_!\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}\bar{\mathcal{L}}_{\mathbf{a}'}[[-1]] \text{ if } a_{r+r'} > a_{r+r'}s'_1;$$

the desired result follows from the induction hypothesis.

If $a_{r+r'-1} = a_{r+r'} < a_{r+r'}s'_1$ and $1 \notin \mathcal{T}'$, then $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} = \emptyset$. If $a_{r+r'-1} = a_{r+r'} > a_{r+r'}s'_1$ and $1 \notin \mathcal{T}'$, then f is a $\mathbf{k}^*$-bundle and $f_!\bar{\mathcal{L}}_{\mathbf{a}} = 0$ (this can be deduced from 28.11) hence $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi^Y_! f_!\bar{\mathcal{L}}_{\mathbf{a}} = 0$.

Assume now that $a_{r+r'-1} = a_{r+r'}$ and $1 \in \mathcal{T}'$. Then $[\mathbf{w}']\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}$ and the sets analogous to $\mathcal{T}, \mathcal{T}'$ (for $\mathbf{w}'$ instead of $\mathbf{w}$) are $\mathcal{T}, \mathcal{T}' \cap [2, r']$. Moreover, $\underline{Z}^{\mathbf{w}',\mathbf{a}'}_{\emptyset,J,\Delta} = Y$; also, f is an isomorphism if $a_{r+r'} < a_{r+r'}s'_1$ and an affine line bundle if $a_{r+r'} > a_{r+r'}s'_1$. In both cases, $\bar{\mathcal{L}}_{\mathbf{a}} = f^*(\bar{\mathcal{L}}_{\mathbf{a}'})$. Hence

$$f_!\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}'}, \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}f_!\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}\bar{\mathcal{L}}_{\mathbf{a}'} \text{ if } a_{r+r'} < a_{r+r'}s'_1,$$
$$f_!\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}'}[[-1]], \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}f_!\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}',\mathbf{a}'!}\bar{\mathcal{L}}_{\mathbf{a}'}[[-1]] \text{ if } a_{r+r'} > a_{r+r'}s'_1;$$

the desired result follows from the induction hypothesis.

Assume next that $r' = 0$. Then $r \ge 1$. Let

$$\mathbf{w}'' = (s_1, s_2, \dots, s_{r-1}, a_{r-1}, a'), [\mathbf{w}''] = s_1 s_2 \dots s_{r-1} a_{r-1} a',$$
$$\mathbf{a}'' = (a_0, a_1, \dots, a_{r-1}),$$

$$Y_1 = \{(B_0, B_1, \dots, B_{r-1}, B'_0, B, xU_{J,B_0}); B_i \in \mathcal{B}(i \in [0, r-1]), B'_0 \in \mathcal{B},$$
$$B \in \mathcal{B}, x \in \Delta, \text{pos}(B_{i-1}, B_i) \in \{s_i, 1\}(i \in \mathcal{T} \cap [1, r-1]),$$
$$\text{pos}(B_{i-1}, B_i) = s_i(i \in [1, r-1] - \mathcal{T}), \text{pos}(B_k, B'_0) = a_k(k \in [0, r-1]),$$
$$\text{pos}(B'_0, B) = a', xB_0x^{-1} = B\}.$$

Define $\pi^{Y_1} : Y_1 \to Z_{J,\Delta}$, $f_1 : \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} \to Y_1$ by

$$\pi^{Y_1} : (B_0, B_1, \dots, B_{r-1}, B'_0, B, xU_{J,B_0}) \mapsto (Q_{J,B_0}, Q_{\epsilon_\Delta(J),B}, xU_{J,B_0}),$$
$$f_1 : (B_0, B_1, \dots, B_r, B'_0, B, xU_{J,B_0}) \mapsto (B_0, B_1, \dots, B_{r-1}, B'_0, B, xU_{J,B_0}).$$

The fibre of f_1 at $(B_0, B_1, \dots, B_{r-1}, B'_0, B, xU_{J,B_0}) \in Y_1$ may be identified with

$$\{B_r \in \mathcal{B}; \text{pos}(B_{r-1}, B_r) \in \{s_r, 1\} \text{ if } r \in \mathcal{T}, \text{pos}(B_{r-1}, B_r) = s_r \text{ if } r \notin \mathcal{T},$$
$$\text{pos}(B_r, B'_0) = a_r\}.$$

If $s_r a_r = a_{r-1}$, then $[\mathbf{w}'']\underline{\Delta} = [\mathbf{w}]\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}$ and the set analogous to $\mathcal{T}$ (for $\mathbf{w}''$ instead of $\mathbf{w}$) is $\mathcal{T} \cap [1, r-1]$. Moreover, $\underline{Z}^{\mathbf{w}'',\mathbf{a}''}_{\emptyset,J,\Delta} = Y_1$ and f_1 is an isomorphism if $a_r < s_r a_r$ and an affine line bundle if $a_r > s_r a_r$. In both cases, $\bar{\mathcal{L}}_{\mathbf{a}} = f_1^*(\bar{\mathcal{L}}_{\mathbf{a}''})$. Hence

$$f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}''}, \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}\bar{\mathcal{L}}_{\mathbf{a}''} \text{ if } a_r < s_r a_r,$$
$$f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}''}[[-1]], \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}\bar{\mathcal{L}}_{\mathbf{a}''}[[-1]] \text{ if } a_r > s_r a_r;$$

the desired result follows from the induction hypothesis.

If $a_{r-1} = a_r < s_r a_r$ and $r \notin \mathcal{T}$, then $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} = \emptyset$. If $a_{r-1} = a_r > s_r a_r$ and $r \notin \mathcal{T}$, then f_1 is a $\mathbf{k}^*$-bundle and $f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = 0$ (this can be deduced from 28.11) hence $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi^{Y_1}_! f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = 0$.

Assume now that $a_{r-1} = a_r$ and $r \in \mathcal{T}$. Then $[\mathbf{w}'']\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}$ and the set analogous to $\mathcal{T}$ (for $\mathbf{w}''$ instead of $\mathbf{w}$) is $\mathcal{T} \cap [1, r-1]$. Moreover, $\underline{Z}^{\mathbf{w}'',\mathbf{a}''}_{\emptyset,J,\Delta} = Y_1$; also, f_1 is an isomorphism if $a_r < s_r a_r$ and an affine line bundle if $a_r > s_r a_r$. In both cases, $\bar{\mathcal{L}}_{\mathbf{a}} = f_1^*(\bar{\mathcal{L}}_{\mathbf{a}''})$. Hence

$$f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}''}, \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}\bar{\mathcal{L}}_{\mathbf{a}''} \text{ if } a_r < s_r a_r,$$
$$f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}''}[[-1]], \pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}f_{1!}\bar{\mathcal{L}}_{\mathbf{a}} = \pi_{\mathbf{w}'',\mathbf{a}''!}\bar{\mathcal{L}}_{\mathbf{a}''}[[-1]] \text{ if } a_r > s_r a_r;$$

the desired result follows from the induction hypothesis. This completes the proof of (d),(e). The previous inductive proof also yields (a),(b),(c). The lemma is proved.

32.5. Let $J \subset \mathbf{I}$. Let D, D', Δ be three connected components of G with $\Delta = D'D$. We write ϵ, ϵ' instead of $\epsilon_D, \epsilon_{D'} : \mathbf{W} \to \mathbf{W}$. We have a diagram

$$Z_{J,D} \times Z_{\epsilon(J),D'} \xleftarrow{b_1} Z_0 \xrightarrow{>b_2} Z_{J,\Delta}$$

where

$$Z_0 = \{(Q, Q', Q'', gU_Q, g'U_{Q'}); Q \in \mathcal{P}_J, Q' \in \mathcal{P}_{\epsilon(J)}, Q'' \in \mathcal{P}_{\epsilon'\epsilon(J)},$$
$$g \in D, g' \in D', gQg^{-1} = Q', g'Q'g'^{-1} = Q''\},$$
$$b_1(Q, Q', Q'', gU_Q, g'U_{Q'}) = ((Q, Q', gU_Q), (Q', Q'', g'U_{Q'})),$$
$$b_2(Q, Q', Q'', gU_Q, g'U_{Q'}) = (Q, Q'', g'gU_Q).$$

Define a functor (*convolution*) $\mathcal{D}(Z_{J,D}) \times \mathcal{D}(Z_{\epsilon(J),D'}) \to \mathcal{D}(Z_{J,\Delta})$ by

$$K, K' \mapsto K * K' = b_{2!}b_1^*(K \boxtimes K').$$

Let $\mathbf{s} = (s_1, s_2, \dots, s_r), \mathbf{s}' = (s'_1, s'_2, \dots, s'_{r'})$ be two sequences in $\mathbf{I}$ and let $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}(\mathbf{T})$ be such that $s_1 s_2 \dots s_r \underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, $s'_1 s'_2 \dots s'_{r'} \underline{D}' \in \mathbf{W}^\bullet_{\mathcal{L}'}$. Then $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} \in \mathcal{D}(Z_{J,D})$, $\bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'} \in \mathcal{D}(Z_{\epsilon(J),D'})$ are defined (see 28.12); hence $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'} \in \mathcal{D}(Z_{J,\Delta})$ is defined.

Let $n \in \mathbf{N}^*_{\mathbf{k}}$ be such that $\mathcal{L} \in \mathfrak{s}_n, \mathcal{L}' \in \mathfrak{s}_n$. Let $\lambda \in \underline{\mathfrak{s}}_n$ (resp. $\lambda' \in \underline{\mathfrak{s}}_n$) be the isomorphism class of $\mathcal{L}$ (resp. $\mathcal{L}'$).

Theorem 32.6. *(a) If* $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'} \neq 0$*, then* $y\lambda' = \underline{D}\lambda$ *for some* $y \in \mathbf{W}_{\epsilon(J)}$.

(b) Let A *be a simple perverse sheaf on* $Z_{J,\Delta}$*. If* $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}$*, then* $A \in \hat{Z}^{\mathcal{L}}_{J,\Delta}$.

(c) Assume that $\mathbf{k}, \mathbf{F}_q, G, F$ *are as in 31.7(b), that* $A \in \hat{Z}_{J,\Delta}$ *and that* $\zeta^A : H_n[\Delta] \to \mathcal{A}$ *is as in 31.7. Then*

$$\chi^A_v(\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}) = (v^2-1)^{\dim \mathbf{T}} v^{\dim G - l(w^0_{\mathbf{I}} w^0_J)}$$
$$\times \zeta^A\Big(\sum_{\substack{y' \in \mathbf{W}_J \\ y'\underline{D}^{-1}\lambda' = \lambda}} v^{2l(w^0_J y')} C^{\mathbf{s}}_{\underline{D}\lambda}[D] T_{y'} [D^{-1}] C^{\mathbf{s}'}_{\underline{D}'\lambda'}[D'][D] T_{y'^{-1}} \Big).$$

(Notation of 31.5, 31.6. We regard $\mathcal{L}, \mathcal{L}'$ *as pure of weight* 0 *and then* $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$, $\bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}$ *and their convolution naturally as mixed complexes.)*

The proof is given in 32.7–32.19.

32.7. With notation of 28.7, let

$$V = \{(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, gU_{J,B_0}, g'U_{\epsilon(J),B'_0}); B_i \in \mathcal{B}(i \in [0,r]),$$
$$B'_j \in \mathcal{B}(j \in [0,r']), g \in D, g' \in D', gB_0g^{-1} = B_r, g'B'_0g'^{-1} = B'_{r'},$$
$$\mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}(i \in [1,r]), \mathrm{pos}(B'_{j-1}, B'_j) \in \{1, s'_j\}(j \in [1,r']),$$
$$\mathrm{pos}(B_r, B'_0) \in \mathbf{W}_{\epsilon(J)}\}.$$

Let $\bar{\mathcal{L}}$ be the constructible sheaf on $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ in 28.10 and let $\bar{\mathcal{L}}'$ be the analogous constructible sheaf on $\bar{Z}^{\mathbf{s}'}_{\emptyset,\epsilon(J),D'}$. The inverse image of $\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'$ under the imbedding $f : V \to \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \times \bar{Z}^{\mathbf{s}'}_{\emptyset,\epsilon(J),D'}$,

$$(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, gU_{J,B_0}, g'U_{\epsilon(J),B'_0})$$
$$\mapsto ((B_0, B_1, \dots, B_r, gU_{J,B_0}), (B'_0, B'_1, \dots, B'_{r'}, g'U_{\epsilon(J),B'_0}))$$

is a constructible sheaf on V denoted again by $\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'$. Define $\rho : V \to Z_{J,\Delta}$ by

$$(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, gU_{J,B_0}, g'U_{\epsilon(J),B'_0}) \mapsto (Q_{J,B_0}, Q_{\epsilon'\epsilon(J),B'_{r'}}, xU_{J,B_0})$$

where $x = g'g \in \Delta$; this is meaningful since $g'U_{\epsilon(J),B'_0}g = g'U_{\epsilon(J),B_r}g = g'gU_{J,B_0}$. We have a commutative diagram in which the left square is cartesian

$$\begin{array}{ccccc} \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \times \bar{Z}^{\mathbf{s}'}_{\emptyset,\epsilon(J),D'} & \xleftarrow{f} & V & \xrightarrow{\rho} & Z_{J,\Delta} \\ h\downarrow & & h_0\downarrow & & 1\downarrow \\ Z_{J,D} \times Z_{\epsilon(J),D'} & \xleftarrow{b_1} & Z_0 & \xrightarrow{b_2} & Z_{J,\Delta} \end{array}$$

where

$$h:((B_0,B_1,\dots,B_r,gU_{J,B_0}),(B'_0,B'_1,\dots,B'_{r'},g'U_{\epsilon(J),B'_0}))$$
$$\mapsto((Q_{J,B_0},Q_{\epsilon(J),B_r},gU_{J,B_0}),(Q_{\epsilon(J),B'_0},Q_{\epsilon'\epsilon(J),B'_{r'}},g'U_{\epsilon(J),B'_0})),$$

$$h_0:(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},gU_{J,B_0},g'U_{\epsilon(J),B'_0})$$
$$\mapsto((Q_{J,B_0},Q_{\epsilon(J),B_r},Q_{\epsilon'\epsilon(J),B'_{r'}},gU_{J,B_0}),g'U_{\epsilon(J),B_r}).$$

Using this commutative diagram and the definitions we have

$$\bar K^{\mathbf{s},\mathcal{L}}_{J,D}*\bar K^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}=\rho_!(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}').$$

Let

$$\mathcal{T}=\{i\in[1,r];s_1s_2\dots s_i\dots s_2s_1\in\mathbf{W}_{\mathcal{L}}\},$$
$$\mathcal{T}'=\{j\in[1,r'];s'_{r'}\dots s'_{j+1}s'_js'_{j+1}\dots s'_{r'}\in\epsilon'(\mathbf{W}_{\mathcal{L}'})\}.$$

(Thus, $\mathcal{T}=\mathcal{J}_{\mathbf{s}}$, $\mathcal{T}'=\mathcal{J}_{\mathbf{s}'}$ with the notation of 28.9.) Let

$$V'=\{(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},gU_{J,B_0},g'U_{\epsilon(J),B'_0});B_i\in\mathcal{B}(i\in[0,r]),$$
$$B'_j\in\mathcal{B}(j\in[0,r']),g\in D,g'\in D',gB_0g^{-1}=B_r,g'B'_0g'^{-1}=B'_{r'},$$
$$\text{pos}(B_{i-1},B_i)\in\{1,s_i\}(i\in\mathcal{T}),\text{pos}(B_{i-1},B_i)=s_i(i\in[1,r]-\mathcal{T}),$$
$$\text{pos}(B'_{j-1},B'_j)\in\{1,s'_j\}(j\in\mathcal{T}'),\text{pos}(B'_{j-1},B'_j)=s'_j(j\in[1,r']-\mathcal{T}'),$$
$$\text{pos}(B_r,B'_0)\in\mathbf{W}_{\epsilon(J)}\},$$

an open subset of V. Let $\rho':V'\to Z_{J,\Delta}$ be the restriction of $\rho:V\to Z_{J,\Delta}$. From 28.10 we see that $(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}')|_{V'}$ is a local system and $(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}')|_{V-V'}=0$. Hence

$$\rho_!(V,\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}')=\rho'_!(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}').$$

For any

(a) $$\mathbf{a}=(a_0,a_1,\dots,a_{r+r'})\in\mathbf{W}^{r+r'}\times\mathbf{W}_{\epsilon(J)},$$

let

$$V'_{\mathbf{a}}=\{(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},gU_{J,B_0},g'U_{\epsilon(J),B'_0})\in V';$$
$$\text{pos}(B_k,B'_{r'})=a_k(k\in[0,r]),\text{pos}(B_r,B'_{r+r'-k})=a_k(k\in[r,r+r'])\}.$$

Let $\rho'_{\mathbf{a}}:V'_{\mathbf{a}}\to Z_{J,\Delta}$ be the restriction of $\rho':V'\to Z_{J,\Delta}$. Then $V'=\cup_{\mathbf{a}}V'_{\mathbf{a}}$ is a partition of V' with $V'_{\mathbf{a}}$ locally closed in V' for all $\mathbf{a}$.

Lemma 32.8. *Let* $\mathbf{a}$ *as in 32.7(a) be such that* $(\underline{D}^{-1})^*\mathcal{L}\not\cong(a^{-1}_{r+r'})^*\mathcal{L}'$. *Then* $\rho'_{\mathbf{a}}(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}')=0$.

Define R, $\tilde\rho:R\to Z_{J,\Delta}$, $\pi:V'_{\mathbf{a}}\to R$ by

$$R=\{(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},xU_{J,B_0});B_i\in\mathcal{B}(i\in[0,r]),$$
$$B'_j\in\mathcal{B}(j\in[0,r']),x\in\Delta,\text{pos}(B_r,B'_0)=a_{r+r'},\text{pos}(B'_{r'},xB_0x^{-1})=\epsilon'(a^{-1}_{r+r'})\},$$

$$\tilde\rho:(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},xU_{J,B_0})\mapsto(Q_{J,B_0},xQ_{J,B_0},xU_{J,B_0}),$$

$$\pi:(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},gU_{J,B_0},g'U_{\epsilon(J),B'_0})$$
(a) $$\mapsto(B_0,B_1,\dots,B_r,B'_0,B'_1,\dots,B'_{r'},g'gU_{J,B_0});$$

this is meaningful since $g'U_{\epsilon(J),B'_0}g = g'U_{\epsilon(J),B_r}g = g'gU_{J,B_0}$ and

$$\begin{aligned}\mathrm{pos}(B'_{r'}, g'gB_0g^{-1}g'^{-1}) &= \epsilon'(\mathrm{pos}(g'^{-1}B'_{r'}g', gB_0g^{-1}))\\ &= \epsilon'(\mathrm{pos}(B'_0, B_r)) = \epsilon'(a^{-1}_{r+r'}).\end{aligned}$$

Since $\rho'_{\mathbf{a}} = \tilde{\rho}\pi$, it suffices to show that $\pi_!(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}') = 0$. Hence it suffices to show that, for any $\xi = (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, xU_{J,B_0}) \in R$, we have $H^e_c(\pi^{-1}(\xi), \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') = 0$ for all e. We may assume that $\pi^{-1}(\xi) \neq \emptyset$. We may identify

$$\pi^{-1}(\xi) = \{gU_{J,B_0}; g \in D, gB_0g^{-1} = B_r, xg^{-1}B'_0gx^{-1} = B'_{r'}\}$$

in an obvious way. We may assume that $B_r = B^*, B'_0 = \dot{a}_{r+r'}B^*\dot{a}^{-1}_{r+r'}$ (notation of 28.5). Write $B_i = h_iB^*h_i^{-1}, B'_j = h'_jB^*h'^{-1}_j$ with $h_i, h'_j \in G^0, h_r = 1, h'_0 = \dot{a}_{r+r'}$. Let

$$\begin{aligned}w &= \mathrm{pos}(B_0, B_1)\mathrm{pos}(B_1, B_2)\dots\mathrm{pos}(B_{r-1}, B_r) \in \mathbf{W},\\ w' &= \mathrm{pos}(B'_0, B'_1)\mathrm{pos}(B'_1, B'_2)\dots\mathrm{pos}(B'_{r'-1}, B'_{r'}) \in \mathbf{W}.\end{aligned}$$

Let T, U^* be as in 28.5. Let $d \in N_DB^* \cap N_DT, d' \in N_{D'}B^* \cap N_{D'}T$. Then T (a maximal torus of $B_r \cap B'_0$) acts freely on $\pi_{\mathbf{a}}^{-1}(\xi)$ by left multiplication and it suffices to show that for any T-orbit θ in $\pi_{\mathbf{a}}^{-1}(\xi)$, we have $H^e_c(\theta, \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') = 0$ for all e. Let $g_0U_{J,B_0} \in \theta$. It suffices to show that the inverse image of $\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}'$ under $t \mapsto tg_0U_{J,B_0}$ (a local system in $\mathfrak{s}(T)$) is $\not\cong \bar{\mathbf{Q}}_l$. Using 28.10 and the definitions in 28.5, 28.8, we see that this inverse image is just $b^*(\mathcal{L}) \otimes c^*(\mathcal{L}')$ where $\mathcal{L}, \mathcal{L}'$ are regarded as local systems on T and $b : T \to T, c : T \to T$ are given by

$$\begin{aligned}b(t) &= d^{-1}\dot{w}^{-1}n_1n_2\dots n_rn_0(n_0^{-1}tn_0),\\ c(t) &= d'^{-1}\dot{w}'n'_1n'_2\dots n'_{r'}n'_0\dot{a}^{-1}_{r+r'}t^{-1}\dot{a}_{r+r'}\end{aligned}$$

where $n_i \in N_{G^0}T(i \in [1, r])$, $n'_j \in N_{G^0}T(j \in [1, r'])$ are given by $h^{-1}_{i-1}h_i \in U^*n_iU^*$, $h'^{-1}_{j-1}h'_j \in U^*n'_jU^*$, and $n_0 \in N_DB^* \cap N_DT$, $n'_0 \in N_{D'}B^* \cap N_{D'}T$ are given by $g_0h_0 \in U^*n_0$, $h'^{-1}_{r'}g_0^{-1}x\dot{a}_{r+r'} \in U^*n'_0$. Since $d^{-1}\dot{w}^{-1}n_1n_2\dots n_rn_0 \in T$, $d'^{-1}\dot{w}'^{-1}n'_1n'_2\dots n'_{r'}n'_0 \in T$, $n_0 \in dT$, we see using 28.1(a) that $b^*\mathcal{L} = \mathrm{Ad}(d^{-1})^*\mathcal{L}$, $c^*\mathcal{L}' = \mathrm{Ad}(\dot{a}^{-1}_{r+r'})^*\check{\mathcal{L}}'$. It then suffices to show that $\mathrm{Ad}(d^{-1})^*\mathcal{L}\otimes\mathrm{Ad}(\dot{a}^{-1}_{r+r'})^*\check{\mathcal{L}}' \not\cong \bar{\mathbf{Q}}_l$. This follows from our assumption. The lemma is proved.

32.9. Until the end of Lemma 32.12 we assume that $\mathbf{a}$ (as in 32.7(a)) is such that

(a) $$(\underline{D}^{-1})^*\mathcal{L} \cong (a^{-1}_{r+r'})^*\mathcal{L}'.$$

We set $\underline{a} = a_{r+r'}, \underline{a}' = \epsilon'(a^{-1}_{r+r'})$. We show that

$$s_1s_2\dots s_r\underline{a}s'_1s'_2\dots s'_{r'}\underline{a}'\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}.$$

Since $s_1s_2\dots s_r\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$, it suffices to show that $\underline{D}^{-1}\underline{a}s'_1s'_2\dots s'_{r'}\underline{D}'\underline{a}^{-1}\underline{D} \in \mathbf{W}^\bullet_{\mathcal{L}}$. Since $s'_1s'_2\dots s'_{r'}\underline{D}' \in \mathbf{W}^\bullet_{\mathcal{L}'} = \mathbf{W}^\bullet_{\check{\mathcal{L}}'}$, it suffices to show that $\underline{a}^*(\underline{D}^{-1})^*\mathcal{L} \cong \check{\mathcal{L}}'$. This holds by our assumption. Let

$$\mathbf{w} = (s_1, s_2, \dots, s_r, \underline{a}, s'_1, s'_2 \dots, s'_{r'}, \underline{a}').$$

Then $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ (see 32.3) is well defined in terms of $\mathcal{T}, \mathcal{T}'$ as in 32.7 (or equivalently as in 32.2). As in 32.3, $\mathcal{L}$ gives rise to a local system $\bar{\mathcal{L}}_{\mathbf{a}}$ on $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$.

32.10. We preserve the setup of 32.9. Let $V''_{\mathbf{a}}$ be the set of all

$$(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, (U_{B_r} \cap U_{B'_0})g, xU_{J,B_0})$$

where $(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, xB_0x^{-1}, xU_{J,B_0}) \in \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ and $g \in D$ satisfies $g^{-1}B_rg = B_0, g^{-1}B'_0g = x^{-1}B'_{r'}x$. (The last equation is meaningful. It suffices to show that if $u \in U_{J,B_0}$, then $ug^{-1}B'_0gu^{-1} = g^{-1}B'_0g$, that is, $gug^{-1} \in N_GB'_0$. We have $gug^{-1} \in U_{\epsilon_D(J),gB_0g^{-1}} = U_{\epsilon_D(J),B_r} = U_{\epsilon_D(J),B'_0} \subset U_{B'_0} \subset N_GB'_0$.) Define $\eta : V''_{\mathbf{a}} \to \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$, $\kappa : V'_{\mathbf{a}} \to V''_{\mathbf{a}}$ by

$$\begin{aligned}
\eta &: (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, (U_{B_r} \cap U_{B'_0})g, xU_{J,B_0}) \\
&\mapsto (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, xB_0x^{-1}, xU_{J,B_0}), \\
\kappa &: (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, gU_{J,B_0}, g'U_{\epsilon(J),B'_0}) \\
&\mapsto (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, (U_{B_r} \cap U_{B'_0})g, g'gU_{J,B_0});
\end{aligned}$$

κ is well defined, by the argument following Lemma 32.8(a). Let $f_{\mathbf{a}} = \eta\kappa : V'_{\mathbf{a}} \to \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. Clearly,

(a) *κ is an affine space bundle with fibres of dimension $l(w^0_{\epsilon(J)}\underline{a})$.*

Now $\mathbf{T}$ acts on $V''_{\mathbf{a}}$ by

$$\begin{aligned}
t &: (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, (U_{B_r} \cap U_{B'_0})g, xU_{J,B_0}) \\
&\mapsto (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, (U_{B_r} \cap U_{B'_0})y_tg, xU_{J,B_0})
\end{aligned}$$

where $y_t \in (B_r \cap B'_0)/(U_{B_r} \cap U_{B'_0})$ is defined by the condition that its image in B_r/U_{B_r} is the image of t under $\mathbf{T} \xrightarrow{\sim} B_r/U_{B_r}$. Then

(b) *η is a principal $\mathbf{T}$-bundle.*

Let $\xi = (B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, B, xU_{J,B_0}) \in \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. Then $\eta^{-1}(\xi)$ may be identified with

$$\{(U_{B_r} \cap U_{B'_0})g; g \in D; g^{-1}B_rg = B_0, g^{-1}B'_0g = x^{-1}B'_{r'}x\}.$$

We show only that $\eta^{-1}(\xi) \cong \mathbf{T}$. It suffices to show that $\eta^{-1}(\xi) \neq \emptyset$. For this it suffices to show that $\mathrm{pos}(B'_0, B_r) = \epsilon(\mathrm{pos}(x^{-1}B'_rx, B_0))$ or that $\mathrm{pos}(B'_0, B_r) = \epsilon\epsilon_\Delta^{-1}(\mathrm{pos}(B'_r, B))$ or that $\underline{a}^{-1} = \epsilon'^{-1}(\underline{a}')$ which is clear.

32.11. We preserve the setup of 32.9. Let w be the product of the sequence $s_1, s_2, \dots, s_r$ in which the factors s_i with $i \in \mathcal{T}_{\mathbf{a}}$ are replaced by 1. Let w' be the product of the sequence $s'_1, s'_2, \dots, s'_{r'}$ in which the factors s'_j with $j \in \mathcal{T}'_{\mathbf{a}}$ are replaced by 1, ($\mathcal{T}_{\mathbf{a}}, \mathcal{T}'_{\mathbf{a}}$ as in 32.3). Let B^*, U^*, T be as in 28.5. Let $d \in N_DB^* \cap N_DT, d' \in N_{D'}B^* \cap N_{D'}T$. We have a commutative diagram

$$\begin{array}{ccccc}
{}^0V'_{\mathbf{a}} & \xleftarrow{f_1} & \tilde{V}' & \xrightarrow{f_2} & T \times T \\
{\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\
{}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta} & \xleftarrow{f_6} & \tilde{Z} & \xrightarrow{f_7} & T
\end{array}$$

Here

$$\begin{aligned}
&{}^0V'_{\mathbf{a}} = \{(B_0, B_1, \dots, B_r, B'_0, B'_1, \dots, B'_{r'}, gU_{J,B_0}, g'U_{\epsilon(J),B'_0}) \in V'_{\mathbf{a}}; \\
&B_{i-1} = B_i (i \in \mathcal{T}_{\mathbf{a}}), \mathrm{pos}(B_{i-1}, B_i) = s_i (i \in [1, r] - \mathcal{T}_{\mathbf{a}}), B'_{j-1} = B'_j (j \in \mathcal{T}'_{\mathbf{a}}), \\
&\mathrm{pos}(B'_{j-1}, B'_j) = s'_j (i \in [1, r'] - \mathcal{T}'_{\mathbf{a}})\},
\end{aligned}$$

$\tilde V' = \{(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},g,g') \in (G^0)^{r+r'+2} \times D \times D';$
$h_{i-1}^{-1}h_i \in B^*(i \in \mathcal{T}_{\mathbf{a}}), h_{i-1}^{-1}h_i \in B^*\dot s_i B^*(i \in [1,r]-\mathcal{T}_{\mathbf{a}}), {h'_{j-1}}^{-1}h'_j \in B^*(j \in \mathcal{T}'_{\mathbf{a}}),$
${h'_{j-1}}^{-1}h'_j \in B^*\dot s'_j B^*(j \in [1,r']-\mathcal{T}'_{\mathbf{a}}), h_r^{-1}gh_0 = d, {h'_{r'}}^{-1}g'h'_0 = d',$
$h_k^{-1}h'_{r'} \in B^*\dot a_k B^*(k \in [0,r-1]), h_r^{-1}h'_{r+r'-k} \in B^*\dot a_k B^*(k \in [r,r+r'-1]),$
$h_r^{-1}h'_0 = \underline{\dot a}\},$

$\tilde Z = \{(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},x) \in (G^0)^{r+r'+2} \times D'D;$
$h_{i-1}^{-1}h_i \in B^*(i \in \mathcal{T}_{\mathbf{a}}), h_{i-1}^{-1}h_i \in B^*\dot s_i B^*(i \in [1,r]-\mathcal{T}_{\mathbf{a}}), {h'_{j-1}}^{-1}h'_j \in B^*(j \in \mathcal{T}'_{\mathbf{a}}),$
${h'_{j-1}}^{-1}h'_j \in B^*\dot s'_j B^*(j \in [1,r']-\mathcal{T}'_{\mathbf{a}}), h_k^{-1}h'_{r'} \in B^*\dot a_k B^*(k \in [0,r-1]),$
$h_r^{-1}h'_{r+r'-k} \in B^*\dot a_k B^*(k \in [r,r+r'-1]), h_r^{-1}h'_0 = \underline{\dot a},$
${h'_{r'}}^{-1}xh_0 \in B^*d'\underline{\dot a}^{-1}d'^{-1}B^*\},$

$$f_1(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},g,g') = (h_0B^*h_0^{-1},\dots,h_rB^*h_r^{-1},$$
$$h'_0B^*{h'_0}^{-1},\dots,h'_{r'}B^*{h'_{r'}}^{-1}, gU_{J,h_0B^*h_0^{-1}}, g'U_{\epsilon_D(J),h'_0B^*{h'_0}^{-1}}),$$

$$f_2(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},g,g')$$
$$= (d^{-1}\dot w^{-1}n_1n_2\dots n_rd, d'^{-1}\dot w'^{-1}n'_1n'_2\dots n'_{r'}d')$$

with $n_i, n'_j \in N_{G^0}T$ given by $h_{i-1}^{-1}h_i \in U^*n_iU^*(i \in [1,r])$, ${h'_{j-1}}^{-1}h'_j \in U^*n'_jU^*(j \in [1,r'])$, f_3 is the restriction of $f_{\mathbf{a}} : V'_{\mathbf{a}} \to \underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$; see 32.10,

$$f_4(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},g,g') = (h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},g'g),$$
$$f_5(t,\tilde t) = d^{-1}\underline{\dot a}d'^{-1}\dot w'^{-1}\underline{\dot a}^{-1}dtd^{-1}\underline{\dot a}\dot w'd'\tilde t\underline{\dot a}^{-1}d = \mathrm{Ad}(d^{-1}\underline{\dot a})(\mathrm{Ad}(d'^{-1}\dot w'^{-1}\underline{\dot a}^{-1}d)(t)\tilde t),$$

$$f_6(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},x)$$
$$= (h_0B^*h_0^{-1},\dots,h_rB^*h_r^{-1},h'_0B^*{h'_0}^{-1},\dots,h'_{r'}B^*{h'_{r'}}^{-1},xU_{J,h_0B^*h_0^{-1}}),$$

$$f_7(h_0,h_1,\dots,h_r,h'_0,h'_1,\dots,h'_{r'},x)$$
$$= (d'd)^{-1}d'\underline{\dot a}d'^{-1}\dot w'^{-1}\underline{\dot a}^{-1}\dot w^{-1}n_1n_2\dots n_r\underline{\dot a}n'_1n'_2\dots n'_{r'}md'd,$$

where n_i, n'_j are as in the definition of f_2, $m \in N_{G^0}T$ is given by ${h'_{r'}}^{-1}xh_0d^{-1}d'^{-1} \in U^*mU^*$.

Lemma 32.12. *We preserve the setup of 32.11. Let* ${}^0\bar{\mathcal{L}}_{\mathbf{a}} = \bar{\mathcal{L}}_{\mathbf{a}}|_{{}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}}$.
(a) We have $f_3^*({}^0\bar{\mathcal{L}}_{\mathbf{a}}) = (\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')|_{{}^0V'_{\mathbf{a}}}$.
(b) We have $f_{\mathbf{a}}^*\bar{\mathcal{L}}_{\mathbf{a}} = (\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')|_{V'_{\mathbf{a}}}$.

From the definitions we have

$$f_6^*({}^0\bar{\mathcal{L}}_{\mathbf{a}}) = f_7^*\mathcal{L}, \quad f_1^*((\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')|_{{}^0V'_{\mathbf{a}}}) = f_2^*(\mathcal{L} \boxtimes \mathrm{Ad}(\underline{\dot a})^*\mathrm{Ad}(d^{-1})^*\check{\mathcal{L}}).$$

Since f_1^* is a smooth morphism with connected fibres, it suffices to show that $f_1^*f_3^*({}^0\bar{\mathcal{L}}_{\mathbf{a}}) = f_1^*((\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')|_{{}^0V'_{\mathbf{a}}})$, or that $f_4^*f_6^*({}^0\bar{\mathcal{L}}_{\mathbf{a}}) = f_2^*(\mathcal{L} \boxtimes \mathcal{L}')$, or that $f_4^*f_7^*\mathcal{L} = f_2^*(\mathcal{L} \boxtimes \mathcal{L}')$, or that $f_2^*f_5^*\mathcal{L} = f_2^*(\mathcal{L} \boxtimes \mathcal{L}')$. It suffices to show that $f_5^*\mathcal{L} = \mathcal{L} \boxtimes \mathcal{L}'$.

Define $f'_5 : T \times T \to T$ by $f'_5(t,\tilde t) = t\tilde t$. Setting $E = \mathrm{Ad}(d^{-1}\underline{\dot a}) : T \to T$, $E' = \mathrm{Ad}(d'^{-1}\dot w'^{-1}\underline{\dot a}^{-1}d) : T \to T$, we have $f_5 = Ef'_5(E' \times 1)$ hence

$$f_5^*\mathcal{L} = (E' \times 1)^*{f'_5}^*E^*\mathcal{L} = (E' \times 1)^*(E^*\mathcal{L} \boxtimes E^*\mathcal{L}) = (EE')^*\mathcal{L} \boxtimes E^*\mathcal{L}.$$

From our assumption we have $\mathcal{L}' \cong E^*\mathcal{L}$. Moreover, $\mathcal{L}' \cong \mathrm{Ad}(d'^{-1}\dot{w}'^{-1})^*\mathcal{L}' = (E'E)^*\mathcal{L}'$. Hence $E^*\mathcal{L} \cong (E'E)^*E^*\mathcal{L} = (EE'E)^*\mathcal{L} = E^*(EE')^*\mathcal{L}$. Since E^* is faithful, it follows that $(EE')^*\mathcal{L} \cong \mathcal{L}$. Thus, $f_5^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}'$. This proves (a).

We prove (b). We may assume that $V'_{\mathbf{a}} \neq \emptyset$. From (a) we see that $f_{\mathbf{a}}^*\bar{\mathcal{L}}_{\mathbf{a}}$, $(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')|_{V'_{\mathbf{a}}}$ are two local systems on $V'_{\mathbf{a}}$ with the same restriction to the subset ${}^0V'_{\mathbf{a}}$. It then suffices to show that $V'_{\mathbf{a}}$ is smooth, irreducible and ${}^0V'_{\mathbf{a}}$ is open dense in $V'_{\mathbf{a}}$. By 32.10(a),(b), $f_{\mathbf{a}}$ is a fibration with connected smooth fibres and ${}^0V'_{\mathbf{a}}$ is the inverse image under $f_{\mathbf{a}}$ of ${}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. Hence it suffices to show that $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ is smooth, irreducible and ${}^0\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$ is open dense in $\underline{Z}^{\mathbf{w},\mathbf{a}}_{\emptyset,J,\Delta}$. This follows from 32.4(c). The lemma is proved.

32.13. Let $\mathbf{S}$ be the set of all $\mathbf{a} = (a_0, a_1, \dots, a_{r+r'}) \in \mathbf{W}^{r+r'} \times \mathbf{W}_{\epsilon(J)}$ such that
(a) $a_k \in \{a_{k-1}, s_k a_{k-1}\}$ for $k \in [1, r]$,
(b) $a_k \in \{a_{k-1}, a_{k-1}s'_{r+r'+1-k}\}$ for $k \in [r+1, r+r']$,
(c) $(\underline{D}^{-1})^*\mathcal{L} \cong (a^{-1}_{r+r'})^*\mathcal{L}'$,
(d) $i \in [1, r], a_{i-1} = a_i \implies i \in \mathcal{T}$,
(e) $j \in [1, r'], a_{-j+r+r'} = a_{-j+1+r+r'} \implies j \in \mathcal{T}'$.

Lemma 32.14. *If* $\mathbf{a} \in (\mathbf{W}^{r+r'} \times \mathbf{W}_{\epsilon(J)}) - \mathbf{S}$, *then* $\rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') = 0$.

If $\mathbf{a}$ does not satisfy 32.13(a) or 32.13(b), then $V'_{\mathbf{a}} = \emptyset$ and the result is trivial. If $\mathbf{a}$ does not satisfy 32.13(c), the desired result follows from Lemma 32.8. Assume now that $\mathbf{a}$ satisfies 32.13(a)–(c) but it does not satisfy 32.13(d) or (e). Using $\rho'_{\mathbf{a}} = \pi_{\mathbf{w},\mathbf{a}} f_{\mathbf{a}}$ and Lemma 32.12(b), we see that it suffices to show that $\pi_{\mathbf{w},\mathbf{a}!} f_{\mathbf{a}!}(f_{\mathbf{a}}^*\bar{\mathcal{L}}_{\mathbf{a}}) = 0$, or that $\pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes f_{\mathbf{a}!}\bar{\mathbf{Q}}_l) = 0$. It suffices to show that $\pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes \mathcal{H}^e(f_{\mathbf{a}!}\bar{\mathbf{Q}}_l)) = 0$ for any e. By 32.10 we have $f_{\mathbf{a}} = \eta\kappa$ and $\kappa_!\bar{\mathbf{Q}}_l = \bar{\mathbf{Q}}_l[[-c]]$ where $c = l(w^0_{\epsilon(J)} a_{r+r'})$. Hence $f_{\mathbf{a}!}\bar{\mathbf{Q}}_l = \eta_!\bar{\mathbf{Q}}_l[[-c]]$. Let $\mathbf{r} = \dim \mathbf{T}$. Since η is a principal $\mathbf{T}$-bundle (see 32.10), the local system $\mathcal{H}^e(\eta_!\bar{\mathbf{Q}}_l)$ admits a filtration whose associated graded is a direct sum of $\binom{\mathbf{r}}{2\mathbf{r}-e}$ copies of $\bar{\mathbf{Q}}_l(\mathbf{r}-e)$. Since $\pi_{\mathbf{w},\mathbf{a}!}\bar{\mathcal{L}}_{\mathbf{a}} = 0$ (see 32.4(d)), we see that $\pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes \mathcal{H}^e(f_{\mathbf{a}!}\bar{\mathbf{Q}}_l)) = 0$ for any e. The lemma is proved.

32.15. We now make a short digression. Let X be an algebraic variety over $\mathbf{k}$. Let $C \in \mathcal{D}(X)$ and let $\{C_n; n \in \mathbf{Z}\}$ be a sequence of objects in $\mathcal{D}(X)$ such that $C_n = 0$ for all but finitely many n. We shall write

$$C \approxeq \{C_n; n \in \mathbf{Z}\}$$

if the following condition is satisfied: there exists a sequence $\{C'_n; n \in \mathbf{Z}\}$ of objects in $\mathcal{D}(X)$ such that $C'_n = 0$ for $n \ll 0$, $C'_n = C$ for $n \gg 0$ and distinguished triangles (C'_{n-1}, C'_n, C_n) for $n \in \mathbf{Z}$.

If X, C, C_n are as above, $C \approxeq \{C_n; n \in \mathbf{Z}\}$ and $X_2 \xrightarrow{f_2} X \xrightarrow{f_1} X_1$ are morphisms of algebraic varieties, we see from definitions that

$$f_{1!}C \approxeq \{f_{1!}C_n; n \in \mathbf{Z}\},$$
$$f_2^*C \approxeq \{f_2^*C_n; n \in \mathbf{Z}\}.$$

Assume now that $C \in \mathcal{D}(X)$ and that $\{C^u; u \in \mathcal{U}\}$ are objects of $\mathcal{D}(X)$ indexed by a finite set $\mathcal{U}$. We shall write

$$C \approxeq \{C_u; u \in \mathcal{U}\}$$

if the following condition is satisfied: there exists a bijection $\mathcal{U} \leftrightarrow [0,m]$ such that, setting $C_n = C^u$ if $u \leftrightarrow n \in [0,m]$ and $C_n = 0$ for $n \notin [0,m]$, we have $C \Bumpeq \{C_n; n \in \mathbf{Z}\}$.

For example, if $C \in \mathcal{D}(X)$, we have $C \Bumpeq \{{}^pH^nC[-n]; n \in \mathbf{Z}\}$; in this case we can take $C'_n = {}^p\tau_{\le n}(C)$ (truncation, as in [BBD]).

Similarly, if $C \in \mathcal{D}(X)$, we have $C \Bumpeq \{\mathcal{H}^n(C)[-n]; n \in \mathbf{Z}\}$.

As another example, assume that we are given a partition $X = \sqcup_{u \in \mathcal{U}} X^u$ with $\mathcal{U}$ finite, where X^u are locally closed subvarieties of X such that for some bijection $\mathcal{U} \leftrightarrow [0,m]$, the union $X'_n = X_n \cup X_{n-1} \cup \dots \cup X_0$ is open in X for any $n \in [0,m]$ (we set $X_n = X^u$ for $u \leftrightarrow n \in [0,m]$). For any $u \in \mathcal{U}$ let $j_u : X^u \to X$ be the inclusion and let $C_u = j_{u!}j_u^*C$. We have $C \Bumpeq \{C_u; u \in \mathcal{U}\}$. Indeed, setting $C_n = C_u$ if $u \leftrightarrow n \in [0,m]$ and $C_n = 0$ for $n \notin [0,m]$, we have $C \Bumpeq \{C_n; n \in \mathbf{Z}\}$. (We can take $C'_n = 0$ for $n < 0$, $C'_n = C$ for $n > m$, $C'_n = j'_{n!}j'^*_nC$ for $n \in [0,m]$, where $j'_n = X'_n \to X$ is the inclusion.)

32.16. Assume that $\mathbf{a} \in \mathbf{S}$. As in the proof of Lemma 32.14 we have $\rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') = \pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes f_{\mathbf{a}!}\bar{\mathbf{Q}}_l)$ and $f_{\mathbf{a}!}\bar{\mathbf{Q}}_l = \eta_!\bar{\mathbf{Q}}_l[[-c]]$. Moreover, $\eta_!\bar{\mathbf{Q}}_l \Bumpeq \{\mathcal{H}^e(\eta_!\bar{\mathbf{Q}}_l)[-e]; e \in \mathbf{Z}\}$ and for any e we have $\mathcal{H}^e(\eta_!\bar{\mathbf{Q}}_l) \Bumpeq \{C^e_{e'}; 1 \le e' \le \binom{\mathbf{r}}{2\mathbf{r}-e}\}$ where $C^e_{e'} = \bar{\mathbf{Q}}_l(\mathbf{r}-e)$. It follows that

$$\text{(a)} \qquad \rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') \Bumpeq \{\pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes \mathcal{H}^e(\eta_!\bar{\mathbf{Q}}_l))[-e][[-c]]; e \in \mathbf{Z}\},$$

$$\text{(b)} \qquad \pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}} \otimes \mathcal{H}^e(\eta_!\bar{\mathbf{Q}}_l)[-e][[-c]] \Bumpeq \{C'^e_{e'}; 1 \le e' \le \binom{\mathbf{r}}{2\mathbf{r}-e}\}$$

where

$$C'^e_{e'} = \pi_{\mathbf{w},\mathbf{a}!}(\bar{\mathcal{L}}_{\mathbf{a}})(\mathbf{r}-e)[-e][[-c]] = K_{J,\Delta}^{(a_0,\epsilon'(a^{-1}_{r+r'})),\mathcal{L}}(\mathbf{r}-e)[-e][[-c]][[-N_{\mathbf{a}}]]$$

(see 32.4(e)). By 32.4(e) we have

$$\text{(c)} \qquad a_0\epsilon'(a^{-1}_{r+r'})\underline{\Delta} \in \mathbf{W}^\bullet_{\mathcal{L}}.$$

From (a),(b) we see that, if A is a simple perverse sheaf on $Z_{J,\Delta}$ such that $A \dashv \rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$, then $A \dashv K_{J,\Delta}^{(a_0,\epsilon'(a^{-1}_{r+r'})),\mathcal{L}}$.

32.17. From the partition $V' = \sqcup_{\mathbf{a}} V'_{\mathbf{a}}$ we get, as in 32.15,

$$\text{(a)} \qquad \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') = \rho'_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') \Bumpeq \{\rho'_{\mathbf{a}}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'); \mathbf{a} \in \mathbf{S}\}$$

(by Lemma 32.14 we can omit the $\mathbf{a} \notin \mathbf{S}$). Thus, if A is a simple perverse sheaf on $Z_{J,D}$ such that $A \dashv \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$, then for some $\mathbf{a} \in \mathbf{S}$ we have $A \dashv \rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$ hence, by 32.16, $A \dashv K_{J,\Delta}^{(a_0,\epsilon'(a^{-1}_{r+r'})),\mathcal{L}}$, so that $A \in \hat{Z}^{\mathcal{L}}_{J,\Delta}$. We also see that $\mathbf{S} \ne \emptyset$; in particular, $(\underline{D}^{-1})^*\mathcal{L} \cong y^*\mathcal{L}'$ for some $\underline{a} \in \mathbf{W}_{\epsilon(J)}$ (see 32.13(c)). Since $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'} = \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$ (see 32.7) we see that Theorem 32.6(b) holds. We also see that 32.6(a) holds since, under the assumption of 32.6(a), we can find an A as above.

32.18. In this and the next subsection we place ourselves in the setup of 32.6(c). Then $V, V', V'_{\mathbf{a}}$ are defined over $\mathbf{F}_q$ and we can regard $\rho'_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$ and $\rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$ (for any $\mathbf{a}$) as mixed complexes on $Z_{J,\Delta}$. Using 32.17(a), 32.16(a),(b) (or rather their variant in the mixed category) and 31.7(c),(e) we see that, with the notation of 31.6, we have

$$\chi^A_v(\rho_!(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}'))=\sum_{\mathbf{a}\in\mathbf{S}}\chi^A_v(\rho'_{\mathbf{a}}(\bar{\mathcal{L}}\boxtimes\bar{\mathcal{L}}'))$$

$$=\sum_{\mathbf{a}\in\mathbf{S}}\sum_{e\in\mathbf{Z}}(-1)^e v^{2N_{\mathbf{a}}+2l(w^0_{\epsilon(J)}a_{r+r'})-2\mathbf{r}+2e}\binom{\mathbf{r}}{2\mathbf{r}-e}\chi^A_v(K^{(a_0,\epsilon'(a^{-1}_{r+r'})),\mathcal{L}}_{J,\Delta})$$

$$=(v^2-1)^{\mathbf{r}}\sum_{\mathbf{a}\in\mathbf{S}}v^{2N_{\mathbf{a}}-2l(w^0_{\epsilon(J)}a_{r+r'})}\chi^A_v(K^{(a_0,\epsilon'(a^{-1}_{r+r'})),\mathcal{L}}_{J,\Delta})$$

(a) $$=(v^2-1)^{\mathbf{r}}\sum_{\mathbf{a}\in\mathbf{S}}v^{2N_{\mathbf{a}}-2l(w^0_{\epsilon(J)}a_{r+r'})}v^{\dim G-l(w^0_{\mathbf{I}}w^0_J)}\zeta^A(T_{a_0}T_{\epsilon'(a^{-1}_{r+r'}}1_{\underline{\Delta}\lambda}[\Delta]).$$

32.19. Let $h\mapsto h^\flat$ be the antiautomorphism of the algebra H_n defined by $T_u\mapsto T_{u^{-1}}$ for $u\in\mathbf{W}$, $1_\lambda\mapsto 1_\lambda$ for $\lambda\in\underline{\mathfrak{s}}_n$. We have $(C^{\tilde{\mathbf{s}}}_\lambda)^\flat=C^{\mathbf{s}}_{s_r\dots s_2s_1\lambda}$ where $\tilde{\mathbf{s}}=(s_r,s_{r-1},\dots,s_1)$. The following identity in the algebra H_n (see 31.2) is a special case of one in 31.11:

(a) $$T_yC^{\mathbf{s}'}_{\lambda_1}=\sum_{\mathbf{y}'}v^{2\delta'(\mathbf{y}')}T_{y'_{r'}}1_{\lambda_1};$$

here $y\in\mathbf{W},\lambda_1\in\underline{\mathfrak{s}}_n$, the sum is taken over all sequences $\mathbf{y}'=(y'_0,y'_1,\dots,y'_{r'})$ in $\mathbf{W}$ such that

$y=y'_0$,
$y'_i\in\{y'_{i-1},y'_{i-1}s'_i\}$ for $i\in[1,r']$,
$i\in[1,r'],y'_{i-1}=y'_i\implies s'_i\in\mathbf{W}_{s'_{i+1}\dots s'_{r'}\lambda_1}$;

moreover, $\delta'(\mathbf{y}')=|\{i\in[1,r'];y'_{i-1}s'_i<y'_{i-1}\}|$. Similarly, we have

(b) $$C^{\mathbf{s}}_{s_r\dots s_2s_1\lambda_2}T_y=\sum_{\mathbf{y}}v^{2\delta(\mathbf{y})}1_{\lambda_2}T_{y_0};$$

here $\lambda_2\in\underline{\mathfrak{s}}_n$, the sum is taken over all sequences $\mathbf{y}=(y_0,y_1,\dots,y_r)$ in $\mathbf{W}$ such that

$y=y_r$,
$y_i\in\{y_{i-1},s_iy_{i-1}\}$ for $i\in[1,r]$,
$i\in[1,r],y_{i-1}=y_i\implies s_i\in\mathbf{W}_{s_{i-1}\dots s_1\lambda_2}$;

moreover, $\delta(\mathbf{y})=|\{i\in[1,r];s_iy_i<y_i\}|$. This can be deduced from (a) using the involution $h\mapsto h^\flat$.

Combining (a),(b) we obtain (for $\lambda_1,\lambda_2\in\underline{\mathfrak{s}}_n$) the identity

$$C^{\mathbf{s}}_{s_r\dots s_2s_1\lambda_2}T_yC^{\mathbf{s}'}_{\lambda_1}=\sum_{\mathbf{y},\mathbf{y}'}v^{2\delta(\mathbf{y})+2\mathbf{d}(\mathbf{y}')}1_{\lambda_2}T_{y_0}1_{\lambda_1};$$

the sum is taken over the pairs $\mathbf{y}=(y_0,y_1,\dots,y_r)$, $\mathbf{y}'=(y'_0,y'_1,\dots,y'_{r'})$ of sequences in $\mathbf{W}$ such that

$y=y'_0,y'_{r'}=y_r$,
$y_i\in\{y_{i-1},s_iy_{i-1}\}$ for $i\in[1,r]$,
$y'_i\in\{y'_{i-1},y'_{i-1}s'_i\}$ for $i\in[1,r']$,
$i\in[1,r],y_{i-1}=y_i\implies s_i\in\mathbf{W}_{s_{i-1}\dots s_1\lambda_2}$,
$i\in[1,r'],y'_{i-1}=y'_i\implies s'_i\in\mathbf{W}_{s'_{i+1}\dots s'_{r'}\lambda_1}$.

We have $s_r\dots s_2s_1\lambda=\underline{D}\lambda$. Take $\lambda_1=\underline{D}'\lambda',\lambda_2=\lambda$. Take $y\in\mathbf{W}_{\epsilon(J)}$ such that

$y\lambda' = \underline{D}\lambda$. We replace $(\mathbf{y}, \mathbf{y}')$ by $\mathbf{a} = (a_0, a_1, \dots, a_{r+r'})$ where $a_k = y_k$ for $k \in [0, r]$, $a_k = y'_{r+r'-k}$ for $k \in [r, r+r']$. Then $\delta(\mathbf{y}) + \delta'(\mathbf{y}') = N_{\mathbf{a}}$. We obtain

$$C^{\mathbf{s}}_{\underline{D}\lambda} T_y C^{\mathbf{s}'}_{\underline{D}'\lambda'} = \sum_{\mathbf{a}} v^{2N_{\mathbf{a}}} 1_\lambda T_{a_0} 1_{\underline{D}'\lambda'};$$

the sum is over all $\mathbf{a} = (a_0, a_1, \dots, a_{r+r'}) \in \mathbf{W}^{r+r'} \times \mathbf{W}_{\epsilon(J)}$ such that
$y = a_{r+r'}$,
$a_i \in \{a_{i-1}, s_i a_{i-1}\}$ for $i \in [1, r]$,
$a_{r+r'-i} \in \{a_{r+r'-i+1}, a_{r+r'-i+1} s'_i\}$ for $i \in [1, r']$,
$i \in [1, r], a_{i-1} = a_i \implies s_i \in \mathbf{W}_{s_{i-1} \dots s_1 \lambda}$,
$i \in [1, r'], a_{r+r'-i+1} = a_{r+r'-i} \implies s'_i \in \mathbf{W}_{s'_{i+1} \dots s'_{r'} \underline{D}'\lambda'}$.
Equivalently,

(c) $$C^{\mathbf{s}}_{\underline{D}\lambda} T_y C^{\mathbf{s}'}_{\underline{D}'\lambda'} = \sum_{\mathbf{a} \in \mathbf{S}; a_{r+r'} = y} v^{2N_{\mathbf{a}}} 1_\lambda T_{a_0} 1_{\underline{D}'\lambda'}.$$

For each $\mathbf{a}$ in the sum we have $a_0 \epsilon'(y^{-1}) \underline{\Delta}\lambda = \lambda$ (see 32.16(a)); combining this with $y\lambda' = \underline{D}\lambda$ we see that $a_0 \underline{D}'\lambda' = \lambda$, hence $1_\lambda T_{a_0} 1_{\underline{D}'\lambda'} = T_{a_0} 1_{\underline{D}'\lambda'}$. We introduce this in (c), then multiply both sides of (c) on the right by

$$(v^2 - 1)^{\mathbf{r}} v^{\dim G - l(w^0_{\mathbf{I}} w^0_J)} v^{-2l(y)} T_{\epsilon'(y^{-1})} 1_{\underline{\Delta}\lambda} [\Delta]$$

and sum over y. We obtain

$$(v^2 - 1)^{\mathbf{r}} v^{\dim G - l(w^0_{\mathbf{I}} w^0_J)} \sum_{y \in \mathbf{W}_{\epsilon(J)}; y\lambda' = \underline{D}\lambda} v^{-2l(y)} C^{\mathbf{s}}_{\underline{D}\lambda} T_y C^{\mathbf{s}'}_{\underline{D}'\lambda'} T_{\epsilon'(y^{-1})} [\Delta]$$
$$= (v^2 - 1)^{\mathbf{r}} v^{\dim G - l(w^0_{\mathbf{I}} w^0_J)} \sum_{\mathbf{a} \in \mathbf{S}} v^{2N_{\mathbf{a}} - 2l(a_{r+r'}} T_{a_0} T_{\epsilon'(a^{-1}_{r+r'})} 1_{\underline{\Delta}\lambda} [\Delta].$$

We apply ζ^A_v (see 31.7) to both sides and use 32.18(a). We obtain

$$\chi^A_v (\rho_! (\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'))$$
$$= (v^2 - 1)^{\mathbf{r}} v^{\dim G - l(w^0_{\mathbf{I}} w^0_J)} \zeta^A \Big(\sum_{\substack{y \in \mathbf{W}_{\epsilon(J)} \\ y\lambda' = \underline{D}\lambda}} v^{2l(w^0_{\epsilon(J)} y)} C^{\mathbf{s}}_{\underline{D}\lambda} T_y C^{\mathbf{s}'}_{\underline{D}'\lambda'} T_{\epsilon(y^{-1})} [\Delta] \Big).$$

We substitute $y = \epsilon(y'), y' \in \mathbf{W}_J$, that is, $T_y = [D] T_{y'} [D]^{-1}$. Since $\chi^A_v (\bar{K}^{\mathbf{s}, \mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}', \mathcal{L}'}_{\epsilon(J), D'}) = \chi^A_v (\rho_! (\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'))$, Theorem 32.6(c) follows. This completes the proof of Theorem 32.6.

32.20. Let $\mathcal{D}^{cs}(Z_{J,D})$ (resp. $\mathcal{D}^{\mathcal{L}}(Z_{J,D})$ with $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$) be the subcategory of $\mathcal{D}(Z_{J,D})$ whose objects are those $K \in \mathcal{D}(Z_{J,D})$ such that for any j, any simple subquotient of ${}^pH^j K$ is in $\hat{Z}_{J,D}$ (resp. in $\hat{Z}^{\mathcal{L}}_{J,D}$). We have the following result.

Corollary 32.21. *(a) If $K \in \mathcal{D}^{cs}(Z_{J,D}), K' \in \mathcal{D}^{cs}(Z_{\epsilon(J), D'})$, then $K * K' \in \mathcal{D}^{cs}(Z_{J,\Delta})$.*
(b) If $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$, $K \in \mathcal{D}^{\mathcal{L}}(Z_{J,D})$, $K' \in \mathcal{D}^{cs}(Z_{\epsilon(J), D'})$, then $K * K' \in \mathcal{D}^{\mathcal{L}}(Z_{J,\Delta})$.

We prove (a). We may assume that $K \in \hat{Z}_{J,D}, K' \in \hat{Z}_{\epsilon(J), D'}$. We can find $\mathbf{s}, \mathbf{s}', \mathcal{L}, \mathcal{L}'$ as in 32.5 and $u, u' \in \mathbf{Z}$ such that $K[u]$ is a direct summand of $\bar{K}^{\mathbf{s}, \mathcal{L}}_{J,D}$ and $K'[u']$ is a direct summand of $\bar{K}^{\mathbf{s}', \mathcal{L}'}_{\epsilon(J), D'}$. Then $K * K'[u + u']$ is a direct summand of $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} * \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}$ which, by 32.6(b), is in $\mathcal{D}^{\mathcal{L}}(Z_{J,\Delta})$. Hence $K * K'[u+u']$ is in $\mathcal{D}^{\mathcal{L}}(Z_{J,\Delta})$. This proves (a). The same argument proves (b).

32.22. If E is a mixed $\bar{\mathbf{Q}}_l$-vector space (that is, a $\bar{\mathbf{Q}}_l$-vector space which, when regarded as a complex over a point, is a mixed complex) we set

$$\chi_v(E) = \sum_j \dim(E_j) v^j \in \mathcal{A}$$

where E_j is the pure subquotient of weight j of E. We preserve the setup in 32.5. Assume that $D' = D^{-1}$ hence $\Delta = G^0$. Let $\mathbf{S}$ be as in 32.13.

Define an $\mathcal{A}$-linear map $\Phi : H_n \to H_n$ by $\xi \mapsto \mathfrak{a}_D(C^{\mathbf{s}}_{\underline{D}\lambda} \xi C^{\mathbf{s}'}_{\underline{D}'\lambda'})$ with $\mathfrak{a}_D$ as in 31.4. For any $y \in \mathbf{W}_{\epsilon(J)}$ such that $y\lambda' = \underline{D}\lambda$ we have

$$\Phi(T_y 1_{\lambda'}) = \sum_{\mathbf{a} \in \mathbf{S}; a_{r+r'} = y} v^{2N_{\mathbf{a}}} 1_{\underline{D}\lambda} T_{\epsilon(a_0)} 1_{\lambda'}.$$

(See 32.19(c).) Define an $\mathcal{A}$-linear map $\Theta^J : H_n \to H_n$ by $\Theta^J(T_w 1_{\lambda_1}) = T_w 1_{\lambda_1}$ if $w \in \mathbf{W}_J, \lambda_1 \in \underline{\mathfrak{s}}_n$, $\Theta^J(T_w 1_{\lambda_1}) = 0$ if $w \in \mathbf{W} - \mathbf{W}_J, \lambda_1 \in \underline{\mathfrak{s}}_n$. Replacing J by $\epsilon(J)$ we obtain an $\mathcal{A}$-linear map $\Theta^{\epsilon(J)} : H_n \to H_n$. Define $\Phi' : H_n \to H_n$ by $\Phi'(\xi') = \Theta^{\epsilon(J)}\Phi(\xi')$. Since H_n is a free $\mathcal{A}$-module and Φ' is $\mathcal{A}$-linear, $\mathrm{tr}(\Phi', H_n) \in \mathcal{A}$ is well defined. From the definitions we have

(a) $$\mathrm{tr}(\Phi', H_n) = \sum_{\mathbf{a} \in \mathbf{S}; a_{r+r'} = \epsilon(a_0)} v^{2N_{\mathbf{a}}} = \sum_{\mathbf{a} \in \mathbf{S}_0} v^{2N_{\mathbf{a}}},$$

where $\mathbf{S}_0 = \{\mathbf{a} \in \mathbf{S}; a_0 = \epsilon'(a_{r+r'})\}$.

Define an $\mathcal{A}$-linear map $\Phi'' : H_n \to H_n$ by $\xi \mapsto \Theta^J(C^{\tilde{\mathbf{s}}'}_{\lambda'^{-1}} \mathfrak{a}_D(\xi) C^{\tilde{\mathbf{s}}}_{\lambda^{-1}})$ where $\tilde{\mathbf{s}} = (s_r, \dots, s_2, s_1)$, $\tilde{\mathbf{s}}' = (s'_{r'}, \dots, s'_2, s'_1)$. Let $\partial : Z_{J,D} \to Z_{\epsilon(J),D'}$ be as in 28.19. Then $\bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} \otimes \partial^* \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'} \in \mathcal{D}(Z_{J,D})$ is well defined. Let

$$\mu(G^0) = (v^2 - 1)^{\mathbf{r}} \sum_{w \in \mathbf{W}} v^{2l(w)} \in \mathcal{A}.$$

The following result is an application of (the proof of) Theorem 32.6.

Corollary 32.23. *Assume that $\mathbf{k}, \mathbf{F}_q, G, F$ are as in 31.7(b). Then*

$$\sum_z (-1)^z \chi_v(H^z_c(Z_{J,D}, \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} \otimes \partial^* \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D^{-1}}) = v^{2l(w^0_J)} \mu(G^0) \mathrm{tr}(\Phi', H_n)$$

(a) $$= v^{2l(w^0_J)} \mu(G^0) \mathrm{tr}(\Phi'', H_n).$$

Let $\mathfrak{Z} = \{(Q, Q', xU_Q) \in Z_{J,\Delta}; Q = Q', x \in U_Q\}$, let $\iota : \mathfrak{Z} \to Z_{J,\Delta}$ be the inclusion and let $p : \mathfrak{Z} \to$ point be the obvious map. From the definitions, for any $A \in \mathcal{D}(\hat{Z}_{J,D}), A' \in \mathcal{D}(Z_{\epsilon(J),D'})$ we have

(b) $$H^z_c(\text{point}, p_! \iota^*(A * A')) = H^z_c(Z_{J,D}, A \otimes \partial^*(A'))$$

for any $z \in \mathbf{Z}$. In particular,

$$H^z_c(Z_{J,D}, \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} \otimes \partial^* \bar{K}^{\mathbf{s}',\mathcal{L}'}_{\epsilon(J),D'}) = H^z_c(\text{point}, p_! \iota^* \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')).$$

Applying $p_! \iota^*$ to 32.17(a) gives

(c) $$p_! \iota^* \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') \approx \{p_! \iota^* \rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}'); \mathbf{a} \in \mathbf{S}\}.$$

Let $\mathbf{a} \in \mathbf{S}$. Applying $p_! \iota^*$ to 32.16(a), 32.16(b) gives

(d) $$p_! \iota^* \rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}') \approx \{K_e; e \in \mathbf{Z}\},$$

(e) $$K_e \approx \{K'^e_{e'}; 1 \le e' \le \binom{\mathbf{r}}{2\mathbf{r} - e}\}$$

where $K_e, K'^e_{e'} \in \mathcal{D}(\text{point})$ and $K'^e_{e'} = p_! \iota^* K_{J,\Delta}^{(a_0, \epsilon'(a_{r+r'}^{-1})), \mathcal{L}}(\mathbf{r} - e)[-e][[-c - N_{\mathbf{a}}]]$ (notation of 32.16). Let

$$X_{\mathbf{a}} = \{(B_0, B'_0) \in \mathcal{B} \times \mathcal{B}; \text{pos}(B_0, B'_0) = a_0, \text{pos}(B'_0, B_0) = \epsilon'(a_{r+r'}^{-1})\}$$

and let $\omega : X_{\mathbf{a}} \to$ point be the obvious map. From the definitions we see that $p_! \iota^* K_{J,\Delta}^{(a_0, \epsilon'(a_{r+r'}^{-1})), \mathcal{L}} = \omega_! \bar{\mathbf{Q}}_l$. If $a_0 \ne \epsilon'(a_{r+r'})$, then $X_{\mathbf{a}} = \emptyset$, hence $\omega_! \bar{\mathbf{Q}}_l = 0$; if $a_0 = \epsilon'(a_{r+r'})$, that is, $\mathbf{a} \in \mathbf{S}_0$, then

(f) $$H^z(\text{point}, \omega_! \bar{\mathbf{Q}}_l) = \oplus_{w \in \mathbf{W}; 2l(w) = z} \bar{\mathbf{Q}}_l(-z - l(a_0)).$$

Using (c),(d),(e),(f) (or rather their variant in the mixed category) we see that

$$\sum_z (-1)^z \chi_v(H^z(\text{point}, p_! \iota^* \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')))$$
$$= \sum_{\mathbf{a} \in \mathbf{S}_0, e \in \mathbf{Z}} (-1)^e v^{2N_{\mathbf{a}} + 2l(w^0_{\epsilon(J)} a_{r+r'}) - 2\mathbf{r} + 2e} \binom{\mathbf{r}}{2\mathbf{r} - e} \sum_{w \in \mathbf{W}} v^{2l(w) + 2l(a_0)}$$
$$= v^{2l(w^0_J)}(v^2 - 1)^{\mathbf{r}} \sum_{w \in \mathbf{W}} v^{2l(w)} \sum_{\mathbf{a} \in \mathbf{S}_0} v^{2N_{\mathbf{a}}} = v^{2l(w^0_J)} \mu(G^0) \text{tr}(\Phi', H_n).$$

It remains to show that $\text{tr}(\Phi', H_n) = \text{tr}(\Phi'', H_n)$. Define $\mathcal{A}$-linear maps $\Psi', \Psi'', \Omega : H_n \to H_n$ by

$$\Psi'(\xi) = C^{\mathbf{s}}_{\underline{D}\lambda} \xi C^{\mathbf{s}'}_{\underline{D}'\lambda'}, \Psi''(\xi) = C^{\tilde{\mathbf{s}}'}_{\lambda'^{-1}} \xi C^{\tilde{\mathbf{s}}}_{\lambda^{-1}}, \Omega(T_w 1_{\lambda_1}) = 1_{\lambda_1^{-1}} T_{w^{-1}}.$$

One checks that
$\Omega \Psi' = \Psi'' \Omega, \mathfrak{a}_D \Theta^J = \Theta^{\epsilon(J)} \mathfrak{a}_D, \mathfrak{a}_D \Theta^J \Omega = \Omega \mathfrak{a}_D \Theta^J$, $\Phi' = \Theta^{\epsilon(J)} \mathfrak{a}_D \Psi'$,
$\Phi'' = \Theta^J \Psi'' \mathfrak{a}_D$.
Hence $\Phi' = \mathfrak{a}_D \Theta^J \Omega^{-1} \Psi'' \Omega = \Omega^{-1} \mathfrak{a}_D \Theta^J \Psi'' \Omega$ and

$$\text{tr}(\Phi', H_n) = \text{tr}(\Omega^{-1} \mathfrak{a}_D \Theta^J \Psi'' \Omega, H_n) = \text{tr}(\mathfrak{a}_D \Theta^J \Psi'', H_n)$$
$$= \text{tr}(\Theta^J \Psi'' \mathfrak{a}_D, H_n) = \text{tr}(\Phi'', H_n),$$

as required. The corollary is proved.

For $\lambda \in \underline{\mathfrak{s}}(\mathbf{T})$ we set $\mathbf{W}_\lambda = \mathbf{W}_{\mathcal{L}}$ where $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ is in the isomorphism class λ; this agrees with the definition in 31.2 when $\lambda \in \underline{\mathfrak{s}}_n$.

Corollary 32.24. *Let $A \in \hat{Z}_{J,D}, \mathcal{L}, \mathcal{L}'' \in \mathfrak{s}(\mathbf{T})$. Let λ (resp. λ'') be the isomorphism class of $\mathcal{L}$ (resp. $\mathcal{L}''$). Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$, $\mathbf{s}'' = (s''_1, s''_2, \dots, s''_{r'})$ be sequences in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D} \lambda = \lambda$, $s''_1 s''_2 \dots s''_{r'} \underline{D} \lambda' = \lambda''$, $A \dashv \bar{K}^{\mathbf{s}, \mathcal{L}}_{J,D}$ and $A \dashv \bar{K}^{\mathbf{s}'', \lambda''}_{J,D}$. Then there exist $b \in \mathbf{W}_{\lambda''}$, $a_0 \in \mathbf{W}_J$ such that*

(a) $$a_0(\lambda'') = \lambda, \quad s_1 s_2 \dots s_r \underline{D} = a_0 s''_1 s''_2 \dots s''_{r'} \underline{D} b a_0^{-1}.$$

Let $A' = \partial_!(\mathfrak{D}(A))$ with ∂ as in 28.19. Then $A' \dashv \bar{K}^{\mathbf{s}', \mathcal{L}'}_{\epsilon_D(J), D'}$ where $\mathbf{s}' = (s'_1, s'_2, \dots, s'_{r'})$ is given by $s'_k = s''_{r'+1-k}$ and $D' = D^{-1}$, $\mathcal{L}' = (\underline{D}')^*(\mathcal{L}'')$ (see

28.17, 28.19); hence $A * A' \in \mathcal{D}(Z_{J,\Delta})$ is well defined with $\Delta = D'D = G^0$. By 32.23(b) we have

$$H_c^0(\text{point}, p_!\iota^*(A * A')) = H_c^0(Z_{J,D}, A \otimes \partial^*(A')) = H_c^0(Z_{J,D}, A \otimes \mathfrak{D}(A)).$$

The last vector space is one-dimensional; see [L3, II, 7.4]. It follows that

$$H_c^0(\text{point}, p_!\iota^*(A * A')) \neq 0.$$

Now some shift of $A * A'$ is a direct summand of $\bar{K}_{J,D}^{\mathbf{s},\mathcal{L}} * \bar{K}_{\epsilon_D(J),D'}^{\mathbf{s}',\mathcal{L}'} = \rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')$ (see Theorem 32.6). Hence $H_c^z(\text{point}, p_!\iota^*(\rho_!(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')) \neq 0$ for some $z \in \mathbf{Z}$. Using this and Corollary 32.23(c) we see that there exists $\mathbf{a} \in \mathbf{S}$ such that $H_c^z(\text{point}, p_!\iota^*\rho'_{\mathbf{a}!}(\bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}')) \neq 0$ for some $z \in \mathbf{Z}$. Using this and 32.23(d) we see that there exists $e \in \mathbf{Z}$ such that $H_c^z(\text{point}, K_e) \neq 0$ for some $z, e \in \mathbf{Z}$. Using this and 32.23(e) we see that there exists $e' \in \mathbf{Z}$ such that $H_c^z(\text{point}, K'^{\,e}_{e'}) \neq 0$ for some $z, e' \in \mathbf{Z}$. As in 32.23 we see that we must have $\mathbf{a} \in \mathbf{S}_0$. Thus, there exists a sequence $a_0, a_1, \dots, a_{r+r'}$ in $\mathbf{W}^{r+r'}$ such that

$a_k \in \{a_{k-1}, s_k a_{k-1}\}$ for $k \in [1, r]$,
$a_k \in \{a_{k-1}, a_{k-1}(s''_{k-r})$ for $k \in [r+1, r+r']$,
$a_0 \in \mathbf{W}_J$, $a_0(\lambda'') = \lambda$, $a_{r+r'} = \epsilon(a_0)$,
$i \in [1, r], a_{i-1} = a_i \implies s_1 s_2 \dots s_i \dots s_2 s_1 \in \mathbf{W}_\lambda$
$j \in [1, r'], a_{j+r-1} = a_{j+r} \implies s''_1 \dots s''_{j-1} s''_j s''_{j-1} \dots s''_1 \in \mathbf{W}_{\lambda''}$.

For $i \in [1, r]$ we set $t_i = s_1 s_2 \dots s_i \dots s_2 s_1$ if $a_{i-1} = a_i$ and $t_i = 1$ if $a_{i-1} \notin a_i$. Then $t_i \in \mathbf{W}_\lambda$ and

$$s_1 s_2 \dots s_i = t_i s_1 s_2 \dots s_{i-1} a_{i-1} a_i^{-1}.$$

It follows that $s_1 s_2 \dots s_r = t_r t_{r-1} \dots t_1 a_0 a_r^{-1}$. Similarly, for $j \in [1, r']$ we set $t''_j = s''_1 s''_2 \dots s''_j \dots s''_2 s''_1$ if $a_{j+r-1} = a_{j+r}$ and $t''_j = 1$ if $a_{j+r-1} \neq a_{j+r}$. Then $t''_j \in \mathbf{W}_{\lambda''}$ and $s''_1 s''_2 \dots s''_j = t''_j s''_1 s''_2 \dots s''_{j-1} a_{j+r-1}^{-1} a_{j+r}$. It follows that $s''_1 s''_2 \dots s''_{r'} = t''_{r'} \dots t''_2 t''_1 a_r^{-1} a_{r+r'}$. Setting

$$\tau = t_r t_{r-1} \dots t_1, \tau'' = t''_{r'} \dots t''_2 t''_1,$$

we have $\tau \in \mathbf{W}_\lambda$, $\tau'' \in \mathbf{W}_{\lambda''}$, $s_1 s_2 \dots s_r = \tau a_0 a_r^{-1}, s''_1 s''_2 \dots s''_{r'} = \tau'' a_r^{-1} \epsilon(a_0)$. Let $b' = (a_0^{-1} \tau a_0) \tau''^{-1}$. Then $b' \in \mathbf{W}_{\lambda''}$ and $s_1 s_2 \dots s_r \underline{D} = a_0 b' s''_1 s''_2 \dots s''_{r'} \underline{D} a_0^{-1}$. We set $b = (s''_1 s''_2 \dots s''_{r'} \underline{D})^{-1} b' s''_1 s''_2 \dots s''_{r'} \underline{D}$. Since $s''_1 s''_2 \dots s''_{r'} \underline{D} \lambda'' = \lambda''$ we have $b \in \mathbf{W}_{\lambda''}$. Moreover, $s_1 s_2 \dots s_r \underline{D} = a_0 s''_1 s''_2 \dots s''_{r'} \underline{D} b a_0^{-1}$. The lemma is proved.

32.25. Given $(w, \lambda), (w', \lambda')$ in $\mathbf{W}^\bullet \times \underline{\mathfrak{s}}(\mathbf{T})$ we say that $(w, \lambda) \asymp_J (w', \lambda')$ if there exist $a \in \mathbf{W}_J, b \in \mathbf{W}_{\lambda'}$ such that $w = a w' b a^{-1}, \lambda = a(\lambda')$. We then have $w' = a^{-1} w (a b^{-1} a^{-1}) a$ where $a^{-1} \in \mathbf{W}_J$, $a^{-1}(\lambda) = \lambda'$, $a b^{-1} a^{-1} \in \mathbf{W}_{a\lambda'} = \mathbf{W}_\lambda$; hence $(w', \lambda') \asymp_J (w, \lambda)$. If, in addition, we have $(w', \lambda') \asymp_J (w'', \lambda'')$, that is, $w' = \tilde{a} w'' \tilde{b} \tilde{a}^{-1}, \lambda' = \tilde{a}(\lambda'')$ with $\tilde{a} \in \mathbf{W}_J, \lambda' = \tilde{a}(\lambda''), \tilde{b} \in \mathbf{W}_{\lambda''}$, then $w = a \tilde{a} w'' (\tilde{b} \tilde{a}^{-1} b \tilde{a}) \tilde{a}^{-1} a^{-1}$ where $a \tilde{a} \in \mathbf{W}_J$, $\lambda = a \tilde{a}(\lambda'')$, $\tilde{b} \tilde{a}^{-1} b \tilde{a} \in \mathbf{W}_{\lambda''} \mathbf{W}_{\tilde{a}^{-1}(\lambda')} = \mathbf{W}_{\lambda''}$, hence $(w, \lambda) \asymp_J (w'', \lambda'')$. We see that $\asymp_J$ is an equivalence relation on $\mathbf{W}^\bullet \times \underline{\mathfrak{s}}(\mathbf{T})$.

We can now reformulate Corollary 32.24 as follows.

(a) *To $A \in \hat{Z}_{J,D}$ we can associate an equivalence class $\mathfrak{C}_A$ under $\asymp_J$ so that the following holds. If $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$, λ is the isomorphism class of $\mathcal{L}$ and $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D} \lambda = \lambda$ and $A \dashv \bar{K}_{J,D}^{\mathbf{s},\mathcal{L}}$, then $(s_1 s_2 \dots s_r \underline{D}, \lambda) \in \mathfrak{C}_A$.*

In particular:

(b) *To $A \in \hat{Z}_{J,D}$ we can associate a $\mathbf{W}_J$-orbit $\mathcal{O}_A$ on $\underline{\mathfrak{s}}(\mathbf{T})$ so that the following holds. If $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$, λ is the isomorphism class of $\mathcal{L}$ and $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D}\lambda = \lambda$ and $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}$, then $\lambda \in \mathcal{O}_A$.*

32.26. Assume now that $J = \mathbf{I}$. We write $\asymp$ instead of $\asymp_{\mathbf{I}}$. Thus, $(w, \lambda) \asymp (w', \lambda')$ if there exist $a \in \mathbf{W}, b \in \mathbf{W}_{\lambda'}$ such that $w = aw'ba^{-1}, \lambda = a(\lambda')$. Let $n \in \mathbf{N}^*_{\mathbf{k}}$.

Let A be a character sheaf on D. Let $\mathfrak{C}_A$ be the equivalence class in $\mathbf{W}^\bullet \times \underline{\mathfrak{s}}(\mathbf{T})$ under $\asymp$ defined by A (see 32.25(a)). Let $\zeta^A : H_n[D] \to \mathcal{A}$ be as in 31.7. We show:

(a) *If $\mathbf{s} = (s_1, s_2, \dots, s_r)$ is a sequence in $\mathbf{I}$, $\lambda \in \underline{\mathfrak{s}}_n$ and $\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) \neq 0$, then $(s_1 s_2 \dots s_r \underline{D}, \lambda) \in \mathfrak{C}_A$.*

Indeed, choose $\mathcal{L} \in \mathfrak{s}(\mathbf{T})$ in the isomorphism class λ. Our assumption implies that $s_1 s_2 \dots s_r \underline{D}\lambda = \lambda$ hence $\bar{K}^{\mathbf{s},\mathcal{L}}_D$ is defined. Moreover, our assumption implies $\sum_j (-v)^j v^{-\dim G}(A : {}^pH^j(\bar{K}^{\mathbf{s},\mathcal{L}}_D)) \neq 0$. In particular, $A \dashv \bar{K}^{\mathbf{s},\mathcal{L}}_D$. Hence (a) follows from 32.25(a).

We show:

(b) *Let $(x, \lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}_n$ be such that $\zeta^A(\tilde{T}_x 1_{\underline{D}\lambda}[D]) \neq 0$. Then $(x\underline{D}, \lambda) \in \mathfrak{C}_A$.*

We argue by induction on $l(x)$. If $x = 1$ we have $tT_x 1_{\underline{D}\lambda} = C^{\mathbf{s}}_{\underline{D}\lambda}$ where $\mathbf{s}$ is the empty sequence and the result follows from (a). Assume now that $l(x) \geq 1$. From our assumption we have $x\underline{D}\lambda = \lambda$. We can find a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ with $x = s_1 s_2 \dots s_r$, $r = l(x)$. From the definitions we have

$$C^{\mathbf{s}}_{\underline{D}\lambda} = \sum_{y \in \mathbf{W}_{\underline{D}\lambda}, xy \leq x} c_y \tilde{T}_{xy} 1_{\underline{D}\lambda}$$

with $c_y \in \mathcal{A}, c_1 = v^r$. Hence

$$\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) = \sum_{y \in \mathbf{W}_{\underline{D}\lambda}, xy \leq x} c_y \zeta^A(\tilde{T}_{xy} 1_{\underline{D}\lambda}[D]).$$

If $\zeta^A(\tilde{T}_{xy} 1_{\underline{D}\lambda}[D]) \neq 0$ for some $y \in \mathbf{W}_{\underline{D}\lambda}, xy < x$, then, by the induction hypothesis, we have $(xy\underline{D}, \lambda) \in \mathfrak{C}_A$; we have $(xy\underline{D}, \lambda) \asymp (x\underline{D}, \lambda)$ so that $(x\underline{D}, \lambda) \in \mathfrak{C}_A$, as required.

We may therefore assume that $\zeta^A(\tilde{T}_{xy} 1_{\underline{D}\lambda}[D]) = 0$ for all $y \in \mathbf{W}_{\underline{D}\lambda}$ such that $xy < x$. Then we have $\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) = v^r \zeta^A(\tilde{T}_x 1_{\underline{D}\lambda}[D])$. Hence $\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]) \neq 0$. Using (a) we see that $(x\underline{D}, \lambda) \in \mathfrak{C}_A$, as required. This proves (b).

33. Disjointness

33.1. We fix an irreducible component D of G. For $(L, S) \in \mathbf{A}$ with $S \subset D$ and $\mathcal{E} \in \mathcal{S}(S)$ we define $\mathfrak{K}$ as in 5.6; we regard $\mathfrak{K}$ as a complex on D, zero outside $\bar{Y}_{L,S}$ and we write $(L, S, \mathcal{E}) \blacktriangleright_G \mathfrak{K}$.

Lemma 33.2. *Let $(L, S) \in \mathbf{A}, (L', S') \in \mathbf{A}$ with $S \subset D, S' \subset D$. Let $\mathcal{E} \in \mathcal{S}(S), \mathcal{E}' \in \mathcal{S}(S')$. Let $(L, S, \mathcal{E}) \blacktriangleright_G \mathfrak{K}$, $(L', S', \mathcal{E}') \blacktriangleright_G \mathfrak{K}'$. Assume that $\mathcal{E}$ (resp. $\mathcal{E}'$) is strongly cuspidal and clean (see 23.3) relative to $N_G L$ (resp. $N_G L'$), that $L = G^0$ hence $Y_{L,S} = S$ and that $Y_{L',S'} \neq S$. Then for any i, the local systems $\check{\mathcal{E}}$, $\mathcal{H}^i \mathfrak{K}'|_S$ have no common irreducible direct summand.*

If $L' = G^0$, then, since $\mathcal{E}'$ is clean, we have $\mathcal{H}^i \mathfrak{K}'|_S = 0$. Assume now that $L' \neq G^0$. By 23.7 we have $H^j_c(G, \mathfrak{K} \otimes \mathfrak{K}') = 0$ for all j. Since $\mathfrak{K} = IC(\bar{S}, \mathcal{E})$ and $\mathcal{E}$ is clean, we have $H^j_c(S, \mathfrak{K} \otimes \mathfrak{K}') = 0$ for all j; hence $H^i_c(S, \mathcal{E} \otimes \mathfrak{K}') = 0$ for all j. We must show that the local system $\mathcal{H}^i(\mathcal{E} \otimes (\mathfrak{K}'|_S))$ on S has no direct summand $\bar{\mathbf{Q}}_l$.

Assume that $\mathcal{H}^{i_0}(\mathcal{E}\otimes(\mathfrak{K}'|_S))$ has a direct summand $\bar{\mathbf{Q}}_l$ and that i_0 is maximum possible with this property. If $a=\dim S'$, we have $H_c^{2a}(S,\mathcal{H}^{i_0}(\mathcal{E}\otimes(\mathfrak{K}'|_S)))\neq 0$. Hence $E_2^{2a,i_0}\neq 0$ in the standard spectral sequence

$$E_2^{p,q}=H_c^p(S,\mathcal{H}^q(\mathcal{E}\otimes(\mathfrak{K}'|_S)))\implies H_c^{p+q}(S,\mathcal{E}\otimes(\mathfrak{K}'|_S)).$$

By the proof of 23.5 we have $H_c^p(S,\mathcal{E}_1)=0$ for any $\mathcal{E}_1\in\mathcal{S}(S)$ which has no direct summand $\bar{\mathbf{Q}}_l$; in particular, taking $\mathcal{E}_1=\mathcal{H}^i(\mathcal{E}\otimes(\mathfrak{K}'|_S))$ with $i>i_0$ we see that $E_2^{p,q}=0$ if $q>i_0$. Clearly, $E_2^{p,q}=0$ if $p>2a$, hence $E_2^{2a,i_0}=E_3^{2a,i_0}=\dots=E_\infty^{2a,i_0}$. Since $E_2^{2a,i_0}\neq 0$, it follows that $H_c^{2a+i_0}(S,\mathcal{E}\otimes(\mathfrak{K}'|_S))\neq 0$, a contradiction. The lemma is proved.

Proposition 33.3. *Let $(L,S)\in\mathbf{A},(L',S')\in\mathbf{A}$ with $S\subset D,S'\subset D$. Let $\mathcal{E}\in\mathcal{S}(S),\mathcal{E}'\in\mathcal{S}(S')$. Let $(L,S,\mathcal{E})\blacktriangleright_G\mathfrak{K}$, $(L',S',\mathcal{E}')\blacktriangleright_G\mathfrak{K}'$. Assume that $\mathcal{E}$ and $\check{\mathcal{E}}$ (resp. $\mathcal{E}'$) are strongly cuspidal and clean relative to N_GL (resp. N_GL'). Let A (resp. A') be an admissible complex on D (see 6.7) which is a direct summand of $\mathfrak{K}$ (resp. of $\mathfrak{K}'$). Assume that $A\not\cong A'$. Let $Y=Y_{L,S}$. Let $\mathcal{F}$ be the local system $A|_Y$. Then for any i, $\mathcal{F}$ is not a direct summand of $\mathcal{H}^i(A')|_Y$ (which is a local system by 25.2).*

Since $\bar{Y}_{L',S'}$ is a union of strata of D, we have either $Y\cap\bar{Y}_{L',S'}=0$ or $Y=Y_{L',S'}$ or $Y\subset\bar{Y}_{L',S'}-Y_{L',S'}$. In the first case we have $\mathcal{H}^i(A')|_Y=0$ and the result is obvious. In the second case we have $\mathcal{H}^i(A')|_Y=0$ unless $i=0$ and since $A\not\cong A'$, the local system $\mathcal{H}^0(A)|_Y$ is irreducible, non-isomorphic to $\mathcal{F}$. Thus, we may assume that $Y\subset\bar{Y}_{L',S'}-Y_{L',S'}$. It is enough to show that for any i, $\mathcal{F}$ is not a direct summand of $\mathcal{H}^i(\mathfrak{K}')|_Y$ (a local system, by 25.2). Let δ be the connected component of N_GL such that $S\subset\delta$. Let $su=us\in S^*$ with s semisimple, u unipotent. Let $\tilde{\delta}$ be the connected component of $Z_G(s)$ such that $u\in\tilde{\delta}$. Since su is isolated in N_GL, we have ${}^{\delta}\mathcal{Z}_L^0={}^{\tilde{\delta}}\mathcal{Z}_{Z_L(s)^0}$; we denote this torus by $\mathcal{T}$. Let R_1 be the subvariety of S consisting of all elements of the form $yzsuy^{-1}$ with $y\in Z_L(s)^0,z\in\mathcal{T}$. Since R_1 is an orbit of a connected group, it is smooth, irreducible. Let $R_1^*=R_1\cap S^*$ be an open dense subset of R_1 (see 25.4, 25.6). Let $\pi_1:\pi^{-1}(R_1^*)\to R_1^*$ be the restriction of $\pi:\tilde{Y}_{L,S}\to Y$ (as in 3.13). Let $\tilde{\mathcal{E}}$ be the local system on $\tilde{Y}_{L,S}$ defined in 5.6; its restriction to $\pi^{-1}(R_1^*)$ is denoted again by $\tilde{\mathcal{E}}$. From the definitions, we have $\mathfrak{K}|_{R_1^*}=\pi_{1!}\tilde{\mathcal{E}}$. By the proof of 3.13(a) we have

$$\pi^{-1}(R_1^*)=\sqcup_{xL\in N(L,S)/L}\{(g,xL);g\in R_1^*\}$$

where $N(L,S)=\{x\in N_{G^0}L,xSx^{-1}=S\}$. Define $\epsilon:s^{-1}R_1\to R_1$ by $g\mapsto sg$. We see that $\epsilon^*\mathfrak{K}|_{R_1^*}=\oplus_{xL\in N(L,S)/L}\mathcal{E}^x|_{s^{-1}R_1^*}$ where $\mathcal{E}^x$ is the local system on $s^{-1}R_1$ obtained by taking the inverse image of $\mathcal{E}$ under $s^{-1}R_1\to S,g\mapsto xsgx^{-1}$. Now $s^{-1}R_1$ is an isolated stratum of $Z_G(s)$ contained in the connected component $\tilde{\delta}$ (it is the stratum containing u). From 23.4 we see that $\mathcal{E}^x$ and $\check{\mathcal{E}}^x$ are strongly cuspidal and clean with respect to $Z_G(s)$. By 16.12 we can find complexes $\mathfrak{K}'_j(j\in[1,m])$ on $\tilde{\delta}$ of the same type as $\mathfrak{K}'$ and an open subset $\mathcal{U}$ of $\tilde{\delta}$ containing all unipotents in $\tilde{\delta}$ such that

(a) $\epsilon'^*(\mathfrak{K}'|_{s\mathcal{U}})\cong\oplus_j\mathfrak{K}'_j|_{\mathcal{U}}$,

where $\epsilon':\mathcal{U}\to s\mathcal{U}$ is $g\mapsto sg$. Note that $R_1^*\cap s\mathcal{U}$ contains su hence is non-empty. Since $s\mathcal{U}$ is open in $s\tilde{\delta}$, and R_1 is an irreducible subset of $s\tilde{\delta}$, we see that $R_1\cap s\mathcal{U}$ is an open dense subset of R_1. Since R_1^* is another open dense subset of R_1, we see that

(b) $R_1^* \cap s\mathcal{U} = (R_1 \cap s\mathcal{U}) \cap R_1^*$ *is open dense in* R_1.
It suffices to show that the local systems $\mathcal{H}^i(\mathfrak{K}')|_{R_1^* \cap s\mathcal{U}}$, $\mathfrak{K}|_{R_1^* \cap s\mathcal{U}}$ have no common irreducible direct summand. Using (a) we see that it suffices to show that for any $j \in [1, m], x \in N(L, S)$,

(c) *the local systems* $\mathcal{H}^i(\mathfrak{K}'_j)|_{s^{-1}R_1^* \cap \mathcal{U}}$, $\mathcal{E}^x_{s^{-1}R_1^* \cap \mathcal{U}}$ *have no common irreducible direct summand.*
Since $s^{-1}R_1$ is an isolated stratum of $Z_G(s)$, $\mathcal{H}^i(\mathfrak{K}'_j)|_{s^{-1}R_1}$ is a local system. Using (b) we see that (c) would follow from the following statement:

(d) *the local systems* $\mathcal{H}^i(\mathfrak{K}'_j)|_{s^{-1}R_1}$, $\mathcal{E}^x$ *have no common irreducible direct summand.*
By 16.12(b) we may assume that there exists $x' \in G^0$ such that $x'^{-1}sx' \in S'_s$ and the following holds. Let $L'' = x'L'x'^{-1}, S'' = x'S'x'^{-1}$, $L'_0 = L'' \cap Z_G(s)^0$,

S'_0 is a stratum of $N_G L'_0$ contained in $\tilde{\delta}$, containing unipotent elements such that $S'_0 \subset s^{-1}S''$,

$\mathcal{E}'_0$ is the local system on S'_0, inverse image of $\mathcal{E}'$ under $S'_0 \to S'$, $g \mapsto x'^{-1}sgx'$, $(L'_0, S'_0, \mathcal{E}'_0) \blacktriangleright_{Z_G(s)} \mathfrak{K}'_j$.
From 23.4 we see that $\mathcal{E}'_0$ is strongly cuspidal and clean with respect to $N_{Z_G(s)}(L'_0)$. We see that (d) follows from Lemma 33.2 (applied to $Z_G(s), \mathfrak{K}'_j, \check{\mathcal{E}}^x$ instead of $G, \mathfrak{K}', \mathcal{E}$) provided we can show that

(e) $Y_{L'_0, S'_0}$ *(defined in terms of* $Z_G(s)$*) is not equal to* $s^{-1}R_1$.
Assume that $Y_{L'_0, S'_0} = s^{-1}R_1$. Since $s^{-1}R_1$ is an isolated stratum of $Z_G(s)$, it follows that $L'_0 = Z_G(s)^0$ and $S'_0 = s^{-1}R_1$, hence $Z_G(s)^0 \subset L''$ and $u \in S'_0$. Since $sS'_0 \subset S''$, we have $su \in S''$. We can find a parabolic P' of G^0 with Levi L' such that $S' \subset N_G P' \cap N_G L'$, hence $S'' \subset N_G(x'P'x'^{-1}) \cap N_G(x'L'x'^{-1})$. We see that $su \in N_G(x'P'x'^{-1}) \cap N_G(x'L'x'^{-1})$. Using 2.1(c) with $g = su, Q = x'P'x'^{-1}$, we see that $L(su) \subset x'L'x'^{-1} = L''$ where $L(su)$ is defined as in 2.1. We can find a parabolic P of G^0 with Levi L such that $S \subset N_G P \cap N_G L$, hence $su \in N_G P \cap N_G L$. Moreover, su is isolated in $N_G P \cap N_G L$. From 3.8(a) we see that $L \subset L(su)$. Combining with $L(su) \subset L''$, we see that $L \subset L''$. Since $Y \subset \bar{Y}_{L',S'} - Y_{L',S'}$, we have $\bar{Y} \subset \bar{Y}_{L',S'}$. Taking images under the map $\sigma : D \to D//G^0$ (see 7.1) we obtain $\dim \sigma(\bar{Y}) \le \dim \sigma(\bar{Y}_{L',S'})$. Using 7.3(b) we can rewrite the last inequality in the form $\dim({}^\delta \mathcal{Z}^0_L) \le \dim({}^{\delta'} \mathcal{Z}^0_{L'})$ where δ' is the connected component of $N_G L'$ that contains S'. Equivalently,

(f) $\dim({}^\delta \mathcal{Z}^0_L) \le \dim({}^{\delta''} \mathcal{Z}^0_{L''})$
where δ'' is the connected component of $N_G L''$ that contains S''. From $L \subset L''$ we deduce $\mathcal{Z}_{L''} \subset \mathcal{Z}_L$. Intersecting both sides with $Z_G(su)$ and noting that $su \in \delta, su \in \delta''$ we see that ${}^{\delta''}\mathcal{Z}_{L''} \subset {}^\delta \mathcal{Z}_L$. Taking identity components we have ${}^{\delta''}\mathcal{Z}^0_{L''} \subset {}^\delta \mathcal{Z}^0_L$. Using (f) we deduce ${}^{\delta''}\mathcal{Z}^0_{L''} = {}^\delta \mathcal{Z}^0_L$. Taking the centralizer of both sides in G^0 and using 1.10(a) we obtain $L = L''$. Now S and S'' are strata of $N_G L = N_G L''$ which contain a common point, su. Hence $S = S''$. Since $(L, S) = (L'', S'')$ we have $Y_{L,S} = Y_{L'',S''}$, hence $Y = Y_{L',S'}$. This contradicts $Y \subset \bar{Y}_{L',S'} - Y_{L',S'}$ and proves (e).

The proposition is proved.

33.4. Let $\mathcal{I}$ be a finite collection of mutually non-isomorphic character sheaves on D and let $A \in \mathcal{I}$. Let $Y = Y_{L,S}$ be the stratum of D such that $\mathrm{supp}(A) = \bar{Y}$. Let $\tilde{Y} = \{(g, xL) \in D \times G^0/L; x^{-1}gx \in S^*\}$ (see 3.13). Define $\pi_1 : \tilde{Y} \to Y$ by $\pi_1(g, xL) = g$. By 25.2, for any $A' \in \mathcal{I}$ and any $i \in \mathbf{Z}$ there exists a local system

$\mathcal{E} \in \mathcal{S}(S)$ such that $\mathcal{H}^i(A')|_Y$ is a local system isomorphic to a direct summand of $\pi_{1!}\tilde{\mathcal{E}}$ with $\tilde{\mathcal{E}}$ as in 5.6. Replacing $\mathcal{E}$ by the direct sum of the local systems $\mathcal{E}$ (for various j, i as above) we see that we may assume that $\mathcal{E}$ is the same for all A', i. We can find $n' \in \mathbf{N}^*_{\mathbf{k}}$ such that $\mathcal{E} \in \mathcal{S}_{n'}(S)$. Let δ be the connected component of $N_G L$ that contains S. Let $g_1 \in S$. Let $H = \{(z_1, l_1) \in {}^\delta\mathcal{Z}^0_L \times L; l_1 z_1^{n'} g_1 l_1^{-1} = g_1\}$. Let

$$V = \{(g, x, z, l) \in D \times G^0 \times {}^\delta\mathcal{Z}^0_L \times L; x^{-1}gx = lz^{n'}g_1 l^{-1} \in S^*\}.$$

Now V is irreducible; it is isomorphic to the product of G^0 with an open dense subset of ${}^\delta\mathcal{Z}^0_L \times L$. We have a commutative diagram with cartesian squares

$$\begin{array}{ccccccc} & & \tilde{Y}' & \xleftarrow{a'} & Z' & \xrightarrow{b'} & S' \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_4} \\ Y & \xleftarrow{\pi_1} & \tilde{Y} & \xleftarrow{a} & Z & \xrightarrow{b} & S \end{array}$$

where

S' is the space of H^0-orbits on ${}^\delta\mathcal{Z}^0_L \times L$ for the free H^0-action by right translation,

$Z = \{(g, x) \in D \times G^0; x^{-1}gx \in S^*\}$,

$\tilde{Y}'$ is the space of $(L \times H^0)$-orbits on V for the free $L \times H^0$-action $(l_0, (z_1, l_1)) : (g, x, z, l) \mapsto (g, xl_0^{-1}, zz_1^{-1}, l_0 l l_1^{-1})$,

Z' is the space of H^0-orbits on V for the free H^0-action $(z_1, l_1)) : (g, x, z, l) \mapsto (g, x, zz_1^{-1}, ll_1^{-1})$,

$a(g, x) = (g, xL), b(g, x) = x^{-1}gx$, a' is the obvious map, $b'(g, x, z, l) \mapsto (z, l)$, $\pi_2(g, x, z, l) = (g, xL), \pi_3(g, x, z, l) = (g, x)$, $\pi_4(z_1, l_1) = l_1^{-1} z_1^{n'} g_1 l_1$.

Now $\tilde{Y}'$ is irreducible since V is irreducible; $\tilde{Y}$ is irreducible since it equals $\pi_2(\tilde{Y}')$. Since $\mathcal{E} \in \mathcal{S}_{n'}(S)$, the local system $\pi_4^*\mathcal{E}$ on S' is $({}^\delta\mathcal{Z}^0_L \times L)$-equivariant (for the action by left translation). Since this action is transitive with connected isotropy groups, we see that $\pi_4^*\mathcal{E} \cong \bar{\mathbf{Q}}_l^e$ for some integer $e \geq 1$. Hence $\pi_3^* b^* \mathcal{E} = b'^* \pi_4^* \mathcal{E} \cong \bar{\mathbf{Q}}_l^e$. By definition, $a^*\tilde{\mathcal{E}} = b^*\mathcal{E}$. Hence $a'^*\pi_2^*\tilde{\mathcal{E}} = \pi_3^* a^* \tilde{\mathcal{E}} = \pi_3^* b^* \mathcal{E} \cong \bar{\mathbf{Q}}_l^e$. Since a' is a principal L-bundle, it follows that $\pi_2^*\tilde{\mathcal{E}} \cong \bar{\mathbf{Q}}_l^e$. Now $\pi_0 := \pi_1\pi_2 : \tilde{Y}' \to Y$ is a composition of two (finite) principal coverings (π_1 is a principal H/H^0-covering since π_4 is a principal H/H^0-covering; π_1 is a principal covering by 3.13(a)), hence it is a not necessarily principal, finite unramified covering. Let $N = |\pi_0^{-1}(y)|$ for some/any $y \in Y$. Let Y'' be the set of all pairs (y, f) where $y \in Y$ and $f : \{1, 2, \dots, N\} \to \pi_0^{-1}(y)$ is a bijection. Then Y'' is an algebraic variety and $\pi_0' : Y'' \to Y, (y, f) \mapsto y$ is a principal covering whose group is the symmetric group $\mathfrak{S}_N$. Moreover, π_0' factors as $Y'' \xrightarrow{\tau} \tilde{Y}' \xrightarrow{\pi_0} Y$ where $\tau(y, f) = f(1)$. Let $\hat{Y}$ be a connected component of Y''. Then $\tau_0 : \hat{Y} \to \tilde{Y}'$ (restriction of τ) is a finite unramified covering. Let $\pi : \hat{Y} \to Y$ be the restriction of π_0'. Then π is a (finite) principal bundle whose group is the group Γ consisting of all elements of $\mathfrak{S}_N$ which map $\hat{Y}$ into itself. Moreover, π factors as $\hat{Y} \xrightarrow{\tau_1} \tilde{Y} \xrightarrow{\pi_1} Y$ where $\tau_1 = \pi_1\tau_0$ is a finite unramified covering. Since $\pi_2^*\tilde{\mathcal{E}} \cong \bar{\mathbf{Q}}_l^e$, we have $\tau_1^*\tilde{\mathcal{E}} \cong \bar{\mathbf{Q}}_l^e$. Hence any irreducible direct summand of the local system $\tilde{\mathcal{E}}$ is a direct summand of $\tau_{1!}\bar{\mathbf{Q}}_l$. Now let $\mathcal{E}_1$ be an irreducible local system on Y which is a direct summand of $\pi_{1!}\tilde{\mathcal{E}}$. We can find an irreducible direct summand $\mathcal{E}_2$ of $\tilde{\mathcal{E}}$ such that $\mathcal{E}_1$ is a direct summand of $\pi_{1!}\mathcal{E}_2$. Then $\mathcal{E}_2$ is a direct summand of $\tau_{1!}\bar{\mathbf{Q}}_l$, hence $\pi_{1!}\mathcal{E}_2$ is a direct summand of $\pi_{1!}\tau_{1!}\bar{\mathbf{Q}}_l = \pi_!\bar{\mathbf{Q}}_l$. Since $\mathcal{E}_1$ is a direct summand of $\pi_{1!}\mathcal{E}_2$ it follows that

(a) $\mathcal{E}_1$ *is a direct summand of* $\pi_!\bar{\mathbf{Q}}_l$.

Let $\mathcal{C}$ be the category whose objects are local systems on Y which are direct sums of irreducible direct summands of $\pi_!\bar{\mathbf{Q}}_l$. Let $\mathcal{C}_\Gamma$ be the category of $\bar{\mathbf{Q}}_l[\Gamma]$-modules of finite dimension over $\bar{\mathbf{Q}}_l$. We have an equivalence of categories $\mathcal{C}_\Gamma \to \mathcal{C}$: it attaches to an object M of $\mathcal{C}_\Gamma$ the local system $[M] = (M^* \otimes \pi_!\bar{\mathbf{Q}}_l)^\Gamma$ in $\mathcal{C}$; here $\pi_!\bar{\mathbf{Q}}_l$ is regarded naturally as a local system with Γ-action, M^* is the dual of M and the superscript denotes Γ-invariants. Using (a) and the definition of $\mathcal{E}$, we see that, for any $A' \in \mathcal{I}, i \in \mathbf{Z}$, we have $\mathcal{H}^i(A')|_Y \in \mathcal{C}$. Hence $\mathcal{H}^i(A')|_Y \cong [M_{A',i}]$ for some $M_{A',i} \in \mathcal{C}_\Gamma$, well defined up to isomorphism. Let $e = \dim Y$. Then $M_{A,-e}$ is an irreducible object of $\mathcal{C}_\Gamma$.

In the remainder of this section we assume that:

(b) *D is clean in the sense that, for any parabolic subgroup P of G^0 such that $N_D P \neq \emptyset$, any cuspidal character sheaf of $N_D P/U_P$ is* 0 *on the complement of some isolated stratum of $N_D P/U_P$.*

We show:

(c) *if $A' \in \mathcal{I}, i \in \mathbf{Z}$ and $A' \neq A$, then $M_{A',i}$ contains no direct summand isomorphic to $M_{A,-e}$.*

This follows from Proposition 33.3 which is applicable in view of (b), the admissibility of character sheaves (30.6), the strong cuspidality of cuspidal character sheaves (31.15) and the fact that $\mathfrak{D}(A)$ is a character sheaf (28.18).

In the remainder of this section we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G has a fixed $\mathbf{F}_q$-rational structure whose Frobenius map F induces the identity map on G/G^0. Replacing $\mathbf{F}_q$ by a finite extension, we may assume that $F(Y) = Y$, that $\hat{Y}$ and $\pi : \hat{Y} \to Y$ are defined over $\mathbf{F}_q$, that the Frobenius map $F : \hat{Y} \to \hat{Y}$ satisfies $F(\gamma\hat{y}) = \gamma F(\hat{y})$ for all $\gamma \in \Gamma, \hat{y} \in \hat{Y}$, that $F^*A' \cong A'$ for all $A' \in \mathcal{I}$ and that for any $\gamma \in \Gamma$ and any integer $m \geq 1$ there exists $\hat{y}_{\gamma,m} \in \hat{Y}$ such that $F^m(\hat{y}_{\gamma,m}) = \gamma\hat{y}_{\gamma,m}$. (We then set $y_{\gamma,m} = \pi(\hat{y}_{\gamma,m})$.)

Let $M \in \mathcal{C}_\Gamma$. The stalk of $[M]$ at $y \in Y$ is the vector space

$$[M]_y = \{f : \pi^{-1}(y) \to M^*; f(\hat{y}) = \gamma(f(\gamma^{-1}\hat{y})) \text{ for all } \gamma \in \Gamma, \hat{y} \in \hat{Y}\}.$$

Let $m \geq 1$. For any $R \in \mathrm{Aut}_{\mathcal{C}_\Gamma}(M^*)$ there is a unique isomorphism of local systems $\tilde{R} : F^{m*}[M] \xrightarrow{\sim} [M]$ such that for any $y \in Y$, $\tilde{R}$ induces on stalks the linear map $\tilde{R}_y : [M]_{F^m(y)} \to [M]_y$ which to a function $f : \pi^{-1}(F^m(y)) \to M^*$ associates the function $f' : \pi^{-1}(y) \to M^*$ given by $f'(\hat{y}) = R(f(F^{-m}(\hat{y})))$. Clearly, any isomorphism $F^{m*}[M] \xrightarrow{\sim} [M]$ is of the form $\tilde{R}$ for a unique R as above.

For $\gamma \in \Gamma$ we have an isomorphism

(d) $[M]_{y_{\gamma,m}} \xrightarrow{\sim} M^*, f \mapsto f(\hat{y}_{\gamma,m})$.

If R is as above, then $\tilde{R}_{y_{\gamma,m}}$ maps $[M]_{y_{\gamma,m}}$ into itself (since $F^m(y_{\gamma,m}) = y_{\gamma,m}$) and it corresponds under (d) to the automorphism $\gamma^{-1}R = R\gamma^{-1} : M^* \to M^*$. Hence

(e) $\mathrm{tr}(\tilde{R}_{y_{\gamma,m}}, [M]_{y_{\gamma,m}}) = \mathrm{tr}(\gamma^{-1}R, M^*) = \mathrm{tr}({}^tR\gamma, M)$.

33.5. Let V be an algebraic variety defined over $\mathbf{F}_q$ with Frobenius map $F : V \to V$. Let $K \in \mathcal{D}(V)$ and let $\phi : F^*K \xrightarrow{\sim} K$ be an isomorphism. For any integer $m \geq 1$ we denote by $\phi^{(m)} : F^{m*}K \xrightarrow{\sim} K$ the composition

$$(F^m)^*K \xrightarrow{(F^{m-1})^*\phi} (F^{m-1})^*K \xrightarrow{(F^{m-2})^*} \dots \xrightarrow{F^*\phi} F^*K \xrightarrow{\phi} K.$$

33.6. For each $A' \in \mathcal{I}$ we choose an isomorphism $\kappa_{A'} : F^*A' \xrightarrow{\sim} A'$. Let $\kappa'_A : F^*\mathfrak{D}(A) \xrightarrow{\sim} \mathfrak{D}(A)$ be the isomorphism such that for any $y \in Y$ the isomorphism

$\mathcal{H}^{-e}\mathfrak{D}(A)_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}\mathfrak{D}(A)_y$ (that is, $\mathcal{H}^{-e}(A)\check{}_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}(A)\check{}_y$) induced by κ'_A is $q^{\dim D-e}$ times the transpose inverse of the isomorphism $\mathcal{H}^{-e}(A)_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}(A)_y$ induced by κ_A.

Proposition 33.7. *Let $A' \in \mathcal{I}$. For any integer $m \geq 1$ we have*

(a) $$q^{-(\dim D-e)m}|\Gamma|^{-1}\sum_{\gamma\in\Gamma}\chi_{A',\kappa_{A'}^{(m)}}(y_{\gamma,m})\chi_{\mathfrak{D}(A),\kappa'_A{}^{(m)}}(y_{\gamma,m}) = \delta_{A,A'}.$$

Under an isomorphism $\mathcal{H}^i(A')|_Y \cong [M_{A',i}]$, the isomorphism $F^{*m}\mathcal{H}^i(A') \xrightarrow{\sim} \mathcal{H}^i(A')$ induced by $\kappa_{A'}^{(m)} : F^{*m}A' \xrightarrow{\sim} A'$ corresponds to an isomorphism $F^{*m}[M_{A',i}] \xrightarrow{\sim} [M_{A',i}]$ which must be of the form $\tilde{R}$ for some $R = R_{m,A',i} \in \mathrm{Aut}_{\mathcal{C}_\Gamma}(M^*_{A',i})$; hence

$$\mathrm{tr}(\kappa_{A'}^{(m)}, \mathcal{H}^i(A')_{y_{\gamma,m}}) = \mathrm{tr}({}^tR_{m,A',i}\gamma, M_{A',i}).$$

Next we have

$$\mathrm{tr}(\kappa'_A{}^{(m)}, \mathcal{H}^{-e}(\mathfrak{D}(A))_{y_{\gamma,m}}) = q^{(\dim D-e)m}\mathrm{tr}((\kappa_A^{(m)})^{-1}, \mathcal{H}^{-e}(A)_{y_{\gamma,m}})$$
$$= q^{(\dim D-e)m}\mathrm{tr}({}^tR^{-1}_{m,A,-e}\gamma^{-1}, M_{A,-e}).$$

Hence the left-hand side of (a) equals

(b) $$\sum_i(-1)^{i+e}|\Gamma|^{-1}\sum_{\gamma\in\Gamma}\mathrm{tr}({}^tR_{m,A',i}\gamma, M_{A',i})\mathrm{tr}({}^tR^{-1}_{m,A,-e}\gamma^{-1}, M_{A,-e}),$$

that is,

$$\sum_i(-1)^{i+e}|\Gamma|^{-1}\mathrm{tr}(({}^tR_{m,A',i}\otimes{}^tR_{m,A,-e})\sum_{\gamma\in\Gamma}(\gamma\otimes\gamma^{-1}), M_{A',i}\otimes M_{A,-e}).$$

Assume first that $A' \neq A$. To show that (b) is zero it is enough to show that for any i, $\sum_{\gamma\in\Gamma}(\gamma\otimes\gamma^{-1})$ acts as 0 on $M_{A',i}\otimes M_{A,-e}$. This follows from the fact that the Γ-invariant part of the Γ-module $M_{A',i}\otimes M^*_{A,-e}$ is zero (see 33.4(c)).

Assume next that $A' = A$. Then we have $M_{A',i} = 0$ unless $i = -e$. We must show that

$$|\Gamma|^{-1}\sum_{\gamma\in\Gamma}\mathrm{tr}({}^tR_{m,A,-e}\gamma, M_{A,-e})\mathrm{tr}({}^tR^{-1}_{m,A,-e}\gamma^{-1}, M_{A,-e}) = 1.$$

Since $M_{A,-e}$ is an irreducible Γ-module, ${}^tR_{m,A,-e}$ acts as on it as a scalar, hence the desired equality follows from the Schur orthogonality relations for irreducible characters of Γ.

34. The structure of H_n^D

34.1. We give (a variant of) some definitions in [L13, 1]. Let $\mathcal{R}$ be a commutative ring with 1. Let $\mathfrak{A}$ be an associative $\mathcal{R}$-algebra with 1 with a given finite basis B as an $\mathcal{R}$-module. We assume that 1 is compatible with B in the following sense: $1 = \sum_\lambda 1_\lambda$ where $1_\lambda \in B$ are distinct, $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'}1_\lambda$ and any $b \in B$ satisfies $1_\lambda b 1_{\lambda'} = b$ for some (uniquely determined) λ, λ'. For $b, b' \in B$ we have $bb' = \sum_{b''\in B} r^{b''}_{b,b'}b''$ where $r^{b''}_{b,b'} \in \mathcal{R}$. We say that $b' \preceq b$ if $b' \in \cap_{K\in\mathcal{F};b\in K}K$ where $\mathcal{F}$ is the collection of all subsets K of B such that $\sum_{b_1\in K}\mathcal{R}b_1$ is a two-sided ideal of $\mathfrak{A}$; we say that $b \sim b'$ if $b' \preceq b$ and $b \preceq b'$. This is an equivalence relation on B and the equivalence classes are the *two-sided cells*. (Replacing two-sided ideals by left ideals in the definition of $\preceq$ and of two-sided cells we obtain the notion of left cells. The left cells form a partition of B finer than that given by two-sided cells.) We say that $b' \prec b$ if $b' \preceq b$ and $b' \not\sim b$. For any $b \in B$ let $\mathfrak{A}_{\prec b} = \oplus_{b' \in B; b' \prec b} \mathcal{R} b$.

Assume now that $\mathcal{R} = \mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $b \in B$. We can find an integer $m \geq 0$ such that $v^{-m} r_{b,b'}^{b''} \in \mathbf{Z}[v^{-1}]$ for any b', b'' in the two-sided cell of b. The smallest such m is denoted by $a(b)$. We say that B satisfies P_1 if $a(b) = a(b')$ whenever b, b' are in the same two-sided cell. Assume that B satisfies P_1. For $b \in B$ we set $\hat{b} = v^{-a(b)} b \in \mathfrak{A}$. Let $\mathfrak{A}^- = \sum_{b \in B} \mathbf{Z}[v^{-1}] \hat{b} \subset \mathfrak{A}$. Then $\mathfrak{A}^-$ is an associative $\mathbf{Z}[v^{-1}]$-algebra for the multiplication $\hat{b} * \hat{b}' = \sum_{b'' \in B; b'' \sim b} v^{-a(b)} r_{b,b'}^{b''} \hat{b}''$ if $b \sim b'$, $\hat{b} * \hat{b}' = 0$ if $b \not\sim b'$. Let $\mathfrak{A}^\infty = \mathfrak{A}^- / v^{-1} \mathfrak{A}^-$ and let $t_b = \hat{b} + v^{-1} \mathfrak{A}^- \in \mathfrak{A}^\infty$. Then $\mathfrak{A}^\infty$ is a ring with $\mathbf{Z}$-basis $\{t_b; b \in B\}$ and with multiplication defined by $t_b t_{b'} = \sum_{b'' \in B} \gamma_{b,b'}^{b''} t_{b''}$ where $\gamma_{b,b'}^{b''} \in \mathbf{Z}$ is given by $v^{-a(b)} r_{b,b'}^{b''} = \gamma_{b,b'}^{b''} \mod v^{-1} \mathbf{Z}[v^{-1}]$ if b, b', b'' are in the same two-sided cell and $\gamma_{b,b'}^{b''} = 0$, otherwise. We say that B satisfies P_2 if $\mathfrak{A}^\infty$ has a unit element compatible with the basis $\{t_b; b \in B\}$. We say that B satisfies P_3 if for any $b_1, b_2, b_3, b_4 \in B$ such that $b_2 \sim b_4$ we have

$$\sum_{\beta \in B; \beta \sim b_2} r_{b_1,b_2}^{\beta}(v) r_{\beta,b_3}^{b_4}(v') = \sum_{\beta \in B; \beta \sim b_2} r_{b_1,\beta}^{\beta_4}(v) r_{\beta_2,b_3}^{\beta}(v')$$

where v' is an indeterminate independent of v. In this case, assuming also that $b_2 \sim b_3 \sim b_4$, we pick the coefficient of $v'^{a(b_2)} = v'^{a(b_4)}$ in both sides and we obtain

(a) $$\sum_{\beta \in B; \beta \sim b_2} r_{b_1,b_2}^{\beta} \gamma_{\beta,b_3}^{b_4} = \sum_{\beta \in B; \beta \sim b_2} r_{b_1,\beta}^{\beta_4} \gamma_{\beta_2,b_3}^{\beta}.$$

Assume that B satisfies P_1, P_2, P_3. The unit element of $\mathfrak{A}^\infty$ is of the form $\sum_{b \in \mathcal{D}} t_b$ where $\mathcal{D} \subset B$. We say that $\mathcal{D}$ is the set of *distinguished* elements of B.

Let $\mathfrak{A}_{\mathcal{A}}^\infty = \mathcal{A} \otimes \mathfrak{A}^\infty$. We define an $\mathcal{A}$-linear map $\Phi : \mathfrak{A} \to \mathfrak{A}_{\mathcal{A}}^\infty$ by

$$\Phi(b) = \sum_{b_1 \in \mathcal{D}, b_2 \in B; b_1 \sim b_2} r_{b,b_1}^{b_2} t_{b_2}$$

for $b \in B$. Then Φ is an $\mathcal{A}$-algebra homomorphism taking 1 to 1. If we identify $\mathfrak{A}, \mathfrak{A}_{\mathcal{A}}^\infty$ as $\mathcal{A}$-modules via $b \leftrightarrow t_b$, the obvious left $\mathfrak{A}_{\mathcal{A}}^\infty$-module structure on $\mathfrak{A}_{\mathcal{A}}^\infty$ becomes the left $\mathfrak{A}_{\mathcal{A}}^\infty$-module structure on $\mathfrak{A}$ given by $t_b * b' = \sum_{b'' \in B} \gamma_{b,b'}^{b''} b''$. For $x \in \mathfrak{A}, b \in B$ we have

(b) $xb = \Phi(x) * b \mod \mathfrak{A}_{\prec b}$.

Indeed, we may assume that $x \in B$. Using (a), we have

$$\begin{aligned}
\Phi(x) * b &= \sum_{b_1 \in \mathcal{D}, b_2 \in B; b_1 \sim b_2} r_{x,b_1}^{b_2} t_{b_2} * b = \sum_{b_1 \in \mathcal{D}, b_2, b'' \in B; b_1 \sim b_2} r_{x,b_1}^{b_2} \gamma_{b_2,b}^{b''} b'' \\
&= \sum_{b_1 \in \mathcal{D}, b_2, b'' \in B; b_1 \sim b \sim b''} r_{x,b_1}^{b_2} \gamma_{b_2,b}^{b''} b'' = \sum_{b_1 \in \mathcal{D}, b_1', b'' \in B; b_1 \sim b \sim b''} r_{x,b_1'}^{b''} \gamma_{b_1,b}^{b_1'} b'' \\
&= \sum_{b_1 \in \mathcal{D}, b'' \in B; b_1 \sim b \sim b''} r_{x,b}^{b''} \gamma_{b_1,b}^{b} b'' = \sum_{b'' \in B; b \sim b''} r_{x,b}^{b''} b'' = xb \mod \mathfrak{A}_{\prec b},
\end{aligned}$$

as required.

Let K be a field and let $\mathcal{A} \to K$ be a homomorphism of rings with 1. Let $\mathfrak{A}_K = K \otimes_{\mathcal{A}} \mathfrak{A}, \mathfrak{A}_K^\infty = K \otimes_{\mathbf{Z}} \mathfrak{A}^\infty, \mathfrak{A}_{K,\prec b} = K \otimes_{\mathcal{A}} \mathfrak{A}_{\prec b}$ ($b \in B$). Then Φ induces a K-algebra homomorphism $\Phi_K : \mathfrak{A}_K \to \mathfrak{A}_K^\infty$. We show:

(c) *If $\mathfrak{A}_K$ is a semisimple algebra, then Φ_K is an (algebra) isomorphism.*

Since $\mathfrak{A}_K, \mathfrak{A}_K^\infty$ have the same (finite) dimension, it suffices to show that Φ_K is injective. The $\mathfrak{A}^\infty_{\mathcal{A}}$-module structure on $\mathfrak{A}$ extends to an $\mathfrak{A}^\infty_K$-module structure on $\mathfrak{A}_K$ denoted again by $*$. From (b) we deduce that $xb = \Phi_K(x) * b \mod \mathfrak{A}_{K,\prec b}$ for any $x \in \mathfrak{A}_K, b \in B$. In particular, if $x \in \mathrm{Ker}\Phi_K, b \in B$, then $xb \in \mathfrak{A}_{K,\prec b}$. Applying this repeatedly, we see that for any $m \geq 1$, any $x_1, x_2, \dots, x_m$ in $\mathrm{Ker}\Phi_K$ and any $b \in B$, $x_1x_2\dots x_m b$ is a K-linear combination of elements $b' \in B$ such that $b' = b_m \prec b_{m-1} \prec \dots \prec b_0 = b$ (with $b_i \in B$). If m is large enough, no such b' exists. Thus for large enough m we have $x_1x_2\dots x_m b = 0$ for all $b \in B$, hence $x_1x_2\dots x_m = 0$. We see that $\mathrm{Ker}\Phi_K$ is a nilpotent two-sided ideal of $\mathfrak{A}_K$. Hence it is 0. Thus Φ_K is injective and (c) is proved.

34.2. Let D be a connected component of G^0. Let $\mathbf{W}^D$ be the subgroup of $\mathbf{W}^\bullet \subset \mathrm{Aut}(\mathbf{T})$ generated by $\mathbf{W}$ and by $\underline{D}$. Now $\mathbf{W}$ is a normal subgroup of $\mathbf{W}^D$ and $\mathbf{W}^D/\mathbf{W}$ is a finite cyclic group.

We fix $n \in \mathbf{N}^*_{\mathbf{k}}$. Let $\lambda \in \underline{\mathfrak{s}}_n$. We write R_λ instead of $R_{\mathcal{L}}$ (see 28.3) where λ is the isomorphism class of $\mathcal{L} \in \mathfrak{s}_n$. Then R_λ is a root system and $R^+_\lambda = R_\lambda \cap R^+$ is a set of positive roots for R_λ. Let Π_λ be the unique set of simple roots for R_λ such that $\Pi_\lambda \subset R^+_\lambda$. Recall that $\mathbf{W}_\lambda$, the subgroup of $\mathbf{W}$ generated by $\{s_\alpha; \alpha \in R_\lambda\}$ is the Weyl group of the root system R_λ. Let $\mathbf{I}_\lambda = \{s_\alpha; \alpha \in \Pi_\lambda\} \subset \mathbf{W}_\lambda$. Then $(\mathbf{W}_\lambda, \mathbf{I}_\lambda)$ is a Coxeter group. Let $\mathbf{W}^D_\lambda = \{w \in \mathbf{W}^D; w\lambda = \lambda\}$. Let $\Omega^D_\lambda = \{w \in \mathbf{W}^D_\lambda; w(R^+_\lambda) = R^+_\lambda\}$. (Here $\mathbf{W}^D$ acts on R by $w : \alpha \mapsto w\alpha, (w\alpha)(t) = \alpha(w^{-1}t)$ for $t \in \mathbf{T}$.) Then $\mathbf{W}_\lambda$ is a normal subgroup of $\mathbf{W}^D_\lambda$, Ω^D_λ is a subgroup of $\mathbf{W}^D_\lambda$ and $\mathbf{W}^D_\lambda$ is the semidirect product of $\mathbf{W}_\lambda$ and Ω^D_λ. Define $l : \mathbf{W}^D \to \mathbf{N}$ by $l(w) = |\{\alpha \in R^+; w(\alpha) \in R^-\}|$. This extends the length function $\mathbf{W} \to \mathbf{N}$. Define $l_\lambda : \mathbf{W}^D \to \mathbf{N}$ by $l_\lambda(w) = |\{\alpha \in R^+_\lambda; w(\alpha) \in R^-\}|$. Then $\Omega^D_\lambda = \{w \in \mathbf{W}^D_\lambda; l_\lambda(w) = 0\}$, $\mathbf{I}_\lambda = \{w \in \mathbf{W}_\lambda; l_\lambda(w) = 1\}$. The standard partial order $\leq_\lambda$ of the Coxeter group $\mathbf{W}_\lambda$ is extended to a partial order $\leq_\lambda$ on $\mathbf{W}^D_\lambda$ as follows: if $w_1, w'_1 \in \Omega^D_\lambda, w_2, w'_2 \in \mathbf{W}_\lambda$, we say that $w_1w_2 \leq_\lambda w'_1w'_2$ if $w_1 = w'_1$ and $w_2 \leq_\lambda w'_2$.

Let H^D_λ be the $\mathcal{A}$-algebra defined by the generators $\tilde{T}^\lambda_w (w \in \mathbf{W}^D_\lambda)$ and relations

(a) $\tilde{T}^\lambda_w \tilde{T}^\lambda_{w'} = \tilde{T}^\lambda_{ww'}$ if $w, w' \in \mathbf{W}^D_\lambda$, $l_\lambda(ww') = l_\lambda(w) + l_\lambda(w')$,

(b) $(\tilde{T}^\lambda_\sigma)^2 = \tilde{T}^\lambda_1 + (v - v^{-1})\tilde{T}^\lambda_\sigma$ for $\sigma \in \mathbf{I}_\lambda$.

Then $\{\tilde{T}^\lambda_w; w \in \mathbf{W}^D_\lambda\}$ is an $\mathcal{A}$-basis of H^D_λ. Let H_λ be the $\mathcal{A}$-submodule of H^D_λ with $\mathcal{A}$-basis $\{\tilde{T}^\lambda_w; w \in \mathbf{W}_\lambda\}$. This is an $\mathcal{A}$-subalgebra of H^D_λ. Let $^- : H^D_\lambda \to H^D_\lambda$ be the unique ring homomorphism such that $\overline{v^m \tilde{T}^\lambda_w} = v^{-m}(\tilde{T}^\lambda_{w^{-1}})^{-1}$ for all $w \in \mathbf{W}^D_\lambda$, $m \in \mathbf{Z}$. From the definitions, for any $w \in \mathbf{W}^D_\lambda$ we have $\overline{\tilde{T}^\lambda_w} - \tilde{T}^\lambda_w \in \sum_{y \in \mathbf{W}^D_\lambda; y \leq_\lambda w, y \neq w} \mathcal{A}\tilde{T}^\lambda_y$. By an argument similar to one in [L12, 5.2] we see that for any $w \in \mathbf{W}^D_\lambda$ there is a unique element $c^\lambda_w \in H^D_\lambda$ such that $\overline{c^\lambda_w} = c^\lambda_w$ and $c^\lambda_w - \tilde{T}^\lambda_w \in \sum_{y \in \mathbf{W}^D_\lambda; y \neq w} v^{-1}\mathbf{Z}[v^{-1}]\tilde{T}^\lambda_y$. Also, $\{c^\lambda_w; w \in \mathbf{W}^D_\lambda\}$ is an $\mathcal{A}$-basis of H^D_λ and $\{c^\lambda_w; w \in \mathbf{W}_\lambda\}$ is an $\mathcal{A}$-basis of H_λ (as in [KL]).

Lemma 34.3. *The $\mathcal{A}$-algebra H^D_λ with its basis $(c^\lambda_w)_{w \in \mathbf{W}^D_\lambda}$ satisfies P_1, P_2, P_3 in 34.1.*

The analogous statement where $H^D_\lambda, \mathbf{W}^D_\lambda$ are replaced by $H_\lambda, \mathbf{W}_\lambda$ holds by [L12, §15]. The proof of the lemma is entirely similar; alternatively, it can be reduced to the case of H_λ using the identities

(a) $c^\lambda_{w_1w_2} = \tilde{T}^\lambda_{w_1} c^\lambda_{w_2}$, $c^\lambda_{w_2w_1} = c^\lambda_{w_2}\tilde{T}^\lambda_{w_1}$ for $w_1 \in \Omega^D_\lambda, w_2 \in \mathbf{W}_\lambda$,

(b) $\tilde{T}^\lambda_{w_1}\tilde{T}^\lambda_{w'_1} = \tilde{T}^\lambda_{w_1w'_1}$ for $w_1 \in \Omega^D_\lambda, w'_1 \in \Omega^D_\lambda$.

The function $a:\{c^\lambda_w; w\in\mathbf{W}^D_\lambda\}\to\mathbf{N}$ (see 34.1) is determined by the analogous function $a:\{c^\lambda_w; w\in\mathbf{W}_\lambda\}\to\mathbf{N}$ (defined in terms of H_λ) by $a(c^\lambda_{w_1w_2})=a(c^\lambda_{w_2w_1})=a(c^\lambda_{w_2})$ for $w_1\in\Omega^D_\lambda, w_2\in\mathbf{W}_\lambda$. The two-sided cells of $\{c^\lambda_w; w\in\mathbf{W}^D_\lambda\}$ are the sets of the form $\tilde{T}^\lambda_{w_1}\mathbf{c}\tilde{T}^\lambda_{w'_1}$ where w_1,w'_1 run through Ω^D_λ and $\mathbf{c}$ is a two-sided cell of $\{c^\lambda_w; w\in\mathbf{W}_\lambda\}$. We show:

(c) *If $c^\lambda_w(w\in\mathbf{W}^D_\lambda)$ is a distinguished basis element of H^D_λ (see 34.1), then $w\in\mathbf{W}_\lambda$ and $w^2=1$.*

By [L12] any left cell of $\mathbf{W}_\lambda$ contains a unique distinguished basis element. By the same argument, any left cell of $\mathbf{W}^D_\lambda$ contains a unique distinguished basis element. Let Γ be the left cell of $\mathbf{W}^D_\lambda$ that contains c^λ_w. (See 3.1.) Write $w=w_1w_2$ with $w_1\in\Omega^D_\lambda, w_2\in\mathbf{W}_\lambda$. From (a) we have $c^\lambda_w=\tilde{T}^\lambda_{w_1}c^\lambda_{w_2}, c^\lambda_{w_2}=\tilde{T}^\lambda_{w_1^{-1}}c_{w^\lambda}$. Hence $c^\lambda_{w_2}\in\Gamma$. We see that, if Γ' is the left cell of $\mathbf{W}_\lambda$ that contains $c^\lambda_{w_2}$, then $\Gamma'\subset\Gamma$. Let $c^\lambda_{w_3}$ be the unique distinguished basis element of $\mathbf{W}_\lambda$ that is contained in Γ'. Then $c^\lambda_{w_3}$ is also a distinguished basis element of $\mathbf{W}^D_\lambda$ contained in Γ hence, by uniqueness, we have $c^\lambda_{w_3}=c^\lambda_w$. We see that $w=w_3\in\mathbf{W}_\lambda$. The fact that $w^2=1$ also follows from [L12].

34.4. Let H^D_n be the $\mathcal{A}$-algebra with 1 defined by the generators $\tilde{T}_w(w\in\mathbf{W}^D)$, $1_\lambda(\lambda\in\underline{\mathfrak{s}}_n)$ and the relations

$1_\lambda 1_\lambda=1_\lambda$ for $\lambda\in\underline{\mathfrak{s}}_n$, $1_\lambda 1_{\lambda'}=0$ for $\lambda\ne\lambda'$ in $\underline{\mathfrak{s}}_n$,
$\tilde{T}_w\tilde{T}_{w'}=\tilde{T}_{ww'}$ for $w,w'\in\mathbf{W}^D$ with $l(ww')=l(w)+l(w')$,
$\tilde{T}_w 1_\lambda=1_{w\lambda}\tilde{T}_w$ for $w\in\mathbf{W}^D,\lambda\in\underline{\mathfrak{s}}_n$,
$\tilde{T}_s^2=\tilde{T}_1+(v-v^{-1})\sum_{\lambda;s\in\mathbf{W}_\lambda}\tilde{T}_s 1_\lambda$ for $s\in\mathbf{I}$,
$\tilde{T}_1=\sum_\lambda 1_\lambda$.

We identify H_n (see 31.2) with the subalgebra of H^D_n generated by $\tilde{T}_w(w\in\mathbf{W})$, $1_\lambda(\lambda\in\underline{\mathfrak{s}}_n)$ by $T_w\mapsto v^{l(w)}\tilde{T}_w(w\in\mathbf{W})$, $1_\lambda\mapsto 1_\lambda(\lambda\in\underline{\mathfrak{s}}_n)$. There is a unique ring homomorphism$^-$: $H^D_n\to H^D_n$ such that $\overline{\tilde{T}_w}=\tilde{T}^{-1}_{w^{-1}}$ for all $w\in\mathbf{W}^D$, $\overline{v^m 1_\lambda}=v^{-m}1_\lambda$ for all λ and all $m\in\mathbf{Z}$. It has square 1. Its restriction to H_n is the involution $^-: H_n\to H_n$ described in 31.3. From the definitions, for any $w\in\mathbf{W}^D$, $\lambda\in\underline{\mathfrak{s}}_n$ we have $\overline{\tilde{T}_w 1_\lambda}-\tilde{T}_w 1_\lambda\in\sum_{y\in\mathbf{W}^D;y\le w,y\ne w}\mathcal{A}\tilde{T}_y 1_\lambda$. By an argument similar to one in [L12, 5.2] we see that for any $w\in\mathbf{W}^D$, $\lambda\in\underline{\mathfrak{s}}_n$, there is a unique element $c_{w,\lambda}\in H^D_n$ such that $\overline{c_{w,\lambda}}=c_{w,\lambda}$ and $c_{w,\lambda}-\tilde{T}_w 1_\lambda\in\sum_{y\in\mathbf{W}^D;y\ne w}v^{-1}\mathbf{Z}[v^{-1}]\tilde{T}_y 1_\lambda$. We have $c_{w,\lambda}\in 1_{w\lambda}H^D_n 1_\lambda$. Also, $\{c_{w,\lambda}; w\in\mathbf{W}^D,\lambda\in\underline{\mathfrak{s}}_n\}$ is an $\mathcal{A}$-basis of H^D_n.

Proposition 34.5. *The $\mathcal{A}$-algebra H^D_n with its basis $(c_{w,\lambda})_{(w,\lambda)\in\mathbf{W}^D\times\underline{\mathfrak{s}}_n}$ satisfies P_1,P_2,P_3 in 34.1.*

The proof is given in 34.10.

34.6. In the setup of 34.2, the $\mathcal{A}$-algebra $1_\lambda H^D_n 1_\lambda$ (a subalgebra of H^D_n) has a unit element 1_λ, an $\mathcal{A}$-basis $\{\tilde{T}_w 1_\lambda; w\in\mathbf{W}^D_\lambda\}$ and an $\mathcal{A}$-basis $\{c_{w,\lambda}; w\in\mathbf{W}^D_\lambda\}$.

Lemma 34.7. *The $\mathcal{A}$-algebra $1_\lambda H^D_n 1_\lambda$ with its basis $\{c_{w,\lambda}; w\in\mathbf{W}^D_\lambda\}$ satisfies P_1,P_2,P_3 in 34.1.*

Define $\vartheta_\lambda: H^D_\lambda\to 1_\lambda H^D_n 1_\lambda$ by $\tilde{T}^\lambda_w\mapsto\tilde{T}_w 1_\lambda$ (an isomorphism of $\mathcal{A}$-modules). Using Lemma 34.3, we see that it suffices to show that ϑ_λ is an isomorphism of $\mathcal{A}$-algebras carrying c^λ_w to $c_{w,\lambda}$ for any $w\in\mathbf{W}^D_\lambda$. We use the notation in 34.2. We show:

(a) *Let* $w \in \mathbf{W}^D, \alpha \in \Pi_\lambda, \sigma = s_\alpha \in \mathbf{I}_\lambda$. *Then* $\tilde{T}_w \tilde{T}_\sigma 1_\lambda = \tilde{T}_{w\sigma} 1_\lambda + \delta(v - v^{-1}) \tilde{T}_w 1_\lambda$ *(in* H_n^D*) with* $\delta \in \{0, 1\}$. *If, in addition,* $w \in \mathbf{W}_\lambda^D$*, then* $\delta = 0$ *if* $l_\lambda(w\sigma) > l_\lambda(w)$ *and* $\delta = 1$*, otherwise.*

The proof has some common features with one in [MS, 3.3.5]. We have $\sigma = s_1 s_2 \dots s_r$ with $s_i \in \mathbf{I}, r = l(\sigma)$. By [L3, I, 5.3], there exists $j \in [1, r]$ such that $s_r \dots s_{j+1} s_j s_{j+1} \dots s_r \in \mathbf{W}_\lambda$ and $s_r \dots s_{i+1} s_i s_{i+1} \dots s_r \notin \mathbf{W}_\lambda$ for $i \in [1, r] - \{j\}$; by [L3, I, 5.6], we have $\sigma = s_r \dots s_{j+1} s_j s_{j+1} \dots s_r$. Hence $s_1 s_2 \dots s_{j-1} s_{j+1} \dots s_r = 1$. From the relations of H_n^D we have

$$\tilde{T}_{ws_1 s_2 \dots s_{j-1}} \tilde{T}_{s_j} 1_{s_{j+1} \dots s_r \lambda} = \tilde{T}_{ws_1 s_2 \dots s_j} 1_{s_{j+1} \dots s_r \lambda} + \delta'(v - v^{-1}) \tilde{T}_{ws_1 s_2 \dots s_{j-1}} 1_{s_{j+1} \dots s_r \lambda}$$

where $\delta' = 0$ if $l(ws_1 s_2 \dots s_j) > l(ws_1 s_2 \dots s_{j-1})$ and $\delta' = 1$ otherwise,

$$\tilde{T}_{ws_1 s_2 \dots s_{i-1}} \tilde{T}_{s_i} 1_{s_{i+1} \dots s_r \lambda} = \tilde{T}_{ws_1 s_2 \dots s_i} 1_{s_{i+1} \dots s_r \lambda}$$

for $i \in [1, r] - \{j\}$,

$$\tilde{T}_{ws_1 s_2 \dots s_{j-1} s_{j+1} \dots s_{i-1}} \tilde{T}_{s_i} 1_{s_{i+1} \dots s_r \lambda} = \tilde{T}_{ws_1 s_2 \dots s_{j-1} s_{j+1} \dots s_i} 1_{s_{i+1} \dots s_r \lambda}$$

for $i \in [j + 1, r]$. From these identities we see that

$$\begin{aligned}
&\tilde{T}_w \tilde{T}_\sigma 1_\lambda = \tilde{T}_w \tilde{T}_{s_1} \tilde{T}_{s_2} \dots \tilde{T}_{s_r} 1_\lambda = \tilde{T}_{ws_1 s_2 \dots s_{j-1}} \tilde{T}_{s_j} \tilde{T}_{s_{j+1}} \dots \tilde{T}_{s_r} 1_\lambda \\
&= \tilde{T}_{ws_1 s_2 \dots s_j} \tilde{T}_{s_{j+1}} \dots \tilde{T}_{s_r} 1_\lambda + \delta'(v - v^{-1}) \tilde{T}_{ws_1 s_2 \dots s_{j-1}} \tilde{T}_{s_{j+1}} \dots \tilde{T}_{s_r} 1_\lambda \\
&= \tilde{T}_{ws_1 s_2 \dots s_j s_{j+1} \dots s_r} 1_\lambda + \delta'(v - v^{-1}) \tilde{T}_{ws_1 s_2 \dots s_{j-1} s_{j+1} \dots s_r} 1_\lambda \\
&= \tilde{T}_{w\sigma} 1_\lambda + \delta'(v - v^{-1}) \tilde{T}_w 1_\lambda.
\end{aligned}$$

Assume now that $w \in \mathbf{W}_\lambda^D$. We show that $\delta = \delta'$. The condition that $\delta = 0$ is equivalent to the condition that $w(\alpha) \in \mathbf{R}_\lambda^+$. The condition that $\delta' = 0$ is equivalent to the condition that $ws_1 s_2 \dots s_{j-1}(\alpha_j) \in R^+$ where $\alpha_j \in R^+$ is defined by $s_j = s_{\alpha_j}$. Since $\alpha = s_1 s_2 \dots s_{j-1}(\alpha_j)$, this completes the proof of (a).

We show:

(b) *Let* $w \in \mathbf{W}_\lambda^D, w' \in \Omega_\lambda^D$. *Then* $\tilde{T}_w \tilde{T}_{w'} 1_\lambda = \tilde{T}_{ww'} 1_\lambda \in H_n^D$.

We write $w' = s_1 s_2 \dots s_r$ with $s_i \in \mathbf{I}, r = l(w')$. Using [L3, I, 5.3], we see that $s_r \dots s_{i+1} s_i s_{i+1} \dots s_r \notin \mathbf{W}_\lambda$ for all $i \in [1, r]$. From the relations of H_n^D we have

$$\tilde{T}_{ws_1 s_2 \dots s_{i-1}} \tilde{T}_{s_i} 1_{s_{i+1} \dots s_r \lambda} = \tilde{T}_{ws_1 s_2 \dots s_i} 1_{s_{i+1} \dots s_r \lambda}$$

for $i \in [1, r]$. Using these identities we see that $\tilde{T}_w \tilde{T}_{w'} 1_\lambda = \tilde{T}_w \tilde{T}_{s_1} \tilde{T}_{s_2} \dots \tilde{T}_{s_r} 1_\lambda = \tilde{T}_{ws_1 s_2 \dots s_r} 1_\lambda$ and (b) follows.

We show that ϑ^λ is an algebra homomorphism. We must check that $(\tilde{T}_\sigma 1_\lambda)^2 = 1_\lambda + (v - v^{-1}) \tilde{T}_\sigma 1_\lambda$ for $s \in \mathbf{I}_\lambda$. This is a special case of (a) (take $w = \sigma$). We must also check that $(\tilde{T}_w 1_\lambda)(\tilde{T}_{w'} 1_\lambda) = \tilde{T}_{ww'} 1_\lambda$ if $w, w' \in \mathbf{W}_\lambda^D$, $l_\lambda(ww') = l_\lambda(w) + l_\lambda(w')$. If $w, w' \in \mathbf{W}_\lambda$, this is proved by induction on $l_\lambda(w')$, the induction step being provided by (a). The general case can be reduced to this special case using (b). We see that ϑ_λ is an $\mathcal{A}$-algebra isomorphism. We show that

$$\overline{\vartheta_\lambda(h)} = \vartheta_\lambda(\overline{h}) \text{ for } h \in H_\lambda^D.$$

Assume first that $h = \tilde{T}_w^\lambda$ where $w \in \Omega_\lambda^D$. Then $\overline{h} = (\tilde{T}_{w^{-1}}^\lambda)^{-1} = \tilde{T}_w^\lambda$. Hence

$$\overline{\vartheta_\lambda(h)} = \overline{\tilde{T}_w 1_\lambda} = \tilde{T}_{w^{-1}}^{-1} 1_\lambda = \tilde{T}_w^{-1} 1_\lambda = \vartheta_\lambda(\tilde{T}_w^l) = \vartheta_\lambda(\overline{h}),$$

as required. Assume next that $h = \tilde{T}^\lambda_\sigma$ where $\sigma \in \mathbf{I}_\lambda$. Then

$$\vartheta_\lambda(\overline{h}) = \vartheta_\lambda((\tilde{T}^\lambda_\sigma)^{-1}) = \vartheta_\lambda(\tilde{T}^\lambda_\sigma + (v^{-1} - v)\tilde{T}^\lambda_1)$$
$$= \tilde{T}_\sigma 1_\lambda + (v^{-1} - v)\tilde{T}_1 1_\lambda = \tilde{T}_\sigma^{-1} 1_\lambda = \overline{\tilde{T}_\sigma 1_\lambda} = \overline{\vartheta_\lambda(h)},$$

as required.

We see that for $w \in \mathbf{W}^D_\lambda$ we have $\overline{\vartheta_\lambda(c^\lambda_w)} = \vartheta_\lambda(c^\lambda_w)$. Hence $\vartheta_\lambda(c^\lambda_w)$ satisfies the defining properties of $c_{w,\lambda}$ so that $\vartheta_\lambda(c^\lambda_w) = c_{w,\lambda}$. The lemma is proved.

Using now Lemma 34.3(c) we see that

(c) *If $c_{w,\lambda} (w \in \mathbf{W}^D_\lambda)$ is a distinguished basis element of $1_\lambda H^D_n 1_\lambda$ (see 34.1), then $w \in \mathbf{W}_\lambda$ and $w^2 = 1$.*

34.8. Let $\underline{\mathfrak{s}}'_n$ be a set of representatives for the $\mathbf{W}^D$-orbits in $\underline{\mathfrak{s}}_n$. For $\lambda \in \underline{\mathfrak{s}}_n$ define $\lambda^0 \in \underline{\mathfrak{s}}'_n$ by $\lambda^0 \in \mathbf{W}^D\lambda$ (the $\mathbf{W}^D$-orbit of λ). Let

$$\Gamma = \{(\lambda_1, \lambda_2) \in \underline{\mathfrak{s}}_n \times \underline{\mathfrak{s}}_n; \mathbf{W}^D\lambda_1 = \mathbf{W}^D\lambda_2\}.$$

Let E^D_n be the set of all formal sums $x = \sum_{(\lambda_1,\lambda_2)\in\Gamma} x_{\lambda_1,\lambda_2}$ where $x_{\lambda_1,\lambda_2} \in 1_{\lambda_1^0} H^D_n 1_{\lambda_2^0}$. Then E^D_n is naturally an $\mathcal{A}$-module and an associative $\mathcal{A}$-algebra where the product xy of $x, y \in E^D_n$ is given by $(xy)_{\lambda_1,\lambda_2} = \sum_{\tilde{\lambda}\in\mathbf{W}^D\lambda_1} x_{\lambda_1,\tilde{\lambda}} y_{\tilde{\lambda},\lambda_2}$. This algebra has a unit element, namely the element 1 such that $1_{\lambda_1,\lambda_2} = \delta_{\lambda_1,\lambda_2} 1_{\lambda_1}$ for $(\lambda_1, \lambda_2) \in \Gamma$. Define a ring involution $^-: E^D_n \to E^D_n$ by $x \mapsto \overline{x}$ where $\overline{x}_{\lambda_1,\lambda_2} = \overline{x_{\lambda_1,\lambda_2}}$. (Note that $^-: H^D_n \to H^D_n$ maps $1_{\lambda_1^0} H^D_n 1_{\lambda_2^0}$ onto itself.)

Let $C = \{(\lambda_1, \lambda_2, w) \in \underline{\mathfrak{s}}_n \times \underline{\mathfrak{s}}_n \times \mathbf{W}^D; w\lambda^0_1 = \lambda^0_1 = \lambda^0_2\}$. For $(\lambda_1, \lambda_2, w) \in C$ define $x^{\lambda_1,\lambda_2,w} \in E^D_n$ by

$$x^{\lambda_1,\lambda_2,w}_{\lambda'_1,\lambda'_2} = \delta_{(\lambda'_1,\lambda'_2),(\lambda_1,\lambda_2)} \tilde{T}_w 1_{\lambda^0_1}.$$

Then $\{x^{\lambda_1,\lambda_2,w}; (\lambda_1, \lambda_2, w) \in C\}$ is an $\mathcal{A}$-basis of H''_n. From the definitions, for $(\lambda_1, \lambda_2, w) \in C$ we have

$$\overline{x^{\lambda_1,\lambda_2,w}} - x^{\lambda_1,\lambda_2,w} \in \sum_{y\in\mathbf{W}^D; y\le w, y\ne w, y\lambda^0_1=\lambda^0_1} \mathcal{A} x^{\lambda_1,\lambda_2,y}.$$

By an argument similar to one in [L12, 5.2] we see that for any $(\lambda_1, \lambda_2, w) \in C$ there is a unique element $c^{\lambda_1,\lambda_2,w} \in E^D_n$ such that $\overline{c^{\lambda_1,\lambda_2,w}} = c^{\lambda_1,\lambda_2,w}$ and

$$c^{\lambda_1,\lambda_2,w} - x^{\lambda_1,\lambda_2,w} \in \sum_{y\in\mathbf{W}^D, y\lambda^0_1=\lambda^0_1, y\ne w} v^{-1}\mathbf{Z}[v^{-1}] x^{\lambda_1,\lambda_2,y}.$$

Also, $\{c^{\lambda_1,\lambda_2,w}; (\lambda_1, \lambda_2, w) \in C\}$ is an $\mathcal{A}$-basis of E^D_n.

Lemma 34.9. *The $\mathcal{A}$-algebra E^D_n with its basis $\{c^{\lambda_1,\lambda_2,w}; (\lambda_1, \lambda_2, w) \in C\}$ satisfies P_1, P_2, P_3 in 34.1.*

For $\lambda \in \underline{\mathfrak{s}}'_n$ let $N_\lambda = |\mathbf{W}^D\lambda|$ and let $M_{N_\lambda}(1_\lambda H^D_n 1_\lambda)$ be the algebra of $N_\lambda \times N_\lambda$ matrices with entries in $1_\lambda H^D_n 1_\lambda$. From the definitions we have a decomposition

$$E^D_n = \oplus_{\lambda\in\underline{\mathfrak{s}}'_n} M_{N_\lambda}(1_\lambda H^D_n 1_\lambda)$$

compatible with the algebra structures and with the natural bases. Using this, the lemma is reduced to the similar statement for $1_\lambda H^D_n 1_\lambda$ where it is known by Lemma 34.7.

The function $a : \{c^{\lambda_1,\lambda_2,w}; (\lambda_1, \lambda_2, w) \in C\}$ (see 34.1) is given by $a(c^{\lambda_1,\lambda_2,w}) = a(c_{w,\lambda^0_1})$ where $a(c_{w,\lambda^0_1})$ is defined as in 34.1 in terms of $1_{\lambda^0_1} H^D_n 1_{\lambda^0_1}$. The two-sided cells of $\{c^{\lambda_1,\lambda_2,w}; (\lambda_1,\lambda_2,w)\in C\}$ are the sets of the form $\{c^{\lambda_1,\lambda_2,w}\}$ where λ_1,λ_2 run through $\mathbf{W}^D\lambda$ (with $\lambda\in\underline{\mathfrak{s}}'_n$ fixed) and w running through a subset X of $\mathbf{W}^D_\lambda$ such that $\{c_{w,\lambda}; w\in X\}$ is a two-sided cell of $\{c_{w,\lambda}; w\in\mathbf{W}^D_\lambda\}$ (see Lemma 34.7).

Using 34.7(c) we obtain:

(a) *If $c^{\lambda_1,\lambda_2,w}$, (where $(\lambda_1,\lambda_2,w)\in C$) is a distinguished basis element of E^D_n, then $\lambda_1=\lambda_2, w\in\mathbf{W}_{\lambda_1^0}, w^2=1$.*

34.10. We prove Proposition 34.5. It is enough to construct an algebra isomorphism $H^D_n\xrightarrow{\sim}E^D_n$ which carries the basis $(c_{w,\lambda})$ onto the basis $(c^{\lambda_1,\lambda_2,w})$.

For each $\lambda\in\underline{\mathfrak{s}}_n$ we choose a sequence $\mathbf{s}_\lambda=(s_1,s_2,\dots,s_r)$ where, for $i\in[1,r]$, s_i is either in $\mathbf{I}$ or is a power of $\underline{D}$ and $\lambda^0=s_1s_2\dots s_r\lambda\ne s_2\dots s_r\lambda\ne\dots\ne s_r\lambda\ne\lambda$ or, equivalently, $\lambda=s_r^{-1}\dots s_2^{-1}s_1^{-1}\lambda^0\ne s_{r-1}^{-1}\dots s_1^{-1}\lambda^0\ne\dots\ne s_1^{-1}\lambda^0\ne\lambda^0$. Let $[\mathbf{s}_\lambda]=s_1s_2\dots s_r$. We set

$$\tau_\lambda=\tilde T_{s_1}\tilde T_{s_2}\dots\tilde T_{s_r}\in H^D_n, \tau'_\lambda=\tilde T_{s_r^{-1}}\dots\tilde T_{s_2^{-1}}\tilde T_{s_1^{-1}}\in H^D_n.$$

We show:

(a) $1_{\lambda^0}\tau_\lambda\tau'_\lambda=1_{\lambda^0}, 1_\lambda\tau'_\lambda\tau_\lambda=1_\lambda$.

We have

$$1_{\lambda^0}\tau_\lambda\tau'_\lambda=1_{\lambda^0}\tilde T_{s_1}\tilde T_{s_2}\dots\tilde T_{s_r}\tilde T_{s_r^{-1}}\dots\tilde T_{s_2^{-1}}\tilde T_{s_1^{-1}}=\tilde T_{s_1}\tilde T_{s_2}\dots\tilde T_{s_r}1_\lambda\tilde T_{s_r^{-1}}\dots\tilde T_{s_2^{-1}}\tilde T_{s_1^{-1}}.$$

Since $s_r\lambda\ne\lambda$, we can replace $\tilde T_{s_r}1_\lambda\tilde T_{s_r^{-1}}$ by $1_{s_r\lambda}$ and we obtain

$$\tilde T_{s_1}\tilde T_{s_2}\dots\tilde T_{s_{r-1}}1_{s_r\lambda}\tilde T_{s_{r-1}^{-1}}\dots\tilde T_{s_2^{-1}}\tilde T_{s_1^{-1}}.$$

Since $s_{r-1}s_r\lambda\ne s_r\lambda$, we can replace $\tilde T_{s_{r-1}}1_{s_r\lambda}\tilde T_{s_{r-1}^{-1}}$ by $1_{s_{r-1}s_r\lambda}$ and we obtain

$$\tilde T_{s_1}\tilde T_{s_2}\dots\tilde T_{s_{r-2}}1_{s_{r-1}s_r\lambda}\tilde T_{s_{r-2}^{-1}}\dots\tilde T_{s_2^{-1}}\tilde T_{s_1^{-1}}.$$

Continuing in this way we find $1_{s_1\dots s_{r-1}s_r\lambda}=1_{\lambda^0}$. This proves the first identity in (a). The second identity is proved in a similar way.

We have

(b) $\overline{\tau_\lambda 1_\lambda}=\tau_\lambda 1_\lambda, \overline{1_\lambda\tau'_\lambda}=1_\lambda\tau'_\lambda$.

The first identity in (b) is equivalent to $\tilde T^{-1}_{s_1^{-1}}\tilde T^{-1}_{s_2^{-1}}\dots\tilde T^{-1}_{s_r^{-1}}1_\lambda=\tau_\lambda 1_\lambda$ or to ${\tau'_\lambda}^{-1}1_\lambda=\tau_\lambda 1_\lambda$, which follows from (a). Similarly, the second identity in (b) follows from (a).

We define an $\mathcal{A}$-linear map $\Psi: H^D_n\to E^D_n$ by

$$\Psi(h)_{\lambda_1,\lambda_2}=\tau_{\lambda_1}1_{\lambda_1}h1_{\lambda_2}\tau'_{\lambda_2}\in 1_{\lambda_1^0}H^D_n1_{\lambda_2^0}.$$

We show that Ψ is a ring homomorphism. Let $h,h'\in H^D_n$, $x=\Psi(h), y=\Psi(h')$, $z=\Psi(hh')$, $z'=\Psi(h)\Psi(h')$. We have

$$\begin{aligned}(\Psi(h)\Psi(h'))_{\lambda_1,\lambda_2}&=\sum_{\tilde\lambda\in\mathbf{W}^D\lambda_1}\Psi(h)_{\lambda_1,\tilde\lambda}\Psi(h')_{\tilde\lambda,\lambda_2}\\&=\sum_{\tilde\lambda\in\mathbf{W}^D\lambda_1}\tau_{\lambda_1}1_{\lambda_1}h1_{\tilde\lambda}\tau'_{\tilde\lambda}\tau_{\tilde\lambda}1_{\tilde\lambda}h'1_{\lambda_2}\tau'_{\lambda_2}=\sum_{\tilde\lambda\in\mathbf{W}^D\lambda_1}\tau_{\lambda_1}1_{\lambda_1}h1_{\tilde\lambda}h'1_{\lambda_2}\tau'_{\lambda_2}\end{aligned}$$

where the last equality comes from (a). Since $1_{\lambda_1}h1_{\tilde\lambda}=0$ if $\tilde\lambda\in\underline{\mathfrak{s}}_n-\mathbf{W}^D\lambda_1$, we see that

$$(\Psi(h)\Psi(h'))_{\lambda_1,\lambda_2}=\tau_{\lambda_1}1_{\lambda_1}h\sum_{\tilde\lambda\in\underline{\mathfrak{s}}_n}1_{\tilde\lambda}h'1_{\lambda_2}\tau'_{\lambda_2}=\tau_{\lambda_1}1_{\lambda_1}hh'1_{\lambda_2}\tau'_{\lambda_2}=\Psi(hh')_{\lambda_1,\lambda_2}.$$

Thus $\Psi(h)\Psi(h')=\Psi(hh')$, as required.

We show that

(c) $\overline{\Psi(h)} = \Psi(\overline{h})$ for $h \in H_n^D$.

We have

$$(\overline{\Psi(h)})_{\lambda_1,\lambda_2} = \overline{\tau_{\lambda_1} 1_{\lambda_1} h 1_{\lambda_2} \tau'_{\lambda_2}}, (\Psi(\overline{h}))_{\lambda_1,\lambda_2} = \tau_{\lambda_1} 1_{\lambda_1} \bar{h} 1_{\lambda_2} \tau'_{\lambda_2}.$$

It suffices to show that $\overline{\tau_{\lambda_1} 1_{\lambda_1}} = \tau_{\lambda_1} 1_{\lambda_1}$, $\overline{1_{\lambda_2} \tau'_{\lambda_2}} = 1_{\lambda_2} \tau'_{\lambda_2}$. This follows from (b).

We show that

(d) $\Psi(\tilde{T}_w 1_\lambda) = x^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}$ for $w \in \mathbf{W}^D, \lambda \in \underline{\mathfrak{s}}_n$.

Indeed, $(\Psi(\tilde{T}_w 1_\lambda))_{\lambda_1,\lambda_2} = \tau_{\lambda_1} 1_{\lambda_1} \tilde{T}_w 1_\lambda 1_{\lambda_2} \tau'_{\lambda_2}$. This is 0 unless $\lambda_2 = \lambda, \lambda_1 = w\lambda$. If $\lambda_2 = \lambda, \lambda_1 = w\lambda$, we see as in the proof of (a) that

$$\tau_{\lambda_1} 1_{\lambda_1} \tilde{T}_w 1_\lambda 1_{\lambda_2} \tau'_{\lambda_2} = \tau_{w\lambda} \tilde{T}_w 1_\lambda \tau'_\lambda = \tilde{T}_{[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}} 1_{\lambda^0}.$$

This proves (d).

(d) shows that Ψ is induced by a map

$$\Psi_0 : \mathbf{W}^D \times \underline{\mathfrak{s}}_n \to C,\ (w,\lambda) \mapsto (w\lambda, \lambda, [\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}).$$

This is a bijection with inverse $(\lambda_1, \lambda_2, w) \mapsto ([\mathbf{s}_{\lambda_1}]^{-1} w [\mathbf{s}_{\lambda_2}], \lambda_2)$. It follows that Ψ is an isomorphism.

Let $w \in \mathbf{W}^D, \lambda \in \underline{\mathfrak{s}}_n$. Since Ψ^{-1} is compatible with $\bar{}$, we have

$$\overline{\Psi^{-1}(c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}})} = \Psi^{-1}(c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}).$$

From (d) we see that

$$\Psi^{-1}(c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}) - \tilde{T}_w 1_\lambda \in \sum_{y \in \mathbf{W}^D, y\lambda = w\lambda; y \neq w} v^{-1}\mathbf{Z}[v^{-1}] \tilde{T}_y 1_\lambda.$$

Since these properties characterize $c_{w,\lambda}$, we see that $\Psi^{-1}(c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}) = c_{w,\lambda}$, that is, $\Psi(c_{w,\lambda}) = c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}$. Thus, Ψ restricts to a bijection between the basis $(c_{w,\lambda})$ of H_n^D and the basis $(c^{\lambda_1,\lambda_2,w})$ of E_n^D, induced by the bijection $\Psi_0 : \mathbf{W}^D \times \underline{\mathfrak{s}}_n \xrightarrow{\sim} C$. Proposition 34.5 is proved.

We show:

(e) *If $c_{w,\lambda}(w \in \mathbf{W}^D, \lambda \in \underline{\mathfrak{s}}_n)$ is a distinguished basis element of H_n^D, then $w \in \mathbf{W}_\lambda, w^2 = 1$.*

Note that, with the notation above, $c^{w\lambda,\lambda,[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1}}$ is a distinguished basis element of E_n^D; hence, by 34.9(a), we have $w\lambda = \lambda$ and $[\mathbf{s}_{w\lambda}]w[\mathbf{s}_\lambda]^{-1} \in \mathbf{W}_{\lambda^0}$ has square 1. Thus, $[\mathbf{s}_\lambda]w[\mathbf{s}_\lambda]^{-1} \in \mathbf{W}_{\lambda^0}$ has square 1. It follows that $w \in \mathbf{W}_{[\mathbf{s}_\lambda]^{-1}\lambda^0} = \mathbf{W}_\lambda$ has square 1. This proves (e).

34.11. The algebra $H_n^{G^0}$ defined as in 34.4 with D replaced by G^0 is the same as H_n; we identify it in an obvious way with a subalgebra of H_n^D (see 34.4). The $\mathcal{A}$-basis of $H_n = H_n^{G^0}$ analogous to the $\mathcal{A}$-basis $\{c_{w,\lambda}; w \in \mathbf{W}^D, \lambda \in \underline{\mathfrak{s}}_n\}$ of H_n^D is the subset of the last basis given by $\{c_{w,\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n\}$. The analogue of Proposition 34.5 holds: the $\mathcal{A}$-algebra H_n with its basis $(c_{w,\lambda})_{(w,\lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}_n}$ satisfies P_1, P_2, P_3 in 34.1.

34.12. Let K be a field of characteristic 0 and let $\mathcal{A} \to K$ be a homomorphism of rings with 1 which carries $v \in \mathcal{A}$ to $v_0 \in K^*$. We show:

(a) *If $\lambda \in \underline{\mathfrak{s}}_n$ and $\sum_{w \in \mathbf{W}_\lambda} v_0^{2l_\lambda(w)} \neq 0$, then the K-algebra $H_{\lambda,K}^D = K \otimes_{\mathcal{A}} H_\lambda^D$ is semisimple.*

Let M be an $H^D_{\lambda,K}$-module of finite dimension over K and let M' be an $H^D_{\lambda,K}$-submodule of M. It is enough to show that there exists an $H^D_{\lambda,K}$-submodule of M complementary to M'. It is well known that under our assumption, the K-algebra $H_{\lambda,K}=K\otimes_{\mathcal{A}}H_\lambda$ is semisimple. Hence there exists an $H_{\lambda,K}$-submodule of M complementary to M', that is, there exists an $H_{\lambda,K}$-linear map $f:M\to M'$ such that $f(x)=x$ for all $x\in M'$. Define $\tilde f:M\to M'$ by $\tilde f(x)=|\Omega^D_\lambda|^{-1}\sum_{z\in\Omega^D_\lambda}\tilde T_{z^{-1}}f(\tilde T_z x)$. For $w\in\mathbf{W}_\lambda, x\in M$ we have

$$|\Omega^D_\lambda|\tilde f(\tilde T_w x)=\sum_{z\in\Omega^D_\lambda}\tilde T_{z^{-1}}f(\tilde T_z\tilde T_w x)=\sum_{z\in\Omega^D_\lambda}\tilde T_{z^{-1}}f(\tilde T_{zwz^{-1}}\tilde T_z x)$$
$$=\sum_{z\in\Omega^D_\lambda}\tilde T_{z^{-1}}\tilde T_{zwz^{-1}}f(\tilde T_z x)=\sum_{z\in\Omega^D_\lambda}\tilde T_w\tilde T_{z^{-1}}f(\tilde T_z x)=|\Omega^D_\lambda|\tilde T_w\tilde f(x).$$

(The third equality holds since f is $H_{\lambda,K}$-linear.) We see that $\tilde f$ is $H_{\lambda,K}$-linear. Since $\tilde f(\tilde T_y x)=\tilde T_y\tilde f(x)$ for $x\in M$, $y\in\Omega^D_\lambda$, we see that $\tilde f$ is $H^D_{\lambda,K}$-linear. If $x\in M'$, we have $\tilde T_z x\in M'$ for $z\in\Omega^D_\lambda$; hence $\tilde f(x)=|\Omega^D_\lambda|^{-1}\sum_{z\in\Omega^D_\lambda}\tilde T_{z^{-1}}\tilde T_z x=x$. It follows that the $\ker(\tilde f)$ is an $H^D_{\lambda,K}$-submodule of M complementary to M'. This proves (a).

Let $\mathfrak{U}$ be the subfield of $\bar{\mathbf{Q}}_l$ generated by the roots of 1.
For any $\kappa\in\mathfrak{U}^*,\lambda\in\underline{\mathfrak{s}}_n$, let

$$H_n^{D,\kappa}=\mathfrak{U}\otimes_{\mathcal{A}}H_n^D,\ H_n^\kappa=\mathfrak{U}\otimes_{\mathcal{A}}H_n, {H_\lambda}^{D,\kappa}=\mathfrak{U}\otimes_{\mathcal{A}}H_\lambda^D,$$

where $\mathfrak{U}$ is regarded as an $\mathcal{A}$-algebra via the ring homomorphism $\mathcal{A}\to\mathfrak{U}, v\mapsto\kappa$. Let

$$H_n^{D,v}=\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n^D,\ H_n^v=\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n$$

where $\mathfrak{U}(v)$ (field of rational functions in v with coefficients in $\mathfrak{U}$) is regarded as an $\mathcal{A}$-algebra via the ring homomorphism $\mathcal{A}\to\mathfrak{U}(v), v\mapsto v$. Now $\Phi:H_n^D\to\mathcal{A}\otimes_{\mathbf{Z}}H_n^{D,\infty}$ (where $H_n^{D,\infty}=(H_n^D)^\infty$ is defined as in 34.1 in terms of the basis $(c_{w,\lambda})$ in 34.4) induces algebra homomorphisms $\Phi^\kappa:H_n^{D,\kappa}\to\mathfrak{U}\otimes_{\mathbf{Z}}H_n^{D,\infty}$ for $\kappa\in\mathfrak{U}^*$ and $\Phi^v:H_n^{D,v}\to\mathfrak{U}(v)\otimes_{\mathbf{Z}}H_n^{D,\infty}$.

Similarly, $\Phi:H_n\to\mathcal{A}\otimes_{\mathbf{Z}}H_n^\infty$ (defined as in 34.1 in terms of the basis $(c_{w,\lambda})$ in 34.11) induces algebra homomorphisms $H_n^\kappa\to\mathfrak{U}\otimes_{\mathbf{Z}}H_n^\infty$ for $\kappa\in\mathfrak{U}^*$ and $H_n^v\to\mathfrak{U}(v)\otimes_{\mathbf{Z}}H_n^\infty$, denoted again by Φ^κ,Φ^v. From the definitions we see that H_n^∞ may be identified with the subgroup of $(H_n^D)^\infty$ spanned by the basis elements of $(H_n^D)^\infty$ indexed by $\{(w,\lambda); w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n\}$ and $\Phi:H_n\to\mathcal{A}\otimes_{\mathbf{Z}}H_n^\infty$ becomes the restriction of $\Phi:H_n^D\to\mathcal{A}\otimes_{\mathbf{Z}}H_n^{D,\infty}$. We show:

(b) *If $\kappa\in\mathfrak{U}^*,\sum_{w\in\mathbf{W}}\kappa^{2l(w)}\ne 0$, then $H_n^{D,\kappa}$ is a semisimple $\mathfrak{U}$-algebra and $\Phi^\kappa:H_n^{D,\kappa}\to\mathfrak{U}\otimes_{\mathbf{Z}}H_n^{D,\infty}$ is an algebra isomorphism. Moreover, H_n^κ is a semisimple $\mathfrak{U}$-algebra and $\Phi^\kappa:H_n^\kappa\to\mathfrak{U}\otimes_{\mathbf{Z}}H_n^\infty$ is an algebra isomorphism.*

(c) *$H_n^{D,v}$ is a split semisimple $\mathfrak{U}(v)$-algebra and $\Phi^v:H_n^{D,v}\to\mathfrak{U}(v)\otimes_{\mathbf{Z}}H_n^{D,\infty}$ is an algebra isomorphism. Moreover, H_n^v is a split semisimple $\mathfrak{U}(v)$-algebra and $\Phi^v:H_n^v\to\mathfrak{U}(v)\otimes_{\mathbf{Z}}H_n^\infty$ is an algebra isomorphism.*

We prove (b). The following statement is easily verified:

If $\lambda\in\underline{\mathfrak{s}}_n$, then $\sum_{w\in\mathbf{W}}v^{2l(w)}=Q\sum_{w\in\mathbf{W}_\lambda}v^{2l_\lambda(w)}$ for some $Q\in\mathbf{Z}[v^2]$.

We see that, if κ is as in (b) and $\lambda\in\underline{\mathfrak{s}}_n$, then $\sum_{w\in\mathbf{W}_\lambda}\kappa^{2l_\lambda(w)}\ne 0$; hence, by (a), $H_\lambda^{D,\kappa}$ is a semisimple algebra. By the arguments in 34.7–34.10, $H_n^{D,\kappa}$ is a direct sum of matrix rings over rings of the form $H_\lambda^{D,\kappa}$. Hence $H_n^{D,\kappa}$ is a semisimple algebra.

Using 34.1(c) we see that $\Phi^\kappa : H_n^{D,\kappa} \to \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ is an algebra isomorphism. It remains to show that $H_n^{D,\kappa}$ is split over $\mathfrak{U}$. Since Φ^κ is an isomorphism, it is enough to show that $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ is split over $\mathfrak{U}$. Since Φ^1 is an isomorphism, it is enough to show that $H_n^{D,1}$ is split over $\mathfrak{U}$. As above, $H_n^{D,1}$ is a direct sum of matrix rings over rings of the form $H_\lambda^{D,1}$. Since $H_\lambda^{D,1}$ is the group algebra of a finite group with coefficients in $\mathfrak{U}$, it is split over $\mathfrak{U}$ by Brauer's theorem. This proves the first sentence in (b). The second sentence in (b) is obtained from the first by replacing D by G^0.

Now the proof of (c) is just like that of (b) except for the splitness assertion. By (b), $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ is a split semisimple $\mathfrak{U}$-algebra, hence $\mathfrak{U}(v) \otimes_{\mathbf{Z}} H_n^{D,\infty}$ is a split semisimple $\mathfrak{U}(v)$-algebra. Since Φ^v is an isomorphism, it follows that $H_n^{D,v}$ is split over $\mathfrak{U}(v)$. Similarly, H_n^v is split over $\mathfrak{U}(v)$. This proves (c).

34.13. Define an $\mathcal{A}$-linear map $\tau : H_n \to \mathcal{A}$ by $\tau(\tilde{T}_w 1_\lambda) = \delta_{w,1}$ for all $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$. Define a bilinear form $(,) : H_n \times H_n \to \mathcal{A}$ by $(x, x') = \tau(xx')$. We show that

(a) $(\tilde{T}_w 1_\lambda, \tilde{T}_{w'} 1_{\lambda'}) = \delta_{w^{-1},w'}\delta_{\lambda,w'\lambda'}$

for $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}_n$. (This shows that $(,)$ is symmetric; indeed, we have $\delta_{w^{-1},w'}\delta_{\lambda,w'\lambda'} = \delta_{w'^{-1},w}\delta_{\lambda',w\lambda}$.) To prove (a) it suffices to show that, for $w, w' \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$, we have $\tau(\tilde{T}_w \tilde{T}_{w'} 1_\lambda) = \delta_{ww',1}$. We argue by induction on $l(w)$. If $l(w) = 0$, then $\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'}$ and the result is clear. Assume now that $l(w) \geq 1$. We can find $s \in \mathbf{I}$ such that $l(w) = l(ws) + 1$. Then $\tau(\tilde{T}_w \tilde{T}_{w'} 1_\lambda) = \tau(\tilde{T}_{ws} \tilde{T}_s \tilde{T}_{w'} 1_\lambda)$. If $l(sw') = l(w') + 1$, then, by the induction hypothesis,

$$\tau(\tilde{T}_{ws} \tilde{T}_s \tilde{T}_{w'} 1_\lambda) = \tau(\tilde{T}_{ws} \tilde{T}_{sw'} 1_\lambda) = \delta_{wssw',1} = \delta_{ww',1},$$

as required. Assume now that $l(sw') = l(w') - 1$. We have

$$\begin{aligned}&\tau(\tilde{T}_{ws} \tilde{T}_s \tilde{T}_{w'} 1_\lambda) = \tau(\tilde{T}_{ws} \tilde{T}_s \tilde{T}_s \tilde{T}_{sw'} 1_\lambda)\\ &= \tau(\tilde{T}_{ws} \tilde{T}_{sw'} 1_\lambda) + (v - v^{-1}) \sum_{\lambda'; s \in \mathbf{W}_{\lambda'}} \tau(\tilde{T}_{ws} \tilde{T}_s 1_{\lambda'} \tilde{T}_{sw'} 1_\lambda).\end{aligned}$$

If $s \notin \mathbf{W}_{sw'\lambda}$, this equals (by the induction hypothesis) $\delta_{wssw',1} = \delta_{ww',1}$, as required; if $s \in \mathbf{W}_{sw'\lambda}$, this equals (by the induction hypothesis)

$$\delta_{wssw',1} + (v - v^{-1})\tau(\tilde{T}_{ws} \tilde{T}_{w'} 1_\lambda) = \delta_{ww',1} + (v - v^{-1})\delta_{wsw',1}.$$

It remains to note that $wsw' \neq 1$ whenever $l(w) = l(ws) + 1$, $l(sw') = l(w') - 1$. This proves (a).

34.14. Let $\mathfrak{C}$ be a finite dimensional semisimple split (associative) algebra with 1 over a field K. Let $\{E_u; u \in \mathcal{U}\}$ be a set or representatives for the isomorphism classes of simple $\mathfrak{C}$-modules. Let $\mathfrak{a} : \mathfrak{C} \to \mathfrak{C}$ be an algebra automorphism. For $u \in \mathcal{U}$, $c : e \mapsto \mathfrak{a}(c)e$ defines a $\mathfrak{C}$-module structure on the K-vector space E_u which is isomorphic to $E_{\bar{u}}$ for a unique $\bar{u} \in \mathcal{U}$. Then $u \mapsto \bar{u}$ is a permutation of $\mathcal{U}$. Let $\mathcal{U}^\mathfrak{a} = \{u \in \mathcal{U}; u = \bar{u}\}$. For $u \in \mathcal{U}^\mathfrak{a}$ we can find a K-linear isomorphism $\mathfrak{a}_u : E_u \to E_u$ such that $\mathfrak{a}_u(ce) = \mathfrak{a}(c)\mathfrak{a}_u e$ for all $c \in \mathfrak{C}, e \in E_u$. Note that $\mathfrak{a}_u$ is uniquely determined up to multiplication by an element in K^*. We show:

(a) *If $c, c' \in \mathfrak{C}$, then the trace of the K-linear map $\mathfrak{C} \to \mathfrak{C}$, $c_1 \mapsto c\mathfrak{a}(c_1)c'$ equals* $\sum_{u \in \mathcal{U}^\mathfrak{a}} \text{tr}(c\mathfrak{a}_u, E_u)\text{tr}(\mathfrak{a}_u^{-1}c', E_u)$.

Under the algebra isomorphism

(b) $\mathfrak{C} \xrightarrow{\sim} \oplus_{u \in \mathcal{U}} \text{End}_K(E_u)$, $c \mapsto [e \mapsto ce, e \in E_u]$,

the linear map $\mathfrak{C} \to \mathfrak{C}$ in (a) corresponds to an endomorphism of $\oplus_{u \in \mathcal{U}} \text{End}_K(E_u)$

which permutes the summands according to $u \mapsto \bar{u}$ and whose restriction to a summand with $u = \bar{u}$ is $\mathrm{End}_K(E_u) \mapsto \mathrm{End}_K(E_u)$, $f \mapsto c\mathfrak{a}_u f \mathfrak{a}_u^{-1} c'$. From this (a) follows easily. (Compare 20.3(b).)

We show:

(c) *If* $y : \mathfrak{C} \to K$ *is* K*-linear and* $y(cc') = y(c'\mathfrak{a}(c))$ *for all* $c, c' \in \mathfrak{C}$*, then there exist* $b_u \in K (u \in \mathcal{U}^{\mathfrak{a}})$ *such that* $y(c) = \sum_{u \in \mathcal{U}^{\mathfrak{a}}} b_u \mathrm{tr}(c\mathfrak{a}_u, E_u)$ *for all* $c \in \mathfrak{C}$.

Let $\mathfrak{C}_u$ be the inverse image of the summand $\mathrm{End}_K(E_u)$ under (b). Then $\mathfrak{C} = \oplus_u \mathfrak{C}_u$ and $\mathfrak{a}(\mathfrak{C}_u) = \mathfrak{C}_{\bar{u}}$ for $u \in \mathcal{U}$. Let $y_u : \mathfrak{C}_u \to K$ be the restriction of y to $\mathfrak{C}_u$. Let $c \in \mathfrak{C}_u$ where $u \neq \bar{u}$. Let $c' \in \mathfrak{C}_u$ be the projection of $1 \in \mathfrak{C}$ onto $\mathfrak{C}_u$. We have $\mathfrak{a}(c) \in \mathfrak{C}_{\bar{u}}$, hence $c'\mathfrak{a}(c) = 0$. Also, $cc' = c \in \mathfrak{C}_u$. Thus, $y_u(c) = y_u(cc') = y(cc') = y(c'\mathfrak{a}(c)) = 0$. We see that $y_u = 0$. We are reduced to the case where $\mathcal{U} = \mathcal{U}^{\mathfrak{a}}$ consists of a single element u and $\mathfrak{C} = \mathrm{End}_K(E_u)$. We can find $h \in \mathfrak{C}$, invertible, such that $\mathfrak{a}(c) = hch^{-1}$ for all $c \in \mathfrak{C}$. We can assume that $\mathfrak{a}_u e = he$ for all $e \in E_u$. We have $y(cc') = y(c'hch^{-1})$ for all $c, c' \in \mathfrak{C}$. Define $\tilde{y} : \mathfrak{C} \to K$ by $\tilde{y}(c) = y(ch^{-1})$. We have $y(cc'h^{-1}) = y(c'h^{-1}hch^{-1}) = y(c'ch^{-1})$, hence $\tilde{y}(cc') = \tilde{y}(c'c)$ for all $y, y' \in \mathfrak{C}$. Thus there exists $b \in K$ such that $\tilde{y}(c) = b\mathrm{tr}(c, E_u)$. Then $y(c) = b\mathrm{tr}(ch : E_u \to E_u) = b\mathrm{tr}(c\mathfrak{a}_u : E_u \to E_u)$. This proves (c).

Assume now that we are given a K-linear map $z : \mathfrak{C} \to K$ such that $(c, c') = z(cc') = z(c'c)$ is a nondegenerate (symmetric) K-bilinear form $\mathfrak{C} \times \mathfrak{C} \to K$. Let $(c_i)_{i \in I}$ be a K-basis of $\mathfrak{C}$. Define a K-basis $(c'_i)_{i \in I}$ of $\mathfrak{C}$ by $(c_i, c'_j) = \delta_{ij}$.

For $u \in \mathcal{U}$ and $c \in \mathfrak{C}$ invertible, $\sum_{i \in I} \mathrm{tr}(c_i c, E_u) c^{-1} c'_i$ is in the center of $\mathfrak{C}$; if $u' \in \mathcal{U}, u' \neq u$, this sum acts on $E_{u'}$ as 0 and on E_u as f_u times the identity, where $f_u \in K^*$ is independent of $(c_i), (c'_i), c$. (We apply [L12, 19.2] to the dual bases $(c_i c), (c^{-1} c'_i)$: we have $(c_i c, c^{-1} c'_j) = (c_i, cc^{-1} c'_j) = (c_i, c'_j) = \delta_{ij}$.) We see that for $u, u' \in \mathcal{U}$ we have

(d) $$\sum_{i \in I} \mathrm{tr}(c_i c, E_u) \mathrm{tr}(c^{-1} c'_i, E_{u'}) = \delta_{u,u'} f_u \dim E_u.$$

Now assume that $u, u' \in \mathcal{U}^{\mathfrak{a}}$. We can pick $c \in \mathfrak{C}$ invertible such that c acts on E_u as $\mathfrak{a}_u$ and on $E_{u'}$ as $\mathfrak{a}_{u'}$. From (d) we deduce

(e) $$\sum_{i \in I} \mathrm{tr}(c_i \mathfrak{a}_u, E_u) \mathrm{tr}(\mathfrak{a}_{u'}^{-1} c'_i, E_{u'}) = \delta_{u,u'} f_u \dim E_u.$$

34.15. We write $\mathfrak{a} : H_n \to H_n$ instead of $\mathfrak{a}_D : H_n \to H_n$ (see 31.4); this is the algebra automorphism given by $h \mapsto \tilde{T}_{\underline{D}} h \tilde{T}_{\underline{D}^{-1}}$ (product in H_n^D) for $h \in H_n$. The same formula defines an algebra automorphism of H_n^κ or H_n^v denoted again by $\mathfrak{a}$. From the definitions we see that $\mathfrak{a} : H_n \to H_n$ takes $c_{w,\lambda}$ to $c_{\epsilon_D(w), \underline{D}\lambda}$ for $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$. Hence it induces a ring automorphism $H_n^\infty \to H_n^\infty$ denoted again by $\mathfrak{a}$. It also induces algebra automorphisms $H_n^\kappa \to H_n^\kappa, \kappa \in \mathfrak{U}^*$ and $H_n^v \to H_n^v$ denoted again by $\mathfrak{a}$. Now $\Phi^\kappa : H_n^\kappa \to \mathfrak{U} \otimes_{\mathbf{Z}} H_n^\infty$ for $\kappa \in \mathfrak{U}^*$ and $\Phi^v : H_n^v \to \mathfrak{U}(v) \otimes_{\mathbf{Z}} H_n^\infty$ (see 34.11) are compatible with $\mathfrak{a}$.

Let $\{E_u; u \in \mathcal{U}\}$ be a set of representatives for the isomorphism classes of simple modules for H_n^1 (a split semisimple $\mathfrak{U}$-algebra, by 31.12(b).) Define $\mathcal{U} \to \mathcal{U}, u \mapsto \bar{u}$ as in 34.14, replacing $(\mathfrak{C}, \mathfrak{a})$ by $(H_n^1, \mathfrak{a})$. Let $\mathcal{U}^{\mathfrak{a}} = \{u \in \mathcal{U}; u = \bar{u}\}$.

Let $u \in \mathcal{U}$. Clearly, if E_u extends to an $H_n^{D,1}$-module, then $u \in \mathcal{U}^{\mathfrak{a}}$. Conversely, we show that, if $u \in \mathcal{U}^{\mathfrak{a}}$, then E_u extends to an $H_n^{D,1}$-module. Since $H_n^1, H_n^{D,1}$ are split over $\mathfrak{U}$, it is enough to prove the analogous statement in which $E_u, H_n^1, H_n^{D,1}$ are replaced by $\mathfrak{U}' \otimes_{\mathfrak{U}} E_u, \mathfrak{U}' \otimes_{\mathfrak{U}} H_n^1, \mathfrak{U}' \otimes_{\mathfrak{U}} H_n^{D,1}$ and $\mathfrak{U}'$ is an algebraic closure of $\mathfrak{U}$.

Since $u \in \mathcal{U}^{\mathfrak{a}}$, we can find a $\mathfrak{U}'$-linear isomorphism $X : \mathfrak{U}' \otimes_{\mathfrak{U}} E_u \to \mathfrak{U}' \otimes_{\mathfrak{U}} E_u$ such that $X(ce) = \mathfrak{a}(c)X(e)$ for all $c \in \mathfrak{U}' \otimes_{\mathfrak{U}} H_n^1, e \in \mathfrak{U}' \otimes_{\mathfrak{U}} E_u$. Let k be the order of $\underline{D} : \mathbf{T} \to \mathbf{T}$. By Schur's lemma, X^k is a scalar times identity. Now $\mathfrak{U}'$ contains a k-th root of this scalar. Dividing X by this root we see that we may assume that $X^k = 1$. We can now define a $\mathfrak{U}' \otimes_{\mathfrak{U}} H_n^{D,1}$-module structure on the vector space $\mathfrak{U}' \otimes_{\mathfrak{U}} E_u$ which extends the $\mathfrak{U}' \otimes_{\mathfrak{U}} H_n^1$-module structure and in which $\tilde{T}_{\underline{D}}$ acts as X.

For any $u \in \mathcal{U}^{\mathfrak{a}}$ we choose an $H_n^{D,1}$-module structure on E_u extending the H_n^1-module structure.

Let $u \in \mathcal{U}$. We regard E_u as a (simple) $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{\infty}$-module E_u^{∞} via $\Phi^1 : H_n^1 \xrightarrow{\sim} \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{\infty}$. (If $u \in \mathcal{U}^{\mathfrak{a}}$ we also regard E_u as a $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$-module E_u^{∞} via $\Phi^1 : H_n^{D,1} \xrightarrow{\sim} \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$. This extends the $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{\infty}$-module structure.)

Now $\mathfrak{U}[v, v^{-1}] \otimes_{\mathfrak{U}} E_u^{\infty}$ is naturally a $\mathfrak{U}[v, v^{-1}] \otimes_{\mathbf{Z}} H_n^{\infty}$-module and also an H_n-module via the homomorphism $H_n \xrightarrow{\Phi} \mathcal{A} \otimes_{\mathbf{Z}} H_n^{\infty} \subset \mathfrak{U}[v, v^{-1}] \otimes_{\mathbf{Z}} H_n^{\infty}$. This H_n-module is denoted by $E_u(v)$. (If $u \in \mathcal{U}^{\mathfrak{a}}$, then $\mathfrak{U}[v, v^{-1}] \otimes_{\mathfrak{U}} E_u^{\infty}$ is naturally a $\mathfrak{U}[v, v^{-1}] \otimes_{\mathbf{Z}} H_n^{D,\infty}$-module and also an H_n^D-module via the homomorphism $H_n^D \xrightarrow{\Phi} \mathcal{A} \otimes_{\mathbf{Z}} H_n^{D,\infty} \subset \mathfrak{U}[v, v^{-1}] \otimes_{\mathbf{Z}} H_n^{D,\infty}$. This extends the H_n-module structure on $E_u(v)$.)

Let $E_u^v = \mathfrak{U}(v) \otimes_{\mathfrak{U}} E_u(v)$. From 34.12(c) we see that $\{E_u^v; u \in \mathcal{U}\}$ is a set of representatives for the isomorphism classes of simple H_n^v-modules. (If $u \in \mathcal{U}^{\mathfrak{a}}$, then the $H_n^{D,v}$-module structure on E_u^v coming from the H_n^D-module structure on $E_u(v)$ extends the H_n^v-module structure.)

For κ as in 34.12(b) let E_u^{κ} be the vector space E_u^{∞} regarded as an H_u^{κ}-module via $\Phi^{\kappa} : H_n^{\kappa} \xrightarrow{\sim} \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{\infty}$. From 34.12(b) we see that $\{E_u^{\kappa}; u \in \mathcal{U}\}$ is a set of representatives for the isomorphism classes of simple H_n^{κ}-modules. Now E_u^{κ} can also be obtained from the H_n-module $E_u(v)$ under the specialization $\mathfrak{U}[v, v^{-1}] \to \mathfrak{U}, v \mapsto \kappa$. Moreover, we have $E_u^1 = E_u$ as H_n^1-modules. (If $u \in \mathcal{U}^{\mathfrak{a}}$, we also regard E_u^{∞} as an $H_u^{D,\kappa}$-module via $\Phi^{\kappa} : H_n^{D,\kappa} \xrightarrow{\sim} \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$. This extends the H_u^{κ}-module structure. This can be also obtained from the H_n^D-module $E_u(v)$ under the specialization $\mathfrak{U}[v, v^{-1}] \to \mathfrak{U}, v \mapsto \kappa$. Moreover, $E_u^1 = E_u$ as $H_u^{D,1}$-modules.)

From the definitions we see that the map $\mathcal{U} \to \mathcal{U}, u \mapsto \bar{u}$ defined as in 34.14, replacing $(\mathfrak{C}, \mathfrak{a})$ by $(H_n^{\kappa}, \mathfrak{a})$ (κ as in 34.12(b)) or by $(H_n^v, \mathfrak{a})$ is the same as the map $u \mapsto \bar{u}$ defined in terms of $(H_n^1, \mathfrak{a})$. We show:

(a) *Let* $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, u \in \mathcal{U}$. *Then* $\mathrm{tr}(\tilde{T}_w 1_\lambda, E_u^v) \in \mathfrak{U}[v, v^{-1}]$ *and, for* κ *as in 34.12(b),* $\mathrm{tr}(\tilde{T}_w 1_\lambda, E_u^{\kappa}) \in \mathfrak{U}$ *is obtained from this element of* $\mathfrak{U}[v, v^{-1}]$ *by the specialization* $\mathfrak{U}[v, v^{-1}] \to \mathfrak{U}, v \mapsto \kappa$. *If, in addition,* $u \in \mathcal{U}^{\mathfrak{a}}$ *and* $j \in \mathbf{Z}$, *then* $\mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}^j, E_u^v) \in \mathfrak{U}[v, v^{-1}]$ *and, for* κ *as in 34.12(b),* $\mathrm{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}^j, E_u^{\kappa}) \in \mathfrak{U}$ *is obtained from this element of* $\mathfrak{U}[v, v^{-1}]$ *by the specialization* $\mathfrak{U}[v, v^{-1}] \to \mathfrak{U}, v \mapsto \kappa$.

This follows immediately from the fact that $\Phi(\tilde{T}_w 1_\lambda)$ (resp. $\Phi(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}^j)$) is an $\mathcal{A}$-linear combination of the standard basis elements of H_n^{∞} (resp.$H_n^{D,\infty}$.)

Combining 34.13(a), 34.14(d) we see that, for $u, u' \in \mathcal{U}$ and for κ as in 34.12(b) we have

$$\sum_{w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n} \mathrm{tr}(\tilde{T}_w 1_\lambda, E_u^v) \mathrm{tr}(1_\lambda \tilde{T}_{w^{-1}}, E_{u'}^v) = \delta_{u,u'} f_u^v \dim E_u,$$

$$\text{(b)} \qquad \sum_{w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n} \mathrm{tr}(\tilde{T}_w 1_\lambda, E_u^{\kappa}) \mathrm{tr}(1_\lambda \tilde{T}_{w^{-1}}, E_{u'}^{\kappa}) = \delta_{u,u'} f_u^{\kappa} \dim E_u,$$

where $f_u^v \in \mathfrak{U}(v)-\{0\}, f_u^\kappa \in \mathfrak{U}-\{0\}$. Using (a) we see that $f_u^v \in \mathfrak{U}[v,v^{-1}]$ and f_u^κ is obtained from f_u^v by the specialization $\mathfrak{U}[v,v^{-1}] \to \mathfrak{U}, v \mapsto \kappa$. Combining 34.13(a), 34.14(e) we see that, for $u,u' \in \mathcal{U}^\mathfrak{a}$, we have

(c) $$\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n} \operatorname{tr}(\tilde{T}_w 1_\lambda \tilde{T}_{\underline{D}}, E_u^v)\operatorname{tr}(\tilde{T}_{\underline{D}}^{-1} 1_\lambda \tilde{T}_{w^{-1}}, E_{u'}^v) = \delta_{u,u'} f_u^v \dim E_u.$$

34.16. Let $x \mapsto x^\spadesuit$ be the automorphism of the field $\mathfrak{U}$ which sends any root of 1 to its inverse. We extend this to an automorphism of the field $\mathfrak{U}(v)$ (denoted by $\xi \mapsto \xi^\spadesuit$) which carries v to itself. For $x' \in \mathfrak{U}$ we say that $x' > 0$ if the image of x' under any imbedding of $\mathfrak{U}$ into the complex numbers is a real number > 0. For example, for $x \in \mathfrak{U}-\{0\}$ we have $xx^\spadesuit > 0$. For $\xi' \in \mathfrak{U}(v)$ we say that $\xi' > 0$ if ξ' can be expanded in a power series $\xi' = a_0 v^n + a_1 v^{n+1} + \dots$ where $a_0, a_1, \alpha_2, \dots \in \mathfrak{U}$ and $a_0 > 0$.

Lemma 34.17. *Let $u \in \mathcal{U}^\mathfrak{a}, w \in \mathbf{W}^D, \lambda \in \underline{\mathfrak{s}}_n$. We have*

(a) $$\operatorname{tr}(1_\lambda \tilde{T}_{w^{-1}}, E_u^v) = \operatorname{tr}(\tilde{T}_w 1_\lambda, E_u^v)^\spadesuit.$$

The antiautomorphism $h \to h^\flat$ of H_n (see 32.19) extends to an antiautomorphism $h \to h^\flat$ of H_n^D given by $\tilde{T}_{w'} \mapsto \tilde{T}_{w'^{-1}}$ for $w' \in \mathbf{W}^D$, $1_{\lambda'} \mapsto 1_{\lambda'}$ for $\lambda' \in \underline{\mathfrak{s}}_n$. Define a ring involution $h \mapsto h^\diamond$ of $H_n^{D,v}$ by $\sum_{w,\lambda} a_{w,\lambda}\tilde{T}_w 1_\lambda \mapsto \sum_{w,\lambda} a_{w,\lambda}^\spadesuit (\tilde{T}_w 1_\lambda)^\flat$ where $a_{w,\lambda} \in \mathfrak{U}(v)$. Assume that there exists a pairing $\langle,\rangle : E_u^v \times E_u^v \to \mathfrak{U}(v)$ such that $\langle,\rangle$ is linear in the second variable, semi-linear (with respect to $\xi \mapsto \xi^\spadesuit$) in the first variable, is non-degenerate, satisfies $\langle e,e'\rangle = \langle e',e\rangle^\spadesuit$ for $e,e' \in E_u^v$ and

(b) $$\langle he,e'\rangle = \langle e,h^\diamond e'\rangle \text{ for } e,e' \in E_u^v, h \in H_n^{D,v}.$$

If $(e_j),(e'_j)$ are bases of E_u^v such that $\langle e_i,e'_j\rangle = \delta_{ij}$, then

$$\operatorname{tr}(h,E_u^v) = \sum_j \langle e_j, he'_j\rangle = \sum_j \langle h^\diamond e_j, e'_j\rangle = \operatorname{tr}(h^\diamond, E_u^v)^\spadesuit.$$

Taking here $h = 1_\lambda \tilde{T}_{w^{-1}}$ we see that (a) would follow. It remains to prove the existence of $\langle,\rangle$ as above.

We can find a pairing $\langle,\rangle' : E_u^v \times E_u^v \to \mathfrak{U}(v)$ which is linear in the second variable, semi-linear (with respect to $\xi \mapsto \xi^\spadesuit$) in the first variable, satisfies $\langle e,e'\rangle' = \langle e',e\rangle'^\spadesuit$ for $e,e' \in E_u^v$ and $\langle e,e\rangle > 0$ for $e \in E_u^v - \{0\}$. (For example, we choose a basis (e_j) of E_u^v and we set $\langle \sum_j a_j e_j, \sum_j a'_j e_j\rangle' = \sum_j a_j^\spadesuit a'_j$ where $a_j, a'_j \in \mathfrak{U}(v)$.) We define a new pairing $\langle,\rangle : E_u^v \times E_u^v \to \mathfrak{U}(v)$ by

$$\langle e,e'\rangle = \sum_{w'\in\mathbf{W}^D,\lambda'\in\underline{\mathfrak{s}}_n} \langle \tilde{T}_{w'} 1_{\lambda'} e, \tilde{T}_{w'} 1_{\lambda'} e'\rangle'.$$

Note that $\langle,\rangle'$ is linear in the second variable, semi-linear (with respect to $\xi \mapsto \xi^\spadesuit$) in the first variable, satisfies $\langle e,e'\rangle = \langle e',e\rangle'^\spadesuit$ for $e,e' \in E_u^v$ and $\langle e,e\rangle > 0$ for $e \in E_u^v - \{0\}$. In particular, $\langle,\rangle$ is non-degenerate. We show that (b) holds. It is enough to show this when h runs through a set of generators of the algebra $H_n^{D,v}$, that is, for $h = 1_\lambda$ or $h = \tilde{T}_{\underline{D}}$ or $h = \tilde{T}_s (s \in \mathbf{I})$. Assume first that $h = 1_\lambda, \lambda \in \underline{\mathfrak{s}}_n$. We must show that

$$\sum_{w'\in\mathbf{W}^D,\lambda'\in\underline{\mathfrak{s}}_n} \langle \tilde{T}_{w'} 1_{\lambda'} 1_\lambda e, \tilde{T}_{w'} 1_{\lambda'} e'\rangle' = \sum_{w'\in\mathbf{W}^D,\lambda'\in\underline{\mathfrak{s}}_n} \langle \tilde{T}_{w'} 1_{\lambda'} e, \tilde{T}_{w'} 1_{\lambda'} 1_\lambda e'\rangle'.$$

Both sides are equal to $\sum_{w'\in\mathbf{W}^D}\langle\tilde T_{w'}1_\lambda e,\tilde T_{w'}1_\lambda e'\rangle'$. Assume next that $h=\tilde T_{\underline{D}}$. We must show that

$$\sum_{\substack{w'\in\mathbf{W}^D\\ \lambda'\in\underline{\mathfrak{s}}_n}}\langle\tilde T_{w'}1_{\lambda'}\tilde T_{\underline{D}}e,\tilde T_{w'}1_{\lambda'}e'\rangle'=\sum_{\substack{w'\in\mathbf{W}^D\\ \lambda'\in\underline{\mathfrak{s}}_n}}\langle\tilde T_{w'}1_{\lambda'}e,\tilde T_{w'}1_{\lambda'}\tilde T_{\underline{D}^{-1}}e'\rangle'$$

that is, $\sum_{\substack{w'\in\mathbf{W}^D\\ \lambda'\in\underline{\mathfrak{s}}_n}}\langle\tilde T_{w'\underline{D}}1_{\underline{D}^{-1}\lambda'}e,\tilde T_{w'}1_{\lambda'}e'\rangle'=\sum_{\substack{y\in\mathbf{W}^D\\ \lambda''\in\underline{\mathfrak{s}}_n}}\langle\tilde T_y1_{\lambda''}e,\tilde T_{y\underline{D}^{-1}}1_{\underline{D}\lambda''}e'\rangle'$, which is clear. Finally, assume that $h=\tilde T_s$, $s\in\mathbf{I}$. We must show that

$$\sum_{w'\in\mathbf{W}^D,\lambda'\in\underline{\mathfrak{s}}_n}\langle\tilde T_{w'}1_{\lambda'}\tilde T_se,\tilde T_{w'}1_{\lambda'}e'\rangle'=\sum_{w'\in\mathbf{W}^D,\lambda'\in\underline{\mathfrak{s}}_n}\langle\tilde T_{w'}1_{\lambda'}e,\tilde T_{w'}1_{\lambda'}\tilde T_se'\rangle'.$$

Both sides are equal to

$$\sum_{\substack{w'\in\mathbf{W}^D\\ \lambda'\in\underline{\mathfrak{s}}_n}}\langle\tilde T_{w's}1_{s\lambda'}e,\tilde T_{w'}1_{\lambda'}e'\rangle'+(v-v^{-1})\sum_{\substack{w'\in\mathbf{W}^D\\ l(w's)=l(w')-1\\ \lambda'\in\underline{\mathfrak{s}}_n,s\in\mathbf{W}_{\lambda'}}}\langle\tilde T_{w'}1_{\lambda'}e,\tilde T_{w'}1_{\lambda'}e'\rangle'.$$

This proves (b). The lemma is proved.

34.18. Using Lemma 34.17(a) we can rewrite 34.15(c) for $u,u'\in\mathcal{U}^{\mathfrak{a}}$ as follows:

(a) $$\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n}\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_u^v)\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_{u'}^v)^\spadesuit=\delta_{u,u'}f_u^v\dim E_u.$$

Specializing this under $\mathfrak{U}[v,v^{-1}]\to\mathfrak{U}$, $v\mapsto\kappa$, we obtain

(b) $$\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n}\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_u^\kappa)\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_{u'}^{\kappa^\spadesuit})^\spadesuit=\delta_{u,u'}f_u^\kappa\dim E_u.$$

34.19. Let A be a character sheaf on D. Define an $\mathcal{A}$-linear map $\hat\zeta:H_n\to\mathcal{A}$ by $h\mapsto\zeta^A(h[D])$ where $\zeta^A:H_n[D]\to\mathcal{A}$ is as in 31.7. From 31.8 we see that $\hat\zeta(hh')=\hat\zeta(h'\mathfrak{a}(h))$ for $h,h'\in H_n$. Applying 34.14(c) to the linear map $\mathfrak{U}(v)\otimes_{\mathcal{A}}H_n\to\mathfrak{U}(v)$ obtained from $\hat\zeta$ by extension of scalars, we see that there exist elements $b_{A,u}^v\in\mathfrak{U}(v)(u\in\mathcal{U}^{\mathfrak{a}})$ such that

(a) $$\zeta^A(\tilde T_w1_{\underline{D}\lambda}[D])=\sum_{u'\in\mathcal{U}^{\mathfrak{a}}}b_{A,u'}^v\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_{u'}^v),$$

for $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n$. We multiply both sides of (a) by $\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_u^v)^\spadesuit$ (with $u\in\mathcal{U}^{\mathfrak{a}}$) and sum over all w,λ. Using 34.18(a), we obtain

(b) $$b_{A,u}^v=\frac{1}{f_u^v\dim E_u}\sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n}\zeta^A(\tilde T_w1_{\underline{D}\lambda}[D])\mathrm{tr}(\tilde T_w1_{\underline{D}\lambda}\tilde T_{\underline{D}},E_u^v)^\spadesuit.$$

Using 28.17(a),(b) and the notation there, we see that $\mathfrak{D}({}^pH^j(\bar K_D^{\mathbf{s},\mathcal{L}}))={}^pH^j(\bar K_D^{\mathbf{s},\check{\mathcal{L}}})$, hence

$$\begin{aligned}\sum_{u\in\mathcal{U}^{\mathfrak{a}}} & b^v_{\mathfrak{D}(A),u}\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde T_{\underline{D}},E^v_u)=\zeta^{\mathfrak{D}(A)}(C^{\mathbf{s}}_{\underline{D}\lambda}[D])\\ &=\sum_j(-v)^jv^{-\dim G}(\mathfrak{D}(A):{}^pH^j(\bar K_D^{\mathbf{s},\mathcal{L}}))\\ &=\sum_j(-v)^jv^{-\dim G}(A:\mathfrak{D}({}^pH^j(\bar K_D^{\mathbf{s},\mathcal{L}})))\\ &=\sum_j(-v)^jv^{-\dim G}(A:{}^pH^j(\bar K_D^{\mathbf{s},\check{\mathcal{L}}}))=\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda^{-1}}[D])\\ &=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}b^v_{A,u}\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda^{-1}}\tilde T_{\underline{D}},E^v_u).\end{aligned}\tag{c}$$

Lemma 34.20. *Let $u\in\mathcal{U}^{\mathfrak{a}}$. Assume that E_u is quasi-rational in the following sense: there exists a function $\eta:\mathbf{W}^D\times\underline{\mathfrak{s}}_n\to\{\text{roots of }1\text{ in }\mathfrak{U}\},(w,\lambda)\mapsto\eta_{w,\lambda}$ such that η is constant on any equivalence class for $\asymp$ in $\mathbf{W}^D\times\underline{\mathfrak{s}}_n$ (see 32.26) and $\mathrm{tr}(\tilde T_w1_\lambda,E_u)\in\eta_{w,\lambda}\mathbf{Z}$ for all $(w,\lambda)\in\mathbf{W}^D\times\underline{\mathfrak{s}}_n$. Then $\mathrm{tr}(\tilde T_w1_\lambda,E^v_u)\in\eta_{w,\lambda}\mathcal{A}$ for all $(w,\lambda)\in\mathbf{W}^D\times\underline{\mathfrak{s}}_n$.*

From the definitions and 34.7–34.10 we see that, for $w\in\mathbf{W}^D,\lambda\in\underline{\mathfrak{s}}_n$, the basis elements $c_{w,\lambda}$ of H_n^D (see Proposition 34.5) satisfy

(a) $c_{w,\lambda}\in\tilde T_{wy}1_\lambda+\sum_{y\in\mathbf{W}_\lambda;wy<w}\mathcal{A}\tilde T_{wy}1_\lambda$.

For $w\in\mathbf{W}^D,\lambda\in\underline{\mathfrak{s}}_n,x\in\mathbf{W}_\lambda$ we have $\tilde T_w1_\lambda\tilde T_x\in\sum_{x'\in\mathbf{W}_\lambda}\mathcal{A}\tilde T_{wx'}1_\lambda$ (in H_n^D). This follows by writing x as product $x=\sigma_1\sigma_2\dots\sigma_m,\sigma_m\in\mathbf{I}_\lambda,m=l_\lambda(x)$ and using repeatedly Lemma 34.7(a). Using this and (a) we see that

(b) $c_{w,\lambda}\tilde T_x\in\sum_{x'\in\mathbf{W}_\lambda}\mathcal{A}\tilde T_{wx'}1_\lambda$.

Now let $c_{w',\lambda}$ be a distinguished basis element of H_n^D (see 34.1). By 34.10(e) we have $w'\in\mathbf{W}_\lambda$, hence $c_{w',\lambda}\in\sum_{x\in\mathbf{W}_\lambda}1_\lambda\tilde T_x$. Hence from (b) we deduce

(c) $c_{w,\lambda}c_{w',\lambda}\in\sum_{x'\in\mathbf{W}_\lambda}\mathcal{A}\tilde T_{wx'}1_\lambda$.

From (a) we deduce by inversion:

(d) $\tilde T_w1_\lambda\in\sum_{y\in\mathbf{W}_\lambda;wy\le w}\mathcal{A}c_{wy,\lambda}$.

Using this for wx' instead of x and using also (c) we deduce

$$c_{w,\lambda}c_{w',\lambda}\in\sum_{x''\in\mathbf{W}_\lambda}\mathcal{A}c_{wx'',\lambda}.$$

Hence, if $\Phi:H_n^D\to\mathcal{A}\otimes_{\mathbf{Z}}H_n^{D,\infty}$ is as in 34.12, then

(e) $\Phi(c_{w,\lambda})\in\sum_{x''\in\mathbf{W}_\lambda}\mathcal{A}t_{wx'',\lambda}$

where $t_{y,\lambda}$ is the basis element of $H_n^{D,\infty}$ corresponding to $c_{y,\lambda}\in H_n^D$ (see 34.1).

Let $c_{w,\lambda;1}$ be the image of $c_{w,\lambda}$ in $H_n^{D,1}$. Let $\Phi':\mathbf{Q}\otimes_{\mathcal{A}}H_n^D\to\mathbf{Q}\otimes_{\mathbf{Z}}H_n^{D,\infty}$ be the homomorphism obtained from Φ under the specialization $\mathcal{A}\to\mathbf{Q},v\mapsto1$. This is an isomorphism of algebras since Φ^1 (see 34.12) is an isomorphism. From (e) we see that for any $(y,\lambda)\in\mathbf{W}^D\times\underline{\mathfrak{s}}_n$, Φ' restricts to a $\mathbf{Q}$-linear map

(f) $\sum_{x\in\mathbf{W}_\lambda}\mathbf{Q}c_{yx,\lambda;1}\to\sum_{x\in\mathbf{W}_\lambda}\mathbf{Q}t_{yx,\lambda}$.

The vector spaces in (f) form direct sum decompositions of $\mathbf{Q}\otimes_{\mathcal{A}}H_n^D,\mathbf{Q}\otimes_{\mathbf{Z}}H_n^{D,\infty}$, hence (f) must be an isomorphism. We deduce that Φ'^{-1} carries $\sum_{x\in\mathbf{W}_\lambda}\mathbf{Q}t_{yx,\lambda}$ onto $\sum_{x\in\mathbf{W}_\lambda}\mathbf{Q}c_{yx,\lambda;1}$. We see that $t_{y,\lambda}$ acts on the $\mathfrak{U}\otimes_{\mathbf{Z}}H_n^{D,\infty}$-module E_u^∞ as a

$\mathbf{Q}$-linear combination of the operators $c_{yx,\lambda;1} : E_u \to E_u$ where $x \in \mathbf{W}_\lambda$, hence also as a $\mathbf{Q}$-linear combination of the operators $\tilde{T}_{yx}1_\lambda : E_u \to E_u$. (From (a) specialized for $v = 1$ we see that $c_{y,\lambda} \in \sum_{x\in\mathbf{W}_\lambda} \mathbf{Z}\tilde{T}_{yx}1_\lambda$.) It follows that $\mathrm{tr}(t_{y,\lambda}, E_u^\infty) \in \eta_{y,\lambda}\mathbf{Q}$. Using (e) we see that $c_{y,\lambda}$ acts on E_u^v as an $\mathcal{A}$-linear combination of the operators $1 \otimes t_{yx,\lambda}$ on $\mathfrak{U}(v) \otimes_{\mathfrak{U}} E_u^\infty$ where $x \in \mathbf{W}_\lambda$. The same holds for $\tilde{T}_{y,\lambda}$ (instead of $c_{y,\lambda}$), by (d). It follows that $\mathrm{tr}(\tilde{T}_y 1_\lambda, E_u^v) \in \eta_{y,\lambda}\mathbf{Q}[v, v^{-1}]$.

Since the algebra $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D,\infty}$ is of finite dimension (say m), with 1, and its structure constants with respect to the basis $(t_{y,\lambda})$ are integers, we see that any basis element $t_{y,\lambda}$ satisfies an equation of the form $t_{y,\lambda}^m + c_1 t_{y,\lambda}^{m-1} + \cdots + c_m = 0$ where $c_1, c_2, \dots, c_m$ are integers. It follows that $\mathrm{tr}(t_{y,\lambda}, E_u^\infty)$ is an algebraic integer (necessarily in $\mathfrak{U}$). Since the definition of Φ involves only coefficients in $\mathcal{A}$, it follows that the coefficients of $\mathrm{tr}(c_{y,w}, E_u^v) \in \mathfrak{U}[v, v^{-1}]$ are algebraic integers in $\mathfrak{U}$. The same holds then for the coefficients of $\mathrm{tr}(\tilde{T}_y 1_\lambda, E_u^v)$. An element of $\eta_{y,\lambda}\mathbf{Q}[v, v^{-1}]$ whose coefficients are algebraic integers in $\mathfrak{U}$ is necessarily in $\eta_{y,\lambda}\mathcal{A}$. We see that $\mathrm{tr}(\tilde{T}_y 1_\lambda, E_u^v) \in \eta_{y,\lambda}\mathcal{A}$, as required.

Lemma 34.21. *Let u, E_u, η be as in Lemma 34.20. Let A be a character sheaf on D. Let $\mathfrak{C}_A \subset \mathbf{W}^D \times \underline{\mathfrak{s}}_n$ be the equivalence class under $\asymp$ attached to A in 32.25(a) (with $J = \mathbf{I}$). Let η_0, a root of 1, be the (constant) value of η on $\mathfrak{C}_A$. We have $b_{A,u}^v \in \eta_0^{-1}\mathbf{Q}(v)$.*

From 34.18(a) we have

$$f_u^v = (\dim E_u)^{-1} \sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n} \mathrm{tr}(\tilde{T}_{w\underline{D}}1_\lambda, E_u^v)\mathrm{tr}(\tilde{T}_{w\underline{D}}1_\lambda, E_u^v)^\spadesuit.$$

By 34.20, for any $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$ we have $\mathrm{tr}(\tilde{T}_{w\underline{D}}1_\lambda, E_u^v) = \eta_{w\underline{D},\lambda}Q_{w,\lambda}$ where $Q_{w,\lambda} \in \mathcal{A}$; hence $\mathrm{tr}(\tilde{T}_{w\underline{D}}1_\lambda, E_u^v)\mathrm{tr}(\tilde{T}_{w\underline{D}}1_\lambda, E_u^v)^\spadesuit = \eta_{w\underline{D},\lambda}Q_{w,\lambda}\eta_{w\underline{D},\lambda}^{-1}Q_{w,\lambda}$. Thus,

(a) $$f_u^v = (\dim E_u)^{-1} \sum_{w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n} Q_{w,\lambda}^2 \in \mathbf{Q}[v, v^{-1}].$$

Using 32.26(b) we rewrite 34.19 as follows

$$b_{A,u}^v = \frac{1}{f_u^v \dim E_u} \sum_{\substack{x\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_n \\ (x\underline{D},\lambda)\in\mathfrak{C}_A}} \zeta^A(\tilde{T}_x 1_{\underline{D}\lambda}[D])\mathrm{tr}(\tilde{T}_{x\underline{D}}1_\lambda, E_u^v)^\spadesuit.$$

For each x, λ in the sum, we have $\zeta^A(\tilde{T}_x 1_{\underline{D}\lambda}[D]) \in \mathcal{A}$ (by definition) and

$$\mathrm{tr}(\tilde{T}_{x\underline{D}}1_\lambda, E_u^v)^\spadesuit \in \eta_0^{-1}\mathcal{A}$$

(by Lemma 34.20). Using this and (a), we see that $b_{A,u}^v \in \eta_0^{-1}\mathbf{Q}(v)$.

35. Functions on G^{0F}/U

35.1. In this section we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ of characteristic p and that G has a fixed $\mathbf{F}_q$-rational structure whose Frobenius map induces the identity map on $\mathbf{W}$ and on G/G^0 and the map $t \mapsto t^q$ on $\mathbf{T}$. For any algebraic variety V over $\mathbf{k}$ with a given $\mathbf{F}_q$-structure we denote by $F : V \to V$ the corresponding Frobenius map. We fix an integer $n \geq 1$ that divides $q - 1$.

35.2. In this section we fix an *épinglage* of G^0 compatible with the $\mathbf{F}_q$-structure. Thus, we fix B^*, T, U^* as in 28.5 such that $F(B^*) = B^*, F(T) = T$ and we fix for each $s \in \mathbf{I}$ an isomorphism $a \mapsto \xi_s(a)$ of $\mathbf{k}$ onto a subgroup of U^* such that $t\xi_s(a)t^{-1} = \xi_s(\alpha_s(t)a)$ for all $t \in T, a \in \mathbf{k}$ and $F(\xi_s(a)) = \xi_s(a^q)$ for all $a \in \mathbf{k}$; here $\alpha_s \in R^+$ and the corresponding coroot $\check{\alpha}_s$ satisfy $t = s(t)\check{\alpha}_s(\alpha_s(t))$ for all $t \in T$. (Clearly, such an épinglage exists and any two such épinglages are conjugate under the action of $(G^0/\mathcal{Z}_{G^0})^F$.) We identify $T = \mathbf{T}$ as in 28.5. For $s \in \mathbf{I}$ let ${}'U_s^*$ be the root subgroup of G^0 corresponding to the root α_s^{-1}. Define $y \in {}'U_s^* - \{1\}$, $\dot{s} \in N_{G^0}T$ by $\dot{s} = \xi_s(1)y\xi_s(1)$; then $\dot{s}$ is a representative of s in $(N_{G^0}T)^F$. For $s \in \mathbf{I} \cup \{1\}$ we define $\check{\alpha}_s : \mathbf{k}^* \to T$ as above if $s \in \mathbf{I}$ and to be 1, if $s = 1$.

Following Tits, we can define uniquely for each $w \in \mathbf{W}$ a representative $\dot{w}$ in $N_{G^0}T$ by the requirements:

(i) if $s \in \mathbf{I}$, then $\dot{s}$ is as above;

(ii) if $w, w' \in \mathbf{W}$ and $l(ww') = l(w) + l(w')$, then $(ww')^{\dot{}} = \dot{w}\dot{w}'$;

(iii) $\dot{1} = 1$.

We have $F(\dot{w}) = \dot{w}$ for any $w \in \mathbf{W}$. Let $\check{T}^F = \mathrm{Hom}(T^F, \bar{\mathbf{Q}}_l^*)$. If $\mathcal{L} \in \mathfrak{s}_{q-1}$ (see 31.2), then $\mathcal{L}^{\otimes(q-1)} \cong \mathcal{L}$; hence $F^*\mathcal{L} \cong \mathcal{L}$ and there is a unique isomorphism $\tau_0 : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ that induces the identity on the stalk at 1. Hence we can form the characteristic function $\chi_{\mathcal{L},\tau_0} : T^F \to \bar{\mathbf{Q}}_l^*$ (a group homomorphism). Let $\lambda \in \underline{\mathfrak{s}}_{q-1}$ be the isomorphism class of $\mathcal{L}$. We set $\theta_F^\lambda = \chi_{\mathcal{L},\tau_0} \in \check{T}^F$. Now $\lambda \mapsto \theta_F^\lambda$ is a bijection $\underline{\mathfrak{s}}_{q-1} \xrightarrow{\sim} \check{T}^F$.

If $\theta \in \check{T}^F, \alpha \in R$ we write $\theta\check{\alpha}$ for the composition $\mathbf{F}_q^* \xrightarrow{\check{\alpha}|_{\mathbf{F}_q^*}} T^F \xrightarrow{\theta} \bar{\mathbf{Q}}_l^*$. For $\alpha \in R, b \in \mathbf{W}^\bullet$ we write $b\check{\alpha}$ for the coroot $a \mapsto b(\check{\alpha}(a))$.

35.3. In this section we write U instead of U^{*F}. Let $\mathfrak{U}$ be as in 34.12. Let $\mathfrak{T}$ be the vector space of all functions $G^{0F} \to \mathfrak{U}$ that are constant on U, U double cosets. Now $\mathfrak{T}$ has a basis $\{k_\nu; \nu \in (N_{G^0}T)^F\}$ where k_ν is 1 on $U\nu U$ and is 0 on $G^{0F} - U\nu U$. We regard $\mathfrak{T}$ as an associative $\mathfrak{U}$-algebra in which the product of h_1, h_2 is given by

$$(h_1h_2)(g) = |U|^{-1} \sum_{g_1,g_2 \in G^{0F}; g_1g_2=g} h_1(g_1)h_2(g_2).$$

This algebra has $1 = k_1$. As in [Y], we have

$$k_{\dot{s}}k_{\dot{s}} = qk_{\check{\alpha}_s(-1)} + \sum_{a \in \mathbf{F}_q^*} k_{\dot{s}}k_{\check{\alpha}_s(a)},$$

$$k_\nu k_{\nu'} = k_{\nu\nu'},$$

where $s \in \mathbf{I}$ and ν, ν' represent w, w' in $\mathbf{W}$ such that $l(ww') = l(w) + l(w')$. For any $\lambda \in \underline{\mathfrak{s}}_{q-1}$ we set

$$1_\lambda = |T^F|^{-1} \sum_{t \in T^F} \theta_F^\lambda(t)k_t \in \mathfrak{T}.$$

Then $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'}1_\lambda$ for $\lambda, \lambda' \in \underline{\mathfrak{s}}_{q-1}$. Let $\sqrt{-1}$ be a fixed element of $\mathfrak{U}^*$ whose square is -1; we set $\sqrt{1} = 1$. For $s \in \mathbf{I}$ we set

$$T_s = k_{\dot{s}} \sum_{\lambda \in \underline{\mathfrak{s}}_{q-1}} \sqrt{\theta_F^\lambda(\check{\alpha}_s(-1))}1_\lambda \in \mathfrak{T}.$$

More generally, for $w \in \mathbf{W}$ we set

$$T_w = \sum_{\lambda \in \underline{\mathfrak{s}}_{q-1}} \prod_{\alpha \in R^+, w^{-1}\alpha \in R^-} \sqrt{\theta_F^\lambda(\check{\alpha}(-1))} k_{\dot{w}} 1_\lambda \in \mathfrak{T}.$$

From the definitions we see that $T_w T_{w'} = T_{ww'}$ for $w, w' \in \mathbf{W}$ with $l(ww') = l(w) + l(w')$ and $T_w 1_\lambda = 1_{w\lambda} T_w$. We have

$$\begin{aligned}
T_s T_s &= \sum_\lambda \theta_F^\lambda(\check{\alpha}_s(-1)) k_{\dot{s}} k_{\dot{s}} 1_\lambda \\
&= q \sum_\lambda \theta_F^\lambda(\check{\alpha}_s(-1)) k_{\check{\alpha}_s(-1)} 1_\lambda + \sum_\lambda \theta_F^\lambda(\check{\alpha}_s(-1)) \sum_{a \in \mathbf{F}_q^*} k_{\dot{s}} k_{\check{\alpha}_s(a)} 1_\lambda \\
&= q + \sum_\lambda \theta_F^\lambda(\check{\alpha}_s(-1)) \sum_{a \in \mathbf{F}_q^*} \theta_F^\lambda(\check{\alpha}_s(a)) k_{\dot{s}} 1_\lambda \\
&= q + (q-1) \sum_{\substack{\lambda \\ \theta_F^\lambda \check{\alpha}_s = 1}} \theta_F^\lambda(\check{\alpha}_s(-1)) k_{\dot{s}} 1_\lambda = q + (q-1) \sum_{\substack{\lambda \\ \theta_F^\lambda \check{\alpha}_s = 1}} \sqrt{\theta_F^\lambda(\check{\alpha}_s(-1))} k_{\dot{s}} 1_\lambda.
\end{aligned}$$

Thus,

$$T_s T_s = q + (q-1) \sum_{\lambda \in \underline{\mathfrak{s}}_{q-1}; \theta_F^\lambda \check{\alpha}_s = 1} T_s 1_\lambda.$$

35.4. We fix a square root $\sqrt{p}$ of p in $\mathfrak{U}$. For any $e \in \mathbf{Z}$ we set $\sqrt{p^e} = (\sqrt{p})^e$. In particular, $\sqrt{q} \in \mathfrak{U}$ is defined. Then $H_{q-1}^{\sqrt{q}}$ is defined as in 34.12. From 35.3 we see that the elements $T_w, 1_\lambda$ of $\mathfrak{T}$ define a $\mathfrak{U}$-algebra homomorphism $H_{q-1}^{\sqrt{q}} \to \mathfrak{T}$. (We use the fact that if $\lambda \in \underline{\mathfrak{s}}_{q-1}$ and $s \in \mathbf{I}$, then $\theta_F^\lambda \check{\alpha}_s = 1$ if and only if $s \in \mathbf{W}_\lambda$.) This is an isomorphism: from the definitions we see that $\{T_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_{q-1}\}$ is a $\mathfrak{U}$-basis of $\mathfrak{T}$; we then use 31.2(a).

35.5. Let $\mathfrak{P}$ be the vector space of all functions $f : G^{0F} \to \mathfrak{U}$ such that $f(xu) = f(x)$ for all $x \in G^{0F}, u \in U$. For $g \in G^F$, $g' \in (N_G B^* \cap N_G T)^F$ such that $gG^0 = g'G^0$ we define a linear map $\rho_{g,g'} : \mathfrak{P} \to \mathfrak{P}$ by $(\rho_{g,g'} f)(x) = f(g^{-1} x g')$. Then $g_0 : f \mapsto \rho_{g_0,1} f$ makes $\mathfrak{P}$ into a G^0F-module.

Any element $c \in \mathfrak{T}$ defines a linear map $\mathfrak{P} \to \mathfrak{P}$:

$$f \mapsto cf, (cf)(x) = |U|^{-1} \sum_{x' \in G^{0F}} c(x') f(xx').$$

Clearly, $c \mapsto [f \mapsto cf]$ is an isomorphism $\mathfrak{T} \xrightarrow{\sim} \mathrm{End}_{G^{0F}}(\mathfrak{P})$ and a left $\mathfrak{T}$-module structure on $\mathfrak{P}$. For $\nu \in (N_{G^0} T)^F, \lambda \in \underline{\mathfrak{s}}_{q-1}, f \in \mathfrak{P}$ we have

$$(k_\nu f)(x) = |U|^{-1} \sum_{x' \in U\nu U} f(xx'),$$

$$(1_\lambda f)(x) = |T^F|^{-1} \sum_{t \in T^F} \theta_F^\lambda(t) f(xt).$$

In the remainder of this section we fix a connected component D of G and an element $d \in (N_D B^* \cap N_D T)^F$. Let $s \in \mathbf{I} \cup \{1\}, \lambda \in \underline{\mathfrak{s}}_{q-1}, f \in \mathfrak{P}$. For $x \in G^{0F}$ we have:

(a) $$(T_s 1_{\underline{D}\lambda} f)(x) = \sqrt{\theta_F^{\underline{D}\lambda}(\check{\alpha}_s(-1))} |B^{*F}|^{-1} \sum_{x' \in U\dot{s}U, t \in T^F} \theta_F^{\underline{D}\lambda}(t) f(xx't).$$

Now let $\mathbf{s}=(s_1,s_2,\dots,s_r)$ be a sequence in $\mathbf{I}\cup\{1\}$, $\lambda\in\underline{\mathfrak{s}}_{q-1}$,

$$T_{\mathbf{s}}1_{\underline{D}\lambda}=T_{s_1}T_{s_2}\dots T_{s_r}1_{\underline{D}\lambda}=T_{s_1}1_{s_2\dots s_r\underline{D}\lambda}T_{s_2}1_{s_3\dots s_r\underline{D}\lambda}\dots T_{s_r}1_{\underline{D}\lambda}.$$

Applying (a) r times gives (for $f\in\mathfrak{P}, x\in G^{0F}, g\in D^F$):

$$(T_{\mathbf{s}}1_{\underline{D}\lambda}\rho_{g,d}f)(x)=a_{\underline{D}\lambda,F,\mathbf{s}}|B^{*F}|^{-r}$$
$$\times\sum_{\substack{g_1,g_2,\dots,g_r\\ t_1,t_2,\dots,t_r\\ g_i\in U\dot{s}_iU\\ t_i\in T^F}}\theta_F^{s_2\dots s_r\underline{D}\lambda}(t_1)\theta_F^{s_3\dots s_r\underline{D}\lambda}(t_2)\dots\theta_F^{\underline{D}\lambda}(t_r)f(g^{-1}xg_1t_1g_2t_2\dots g_rt_rd)$$
$$=a_{\underline{D}\lambda,F,\mathbf{s}}|B^{*F}|^{-r}$$

(a′)
$$\times\sum_{\substack{g_1,g_2,\dots,g_r\\ t_1,t_2,\dots,t_r\\ g_i\in U\dot{s}_iU\\ t_i\in T^F}}\theta_F^{\underline{D}\lambda}((s_r\dots s_2t_1)(s_r\dots s_3t_2)\dots(t_r))f(g^{-1}xg_1t_1g_2t_2\dots g_rt_rd)$$

where

$$a_{\underline{D}\lambda,F,\mathbf{s}}$$
$$=\sqrt{\theta_F^{\underline{D}\lambda}(s_r\dots s_2(\check{\alpha}_{s_1}(-1)))}\sqrt{\theta_F^{\underline{D}\lambda}(s_r\dots s_3(\check{\alpha}_{s_2}(-1)))}\dots\sqrt{\theta_F^{\underline{D}\lambda}(\check{\alpha}_{s_r}(-1))}.$$

Let $J\subset\mathbf{I}$ and let $Q\in\mathcal{P}_J$ be such that $F(Q)=Q$. Define a linear map $\mathrm{pr}_Q:\mathfrak{P}\to\mathfrak{P}$ by $(\mathrm{pr}_Qf)(x)=f(x)$ if $x\in G^{0F}, xQ_{J,B^*}x^{-1}=Q$ and $(\mathrm{pr}_Qf)(x)=f(x)$ if $x\in G^{0F}, xQ_{J,B^*}x^{-1}\neq Q$. We compute the trace of the linear map

(b)
$$T_{\mathbf{s}}1_{\underline{D}\lambda}\rho_{g,d}\mathrm{pr}_Q:\mathfrak{P}\to\mathfrak{P}$$

using the $\mathfrak{U}$-basis of $\mathfrak{P}$ consisting of the characteristic functions of the various right U-cosets in G^{0F}. Using (a′) and the definitions we see that this trace equals

(c)
$$\frac{a_{\underline{D}\lambda,F,\mathbf{s}}}{|B^{*F}|^r|U|}\sum_{\substack{g_1,g_2,\dots,g_r,\\ t_1,t_2,\dots,t_r;\\ g_i\in U\dot{s}_iU,t_i\in T^F,x\in D^F;\\ g^{-1}xg_1t_1g_2t_2\dots g_rt_rd\in xU;\\ xQ_{J,B^*}x^{-1}=Q}}\theta_F^{\lambda}(d^{-1}(s_r\dots s_2t_1)(s_r\dots s_3t_2)\dots(t_r)d).$$

35.6. Let $\mathcal{L},\lambda,\tau_0:F^*\mathcal{L}\xrightarrow{\sim}\mathcal{L}$ be as in 35.2. Let $\mathbf{s}=(s_1,s_2,\dots,s_r)$ be a sequence in $\mathbf{I}\cup\{1\}$ such that $s_1s_2\dots s_r\underline{D}\lambda=\lambda$. In 28.7 we have defined a local system $\tilde{\mathcal{L}}$ on $Z^{\mathbf{s}}_{\emptyset,J,D}$ in terms of d and a representative for $s_1s_2\dots s_r$ in $N_{G^0}T$. We now take as a representative for $s_1s_2\dots s_r$ the element $\dot{s}_1\dot{s}_2\dots\dot{s}_r$ with $\dot{s}_i$ as in 35.2. We reformulate the definition of $\tilde{\mathcal{L}}$ as follows (see also the proof of 28.10). Define $\gamma:Z'^{\mathbf{s}}\to Z^{\mathbf{s}}_{\emptyset,J,D}$ by the formula 28.10(a). Define $\psi:Z'^{\mathbf{s}}\to T$ by

$$(h_0,h_1,\dots,h_r,g)\mapsto d^{-1}(\dot{s}_1\dot{s}_2\dots\dot{s}_r)^{-1}n_1n_2\dots n_rn_0$$

with $n_i\in N_{G^0}T$ given by $h_{i-1}^{-1}h_i\in U^*n_iU^*$ and $n_0\in N_GB^*\cap N_GT$ given by $h_r^{-1}gh_0\in U^*n_0$. Then $\tilde{\mathcal{L}}$ is the local system on $Z^{\mathbf{s}}_{\emptyset,J,D}$ such that $\gamma^*\tilde{\mathcal{L}}=\psi^*\mathcal{L}$. Note that γ,ψ are naturally defined over $\mathbf{F}_q$. Let $\tilde{\tau}:F^*\psi^*\mathcal{L}\xrightarrow{\sim}\psi^*\mathcal{L}$ be the isomorphism induced by τ_0. There is a well-defined isomorphism $\tau:F^*\tilde{\mathcal{L}}\xrightarrow{\sim}\tilde{\mathcal{L}}$ such that τ induces via γ the isomorphism $\tilde{\tau}$.

Let $\pi_{\mathbf{s}}:Z^{\mathbf{s}}_{\emptyset,J,D}\to Z_{J,D}$ be as in 28.12(a) and let $K=K^{\mathbf{s},\mathcal{L}}_{J,D}=\pi_{\mathbf{s}!}\tilde{\mathcal{L}}\in\mathcal{D}(Z_{J,D})$. Now $\pi_{\mathbf{s}}$ is naturally defined over $\mathbf{F}_q$. Hence $\tau:F^*\tilde{\mathcal{L}}\xrightarrow{\sim}\tilde{\mathcal{L}}$ induces an isomorphism

$\omega : F^*K \xrightarrow{\sim} K$ and $\chi_{K,\omega} : Z^F_{J,D} \to \bar{\mathbf{Q}}_l$ is defined. Let $\xi = (Q, Q', gU_Q) \in Z^F_{J,D}$ (we can take $g \in D^F$). Using the definitions and the Grothendieck trace formula we have

$$\chi_{K,\omega}(\xi) = \sum_i (-1)^i \mathrm{tr}(\tau^*, H^i_c(\pi_{\mathbf{s}}^{-1}(\xi), \tilde{\mathcal{L}})) = \sum_{\eta \in \pi_{\mathbf{s}}^{-1}(\xi)^F} \mathrm{tr}(\tau, \tilde{\mathcal{L}}_\eta)$$

where $\tilde{\mathcal{L}}_\eta$ is the stalk of $\tilde{\mathcal{L}}$ at η. Now γ induces a map $Z'^{\mathbf{s}F} \to {Z^{\mathbf{s}}_{\emptyset,J,D}}^F$ all of whose fibres have cardinal $|B^{*F}|^{r+1}|U^F_{J,B^*}|$. It follows that

$$|B^{*F}|^{r+1}|U^F_{J,B^*}|\chi_{K,\omega}(\xi) = \sum_{\substack{\tilde{\eta} \in Z'^{\mathbf{s}F} \\ \pi_{\mathbf{s}}\gamma(\tilde{\eta}) = \xi}} \mathrm{tr}(\tilde{\tau}, (\psi^*\mathcal{L})_{\tilde{\eta}}) = \sum_{\substack{\tilde{\eta} \in Z'^{\mathbf{s}F} \\ \pi_{\mathbf{s}}\gamma(\tilde{\eta}) = \xi}} \mathrm{tr}(\tau_0, \mathcal{L}_{\psi(\tilde{\eta})}).$$

Let

$$\Xi = \{(h_0, h_1, \dots, h_r) \in (G^{0F})^{r+1}; h_{i-1}^{-1}h_i \in B^*\dot{s}_iB^* (i \in [1, r]), \\ h_r^{-1}gh_0 \in N_GB^*, h_0Q_{J,B^*}h_0^{-1} = Q\}.$$

Then $\{\tilde{\eta} \in Z'^{\mathbf{s}F}; \pi_{\mathbf{s}}\gamma(\tilde{\eta}) = \xi\}$ may be identified with $\Xi \times (gU^F_Q)$. Hence

$$\chi_{K,\omega}(\xi) = |B^{*F}|^{-r-1} \sum_{(h_0,h_1,\dots,h_r) \in \Xi} \chi_{\mathcal{L},\tau_0}(\delta(h_0, h_1, \dots, h_r))$$

with $\delta : \Xi \to T$ given by $(h_0, h_1, \dots, h_r) \mapsto d^{-1}(\dot{s}_1\dot{s}_2 \dots \dot{s}_r)^{-1}n_1n_2 \dots n_rn_0$ (n_i, n_0 as above). For any $(h_0, h_1, \dots, h_r) \in \Xi$ we define $g_i \in U\dot{s}_iU$, $t_i \in T^F (i \in [1, r])$ by $h_{i-1}^{-1}h_i = g_it_i$. We also set $h_0 = x$. Then Ξ becomes

$$\{(x, g_1, g_2, \dots, g_r, t_1, t_2, \dots, t_r); x \in G^{0F}, g_i \in U\dot{s}_iU, t_i \in T^F; \\ gx = xg_1t_1g_2t_2 \dots g_rt_rtdu \text{ for some } t \in T^F, u \in U; xQ_{J,B^*}x^{-1} = Q\}$$

and $\delta : \Xi \to T$ becomes

$$(x, g_1, g_2, \dots, g_r, t_1, t_2, \dots, t_r) \mapsto d^{-1}(\dot{s}_1\dot{s}_2 \dots \dot{s}_r)^{-1}ds_1t_1\dot{s}_2t_2 \dots \dot{s}_rt_rtd \\ = d^{-1}(s_r \dots s_2t_1)(s_r \dots s_3t_2) \dots (s_rt_{r-1})(t_rt)d.$$

We make the change of variable $t_rt \mapsto t_r, t \mapsto t$. Then t no longer appears explicitly; it only introduces a factor $|T^F|$. We see that

$$\chi_{K,\omega}(\xi) \\ = \frac{|T^F|}{|B^{*F}|^{r+1}} \sum_{\substack{g_1,g_2,\dots,g_r, \\ t_1,t_2,\dots,t_r, \\ g_i \in U\dot{s}_iU, t_i \in T^F, x \in D^F, \\ g^{-1}xg_1t_1g_2t_2\dots g_rt_rd \in xU \\ xQ_{J,B^*}x^{-1} = Q}} \theta^\lambda_F(d^{-1}(s_r \dots s_2t_1)(s_r \dots s_3t_2) \dots (t_r)d).$$

This is the same (up to the factor $a_{\underline{D}\lambda,F,\mathbf{s}}$) as the expression 35.5(c). Using the equality between 35.5(c) and the trace of 35.5(b), we see that

(a) $$\chi_{K,\omega}(Q, gQg^{-1}, gU_Q) = a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} \mathrm{tr}(T_{\mathbf{s}} 1_{\underline{D}\lambda} \rho_{g,d} \mathrm{pr}_Q, \mathfrak{P}).$$

35.7. In the setup of 35.6, let $\mathcal{J}_{\mathbf{s}} \subset \mathcal{J}^0 \subset [1,r]$ be as in 28.9. For any $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$ let $\mathbf{s}_{\mathcal{J}}$ be as in 28.9. We have $\mathbf{s}_\emptyset = \mathbf{s}$. Define $\gamma_{\mathcal{J}} : Z'^{\mathbf{s}_{\mathcal{J}}} \to Z^{\mathbf{s}_j}_{\emptyset,J,D}$ by the formula 28.10(a). Define $\psi_{\mathcal{J}} : Z'^{\mathbf{s}_{\mathcal{J}}} \to T$ as in 28.10 (with $\dot{s}$ as in 35.2). Let $\tilde{\mathcal{L}}_{\mathcal{J}}$ be the local system on $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$ such that $\gamma^*_{\mathcal{J}}\tilde{\mathcal{L}}_{\mathcal{J}} = \psi^*_{\mathcal{J}}\mathcal{L}$. Let $\tilde{\tau}^{\mathcal{J}} : F^*\psi^*_{\mathcal{J}}\mathcal{L} \xrightarrow{\sim} \psi^*_{\mathcal{J}}\mathcal{L}$ be the isomorphism induced by τ_0. There is a well-defined isomorphism $\tau^{\mathcal{J}} : F^*\tilde{\mathcal{L}}_{\mathcal{J}} \xrightarrow{\sim} \tilde{\mathcal{L}}_{\mathcal{J}}$ such that $\tau^{\mathcal{J}}$ induces via $\gamma_{\mathcal{J}}$ the isomorphism $\tilde{\tau}^{\mathcal{J}}$. Note that for $\mathcal{J} = \emptyset$, $\tilde{\mathcal{L}}_{\mathcal{J}}, \tilde{\tau}^{\mathcal{J}}, \tau^{\mathcal{J}}$ reduce to $\tilde{\mathcal{L}}, \tilde{\tau}, \tau$ of 35.6.

Let $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ be as in 28.9. This is a smooth irreducible variety, naturally defined over $\mathbf{F}_q$ and $Z^{\mathbf{s}_\emptyset}_{\emptyset,J,D}$ is open dense in $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$. Hence $\bar{\mathcal{L}} = IC(\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}, \tilde{\mathcal{L}}_\emptyset)$ is defined and the isomorphism $\tau^\emptyset : F^*\tilde{\mathcal{L}}_\emptyset \xrightarrow{\sim} \tilde{\mathcal{L}}_\emptyset$ of local systems on $Z^{\mathbf{s}_\emptyset}_{\emptyset,J,D}$ extends canonically to an isomorphism $\tau^\sharp : F^*\bar{\mathcal{L}} \xrightarrow{\sim} \bar{\mathcal{L}}$ (of constructible sheaves, see 28.10). Restricting this isomorphism to the subset $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$ of $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ (with $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$) we obtain an isomorphism $\tau^{\sharp\mathcal{J}} : F^*\mathcal{E}^{\mathcal{J}} \xrightarrow{\sim} \mathcal{E}^{\mathcal{J}}$ where $\mathcal{E}^{\mathcal{J}} = \bar{\mathcal{L}}|_{Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}}$. From 28.10 we see that $\mathcal{E}^{\mathcal{J}}$ is a local system isomorphic to $\tilde{\mathcal{L}}_{\mathcal{J}}$.

Lemma 35.8. *The isomorphisms $\tau^{\sharp\mathcal{J}}, \tau^{\mathcal{J}}$ correspond to each other under some/any isomorphism of local systems $\mathcal{E}^{\mathcal{J}} \xrightarrow{\sim} \tilde{\mathcal{L}}_{\mathcal{J}}$ on $Z^{\mathbf{s}_{\mathcal{J}}}_{\emptyset,J,D}$.*

The proof is a refinement of that of Lemma 28.10. Note that $Z'^{\mathbf{s}_\emptyset}$ is an open dense subvariety of the smooth irreducible variety $\tilde{Z}^{\mathbf{s}}$ (as in 28.10). Hence $IC(\tilde{Z}^{\mathbf{s}}, \psi^*_\emptyset\mathcal{L})$ is defined and the isomorphism $\tilde{\tau}^\emptyset : F^*\psi^*_\emptyset\mathcal{L} \xrightarrow{\sim} \psi^*_\emptyset\mathcal{L}$ of local systems on $Z'^{\mathbf{s}_\emptyset}$ extends canonically to an isomorphism

$$\tilde{\tau}^\sharp : F^*IC(\tilde{Z}^{\mathbf{s}}, \psi^*_\emptyset\mathcal{L}) \xrightarrow{\sim} IC(\tilde{Z}^{\mathbf{s}}, \psi^*_\emptyset\mathcal{L})$$

which may be identified with the isomorphism induced by $\tau^\sharp$ through the fibration $\tilde{Z}^{\mathbf{s}} \to \bar{Z}^{\mathbf{s}}_{\emptyset,J,D}$ (see 28.10(a)) whose fibres are smooth and connected. Let $\tilde{\mathcal{E}}^{\mathcal{J}} = IC(\tilde{Z}^{\mathbf{s}}, \psi^*_\emptyset\mathcal{L})|_{Z'^{\mathbf{s}_{\mathcal{J}}}}$ and let $\tilde{\tau}^{\sharp\mathcal{J}} : F^*\tilde{\mathcal{E}}^{\mathcal{J}} \xrightarrow{\sim} \tilde{\mathcal{E}}^{\mathcal{J}}$ be the isomorphism induced by $\tilde{\tau}^\sharp$ by restriction. It suffices to prove the following statement:

The isomorphisms $\tilde{\tau}^{\sharp\mathcal{J}}, \tilde{\tau}^{\mathcal{J}}$ correspond to each other under some/any isomorphism of local systems $\tilde{\mathcal{E}}^{\mathcal{J}} \xrightarrow{\sim} \psi^*_{\mathcal{J}}\mathcal{L}$.

Let $\mathcal{L}' = (\underline{D}^{-1})^*\mathcal{L} \in \mathfrak{s}(\mathbf{T}) = \mathfrak{s}(T)$. Let $\tau'_0 : F^*\mathcal{L}' \xrightarrow{\sim} \mathcal{L}'$ be the unique isomorphism which induces the identity map on the stalk of $\mathcal{L}'$ at 1. Let $'\bar{Z}^{\mathbf{s}}, 'Z^{\mathbf{s}_{\mathcal{J}}}$ be as in 28.10. Define $'\psi_{\mathcal{J}} : 'Z^{\mathbf{s}_{\mathcal{J}}} \to T$ as in 28.10 (with $\dot{s}$ as in 35.2). Then $'\psi_{\mathcal{J}}$ is compatible with the natural $\mathbf{F}_q$-structures on $'Z^{\mathbf{s}_{\mathcal{J}}}, T$; hence $\tau'_0 : F^*\mathcal{L}' \xrightarrow{\sim} \mathcal{L}'$ induces an isomorphism of local systems $'\tau^{\mathcal{J}} : F^*{'\psi^*_{\mathcal{J}}}\mathcal{L}' \xrightarrow{\sim} {'\psi^*_{\mathcal{J}}}\mathcal{L}'$. Let

$$'\tau^\sharp : F^*IC('\bar{Z}^{\mathbf{s}}, {'\psi^*_\emptyset}\mathcal{L}') \xrightarrow{\sim} IC('\bar{Z}^{\mathbf{s}}, {'\psi^*_\emptyset}\mathcal{L}')$$

be the isomorphism induced by $'\tau^\emptyset : F^*{'\psi^*_\emptyset}\mathcal{L}' \xrightarrow{\sim} {'\psi^*_\emptyset}\mathcal{L}'$. Let

$$'\mathcal{E}^{\mathcal{J}} = IC('\bar{Z}^{\mathbf{s}}, {'\psi^*_\emptyset}\mathcal{L}')|_{'Z^{\mathbf{s}_{\mathcal{J}}}}$$

and let $'\tau^{\sharp\mathcal{J}} : F^*{'\mathcal{E}^{\mathcal{J}}} \xrightarrow{\sim} {'\mathcal{E}^{\mathcal{J}}}$ be the isomorphism induced by $'\tau^\sharp$ by restriction. As in the proof of 28.10, it suffices to prove the following statement:

(a) *The isomorphisms $'\tau^{\sharp\mathcal{J}}, {'\tau^{\mathcal{J}}}$ correspond to each other under some/any isomorphism of local systems $'\mathcal{E}^{\mathcal{J}} \xrightarrow{\sim} {'\psi^*_{\mathcal{J}}}\mathcal{L}'$.*

Assume that (a) is known in the case where $|\mathcal{J}| = 1$. We now consider a general $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$. We prove (a) by induction on $|\mathcal{J}|$. If $\mathcal{J} = \emptyset$, (a) is obvious. Assume that $\mathcal{J} \neq \emptyset$. Let $j \in \mathcal{J}$ be the largest number in $\mathcal{J}$. Let $\mathcal{J}' = \mathcal{J} - \{j\}$. Let $'\bar{Z}^{\mathbf{s}_{\mathcal{J}'}}$ be the variety analogous to $'\bar{Z}^{\mathbf{s}}$ when $\mathbf{s}$ is replaced by $\mathbf{s}_{\mathcal{J}'}$ (this is the same as the closure of $'Z^{\mathbf{s}_{\mathcal{J}'}}$ in $'\bar{Z}^{\mathbf{s}}$). Let

$$''\tau^\sharp : F^*IC('\bar{Z}^{\mathbf{s}_{\mathcal{J}'}}, '\psi^*_{\mathcal{J}'}\mathcal{L}') \xrightarrow{\sim} IC('\bar{Z}^{\mathbf{s}_{\mathcal{J}'}}, '\psi^*_{\mathcal{J}'}\mathcal{L}')$$

be the isomorphism induced by $'\tau^{\mathcal{J}'} : F^*{'\psi^*_{\mathcal{J}'}}\mathcal{L}' \xrightarrow{\sim} {'\psi^*_{\mathcal{J}'}}\mathcal{L}'$ and let $''\tau^{\sharp\mathcal{J}}$ be its restriction to $'Z^{\mathbf{s}_{\mathcal{J}}}$. By the induction hypothesis, the isomorphisms $'\tau^{\sharp\mathcal{J}'}, '\tau^{\mathcal{J}'}$ correspond to each other under some/any isomorphism of local systems $'\mathcal{E}^{\mathcal{J}'} \xrightarrow{\sim} {'\psi^*_{\mathcal{J}'}}\mathcal{L}'$. It follows that the isomorphisms $''\tau^{\sharp\mathcal{J}}, '\tau^{\sharp\mathcal{J}}$ correspond to each other under some/any isomorphism of local systems

$$IC('\bar{Z}^{\mathbf{s}_{\mathcal{J}'}}, '\psi^*_{\mathcal{J}'}\mathcal{L}')|_{'Z^{\mathbf{s}_{\mathcal{J}}}} \xrightarrow{\sim} {'\mathcal{E}^{\mathcal{J}}}.$$

By our assumption (applied to $\mathbf{s}_{\mathcal{J}'}$ instead of $\mathbf{s}$), the isomophisms $''\tau^{\sharp\mathcal{J}}, '\tau^{\mathcal{J}}$ correspond to each other under some/any isomorphism of local systems

$$IC('\bar{Z}^{\mathbf{s}_{\mathcal{J}'}}, '\psi^*_{\mathcal{J}'}\mathcal{L}')|_{'Z^{\mathbf{s}_{\mathcal{J}}}} \xrightarrow{\sim} {'\psi^*_{\mathcal{J}}}\mathcal{L}'.$$

It follows that the isomorphisms $'\tau^{\sharp\mathcal{J}}, '\tau^{\mathcal{J}}$ correspond to each other under some/any isomorphism of local systems $'\mathcal{E}^{\mathcal{J}} \xrightarrow{\sim} {'\psi^*_{\mathcal{J}}}\mathcal{L}'$. Thus, (a) holds for $\mathcal{J}$.

We now consider the remaining case, where $\mathcal{J}$ consists of a single element j. Note that $j \in \mathcal{J}_{\mathbf{s}}$ where $\mathcal{J}_{\mathbf{s}}$ is defined in terms of $D, \mathcal{L}$ or equivalently, in terms of $G^0, \mathcal{L}'$. The statement (a) involves only G^0. Hence to prove it, we may assume that $G = G^0$. We write Z', Z'' instead of $'Z^{\mathbf{s}_\emptyset}, 'Z^{\mathbf{s}_{\{j\}}}$ and we set $Z = Z' \cup Z''$, a subvariety of $'\bar{Z}^{\mathbf{s}}$. We write f', f'' instead of $'\psi_\emptyset, '\psi_{\{j\}}$. Let $b = \dot{s}_{j+1}\dot{s}_{j+2}\dots\dot{s}_r$.

We have $\mathcal{L}' = \kappa^*\mathcal{L}_1$ where $\kappa \in \mathrm{Hom}(T, \mathbf{k}^*)$ and $\mathcal{L}_1 \in \mathfrak{s}(\mathbf{k}^*)$. Since $\mathcal{L}'^{\otimes(q-1)} \cong \bar{\mathbf{Q}}_l$, we may assume that $\mathcal{L}_1^{\otimes(q-1)} \cong \bar{\mathbf{Q}}_l$. Hence there is a unique isomorphism $\tau_1 : F^*\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_1$ which induces the identity on the stalk of $\mathcal{L}_1$ at 1. Then $\tau'_0 : F^*\mathcal{L}' \xrightarrow{\sim} \mathcal{L}'$ is induced by τ_1, via κ^*.

We continue the proof assuming that G has simply connected derived subgroup. Let $\check{\beta}$ be as in the proof of 28.10. As in that proof, we may assume that $\langle\check{\beta}, \kappa\rangle = 0$. Hence there exists a homomorphism of algebraic groups $\chi : B^*\dot{s}_jB^* \cup B^* \to \mathbf{k}^*$ such that $\chi(t) = \kappa(b^{-1}tb)$ for all $t \in T$. Let $\tilde{f} : Z \to \mathbf{k}^*$ be as in 28.10. If $y_j \in B^*\dot{s}_jB^*$, we have $\tilde{f}(y_1, \dots, y_r) = \kappa(f'(y_1, \dots, y_r))$; if $y_j \in B^*$ we have $\tilde{f}(y_1, \dots, y_r) = \kappa(f''(y_1, \dots, y_r))$. (See 28.10.) We show that

$$\chi(F(g)) = \chi(g)^q \text{ for all } g \in B^*\dot{s}_jB^* \cup B^*.$$

We may assume that $g \in T$. Then

$$\chi(F(g)) = \kappa(b^{-1}F(g)b) = \kappa(F(b^{-1}gb)) = \kappa((b^{-1}gb)^q) = \kappa(b^{-1}gb)^q = \chi(g)^q,$$

as required. It follows that for any $(y_1, \dots, y_r) \in Z$ we have

(b) $$\tilde{f}(F(y_1), \dots, F(y_r)) = (\tilde{f}(y_1, \dots, y_r))^q.$$

Let $\mathcal{F} = \tilde{f}^*(\mathcal{L}_1)$. We have canonically $\mathcal{F}_{Z'} = f'^*\mathcal{L}', \mathcal{F}|_{Z''} = f''^*\mathcal{L}'$. From (b) we see that the isomorphism $\tau_1 : F^*\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_1$ induces via $\tilde{f}^*$ an isomorphism $F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ and this gives rise upon restriction to Z', Z'' to the isomorphisms $'\tau^\emptyset, '\tau^{\{j\}}$. Since $\mathcal{F}$ is a local system on Z such that $\mathcal{F}|_{Z'} = f'^*\mathcal{L}'$, we have canonically $\mathcal{F} = IC(Z, f'^*\mathcal{L}')$. Hence (a) holds in our case.

We now drop the assumption that G has simply connected derived subgroup. Let $\pi : \hat{G} \to G$ be a surjective homomorphism of connected reductive groups whose kernel is a central torus in $\hat{G}$ and such that $\hat{G}$ has simply connected derived subgroup. We may assume that $\hat{G}$ and π are defined over $\mathbf{F}_q$. Then π restricts to a surjective homomorphism $\hat{G}^F \to G^F$. Since the set of épinglages of $G, \hat{G}$ are in natural bijection, the épinglage of G fixed in 35.2 gives rise to an épinglage of $\hat{G}$ (the associated Borel subgroup and maximal torus are $\hat{B}^* = \pi^{-1}(B^*)$, $\hat{T} = \pi^{-1}(T)$ and the analogue of $\xi_s : \mathbf{k} \to U^*$ is the obvious one). For $s \in \mathbf{I}$ let $\tilde{s} \in (N_{\hat{G}}\hat{T})^F$ be associate to this épinglage of $\hat{G}$ in the same way as $\dot{s}$ was associated to the épinglage of G. We set $\tilde{1} = 1 \in \tilde{G}$. Define $\hat{Z}, \hat{Z}', \hat{Z}'', \hat{f}' : \hat{Z}' \to \hat{T}, \hat{f}'' : \hat{Z}'' \to \hat{T}$ in terms of $\hat{G}, \hat{B}^*, \hat{T}, \tilde{s}_i$ in the same way as $Z, Z', Z'', f' : Z' \to T, f'' : Z'' \to T$ are defined in terms of $G, B^*, T, \dot{s}_i$. Let $\hat{\mathcal{L}}' \in \mathfrak{s}(\hat{T})$ be the inverse image of $\mathcal{L}'$ under $\pi : \hat{T} \to T$. There is a unique isomorphism $F^*\hat{\mathcal{L}}' \cong \hat{\mathcal{L}}'$ which induces the identity map on the stalk of $\hat{\mathcal{L}}'$ at 1. This induces isomorphisms

$$'\hat{\tau}^{\emptyset} : F^*\hat{f}'^*\hat{\mathcal{L}}' \to \hat{f}'^*\hat{\mathcal{L}}', {'\hat{\tau}^{\{j\}}} : F^*\hat{f}''^*\hat{\mathcal{L}}' \to \hat{f}''^*\hat{\mathcal{L}}'.$$

Now $'\hat{\tau}^{\emptyset}$ induces an isomorphism $F^*IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}') \xrightarrow{\sim} IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')$ and this restricts to an isomorphism of local systems

$$'\hat{\tau}^{\sharp\{j\}} : F^*IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')|_{\hat{Z}''} \xrightarrow{\sim} IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')|_{\hat{Z}''}.$$

By an earlier part of the proof, the isomorphisms $'\hat{\tau}^{\sharp\{j\}}, {'\hat{\tau}^{\{j\}}}$ correspond to each other under some/any isomorphism of local systems $IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')|_{\hat{Z}''} \xrightarrow{\sim} \hat{f}''^*\hat{\mathcal{L}}'$. Now the map $\hat{Z} \to Z$ induced by π is a fibration with smooth connected fibres and $IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')$ is canonically the inverse image under this map of $IC(Z, f'^*\mathcal{L}')$. Hence $IC(\hat{Z}, \hat{f}'^*\hat{\mathcal{L}}')|_{\hat{Z}''}$ is the inverse image under $\hat{Z}'' \to Z''$ of $IC(Z, f'^*\mathcal{L}')|_{Z''}$. Similarly, $\hat{f}''^*\hat{\mathcal{L}}'$ is the inverse image under $\hat{Z}'' \to Z''$ of $f''^*\mathcal{L}'$. Also, $'\hat{\tau}^{\sharp\{j\}}, {'\hat{\tau}^{\{j\}}}$ are obtained from $'\tau^{\sharp\{j\}}, {'\tau^{\{j\}}}$ by inverse image under $\hat{Z}'' \to Z''$. Therefore the required statement about $'\tau^{\sharp\{j\}}, {'\tau^{\{j\}}}$ is a consequence of the analogous, already known statement about $'\hat{\tau}^{\sharp\{j\}}, {'\hat{\tau}^{\{j\}}}$ (we use the faithfulness of the inverse image functor under the fibration $\hat{Z}'' \to Z''$ with smooth connected fibres). The lemma is proved.

35.9. We preserve the setup of 35.7. Let $\bar{\pi}_{\mathbf{s}} : \bar{Z}^{\mathbf{s}}_{\emptyset,J,D} \to Z_{J,D}$ be as in 28.12 and let $\bar{K} = \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D} = \bar{\pi}_{\mathbf{s}!}\bar{\mathcal{L}} \in \mathcal{D}(Z_{J,D})$ (see 28.12, 35.7). Now $\bar{\pi}_{\mathbf{s}}$ is naturally defined over $\mathbf{F}_q$. Hence the isomorphism $\tau^\sharp : F^*\bar{\mathcal{L}} \xrightarrow{\sim} \bar{\mathcal{L}}$ in 35.7 induces an isomorphism $\bar{\omega} : F^*\bar{K} \to \bar{K}$ and $\chi_{\bar{K},\bar{\omega}} : Z^F_{J,D} \to \bar{\mathbf{Q}}_l$ is defined.

Proposition 35.10. *Let $(Q, Q', gU_Q) \in Z^F_{J,D}$ (we take $g \in D^F$). Let $C^{\mathbf{s}}_{\underline{D}\lambda} \in H_{q-1}$ be as in 31.5. We have*

$$\chi_{\bar{K},\bar{\omega}}(Q, Q', gU_Q) = a^{-1}_{\underline{D}\lambda,F,\mathbf{s}}\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{g,d}\mathrm{pr}_Q, \mathfrak{P}).$$

Consider the partition $\bar{Z}^{\mathbf{s}}_{\emptyset,J,D} = \sqcup_{\mathcal{J}\subset\mathcal{J}^0} Z^{\mathbf{s}_\mathcal{J}}_{\emptyset,J,D}$ (see 28.9). For each $\mathcal{J} \subset \mathcal{J}^0$ let $\pi_\mathcal{J} : Z^{\mathbf{s}_\mathcal{J}}_{\emptyset,J,D} \to Z_{J,D}$ be the restriction of $\bar{p}_{\mathbf{s}}$, let $K_\mathcal{J} = \pi_{\mathcal{J}!}(\bar{\mathcal{L}}|_{Z^{\mathbf{s}_\mathcal{J}}_{\emptyset,J,D}})$ and let $\omega_\mathcal{J} : F^*K_\mathcal{J} \to K_\mathcal{J}$ be the isomorphism induced by $\tau^\sharp$. Using the additivity property of characteristic functions, we see that $\chi_{\bar{K},\bar{\omega}} = \sum_{\mathcal{J}\subset\mathcal{J}^0} \chi_{K_\mathcal{J},\omega_\mathcal{J}}$. By 28.10, we have $K_\mathcal{J} = 0$ unless $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$. By Lemma 35.8, if $\mathcal{J} \subset \mathcal{J}_{\mathbf{s}}$, $\chi_{K_\mathcal{J},\omega_\mathcal{J}}$ is just like $\chi_{K,\omega}$ in 35.6, with $\mathbf{s}$ replaced by $\mathbf{s}_\mathcal{J}$. Hence 35.6(a) can be applied and it yields

$$\chi_{K_\mathcal{J},\omega_\mathcal{J}}(Q, Q', gU_Q) = a^{-1}_{\underline{D}\lambda,F,\mathbf{s}_\mathcal{J}}\mathrm{tr}(T_{\mathbf{s}_\mathcal{J}}1_{\underline{D}\lambda}\rho_{g,d}\mathrm{pr}_Q, \mathfrak{P}).$$

We will verify below that

(a) $$a_{\underline{D}\lambda,F,\mathbf{s}} = a_{\underline{D}\lambda,F,\mathbf{s}_\mathcal{J}} \text{ for any } \mathcal{J} \subset \mathcal{J}_{\mathbf{s}}.$$

We see that

$$\chi_{\bar{K},\bar{\omega}}(Q, gQg^{-1}, gU_Q) = a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} \sum_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}} \mathrm{tr}(T_{\mathbf{s}_{\mathcal{J}}} 1_{\underline{D}\lambda}\rho_{g,d}\mathrm{pr}_Q, \mathfrak{P})$$

is as desired. (We have used the identity $C^{\mathbf{s}}_{\underline{D}\lambda} = \sum_{\mathcal{J}\subset\mathcal{J}_{\mathbf{s}}} T_{\mathbf{s}_{\mathcal{J}}} 1_{\underline{D}\lambda}$ which follows easily from the definitions.)

We now verify (a). We may assume that $\mathcal{J}$ has a single element $j \in \mathcal{J}_{\mathbf{s}}$ (the general case can then be obtained by iteration). We have $\mathbf{s}_{\mathcal{J}} = (s'_1, s'_2, \dots, s'_r)$ where $s'_i = s_i$ for $i \neq j$ and $s'_j = 1$. It suffices to show that, for any $k \in [1, r]$ we have

(b) $$\sqrt{\theta^{\underline{D}\lambda}_F(s_r \dots s_{k+1}\check{\alpha}_{s_k}(-1))} = \sqrt{\theta^{\underline{D}\lambda}_F(s'_r \dots s'_{k+1}\check{\alpha}_{s'_k}(-1))}.$$

If $s_k = 1$, then $s'_k = 1$ and both sides are 1. If $s_k \in \mathbf{I}$ and $k > j$, then $s_{k'} = s'_{k'}$ for $k' \geq k$ and (b) is obvious. If $k = j$, then (b) states that

$$\sqrt{\theta^{\underline{D}\lambda}_F(s_r \dots s_{j+1}\check{\alpha}_{s_j}(-1))} = 1;$$

this follows from

(c) $$\theta^{\underline{D}\lambda}_F(s_r \dots s_{j+1}\check{\alpha}_{s_j}(a)) = 1 \text{ for all } a \in \mathbf{F}^*_q$$

which comes from $j \in \mathcal{J}_{\mathbf{s}}$. Assume now that $s_k \in \mathbf{I}$ and $k < j$. Then for some $m \in \mathbf{Z}$ we have

$$s_j s_{j-1} \dots s_{k+1}\check{\alpha}_{s_k} = (s_{j-1} \dots s_{k+1}\check{\alpha}_{s_k})\check{\alpha}^m_{s_j}.$$

Applying $s_r s_{r-1} \dots s_{j+1}$ to both sides gives

$$s_r s_{r-1} \dots s_{k+1}\check{\alpha}_{s_k} = (s_r s_{r-1} \dots s_{j+1} s_{j-1} \dots s_{k+1}\check{\alpha}_{s_k})(s_r s_{r-1} \dots s_{j+1}\check{\alpha}_{s_j})^m.$$

Applying $\theta^{\underline{D}\lambda}_F$ to both sides and using (c) gives

$$\theta^{\underline{D}\lambda}_F(s_r s_{r-1} \dots s_{k+1}\check{\alpha}_{s_k}(a)) = \theta^{\underline{D}\lambda}_F(s_r s_{r-1} \dots s_{j+1} s_{j-1} \dots s_{k+1}\check{\alpha}_{s_k}(a))$$

for all $a \in \mathbf{F}^*_q$. We set $a = -1$ and we see that (b) holds for this k. This proves (a). The proposition is proved.

35.11. Let $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}_n$. Then $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}_{q-1}$. Let $\tau_0 : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ be as in 35.2; let $\tau'_0 : F^*\mathcal{L}' \xrightarrow{\sim} \mathcal{L}'$ be the analogous isomorphism. Let $\lambda \in \underline{\mathfrak{s}}_n$ (resp. $\lambda' \in \underline{\mathfrak{s}}_n$) be the isomorphism class of $\mathcal{L}$ (resp. $\mathcal{L}'$). We have $\lambda \in \underline{\mathfrak{s}}_{q-1}$, $\lambda' \in \underline{\mathfrak{s}}_{q-1}$. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$, $\mathbf{s}' = (s'_1, s'_2, \dots, s'_{r'})$ be sequences in $\mathbf{I}$ such that $s_1 s_2 \dots s_r \underline{D}\lambda = \lambda$, $s'_1 s'_2 \dots s'_{r'}\underline{D}\lambda' = \lambda'$. Let $\bar{K} = \bar{K}^{\mathbf{s},\mathcal{L}}_{J,D}, \bar{\omega} : F^*\bar{K} \xrightarrow{\sim} \bar{K}$ be as in 35.9. Let $\bar{K}' = \bar{K}^{\mathbf{s}',\mathcal{L}'}_{J,D}$ and let $\bar{\omega}' : F^*\bar{K}' \xrightarrow{\sim} \bar{K}'$ be the analogue of $\bar{\omega}$ (defined in terms of τ'_0). Then $\chi_{\bar{K},\bar{\omega}} : Z^F_{J,D} \to \bar{\mathbf{Q}}_l$, $\chi_{\bar{K}',\bar{\omega}'} : Z^F_{J,D} \to \bar{\mathbf{Q}}_l$ are defined. Let

$$E = \sum_{(Q,Q',gU_Q)\in Z^F_{J,D}} \chi_{\bar{K},\bar{\omega}}(Q, Q', gU_Q)\chi_{\bar{K}',\bar{\omega}'}(Q, Q', gU_Q).$$

Using Proposition 35.10 for $\bar{K}$ and for $\bar{K}'$, we see that

$$E = |U^F_{J,B^*}|^{-1} a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} a^{-1}_{\underline{D}\lambda',F,\mathbf{s}'} \sum_{Q\in\mathcal{P}^F_J} \sum_{g\in D^F} \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{g,d}\mathrm{pr}_Q, \mathfrak{P})\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\rho_{g,d}\mathrm{pr}_Q, \mathfrak{P}).$$

Setting $g = dg_0$ where $g_0 \in G^{0F}$ we have $\rho_{g,d} = \rho_{d,d}\rho_{g_0,1}$. Hence

$$E = |U^F_{J,B^*}|^{-1} a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} a^{-1}_{\underline{D}\lambda',F,\mathbf{s}'} \sum_{\substack{Q\in\mathcal{P}^F_J\\ g_0\in G^{0F}}} \operatorname{tr}(\operatorname{pr}_Q C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{d,d}\rho_{g_0,1},\mathfrak{P})$$
$$\times \operatorname{tr}(\operatorname{pr}_Q C^{\mathbf{s}'}_{\underline{D}\lambda'}\rho_{d,d}\rho_{g_0,1},\mathfrak{P}) = |G^{0F}||U^F_{J,B^*}|^{-1} a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} a^{-1}_{\underline{D}\lambda',F,\mathbf{s}'} \operatorname{tr}(XY,\mathfrak{P}\otimes\mathfrak{P})$$

where

$$X = \sum_{Q\in\mathcal{P}^F_J} (\operatorname{pr}_Q \otimes \operatorname{pr}_Q)((C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{d,d}) \otimes (C^{\mathbf{s}'}_{\underline{D}\lambda'}\rho_{d,d})) : \mathfrak{P}\otimes\mathfrak{P} \to \mathfrak{P}\otimes\mathfrak{P},$$

$$Y = |G^{0F}|^{-1} \sum_{g_0\in G^{0F}} (\rho_{g_0,1}\otimes\rho_{g_0,1}) : \mathfrak{P}\otimes\mathfrak{P} \to \mathfrak{P}\otimes\mathfrak{P}.$$

We have $XY = YX$ and Y is a projection of $\mathfrak{P}\otimes\mathfrak{P}$ onto the subspace $(\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}}$ of G^{0F}-invariants for the G^{0F}-action in which g_0 acts as $\rho_{g_0,1}\otimes\rho_{g_0,1}$. Hence X maps $(\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}}$ into itself and

$$E = |G^{0F}||U^F_{J,B^*}|^{-1} a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} a^{-1}_{\underline{D}\lambda',F,\mathbf{s}'} \operatorname{tr}(X,(\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}}).$$

The non-singular symmetric bilinear form

$$(,) : \mathfrak{P}\times\mathfrak{P} \to \mathfrak{U}, (f,f') = \sum_{x\in G^{0F}} f(x)f'(x)$$

gives rise to an isomorphism $\mathfrak{P}\otimes\mathfrak{P} \xrightarrow{\sim} \operatorname{End}(\mathfrak{P})$, $f'\otimes f'' \mapsto [f\mapsto (f,f')f'']$. Under this isomorphism, X corresponds to a linear map $X' : \operatorname{End}(\mathfrak{P}) \to \operatorname{End}(\mathfrak{P})$,

$$\phi \mapsto \sum_{Q\in\mathcal{P}^F_J} (\operatorname{pr}_Q C^{\mathbf{s}'}_{\underline{D}\lambda'}\rho_{d,d})\phi({}^t(\operatorname{pr}_Q C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{d,d}))$$

where ^{t} denotes taking transpose with respect to $(,)$. We have $(\rho_{g_0,1}f,\rho_{g_0,1}f') = (f,f')$ for all $f,f'\in\mathfrak{P}, g_0\in G^{0F}$. Hence $\mathfrak{P}\otimes\mathfrak{P} \xrightarrow{\sim} \operatorname{End}(\mathfrak{P})$ restricts to an isomorphism $(\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}} \xrightarrow{\sim} \operatorname{End}_{G^{0F}}(\mathfrak{P})$ under which

$$X : (\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}} \to (\mathfrak{P}\otimes\mathfrak{P})^{G^{0F}}$$

corresponds to the restriction of X' to $\operatorname{End}_{G^{0F}}(\mathfrak{P})$. It follows that

$$E = |G^{0F}||U^F_{J,B^*}|^{-1} a^{-1}_{\underline{D}\lambda,F,\mathbf{s}} a^{-1}_{\underline{D}\lambda',F,\mathbf{s}'} \operatorname{tr}(X',\operatorname{End}_{G^{0F}}(\mathfrak{P})).$$

From the definitions we have

$${}^t\rho_{d,d} = \rho_{d^{-1},d^{-1}} : \mathfrak{P}\to\mathfrak{P},$$
$${}^t(\operatorname{pr}_Q) = \operatorname{pr}_Q \text{ for all } Q\in\mathcal{P}_J,$$
$${}^t(k_\nu) = k_{\nu^{-1}} : \mathfrak{P}\to\mathfrak{P} \text{ for all } \nu\in N_{G^0}T.$$

In particular,

$${}^tk_{\dot{s}} = k_{\dot{s}^{-1}} = k_{\dot{s}}k_{\check{\alpha}_s(-1)} : \mathfrak{P}\to\mathfrak{P} \text{ for all } s\in\mathbf{I}.$$

We also see that

$${}^t(1_{\lambda_1}) = 1_{\lambda_1^{-1}} : \mathfrak{P}\to\mathfrak{P} \text{ for all } \lambda_1\in\underline{\mathfrak{s}}_{q-1}.$$

For $s\in\mathbf{I}, \lambda_1\in\underline{\mathfrak{s}}_{q-1}$ we have

$${}^t(T_s1_{\lambda_1}) = \sqrt{\theta^{\lambda_1}_F(\check{\alpha}_s(-1))}1_{\lambda_1^{-1}}k_{\dot{s}}k_{\check{\alpha}_s(-1)} = \sqrt{\theta^{\lambda_1}_F(\check{\alpha}_s(-1))}\theta^{\lambda_1}_F(\check{\alpha}_s(-1))1_{\lambda_1^{-1}}k_{\dot{s}}$$
$$= \theta^{\lambda_1}_F(\check{\alpha}_s(-1))T_s1_{s\lambda_1^{-1}} : \mathfrak{P}\to\mathfrak{P},$$

hence ${}^t(C^s_{\lambda_1}) = \theta_F^{\lambda_1}(\check\alpha_s(-1))C^s_{s\lambda_1^{-1}} : \mathfrak{P} \to \mathfrak{P}$. It follows that

$$ {}^t(C^{\mathbf{s}}_{\underline{D}\lambda}) = \delta_0 C^{\tilde{\mathbf{s}}}_{s_1 s_2 \dots s_r(\underline{D}\lambda)^{-1}} = \delta_0 C^{\tilde{\mathbf{s}}}_{\lambda^{-1}} $$

where $\tilde{\mathbf{s}} = (s_r, \dots, s_2, s_1)$ and

$$ \delta_0 = (\theta_F^{\underline{D}\lambda}(\check\alpha_{s_r}(-1)))(\theta_F^{\underline{D}\lambda}(s_r\check\alpha_{s_{r-1}}(-1)))\dots(\theta_F^{\underline{D}\lambda}(s_r\dots s_2\check\alpha_{s_1}(-1))) = a^2_{\underline{D}\lambda,F,\mathbf{s}}. $$

We see that $X'(\phi) = \delta_0 \sum_{Q\in\mathcal{P}_J^F} \mathrm{pr}_Q C^{\mathbf{s}'}_{\underline{D}\lambda'}\rho_{d,d}\phi\rho_{d^{-1},d^{-1}}C^{\tilde{\mathbf{s}}}_{\lambda^{-1}}\mathrm{pr}_Q$ for $\phi \in \mathrm{End}(\mathfrak{P})$.

35.12. For $w \in \mathbf{W}$ we set

$$ t_{d,w} = ((\epsilon_D(w))\dot{})^{-1} d\dot{w} d^{-1} \in T^F. $$

We show that, for $\lambda_1 \in \underline{\mathfrak{s}}_{q-1}$, we have

$$ \text{(a)} \qquad \rho_{d,d}T_w 1_{\lambda_1}\rho_{d^{-1},d^{-1}} = \theta_F^{\lambda_1}(d^{-1}t_{d,w}d)^{-1}T_{\epsilon_D(w)}1_{\underline{D}\lambda_1} : \mathfrak{P} \to \mathfrak{P}. $$

(We regard $T_w 1_{\lambda_1}$ as an element of $\mathrm{End}_{G^{0F}}\mathfrak{P} = \mathfrak{T} = H^{\sqrt{q}}_{q-1}$; see 35.4, 35.5.) We first show that for $\nu \in (N_{G^0}T)^F$ we have

$$ \rho_{d,d}k_\nu\rho_{d^{-1},d^{-1}} = k_{d\nu d^{-1}} : \mathfrak{P} \to \mathfrak{P}. $$

Indeed, for $f \in \mathfrak{P}, x \in G^{0F}$, we have

$$ \begin{aligned} &(\rho_{d,d}k_\nu\rho_{d^{-1},d^{-1}}f)(x) = (k_\nu\rho_{d^{-1},d^{-1}}f)(d^{-1}xd) \\ &= |U|^{-1}\sum_{x'\in U\nu U}(\rho_{d^{-1},d^{-1}}f)(d^{-1}xdx') = |U|^{-1}\sum_{x'\in U\nu U} f(dd^{-1}xdx'd^{-1}) \\ &= |U|^{-1}\sum_{x''\in Ud\nu d^{-1}U} f(xx'') = (k_{d\nu d^{-1}}f)(x), \end{aligned} $$

as required. Using the equality

$$ \begin{aligned} T_w 1_{\lambda_1} &= \prod_{\alpha\in R^+, w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\lambda_1}(\check\alpha(-1))}k_{\dot w}1_{\lambda_1} \\ &= |T^F|^{-1}\prod_{\alpha\in R^+, w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\lambda_1}(\check\alpha(-1))}\sum_{t\in T^F}\theta_F^{\lambda_1}(t)k_{\dot w t}, \end{aligned} $$

we have

$$\begin{aligned}
&\rho_{d,d}T_w 1_{\lambda_1}\rho_{d^{-1},d^{-1}}\\
&=|T^F|^{-1}\prod_{\alpha\in R^+,w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\lambda_1}(\check\alpha(-1))}\sum_{t\in T^F}\theta_F^{\lambda_1}(t)\rho_{d,d}k_{\dot wt}\rho_{d^{-1},d^{-1}}\\
&=|T^F|^{-1}\prod_{\alpha\in R^+,w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\lambda_1}(\check\alpha(-1))}\sum_{t\in T^F}\theta_F^{\lambda_1}(t)k_{d\dot wtd^{-1}}\\
&=|T^F|^{-1}\prod_{\alpha\in R^+,w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\lambda_1}(\check\alpha(-1))}\sum_{t\in T^F}\theta_F^{\lambda_1}(t)k_{d\dot wd^{-1}}k_{dtd^{-1}}\\
&=\prod_{\alpha\in R^+,w^{-1}\alpha\in R^-}\sqrt{\theta_F^{\underline{D}\lambda_1}(\underline{D}\check\alpha(-1))}k_{(\epsilon_D(w))^\cdot}k_{t_{d,w}}1_{\underline{D}\lambda_1}\\
&=\theta_F^{\underline{D}\lambda_1}(t_{d,w}^{-1})\prod_{\alpha'\in R^+,\epsilon_D(w)^{-1}\alpha'\in R^-}\sqrt{\theta_F^{\underline{D}\lambda_1}(\check\alpha'(-1))}k_{(\epsilon_D(w))^\cdot}1_{\underline{D}\lambda_1}\\
&=\theta_F^{\lambda_1}(d^{-1}t_{d,w}d)^{-1}T_{\epsilon_D(w)}1_{\underline{D}\lambda_1}
\end{aligned}$$

and (a) is proved.

35.13. We write $\Theta_n^J : H_n \to H_n$ instead of $\Theta^J : H_n \to H_n$; see 32.22. Let $\Theta_{n,q}^J : H_n^{\sqrt q} \to H_n^{\sqrt q}$ be the linear map defined by Θ_n^J by extension of scalars. Replacing n by $q-1$ we obtain a linear map $\Theta_{q-1,q}^J : H_{q-1}^{\sqrt q} \to H_{q-1}^{\sqrt q}$. We identify $\mathrm{End}_{G^{0F}}\mathfrak{P} = \mathfrak{T} = H_{q-1}^{\sqrt q}$; see 35.4, 35.5. We show:

(a) $$\Theta_{q-1,q}^J(\phi)=\sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q\phi\mathrm{pr}_Q:\mathfrak{P}\to\mathfrak{P}$$

for any $\phi\in\mathrm{End}_{G^{0F}}\mathfrak{P}$. First we show that for $\nu\in(N_{G^0}T)^F$ we have

(b) $$\sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q k_\nu\mathrm{pr}_Q=k_\nu \text{ if } \nu\in Q_{J,B^*};\ \sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q k_\nu\mathrm{pr}_Q=0 \text{ if } \nu\notin Q_{J,B^*}.$$

Let $f\in\mathfrak{P}, x\in G^{0F}$. We have

$$\begin{aligned}
&(\sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q k_\nu\mathrm{pr}_Q f)(x)=\sum_{\substack{Q\in\mathcal{P}^J;\\ xQ_{J,B^*}x^{-1}=Q}}(k_\nu\mathrm{pr}_Q f)(x)\\
&=|U|^{-1}\sum_{\substack{Q\in\mathcal{P}^J;\\ xQ_{J,B^*}x^{-1}=Q;\\ x'\in U\nu U}}(\mathrm{pr}_Q f)(xx')=|U|^{-1}\sum_{\substack{Q\in\mathcal{P}^J;\\ xQ_{J,B^*}x^{-1}=Q;\\ x'\in U\nu U;xx'Q_{J,B^*}x'^{-1}x^{-1}=Q}}f(xx')\\
&=|U|^{-1}\sum_{\substack{x'\in U\nu U;\\ x'Q_{J,B^*}x'^{-1}=Q_{J,B^*}}}f(xx')=|U|^{-1}\sum_{x'\in U\nu U\cap Q_{J,B^*}}f(xx'),
\end{aligned}$$

and (b) follows.

It suffices to prove (a) for $\phi=T_w1_\lambda$ where $w\in\mathbf{W},\lambda\in\underline{\mathfrak{s}}_{q-1}$. We have

$$\sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q T_w 1_\lambda\mathrm{pr}_Q=|T^F|^{-1}\prod_{\substack{\alpha\in R^+\\ w^{-1}\alpha\in R^-}}\sqrt{\theta_F^\lambda(\check\alpha(-1))}\sum_{t\in T^F}\theta_F^\lambda(t)\sum_{Q\in\mathcal{P}_J^F}\mathrm{pr}_Q k_{\dot wt}\mathrm{pr}_Q.$$

By (b) this is zero, unless $\dot{w} \in Q_{J,B^*}$ that is, $w \in W_J$; assuming that $w \in W_J$, this equals

$$|T^F|^{-1} \prod_{\alpha \in R^+, w^{-1}\alpha \in R^-} \sqrt{\theta_F^\lambda(\check{\alpha}(-1))} \sum_{t \in T^F} \theta_F^\lambda(t) k_{\dot{w}t} = T_w 1_\lambda;$$

(a) is proved.

35.14. Let m be an integer ≥ 1. Let $\mathbf{F}_{q^m}$ be the subfield of $\mathbf{k}$ consisting of q^m elements. Then $F^m : G \to G$ is the Frobenius map for an $\mathbf{F}_{q^m}$-rational structure on G. The épinglage in 35.2 relative to $\mathbf{F}_q$ can be also regarded as an épinglage relative to $\mathbf{F}_{q^m}$. In the setup of 35.11, define $\tau_0^{(m)} : F^{m*}\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, $\tau_0'^{(m)} : F^{m*}\mathcal{L}' \xrightarrow{\sim} \mathcal{L}'$ in terms of τ_0, τ_0' as in 33.5. We have $\chi_{\mathcal{L},\tau_0^{(m)}} = \theta_{F^m}^\lambda$, $\chi_{\mathcal{L}',\tau_0'^{(m)}} = \theta_{F^m}^{\lambda'}$ where

$$\theta_{F^m}^\lambda(t) = \theta_F^\lambda(t^{1+q+\cdots+q^{m-1}}),\ \theta_{F^m}^{\lambda'}(t) = \theta_F^{\lambda'}(t^{1+q+\cdots+q^{m-1}})$$

for all $t \in T^{F^m}$. Now $\bar{\omega}^{(m)} : F^{m*}\bar{K} \xrightarrow{\sim} \bar{K}$ (see 33.5) has the same relation to $\tau_0^{(m)}$ as $\bar{\omega} : F^*\bar{K} \xrightarrow{\sim} \bar{K}$ (see 35.9) to τ_0. Let $\bar{\omega}'^{(m)} : F^{m*}\bar{K}' \xrightarrow{\sim} \bar{K}'$ be the analogous isomorphism defined in terms of $\tau_0'^{(m)}$. Then $\chi_{\bar{K},\bar{\omega}^{(m)}} : Z_{J,D}^{F^m} \to \bar{\mathbf{Q}}_l$, $\chi_{\bar{K}',\bar{\omega}'^{(m)}} : Z_{J,D}^{F^m} \to \bar{\mathbf{Q}}_l$ are well defined.

We choose (as we may) $m_0 \in \mathbf{N}_{\mathbf{k}}^*$ such that $(-1)^{m_0} = 1$ (in $\mathbf{k}^*$) and $t_{d,w}^{m_0} = 1$ (in T) for all $w \in \mathbf{W}$.

We show that if $m \in m_0\mathbf{Z}, m \geq 1$, then $a_{\underline{D}\lambda,F^m,\mathbf{s}} = 1$. It suffices to show that $\theta_{F^m}^{\underline{D}\lambda}\check{\alpha}(-1) = 1$ for any $\alpha \in R$. Since $\underline{D}\lambda \in \underline{\mathfrak{s}}_{q-1}$ we have $\theta_{F^m}^{\underline{D}\lambda}(t) = \theta_F^\lambda(t^{1+q+\cdots+q^{m-1}})$. Hence it suffices to show that $\check{\alpha}((-1)^{1+q+\cdots+q^{m-1}}) = 1$ for any $\alpha \in R$ or that $(-1)^{1+q+\cdots+q^{m-1}} = 1$ (in $\mathbf{k}^*$). Since $(-1)^q = -1$ (in $\mathbf{k}^*$), it suffices to show that $(-1)^m = 1$. This follows from our assumption on m and m_0.

Similarly, we see that if $m \in m_0\mathbf{Z}, m \geq 1$, then $a_{\underline{D}\lambda',F^m,\mathbf{s}'} = 1$.

We show that, if $\tilde{\lambda} \in \underline{\mathfrak{s}}_n, w \in \mathbf{W}$ and $m \in m_0\mathbf{Z}, m \geq 1$, then $\theta_{F^m}^{\tilde{\lambda}}(d^{-1}t_{d,w}d) = 1$. Since $\tilde{\lambda} \in \underline{\mathfrak{s}}_{q-1}$, we have $\theta_{F^m}^{\tilde{\lambda}}(d^{-1}t_{d,w}d) = \theta_F^{\tilde{\lambda}}(d^{-1}t_{d,w}d)^{1+q+\cdots+q^{m-1}}$. Hence it suffices to show that $t_{d,w}^{1+q+\cdots+q^{m-1}} = 1$. Since $t_{d,w}^q = t_{d,w}$, it suffices to show that $t_{d,w}^m = 1$. This follows from our assumption on m_0, m.

We replace $\mathbf{F}_q$ in 35.1 by $\mathbf{F}_{q^{m_0}}$ which we rename as $\mathbf{F}_q$. The results above can be reformulated as follows:

(a) *If* $m \in \mathbf{Z}, m \geq 1$, *then* $a_{\underline{D}\lambda,F^m,\mathbf{s}} = 1$, $a_{\underline{D}\lambda',F^m,\mathbf{s}'} = 1$.

(b) *If* $\tilde{\lambda} \in \underline{\mathfrak{s}}_n, w \in \mathbf{W}$ *and* $m \in \mathbf{Z}, m \geq 1$, *then* $\theta_{F^m}^{\tilde{\lambda}}(d^{-1}t_{d,w}d) = 1$.

Proposition 35.15. *Let* $\mathfrak{a} = \mathfrak{a}_D : H_n \to H_n$ *be as in 34.15. Define* $\Phi'' : H_n \to H_n$ *by* $h \mapsto \Theta_n^J(C_{\underline{D}\lambda'}^{\mathbf{s}'}\mathfrak{a}(h)C_{\lambda^{-1}}^{\tilde{\mathbf{s}}})$ *(an $\mathcal{A}$-linear map) and let* $\mu(G^0)$ *be as in 32.22. If* $m \in \mathbf{Z}, m \geq 1$, *then*

$$E_m = \sum_{(Q,Q',gU_Q) \in Z_{J,D}^{F^m}} \chi_{\bar{K},\bar{\omega}^{(m)}}(Q,Q',gU_Q)\chi_{\bar{K}',\bar{\omega}'^{(m)}}(Q,Q',gU_Q)$$

is obtained by substituting $v^2 = q^m$ *in* $v^{2l(w_J^0)}\mu(G^0)\mathrm{tr}(\Phi'', H_n)$, *which is a polynomial in* $\mathbf{N}[v^2]$.

By the arguments in 35.11–35.13 applied with F^m instead of F we see that

$$E_m = |G^{0F^m}||U_{J,B^*}^{F^m}|^{-1} a_{\underline{D}\lambda,F^m,\mathbf{s}} a_{\underline{D}\lambda',F^m,\mathbf{s}'}^{-1} \mathrm{tr}(X'_m, H_{q^m-1}^{\sqrt{q^m}})$$

where $X'_m(h) = \Theta^J_{q^m-1,q^m}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\xi'_m(h)C^{\tilde{\mathbf{s}}}_{\lambda^{-1}})$ for $h \in H^{\sqrt{q^m}}_{q^m-1}$ and $\xi'_m : H^{\sqrt{q^m}}_{q^m-1} \to H^{\sqrt{q^m}}_{q^m-1}$ is the linear map given by

$$T_w 1_{\tilde{\lambda}} \mapsto \theta^{\tilde{\lambda}}_{F^m}(d^{-1}t_{d,w}d)^{-1}T_{\epsilon_D(w)}1_{\underline{D}\tilde{\lambda}}$$

for $w \in \mathbf{W}, \tilde{\lambda} \in \underline{\mathfrak{s}}_{q^m-1}$. Clearly, $X'_m(T_w 1_{\tilde{\lambda}}) = 0$ unless $\tilde{\lambda} \in \underline{\mathfrak{s}}_n$ and, if this condition is satisfied, then $X'_m(T_w 1_{\tilde{\lambda}})$ is a linear combination of elements of the form $T_{w'}1_{\tilde{\lambda}'}$ with $\tilde{\lambda}' \in \underline{\mathfrak{s}}_n$. It follows that

$$\mathrm{tr}(X'_m, H^{\sqrt{q^m}}_{q^m-1}) = \mathrm{tr}(\Phi''_m, H^{\sqrt{q^m}}_n)$$

where $\Phi''_m(h) = \Theta^J_{n,q^m}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\xi''_m(h)C^{\tilde{\mathbf{s}}}_{\lambda^{-1}})$ and $\xi''_m : H^{\sqrt{q^m}}_n \to H^{\sqrt{q^m}}_n$ is the restriction of ξ'_m. We now use 35.14(a),(b). The proposition follows, except for the assertion "which is a polynomial in $\mathbf{N}[v^2]$". That assertion follows from 32.22(a) and the second equality in Corollary 32.23(a).

35.16. We now assume that $J = \mathbf{I}$. Let $\mathcal{I}_n$ be a set of representatives for the isomorphism classes of character sheaves contained in $\hat{D}^{\mathcal{L}}$ for some $\mathcal{L} \in \mathfrak{s}_n$. Then $\mathcal{I}_n$ is finite and we can find an integer $m_1 \geq 1$ such that $F^{m_1 *}A \cong A$ for any $A \in \mathcal{I}_n$. We replace $\mathbf{F}_q$ in 35.14 by $\mathbf{F}_{q^{m_1}}$ which we rename as $\mathbf{F}_q$. We now have $F^*A \cong A$ for any $A \in \mathcal{I}_n$. For $A \in \mathcal{I}_n$, the Verdier dual $\mathfrak{D}(A)$ is isomorphic to an object in $\mathcal{I}_n$. (See 28.18(a).) For each $A \in \mathcal{I}_n$ we choose isomorphisms $\kappa_A : F^*A \xrightarrow{\sim} A$, $\kappa'_A : F^*\mathfrak{D}(A) \xrightarrow{\sim} \mathfrak{D}(A)$ so that the following holds: if $\mathcal{O}$ is an open dense F-stable subset of $\mathrm{supp}(A) = \mathrm{supp}(\mathfrak{D}(A))$ on which $\mathcal{H}^{-e}(A), \mathcal{H}^{-e}(\mathfrak{D}(A))$ are local systems ($e = \dim \mathcal{O}$), then for any $y \in \mathcal{O}$ and any $m \geq 1$ such that $F^m(y) = y$,

(i) $\sqrt{q}^{m(e-\dim D)}\kappa^{(m)}_A : \mathcal{H}^{-e}(A)_y \to \mathcal{H}^{-e}(A)_y$ is of finite order;

(ii) $\sqrt{q}^{m(e-\dim D)}\kappa'^{(m)}_A : \mathcal{H}^{-e}(\mathfrak{D}(A))_y \to \mathcal{H}^{-e}(\mathfrak{D}(A))_y$ is of finite order;

(iii) the isomorphism $\mathcal{H}^{-e}\mathfrak{D}(A)_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}\mathfrak{D}(A)_y$ (that is, $\mathcal{H}^{-e}(A)\check{}_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}(A)\check{}_y$) induced by κ'_A is $q^{\dim D - e}$ times the transpose inverse of the isomorphism $\mathcal{H}^{-e}(A)_{F(y)} \xrightarrow{\sim} \mathcal{H}^{-e}(A)_y$ induced by κ_A.

Note that κ'_A is uniquely determined by κ_A and that (ii) follows from (i) and (iii).

Let $\mathcal{L}, \mathcal{L}', \mathbf{s}, \mathbf{s}', \bar{K}, \bar{K}', \bar{\omega}, \bar{\omega}'$ be as in 35.11. Since $\bar{K}, \bar{K}'$ are semisimple, we have canonically (for $i, i' \in \mathbf{Z}$):

(a) $$ {}^pH^i(\bar{K}) = \oplus_{A \in \mathcal{I}_n}(A \otimes V_{A,i,\mathbf{s},\mathcal{L}}),$$

(b) $$ {}^pH^{i'}(\bar{K}') = \oplus_{A' \in \mathcal{I}_n}(\mathfrak{D}(A') \otimes V'_{A',i',\mathbf{s}',\mathcal{L}'})$$

where $V_{A,i,\mathbf{s},\mathcal{L}}, V'_{A',i',\mathbf{s}',\mathcal{L}'}$ are finite dimensional $\bar{\mathbf{Q}}_l$-vector spaces endowed with endomorphisms

$$\psi_A : V_{A,i,\mathbf{s},\mathcal{L}} \to V_{A,i,\mathbf{s},\mathcal{L}},\ \psi'_{A'} : V'_{A',i',\mathbf{s}',\mathcal{L}'} \to V'_{A',i',\mathbf{s}',\mathcal{L}'}$$

such that under (a) (resp. (b)) the map $\oplus_A(\kappa_A \otimes \psi_A)$ (resp. $\oplus_{A'}(\kappa'_{A'} \otimes \psi'_{A'})$) corresponds to the isomorphism $F^*({}^pH^i(\bar{K})) \xrightarrow{\sim} {}^pH^i(\bar{K})$ (resp. $F^*({}^pH^{i'}(\bar{K}')) \xrightarrow{\sim} {}^pH^{i'}(\bar{K}')$) induced by $\bar{\omega}$ (resp. $\bar{\omega}'$).

In the remainder of this section we assume that D is clean (see 33.4(b)).

Proposition 35.17. *(a) With each $A \in \mathcal{I}_n$ one can associate $\mathrm{sgn}_A \in \{1, -1\}$ with the following property: if A is a direct summand of ${}^pH^i(\bar{K}^{\mathbf{s},\mathcal{L}}_D)$ where $\mathbf{s}, \mathcal{L}$ are as in 35.11, then $(-1)^{i+\dim G} = \mathrm{sgn}_A$.*

(b) With each $A \in \mathcal{I}_n$ one can attach an element $\xi_A \in \bar{\mathbf{Q}}_l^*$ such that the following hold: ξ_A is an algebraic number all of whose complex conjugates have absolute value 1; for any $\mathbf{s}, \mathcal{L}$ as in 35.11 and any $i \in \mathbf{Z}$, ψ_A is equal to $\xi_A \sqrt{q}^{i-\dim G}$ times a unipotent automorphism of $V_{A,i,\mathbf{s},\mathcal{L}}$; for any $\mathbf{s}', \mathcal{L}'$ as in 35.11 and any $i \in \mathbf{Z}$, ψ'_A is equal to $\xi_A^{-1} \sqrt{q}^{i-\dim G}$ times a unipotent automorphism of $V'_{A,i,\mathbf{s}',\mathcal{L}'}$.

From the definitions we see that, in the setup of 35.16, we have

$$\mathrm{tr}(\bar{\omega}, \mathcal{H}_g^j({}^pH^i(\bar{K}))) = \sum_{A \in \mathcal{I}_n} \mathrm{tr}(\kappa_A, \mathcal{H}_g^j A)\mathrm{tr}(\psi_A, V_{A,i,\mathbf{s},\mathcal{L}}),$$

$$\mathrm{tr}(\bar{\omega}', \mathcal{H}_g^{j'}({}^pH^{i'}(\bar{K}'))) = \sum_{A' \in \mathcal{I}_n} \mathrm{tr}(\kappa'_{A'}, \mathcal{H}_g^{j'} \mathfrak{D}(A'))\mathrm{tr}(\psi'_{A'}, V'_{A',i',\mathbf{s}',\mathcal{L}'}),$$

for all $g \in D^F$ and all i, j, i', j'. Taking alternating sums over i, j or i', j' and using

$$\chi_{\bar{K},\bar{\omega}}(g) = \sum_{i,j} (-1)^{i+j} \mathrm{tr}(\bar{\omega}, \mathcal{H}_g^j({}^pH^i(\bar{K}))),$$

$$\chi_{\bar{K}',\bar{\omega}'}(g) = \sum_{i',j'} (-1)^{i'+j'} \mathrm{tr}(\bar{\omega}', \mathcal{H}_g^{j'}({}^pH^{i'}(\bar{K}'))),$$

we obtain

$$\chi_{\bar{K},\bar{\omega}}(g) = \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \sum_i (-1)^i \mathrm{tr}(\psi_A, V_{A,i,\mathbf{s},\mathcal{L}}),$$

$$\text{(c)} \qquad \chi_{\bar{K}',\bar{\omega}'}(g) = \sum_{A' \in \mathcal{I}_n} \chi_{\mathfrak{D}(A'),\kappa'_{A'}}(g) \sum_{i'} (-1)^{i'} \mathrm{tr}(\psi'_{A'}, V'_{A',i',\mathbf{s}',\mathcal{L}'}).$$

Since $\bar{K}$ (resp. $\bar{K}'$) is pure of weight 0, we see that ${}^pH^i(\bar{K})$ (resp. ${}^pH^{i'}(\bar{K}')$) is pure of weight i (resp. i'). By our choice of κ_A (resp. κ'_A) we see that (A, κ_A) and $(\mathfrak{D}(A), \kappa'_A)$ are pure of weight $\dim G$ for $A \in \mathcal{I}_n$. Using 35.16(a),(b), we deduce that

(d) *$(V_{A,i,\mathbf{s},\mathcal{L}}, \psi_A)$ is pure of weight $i - \dim G$ and $(V'_{A,i',\mathbf{s}',\mathcal{L}'}, \psi'_A)$ is pure of weight $i' - \dim G$.*

Using (c) we have

$$|G^{0F}|^{-1} \sum_{g \in D^F} \chi_{\bar{K},\bar{\omega}}(g)\chi_{\bar{K}',\bar{\omega}'}(g) = \sum_{A,A' \in \mathcal{I}_n} |G^{0F}|^{-1} \sum_{g \in D^F} \chi_{A,\kappa_A}(g)\chi_{\mathfrak{D}(A'),\kappa'_{A'}}(g)$$

$$\text{(e)} \quad \times \sum_{i,i'} (-1)^{i+i'} \mathrm{tr}(\psi_A, V_{A,i,\mathbf{s},\mathcal{L}})\mathrm{tr}(\psi'_{A'}, V'_{A',i',\mathbf{s}',\mathcal{L}'}).$$

Using 24.18 (which is applicable by our cleanness assumption and by Corollary 31.15) we see that for any $A, A' \in \mathcal{I}_n$ we have

$$|G^{0F}|^{-1} \sum_{g \in D^F} \chi_{A,\kappa_A}(g)\chi_{\mathfrak{D}(A'),\kappa'_{A'}}(g) = \delta_{A,A'}.$$

Hence (e) becomes

$$|G^{0F}|^{-1} \sum_{g \in D^F} \chi_{\bar{K},\bar{\omega}}(g)\chi_{\bar{K}',\bar{\omega}'}(g) = \sum_{\substack{A \in \mathcal{I}_n \\ i,i'}} (-1)^{i+i'} \mathrm{tr}(\psi_A, V_{A,i,\mathbf{s},\mathcal{L}})\mathrm{tr}(\psi'_A, V'_{A,i',\mathbf{s}',\mathcal{L}'}).$$

This remains true if F is replaced by F^m, where $m\ge 1$. Thus we have

$$|G^{0F^m}|^{-1}\sum_{g\in D^{F^m}}\chi_{\bar K,\bar\omega^{(m)}}(g)\chi_{\bar K',\bar\omega'^{(m)}}(g)$$

$$=\sum_j(-1)^j\sum_{A\in\mathcal{I}_n}\sum_{i,i';i+i'=j}\mathrm{tr}(\psi_A^m,V_{A,i,\mathbf{s},\mathcal{L}})\mathrm{tr}({\psi'_A}^m,V'_{A,i',\mathbf{s}',\mathcal{L}'}) \tag{f}$$

with $\bar\omega^{(m)},\bar\omega'^{(m)}$ as in 35.14. Using 35.15, we may rewrite the previous equality as follows:

$$\sum_j(-1)^j\sum_f a_{j,f}^m=\Pi(q^m)$$

where $a_{j,f}$ are the eigenvalues of $\psi_A\otimes\psi'_A$ on $\oplus_{i,i';i+i'=j}V_{A,i,\mathbf{s},\mathcal{L}}\otimes V'_{A,i',\mathbf{s}',\mathcal{L}'}$ and Π is a polynomial with coefficients in $\mathbf{N}$. By (d), each $a_{j,f}$ is an algebraic number all of whose complex conjugates have absolute value squared equal to $q^{j-2\dim G}$. It follows that, for fixed j, the set $\{a_{j,f}\}$ is empty if j is odd and that each $a_{j,f}$ is equal to $q^{j/2-\dim G}$ if j is even. This implies that, for any $A\in\mathcal{I}_n$, $V_{A,i,\mathbf{s},\mathcal{L}}\otimes V'_{A,i',\mathbf{s}',\mathcal{L}'}$ is 0 if $i+i'$ is odd and, if $i+i'$ is even, any eigenvalue of ψ_A on $V_{A,i,\mathbf{s},\mathcal{L}}$ multiplied by any eigenvalue of ψ'_A on $V'_{A,i',\mathbf{s}',\mathcal{L}'}$ gives $q^{(i+i')/2-\dim G}$. Since for $A\in\mathcal{I}_n$ we have $V'_{A,i',\mathbf{s}',\mathcal{L}'}\ne 0$ for some $\mathbf{s}',\mathcal{L}'$ as in 35.11, we see that the parity of i such that $V_{A,i,\mathbf{s},\mathcal{L}}\ne 0$ for some $\mathbf{s},\mathcal{L}$ as in 35.11 is an invariant of A and that, for any eigenvalue ξ of ψ_A on $V_{A,i,\mathbf{s},\mathcal{L}}$, the product $\xi\sqrt{q}^{\dim G-i}$ is also an invariant of A. The proposition follows.

35.18. As in 34.15, let $\{E_u;u\in\mathcal{U}\}$ be a set of representatives for the isomorphism classes of simple modules for H_n^1. Let $m\ge 1$. Using 35.17, we can rewrite 35.17(f) as follows:

$$|G^{0F^m}|^{-1}\sum_{g\in D^{F^m}}\chi_{\bar K,\bar\omega^{(m)}}(g)\chi_{\bar K',\bar\omega'^{(m)}}(g)$$

$$=\sum_{A\in\mathcal{I}_n}\sum_{i,i'}(-1)^{i+i'}\dim V_{A,i,\mathbf{s},\mathcal{L}}\dim V'_{A,i',\mathbf{s}',\mathcal{L}'}q^{m(i-\dim G)/2}q^{m(i'-\dim G)/2}. \tag{a}$$

By Proposition 35.15 (with $J=\mathbf{I}$), the left-hand side of (a) is $\mathrm{tr}(\Phi'',H_n)|_{v=\sqrt{q^m}}$. We have

$$\sum_{A\in\mathcal{I}_n}(\sum_i(-v)^iv^{-\dim G}\dim V_{A,i,\mathbf{s},\mathcal{L}})(\sum_{i'}(-v)^{i'}v^{-\dim G}\dim V'_{A,i',\mathbf{s}',\mathcal{L}'})$$

$$=\mathrm{tr}(\Phi'',H_n)=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\tilde T_{\underline{D}},E_u^v)\mathrm{tr}(\tilde T_{\underline{D}}^{-1}C^{\tilde{\mathbf{s}}}_{\lambda^{-1}},E_u^v)$$

$$=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\tilde T_{\underline{D}},E_u^v)\mathrm{tr}((C^{\tilde{\mathbf{s}}}_{\lambda^{-1}})^\flat\tilde T_{\underline{D}},E_u^v)^\spadesuit$$

$$=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\tilde T_{\underline{D}},E_u^v)\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda^{-1}}\tilde T_{\underline{D}},E_u^v)^\spadesuit. \tag{b}$$

(The first equality in (b) comes from the fact that (a) holds for any integer $m\ge 1$. Here $\mathrm{tr}(\Phi'',H_n)$, as in 35.15 with $J=\mathbf{I}$, can be replaced by a sum of products of traces, see 34.14(a). This gives the second equality in (b) where the notation of 34.15 is used. The third equality in (b) comes from Lemma 34.17 since $\tilde T_{\underline{D}}^{-1}C^{\tilde{\mathbf{s}}}_{\lambda^{-1}}$ is an $\mathcal{A}$-linear combination of elements of the form $\tilde T_w1_{\lambda^{-1}},w\in\mathbf{W}^D$. The fourth equality in (b) follows from the definitions using $s_r\dots s_2s_1\lambda^{-1}=\underline{D}\lambda^{-1}$.) Using the definitions and 34.19(a),(c), we obtain for $A\in\mathcal{I}_n$,

$$\sum_i(-v)^i v^{-\dim G}\dim V_{A,i,\mathbf{s},\mathcal{L}}=(\sum_i(-v)^i v^{-\dim G}\dim V_{A,i,\mathbf{s},\mathcal{L}})^{\spadesuit}$$

$$\text{(b)}\qquad =(\zeta^A(C^{\mathbf{s}}_{\underline{D}\lambda}[D]))^{\spadesuit}=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}(b^v_{A,u})^{\spadesuit}\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit},$$

$$\sum_{i'}(-v)^{i'}v^{-\dim G}\dim V'_{A,i',\mathbf{s}',\mathcal{L}'}=\zeta^{\mathfrak{D}(A)}(C^{\mathbf{s}'}_{\underline{D}\lambda'}[D])$$

$$\text{(c)}\qquad =\sum_{u'\in\mathcal{U}^{\mathfrak{a}}}b^v_{\mathfrak{D}(A),u'}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\tilde{T}_{\underline{D}},E^v_{u'})=\sum_{u'\in\mathcal{U}^{\mathfrak{a}}}b^v_{A,u'}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'}).$$

Introducing (b),(c) in (a) we obtain

$$\sum_{A\in\mathcal{I}_n}(\sum_{u'\in\mathcal{U}^{\mathfrak{a}}}b^v_{A,u'}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'}))(\sum_{u\in\mathcal{U}^{\mathfrak{a}}}(b^v_{A,u})^{\spadesuit}\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit})$$
$$=\sum_{u\in\mathcal{U}^{\mathfrak{a}}}\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'}\tilde{T}_{\underline{D}},E^v_u)\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda^{-1}}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit},$$

that is,

$$\text{(d)}\qquad \sum_{u,u'\in\mathcal{U}^{\mathfrak{a}}}(\sum_{A\in\mathcal{I}_n}b^v_{A,u'}(b^v_{A,u})^{\spadesuit}-\delta_{u,u'})\mathrm{tr}(C^{\mathbf{s}'}_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'})\mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit}=0.$$

Recall that here $\mathbf{s}$ is assumed to satisfy $s_1s_2\dots s_r\underline{D}\lambda=\lambda$. The $\mathcal{A}$-submodule of H_n^D spanned by the elements $C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}$ with $\mathbf{s}$ as above (and λ fixed) is just $1_\lambda H_n\tilde{T}_{uD}1_\lambda$; hence in (d) we may replace $C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}$ by any element in $1_\lambda H_n\tilde{T}_{uD}1_\lambda$ and the equality remains true. Similarly, we may replace $C^{\mathbf{s}'}_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}}$ by any element in $1_{\lambda'^{-1}}H_n\tilde{T}_{\underline{D}}1_{\lambda'^{-1}}$ and the equality remains true. Thus we have

$$\text{(e)}\quad \sum_{u,u'\in\mathcal{U}^{\mathfrak{a}}}(\sum_{A\in\mathcal{I}_n}b^v_{A,u'}(b^v_{A,u})^{\spadesuit}-\delta_{u,u'})\mathrm{tr}(\tilde{T}_{x'}1_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'})\mathrm{tr}(\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit})=0$$

for any $x,x'\in\mathbf{W}$ such that

(f) $x\underline{D}\lambda=\lambda, x'\underline{D}\lambda'^{-1}=\lambda'^{-1}$.

Now (e) remains true even if (f) does not hold. (If, for example, $x\underline{D}\lambda=\lambda_1\neq\lambda$, then for any $u\in\mathcal{U}^{\mathfrak{a}}$ we have

$$\mathrm{tr}(\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)=\mathrm{tr}(1_{\lambda_1}\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)=\mathrm{tr}(\tilde{T}_x\tilde{T}_{\underline{D}}1_\lambda1_{\lambda_1},E^v_u)=0,$$

hence the left-hand side of (e) is zero.) We now multiply both sides of (e) by

$$\mathrm{tr}(\tilde{T}_{x'}1_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'_1})^{\spadesuit}\mathrm{tr}(\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_{u_1})$$

where $u_1,u'_1\in\mathcal{U}^{\mathfrak{a}}$ and sum the resulting equalities over all $x,x'\in\mathbf{W}$ and $\lambda,\lambda'\in\underline{\mathfrak{s}}_n$. We obtain

$$\sum_{u,u'\in\mathcal{U}^{\mathfrak{a}}}(\sum_{A\in\mathcal{I}_n}b^v_{A,u'}(b^v_{A,u})^{\spadesuit}-\delta_{u,u'})(\sum_{x,\lambda}\mathrm{tr}(\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_{u_1})\mathrm{tr}(\tilde{T}_x1_{\underline{D}\lambda}\tilde{T}_{\underline{D}},E^v_u)^{\spadesuit})$$
$$\times(\sum_{x',\lambda'}\mathrm{tr}(\tilde{T}_{x'}1_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'})\mathrm{tr}(\tilde{T}_{x'}1_{\underline{D}\lambda'^{-1}}\tilde{T}_{\underline{D}},E^v_{u'_1})^{\spadesuit})=0.$$

Using 34.18(a) we deduce

$$\sum_{u,u'\in\mathcal{U}^{\mathfrak{a}}}(\sum_{A\in\mathcal{I}_n} b^v_{A,u'}(b^v_{A,u})^{\spadesuit}-\delta_{u,u'})\delta_{u',u'_1}f^v_{u'_1}\dim E_{u'_1}\delta_{u,u_1}f^v_{u_1}\dim E_{u_1}=0,$$

that is,

$$(\sum_{A\in\mathcal{I}_n} b^v_{A,u'_1}(b^v_{A,u_1})^{\spadesuit}-\delta_{u_1,u'_1})f^v_{u'_1}\dim E_{u'_1}f^v_{u_1}\dim E_{u_1}=0.$$

Since $f^v_{u'_1}\dim E_{u'_1}f^v_{u_1}\dim E_{u_1}\neq 0$, we deduce

(g) $$\sum_{A\in\mathcal{I}_n} b^v_{A,u'_1}(b^v_{A,u_1})^{\spadesuit}=\delta_{u_1,u'_1}$$

for any $u_1,u'_1\in\mathcal{U}^{\mathfrak{a}}$.

35.19. Let $A_0\in\mathcal{I}_n$. Then $\mathrm{supp}(A_0)$ is the closure of a stratum $Y=Y_{L,S}$ of D. Let $e=\dim Y$. Replacing $\mathbf{F}_q$ by a finite extension, we may assume that the following holds:

(a) *there exists a finite group* Γ *and a sequence of maps* $\Gamma\to Y^{F^m}$, $\gamma\mapsto y_{\gamma,m}$ $(m=1,2,3,\dots)$ *such that for any* $m\geq 1$ *and any* $A\in\mathcal{I}_n$ *we have*

$$q^{-(\dim D-e)m}|\Gamma|^{-1}\sum_{\gamma\in\Gamma}\chi_{A,\kappa_A^{(m)}}(y_{\gamma,m})\chi_{\mathfrak{D}(A_0),\kappa'_{A_0}{}^{(m)}}(y_{\gamma,m})=\delta_{A_0,A'}.$$

(We apply Proposition 33.7 with $\mathcal{I}=\mathcal{I}_n$.)

35.20. We identify H_n with a subalgebra of H_{q-1} by $\tilde{T}_w 1_\lambda\mapsto\tilde{T}_w 1_\lambda$ for $w\in\mathbf{W}$, $\lambda\in\underline{\mathfrak{s}}_n$. (We have $\underline{\mathfrak{s}}_n\subset\underline{\mathfrak{s}}_{q-1}$ since n divides $q-1$.) Similarly, for any $\kappa\in\mathfrak{U}$ we identify H^κ_n with a subalgebra of H^κ_{q-1}.

Let $\{\tilde{E}_u;u\in\tilde{\mathcal{U}}\}$ be a set of representatives for the isomorphism classes of simple modules for H^1_{q-1}. We may assume that $\mathcal{U}\subset\tilde{\mathcal{U}}$, that for $u\in\mathcal{U}$ we have $\tilde{E}_u=E_u$ as an H^1_n-module with 1_λ acting as 0 for $\lambda\in\underline{\mathfrak{s}}_{q-1}-\underline{\mathfrak{s}}_n$, and for $u\in\tilde{\mathcal{U}}-\mathcal{U}$, H^1_n acts on $\tilde{E}_u$ as zero. For $u\in\tilde{\mathcal{U}}$, the $H^{\sqrt{q}}_{q-1}$-module $\tilde{E}^{\sqrt{q}}_u$ is defined as in 34.15. If $u\in\mathcal{U}$, we have again $\tilde{E}^{\sqrt{q}}_u=E^{\sqrt{q}}_u$ as an $H^{\sqrt{q}}_n$-module with 1_λ acting as 0 for $\lambda\in\underline{\mathfrak{s}}_{q-1}-\underline{\mathfrak{s}}_n$. For $u\in\tilde{\mathcal{U}}$ we set $V_u=\mathrm{Hom}_{H^{\sqrt{q}}_{q-1}}(\tilde{E}^{\sqrt{q}}_u,\mathfrak{P})$ where $\mathfrak{P}$ is regarded as an $H^{\sqrt{q}}_{q-1}=\mathfrak{T}$-module as in 35.4, 35.5. Since $\mathfrak{T}=\mathrm{End}_{G^{0F}}(\mathfrak{P})$ (see 35.5), the G^{0F}-module structure on $\mathfrak{P}$ makes V_u into an irreducible G^{0F}-module and we have an isomorphism

$$\vartheta:\oplus_{u\in\tilde{\mathcal{U}}}(\tilde{E}^{\sqrt{q}}_u\otimes V_u)\xrightarrow{\sim}\mathfrak{P}, e\otimes x\mapsto x(e), e\in\tilde{E}^{\sqrt{q}}_u, x\in V_u.$$

Hence $\mathfrak{P}=\oplus_{u\in\tilde{\mathcal{U}}}\mathfrak{P}_u$ where $\mathfrak{P}_u=\vartheta(\tilde{E}^{\sqrt{q}}_u\otimes V_u)$. For $u\in\mathcal{U}^{\mathfrak{a}}$ and $x\in V_u$ we define $R_{u,x}:\tilde{E}^{\sqrt{q}}_u\to\mathfrak{P}$ by $R_{u,x}(e)=\rho_{d,d}(x(\tilde{T}^{-1}_{\underline{D}}e))$. We show that

(a) $R_{u,x}(he)=hR_{u,x}(e)$ for $h\in H^{\sqrt{q}}_{q-1}$, $e\in\tilde{E}^{\sqrt{q}}_u$.

If $h\in H^{\sqrt{q}}_{q-1}1_\lambda$ with $\lambda\in\underline{\mathfrak{s}}_{q-1}-\underline{\mathfrak{s}}_n, w\in\mathbf{W}$, then both sides of (a) are zero (we use 35.12(a)). Hence we may assume that $h\in H^{\sqrt{q}}_n$. Recall that $\rho_{d,d}(\mathfrak{a}^{-1}(h))=h\rho_{d,d}:\mathfrak{P}\to\mathfrak{P}$ (see 35.12(a) and 35.14(b)). Hence

$$\begin{aligned}R_{u,x}(he)&=\rho_{d,d}(x(\tilde{T}^{-1}_{\underline{D}}(he)))=\rho_{d,d}(x((\mathfrak{a}^{-1}(h))(\tilde{T}^{-1}_{\underline{D}}e)))=\rho_{d,d}(\mathfrak{a}^{-1}(h))(x(\tilde{T}^{-1}_{\underline{D}}e))\\&=h\rho_{d,d}(x(\tilde{T}^{-1}_{\underline{D}}e))=hR_u(x)(e),\end{aligned}$$

as required. We see that $R_{u,x} \in V_u$. Thus, we have a map $R_u : V_u \to V_u$, $x \mapsto R_{u,x}$. From the definitions we have

$$\vartheta(\tilde{T}_{\underline{D}} e \otimes R_{u,x}) = \rho_{d,d}\vartheta(e \otimes x) \text{ for } e \in E_u^{\sqrt{q}}, x \in V_u.$$

In particular, $\rho_{d,d} : \mathfrak{P} \to \mathfrak{P}$ maps $\mathfrak{P}_u$ into itself. On the other hand, it maps $\mathfrak{P}_u$ for $u \in \mathcal{U} - \mathcal{U}^{\mathfrak{a}}$ into $\mathfrak{P}_{u'}$ for some $u' \in \mathcal{U}, u' \neq u$. It also maps $\oplus_{u \in \tilde{\mathcal{U}} - \mathcal{U}} \mathfrak{P}_u$ into itself. Hence, if $h \in H_n^{\sqrt{q}}$ and $g_0 \in G^{0F}$, then $h\rho_{d,d}\rho_{g_0,1}$ acts as zero on $\oplus_{u \in \tilde{\mathcal{U}} - \mathcal{U}} \vartheta(E_u^{\sqrt{q}} \otimes V_u)$ and we have

$$\mathrm{tr}(h\rho_{d,d}\rho_{g_0,1}, \mathfrak{P}) = \sum_{u \in \mathcal{U}^{\mathfrak{a}}} \mathrm{tr}(h\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}})\mathrm{tr}(R_u g_0, V_u).$$

Now, from Proposition 35.10 (with $J = \mathbf{I}$) we have $\chi_{\bar{K},\bar{\omega}}(g) = \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\rho_{g,d}, \mathfrak{P})$ for any $g \in D^F$. (Recall that $a_{\underline{D}\lambda,F,\mathbf{s}} = 1$; see 35.14(a).) Hence

$$\text{(b)} \qquad \chi_{\bar{K},\bar{\omega}}(g) = \sum_{u \in \mathcal{U}^{\mathfrak{a}}} \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}})\mathrm{tr}(R_u d^{-1} g, V_u).$$

We have

$$\begin{aligned} \chi_{\bar{K},\bar{\omega}}(g) &= \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \sum_i (-\sqrt{q})^i \sqrt{q}^{-\dim G} \xi_A \dim V_{A,i,\mathbf{s},\mathcal{L}} \\ \text{(c)} \qquad &= \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g)\xi_A \sum_{u \in \mathcal{U}^{\mathfrak{a}}} b_{A,u}^{\sqrt{q}} \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}}). \end{aligned}$$

(The first equality follows from 35.17(b),(c). The second equality is obtained from the identity

$$\sum_i (-v)^i v^{-\dim G} \dim V_{A,i,\mathbf{s},\mathcal{L}} = \sum_{u \in \mathcal{U}^{\mathfrak{a}}} b^v_{A,u} \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E^v_u)$$

(see 34.19), under the specialization $v \mapsto \sqrt{q}$. Note that the rational function $b^v_{A,u}$ does not have a pole at $v = \sqrt{q}$; indeed, in 34.19(b), we have $\zeta^A(\tilde{T}_w 1_{\underline{D}\lambda}[D]) \in \mathcal{A}$, $\mathrm{tr}(\tilde{T}_w 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E^v_u)^{\spadesuit} \in \mathfrak{U}[v, v^{-1}]$ by 34.15(a), while $f^v_u|_{v=\sqrt{q}} = f_u^{\sqrt{q}} \neq 0$ by 34.15(b). Hence $b^v_{A,u}$ can be specialized for $v = \sqrt{q}$ and yields a value $b_{A,u}^{\sqrt{q}} \in \mathfrak{U}$.) From (b),(c) we deduce

$$\sum_{u \in \mathcal{U}^{\mathfrak{a}}} \mathrm{tr}(C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}})(\mathrm{tr}(R_u d^{-1} g, V_u) - \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g)\xi_A b_{A,u}^{\sqrt{q}}) = 0.$$

As in the paragraph preceding 35.18(e) we see that here we may replace $C^{\mathbf{s}}_{\underline{D}\lambda}\tilde{T}_{\underline{D}}$ by any element in $1_\lambda H_n \tilde{T}_{\underline{D}} 1_\lambda$. Thus we have

$$\text{(d)} \qquad \sum_{u \in \mathcal{U}^{\mathfrak{a}}} \mathrm{tr}(\tilde{T}_x 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}})(\mathrm{tr}(R_u d^{-1} g, V_u) - \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g)\xi_A b_{A,u}^{\sqrt{q}}) = 0$$

for any $x \in \mathbf{W}$ such that $x\underline{D}\lambda = \lambda$; moreover, if $x\underline{D}\lambda \neq \lambda$, then $\mathrm{tr}(\tilde{T}_x 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_u^{\sqrt{q}}) = 0$ so that (d) holds again. We see that (d) holds for any $x \in \mathbf{W}$ and any $\lambda \in \underline{\mathfrak{s}}_n$. We now multiply both sides of (d) by $\mathrm{tr}(\tilde{T}_x 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}, E_{u_1}^{\sqrt{q}})^{\spadesuit} = 0$ where $u_1 \in \mathcal{U}^{\mathfrak{a}}$ and sum the resulting equalities over all $x \in \mathbf{W}$ and $\lambda \in \underline{\mathfrak{s}}_n$. We obtain

$$\sum_{u \in \mathcal{U}^{\mathfrak{a}}} (\sum_{x,\lambda} \mathrm{tr}(\tilde{T}_x 1_{\underline{D}\lambda} \tilde{T}_{\underline{D}}, E_u^{\sqrt{q}}) \mathrm{tr}(\tilde{T}_x 1_{\underline{D}\lambda} \tilde{T}_{\underline{D}}, E_{u_1}^{\sqrt{q}})^{\spadesuit})$$
$$\times (\mathrm{tr}(R_u d^{-1} g, V_u) - \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \xi_A b_{A,u}^{\sqrt{q}}) = 0.$$

Using 34.18(b) with $\kappa = \sqrt{q} = \sqrt{q}^{\spadesuit}$ we deduce that

$$\sum_{u \in \mathcal{U}^{\mathfrak{a}}} \delta_{u,u_1} f_u^{\sqrt{q}} \dim E_u (\mathrm{tr}(R_u d^{-1} g, V_u) - \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \xi_A b_{A,u}^{\sqrt{q}}) = 0,$$

that is,

$$f_{u_1}^{\sqrt{q}} \dim E_{u_1} (\mathrm{tr}(R_{u_1} d^{-1} g, V_{u_1}) - \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \xi_A b_{A,u_1}^{\sqrt{q}}) = 0.$$

Since $f_{u_1}^{\sqrt{q}} \dim E_{u_1} \neq 0$, we deduce

(e) $$\mathrm{tr}(R_{u_1} d^{-1} g, V_{u_1}) = \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \xi_A b_{A,u_1}^{\sqrt{q}} = 0.$$

We show:

(f) $R_u d^{-1} g : V_u \to V_u$ *has finite order.*

For any $t \geq 1$ we have $R_u^t(x) = \rho_{d,d}^t(x(\tilde{T}_{\underline{D}}^{-t} e))$, hence $R_u^c = 1$ for some $c \geq 1$. For $g_0 \in G^{0F}$ we have

$$R_u g_0(x)(e) = \rho_{d,d} \rho_{g_0,1} x(\tilde{T}_{\underline{D}}^{-1} e) = \rho_{d g_0 d^{-1},1} \rho_{d,d} x \tilde{T}_{\underline{D}}^{-1} e,$$

hence $R_u g_0 = (d g_0 d^{-1}) R_u$. From this we deduce

$$(R_u g_0)^t = (d g_0 d^{-1})(d^2 g_0 d^{-2}) \dots (d^t g_0 d^{-t}) R_u^t$$

for $t \geq 1$. We have $d^a = 1$ for some $a \geq 1$. Let

$$g_1 = (d g_0 d^{-1})(d^2 g_0 d^{-2}) \dots (d^a g_0 d^{-a}).$$

We have $g_1^b = 1$ for some $b \geq 1$. We have $(R_u g_0)^{ab} = g_1^b R_u^{ab} = R_u^{ab}$. Thus $(R_u g_0)^{abc} = R_u^{abc} = 1$. Taking $g_0 = d^{-1} g$ we see that (f) holds.

From (f) we see that $\mathrm{tr}(R_u d^{-1} g, V_u)$ is a cyclotomic integer. Introducing this in (e) we see that

(g) $\sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A}(g) \xi_A b_{A,u}^{\sqrt{q}}$ *is a cyclotomic integer for any* $u \in \mathcal{U}^{\mathfrak{a}}, g \in D^F$.

Lemma 35.21. *In the setup of 35.19, let* $u \in \mathcal{U}^{\mathfrak{a}}$. *Assume that* E_u *is quasi-rational (see Lemma 34.20). Then* $b_{A_0,u}^v \in \eta \mathbf{Q}[v, v^{-1}]$ *where* η *is a root of* 1.

Note that 35.20(g) remains true if $\mathbf{F}_q$ is replaced by $\mathbf{F}_{q^m}$ where $m \geq 1$. Thus:

(a) $$\sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A^{(m)}}(g) \xi_A^m b_{A,u}^{\sqrt{q^m}}$$

is a cyclotomic integer for any $m \geq 1, g \in D^{F^m}$. We use the notation of 35.19. Assume that $g \in Y^{F_m}$. We multiply (a) by $\sqrt{q}^{m(e - \dim D)} \chi_{\mathfrak{D}(A_0),\kappa'_{A_0}{}^{(m)}}(g)$ which is a cyclotomic integer by 35.16(ii). We obtain again a cyclotomic integer. Now take $g = y_{\gamma,m}$ (see 35.19) and sum over all $\gamma \in \Gamma$ (see 35.19). We see that

(b) $$\sqrt{q}^{m(e - \dim D)} \sum_{\gamma \in \Gamma} \sum_{A \in \mathcal{I}_n} \chi_{A,\kappa_A^{(m)}}(y_{\gamma,m}) \chi_{\mathfrak{D}(A_0),\kappa'_{A_0}{}^{(m)}}(y_{\gamma,m}) \xi_A^m b_{A,u}^{\sqrt{q^m}}$$

is a cyclotomic integer for any $m \geq 1$. Using 35.19(a) we see that (b) equals

$$\text{(c)} \qquad \sqrt{q}^{m(\dim D-e)}|\Gamma|\xi^m_{A_0} b^{\sqrt{q^m}}_{A_0,u}$$

which is therefore a cyclotomic integer for any $m \geq 1$. By 35.21 we have $b^v_{A_0,u} = \eta Q(v)$ where $\eta \in \mathfrak{U}$ is a root of 1 and $Q(v) \in \mathbf{Q}(v)$. Let K be a subfield of $\bar{\mathbf{Q}}_l$ such that K is a finite Galois extension of $\mathbf{Q}$ of degree a which contains η and ξ_{A_0}. Let $N : K \to \mathbf{Q}$ be the norm map. Since all complex conjugates of ξ_{A_0} have absolute value 1 we see that $N(\xi_{A_0}) = \pm 1$. We have also $N(\eta) = \pm 1$. Hence applying N to (c) (with $m = 2m'$) we see that $|\Gamma|^a q^{am'(\dim D-e)}Q(q^{m'})^a$ is a cyclotomic integer. This being also a rational number, is an ordinary integer. Let

$$R(v) = |\Gamma|^a v^{a(\dim D-e)}Q(v)^a \in \mathbf{Q}(v).$$

We see that $R(q^{m'}) \in \mathbf{Z}$ for any integer $m' \geq 1$. This forces $R(v) \in \mathbf{Q}[v]$. Thus, $(v^{\dim D-e}Q(v))^a \in \mathbf{Q}[v]$. It follows that $v^{\dim D-e}Q(v) \in \mathbf{Q}[v]$, hence $Q(v) \in \mathbf{Q}[v, v^{-1}]$. The lemma is proved.

Theorem 35.22. *Assume that D is clean (see 33.4(b)). Let $A \in \mathcal{I}_n$. Let $u \in \mathcal{U}^{\mathfrak{a}}$ be such that E_u is quasi-rational (see Lemma 34.20). Then $b^v_{A,u} \in \eta\mathbf{Q}$ for some η, a root of 1.*

Using 35.18(g) with $u_1 = u'_1 = u$ we see that $\sum_{A\in\mathcal{I}_n} b^v_{A,u}(b^v_{A,u})^{\spadesuit} = 1$. Using 35.21 we write $b^v_{A,u} = \eta_A Q_A$ for $A \in \mathcal{I}_n$ where η_A is a root of 1 and $Q_A \in \mathbf{Q}[v, v^{-1}]$. Then $(b^v_{A,u})^{\spadesuit} = \eta_A^{-1}Q_A$ so that $\sum_{A\in\mathcal{I}_n} Q_A^2 = 1$. Since $Q_A \in \mathbf{Q}[v, v^{-1}]$, this forces each Q_A to be a constant. The theorem is proved.

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