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CONJUGACY CLASS ASYMPTOTICS, ORBITAL INTEGRALS, AND THE BERNSTEIN CENTER: THE CASE OF $SL(2)$

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Abstract. The Bernstein center of a reductive p-adic group is the algebra of conjugation invariant distributions on the group which are essentially compact, i.e., invariant distributions whose convolution against a locally constant compactly supported function is again locally constant complactly supported. In the case of $SL(2)$, we show that certain combinations of orbital integrals belong to the Bernstein center and reveal a geometric reason for this phenomenon.

1. Introduction

1.1. Suppose F is a non-archimedean local field and $G = \mathsf{G}(F)$ the F-rational points of a reductive group G. The Bernstein center $\mathcal{Z}(G)$ of G, introduced by J. Bernstein, has a formulation as the space of G-invariant distributions on G which are essentially compact. A G-invariant distribution D is in the center if for all $f \in \mathcal{C}_c^\infty(G)$, the convolution of D and f,

$$D \star f \ := \ x \to D(\lambda_x(\check{f}))$$

is in $\mathcal{C}_c^\infty(G)$. Here $\check{f}(g) := f(g^{-1})$, and $\lambda_x(f)(g) := f(x^{-1}g)$ is left translation by x. An elementary example of such a distribution is the delta distribution associated to a central element of G. The space of G-invariant essentially compact distributions is vast. For example, if G is semisimple and π is an irreducible supercuspidal representation of G, then the character Θ_π of π belongs to $\mathcal{Z}(G)$. But supercuspidal characters are rather mysterious objects, and indeed so too is the Bernstein center. Besides the delta distributions, and characters of supercuspidal representations, only one other explicit distribution can be found in the literature. Suppose ψ is a nontrivial additive character of the p-adic field F. In the notes [Bn], Bernstein mentions that the distribution on $SL(n, F)$ represented by the function

$$g \mapsto \psi(\mathrm{Trace}(g))$$

is essentially compact and thus lies in the Bernstein center.

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1.2. In [MT], the authors gave a description of the Bernstein center in terms of G-invariant functions which are locally L^1 with respect to the Haar measure of G. In the case of $SL(2, F)$, we computed, via the Plancherel measure, a basis of such functions. However, a natural question, already asked by Bernstein in [Bn], is the following: Is there a natural source of G-invariant essentially compact distributions?

It is known distributions in the Bernstein center are tempered. A very natural, important, and relatively simple source of G-invariant tempered distributions on reductive groups are orbital integrals. Such integrals are important for harmonic analysis of the group, and applications to automorphic forms. It is elementary that an orbital integral is essentially compact if and only if the orbit is compact. In particular, if G has no compact factors, then aside from the delta distributions on central elements, orbital integrals do not belong to the Bernstein center.

1.3. Suppose F is a non-archimedean local field of characteristic zero, i.e., a p-adic field, and also of odd residual characteristic. Let $G = SL(2, F)$. One striking discovery we announce here is that certain linear combinations of orbital integrals, in particular certain differences, are essentially compact and therefore in the Bernstein center.

We give a consequence. Let $\mathcal{OI}$ be the space of G-invariant distributions spanned by the orbital integrals of regular elements. Then

(i) $\dim_{\mathbb{C}} \mathcal{OI}/(\mathcal{OI} \cap \mathcal{Z}(G)) \leq 4$.

(ii) The unipotent orbital integrals generate $\mathcal{OI}$ over $\mathcal{OI} \cap \mathcal{Z}(G)$.

Thus, unipotent orbital integrals play a very important role in describing other orbital integrals. This is reminiscent of a similar role they play in the Shalika germ expansion.

1.4. In considering when a difference of orbital integrals lies in the Bernstein center a crucial notion is that of asymptotic conjugacy classes. Two conjugacy classes $\mathcal{O}_1$ and $\mathcal{O}_2$ of a reductive group $\mathsf{G}(F)$ are called asymptotic, if there exist sequences $\{g_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}_1$ and $\{h_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}_2$, tending to infinity, such that

$$\lim_{i\to\infty} g_i h_i^{-1} \;=\; 1.$$

Two conjugacy classes $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathsf{G}(F)$ are said to have the same asymptotic behavior at infinity, if given any sequence $\{g_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}_1$ which tends to infinity, there exists a sequence $\{h_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}_2$, so that $\lim\limits_{i\to\infty} g_i h_i^{-1} = 1$, and vice versa. An equivalent formulation is given any open subgroup J, there exists a bounded set $M = M(\mathcal{O}_1, \mathcal{O}_2, J) \subset \mathsf{G}(F)$ such that if $g_1 \in \mathcal{O}_1 \cap (\mathsf{G}(F)\backslash M)$ (resp. $g_2 \in \mathcal{O}_2 \cap (\mathsf{G}(F)\backslash M)$), then $g_1 J \cap \mathcal{O}_2$ (resp. $g_2 J \cap \mathcal{O}_1$) is nonempty.

In the case of $SL(2, F)$, we show two regular conjugacy classes have the same asymptotical behavior at infinity precisely when the two conjugacy classes have the same set of asymptotic unipotent classes. In particular, this allows us to show the conjugacy classes of two regular elements of non-conjugate maximal tori cannot have the same asymptotic behaviour at infinity. When the conjugacy classes of two regular elements of a fixed maximal torus have the same asymptotic behaviour at infinity, we show the difference of their normalized orbital integrals lies in the Bernstein center. For example, any two hyperbolic regular elements have the same asymptotic behaviour at infinity. Therefore, the difference of their normalized orbital integrals lies in the Bernstein center. We in fact prove stronger results. Their statements can be found in Sections 4, 5, and 6.

1.5. Our study of orbital integrals is based on the computation of the Fourier transforms of orbital integrals by Sally and Shalika in [SS3].

1.6. Now we describe the paper according to its sections. In Section 2 we introduce the two notions of orbits being asymptotic at infinity and having the same asymptotic behavior at infinity. We then study asymptotic relations between semisimple and unipotent conjugacy classes. In Section 3 we explain a simple criterion for an invariant tempered distribution to belong to the Bernstein center. In Section 4 we study differences of orbital integrals on the hyperbolic (split) torus. In Section 5 we deal with elliptic tori. Section 6 is devoted to unipotent orbital integrals, and their relationship to semisimple orbital integrals. In Section 7, we reprove some of our main results on certain orbital integral differences being in the Bernstein center in a different rather elegant geometric way which we believe partly explains the situation, in particular, why it seems to be a p–adic phenomenon. This last section is part of some joint work with Dan Barbasch.

2. The asymptotic behavior of conjugacy classes

2.1. Two notions of asymptotical behavior at infinity.

2.1.1. Let F be a non-archimedean local field with modulus character $|\ |_F$ (so that $d(ax) = |a|_F dx$ for any $a \in F^\times$ and dx a Haar measure on F). Let $G = \mathsf{G}(F)$ be the F-rational points of a reductive group G. For $\gamma \in G$, let $\mathcal{O}(\gamma)$ denote the conjugacy class of γ.

2.1.2. Definition. (i) A sequence $\{g_i\}_{i\in\mathbb{N}} \subset G$ approaches infinity if, given any bounded, i.e., precompact, set C in G, there exists $n(C) \in \mathbb{N}$, so that $g_i \in G\backslash C$ for $i > n(C)$.

(ii) Suppose $\mathcal{Q}_x \subset G$ and $\mathcal{Q}_y \subset G$ is each a finite union of conjugacy classes of G. We say $\mathcal{Q}_x$ and $\mathcal{Q}_y$ are asymptotic (at infinity) if there exist sequences $g_i \in \mathcal{Q}_x$ and $h_i \in \mathcal{Q}_y$, such that each sequence tends to infinity and

$$\lim_{i\to\infty} g_i h_i^{-1} = 1$$

(iii) Two subsets $\mathcal{Q}_x \subset G$ and $\mathcal{Q}_y \subset G$, each a finite union of conjugacy classes of G, have the same asymptotic behavior at infinity if given any open compact subgroup $J \subset G$, there exists a compact set $M = M(J, \mathcal{Q}_x, \mathcal{Q}_y) \subset G$ so that if

$$g \in \mathcal{Q}_x \cap (G\backslash M) \quad (\text{resp. } h \in \mathcal{Q}_y \cap (G\backslash M)),$$

then $gJ \cap \mathcal{Q}_y$ (resp. $hJ \cap \mathcal{Q}_x$) is nonempty.

2.1.3. *Remarks.* (i) Let $G = KAK$ be a Cartan decomposition of G. Then, a sequence $\{g_i\}_{i\in\mathbb{N}} \subset G$ approaches infinity if the Cartan decompositions $g_i = k_i a_i k_i'$, $k_i, k_i' \in K$ and $a_i \in A$ have the property that the a_i approach infinity in A as $i \to \infty$, i.e., if C is any compact subset of A, there exists $n(C) \in \mathbb{N}$, so that $a_i \in A\backslash C$ for $i > n(C)$.

(ii) It is elementary that a sequence $\{g_i\}_{i\in\mathbb{N}} \subset SL(n, F)$ tends to infinity if and only if $\max\{|a_i|_F, |b_i|_F, |c_i|_F, |d_i|_F\}$ tends to infinity (in $\mathbb{R}$), where $g_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$.

(iii) We shall shortly see that for $G = SL(2)$, any regular hyperbolic conjugacy class is asymptotic to any nontrivial unipotent class. However, a non-compact hyperbolic conjugacy class does not have the same asymptotic behavior at infinity as a nontrivial unipotent class. Intuitively, the two classes are asymptotic to each other in certain directions but not others.

2.2. Notation.

2.2.1. For the rest of this paper we assume F is a non-archimedean local field of characteristic zero, i.e., a p-adic field. Set

$$G = SL(2, F).$$

We establish some notation. Set

$$\begin{aligned} \mathcal{R}_F &:= \text{ ring of integers of } F, \\ \mathfrak{p}_F &:= \text{maximal ideal of } \mathcal{R}_F, \\ \varpi_F &:= \text{generator of } \mathfrak{p}_F, \\ q_F &:= \text{card } (\mathcal{R}_F/\mathfrak{p}_F), \\ K &:= SL(2, \mathcal{R}_F), \\ h(a) &:= \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \\ A_\emptyset &:= \{ h(a) \,|\, a \in F^\times \}, \\ n(x) &:= \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \\ u(y) &:= \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}, \\ N_\emptyset &:= \{ n(x) \,|\, x \in F \} \ . \end{aligned}$$

2.2.2. We have the Iwasawa decomposition

$$G \ = \ BK \ = \ N_\emptyset A_\emptyset K \ .$$

2.2.3. Let ξ run over representatives of $F^\times/(F^\times)^2$. The elements $n(\xi)$, form a set of representatives for the nontrivial unipotent classes of $SL(2, F)$. The centralizer of $n(\xi)$ $(\xi \neq 0)$ in $SL(2, F)$ is $\{\pm I\}N_\emptyset$; thus,

$$g\{\pm I\}N_\emptyset \leftrightarrow gn(x)g^{-1}$$

identifies the coset space $G/\{\pm I\}N_\emptyset$ with the conjugacy class $\mathcal{O}(n(x))$ of $n(x)$.

2.2.4. Suppose $A \in F$. Let $p(t) = t^2 - At + 1$, and let $\mathcal{Q}_{p(t)}$ denote the union of the conjugacy classes of G whose characteristic polynomials equal $p(t)$. The assumption that F is p-adic—characteristic zero—means $\mathcal{Q}_{p(t)}$ is a finite union of G conjugacy classes.

Recall the elementary fact that the characteristic polynomial $p_g(t)$ of an element $g \in G$ is completely determined by its trace $\mathrm{tr}(g)$, i.e., $p_g(t) = t^2 - \mathrm{tr}(g)t + 1$.

By definition, g is hyperbolic if the roots λ, λ^{-1} of $p_g(t)$ belong to F and g is diagonalizable. An element g is elliptic if its characteristic polynomial $p_g(t)$ is irreducible in $F[t]$. In the next subsection, we wish to analyze conjugacy class asymptotics in G. Suppose $\gamma \in G$ is a regular element, i.e., the F-dimension of the centralizer $C_G(\gamma)$ of γ in G is 1. We shall see that G has a decomposition as $G = C_G(\gamma)HK$ where H is either a subgroup or the product of a subgroup and a finite set. Such a decomposition will be useful to us because it satisfies the following property: a sequence $\{g_i\}_{i\in\mathbb{N}} \subset \mathcal{O}(\gamma)$ tends to infinity precisely if the decompositions $g_i = \gamma_i' h_i k_i$ with $\gamma_i' \in C_G(\gamma)$, $h_i \in H$ and $k_i \in K$ have the sequence $\{h_i\}_{i\in\mathbb{N}} \subset H$ tending to infinity.

2.3. Unipotent classes.

2.3.1. In the group $GL(2,F)$, there is a single nontrivial unipotent conjugacy class, i.e., $\mathcal{O}(n(x)) = \mathcal{O}(n(y))$ $(x, y \in F^\times)$. The $SL(2,F)$ nontrivial unipotent classes are in a natural bijection with the double cosets $SL(2,F)\backslash GL(2,F)/ZN_\emptyset$, where Z is the center of $GL(2,F)$. The determinant map is a bijection of the double cosets to $F^\times/(F^\times)^2$.

2.3.2. Suppose $\nu = n(z)$ is a nontrivial unipotent element. For $g \in G = K\,A_\emptyset\,N_\emptyset$, let $g = k\,h(\alpha)\,n(x)$ be an Iwasawa decomposition of g. Then,

$$gn(z)g^{-1} \;=\; k\,h(\alpha)\,n(x)\,n(z)\,n(-x)\,h(\alpha^{-1})\,k^{-1} \;=\; k\,n(\alpha^2 z)\,k^{-1}\ .$$

In particular, a sequence $g_i n(z) g_i^{-1} \subset \mathcal{O}(\nu)$ tends to infinity precisely if the Iwasawa decompositions $g_i = k_i h(\alpha_i) n(x_i)$ of the g_i's have the property that $\alpha_i \to \infty$ (in $F^\times$) as $i \to \infty$.

2.3.3. Proposition. (i) *Fix $x \in F^\times$, and let $\nu = n(x)$. Suppose $A \in F$, and $p(t) \in F[t]$ is the quadratic polynomial $p(t) := t^2 - At + 1$. If $\epsilon > 0$, then there exists $N = N(x,\epsilon)$ with the following property: If $|\alpha|_F > N$ $(\alpha \in F)$, then there exists $y \in F$ with $|y|_F < \epsilon$ so that $\kappa := h(\alpha)\,\nu\,h(\alpha)^{-1}\,u(y)$ has characteristic polynomial $p(t)$, i.e., κ has trace A. Rephrased: Let $\mathcal{Q}_{p(t)}$ be as in paragraph (2.2.4). Suppose $\mathcal{O}$ is a unipotent conjugacy class and J is an open subgroup of G. Then there exists a bounded subset $M = M(\mathcal{O}, \mathcal{Q}_{p(t)}, J) \subset G$ such that if $g \in \mathcal{O} \cap (G\backslash M)$, then $gJ \cap \mathcal{Q}_{p(t)}$ is nonempty.*

(ii) *Two distinct nontrivial unipotent conjugacy classes $\mathcal{O}(n(x))$ and $\mathcal{O}(n(y))$ in G are not asymptotic at infinity.*

(iii) *Let $-I \in SL(2,F)$ be minus the identity. Suppose $x \in F^\times$, then the conjugacy classes $\mathcal{O}(n(x))$ and $\mathcal{O}(-I\cdot n(-x))$ have the same asymptotic behavior.*

Proof. We note that

$$\kappa \;:=\; h(\alpha)n(x)h(\alpha)^{-1}u(y) \;=\; \begin{bmatrix} 1+\alpha^2 xy & \alpha^2 x \\ y & 1 \end{bmatrix}\ .$$

The trace $\mathrm{tr}(\kappa)$ of κ equals A when

$$y \;=\; \frac{A-2}{x\alpha^2}\ .$$

Assertion (i) follows immediately.

To prove statement (ii), we argue by contradiction. Suppose $\mathcal{O}(n(x))$ and $\mathcal{O}(n(y))$ are distinct unipotent classes and they are asymptotic. Then, there exist sequences $\{v_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}(n(x))$ and $\{w_i\}_{i\in\mathbb{N}}$ in $\mathcal{O}(n(y))$, both unbounded, such that

$$\lim_{i\to\infty} v_i w_i^{-1} \;=\; 1.$$

Use the Iwasawa decomposition to write

$$v_i \;=\; g_i n(x) g_i^{-1} =\; k_i h(a_i) n(x) h(a_i^{-1}) k_i^{-1} =\; k_i \begin{bmatrix} 1 & a_i^2 x \\ 0 & 1 \end{bmatrix} k_i^{-1}.$$

The hypothesis $v_i \to \infty$ as $i \to \infty$ is equivalent to $|a_i| \to \infty$ as $i \to \infty$. Similarly write

$$w_i \;=\; \gamma_i n(y)\gamma_i{}^{-1} =\; \kappa_i h(\alpha_i) n(y) h(\alpha_i{}^{-1})\kappa_i{}^{-1} =\; \kappa_i \begin{bmatrix} 1 & \alpha_i^2 y \\ 0 & 1 \end{bmatrix} \kappa_i^{-1},$$

where $|\alpha_i| \to \infty$ as $i \to \infty$.

By passing to subsequences (first of $\{v_i\}_{i\in\mathbb{N}}$, and then of $\{w_i\}_{i\in\mathbb{N}}$), we may assume that the sequences $\{k_i\}_{i\in\mathbb{N}}$ and $\{\kappa_i\}_{i\in\mathbb{N}}$ converge. Denote

$$\begin{aligned} \ell_i \; &:= \; v_i w_i^{-1} \;=\; k_i \begin{bmatrix} 1 & a_i^2 x \\ 0 & 1 \end{bmatrix} k_i^{-1} \; \kappa_i \begin{bmatrix} 1 & -\alpha_i^2 y \\ 0 & 1 \end{bmatrix} \kappa_i^{-1}, \\ m_i \; &= \; \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \; := \; k_i^{-1}\kappa_i \;, \\ m \; &= \; \lim_{i\to\infty} \, m_i \; = \; \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \; \in \; K. \end{aligned}$$

Since $\ell_i \to 1$, we have

$$\begin{bmatrix} 1 & a_i^2 x \\ 0 & 1 \end{bmatrix} m_i \begin{bmatrix} 1 & -\alpha_i^2 y \\ 0 & 1 \end{bmatrix} \to m \;=\; \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} A_i + a_i^2 x C_i & -\alpha_i^2 y A_i - a_i^2 x \alpha_i^2 y C_i + B_i + a_i^2 x D_i \\ C_i & -\alpha_i^2 y C_i + D_i \end{bmatrix} \to \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \;.$$

From the diagonal entries of the limit we see that $a_i^2\, x\, C_i \to 0$ and $\alpha_i^2\, y\, C_i \to 0$. In particular, $C_i \to 0$, so $C' \,=\, 0$. If we combine the latter with the condition $m \in K$ we conclude A' and D' are units in $\mathcal{R}_F$ and $A'D' = 1$. If we now consider the (1,2) entry, we see that

$$y a_i^2 (\, A_i \;+\; a_i^2 x C_i) \left(- \left(\frac{\alpha_i}{a_i}\right)^2 \;+\; \frac{x}{y} \frac{D_i}{A_i \;+\; a_i^2 x C_i} \right) \;\to\; 0 \;.$$

Since $y \neq 0$, $|a_i|_F \to \infty$ and $A_i \;+\; a_i^2 x C_i \to A'$ ($\neq 0$), we get

$$- \left(\frac{\alpha_i}{a_i}\right)^2 \;+\; \frac{x}{y} \frac{D_i}{A_i \;+\; a_i^2 x C_i} \;\to\; 0 \;.$$

Now, $\frac{x}{y}\frac{D_i}{A_i + a_i^2 x C_i} \;\to\; \frac{x}{y}\frac{D'}{A'} \;=\; \frac{x}{y} {D'}^2$. Therefore, $\left(\frac{\alpha_i}{a_i}\right)^2 \;\to\; \frac{x}{y}{D'}^2$. Since the squares in $F^\times$ form a closed subset of $F^\times$, $\frac{x}{y}{D'}^2$ is a square, i.e., $\frac{x}{y}$ is a square. This contradicts the initial assumption that $\mathcal{O}(n(x))$ and $\mathcal{O}(n(y))$ are distinct orbits.

To prove assertion (iii), we use the following identity:

$$\begin{bmatrix} 1 & a^2 x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ \frac{\frac{1}{y} - x}{a^2 x} & \frac{1}{x} \end{bmatrix} \begin{bmatrix} -1 & a^2 y \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x & 0 \\ \frac{\frac{1}{y} - x}{a^2 x} & \frac{1}{x} \end{bmatrix}^{-1} \;=\; \begin{bmatrix} \frac{-1}{xy} & 0 \\ \frac{-1 + 2xy - x^2 y^2}{a^2 x^2 y} & -xy \end{bmatrix} \;.$$

Suppose J is an open compact subgroup. If we take $xy = -1$, then for a sufficiently large the right-hand side belongs to J. This implies there is a bounded set $M = M(\mathcal{O}(n(x)), \mathcal{O}(-I \cdot n(-1/x)), J)$ such that if $h_1 \in \mathcal{O}(n(x)) \backslash M$ (resp. $h_2 \in \mathcal{O}(-I \cdot n(-1/x)) \backslash M$), then $h_1 J \cap \mathcal{O}(-I \cdot n(-1/x))$ (resp. $h_2 J \cap \mathcal{O}(n(x))$) is nonempty. Thus, $\mathcal{O}(n(x))$ and $\mathcal{O}(-I \cdot n(-x))$ have the same asymptotical behavior. $\square$

2.3.4. *Remarks.* The claim (ii) of Proposition 2.3.3 can be rephrased as follows: Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct nontrivial unipotent conjugacy classes. Then, there exists an open compact subgroup $J \subset G$ and a bounded set $M = M(J, \mathcal{O}_1, \mathcal{O}_2)$ so that if $\nu_1 \in \mathcal{O}_1 \backslash M$ (resp. $\nu_2 \in \mathcal{O}_2 \backslash M$), then $\nu_1 J \cap \mathcal{O}_2 = \emptyset$. (resp. $\nu_2 J \cap \mathcal{O}_1 = \emptyset$).

2.4. Hyperbolic classes.

2.4.1. The conjugacy class $\mathcal{O}(g)$ of a hyperbolic element g is completely determined by its characteristic polynomial, i.e., $\mathcal{O}(g) = \mathcal{Q}_{p_g(t)}$, where $p_g(t) = t^2 - \mathrm{tr}(g)t + 1$ is the characteristic polynomial of g.

2.4.2. Suppose $s = h(a)$ is a regular hyperbolic element in $A_\emptyset$, so $C_G(s) = A_\emptyset$. For $g \in G$, we use the Iwasawa decomposition to write it as $g = kn(x)h(\alpha)$. Then, $gsg^{-1} = kn(x)sn(-x)k^{-1}$. In particular, a sequence $g_i s g_i^{-1} \subset \mathcal{O}(s)$ approaches infinity precisely if the Iwasawa decompositions $g_i = k_i n(x_i)h(\alpha_i)$ of the g_i's have the property $x_i \to \infty$ (in F) as $i \to \infty$.

2.4.3. Proposition. *Suppose*

$$s \;=\; h(a) \;= \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

is a regular element (i.e., $a \neq \pm 1$). Then,

(i) *Suppose $A \in F$, and $p(t) \in F[t]$ is the quadratic polynomial $p(t) := t^2 - At + 1$. If $\epsilon > 0$, then there exists $N = N(A, \epsilon)$ with the following property: If $|x|_F > N$ ($x \in F$), then there exists $y \in F$ with $|y|_F < \epsilon$ so that $\kappa = n(x)\, s\, n(-x)\, u(y)$ has characteristic polynomial $p(t)$, i.e., κ has trace A. Rephrased: Suppose J is an open subgroup of G. Let $\mathcal{Q}_{p(t)}$ be as in (2.2.4). Then there exists a bounded subset $M = M(\mathcal{O}(s), \mathcal{Q}_{p(t)}, J) \subset G$ such that if $g \in \mathcal{O}(s) \cap (G \backslash M)$, then $gJ \cap \mathcal{Q}_{p(t)}$ is nonempty.*

(ii) *If $\epsilon > 0$, then there exists $N = N(s, \epsilon)$ with the following property: If $|x|_F > N$ ($x \in F$), then there exists $y \in F$ with $|y|_F < \epsilon$ so that $\kappa = n(x)\, s\, n(-x)\, u(y)$ is conjugate to the unipotent element $n(x(a^{-1} - a))$. Rephrased: Suppose J is an open subgroup of G. Then there exists a bounded subset $M = M(\mathcal{O}(s), J) \subset G$ such that if $g \in \mathcal{O}(s) \cap (G \backslash M)$, then gJ contains a unipotent element. The conjugacy class of the unipotent element depends on the choice of g, and it is possible, by varying g, to get all nontrivial unipotent conjugacy classes.*

(iii) *Suppose $\nu = n(z)$ is a nontrivial unipotent element. If $\epsilon > 0$, then there exists $N = N(z, \epsilon)$ with the following property: If $|\alpha|_F > N$ ($\alpha \in F^\times$), then there exists $y \in F$ with $|y|_F < \epsilon$ so that*

$$\tau := h(\alpha)\nu h(\alpha)^{-1} u(y) \;=\; \begin{bmatrix} 1 & \alpha^2 z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \text{ lies in } \; \mathcal{O}(s)\;.$$

Rephrased: Suppose J is an open subgroup of G. Then there exists a bounded subset $M = M(\mathcal{O}(\nu), J) \subset G$ such that if $g \in \mathcal{O}(\nu) \cap (G \backslash M)$, then $gJ \cap \mathcal{O}(s)$ is nonempty.

2.4.4. *Remarks.* (i) Part (i), in particular, implies the any two regular hyperbolic conjugacy classes have the same asymptotic behavior at infinity.

(ii) Part (ii) can be rephrased intuitively by saying as an element moves in a regular hyperbolic orbit towards infinity, the element will become close to a nontrivial unipotent element, and the conjugacy class of the unipotent element depends on how one moves to infinity.
(iii) Parts (ii) and (iii) imply any regular conjugacy class $\mathcal{O}(s)$ has the same asymptotic behavior at infinity as the variety $\mathcal{Q}_{(t-1)^2}$ of all unipotent elements.

Proof. We note that

$$n(x)sn(-x) \;=\; \begin{bmatrix} a & x(a^{-1}-a) \\ 0 & a^{-1} \end{bmatrix}$$

and

$$n(x)sn(-x)u(y) \;=\; \begin{bmatrix} a+yx(a^{-1}-a) & x(a^{-1}-a) \\ a^{-1}y & a^{-1} \end{bmatrix} .$$

The trace of $\kappa := n(x)sn(-x)u(y)$ is

$$\mathrm{tr}(\kappa) \;=\; (\,a+\frac{1}{a}\,) \;+\; x\,y\,(\frac{1}{a}-a)\;,$$

so $\mathrm{tr}(\kappa)=A$ if and only if

$$y \;=\; \frac{1}{x}\;\frac{A-(a^{-1}+a)}{a^{-1}-a}\;.$$

Statement (i) follows immediately.

To prove (ii), note that κ is unipotent if and only if $\mathrm{tr}(\kappa)=2$. This occurs precisely when

$$y \;=\; \frac{1}{x}\;\frac{2-(a^{-1}+a)}{a^{-1}-a}\;.$$

Note that if

$$v=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2,F)\;,$$

then

$$v\,n(t)\,v^{-1}=\begin{bmatrix} 1-\alpha\gamma t & \alpha^2 t \\ -\gamma^2 t & 1+\alpha\gamma t \end{bmatrix}.$$

In particular, if $g\in SL(2,F)$ is conjugate to a nontrivial $n(t)$, and the (1,2)-entry $g_{1,2}$ of g is nonzero, then $g_{1,2}t^{-1}$ is a square. It follows that when $\kappa = n(x)sn(-x)u(y)$ is unipotent, it is conjugate to $n(x(a^{-1}-a))$. Statement (ii) follows.

To prove (iii), note that

$$\tau \;:=\; h(\alpha)\nu h(\alpha)^{-1}u(y) \;=\; \begin{bmatrix} 1 & z\alpha^2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \;=\; \begin{bmatrix} 1+z\alpha^2 y & z\alpha^2 \\ y & 1 \end{bmatrix}\;.$$

So, we see τ belongs to $\mathcal{O}(s)$ if and only if $\mathrm{tr}(\tau)=2+z\alpha^2y$ is equal to $a+a^{-1}$, i.e.,

$$y \;=\; \frac{a+a^{-1}-2}{z\alpha^2}\;.$$

Statement (iii) follows immediately from this. □

2.5. Elliptic classes.

2.5.1. To determine the asymptotic properties of an elliptic orbit in $SL(2,F)$ we simplify the situation. We assume F is p-adic and has odd residual characteristic.

2.5.2. Let ϵ_F be a primitive $(q-1)$-th root of unity in F. Then

$$F^\times \;=\; \{\,\varpi_F^k\,|\,k\in\mathbb{Z}\,\}\times(1+\mathfrak{p}_F)\times\{\,\epsilon_F^i\,|\,0\le i\le q_F-1\,\},$$

and the four cosets of $(F^\times)^2$ in $F^\times$ have representatives 1, ϵ_F, ϖ_F and $\epsilon_F\varpi_F$.

For $v\in\{\,\epsilon_F\,,\ \varpi_F\ \ \epsilon_F\varpi_F\,\}$, let $E_v := F[\sqrt{v}]$, a quadratic extension of F. The field F has exactly three quadratic extensions (up to F-isomorphisms), and they are these E_v's.

Let

$$t_v(x,y) \;:=\; \begin{bmatrix} x & y \\ vy & x \end{bmatrix}\ .$$

Then, $x+y\sqrt{v}\to t_v(x,y)$ is an embedding of E_v (resp. $E_v^\times$) into the 2×2 matrices $M(2,F)$ (resp. $GL(2,F)$), and any elliptic torus of $GL(2,F)$ is conjugate to precisely one of these three $E_v^\times$'s. In $SL(2,F)$, a conjugacy class of elliptic torus is determined by an elliptic torus $E_v^\times$ in $GL(2,F)$ and an element of the double coset $SL(2,F)\backslash GL(2,F)/N_{GL(2,F)}(E_v^\times)$. Here, $N_{GL(2,F)}(E_v^\times)$ is the normalizer of $E_v^\times$. The determinant maps the double cosets bijectively with the cosets in $F^\times$ of the subgroup generated by -1 and the norms $N_{E_v/F}(E_v^\times)$. If E_v is an unramified extension, the double coset space has two elements. If E_v is a ramified extension, the double coset space has two (resp. one) elements precisely when -1 is a square (resp. non-square).

2.5.3. We conclude an elliptic torus in $SL(2,F)$ has the form

$$T_v = \{\,t_v(x,y)\ |\ x,y\in F\,,\ x^2-vy^2=1\,\},$$

where $v\in F^\times\backslash(F^\times)^2$, and furthermore, there are six (resp. four) conjugacy classes of elliptic tori if $-1\in(F^\times)^2$ (resp. $-1\notin(F^\times)^2$).

Case $-1\in(F^\times)^2$**:** The six conjugacy classes of elliptic tori correspond to

$$v\ \in\ \{\,\varpi_F,\,\epsilon_F^2\varpi_F,\,\epsilon_F\varpi_F,\,\epsilon_F^{-1}\varpi_F,\,\epsilon_F,\,\epsilon_F\varpi_F^2\,\}\ . \tag{2.5.3a}$$

The extension $F[\sqrt{v}]$ is unramified when $v\in\{\epsilon_F,\,\epsilon_F\varpi_F^2\}$ and ramified when $v\in\{\varpi_F,\,\epsilon_F\varpi_F,\,\epsilon_F^2\varpi_F,\,\epsilon_F^{-1}\varpi_F\}$.

For any of these tori T, the $SL(2,F)$ Weyl group $N_{SL(2,F)}(T)/T$ has order two.

Case $-1\notin(F^\times)^2$**:** Then $-\epsilon_F\in(F^\times)^2$. The four conjugacy classes of elliptic tori correspond to

$$v\ =\ \varpi_F\ (\text{or }\epsilon_F^2\varpi_F),\ \epsilon_F\varpi_F\ (\text{or }\epsilon_F^{-1}\varpi_F),\ \epsilon_F\text{ and }\epsilon_F\varpi_F^2\ .$$

Indeed, if $-\epsilon_F=a^{-2}$, then

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 0 & \epsilon_F \\ \varpi_F\epsilon_F & 0 \end{bmatrix}\begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\epsilon_F^2\varpi_F & 0 \end{bmatrix}\ .$$

In particular, $T_{\varpi_F}=T_{\epsilon_F^2\varpi_F}$. Similarly, $T_{\epsilon_F\varpi_F}=T_{\epsilon_F^{-1}\varpi_F}$.

For any of these tori T, the $SL(2,F)$ Weyl group $N_{SL(2,F)}(T)/T$ has order two when the quadratic extension E_v is unramified and is trivial when the E_v is ramified.

We note, in both cases, that when the $SL(2,F)$ Weyl group $N_{SL(2,F)}(T)/T$ is of order two, then the Weyl action of the torus T is the Galois action $x+y\sqrt{v}\to x-y\sqrt{v}$.

2.5.4. Suppose $g = t_v(x,y)$ is a nontrivial elliptic element in G. Let $p_g(t)$ be the characteristic polynomial of g, and in the notation of paragraph (2.2.4), let $\mathcal{Q}_{p_g(t)}$ be the elements in G with characteristic polynomial $p_g(t)$. Let E denote the splitting field of $p_g(t)$. The group $GL(2,F)$ acts, by conjugation, transitively on $\mathcal{Q}_{p_g(t)}$. The set $\mathcal{Q}_{p_g(t)}$ consists of two G conjugacy classes, and a conjugacy class is determined by a coset $G\backslash GL(2,F)/C_{GL(2,F)}(g)$. Here, $C_{GL(2,F)}(g)$, an elliptic torus in $GL(2,F)$ is the centralizer of g. The determinant maps these cosets bijectively with $F^\times/N_{E/F}(E^\times)$. In particular, if $p(t) = t^2 - At + 1$ is an irreducible quadratic polynomial, then there are two G conjugacy classes with characteristic polynomial $p(t)$. If $c \in F^\times \backslash N_{E/F}(E^\times)$, the two G conjugacy classes are related in $GL(2,F)$ by conjugation by the element

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}.$$

2.5.5. Proposition. *Suppose F is characteristic zero, odd residual characteristic, and $s = t_v(x,y) \in G$ is a regular elliptic element. Let $E = F[\sqrt{v}]$, and $N_{E/F}(E^\times) = (F^\times)^2 \cup \beta(F^\times)^2$.*

(i) *Suppose $A \in F$, and $p(t) \in F[t]$ is the quadratic polynomial $p(t) := t^2 - At + 1$, and suppose J is an open subgroup of G. Then there exists a bounded subset $M = M(\mathcal{O}(s), J) \subset G$ such that if $g \in \mathcal{O}(s) \cap (G\backslash M)$, then gJ contains an element with characteristic polynomial $p(t)$.*

(ii) *Suppose J is an open subgroup of G. Then, there exists a bounded set $M = M(\mathcal{O}(s), J)$ so that if $g \in \mathcal{O}(s) \cap (G\backslash M)$, then the intersection of gJ with the unipotent variety is contained in the union $\mathcal{O}(n(y)) \cup \mathcal{O}(n(\beta y))$. Furthermore, for any bounded set K, there exists $h_1, h_2 \in \mathcal{O}(s) \cap (G\backslash K)$ such that $h_1 J \cap \mathcal{O}(n(y))$ (resp. $h_2 J \cap \mathcal{O}(n(\beta y))$) is nonempty. Thus, the conjugacy class $\mathcal{O}(s)$ is asymptotic to the unipotent orbits $\mathcal{O}(n(y))$ and $\mathcal{O}(n(\beta y))$ and no other unipotent orbits.*

(iii) *Conversely, suppose $\eta = n(z)$ is a nontrivial unipotent element, J is an open compact subgroup, and $s = t_v(x,y)$ is an elliptic element such that $y \in zN_{E/F}(E^\times)$. Then, there exists a bounded set $M = M(\mathcal{O}(\eta), \mathcal{O}(s), J)$ so that if $g \in \mathcal{O}(\eta) \cap (G\backslash M)$, then $gJ \cap \mathcal{O}(s)$ is nonempty.*

(iv) *Two elliptic elements $t_v(x,y)$, and $t'_v(x',y')$ is G have the same asymptotic behavior at infinity if and only if $v/v'' \in F^\times)^2$, and y/y' is a norm of $E_v = F[\sqrt{v}]$.*

2.5.6. Before we proceed with the proof, we observe two types of decompositions of $SL(2,F)$.

Let $\mathcal{B}$ denote the Bruhat-Tits building of $SL(2,F)$. The fixed point set of T_v in $\mathcal{B}$ is either a singleton point $\{x\}$, when T_v is an unramified torus, or an alcove, when T_v is a ramified torus. For the latter, take $\{x_1, x_2\}$ to be the two vertices of the alcove. When T_v is unramified, let $K = \mathrm{Stab}(x)$ denote the stabilizer of x in $SL(2,F)$. When T_v is a ramified torus, let K denote either $\mathrm{Stab}(x_1)$ or $\mathrm{Stab}(x_2)$. Let A be any split torus of $SL(2,F)$ whose associated apartment in $\mathcal{B}$ contains the fixed point of K, and when T_v is ramified the alcove. Let $N(T_v)$ denote the normalizer of T_v. The Weyl group $N(T_v)/T_v$ is order at most two and its conjugation action on T_v is the Galois automorphism $a + b\sqrt{v} \mapsto a - b\sqrt{v}$ (i.e., $t_v(a,b) \mapsto t_v(a,-b)$).

We have the following decompositions of $G = SL(2,F)$:

If T_v is unramified, then

$$G = N_G(T_v)AK. \tag{2.5.6a}$$

If T_v is ramified, then

$$G = T_v AK. \tag{2.5.6b}$$

The proof of the two decompositions is elementary. Let x be the vertex in $\mathcal{B}$ fixed by K. The point x is fixed by $N_G(T_v)$. If $g \in G$, consider the point $g \cdot x$. We can choose $h \in N_G(T_v)$ such that $h \cdot (g \cdot x)$ lies in the apartment of A, and then take $a \in A$ so that $a \cdot (h \cdot (g \cdot x)) = x$. Obviously $ahg \in K$, i.e., $g \in N_G(T_v)AK$. When T_v is ramified, the element h can be taken in T_v. This proves the two decompositions.

2.5.7. We now prove Proposition 2.5.5.

Proof. We use the decompositions (2.5.6a) and (2.5.6b) to determine the behavior of $g^{-1}sg$ at infinity. For convenience, we choose a basis so that the split torus is the group of diagonal matrices $A_\emptyset$. Decompose $g \in G$ as

$$g = h\, h(\alpha)\, k\ , \quad \text{where } h \in N_G(T_v)\,,\ k \in K\ .$$

We make some observations:

(1) Conjugation of $s = t_v(x,y)$ by h results in either s or $t_v(x,-y)$.
(2) As a function of g, the conjugate $g^{-1}sg$ tends to infinity if and only if

$$\max\{\, |\alpha|_F\, ,\, |\alpha^{-1}|_F\,\} \to \infty\ .$$

We have

$$h(\alpha)^{-1}t_v(x,y)h(\alpha) \ = \ \begin{bmatrix} x & \alpha^{-2}y \\ \alpha^2 vy & x \end{bmatrix}.$$

(3) The product

$$\begin{aligned} P \ &= \ h(\alpha)^{-1}t_v(x,y)h(\alpha)\, u(p) \\ &= \ \begin{bmatrix} x & \alpha^{-2}y \\ \alpha^2 vy & x \end{bmatrix} u(p) \\ &= \ \begin{bmatrix} x + p\alpha^{-2}y & \alpha^{-2}y \\ \alpha^2 vy + px & x \end{bmatrix}\ , \end{aligned}$$

has trace A precisely when

$$p = \alpha^2 \frac{A - 2x}{y}.$$

In the case when $A = 2$, and p is specified as in the previous line, then the product P is unipotent and is conjugate to $n(y)$.
(4) We have

$$\begin{aligned} P \ &= \ h(\alpha)^{-1}t_v(x,y)h(\alpha)\, n(r) \ = \ \begin{bmatrix} x & \alpha^{-2}y \\ \alpha^2 vy & x \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \\ &= \ \begin{bmatrix} x & rx + \alpha^{-2}y \\ \alpha^2 vy & r\alpha^2 vy + x \end{bmatrix}\ , \end{aligned}$$

so $\mathrm{tr}(P) = A$ precisely when

$$r = \frac{A - 2x}{\alpha^2 vy} \ .$$

When P is unipotent, it is conjugated to $u(vy)$, i.e., to $n(-vy)$.

Statement (i) of Proposition 2.5.5 is a consequence of the above four observations. By (2), $g^{-1}sg$ tends to infinity if and only if $\max\{|\alpha|_F, |\alpha^{-1}|_F\} \to \infty$. Then, observations (3) and (4) treat $|\alpha^{-1}|_F \to \infty$ and $|\alpha|_F \to \infty$ respectively.

To prove statement (ii), we show that if $\mathcal{O}(s)$ is asymptotic to a unipotent orbit $\mathcal{O}(n(t))$, then $t \in yN_{E/F}(E^\times)$. We show this in two stages. The first is to show that for any open compact subgroup J, there is a bounded set M so that if $h \in \mathcal{O}(s) \cap (G\backslash M)$, then $hJ \cap (\mathcal{O}(n(y)) \cup \mathcal{O}(\beta n(y)))$ is nonempty. The second stage is to assert that distinct nontrivial unipotent classes are not asymptotic at infinity (see Proposition 2.3.3), implies we can choose M bounded so that if $h \in \mathcal{O}(s) \cap (G\backslash M)$, then hJ does not meet any of the other two nontrivial unipotent conjugacy classes.

Write $g = hh(\alpha)k$ as in (2.5.6a). Now, hsh^{-1} is either s or $t_v(x, -y)$, and the latter can happen only when E is unramified or $-1 \in (F^\times)^2$. As already mentioned $g^{-1}sg$ tends to infinity if and only if either $|\alpha|_F$ or $|\alpha^{-1}|_F$ tends to infinity.

Case $hsh^{-1} = s$: If $|\alpha|_F$ is sufficiently large, then observation (4) says $g^{-1}sgJ$ meets $\mathcal{O}(n(-vy))$. If $|\alpha^{-1}|_F$ is sufficiently large, then observation (3) says $g^{-1}sgJ$ meets $\mathcal{O}(n(y))$. In particular, $-vy, y \in yN_{E/F}(E^\times)$.

Case $hsh^{-1} = t_v(x, -y)$: If $|\alpha|_F$ is sufficiently large, then observation (4) says $g^{-1}sgJ$ meets $\mathcal{O}(n(vy))$. If $|\alpha^{-1}|_F$ is sufficiently large, then observation (3) says $g^{-1}sgJ$ meets $\mathcal{O}(n(-y))$. If E/F is unramified or $-1 \in (F^\times)^2$, then $vy, -y \in yN_{E/F}(E^\times)$.

Thus, we have established our first assertion. As already mentioned, our second assertion that there exists a bounded set M such that if $h \in \mathcal{O}(s) \cap (G\backslash M)$, then hJ does not meet any of the other two nontrivial unipotent conjugacy classes now follows from our first assertion and Proposition 2.3.3. This proves statement (ii).

To prove statement (iii), let $p(t) = t^2 - \mathrm{tr}(s)t + 1$ be the characteristic polynomial of s. By Proposition 2.3.3(ii), there is a bounded set L, dependent on $p(t)$, $\mathcal{O}(n(z))$, and J so that if $g \in \mathcal{O}(n(z)) \cap (G\backslash L)$, then $gJ \cap \mathcal{Q}_{p(t)}$ is nonempty. Now $\mathcal{Q}_{p(t)}$ is a union of two conjugacy classes. Let $c \in F^\times \backslash N_{E/F}(E^\times)$. Representatives for the two classes are

$$s = t_v(x, y) \quad \text{and,} \quad s' = t_{\frac{v}{c^2}}(x, cy) \ .$$

Since $z/y \in N_{E/F}(E^\times)$, then $z/(cy) \notin N_{E/F}(E^\times)$, and therefore, by statement (ii), the conjugacy class $\mathcal{O}(s')$ is not asymptotic to $\mathcal{O}(n(z))$. Statement (iii) follows.

Statement (iv) follows from statements (i), (ii), (iii). □

2.6. *Remark.* It follows from Propositions 2.3.3, 2.4.3 and 2.5.5 that if γ_1 and γ_2 are two semisimple elements in G, then the conjugacy classes $\mathcal{O}(\gamma_1)$ and $\mathcal{O}(\gamma_2)$ have the same asymptotic behavior at infinity, if and only if the unipotent classes to which they are asymptotic are the same.

2.7. Shalika germs.

2.7.1. For an elliptic torus T and unipotent conjugacy class $\mathcal{O}$, the Shalika germ corresponding to T and $\mathcal{O}$ will be denoted $c^T_{\mathcal{O}}$. For a quadratic extension E of F, the image of the norm mapping from $E^\times$ is a subgroup of index 2 in $F^\times$, and therefore determines a character of order 2 of $F^\times$, which will be denoted by sgn_E. Now Lemma 2.4 in [SS3] shows that on regular elements of an elliptic torus T_v and $\xi \in F^\times$ we have

$$c^{T_v}_{\mathcal{O}(n(\xi))}(t_v(x,y)) = (1/2)(1 + \mathrm{sgn}_{F[\sqrt{v}]}(y\xi)).$$

2.7.2. Corollary. *Suppose T is an elliptic torus, and $\mathcal{O}$ is a nontrivial unipotent orbit in the group $SL(2,F)$. For $t \in T_{\mathrm{reg}}$, the value $c^T_{\mathcal{O}}(t)$ of the Shalika germ $c^T_{\mathcal{O}}$ equals 1 if the orbits $\mathcal{O}(t)$ and $\mathcal{O}$ are asymptotic, and 0 if they are not.*

Proof. We can replace T with one of its conjugates, and assume $T = T_v$. By Lemma 2.4 in [SS3] (see above), $c^{T_v}_{\mathcal{O}(n(\xi))}(t_v(x,y))$ is either 0 or 1, and it is 1 if and only if

$$\mathrm{sgn}_{F[\sqrt{v}]}(y) = \mathrm{sgn}_{F[\sqrt{v}]}(\xi).$$

By Proposition 2.5.5, the conjugacy class $\mathcal{O}(t_v(x,y))$ is asymptotic to $\mathcal{O}(n(\xi))$ if and only if $\xi \in yN_{E/F}(E^\times)$. This proves the Corollary. □

3. Fourier transform of invariant tempered distributions

3.1. The Bernstein Center.

3.1.1. In this subsection, let $G = \mathsf{G}(F)$ be the F-rational points of a reductive group G. Let $\mathcal{Z}(G)$, denote the Bernstein center of G, i.e., the algebra of G-invariant essentially compact distributions. Recall a distribution T is essentially compact if for all $f \in \mathcal{C}^\infty_c(G)$, the convolution

$$D \star f \ := \ x \to D(\lambda_x \check{f}) \qquad (\text{here } \check{f}(g) \ := \ f(g^{-1}))$$

belongs to $\mathcal{C}^\infty_c(G)$. The Bernstein center acts on any smooth representation (π, V_π) as follows: If $v \in V_\pi$, let J be an open compact subgroup of $\mathrm{Stab}(v)$, and $e_J \in \mathcal{C}^\infty_c(G)$ the idempotent which is the characteristic function of J divided by the Haar measure of J. In particular, $\pi(e_J)v = v$. If $T \in \mathcal{Z}(G)$, then $T \star e_J \in \mathcal{C}^\infty_c(G)$, and we set

$$\pi(T)v \ := \ \pi(T \star e_J)\, v \ .$$

The definition is well defined. To see this, suppose L is an open subgroup of J, then $e_J = e_J \star e_L = e_L \star e_J$; in particular, $(e_J - e_L)$ is an idempotent. Thus

$$\begin{aligned} \pi(T \star (e_J - e_L))\, v \ &= \ \pi(T \star (e_J - e_L) \star (e_J - e_L))\, v \\ &= \ \pi(T \star (e_J - e_L))\, \pi(e_J - e_L)\, v \\ &= \ 0 \ . \end{aligned}$$

It follows $\pi(T \star e_J)v = \pi(T \star e_L)v$ for any open subgroup L of J. That $\pi(T)v$ is well defined is then clear.

3.1.2. Let $\tilde{G}$ denote the smooth dual of G. Suppose $\pi \in \tilde{G}$ and $T \in \mathcal{Z}(G)$. Then $\pi(T)$ acts as a scalar $\chi_\pi(T)$ on V_π. If we fix π and vary T, the ring homomorphism $\chi_\pi : \mathcal{Z}(G) \to \mathbb{C}$ is called the infinitesimal character of π. Alternatively, if we fix T and vary π, we get a function

$$\hat{T} : \tilde{G} \to \mathbb{C}$$

$$\hat{T}(\pi) := \chi_\pi(T) \ .$$

For $\pi \in \tilde{G}$ and $f \in \mathcal{C}^\infty_c(G)$, set

$$\hat{f}(\pi) \ := \ \mathrm{trace}(\pi(f)) \ = \ \Theta_\pi(f) \ .$$

Here, Θ_π is the character of π (considered as a distribution). Let $\hat{G}_t$ denote the tempered dual of G, and let μ_{PL} denote the Plancherel measure on $\hat{G}_t$. From [BD] we know

$$T(f) \;=\; \int_{\hat{G}_t} \hat{f}(\tilde{\pi})\, \hat{T}(\pi)\, d\mu_{PL}(\pi).$$

Thus, the function $\pi \mapsto \hat{T}(\pi) := \chi_\pi(T)$ on $\hat{G}_t$ is just the Fourier transform of T.

More generally, a measure μ_τ on $\hat{G}_t$ is the Fourier transform (measure) of a G-invariant distribution T on G if

$$T(f) \;=\; \int_{\hat{G}_t} \hat{f}(\tilde{\pi})\, d\mu_\tau(\pi) \quad \text{for all} \quad f \in \mathcal{C}_c^\infty(G)\ .$$

A function $\tau : \hat{G}_t \to \mathbb{C}$ which is locally integrable with respect to the Plancherel measure, is called the Fourier transform of a G-invariant distribution T on G if

$$T(f) \;=\; \int_{\hat{G}_t} \hat{f}(\tilde{\pi})\, \tau(\pi)\, d\mu_{PL}(\pi) \quad \text{for all} \quad f \in \mathcal{C}_c^\infty(G)\ .$$

In particular, the constant function $\tau = 1$ ($\mu_\tau = \mu_{PL}$), is the Fourier transform (Fourier transform measure) for the G-invariant distribution of the delta function δ_1 at the identity of G.

3.1.3. Denote the space of all infinitesimal characters of representations in $\tilde{G}$ by $\Omega(G)$. We recall [BD]:

- (i) The space $\Omega(G)$ is also naturally the quotient of $\tilde{G}$ by the equivalence relation $\pi \sim \pi'$ if $\chi_\pi = \chi_{\pi'}$. Let χ denote the quotient map $\tilde{G} \to (\tilde{G}/\sim) = \Omega(G)$.
- (ii) The quotient $\Omega(G)$ is naturally a complex algebraic variety. If $T \in \mathcal{Z}(G)$, the Fourier transform $\hat{T} : \pi \to \chi_\pi(T)$, factors to a function $\Omega(G) \to \mathbb{C}$. For convenience, we also denote it as $\hat{T}$, so
$$\hat{T} : \Omega(G) \to \mathbb{C}\ .$$
- (iii) If $T \in \mathcal{Z}(G)$, the Fourier transform $\hat{T}$, as a function on the complex variety $\Omega(G)$, is a regular function. Furthermore, $T \mapsto \hat{T}$ is an isomorphism of $\mathcal{Z}(G)$ onto the space of all regular functions on $\Omega(G)$.
- (iv) The image of $\hat{G}_t$ under the quotient map $\chi : \tilde{G} \to (G/\sim) = \Omega(G)$ is Zariski dense in $\Omega(G)$. In particular, $\hat{T} : \Omega(G) \to \mathbb{C}$ for $T \in \mathcal{Z}(G)$ is completely determined by its restriction on $\chi(\hat{G}_t)$.

3.1.4. Assume now

$$G = SL(2, F).$$

If Ω is a connected component of $\Omega(G)$, let $\hat{G}_t(\Omega)$ denote the tempered irreducible representations with infinitesimal character belonging to Ω. Recall that space of functions $\hat{f}$, $f \in C_c^\infty(G)$, is dense, with respect to the supp norm, in the space of continuous (bounded) functions on $\hat{G}_t$ with support in $\hat{G}_t(\Omega)$ without reducible principal series (recall that their Plancherel measure is zero). Therefore, if we have an invariant tempered distribution T on G, and if its Fourier transform is given by a function $\hat{T}$, then T determines function $\hat{T}$ uniquely as a locally integrable function on $\hat{G}_t$.

The group G, is somewhat special in that $\hat{G}_t$ has the property that an irreducible tempered representation is essentially determined by its infinitesimal character. Indeed, the only instances when different irreducible tempered representations have the same infinitesimal character are constituents of reducible unitary principal series. For convenience, we let $\hat{G}'_t$ denote the subset of $\hat{G}_t$ obtained by removing these irreducible tempered representations. In particular, since the constituents of reducible unitary principal series has Plancherel measure zero, a locally integrable function on $\hat{G}_t$ is determined by its restriction to $\hat{G}'_t$. The utility of $\hat{G}'_t$ is that it maps bijectively, under the infinitesimal character map, to its image in $\Omega(G)$. Denote the image as $\Omega(G)'_t$. We now observe that if T is an invariant tempered distribution on G, and $\hat{T}$ is a locally integrable function on $\hat{G}_t$, then T is in fact uniquely determined by the restriction $\hat{T}|\hat{G}'_t$.

Therefore, summing up the above observations we get that the following criterion holds for $G = SL(2, F)$:

3.1.5. Criterion. (i) *Suppose a distribution T lies in the Bernstein center. Then the restriction of its Fourier transform $\hat{T}$ to $\Omega(G)'_t$ is a continuous function. Furthermore, the function $\hat{T}|\Omega(G)'_t$ extends uniquely to a regular function on $\Omega(G)$.*

(ii) *Conversely, if T is an invariant tempered distribution on G such that the restriction of its Fourier transform $\hat{T}$ to $\Omega(G)'_t$ is a continuous function, and $\hat{T}|\Omega(G)'_t$ extends to a regular function on $\Omega(G)$, then T is in the Bernstein center.*

3.2. Haar measure and the Plancherel Formula.

3.2.1. We recall the Plancherel Formula calculated by Sally and Shalika. Following them, normalize additive Haar measure on F so that the measure of $\mathcal{R}_F$ equals one, and so $\mathfrak{p}_F$ has measure $\frac{1}{q_F}$. Then, the first congruence subgroup K_1 of $K = SL(2, \mathcal{R}_F)$ has measure $\frac{1}{q_F^3}$ and consequently the measure of $K = SL(2, \mathcal{R}_F)$ equals

$$\begin{aligned} \operatorname{meas}(K) \ &= \ |K/K_1| \cdot \operatorname{meas}(K_1) \ = \ |SL(2, \mathbb{F}_{q_F})| \cdot \frac{1}{q_F^3} \\ &= \ (q_F + 1)(q_F - 1)q_F \cdot \frac{1}{q_F^3} \ = \ 1 - \frac{1}{q_F^2} \ . \end{aligned}$$

3.2.2. The gamma function of F (which is a function on $\hat{F}$) will be denoted by Γ. Since $F^\times = \{\, \varpi_F^k \,|\, k \in \mathbb{Z}\,\} \times \mathcal{R}_F^\times$, the mapping

$$\begin{aligned} \widetilde{F^\times} \ &\to \quad \mathbb{C}^\times \times \widehat{\mathcal{R}_F^\times} \\ \xi &\mapsto (\xi(\varpi_F), \xi\big|_{\mathcal{R}_F^\times}) \end{aligned}$$

is an isomorphism. Let $\xi = \xi(s, \theta)$ be the character of $F^\times$ corresponding to $(s, \theta) \in \mathbb{C}^\times \times \widehat{\mathcal{R}_F^\times}$ for the above isomorphism. If $1 + \mathfrak{p}_F^m$ $(m \geq 1)$ is the conductor of θ, then

$$\Gamma(s, \theta) = c_\theta s^{-m} \quad \text{where} \quad c_\theta \in \mathbb{C}^\times \quad \text{satisfies} \quad |c_\theta| = q_F^{-m/2}.$$

For the trivial character $1_{\mathcal{R}_F^\times}$ of $\mathcal{R}_F^\times$ we have

$$\Gamma(s, 1_{\mathcal{R}_F^\times}) = \frac{1 - q_F^{-1} s^{-1}}{1 - s}.$$

Denote the set of all classes of irreducible square integrable (resp. supercuspidal) representations of $G = SL(2, F)$ by DS (resp. SC). Then the only representation which is in DS but not in SC, is the Steinberg representation π_{St}.

3.2.3. Plancherel Formula. *For $f \in \mathcal{C}_c^\infty(G)$, we have*

$$\begin{aligned}(1 - \frac{1}{q_F^2})\, f(1) \;&=\; \sum_{\pi \in \mathrm{DS}} d(\pi)\, \hat{f}(\pi) \;+\; \frac{1}{2}\,(1 - \frac{1}{q_F^2}) \int_{\xi \in \widehat{F^\times}} \frac{1}{|\Gamma(\xi)|^2}\, \hat{f}(\pi(\xi))\, d\xi \\ &=\; \sum_{\pi \in \mathrm{SC}} d(\pi)\, \hat{f}(\pi) \;+\; d(\pi_{\mathrm{St}})\, \hat{f}(\pi_{\mathrm{St}}) \\ &\qquad + \frac{1}{2}\,(1 - \frac{1}{q_F^2}) \int_{\xi \in \widehat{F^\times}} \frac{1}{|\Gamma(\xi)|^2}\, \hat{f}(\pi(\xi))\, d\xi,\end{aligned}$$

where $\pi(\xi)$ denotes the principal series representation $\mathrm{Ind}(\xi)$*.*

In the above formula the sum over DS (resp. SC) means the sum over all the irreducible square integrable (resp. supercuspidal) classes of G. Furthermore, $d(\pi)$ denotes the formal degree of π (recall, Sally and Shalika normalize the Haar measure so $\mathrm{meas}(K) = 1 - \frac{1}{q_F^2}$), $\hat{f}(\pi)$ denotes the trace of $\pi(f)$, i.e.,

$$\hat{f}(\pi) \;=\; \Theta_\pi(f) \;:=\; \int_G f(g)\, \Theta_\pi(g)\, dg \;,$$

π_{St} denotes the Steinberg representation of G, and $d\xi$ is the Haar measure of the multiplicative group $\widehat{F^\times}$ normalized so $\mathrm{meas}(\mathcal{R}_F^\times) = 1 - \frac{1}{q_F}$. The formal degree, $d(\pi_{\mathrm{St}})$, of the Steinberg representation is $(q_F - 1)$.

3.2.4. On Bernstein components which are not unramified, the Plancherel measure is constant. For these components, verification of Criterion 3.1.5 simplifies to showing $\hat{T}$ is regular on these components.

3.2.5. The Plancherel measure on the unramified Bernstein component is

$$\begin{aligned}&\delta_{\pi_{\mathrm{St}}}\; d(\pi_{\mathrm{St}}) \;+\; \frac{1}{2}\, \frac{q_F^2 - 1}{q_F^2}\, \frac{1}{|\Gamma(\xi)|^2}\, d\xi \\ &= \delta_{\pi_{\mathrm{St}}}\; d(\pi_{\mathrm{St}}) \;+\; \frac{1}{2}\,(q_F^2 - 1)\, \frac{(1 - \xi(\varpi_F))(1 - \xi(\varpi_F)^{-1})}{(q_F - \xi(\varpi_F))(q_F - \xi(\varpi_F)^{-1})} d\xi \;.\end{aligned}$$

4. Hyperbolic orbital integrals

4.1. *Orbital integrals.* Suppose $t \in G = SL(2, F)$. Let $C_G(t)$ be the centralizer of t in G. For $f \in \mathcal{C}_c^\infty$, let

$$\mathcal{O}_f(t) := \int_{G/C_G(t)} f(gtg^{-1}) dg$$

be the orbital integral. Then $f \mapsto \mathcal{O}_f(t)$ is an invariant tempered distribution on G. If $t = \pm I$ is in the center of G, then $\mathcal{O}_f(t) = f(t)$ is the delta distribution at t, which is obviously an element of the Bernstein center of G.

Suppose T is a maximal torus of G. Fix Haar measures on G and T, and by consequence, a G-invariant measure on G/T. Suppose $t \in T^{\mathrm{reg}}$ is a regular element of T. Set,

$$I_f^T(t) \;:=\; \mathcal{O}_f(t)\,.$$

4.2. Let $D : G \to F$ denote the Weyl discriminant of $G = SL(2, F)$. For $h(a) \in A_\emptyset$, we have

$$D(h(a)) = (a - a^{-1})^2.$$

Note that

$$I_f^{A_\emptyset}(h(a)) = I_f^{A_\emptyset}(h(a)^{-1}).$$

Recall that by Weyl's integration formula, the hyperbolic (or split) invariant integral $|D(h(a))|_F^{1/2}\ I_f^{A_\emptyset}(h(a))$ equals

$$|D(h(a))|_F^{1/2}\ I_f^{A_\emptyset}(h(a)) = C \int_{(F^\times)\hat{}} \hat{f}(\pi(\xi))\, \xi(a) d\xi \tag{4.2a}$$

where $d\xi$ is Haar measure on $\widehat{F^\times}$, $\pi(\xi) = \mathrm{Ind}(\xi)$ is the representation of G induced from the character $h(a) \mapsto \xi(a)$ of $B = A_\emptyset N_\emptyset$, $\hat{f}(\pi(\xi)) = \Theta_\pi(f)$ and C is constant (coming from the Weyl integration formula; it depends only on Haar measures, which we have fixed). For $a_1, a_2 \in F^\times$, we have

$$\begin{aligned} &|D(h(a_1))|_F^{1/2}\ I_f^{A_\emptyset}(h(a_1)) - |D(h(a_2))|_F^{1/2}\ I_f^{A_\emptyset}(h(a_2)) \\ &\quad = C \int_{(F^\times)\hat{}} \hat{f}(\pi(\xi))(\xi(a_1) - \xi(a_2)) d\xi\ . \end{aligned} \tag{4.2b}$$

4.3. Proposition (Hyperbolic orbital integral expansion)**.** *Let $A_\emptyset$ be the split diagonal subgroup of G and t_1, $t_2 \in (A_\emptyset)_{\mathrm{reg}}$. Then, the invariant distribution which is the difference*

$$f \ \mapsto\ |D(t_1)|_F^{\frac{1}{2}} I_f^{A_\emptyset}(t_1)\ -\ |D(t_2)|_F^{\frac{1}{2}} I_f^{A_\emptyset}(t_2)$$

lies in the Bernstein center.

4.4. *Proof of Proposition* 4.3. We prove Proposition 4.3 by verifying the distribution satisfies Criterion 3.1.5.

Proof. Recall that on connected components that are not unramified, the Plancherel measure is constant (on each of them) with respect to Haar measure of $F^\times$ (after the obvious identifications) and functions $\xi \mapsto \xi(a_i)$ are regular on non-supercuspidal components of $\Omega(G)$. Thus, Criterion 3.1.5 holds for the above distribution for components which are not unramified. It remains to check that the criterion holds for the unramified component.

Suppose $|a_1|_F = |a_2|_F$. On the unramified component the distribution (4.2b) is represented by 0. Therefore, if $|a_1|_F = |a_2|_F^{\pm 1}$, Criterion 3.1.5. implies the above distribution is in Bernstein center.

Suppose now $|a_1|_F \neq |a_2|_F^{\pm 1}$. It remains to see that Criterion 3.1.5 holds for the unramified component also in this case. On this component we identify ξ with $\xi(\varpi_F)$. Therefore, we need to examine the integral

$$f \ \mapsto\ \frac{1}{2\pi i} \int_{s \in \mathbb{C}, |s|=1} \hat{f}(s) \left((s^{n_1} + s^{-n_1}) - (s^{n_2} + s^{-n_2}) \right) \frac{ds}{s} \tag{4.4a}$$

for different nonnegative integers n_1 and n_2. Without lost of generality, we can suppose $n_1 > n_2$. Recall that the Plancherel measure on this component is

$$f \mapsto \frac{q_F^2 - 1}{2} \frac{1}{2\pi i} \int_{|s|=1} \hat{f}(s)\, \frac{(1-s)(1-s^{-1})}{(q_F - s)(q_F - s^{-1})}\, \frac{ds}{s} + \hat{f}(\pi_{St})\,(q_F - 1)\ .$$

Therefore, the distribution (4.4a) is represented on the unramified component as an integration, with respect to Plancherel measure μ_{PL} restricted to $\{s \in \mathbb{C} \mid |s| = 1\}$, against the function

$$\begin{aligned} r(s) \ &:= \ C \cdot ((s^{n_1} + s^{-n_1}) - (s^{n_2} + s^{-n_2})) \frac{(q_F - s)(q_F - s^{-1})}{(1-s)(1-s^{-1})} \\ &= \ C \cdot s^{n_2} \left(\sum_{k=0}^{n_1 - n_2 - 1} s^k \right) \left(\sum_{k=0}^{n_1 + n_2 - 1} (s^{-1})^k \right) (q_F - s)(q_F - s^{-1}), \end{aligned}$$

which is a Laurent polynomial, and so is regular on $\mathbb{C}^\times$. Since $r(s)$ vanishes at $s = q_F$, the distribution (4.4a) is given as integration against $r(s)$ on the whole unramified component. So, we have verified Criterion 3.1.5 for the unramified component. This completes the proof of Proposition 4.3. □

5. Elliptic orbital integrals

5.1. Reducible principal series. We first review the reducible principal series of $G = SL(2, F)$ from [SS3, §3]. A unitary principal series $\mathrm{Ind}(\chi)$ of G is irreducible if and only if χ is precisely of order 2. Recall that characters of order 2 of $F^\times$ are canonically associated to quadratic extensions of V of F by $\ker(\chi) \leftrightarrow N_{V/F}(V^\times)$. Let $V = F[\sqrt{v}]$ ($v \in \{\epsilon_F, \varpi_F, \epsilon_F\varpi_F\}$) be the quadratic extension of F such that $N_{V/F}(V^\times) = \ker(\chi)$. We use the mnemonic notation of sgn_V to denote the non-trivial character of $F^\times$ with kernel $N_{V/F}(V^\times)$.

Let Φ be a nontrivial additive character of F. For $b \in F^\times$, let Φ_b be the additive character $x \mapsto \Phi(bx)$. Sally and Shalika [SS3, §3] single out an irreducible component $\pi_{\Phi_b,V}$ of $\mathrm{Ind}(\mathrm{sgn}_V)$, with $\pi_{\Phi_b,V}$ equivalent to $\pi_{\Phi_c,V}$ if and only if b and c belong to the same coset of $N_{V/F}(V^\times)$ in $F^\times$.

Suppose $v \in F^\times \backslash (F^\times)^2$, and $V = F[\sqrt{v}]$. If $t = t_v(x, y) \in T_v$, define

$$\mathrm{sgn}_V(t) = \mathrm{sgn}_V(y) \ .$$

We recall an important constant [SS3, A.1]:

$$\kappa(\Phi_b, V) \ := \ \text{principal value of } \int_V \Phi_b(N_{V/F}(x))\, dx \ .$$

We have from [SS3, A.2]: Suppose $b, c \in F^\times$.

(i) If $b/c \in N_{V/F}(V^\times)$, then $\kappa(\Phi_b, V) = \kappa(\Phi_c, V)$.

(ii) If $b/c \notin N_{V/F}(V^\times)$, i.e., b and c are representatives for the two cosets of $N_{V/F}(V^\times)$ in $F^\times$, then $\kappa(\Phi_b, V) = -\kappa(\Phi_c, V)$. In this case we have

$$\mathrm{Ind}(\chi) = \pi_{\Phi_b,V} \oplus \pi_{\Phi_c,V} .$$

The character $\Theta_{\Phi,V}$ of $\pi_{\Phi,V}$, as a locally constant function on the set of regular semisimple elements, satisfies

$$\Theta_{\Phi,V}(g) \;=\; \begin{cases} \dfrac{\mathrm{sgn}_V(\lambda)}{|\lambda-\lambda^{-1}|_F} & \text{if } g \text{ is conjugated to } h(\lambda)\in A_{\emptyset}, \\[2ex] \dfrac{\kappa(\Phi,V)\,\mathrm{sgn}_V(t)}{|D(t)|_F^{\frac12}} & \text{if } g \text{ is regular elliptic and conjugate to} \\ & t\in T_v\cup T_{v\epsilon_F^2} \text{ (resp. } T_v\cup T_{v\varpi_F^{-2}}\text{) if } F[\sqrt{v}] \text{ is} \\ & \text{ramified (resp. unramified) extension of } F; \\ 0 & \text{otherwise.} \end{cases}$$

Let RPS (resp. RPS_V) denote the set of all irreducible subquotients of the reducible unitary principal series (resp. $\mathrm{Ind}(\mathrm{sgn}_V)$). For our situation that F has odd residual characteristic, RPS has six elements.

5.2. Constants, filtrations and measures. If T is a compact Cartan subgroup of G, define

$$\kappa_T \;=\; \begin{cases} \dfrac{q_F+1}{q_F} & \text{if } T \text{ is unramified ,} \\[2ex] 2q_F^{-\frac12} & \text{if } T \text{ is ramified.} \end{cases} \tag{5.2a}$$

Suppose L/F is a finite extension. Recall that the modulus characters $|\;|_L$ and $|\;|_F$ are related by

$$|\;|_L\big|_F \;=\; |\;|_F^{[L:F]}\ .$$

The normalized absolute value $|\;|_{L/F}$ on L, with respect to F, is defined as

$$|\;|_{L/F} \;:=\; |\;|_L^{\frac{1}{[L:F]}}\ .$$

Clearly, $|\;|_{L/F}$ restricted to F equals $|\;|_F$, and it is the unique absolute value on L with this property. We have a canonical filtration of the units $\mathcal{R}_L^\times$ of $\mathcal{R}_L$: If e is the ramification index of L/F, then for $k\ge 0$,

$$(\mathcal{R}_L^\times)_{k/e} \;=\; \begin{cases} \mathcal{R}_L^\times & \text{if } k=0, \\ \{\, x\in\mathcal{R}_L \mid |x-1|_{L/F}\le \dfrac{1}{q_F^{k/e}}\,\} & \text{if } k>0\ . \end{cases}$$

In particular, $(\mathcal{R}_F^\times)_n \;=\; 1+\mathfrak{p}_F^n$ for $n\ge 1$. We transfer the filtration subgroups $(\mathcal{R}_F^\times)_n$ (resp. $(\mathcal{R}_V^\times)_{\frac{k}{e}}$, $V=F[\sqrt{v}]$), to the split torus $A_{\emptyset}$ (resp. elliptic torus T_v) as follows:

$$(A_{\emptyset})_n \;:=\; h((\mathcal{R}_F^\times)_n) \;=\; \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1}\end{bmatrix} \mid a\in(\mathcal{R}_F^\times)_n \right\},$$
$$T_{v,\frac{k}{e}} \;:=\; (\mathcal{R}_V^\times)_{\frac{k}{e}}\cap T_v\ .$$

The matrix $t_v(x,y)$ corresponds to $x+\sqrt{v}y\in V$ in the above formula. Sally and Shalika [SS1, p. 662] define

$$C_v^{(n)} \;:=\; \begin{cases} T_{v,n} & \text{if } V \text{ is unramified,} \\ T_{v,n+\frac12} & \text{if } V \text{ is ramified.} \end{cases}$$

Recall that, following Sally and Shalika [SS1], we have taken Haar measure on additive F so that $\mathrm{meas}(\mathcal{R}_F) = 1$. For an elliptic torus T_v, set $T_{v,0^+}$ to be $T_{v,1}$ (resp. $T_{v,\frac{1}{2}}$) if T_v is unramified (resp. ramified). Then

$$\begin{aligned}\mathrm{meas}(K \cap N_\emptyset) &= 1,\\ \mathrm{meas}((A_\emptyset)_1) &= \frac{1}{q_F},\\ \mathrm{meas}(T_{v,0^+}) &= \frac{1}{q_F}.\end{aligned}$$

5.3. Elliptic orbital integral expansion ([SS3, Theorem 5.1])**.** *Suppose $T = T_v$ is an elliptic torus of $G = SL(2,F)$. Denote $V = F[\sqrt{v}]$. Suppose $h \geq 0$ and $t \in C_v^{(h)}/C_v^{(h+1)}$. Then, for $f \in \mathcal{C}_c^\infty(G)$, we have*

$$\begin{aligned}|D(t)|_F^{\frac{1}{2}} I_f^T(t) &= \sum_{\pi \in \mathrm{SC}} \overline{\Theta}_\pi(t) |D(t)|_F^{\frac{1}{2}}\ \hat{f}(\pi)\\ &+ \overline{\Theta}_{\pi_{St}}(t)|D(t)|_F^{\frac{1}{2}}\ \hat{f}(\pi_{St})\\ &+ \sum_{\pi \in \mathrm{RPS}_V} \frac{1}{2}\overline{\Theta}_\pi(t)|D(t)|_F^{\frac{1}{2}}\ \hat{f}(\pi)\\ &+ \frac{1}{2}\ \kappa_T \int_{\substack{\xi \in \widehat{F^\times}\\ \xi|_{(A_\emptyset)_{h+1}} \equiv 1}} \hat{f}(\pi(\xi))\ d\xi\\ &- \frac{q_F+1}{2q_F^2}\ |D(t)|_F^{\frac{1}{2}} \int_{\substack{\xi \in \widehat{F^\times}\\ \xi|_{(A_\emptyset)_{h+1}} \equiv 1}} \frac{1}{|\Gamma(\xi)|^2}\ \hat{f}(\pi(\xi))\ d\xi,\end{aligned} \tag{5.3a}$$

where Θ_π denotes the character of π as a locally constant function on the set of regular semisimple elements.

5.4. *Remarks.*

(i) In the third line of (5.3a) we can replace the sum over RPS_V by the sum over RPS because the other characters in RPS vanish on t.

(ii) Observe that the Fourier transform of the orbital integral at an elliptic element t on a non-supercuspidal connected component of $\Omega(G)$, which is regular (i.e., $\xi\big|_{h(\mathcal{R}^\times)} \neq \xi^{-1}\big|_{h(\mathcal{R}^\times)}$, or a supercuspidal connected component, is given by integration against a constant with respect to the Plancherel measure. Since constants are regular functions, they automatically satisfy Criterion 3.1.5. Thus, to verify whether a linear combination of elliptic orbital integrals belongs to the Bernstein center, it is sufficient to verify Criterion 3.1.5 on the two irregular connected components only. We now analyze these two components.

(iii) Note that $\overline{\Theta}_{\pi_{\mathrm{St}}}(t) = -1$, so the measure on the unramified Bernstein component is

$$\begin{aligned} & -|D(t)|_F^{\frac{1}{2}} \left\{ \delta_{\mathrm{St}} + \frac{1}{2}\frac{q_F+1}{q_F^2} \frac{1}{|\Gamma(\xi)|^2}\, d\xi \right\} + \frac{1}{2}\kappa_T\, d\xi \\ (5.4a) \qquad & = \frac{-|D(t)|_F^{\frac{1}{2}}}{q_F-1} \left\{ (q_F-1)\delta_{\mathrm{St}} + \frac{1}{2}\frac{q_F^2-1}{q_F^2} \frac{1}{|\Gamma(\xi)|^2}\, d\xi \right\} + \frac{1}{2}\kappa_T\, d\xi \ , \end{aligned}$$

The term $(q_F-1)\delta_{\mathrm{St}} + \frac{1}{2}\frac{q_F^2-1}{q_F^2} \frac{1}{|\Gamma(\xi)|^2}\, d\xi$ is obviously the Plancherel measure on this component.

(iv) We analyze now the part of the Fourier transform measure of elliptic orbital integral supported in the ramified irregular component. In (5.3a), we see the contributions to the measure are from lines 3, 4 and 5. For the latter two, the contribution is an integration against a multiple of the Plancherel measure. In particular, these contributions satisfy Criterion 3.1.5. Consider the contribution from the line 3. Recall that V is the quadratic extension obtained by adjoining the eigenvalues of t to F. The character value of an irreducible subrepresentation of a reducible unitary principal series representation is nonzero at t precisely when that subrepresentation occurs as a subrepresentation in $\mathrm{Ind}(\mathrm{sgn}_V)$. Let $\pi_{\Phi,V}$ and $\pi_{\Phi',V}$ be the two components of $\mathrm{Ind}(\mathrm{sgn}_V)$. The Fourier Transform measure of the distribution $f \mapsto |D(t)|_F^{\frac{1}{2}} I_f^T(t)$ on the union of the two irreducible subrepresentations of $\mathrm{Ind}(\mathrm{sgn}_V)$ is

$$(5.4b) \qquad \frac{\mathrm{sgn}_V(t)}{2} \left\{ \overline{\kappa}_{\Phi,V}\, \delta_{\pi_{\Phi,V}} + \overline{\kappa}_{\Phi',V}\, \delta_{\pi_{\Phi',V}} \right\} .$$

5.5. Corollary. *Suppose T is an elliptic torus of G, $t_1,\, t_2 \in T^{\mathrm{reg}}$ and $\mathcal{O}(t_1)$ and $\mathcal{O}(t_2)$ have the same asymptotic behavior at infinity. Then, the invariant distribution, which is the difference*

$$(5.5a) \qquad f \ \mapsto \ |D(t_1)|_F^{\frac{1}{2}} I_f^T(t_1) \ - \ |D(t_2)|_F^{\frac{1}{2}} I_f^T(t_2),$$

lies in the Bernstein center.

Proof. We need to analyze the Fourier transform measure of (5.5a). By remark (5.4)(ii), we need to consider only the two irregular components. Consider first the unramified connected component. From remark (5.4)(iii), we see that the terms $\frac{1}{2}\kappa_T$ of (5.4a) cancel and so leave a multiple of the Plancherel measure. Thus, Criterion 3.1.5 is verified for the unramified Bernstein component. Note we have not used the hypothesis that $\mathcal{O}(t_1)$ and $\mathcal{O}(t_2)$ are asymptotic.

Consider now the ramified irregular component. Suppose that t_1 and t_2 have the same asymptotic behavior at infinity, i.e., they have the same set of asymptotically close unipotent orbits.

Suppose first that $T = T_v$ is a ramified torus (then $V = F[\sqrt{v}]$). Then the asymptotic unipotent orbits to $t_1 = t_v(x_1, y_1)$ are $\mathcal{O}(n(y_1))$ and $\mathcal{O}(n(-vy_1))$. Further, asymptotic unipotent orbits to $t_2 = t_v(x_2, y_2)$ are $\mathcal{O}(n(y_2))$ and $\mathcal{O}(n(-vy_2))$. We have two possibilities. The first is that $\mathcal{O}(n(y_1)) = \mathcal{O}(n(y_2))$, which implies that $y_1/y_2 \in (F^\times)^2$. Now clearly $\mathrm{sgn}_V(t_1) = \mathrm{sgn}_V(t_2)$. Therefore, the terms (5.4b) cancel. So we have verified Criterion 3.1.5. This shows that the distribution (5.5a) is in the Bernstein center. Suppose now $\mathcal{O}(n(-vy_1)) = \mathcal{O}(n(y_2))$.

Then, $-vy_1/y_2 \in (F^\times)^2$. Since $\sqrt{v}$ is in V, and thus $\mathrm{sgn}_V(-v) = 1$, we conclude $\mathrm{sgn}_V(t_1) = \mathrm{sgn}_V(t_2)$, and by the same argument as above we see the distribution (5.5a) is in the Bernstein center.

Consider now the case when $T = T_v$ is an unramified torus. The asymptotic unipotent orbits to $t_1 = t_v(x_1, y_1)$ are $\mathcal{O}(n(y_1))$ and $\mathcal{O}(n(\epsilon_F y_1))$, and the asymptotic unipotent orbits to $t_2 = t_v(x_2, y_2)$ are $\mathcal{O}(n(y_2))$ and $\mathcal{O}(n(\epsilon_F y_2))$. Of these two possibilities, the first $\mathcal{O}(n(y_1)) = \mathcal{O}(n(y_2))$ implies that $y_1/y_2 \in (F^\times)^2$, which in turn means $\mathrm{sgn}_V(t_1) = \mathrm{sgn}_V(t_2)$. From this, we deduce, as above, the distribution (5.5a) is in the Bernstein center. The remaining case is $\mathcal{O}(n(\epsilon_F y_1)) = \mathcal{O}(n(y_2))$. Thus, $\epsilon_F y_1/y_2 \in (F^\times)^2$. We know that ϵ_F is in the norm group of V, so $\mathrm{sgn}_V(t_1) = \mathrm{sgn}_V(t_2)$, and the claim follows. The proof is now complete. □

6. Unipotent orbital integrals

6.1. Constants. Suppose V is a quadratic extension of F and $T = T_v$ is an elliptic torus of $G = SL(2, F)$ associated to V. Define

$$\kappa_V = \kappa_T \quad \text{(see (5.2a))} .$$

If an irreducible constituent π of a reducible unitary principal series is parametrized by $\Phi = \Phi_1$ and V as in Section 5, let

$$\kappa(\pi) = \kappa(\Phi, V) .$$

Shalika, in his thesis [Sh] attaches to each nontrivial character ψ of the norm one elements of V and Φ, an irreducible supercuspidal representation

$$\pi = \pi(\Phi, \psi, V).$$

Denote by SC_V the set of these classes in SC, and set $\kappa(\pi) = \kappa(\Phi, V)$ also in this case.

6.2. Unipotent orbital integral expansion ([SS3, Theorem 7.1])**.** *For $\zeta \in F^\times/(F^\times)^2$, let Λ_ζ denote the orbital integral over the conjugacy class of the unipotent element $n(\zeta)$. Then, for $f \in \mathcal{C}_c^\infty(G)$, we have*

$$\begin{aligned} 4\frac{q_F}{q_F - 1}\,\Lambda_\zeta(f) \;=\; & \sum_V \frac{2}{\kappa_V}\,\mathrm{sgn}_V(\zeta) \sum_{\pi \in \mathrm{SC}_V} \overline{\kappa(\pi)}\;\hat{f}(\pi) \\ & + \sum_V \frac{1}{\kappa_V}\,\mathrm{sgn}_V(\zeta) \sum_{\pi \in \mathrm{RPS}_V} \overline{\kappa(\pi)}\;\hat{f}(\pi) \\ & + \int_{\xi \in \widehat{F^\times}} \Theta_{\pi(\xi)}(f)\,d\xi. \end{aligned}$$

6.3. Corollary.

$$\frac{q_F}{q_F - 1} \sum_{\zeta \in F^\times/(F^\times)^2} \Lambda_\zeta(f) \;=\; \int_{\xi \in \widehat{F^\times}} \Theta_{\pi(\xi)}(f)\,d\xi.$$

Proof. Since

$$4\frac{q_F}{q_F-1}\sum_{\zeta\in F^\times/(F^\times)^2}\Lambda_\zeta(f)=\sum_V\ \frac{2}{\kappa_V}\left(\sum_{\zeta\in F^\times/(F^\times)^2}\mathrm{sgn}_V(\zeta)\right)\sum_{\pi\in\mathrm{SC}_V}\overline{\kappa(\pi)}\ \hat{f}(\pi)$$
$$+\sum_V\ \frac{1}{\kappa_V}\left(\sum_{\zeta\in F^\times/(F^\times)^2}\mathrm{sgn}_V(\zeta)\right)\sum_{\pi\in\mathrm{RPS}_V}\overline{\kappa(\pi)}\ \hat{f}(\pi)$$
$$+\,4\int_{\xi\in\widehat{F^\times}}\Theta_{\pi(\xi)}(f)\,d\xi,$$

the corollary follows immediately using $\sum_{\zeta\in F^\times/(F^\times)^2}\mathrm{sgn}_V(\zeta)=0$. □

6.4. Corollary. *Let C be the constant of (4.2a). Suppose $t\in A_\emptyset$ is regular. Then, the invariant distribution*

$$f\ \mapsto\ |D(t)|^{\frac12}I_f^{A_\emptyset}(t)\ -\ C\,\frac{q_F}{q_F-1}\sum_{\zeta\in F^\times/(F^\times)^2}\Lambda_\zeta(f)$$

lies in the Bernstein center.

Proof. This follows from the split orbital integral expansion from Section 4, the above corollary and Criterion 3.1.5. □

Consider an elliptic torus T_v with v as in (2.5.3a). Let $E=F[\sqrt{v}]$ and $N_{E/F}(E^\times)=(F^\times)^2\cup\beta(F^\times)^2$. If E is unramified (resp. ramified), we can take $\beta=\epsilon_F$ (resp. $\beta=-v$). The regular element

$$t_v(\alpha,\zeta)\ =\ \begin{bmatrix}\alpha & \zeta\\ v\zeta & \alpha\end{bmatrix}\ \in T_v$$

is asymptotic to the two unipotent orbits of

$$n(\zeta)=\begin{bmatrix}1 & \zeta\\ 0 & 1\end{bmatrix}\quad\text{and}\quad n(\beta\zeta)=\begin{bmatrix}1 & \beta\zeta\\ 0 & 1\end{bmatrix}\ .$$

6.5. Corollary. *With the above notation, the orbit of $t=t_v(\alpha,\zeta)$ is asymptotic to the two unipotent classes of $n(\zeta)$ and $n(\beta\zeta)$, and the invariant distribution*

$$f\ \mapsto\ |D(t)|^{\frac12}I_f^{T}(t)\ -\ \kappa_E\frac{q_F}{q_F-1}\ \{\ \Lambda_\zeta(f)\ +\ \Lambda_{\beta\zeta}(f)\ \}\tag{6.5a}$$

lies in the Bernstein center.

Proof. Set $X := \{1, \beta\}$. Note that $\mathrm{sgn}_V(\zeta) + \mathrm{sgn}_V(\beta\zeta) = 0$ for any quadratic extension V/F different from E. Therefore, we have

$$\begin{aligned} \frac{q_F}{q_F - 1} \sum_{\psi \in X} \Lambda_{\psi\zeta}(f) \;=\; & \sum_V \frac{1}{2\kappa_V} \Big(\sum_{\psi \in X} \mathrm{sgn}_V(\psi\zeta) \Big) \sum_{\pi \in \mathrm{SC}_V} \overline{\kappa(\pi)}\; \hat{f}(\pi) \\ & + \sum_V \frac{1}{4\kappa_V} \Big(\sum_{\psi \in X} \mathrm{sgn}_V(\psi\zeta) \Big) \sum_{\pi \in \mathrm{RPS}_V} \overline{\kappa(\pi)}\; \hat{f}(\pi) \\ & + \frac{1}{2} \int_{\xi \in \widehat{F^\times}} \Theta_{\pi(\xi)}(f)\, d\xi \\ = \; & \frac{\mathrm{sgn}_E(\zeta)}{\kappa_E} \Big\{ \sum_{\pi \in \mathrm{SC}_E} \overline{\kappa(\pi)}\; \hat{f}(\pi) \;+\; \frac{1}{2} \sum_{\pi \in \mathrm{RPS}_E} \overline{\kappa(\pi)}\; \hat{f}(\pi) \Big\} \\ & + \frac{1}{2} \int_{\xi \in \widehat{F^\times}} \Theta_{\pi(\xi)}(f)\, d\xi . \end{aligned}$$

Thus

$$\begin{aligned} & \kappa_E \frac{q_F}{q_F - 1} \sum_{\psi \in N_{E/F}(E^\times)/(F^\times)^2} \Lambda_\psi(f) \\ & = \mathrm{sgn}_E(\zeta) \Big\{ \sum_{\pi \in \mathrm{SC}_E} \overline{\kappa(\pi)}\; \hat{f}(\pi) + \frac{1}{2} \sum_{\pi \in \mathrm{RPS}_E} \overline{\kappa(\pi)}\; \hat{f}(\pi) \Big\} \;+\; \frac{\kappa_E}{2} \int_{\xi \in \widehat{F^\times}} \Theta_{\pi(\xi)}(f)\, d\xi . \end{aligned}$$

From the above formula and (5.4a), we see that on the unramified connected component, Criterion 3.1.5 holds for the linear combination (6.5a). By Remark (5.4)(iv) we see that Criterion 3.1.5 holds on irregular ramified components. The remaining components obviously satisfy Criterion 3.1.5. This completes the proof. □

7. A geometric proof

7.1. In this section we give an alternative proof to some of our results that certain differences of normalized orbital integrals belong to the Bernstein center. Our alternative proof is elegant and it is intriguing whether it can be extrapolated in some form to higher rank groups. This section is based on joint work with Dan Barbasch.

7.2. Invariant forms. Let a, b, c, $d : G \to F$ be the four coordinates of a matrix in $G = SL(2, F)$, so

$$g \;=\; \begin{bmatrix} a(g) & b(g) \\ c(g) & d(g) \end{bmatrix} \quad \text{and} \quad ad - bc \equiv 1 \text{ on } \; G \; .$$

Let da, db, dc, and dd be the differentials of a, b, c, and d respectively. It is an elementary calculation that

$$\omega_3 \;=\; da \wedge db \wedge dc$$

defines, up to nonzero scalar multiple, the unique left and right translation G-invariant 3-form on G. Furthermore,

$$-\frac{db \wedge dc}{a - d} \;=\; \frac{da \wedge db}{b} \;=\; \frac{da \wedge dc}{c} \; . \tag{7.2a}$$

We deduce that (7.2a) defines a global 2-form ω_2 on G. The form ω_2 is Ad-invariant. Its restriction to a regular conjugacy class $\mathcal{O}$ yields a G-invariant measure on $\mathcal{O}$. For C a compact measurable set in $\mathcal{O}$, let $\mathrm{meas}_{\mathcal{O}}(C)$ denote its measure with respect to ω_2. Note also, that since $a+d : G \to F$ is the trace, we have

$$\frac{db \wedge dc}{a-d} \;=\; \frac{db \wedge dc}{2a - \mathrm{trace}} \;=\; \frac{db \wedge dc}{\mathrm{trace} - 2d}\ .$$

7.3. Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two conjugacy classes in $G = SL(2,F)$ with the same asymptotic behavior at infinity, and J is an open compact subgroup of G. It is a consequence of Propositions 2.3.3, 2.4.3, and 2.5.5 that if $g \in G$ is sufficiently large, i.e., a Cartan decomposition $g = k_1 a k_2$ has a large, then

$$gJ \cap \mathcal{O}_1 \quad \text{and} \quad gJ \cap \mathcal{O}_2.$$

are both empty or both nonempty.

7.4. Proposition. *Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two conjugacy classes in $G = SL(2,F)$ with the same asymptotic behavior at infinity. Then, for $g \in G$ sufficiently large*

$$\mathrm{meas}_{\mathcal{O}_1}(gJ \cap \mathcal{O}_1) = \mathrm{meas}_{\mathcal{O}_2}(gJ \cap \mathcal{O}_2).$$

Proof. As remarked immediately above, we may assume g is sufficiently large so that $gJ \cap \mathcal{O}_1$ and $gJ \cap \mathcal{O}_2$ are both empty or both nonempty. The assertion is trivially true if the two intersections are empty, so we assume the two intersections are nonempty. Also, it is enough to prove the proposition in the special case when $J = K_m$ is an arbitrary m-th congruence subgroup of $K = SL(2,\mathcal{R}_F)$. We can choose local coordinates for the two intersections $gK_m \cap \mathcal{O}_1$ and $gK_m \cap \mathcal{O}_2$ according to the following rule: Select the coordinate of g with the largest size. It is elementary that for any $gk \in gK_m$ the same coordinate is also the largest coordinate of gk. We consider the obvious four possible cases.

(i) $|b(g)|_F$ is maximal. Write $k \in K_m$ as

$$k \;=\; \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;=\; \begin{bmatrix} 1+\alpha\varpi^m & \beta\varpi^m \\ \gamma\varpi^m & 1+\delta\varpi^m \end{bmatrix}\ ,$$

where α, β, γ, $\delta \in \mathcal{R}_F$, and

$$gk = \begin{bmatrix} a(g) & b(g) \\ c(g) & d(g) \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;=\; \begin{bmatrix} a(g)A + b(g)C & a(g)B + b(g)D \\ c(g)A + d(g)C & c(g)B + d(g)D \end{bmatrix}\ .$$

Let $c_{1,1} = a(gk)$, $c_{1,2} = b(gk)$, $c_{2,1} = c(gk)$, $c_{2,2} = d(gk)$ denote the entries of the product. The assumption that $b(g)$ has the largest absolute value means

$$\begin{aligned} c_{1,1} \;&=\; a(g)A + b(g)C \;=\; a(g)(1+\alpha\varpi^m) + b(g)\gamma\varpi^m \\ &=\; a(g) + (a(g)\alpha + b(g)\gamma)\varpi^m \\ &\in\; a(g) + b(g)\mathfrak{p}^m\ . \end{aligned}$$

Similarly,

$$\begin{aligned} c_{1,2} \;&=\; a(g)B + b(g)D \;=\; a(g)\beta\varpi^m + b(g)(1+\delta\varpi^m) \\ &=\; b(g) + (b(g)\delta + a(g)\beta)\varpi^m \\ &\in\; b(g) + b(g)\mathfrak{p}^m\ . \end{aligned}$$

Set $C_{1,1} := a(g) + b(g)\mathfrak{p}^m$ and $C_{1,2} := b(g) + b(g)\mathfrak{p}^m$. As we vary $k \in K_m$, both γ and δ run over $\mathcal{R}_F$ independently of one another. It follows that in the coset gK_m, we can allow the (1,1) and (1,2) coordinates of an element to run over the two cosets $C_{1,1}$ and $C_{1,2}$ independently of one another. From this, we deduce that the cartesian product $C_{1,1} \times C_{1,2}$ can be used as coordinates for the intersection of each conjugacy class $\mathcal{O}_i$ with gK_m. The G-invariant measure is given by

$$\frac{da \wedge db}{|b|_F}\ .$$

The value $|b|_F$ is constant on the parametrization set. We conclude the G-invariant measure of the two intersections $gJ \cap \mathcal{O}_1$ and $gJ \cap \mathcal{O}_2$ are equal.

(ii) $|a(g)|_F$ is maximal. Let $\mathrm{tr}(\mathcal{O}_i)$ denote the trace of any element of $\mathcal{O}_i$. We use the assumption that the intersection $gK_m \cap \mathcal{O}_i$ is nonempty. Suppose $gk \in gK_m \cap \mathcal{O}_i$. Then $a(gk) + d(gk) = \mathrm{tr}(\mathcal{O}_i)$. The assumption that g is sufficiently large and the (1,1) coordinate of g has maximal size means $d(g)$ must have the same absolute value as $a(g)$. If we combine these facts with $\det(g) = 1$, we deduce $a(g)$, $b(g)$, $c(g)$, and $d(g)$ all have the same absolute value. Thus, this case reduces to the previous case (i).

The remaining two cases when $|c(g)|_F$ and $|d(g)|_F$ are maximal can obviously be treated in similar fashions to (i) and (ii) respectively. This completes the proof. □

7.5. Corollary. *Suppose T is a maximal torus of $G = SL(2,F)$ and $\gamma_1, \gamma_2 \in T^{\mathrm{reg}}$ such that their conjugacy classes $\mathcal{O}(\gamma_1)$ and $\mathcal{O}(\gamma_2)$ have the same asymptotic behavior at infinity. Then, the G-invariant distribution which is the difference*

$$f \in \mathcal{C}_c^\infty(G)\ :\ f \ \mapsto\ |D(\gamma_1)|_F^{\frac{1}{2}}\, I_f^T(\gamma_1) - |D(\gamma_1)|_F^{\frac{1}{2}}\, I_f^T(\gamma_2) \tag{7.5a}$$

is essentially compact and so belongs to the Bernstein center.

Proof. Let μ_1, μ_2 be the G-invariant measures on $\mathcal{O}(\gamma_1)$ and $\mathcal{O}(\gamma_2)$ obtained from the 2-form ω_2 of (7.2a), and let D be the distribution

$$f \in \mathcal{C}_c^\infty(G)\ :\ f \mapsto \int_{\mathcal{O}(\gamma_1)} f\, d\mu_1 \ -\ \int_{\mathcal{O}(\gamma_2)} f\, d\mu_2\ . \tag{7.5b}$$

Claim: D is essentially compact.

To prove the claim, it is sufficient to prove the special situation when f is the characteristic function of an arbitrary congruence subgroup K_m. We have

$$\begin{aligned} D \star 1_{K_m}(x) \ &=\ \int_{\mathcal{O}(\gamma_1)} \lambda_x(1_{K_m})\, d\mu_1 \ -\ \int_{\mathcal{O}(\gamma_2)} \lambda_x(1_{K_m})\, d\mu_2 \\ &= \int_{\mathcal{O}(\gamma_1)} 1_{xK_m}\, d\mu_1 \ -\ \int_{\mathcal{O}(\gamma_2)} 1_{xK_m}\, d\mu_2 \\ &= \mathrm{meas}(\mathcal{O}(\gamma_1) \cap xK_m) \ -\ \mathrm{meas}(\mathcal{O}(\gamma_2) \cap xK_m) \end{aligned}$$

By Proposition 7.4., the last line is zero provided x is sufficiently large. This proves the claim and thus D belongs to the Bernstein center. The corollary follows when we observe the distributions of (7.5a) and (7.5b) are the same. □

References

[Bn] Bernstein, J. (written by K. Rumelhart), *Draft of: Representations of p-adic groups*, preprint.

[BD] Bernstein, J., *Le "centre" de Bernstein* (Edited by Deligne, P.) in "Représentations des Groupes Réductifs sur un Corps Local" written by J.-N. Bernstein, P. Deligne, D. Kazhdan, M.-F. and Vignéras,, Hermann, Paris, 1984. MR0771671 (86e:22028)

[BDK] Bernstein, J., Deligne, P. and Kazhdan, D., *Trace Paley-Wiener theorem for reductive p-adic groups*, J. Analyse Math. **47** (1986), 180-192. MR0874050 (88g:22016)

[Di] Dijk, G. van, *Computation of certain induced characters of $\mathfrak{p}$-adic groups*, Math. Ann. **199** (1972), 229-240. MR0338277 (49 #3043)

[HC1] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Symp. Pure Math. 26,, Amer. Math. Soc., Providence, Rhode Island, 1973, pp. 167-192. MR0340486 (49 #5238)

[HC2] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Lie theories and their applications (Proc. Ann. Sem. Canad. Math. Congr., Queen's Univ., Kingston, Ont., 1977). Queen's Papers in Pure Appl. Math., No. 48, Queen's Univ., Kingston, Ont., 1978, pp. 281–347. MR0579175 (58 #28313)

[MT] Moy, A., Tadić, M., *The Bernstein center in terms of invariant locally integrable functions.*, Represent. Theory **6** (2002), 313–329. MR1979109 (2004f:22019)

[Sy] Sally, P. J. Jr., *Character formulas for SL_2.*, Harmonic analysis on homogeneous spaces Proc. Sympos. Pure Math., Williams Coll., Williamstown Mass., 1972, Vol. XXVI, (1973), 395–400. MR0338281 (49 #3047)

[Sh] Shalika, J. A., *Representations of the Two by Two Unimodular Group over Local Fields*, thesis, The John Hopkins University (1966).

[SS1] Sally, P. J. Jr., Shalika, J. A., *The Plancherel formula for $SL(2)$ over a local field*, Proceedings of Nat. Acad. Sci. U.S.A. **63** (1969), 661–667. MR0364559 (51 #813)

[SS2] Sally, P. J. Jr., Shalika, J. A., *Characters of the discrete series of representations of $SL(2)$ over a local field*, Proceedings of Nat. Acad. Sci. U.S.A. **61** (1968), 1231–1237. MR0237713 (38 #5994)

[SS3] Sally, Paul J. Jr., Shalika, Joseph A., *The Fourier transform of orbital integrals on SL_2 over a p-adic field.*, Lie group representations, II (College Park, Md., 1982/1983), Lecture Notes in Math., 1041, Springer **1041** (1984), 303–340,. MR0748512 (86a:22017)

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