

## LEFT CELLS AND CONSTRUCTIBLE REPRESENTATIONS

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ABSTRACT. We consider the partition of a finite Coxeter group  $W$  into left cells with respect to a weight function  $L$ . In the equal parameter case, Lusztig has shown that the representations carried by the left cells are precisely the so-called constructible ones. We show that this holds for general  $L$ , assuming that the conjectural properties (P1)–(P15) in Lusztig’s book on Hecke algebras with unequal parameters hold for  $W, L$ . Our proofs use the idea (Gyoja, Rouquier) that left cell representations are projective in the sense of modular representation theory. This also gives partly new proofs for Lusztig’s result in the equal parameter case.

### 1. INTRODUCTION

Let  $W$  be a finite Coxeter group and  $L$  a weight function on  $W$ , as in [27]. Thus,  $L$  is a function  $L: W \rightarrow \mathbb{Z}$  such that  $L(ww') = L(w) + L(w')$  whenever  $l(ww') = l(w) + l(w')$  where  $l$  is the length function on  $W$ . We assume that  $L(w) > 0$  for all  $w \neq 1$ . The choice of such an  $L$  gives rise to a partition of  $W$  into left cells; each left cell naturally carries a representation of  $W$ . These representations play an important role, for example, in the representation theory of reductive groups over finite or  $p$ -adic fields; see [24], [27, Chap. 0]. In the case where  $L = al$  for some  $a > 0$ , the representations carried by the left cells are explicitly known: by Lusztig [25], they are precisely the constructible representations which were defined (and determined) in [23].

Let us now consider a general weight function  $L$  and let us assume that the conjectural properties (P1)–(P15) in [27, Chap. 14] hold; we recall these properties and some of their consequences in Section 2. The purpose of this paper is to prove that, in this setting, the left cell representations are again the constructible ones, as conjectured by Lusztig [27, 22.29]. Our arguments also cover the case of equal parameters, so that we obtain partly new proofs for Lusztig’s results in [25].

The main theme of this paper is to use a generalization of a result of Rouquier [30] which shows that left cell representations can be interpreted as projective indecomposable representations in the sense of modular representation theory. Rouquier’s original result was concerned with the equal parameter case. The proof of its generalisation to arbitrary weight functions relies on (P1)–(P15); see Section 3. In this context, a crucial role is played by a general result from [15] which shows that the columns of the “decomposition matrix” are linearly independent; see (3.9). This result yields, for example, the fact that the representations carried by the various

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left cells of  $W$  are linearly independent in the appropriate Grothendieck group; see Remark 3.12.

See Gyoja [17] where the connection between left cells and modular representations first appeared.

Further general results on left cells, constructible representations and families will be discussed in Sections 4 and 5.

In order to deal with type  $B_m$  (Section 6), we need a purely combinatorial identity about the constructible representations; see Proposition 6.2. We shall give here an elementary proof of that identity which uses the fact that the constructible representations can be identified with the canonical basis of a certain irreducible representation of the Lie algebra  $\mathfrak{sl}_{n+1}(\mathbb{C})$ , as shown by Leclerc and Miyachi [20]. (I wish to thank Bernard Leclerc for his help with the proof of Proposition 6.2.)

Finally, in Section 7, we discuss the computation of left cell representations in groups of exceptional type. For groups of type  $E_6$ ,  $E_7$ ,  $E_8$  and  $F_4$  with equal parameters, we show that Lusztig’s proof in [25], which partly relies on deep results about representations of reductive groups over finite fields [24], can be replaced by arguments which only rely on (P1)–(P15) and some explicit computations with the character tables in [14].

## 2. LUSZTIG’S CONJECTURES

Let  $(W, S)$  be a Coxeter system where  $W$  is finite; let  $L: W \rightarrow \mathbb{Z}$  be a weight function such that  $L(s) > 0$  for all  $s \in S$ . Let  $A = \mathbb{Z}[v, v^{-1}]$  where  $v$  is an indeterminate. We set  $v_w = v^{L(w)}$  for all  $w \in W$ . Let  $H$  be the generic Iwahori–Hecke algebra corresponding to  $(W, S)$  with parameters  $\{v_s \mid s \in S\}$ . Thus,  $H$  has an  $A$ -basis  $\{T_w \mid w \in W\}$  and the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } l(sw) = l(w) - 1, \end{cases}$$

where  $s \in S$  and  $w \in W$ . Let  $\{c_w \mid w \in W\}$  be the “new” basis of  $H$  defined in [27, Theorem 5.2]. We have  $c_w = T_w + \sum_y p_{y,w} T_y$  where  $p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$  and  $p_{y,w} = 0$  unless  $y < w$  in the Bruhat–Chevalley order. Given  $x, y \in W$ , we can write

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z \quad \text{where } h_{x,y,z} \in A.$$

We usually work with the elements  $c_w^\dagger$  obtained by applying the unique  $A$ -algebra involution  $H \rightarrow H$ ,  $h \mapsto h^\dagger$  such that  $T_s^\dagger = -T_s^{-1}$  for any  $s \in S$ ; see [27, 3.5].

We refer to [27, Chap. 8] for the definition of the preorders  $\leq_{\mathcal{L}}$ ,  $\leq_{\mathcal{R}}$ ,  $\leq_{\mathcal{LR}}$  and the corresponding equivalence relations  $\sim_{\mathcal{L}}$ ,  $\sim_{\mathcal{R}}$ ,  $\sim_{\mathcal{LR}}$  on  $W$ , induced by  $L$ . (See also Remark 2.4 below.) The equivalence classes with respect to these relations are called left, right and two-sided cells of  $W$  with respect to  $L$ , respectively.

Let  $\Gamma$  be a left cell of  $W$ . As in [27, 21.1], the  $A$ -submodule

$$[\Gamma]_A := \sum_{y \in \Gamma} A c_y^\dagger$$

of  $H$  can be regarded as an  $H$ -module by the rule  $c_x^\dagger \cdot c_y^\dagger = \sum_{z \in \Gamma} h_{x,y,z} c_z^\dagger$  where  $x \in W$ ,  $y \in \Gamma$ . By change of scalars ( $v \mapsto 1$ ) this gives rise to a  $\mathbb{C}[W]$ -module which we simply denote by  $[\Gamma]$ . The representation  $[\Gamma]$  is called a *left cell representation* of  $W$ .

**Conjecture 2.1** (Lusztig [27, 22.29]). *Let  $\Gamma$  be a left cell of  $W$ . Then  $[\Gamma]$  is a constructible representation in the sense of [27, 22.1]. Furthermore, all constructible representations arise in this way.*

We recall the definition of constructible representations in (4.1). Conjecture 2.1 is already known to be true if  $L = al$  for some  $a > 0$ , see [25] (for  $W$  a finite Weyl group) and [1] (for  $W$  of type  $H_4$ ).

We will show in Sections 6 and 7 that Conjecture 2.1 is true, if the general conjectural properties (P1)–(P15) formulated by Lusztig in [27, Chap. 14] hold. This also yields a partly new proof for finite Weyl groups and  $L = al$  for some  $a > 0$  (where (P1)–(P15) are known to hold; see [27, Chap. 15].) Assuming (P1)–(P15), we already know that every constructible representation is of the form  $[\Gamma]$  for some left cell  $\Gamma$ ; see [27, Lemma 22.2]. So it remains to show that  $[\Gamma]$  is constructible for every left cell  $\Gamma$ .

To state (P1)–(P15), we have to introduce some further notation.

For  $z \in W$ , there is a unique integer  $\mathbf{a}(z) \geq 0$  such that

$$\begin{aligned} h_{x,y,z} &\in v^{\mathbf{a}(z)}\mathbb{Z}[v^{-1}] && \text{for all } x, y \in W, \\ h_{x,y,z} &\notin v^{\mathbf{a}(z)-1}\mathbb{Z}[v^{-1}] && \text{for some } x, y \in W; \end{aligned}$$

see [27, 13.6]. Given  $x, y, z \in W$ , we define  $\gamma_{x,y,z^{-1}} \in \mathbb{Z}$  by the condition

$$h_{x,y,z} = \gamma_{x,y,z^{-1}}v^{\mathbf{a}(z)} + \text{strictly smaller powers of } v.$$

Finally, for  $z \in W$ , we define an integer  $\Delta(z) \geq 0$  and a non-zero integer  $n_z \in \mathbb{Z}$  by the condition

$$p_{1,z} = n_z v^{-\Delta(z)} + \text{strictly smaller powers of } v; \quad \text{see [27, 14.1].}$$

**Conjecture 2.2** (Lusztig [27, 14.2]). *Let  $\mathcal{D} = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\}$ . Then the following properties hold.*

- P1.** *For any  $z \in W$  we have  $\mathbf{a}(z) \leq \Delta(z)$ .*
- P2.** *If  $d \in \mathcal{D}$  and  $x, y \in W$  satisfy  $\gamma_{x,y,d} \neq 0$ , then  $x = y^{-1}$ .*
- P3.** *If  $y \in W$ , there exists a unique  $d \in \mathcal{D}$  such that  $\gamma_{y^{-1},y,d} \neq 0$ .*
- P4.** *If  $z' \leq_{\mathcal{LR}} z$ , then  $\mathbf{a}(z') \geq \mathbf{a}(z)$ . Hence, if  $z' \sim_{\mathcal{LR}} z$ , then  $\mathbf{a}(z) = \mathbf{a}(z')$ .*
- P5.** *If  $d \in \mathcal{D}$ ,  $y \in W$ ,  $\gamma_{y^{-1},y,d} \neq 0$ , then  $\gamma_{y^{-1},y,d} = n_d = \pm 1$ .*
- P6.** *If  $d \in \mathcal{D}$ , then  $d^2 = 1$ .*
- P7.** *For any  $x, y, z \in W$ , we have  $\gamma_{x,y,z} = \gamma_{y,z,x}$ .*
- P8.** *Let  $x, y, z \in W$  be such that  $\gamma_{x,y,z} \neq 0$ . Then  $x \sim_{\mathcal{L}} y^{-1}$ ,  $y \sim_{\mathcal{L}} z^{-1}$ ,  $z \sim_{\mathcal{L}} x^{-1}$ .*
- P9.** *If  $z' \leq_{\mathcal{L}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{L}} z$ .*
- P10.** *If  $z' \leq_{\mathcal{R}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{R}} z$ .*
- P11.** *If  $z' \leq_{\mathcal{LR}} z$  and  $\mathbf{a}(z') = \mathbf{a}(z)$ , then  $z' \sim_{\mathcal{LR}} z$ .*
- P12.** *Let  $I \subset S$  and  $W_I$  be the parabolic subgroup generated by  $I$ . If  $y \in W_I$ , then  $\mathbf{a}(y)$  computed in terms of  $W_I$  is equal to  $\mathbf{a}(y)$  computed in terms of  $W$ .*
- P13.** *Any left cell  $\Gamma$  of  $W$  contains a unique element  $d \in \mathcal{D}$ . We have  $\gamma_{x^{-1},x,d} \neq 0$  for all  $x \in \Gamma$ .*
- P14.** *For any  $z \in W$ , we have  $z \sim_{\mathcal{LR}} z^{-1}$ .*
- P15.** *Let  $v'$  be a second indeterminate and let  $h'_{x,y,z} \in \mathbb{Z}[v', v'^{-1}]$  be obtained from  $h_{x,y,z}$  by the substitution  $v \mapsto v'$ . If  $x, x', y, w \in W$  satisfy  $\mathbf{a}(w) = \mathbf{a}(y)$ ,*

then

$$\sum_{y' \in W} h'_{w,x',y'} h_{x,y',y} = \sum_{y' \in W} h_{x,w,y'} h'_{y',x',y}$$

By [27, Chap. 15], the above conjectures hold if  $L = al$  for some  $a > 0$ . We shall assume from now on that the above conjectures hold for  $W, L$ . Then we can define a new algebra  $J$  over  $\mathbb{Z}$  as in [27, Chap. 18]. As a  $\mathbb{Z}$ -module,  $J$  is free with a basis  $\{t_w \mid w \in W\}$ . The multiplication is defined by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z} t_{z^{-1}}.$$

This multiplication is associative and we have an identity element given by  $1_J = \sum_{d \in \mathcal{D}} n_d t_d$ . Furthermore, we have a homomorphism of  $A$ -algebras  $\phi: H \rightarrow J_A = A \otimes_{\mathbb{Z}} J$  given by

$$\phi(c_w^\dagger) = \sum_{\substack{z \in W, d \in \mathcal{D} \\ \mathbf{a}(z) = \mathbf{a}(d)}} h_{w,d,z} \hat{n}_z t_z \quad (w \in W),$$

where  $\hat{n}_z$  is defined as follows. For any  $z \in W$ , we set  $\hat{n}_z = n_d$  where  $d$  is the unique element of  $\mathcal{D}$  such that  $d \sim_{\mathcal{L}} z^{-1}$  and  $n_d = \pm 1$ ; see (P5).

*Remark 2.3.* Let  $w, z \in W$  and assume that there exists some  $d \in \mathcal{D}$  such that  $\mathbf{a}(z) = \mathbf{a}(d)$  and  $h_{w,d,z} \neq 0$ . Then we have  $z \leq_{\mathcal{L}} d$ . So (P9) implies that  $z \sim_{\mathcal{L}} d$ . But then (P13) shows that  $d$  is the unique element of  $\mathcal{D}$  in the same left cell as  $z$ . Thus, we have in fact

$$\phi(c_w^\dagger) = \sum_{z \in W} \hat{n}_z h_{w,d_z,z} t_z$$

where, for any  $z \in W$ , we denote by  $d_z$  the unique element of  $\mathcal{D}$  in the same left cell as  $z$ . Now  $\phi$  is a linear map between two  $A$ -modules which have bases indexed by the same set; we denote by  $\det(\phi)$  the determinant of the matrix of  $\phi$  with respect to those bases. We have

$$\pm v^e \det(\phi) \in 1 + v\mathbb{Z}[v] \quad \text{where } e = \sum_{w \in W} \mathbf{a}(w).$$

Indeed, let  $w \in W$ . Then, by (P2), (P4), (P5) and (P12), we have

$$v^{-\mathbf{a}(w)} \phi(c_w^\dagger) = \pm t_w + v\mathbb{Z}[v]\text{-combination of elements } t_z \text{ where } \mathbf{a}(z) = \mathbf{a}(w) \\ + A\text{-combination of elements } t_x \text{ where } \mathbf{a}(x) > \mathbf{a}(w).$$

Hence we see that the matrix of  $\phi$  has a block triangular shape, when we order the elements of  $W$  according to increasing value of  $\mathbf{a}$ . Furthermore, inside a diagonal block, the coefficients have leading term  $\pm v^{\mathbf{a}(w)}$  on the diagonal and strictly smaller leading term off the diagonal.

*Remark 2.4.* Using  $J$ , the relations  $\sim_{\mathcal{L}}, \sim_{\mathcal{R}}, \sim_{\mathcal{LR}}$  can be characterized as follows (see [27, Prop. 18.4]):

- We have  $x \sim_{\mathcal{L}} y$  if and only if  $t_x t_{y^{-1}} \neq 0$ .
- We have  $x \sim_{\mathcal{R}} y$  if and only if  $t_{x^{-1}} t_y \neq 0$ .
- We have  $x \sim_{\mathcal{LR}} y$  if and only if  $t_x t_w t_y \neq 0$  for some  $w \in W$ .

*Remark 2.5.* All of the above statements are known to hold if  $L = al$  for some  $a > 0$ ; see [27, Chap. 15]. Although this is not mentioned in [*loc. cit.*], there is a slightly more general situation where this is still the case. Indeed, let us assume that we are given a homomorphism  $\pi: \Omega \rightarrow \text{Aut}(W, S)$  where  $\Omega$  is a finite group

and  $\text{Aut}(W, S)$  is the group of all automorphisms of  $W$  which leave the set  $S$  invariant. We then form the semidirect product  $\tilde{W} = W \rtimes \Omega$  so that, in  $\tilde{W}$ , we have the identity  $\omega w \omega^{-1} = \pi(\omega)(w)$  for  $w \in W$  and  $\omega \in \Omega$ . The group  $\tilde{W}$  is called an *extended Coxeter group*. We have a length function  $\tilde{l}: \tilde{W} \rightarrow \mathbb{N}_0$  given by the formula  $\tilde{l}(w\omega) := l(w)$  where  $w \in W$ ,  $\omega \in \Omega$ . Thus, we have  $\tilde{l}(\omega) = 0$  for all  $\omega \in \Omega$  and  $\tilde{l}(s) = 1$  for all  $s \in S$ . Then all of the above constructions and results can be extended to  $\tilde{W}$ ; see [26, §3]. We shall use this set-up in §6, where a Coxeter group of type  $B_n$  is considered as an extended Coxeter group of type  $D_n$ .

3. CELLS AND IDEMPOTENTS

The purpose of this section is to generalize the main result of Rouquier [30] to the unequal parameter case, assuming that (P1)–(P15) hold.

Let  $B_0$  be the set of two-sided cells in  $W$ . We set

$$t_{\mathbf{c}} = \sum_{d \in \mathcal{D} \cap \mathbf{c}} n_d t_d \quad \text{for } \mathbf{c} \in B_0.$$

**Lemma 3.1.** *We have  $1_J = \sum_{\mathbf{c} \in B_0} t_{\mathbf{c}}$ . Furthermore,  $t_{\mathbf{c}} t_{\mathbf{c}' } = \delta_{\mathbf{c}\mathbf{c}' } t_{\mathbf{c}}$  for any  $\mathbf{c}, \mathbf{c}' \in B_0$ . Thus,  $\{t_{\mathbf{c}} \mid \mathbf{c} \in B_0\}$  is a set of mutually orthogonal, central idempotents in  $J$ .*

*Proof.* For  $\mathbf{c} \in B_0$ , let  $J_{\mathbf{c}} = \langle t_w \mid w \in \mathbf{c} \rangle_{\mathbb{Z}}$ . By (P8),  $J_{\mathbf{c}}$  is a two-sided ideal in  $J$ ; we have  $J = \bigoplus_{\mathbf{c} \in B_0} J_{\mathbf{c}}$ . This yields a unique decomposition  $1_J = \sum_{\mathbf{c} \in B_0} e_{\mathbf{c}}$  where  $e_{\mathbf{c}} \in J_{\mathbf{c}}$ . Here,  $\{e_{\mathbf{c}} \mid \mathbf{c} \in B_0\}$  is a set of mutually orthogonal, central idempotents. For  $\mathbf{c} \in B_0$ , let us write  $e_{\mathbf{c}} = \sum_{w \in \mathbf{c}} a_{\mathbf{c},w} t_w$  where  $a_{\mathbf{c},w} \in \mathbb{Z}$ . Since  $1_J = \sum_{d \in \mathcal{D}} n_d t_d$ , we conclude that  $a_{\mathbf{c},w} = n_w$  for  $w \in \mathcal{D} \cap \mathbf{c}$ , and 0 otherwise. Thus, we have  $e_{\mathbf{c}} = t_{\mathbf{c}}$  as required.  $\square$

**Lemma 3.2.** *Let  $\Gamma$  be a left cell and let  $\mathcal{D} \cap \Gamma = \{d\}$ . Then we have  $n_d t_w t_d = t_w$  for any  $w \in \Gamma$ . Furthermore, we have*

$$J t_d = \langle t_y \mid y \in \Gamma \rangle_{\mathbb{Z}} \quad \text{and} \quad (n_d t_d)^2 = n_d t_d.$$

*Proof.* Let  $y \in W$ . We have  $t_y t_d = \sum_{x \in W} \gamma_{y,d,x} t_{x^{-1}}$  where  $\gamma_{y,d,x} \in \mathbb{Z}$ . If  $x = y^{-1}$  and  $y \in \Gamma$ , then  $\gamma_{y,d,x} = \gamma_{x,y,d} = n_d = \pm 1$  by (P5), (P7), (P12). Now consider any  $x \in W$  and assume that  $\gamma_{y,d,x} \neq 0$ . We must show that  $x = y^{-1}$  and  $y \in \Gamma$ . Now, by (P7), we have  $\gamma_{x,y,d} = \gamma_{y,d,x} \neq 0$ . By (P2) and (P8), this implies  $x = y^{-1}$  and  $y \in \Gamma$ , as required.  $\square$

We shall see that each  $n_d t_d$  actually is a primitive idempotent in  $J$ . In fact, this will even work when we extend scalars from  $\mathbb{Z}$  to suitable larger rings. In order to describe the required conditions on such a larger ring, we recall the following constructions.

**3.3. The simple  $J_{\mathbb{C}}$ -modules.** Upon substituting  $v \mapsto 1$ , the algebra  $H$  specialises to  $\mathbb{Z}[W]$ . Hence, extending scalars from  $\mathbb{Z}$  to  $\mathbb{C}$ , we obtain an isomorphism of  $\mathbb{C}$ -algebras

$$\phi_1: \mathbb{C}[W] \rightarrow J_{\mathbb{C}}, \quad \text{where } J_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} J;$$

see [27, 20.1]. Since  $\mathbb{C}[W]$  is split semisimple, we can now conclude that  $J_{\mathbb{C}}$  also is split semisimple.

Let  $\text{Irr}(W)$  be the set of simple  $\mathbb{C}[W]$ -modules up to isomorphism. For any  $\mathbb{C}[W]$ -module  $E$ , we denote the corresponding  $J_{\mathbb{C}}$ -module by  $E_{\spadesuit}$ . Thus,  $E_{\spadesuit}$  coincides with

$E$  as a  $\mathbb{C}$ -vector space and the action of  $a \in J_{\mathbb{C}}$  on  $E_{\blacklozenge}$  is the same as the action of  $\phi_1^{-1}(a)$  on  $E$ ; see [27, 20.2]. Then we have

$$\text{Irr}(J_{\mathbb{C}}) = \{E_{\blacklozenge} \mid E \in \text{Irr}(W)\},$$

where  $\text{Irr}(J_{\mathbb{C}})$  is the set of simple  $J_{\mathbb{C}}$ -modules up to isomorphism. We shall also need the fact that  $J_{\mathbb{C}}$  is a symmetric algebra, with trace form  $\tau: J_{\mathbb{C}} \rightarrow \mathbb{C}$  given by  $\tau(t_z) = n_z$  if  $z \in \mathcal{D}$  and  $\tau(t_z) = 0$  otherwise. We have  $\tau(t_x t_y) = \delta_{xy,1}$  for any  $x, y \in W$ ; see [27, 20.1]. By general results on split semisimple symmetric algebras (see [14, Chap. 7]), we have

$$\tau(t_w) = \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} \text{tr}(t_w, E_{\blacklozenge}) \quad \text{for all } w \in W,$$

where  $0 \neq f_E \in \mathbb{C}$  for all  $E \in \text{Irr}(W)$ . By [27, Lemma 20.13], we have in fact

$$f_E \in \mathbb{R} \quad \text{and} \quad f_E > 0.$$

**3.4. The simple  $H_K$ -modules.** Let us extend scalars from  $A$  to  $K = \mathbb{C}(v)$ . Then we obtain an isomorphism of  $K$ -algebras

$$\phi_K: H_K \rightarrow J_K \quad \text{where } H_K = K \otimes_A H \text{ and } J_K = K \otimes_{\mathbb{Z}} J;$$

see [27, 20.1]. Given a  $\mathbb{C}[W]$ -module  $E$ , the  $J_{\mathbb{C}}$ -module structure on  $E_{\blacklozenge}$  extends in a natural way to a  $J_K$ -module structure on  $E_v := K \otimes_{\mathbb{C}} E_{\blacklozenge}$ . Then we can also regard  $E_v$  as an  $H_K$ -module via  $\phi_K$ . We have

$$\text{Irr}(H_K) = \{E_v \mid E \in \text{Irr}(W)\},$$

where  $\text{Irr}(H_K)$  is the set of simple  $H_K$ -modules up to isomorphism. We have  $\text{tr}(T_w, E_v) \in \mathbb{C}[v, v^{-1}]$  for all  $w \in W$  and every  $\mathbb{C}[W]$ -module  $E$ . This yields the following direct relation between  $E$  and  $E_v$  (see [27, 20.3]):

$$\text{tr}(w, E) = \text{tr}(T_w, E_v) \Big|_{v=1} \quad \text{for all } w \in W.$$

The above direct relation between  $\text{Irr}(W)$  and  $\text{Irr}(H_K)$  can also be established without reference to the algebra  $J$  and (P1)–(P15); see [14, 8.1.7 and 9.3.5].

Let  $E \in \text{Irr}(W)$ . As in [27, Prop. 20.6], define an integer  $\mathbf{a}_E \geq 0$  by the condition

$$\begin{aligned} v^{\mathbf{a}_E} \text{tr}(T_w, E_v) &\in \mathbb{C}[v] && \text{for all } w \in W, \\ v^{\mathbf{a}_E - 1} \text{tr}(T_w, E_v) &\notin \mathbb{C}[v] && \text{for some } w \in W. \end{aligned}$$

Note that  $\mathbf{a}_E$  depends on the choice of  $L$ . Now let

$$c_E := \frac{1}{\dim E} \sum_{w \in W} \text{tr}(T_w, E_v) \text{tr}(T_{w^{-1}}, E_v) \in \mathbb{C}[v, v^{-1}].$$

In [14], this is called the *Schur element* associated to  $E$ . Then, by [27, Cor. 20.11], we have

$$c_E = f_E v^{-2\mathbf{a}_E} + \text{combination of higher powers of } v,$$

where  $f_E$  is the positive real number introduced in (3.3).

**Definition 3.5.** By Lemma 3.1, we have a partition

$$\text{Irr}(W) = \coprod_{\mathbf{c} \in B_0} \text{Irr}(W, \mathbf{c})$$

where  $\text{Irr}(W, \mathbf{c}) = \{E \in \text{Irr}(W) \mid t_{\mathbf{c}}E_{\blacklozenge} \neq 0\}$ . The sets  $\text{Irr}(W, \mathbf{c})$  may be called the “blocks” of  $\text{Irr}(W)$ . Now fix  $\mathbf{c} \in B_0$  and let  $R \subseteq \mathbb{C}(v)$  be a noetherian subring. We say that  $R$  is  $\mathbf{c}$ -adapted if the following condition holds:

$$(*_{\mathbf{c}}) \quad R \cap \left\{ \sum_{E \in \text{Irr}(W, \mathbf{c})} \frac{n_E}{f_E} \mid n_E \in \mathbb{Z} \right\} \subseteq \mathbb{Z}.$$

**Example 3.6.** Here are some examples of  $\mathbf{c}$ -adapted rings  $R$ .

- (a) The subring  $\mathbb{Z} \subseteq \mathbb{C}(v)$  clearly satisfies  $(*_{\mathbf{c}})$  for any  $\mathbf{c} \in B_0$ .
- (b) Following Rouquier [30], let  $\mathfrak{J}_+ = \{1 + vf \mid f \in \mathbb{Z}[v]\}$  and  $\mathcal{O} = \{f/g \in \mathbb{C}(v) \mid f \in A, g \in \mathfrak{J}_+\}$ . Then  $\mathcal{O}$  is a principal ideal domain; see [30, p. 1040]. It is easily checked that  $\mathcal{O}$  satisfies  $(*_{\mathbf{c}})$  for any  $\mathbf{c} \in B_0$ .
- (c) Let  $\mathbf{c} \in B_0$  and assume that there is a prime number  $p$  such that  $f_E$  is a power of  $p$  for any  $E \in \text{Irr}(W, \mathbf{c})$ . Let  $A_p = \{f/g \in \mathbb{C}(v) \mid f, g \in A, g \notin pA\}$ . Then  $A_p$  is a noetherian local ring whose maximal ideal is the principal ideal generated by  $p$ . It is readily checked that  $(*_{\mathbf{c}})$  holds for  $A_p$ . Rings of this type have been used by Gyoja [17] in the study of left cells in the equal parameter case.

**Theorem 3.7** (See Rouquier [30] in the equal parameter case). *Let  $\mathbf{c} \in B_0$  and  $R \subseteq \mathbb{C}(v)$  be a  $\mathbf{c}$ -adapted subring. By extension of scalars, we obtain an  $R$ -algebra  $J_R = R \otimes_{\mathbb{Z}} J$ . Then the following hold.*

- (a) *Let  $d \in \mathcal{D} \cap \mathbf{c}$ . Then  $n_d t_d$  is a primitive idempotent in  $J_R$ .*
- (b)  *$t_{\mathbf{c}}$  is a primitive idempotent in the center of  $J_R$ .*

*Proof.* Let  $K = \mathbb{C}(v)$  and consider  $J_R$  as a subalgebra of  $J_K = K \otimes_{\mathbb{Z}} J$ . Recall that, for any  $E \in \text{Irr}(W)$ , we have a  $J_K$ -module structure on  $E_v = K \otimes_{\mathbb{C}} E_{\blacklozenge}$ . The trace form  $\tau$  also extends to a trace form on  $J_K$  which we denote by the same symbol. Thus, we have

$$\tau(h) = \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} \text{tr}(h, E_v) \quad \text{for any } h \in J_K.$$

(a) We begin with the following general remark. Let  $e \in J_R$  be any primitive idempotent such that  $e = et_{\mathbf{c}}$ . We regard  $e$  as an element of  $J_K$ . Now, for  $E \in \text{Irr}(W)$ , we have  $eE_{\blacklozenge} = 0$  unless  $E \in \text{Irr}(W, \mathbf{c})$ . Furthermore, the term  $\text{tr}(e, K \otimes_{\mathbb{C}} E_{\blacklozenge})$  is a non-negative integer which is less than or equal to  $\dim E$ . Since  $e = et_{\mathbf{c}}$ , there is at least one  $E \in \text{Irr}(W, \mathbf{c})$  such that  $\text{tr}(e, E_v) > 0$ . Now recall from the discussion in (3.3) that  $f_E$  is a positive real number for all  $E \in \text{Irr}(W)$ . Combining all these pieces of information, the above formula shows that  $\tau(e)$  is a positive real number.

On the other hand, the defining formula for  $\tau$  and the fact that  $e$  lies in  $J_R$  show that  $\tau(e) \in R$ . Hence, using  $(*_{\mathbf{c}})$ , we conclude that

$$\tau(e) > 0 \quad \text{and} \quad \tau(e) \in R \cap \left\{ \sum_{E \in \text{Irr}(W, \mathbf{c})} \frac{n_E}{f_E} \mid n_E \in \mathbb{Z} \right\} \subseteq \mathbb{Z}.$$

In particular, this proves that  $\tau(e) \geq 1$  for any primitive idempotent  $e \in J_R$  such that  $e = et_{\mathbf{c}}$ . Now let  $d \in \mathcal{D}$ . By Lemma 3.2,  $n_d t_d$  is an idempotent; furthermore, we have  $\tau(n_d t_d) = 1$ . Let us write  $n_d t_d = e_1 + \dots + e_n$  where  $e_i$  are orthogonal primitive idempotents in  $J_R$ . (This is possible since  $R$  is noetherian.) Since  $d \in \mathbf{c}$ , we have  $e_i t_{\mathbf{c}} = e_i$  for all  $i$ . The above discussion shows that  $1 = \tau(n_d t_d) = \tau(e_1) + \dots + \tau(e_n) \geq n$ . Thus, we must have  $n = 1$  and so  $n_d t_d$  is primitive.

(b) Let  $\mathbf{c} \in B_0$  and take some  $d \in \mathcal{D} \cap \mathbf{c}$ . We consider a decomposition  $1_J = e_1 + \cdots + e_r$ , where the  $e_i$  are mutually orthogonal, central primitive idempotents in  $J_R$ . (This exists since  $R$  is noetherian.) We have  $n_d t_d = n_d t_d 1_J = \sum_i n_d t_d e_i$ . Since  $n_d t_d$  is a primitive idempotent, there exists a unique  $j$  such that  $n_d t_d e_j = n_d t_d$  and  $n_d t_d e_i = 0$  for  $i \neq j$ . Thus, we have

$$t_d \in J_R e_j \quad \text{where} \quad t_d e_j \neq 0.$$

Now consider any element  $d' \in \mathcal{D} \cap \mathbf{c}$ . The fact that  $d, d'$  lie in the same two-sided cell means that there exists some  $w \in W$  such that  $t_d t_w t_{d'} \neq 0$ ; see Remark 2.4. Since  $t_d t_w t_{d'} \in J_R e_j$ , we also have  $t_d t_w t_{d'} e_j \neq 0$  and so  $t_{d'} e_j \neq 0$ . By the previous argument, this implies  $t_{d'} \in J_R e_j$ .

Thus, we have shown that  $t_{d'} \in J_R e_j$  for all  $d' \in \mathcal{D} \cap \mathbf{c}$  and so  $t_{\mathbf{c}} \in J_R e_j$ . Since  $e_j$  is a primitive idempotent, we conclude that  $t_{\mathbf{c}} = e_j$ , as required.  $\square$

Recall that, given a left cell  $\Gamma$  of  $W$ , we have a corresponding  $H$ -module  $[\Gamma]_A = \sum_{y \in \Gamma} A c_y^\dagger$ . If  $R$  is any (commutative) ring containing  $A$ , let  $H_R = R \otimes_A H$ . Then the  $H$ -module structure on  $[\Gamma]_A$  extends in a natural way to an  $H_R$ -module structure on  $[\Gamma]_R := \sum_{y \in \Gamma} R c_y^\dagger$ .

**Corollary 3.8.** *Let  $R \subseteq \mathbb{C}(v)$  be a noetherian subring such that  $\mathcal{O} \subseteq R$ . Let  $\Gamma$  be a left cell of  $W$ . Then  $[\Gamma]_R$  is a projective  $H_R$ -module. Furthermore, if  $\Gamma$  is contained in  $\mathbf{c} \in B_0$  and  $R$  is  $\mathbf{c}$ -adapted, then  $[\Gamma]_R$  is indecomposable.*

*Proof.* By extension of scalars, we obtain a homomorphism  $\phi_R: H_R \rightarrow J_R$  of  $R$ -algebras. By Remark 2.3,  $\det(\phi)$  is invertible in  $\mathcal{O} \subseteq R$ ; hence  $\phi_R$  is an isomorphism. Now let  $\mathcal{D} \cap \Gamma = \{d\}$ . By Lemma 3.2, we have

$$J_R t_d = \langle t_y \mid y \in \Gamma \rangle_R.$$

Now, since  $n_d t_d$  is an idempotent,  $J_R t_d$  is a projective  $J_R$ -module. Using  $\phi_R$ , we may also regard  $J_R t_d$  as an  $H_R$ -module. Thus, the action of  $h \in H_R$  on  $J_R t_d$  is given by  $h * t_y := \phi(h) t_y$  ( $y \in \Gamma$ ). Since  $\phi_R$  is an isomorphism, the resulting  $H_R$ -module is projective. Finally, consider the  $R$ -linear bijection

$$\theta: J_R t_d \rightarrow [\Gamma]_R, \quad t_y \mapsto \hat{n}_y c_y^\dagger.$$

By a computation analogous to that in [27, 18.10], we obtain

$$c_w^\dagger * t_y = \phi(c_w^\dagger) t_y = \hat{n}_y \sum_{u \in \Gamma} h_{w,y,u} \hat{n}_u t_u \quad \text{for any } w \in W, y \in \Gamma.$$

Applying  $\theta$  yields  $\theta(c_w^\dagger * t_y) = c_w^\dagger \cdot \theta(t_y)$ . Thus,  $\theta$  is an  $H_R$ -module isomorphism. Consequently,  $[\Gamma]_R$  is a projective  $H_R$ -module.

If  $\Gamma \subseteq \mathbf{c}$  and  $R$  is  $\mathbf{c}$ -adapted, then  $n_d t_d$  is a primitive idempotent by Theorem 3.7 and so  $[\Gamma]_R$  is seen to be indecomposable.  $\square$

Note that the statement of Corollary 3.8 only involves the notion of a left cell of  $W$ . The ring  $J$  and the properties (P1)–(P15) are needed in the proof. It would be very interesting to find a more elementary proof.

**3.9. Left cells and decomposition numbers.** Assume that we have a discrete valuation ring  $R \subseteq \mathbb{C}(v)$  such that  $\mathcal{O} \subseteq R$ , where  $\mathcal{O}$  is the ring in Example 3.6(b). Assume, furthermore, that

- (0)  $H_F = F \otimes_R H_R$  is split semisimple and  $H_k = k \otimes_R H_R$  is split,



where  $F$  is the field of fractions of  $R$  and  $k$  is the residue field of  $R$ . As pointed out in [16, §3], these assumptions imply that the Krull–Schmidt Theorem holds for  $H_R$ -modules which are finitely generated and free over  $R$ ; furthermore, idempotents can be lifted from  $H_k$  to  $H_R$ . (That is, we don't have to pass to a completion of  $R$  as is usually done in the modular representation theory of finite groups and associative algebras.) Now we are in the general setting of [14, §7.5]. The canonical map  $R \rightarrow k$  induces a decomposition map

$$d_R: R_0(H_F) \rightarrow R_0(H_k)$$

between the Grothendieck groups of finite-dimensional representations of  $H_F$  and  $H_k$ , respectively. Let  $D_R$  be the corresponding decomposition matrix. Thus,  $D_R$  has rows labelled by  $\text{Irr}(H_F)$  and columns labelled by  $\text{Irr}(H_k)$ . Note that the extension of scalars from  $F$  to  $K = \mathbb{C}(v)$  induces a natural bijection  $\text{Irr}(H_F) \xrightarrow{\sim} \text{Irr}(H_K)$  (since  $H_F$  is assumed to be split semisimple). Thus, as in (3.4), we also have a bijection  $\text{Irr}(H_F) \xrightarrow{\sim} \text{Irr}(W)$ .

Now the entries of  $D_R$  are given as follows. Consider the projective indecomposable  $H_R$ -modules (PIM's for short). Every PIM has a unique simple quotient, which is a simple  $H_k$ -module. In fact, associating to each PIM its simple quotient defines a bijection between isomorphism classes of PIM's and  $\text{Irr}(H_k)$ . For each  $V \in \text{Irr}(H_k)$ , choose a PIM  $P_V$  with simple quotient  $V$ . Thus,  $\{P_V \mid V \in \text{Irr}(H_k)\}$  is the set of all PIM's of  $H_R$ , up to isomorphism. By Brauer reciprocity (see [14, Theorem 7.5.2]), the coefficients in a fixed column of  $D_R$  give the expansion of the corresponding PIM (viewed as an  $H_F$ -module by extension of scalars from  $R$  to  $F$ ) in terms of the irreducible ones. Thus, we have

$$(1) \quad D_R = \left( [E_v : P_V] \right)_{E \in \text{Irr}(W), V \in \text{Irr}(H_k)}$$

where  $\text{Irr}(W)$  indexes the rows and  $\text{Irr}(H_k)$  indexes the columns. Here, we denote by  $[E_v : P_V]$  the multiplicity of  $E_v \in \text{Irr}(H_K)$  as a simple component of  $P_V$  (viewed as an  $H_K$ -module by extension of scalars). Now, since  $H$  is a symmetric algebra with a “reduction-stable” center in the sense of [14, Def. 7.5.5], we know by a general argument (due to Rouquier and the author, see [14, Theorem 7.5.6]) that

$$(2) \quad \text{the columns of } D_R \text{ are linearly independent over } \mathbb{Q}.$$

Now assume, moreover, that  $R$  is  $\mathbf{c}$ -adapted for every two-sided cell  $\mathbf{c}$  of  $W$ . By Corollary 3.8,  $[\Gamma]_R$  is a PIM for every left cell  $\Gamma$ . Since  $W$  is the union of all left cells, we obtain all PIM's in this way. For each  $V \in \text{Irr}(H_k)$ , choose a left cell  $\Gamma_V$  such that  $[\Gamma_V]_R$  has simple quotient  $V$ . Thus,  $\{[\Gamma_V]_R \mid V \in \text{Irr}(H_k)\}$  is the set of all PIM's of  $H_R$ , up to isomorphism. Thus, we finally obtain

$$(3) \quad D_R = \left( [E : [\Gamma_V]] \right)_{E \in \text{Irr}(W), V \in \text{Irr}(H_k)}.$$

Here, we denote by  $[E : [\Gamma]]$  the multiplicity of  $E$  as a simple component of  $[\Gamma]$ ; note that  $[E_v : [\Gamma]_K] = [E : [\Gamma]]$  for any  $E \in \text{Irr}(W)$  and any left cell  $\Gamma$  (see, for example, the argument in [27, 20.5]).

The above discussion also shows that, given any left cell  $\Gamma$ , there exists a unique  $V \in \text{Irr}(H_k)$  such that  $[\Gamma]_R \cong [\Gamma_V]_R$ . Thus, we also have  $[\Gamma] \cong [\Gamma_V]$  as  $\mathbb{C}[W]$ -modules.

The above idea of relating left cells and decomposition matrices has already been considered by Gyoja [17] in the case of equal parameters.

**3.10. Construction of  $(F, R, k)$ .** We now give a general construction of a discrete valuation ring  $R \supseteq \mathcal{O}$  satisfying the hypotheses in (3.9)(0), where  $k$  has characteristic  $p > 0$ .

Let  $F_0$  be a finite extension of  $\mathbb{Q}$ ; let  $R_0$  be the ring of algebraic integers in  $F_0$ . Let  $\mathfrak{p}$  be a prime ideal such that  $p \in \mathfrak{p}$ . Then  $k = R_0/\mathfrak{p}$  is a finite field of characteristic  $p$ . Now we may choose  $F_0$  large enough so that  $k_0 \otimes_{\mathbb{Z}} J$  is split. (This is clearly possible.) Then  $J_k = k \otimes_{\mathbb{Z}} J$  also is split where  $k = k_0(v)$ . Now  $\phi_k: H_k \rightarrow J_k$  is an isomorphism (see Remark 2.3). So we conclude that  $H_k$  is split. On the other hand,  $H_F$  is split semisimple where  $F = F_0(v)$  (see [14, 9.3.5]). We now take the ring

$$R = \{f/g \mid f, g \in R_0[v, v^{-1}], g \notin \mathfrak{p}[v, v^{-1}]\} \supseteq A_p \supseteq \mathcal{O}.$$

Then  $R$  is a discrete valuation ring with field of fractions  $F$  and residue field  $k$ . The hypotheses in (3.9)(0) are satisfied and we have a decomposition matrix  $D_R$ .

**Corollary 3.11.** *Assume that there exists a discrete valuation ring  $R$  as in (3.9) which is  $\mathbf{c}$ -adapted for every  $\mathbf{c} \in B_0$ . Then the set*

$$\left\{ \sum_{E \in \text{Irr}(W)} [E : [\Gamma]] E \mid \Gamma \text{ any left cell of } W \right\} \subseteq \mathbb{Q}[\text{Irr}(W)]$$

*is linearly independent.*

*Proof.* This is just another way of saying that the columns of the decomposition matrix  $D_R$  are linearly independent. □

*Remark 3.12.* If  $W$  is of classical type, then  $R = A_2$  satisfies the above hypothesis; see Section 6. For groups of exceptional type, a ring  $R$  satisfying the hypothesis of Corollary 3.11 no longer exists. However, one can just check by an explicit computation (using the results in Section 7) that the conclusion of Corollary 3.11 holds. Thus, the statement in Corollary 3.11 is seen to hold for any  $W, L$  (assuming (P1)–(P15)).

#### 4. CONSTRUCTIBLE REPRESENTATIONS AND FAMILIES

We preserve the set-up of the previous sections. Now we turn to constructible representations and families. The definition relies on Lusztig’s notion of *truncated induction*.

First recall the definition of the invariants  $\mathbf{a}_E$  for any  $E \in \text{Irr}(W)$ ; see (3.4). Now, given any  $\mathbb{C}[W]$ -module  $E'$ , we can write uniquely  $E' \cong E'_{(0)} \oplus E'_{(1)} \oplus E'_{(2)} \oplus \dots$  where, for any integer  $i$ , we define

$$E'_{(i)} := \bigoplus_E [E : E'] E \quad (\text{sum over all } E \in \text{Irr}(W) \text{ such that } \mathbf{a}_E = i);$$

here,  $[E : E']$  denotes the multiplicity of  $E$  as a simple component of  $E'$ . Now let  $I \subset S$  and consider the parabolic subgroup  $W_I \subset W$ . Let  $E'$  be any  $\mathbb{C}[W_I]$ -module and denote by  $\text{Ind}_I^S(E')$  the corresponding  $\mathbb{C}[W]$ -module obtained by induction. Then, as above, we can write uniquely

$$\text{Ind}_I^S(E') \cong E_{(0)} \oplus E_{(1)} \oplus E_{(2)} \oplus \dots, \quad \text{where } E = \text{Ind}_I^S(E').$$

It is shown in [27, 20.15] that if  $E' = E'_{(a)} \neq \{0\}$  for some  $a$ , then we have

$$\text{Ind}_I^S(E') \cong E_{(a)} \oplus E_{(a+1)} \oplus E_{(a+2)} \oplus \dots, \quad \text{where } E_{(a)} \neq \{0\}.$$

In this case, we set  $J_I^S(E') := E_{(a)}$ . Thus,  $J_I^S(E')$  is a  $\mathbb{C}[W]$ -module such that

$$\text{Ind}_I^S(E') \cong J_I^S(E') \oplus \text{higher terms,}$$

where “higher terms” means a direct sum of simple  $W$ -modules with  $\mathbf{a}$ -invariant strictly bigger than  $a$ . This is precisely Lusztig’s *truncated induction* in [27, 20.15].

**4.1. Constructible representations and families.** Following Lusztig [27, 22.1], we define the set  $\text{Con}(W)$  of constructible representations of  $W$  (with respect to the fixed weight function  $L$ ). This is done by induction on  $|W|$ . If  $W = \{1\}$ , then  $\text{Con}(W)$  consists of the unit representation. If  $W \neq \{1\}$ , then  $\text{Con}(W)$  consists of the  $\mathbb{C}[W]$ -modules of the form  $J_I^S(E')$  or  $J_I^S(E') \otimes \text{sgn}$ , for all subsets  $I \subsetneq S$  and all  $E' \in \text{Con}(W_I)$ .

We define a corresponding “decomposition matrix”  $\mathbf{D}$  as follows. The rows are labelled by  $\text{Irr}(W)$  and the columns are labelled by  $\text{Con}(W)$ ; the coefficients in a fixed column of  $\mathbf{D}$  give the expansion of the corresponding constructible representation in terms of the irreducible ones.

The matrix  $\mathbf{D}$  has been computed explicitly (for all  $W, L$ ) by Lusztig [23], [27, Chap. 22], and Alvis and Lusztig [2] (type  $H_4$ ).

Following Lusztig [27, 23.1], we define a graph  $\mathcal{G}_W$  as follows. The vertices of  $\mathcal{G}_W$  are labelled by  $\text{Irr}(W)$ . Given  $E \neq E'$  in  $\text{Irr}(W)$ , the corresponding vertices in  $\mathcal{G}_W$  are joined if  $E, E'$  both appear as simple components of some constructible representation of  $W$ . We say that  $E, E' \in \text{Irr}(W)$  belong to the same *family* if the corresponding vertices are in the same connected component of  $\mathcal{G}_W$ .

The partition of  $\text{Irr}(W)$  and  $\text{Con}(W)$  according to families gives rise to a block diagonal shape of  $\mathbf{D}$ , with one block on the diagonal for each family.

Note that all these notions depend on the weight function  $L$ .

*Remark 4.2.* One may also consider the following set  $\text{Con}'(W)$  of representations of  $W$  (with respect to the fixed weight function  $L$ ). The definition of  $\text{Con}'(W)$  appears, in a slightly refined form, in Malle and Rouquier [28, §2].

Again, we proceed by induction on  $|W|$ . If  $W = \{1\}$ , then  $\text{Con}'(W)$  consists of the unit representation. If  $W \neq \{1\}$ , then  $\text{Con}'(W)$  consists of all  $\mathbb{C}[W]$ -modules of the form  $E_{(i)}$ , where  $E = \text{Ind}_I^S(E')$ , for all subsets  $I \subseteq S$  such that  $|I| = |S| - 1$ , all  $E' \in \text{Con}'(W_I)$  and  $i \in \mathbb{Z}$ . It is clear that

$$\text{Con}(W) \subseteq \text{Con}'(W).$$

Note that we certainly do not have  $\text{Con}(W) = \text{Con}'(W)$  in general. For example, for  $W$  of type  $E_7$ , we have  $2 \cdot (512_a \oplus 512'_a) \in \text{Con}'(W) \setminus \text{Con}(W)$  (see the discussion in (7.6) below). In Remark 5.4, we shall see that  $\text{Con}'(W) \subseteq \mathbb{N}[\text{Con}(W)]$ .

**Theorem 4.3** (Lusztig, Rouquier). *Let  $\mathcal{O} \subseteq \mathbb{C}(v)$  be the subring defined in Example 3.6(b). Assume that (P1)–(P15) hold for  $W, L$ . Let  $E, E' \in \text{Irr}(W)$ . Then the following conditions are equivalent:*

- (a)  $E, E'$  belong to the same family, in the sense of (4.1).
- (b)  $E, E'$  belong to the same two-sided cell, that is, there exists some  $\mathbf{c} \in B_0$  such that  $t_{\mathbf{c}}E_{\blacklozenge} \neq 0$  and  $t_{\mathbf{c}}E'_{\blacklozenge} \neq 0$ ; see [27, 20.2].
- (c)  $E, E'$  belong to the same block of  $H_{\mathcal{O}}$ , that is, there exists a primitive idempotent  $e$  in the center of  $H_{\mathcal{O}}$  such that  $eE_v \neq 0$  and  $eE'_v \neq 0$ .

Furthermore, if  $E, E'$  belong to the same family, then  $\mathbf{a}_E = \mathbf{a}_{E'}$ .

*Proof.* For the equivalence of (a) and (b), see the argument in the proof of [27, Prop. 23.3]. The equivalence of (b) and (c) is an immediate consequence of Theorem 3.7(b). For the statement concerning  $\mathbf{a}_E$  and  $\mathbf{a}_{E'}$ , see [27, Prop.20.6(c)].  $\square$

The equivalence of (a) and (b) was first proved by Barbasch and Vogan in the equal parameter case; see [24, Theorem 5.25]. The equivalence of (b) and (c) has been established by Rouquier [30] in the equal parameter case. The important point about that equivalence is that the statement in (c) is independent of the Kazhdan–Lusztig basis  $\{c_w\}$  of  $H$ . As such, it applies to a wider class of algebras; see Broué and Kim [7] and Malle and Rouquier [28].

We now establish some results which will be helpful for the computation of the left cell representations. The discussion mainly follows [27]. Note that the *statements* in Lemmas 4.4 and 4.6 (and the corollaries following them) do not involve the properties (P1)–(P15); these are only implicit in the proofs. It would be very interesting to prove any of those statements by elementary methods.

**Lemma 4.4.** *Let  $\Gamma$  be a left cell of  $W$ . Then all simple components of  $[\Gamma]$  belong to a fixed family of  $\text{Irr}(W)$ . In particular, all simple components of  $[\Gamma]$  have the same  $\mathbf{a}$ -invariant.*

*Proof.* Let  $\mathcal{D} \cap \Gamma = \{d\}$ . If  $E, E'$  are simple components of  $[\Gamma]$ , then  $t_d$  acts non-trivially on  $E_\spadesuit$  and on  $E'_\spadesuit$ ; see [27, Prop. 21.4]. Consequently, we also have  $t_c E_\spadesuit \neq 0$  and  $t_c E'_\spadesuit \neq 0$ , where  $c$  is the two-sided cell containing  $d$ . Thus,  $E, E'$  belong to the same two-sided cell. Now the assertion follows from Theorem 4.3.  $\square$

**Lemma 4.5.** *Let  $\Gamma$  be a left cell of  $W$ . Then  $\Gamma w_0$  also is a left cell of  $W$ , where  $w_0$  is the unique element of maximal length of  $W$ . We have  $[\Gamma w_0] \cong [\Gamma] \otimes \text{sgn}$  where  $\text{sgn}$  is the sign representation of  $W$ .*

*Proof.* See [27, Cor. 11.7] and [27, Prop. 21.5]. Here, (P1)–(P15) are not needed.  $\square$

The following result is an extremely strong condition on the expansion of a left cell representation in terms of the irreducible ones.

**Lemma 4.6.** *Let  $\Gamma$  be a left cell of  $W$ . Then we have*

$$1 = \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} [E : [\Gamma]].$$

Here,  $[E : [\Gamma]]$  denotes the multiplicity of  $E$  as a simple component of  $[\Gamma]$ .

*Proof.* Let  $\Gamma \cap \mathcal{D} = \{d\}$  and  $n_d = \pm 1$ . By [27, 19.2 and 20.1(b)], we have

$$n_d = \tau(t_d) = \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} \text{tr}(t_d, E_\spadesuit).$$

On the other hand, by [27, Prop. 21.4], we have  $n_d \text{tr}(t_d, E_\spadesuit) = [E : [\Gamma]]$  for every  $E \in \text{Irr}(W)$ . This yields the desired assertion.  $\square$

**Corollary 4.7.** *Let  $\Gamma$  be a left cell of  $W$ . Assume that there exists some  $E \in \text{Irr}(W)$  such that  $[E : [\Gamma]] \neq 0$  and  $f_E = 1$ . Then  $[\Gamma] \cong E \in \text{Irr}(W)$  and  $[\Gamma]$  is constructible.*

*Proof.* Let  $\text{Irr}(W) = \{E_1, \dots, E_r\}$  where  $E_1 = E$ . We write  $[\Gamma] \cong \bigoplus_{i=1}^r n_i E_i$  where  $n_i \in \mathbb{Z}_{\geq 0}$ . Then, by Lemma 4.6, we have

$$1 = \sum_{i=1}^r \frac{1}{f_{E_i}} n_i = n_1 + \sum_{i=2}^r \frac{1}{f_{E_i}} n_i.$$

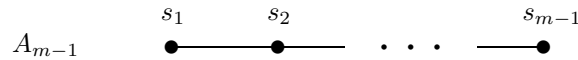
Now note that  $n_1 = [E : [\Gamma]]$  is a positive integer and that  $f_{E_i} > 0$  for all  $i$ . Hence we must have  $n_1 = 1$  and  $n_i = 0$  for all  $i > 1$ .

To prove that  $E \cong [\Gamma]$  is constructible, we argue as follows. By [27, Prop. 22.3], there exists some constructible representation  $E'$  of  $W$  which contains a simple component isomorphic to  $E$ . By [27, Lemma 22.2], there exists a left cell  $\Gamma'$  such  $E' \cong [\Gamma']$ . Thus,  $E$  is a simple component of  $[\Gamma']$ . By the previous argument, we have  $E \cong [\Gamma']$  and so  $E \cong E'$ , as desired.  $\square$

**Corollary 4.8.** *Assume that  $f_E = 1$  for all  $E \in \text{Irr}(W)$ . Then  $[\Gamma]$  is irreducible and constructible for every left cell  $\Gamma$ .*

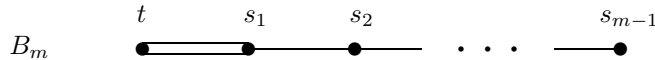
*Proof.* Immediate by Corollary 4.7.  $\square$

**Example 4.9.** Assume that  $W$  is of type  $A_{m-1}$  ( $m \geq 2$ ) with generators and relations given by the following diagram:



All generators are conjugate in  $W$ . So every weight function is of the form  $L = aI$  where  $a > 0$ . We have  $f_E = 1$  for all  $E \in \text{Irr}(W)$ ; see [27, 22.4]. Now Corollary 4.8 shows that  $[\Gamma]$  is irreducible and constructible for every left cell  $\Gamma$ . On the other hand, by [27, 22.5], the irreducible representations are precisely the constructible ones. The fact that  $[\Gamma]$  is irreducible for every left cell  $\Gamma$  has been already established by Kazhdan and Lusztig [19, §5]; for a more detailed and completely elementary proof, see Ariki [4].

**Example 4.10.** Let  $W$  be of type  $B_m$  ( $m \geq 2$ ), with generators and relations given by the following diagram:



Let  $L$  be a weight function such that

$$L(t) = b > 0 \quad \text{and} \quad L(s_1) = L(s_2) = \dots = L(s_{m-1}) = a > 0.$$

Assume that (P1)–(P15) hold for  $W, L$ . As in [27, 22.6], we write  $b = ar + b'$  where  $r, b' \in \mathbb{Z}$  such that  $r \geq 0$  and  $0 \leq b' < a$ .

Assume first that  $b' > 0$ . Then we have  $f_E = 1$  for all  $E \in \text{Irr}(W)$ ; see [27, Prop. 22.14(a)]. So Corollary 4.8 shows that  $[\Gamma] \in \text{Irr}(W)$  for all left cells  $\Gamma$ . On the other hand, the constructible representations are precisely the irreducible ones by [27, Prop. 22.25]. In particular, Conjecture 2.1 holds for  $W, L$  if  $b' > 0$ . The case where  $b' = 0$  will be considered in Section 6.

*Remark 4.11.* Let  $W$  be of type  $B_m$  with parameters given as above. Assume that  $b/a$  is “large”. Then the left cells are explicitly determined by Bonnafé and Iancu [6] (without using the assumption that (P1)–(P15) hold); the corresponding representations are irreducible and constructible. There is some hope that similar

arguments might be found to deal with arbitrary values of  $a$  and  $b$ , as long as  $a$  does not divide  $b$ .

Finally, we present some further conditions on the coefficients in the expansion  $[\Gamma] \cong \bigoplus_E [E : [\Gamma]] E$  for a left cell  $\Gamma$ . This will also be needed in the discussion of type  $E_8$  in (7.7). The desired conditions follow from results in [15] concerning the center of  $H$ ; we shall follow the exposition in [14].

**4.12. Central characters.** Let  $Z(H)$  be the center of  $H$ . An element  $z \in Z(H)$  acts by a scalar in every simple  $H_K$ -modules  $E_v$ , where  $E \in \text{Irr}(W)$ ; we denote that scalar by  $\omega_E(z)$ . For technical simplicity, let us now assume that  $W$  is a Weyl group; then  $\mathbb{Q}$  is a splitting field for  $W$  and  $\mathbb{Q}(v)$  is a splitting field for  $H$ ; see [14, Theorem 9.3.5]. Consequently, since  $A$  is integrally closed in  $\mathbb{Q}(v)$ , we have

$$(a) \quad \omega_E(z) \in A \quad \text{for any } E \in \text{Irr}(W) \text{ and } z \in Z(H).$$

Now let  $R \subseteq \mathbb{C}(v)$  be a noetherian subring such that  $\mathcal{O} \subseteq R$ , where  $\mathcal{O}$  is the ring in Example 3.6(b). Let  $e \in H_R$  be an idempotent and consider the corresponding projective  $H_R$ -module  $P := H_R e$ . Extending scalars from  $R$  to  $K$ , we obtain an  $H_K$ -module  $P_K$ . We denote

$$[E : P] = [E_v : P_K] := \text{multiplicity of } E_v \text{ in } P_K, \text{ for any } E \in \text{Irr}(W).$$

Then the argument in the proof of [14, Theorem 7.5.3] shows that

$$(b) \quad \sum_{E \in \text{Irr}(W)} \frac{1}{c_E} [E : P] \omega_E(z) \in R \quad \text{for any } z \in Z(H),$$

where  $c_E$  is defined in (3.4). This yields rather restrictive conditions on the coefficients  $[E : P]$ .

By Corollary 3.8, we can apply this, in particular, to the module  $P = [\Gamma]_R$  for a left cell  $\Gamma$  of  $W$ . Note that, for  $z = 1$ , we have  $\omega_E(1) = 1$  for all  $E$  and so the above condition bears some resemblance to that in Lemma 4.6.

In order to be able to use the formula (b), we shall need to compute  $\omega_E(z)$  for some elements  $z \in Z(H)$ .

Now, by [14, Cor. 8.2.5], there is a distinguished  $A$ -basis of  $Z(H)$ , denoted by  $\{z_C \mid C \in \text{Cl}(W)\}$  where  $\text{Cl}(W)$  is the set of conjugacy classes of  $W$ . The scalars  $\omega_E(z_C)$  are determined by the following identity:

$$(c) \quad \sum_{C \in \text{Cl}(W)} v_{w_C} \text{tr}(T_{w_C}, E_v) \omega_{E'}(z_C) = \begin{cases} c_E & \text{if } E \cong E', \\ 0 & \text{otherwise;} \end{cases}$$

see [14, Exc. 9.5]. Here,  $w_C$  is an element of minimal length in  $C \in \text{Cl}(W)$ . Note that, by [14, Cor. 8.2.6], the value  $\text{tr}(T_{w_C}, E_v)$  does not depend on the choice of  $w_C$ . The ‘‘character tables’’

$$\left( \text{tr}(T_{w_C}, E_v) \right)_{E \in \text{Irr}(W), C \in \text{Cl}(W)}$$

are known for all  $W, L$ ; see [14]. As explained in the proof of [14, Prop. 11.5.13], the identities (c) can be used to compute the scalars  $\omega_E(z_C)$ .

## 5. INDUCTION AND RESTRICTION OF LEFT CELLS

Let  $I \subseteq S$  and consider the parabolic subgroup  $W_I$ . The restriction of  $L$  to  $W_I$  is a weight function on  $W_I$ . Thus, we have a partition of  $W_I$  into left cells with respect to  $L|_{W_I}$ . We shall now consider the compatibility of the corresponding left cell representations with respect to induction and restriction. All the results that we are going to present here were already known in the case of equal parameters. However, new arguments are needed in the proofs for some of these results in the unequal parameter case (most notably Lemmas 5.2, 5.5 and 5.6).

**Lemma 5.1** (See Barbasch and Vogan [5, Prop. 3.11] in the equal parameter case). *Let  $\Gamma$  be a left cell of  $W$  and  $I \subset S$ . Then there exist (pairwise different) left cells  $\Gamma'_1, \dots, \Gamma'_r$  of  $W_I$  and positive integers  $n_1, \dots, n_r$  such that*

$$\text{Res}_I^S([\Gamma]) \cong n_1[\Gamma'_1] \oplus \cdots \oplus n_r[\Gamma'_r].$$

*Proof.* We follow the proof given by Roichman [29, Theorem 5.2] in the equal parameter case.

Recall the definition of the relation  $\leq_{\mathcal{L}}$  on  $W$ : this is the transitive closure of the relation “ $y \leftarrow_{\mathcal{L}} w$  if  $h_{s,w,y} \neq 0$  for some  $s \in S$ ”; see [27, 8.1].

We define a relation  $\leq_{\mathcal{L},I}$  on  $W$  as the transitive closure of the relation “ $y \leftarrow_{\mathcal{L},I} w$  if  $h_{s,w,y} \neq 0$  for some  $s \in I$ ”. Let  $\sim_{\mathcal{L},I}$  be the corresponding equivalence relation on  $W$ . Now note that  $\{C_w \mid w \in W_I\}$  is the Kazhdan–Lusztig basis of the parabolic subalgebra  $H_I$ . Hence, the restriction of  $\leq_{\mathcal{L},I}$  to  $W_I$  gives precisely the left cells of  $W_I$  with respect to  $L|_{W_I}$ .

Let  $Y_I$  be the set of all  $w \in W$  such that  $w$  has minimal length in the right coset  $W_I w$ . Let  $u, u' \in W_I$  and  $x, x' \in Y_I$ . Now, if  $ux \leftarrow_{\mathcal{L},I} u'x'$ , then there exists some  $s \in I$  such that  $h_{s,u'x',ux} \neq 0$ . By the formula in [27, Prop. 6.3], we have either  $ux = su'x' > u'x'$  or  $ux < u'x'$ . In the first case, we have  $u = su'$  and  $x = x'$ ; in the second case, we have  $x \leq x'$ , see [27, 9.10(f)]. Thus, we have the implication

$$ux \leq_{\mathcal{L},I} u'x' \quad \Rightarrow \quad x \leq x'.$$

Consequently, we have the implication

$$ux \sim_{\mathcal{L},I} u'x' \quad \Rightarrow \quad x = x'.$$

On the other hand, we have the implication “ $u \sim_{\mathcal{L},I} u' \Rightarrow ux \sim_{\mathcal{L},I} u'x'$ ” for any  $u, u' \in W_I$  and  $x \in Y_I$ ; see [27, 9.11]. This shows that there exist (pairwise different) left cells  $\Gamma'_1, \dots, \Gamma'_r$  of  $W_I$  and subsets  $R_1, \dots, R_r$  of  $Y_I$  such that

$$\Gamma = \bigcup_{i=1}^r \bigcup_{y \in R_i} \Gamma'_i y.$$

Now consider the restriction of the  $H$ -module  $[\Gamma]_A$  to  $H_I$ . The formula in [27, Lemma 9.10(e)] shows that, for fixed  $i \in \{1, \dots, r\}$  and  $y \in R_i$ , we have

$$c_s^\dagger \cdot c_{uy}^\dagger = \sum_{u' \in \Gamma'_i} h_{s,u,u'} c_{u'y}^\dagger \quad \text{for any } s \in I \text{ and } u \in \Gamma'_i.$$

Thus, as an  $H_I$ -module, we have  $\text{Res}_I^S([\Gamma]_A) \cong n_1[\Gamma'_1]_A \oplus \cdots \oplus n_r[\Gamma'_r]_A$ , where  $n_i = |R_i|$  for all  $i$ . Upon setting  $v \mapsto 1$ , we obtain the required assertion concerning  $W_I$ -modules.  $\square$

**Lemma 5.2** (See Barbasch and Vogan [5, Prop. 3.15] in the equal parameter case). *Let  $I \subset S$  and  $\Gamma'$  be a left cell of  $W_I$ . Then there exist (pairwise different) left cells  $\Gamma_1, \dots, \Gamma_r$  of  $W$  such that*

$$\mathrm{Ind}_I^S([\Gamma']) \cong [\Gamma_1] \oplus \cdots \oplus [\Gamma_r].$$

*Proof.* Let  $X_I$  be the set of all  $x \in W$  such that  $x$  has minimal length in the left coset  $xW_I$ . Then, by [11], we have  $X_I\Gamma' = \Gamma_1 \Pi \cdots \Pi \Gamma_r$  where  $\Gamma_i$  are (distinct) left cells of  $W$ . So we have an  $H_A$ -module

$$[X_I\Gamma']_A := \sum_{x \in X_I} \sum_{u \in \Gamma'} Ac_{xu}^\dagger \cong [\Gamma_1]_A \oplus \cdots \oplus [\Gamma_r]_A.$$

Here,  $c_w^\dagger$  ( $w \in W$ ) acts by the rule  $c_w^\dagger \cdot c_y^\dagger = \sum_{z \in X_I\Gamma'} h_{x,y,z} c_z^\dagger$  for any  $y \in X_I\Gamma'$ . To identify this module with an induced module, we set

$$\begin{aligned} \mathcal{I} &:= \langle T_x c_u^\dagger \mid x \in X_I, u \in W_I, u \leq_{\mathcal{L},I} u' \text{ for some } u' \in \Gamma' \rangle_A, \\ \hat{\mathcal{I}} &:= \langle T_x c_u^\dagger \mid x \in X_I, u \in W_I \setminus \{\Gamma'\}, u \leq_{\mathcal{L},I} u' \text{ for some } u' \in \Gamma' \rangle_A. \end{aligned}$$

Then  $\hat{\mathcal{I}} \subseteq \mathcal{I}$  are left ideals in  $H_A$ ; see [11, Lemma 2.2]. Thus,  $\mathcal{I}/\hat{\mathcal{I}}$  is an  $H_A$ -module; it is free as an  $A$ -module with a basis given by the residue classes of the elements  $T_x c_u^\dagger$  ( $x \in X_I, u \in \Gamma'$ ). By the definition of  $\mathrm{Ind}_I^S$  (see [14, §9.1]), we have an isomorphism of  $H_A$ -modules

$$\mathrm{Ind}_I^S([\Gamma']) \xrightarrow{\sim} \mathcal{I}/\hat{\mathcal{I}}, \quad T_x \otimes c_u^\dagger \mapsto T_x c_u^\dagger + \hat{\mathcal{I}}.$$

On the other hand, by [11, Prop. 3.3], we also have

$$\begin{aligned} \mathcal{I} &= \langle c_{xu}^\dagger \mid x \in X_I, u \in W_I, u \leq_{\mathcal{L},I} u' \text{ for some } u' \in \Gamma' \rangle_A, \\ \hat{\mathcal{I}} &= \langle c_{xu}^\dagger \mid x \in X_I, u \in W_I \setminus \{\Gamma'\}, u \leq_{\mathcal{L},I} u' \text{ for some } u' \in \Gamma' \rangle_A. \end{aligned}$$

Thus,  $\mathcal{I}/\hat{\mathcal{I}}$  also has an  $A$ -basis given by the residue classes of the elements  $c_{xu}^\dagger$  ( $x \in X_I, u \in \Gamma'$ ). Hence there is an isomorphism of  $H_A$ -modules

$$[X_I\Gamma']_A \xrightarrow{\sim} \mathcal{I}/\hat{\mathcal{I}}, \quad c_{xu}^\dagger \mapsto c_{xu}^\dagger + \hat{\mathcal{I}}.$$

Upon setting  $v \mapsto 1$ , we obtain  $\mathrm{Ind}_I^S([\Gamma']) \cong [\Gamma_1] \oplus \cdots \oplus [\Gamma_r]$  as desired.  $\square$

*Remark 5.3.* The proofs of Lemma 5.1 and Lemma 5.2 do not require the assumption that (P1)–(P15) hold.

*Remark 5.4.* We can now clarify the relation between  $\mathrm{Con}(W)$  and the set  $\mathrm{Con}'(W)$  defined in Remark 4.2. (See Malle and Rouquier [28, Prop. 2.5] for the case of equal parameters.) Assume that Conjecture 2.1 holds for  $W, L$  and all parabolic subgroups of  $W$ . Then we have:

- (a)  $\mathrm{Con}'(W) \subseteq \mathbb{N}[\mathrm{Con}(W)]$  and
- (b) the regular representation of  $W$  is a sum of constructible representations.

Indeed, using the notation in Remark 4.2, let  $E_i \in \mathrm{Con}'(W)$  where  $E = \mathrm{Ind}_I^S(E')$  for some  $E' \in \mathrm{Con}'(W_I)$  ( $I \subsetneq S$ ) and some integer  $i \geq 0$ . By induction,  $E'$  is a sum of constructible representations. Hence, by Conjecture 2.1,  $E'$  is a sum of left cell representations. Using Lemma 4.4 and Lemma 5.2, we conclude that  $E_i$  is a sum of left cell representations and, hence (via Conjecture 2.1), a sum of constructible representations. This proves (a). Now (b) is an easy consequence: we just have to note that the regular representation of  $W$  (which can be obtained by inducing the



regular representation of any proper parabolic subgroup) is a sum of representations in  $\text{Con}'(W)$ .

**Lemma 5.5.** *Let  $I \subset S$  and  $\Gamma'$  be a left cell of  $W_I$ . Then we have*

$$J_I^S([\Gamma']) \cong [\Gamma],$$

where  $\Gamma$  is the left cell of  $W$  such that  $\Gamma' \subseteq \Gamma$ .

*Proof.* See the argument in the proof of Case 1 in [27, Lemma 22.2]. Note that this argument is much simpler than the one given in [24, 5.28.1] in the case of equal parameters, which uses results from Barbasch and Vogan [5].  $\square$

**Lemma 5.6** (See Lusztig [25, §3] in the equal parameter case). *Let  $\mathcal{F} \subseteq \text{Irr}(W)$  be a family and assume that there exists a proper subset  $I \subset S$  and a family  $\mathcal{F}' \subset \text{Irr}(W_I)$  such that  $J_I^S$  defines a bijection  $\mathcal{F}' \xrightarrow{\sim} \mathcal{F}$ .*

*Let  $\Gamma$  be a left cell of  $W$  and assume that all simple components of  $[\Gamma]$  belong to  $\mathcal{F}$ . Then there exists a left cell  $\Gamma'$  of  $W_I$  such that  $[\Gamma] \cong J_I^S([\Gamma'])$ . In particular, if  $[\Gamma']$  is constructible, then so is  $[\Gamma]$ .*

*Proof.* Let  $\mathcal{F} = \{E_1, \dots, E_f\}$ . By assumption, we have  $\mathcal{F}' = \{E'_1, \dots, E'_f\}$  where  $E_i \cong J_I^S(E'_i)$  for all  $i$ . Using Frobenius reciprocity, we conclude that

$$\text{Res}_I^S(E_i) \cong E'_i \oplus \text{combination of } E' \in \text{Irr}(W_I) \text{ with } E' \notin \mathcal{F}'.$$

Now let  $\Gamma$  be a left cell belonging to  $\mathcal{F}$ ; we have  $[\Gamma] \cong \bigoplus_i [E_i : [\Gamma]] E_i$ . Upon restriction to  $W_I$ , we obtain

$$\text{Res}_I^S([\Gamma]) \cong \bigoplus_{i=1}^f [E_i : [\Gamma]] E'_i \oplus \text{combination of } E' \in \text{Irr}(W_I) \text{ with } E' \notin \mathcal{F}'.$$

On the other hand, by Lemma 5.1, we know that  $\text{Res}_I^S([\Gamma])$  is a sum of left cell representations of  $W_I$ . Since all simple components of a left cell representation belong to a fixed family, we conclude that

$$\bigoplus_{i=1}^f [E_i : [\Gamma]] E'_i \cong \text{sum of left cell representations of } W_I.$$

Applying  $J_I^S$  and using Lemma 5.5, we deduce that

$$[\Gamma] \cong \bigoplus_{i=1}^f [E_i : [\Gamma]] E_i \cong \text{sum of left cell representations of } W,$$

where each representation occurring in the sum is obtained by J-induction of a left cell representation of  $W_I$ . Thus, to complete the proof, it is sufficient to show that only one left cell representation occurs in the above sum. But this follows from the following general statement:

*Let  $\Gamma, \Gamma_1$  be left cells and assume that  $[\Gamma_1] \cong [\Gamma] \oplus E'$  for some  $\mathbb{C}[W]$ -module  $E'$ . Then we automatically have  $[\Gamma] \cong [\Gamma_1]$ .*

Indeed, the assumption implies that  $[E : [\Gamma]] \leq [E : [\Gamma_1]]$  for any  $E \in \text{Irr}(W)$ . Hence Lemma 4.6 yields

$$1 = \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} [E : [\Gamma]] \leq \sum_{E \in \text{Irr}(W)} \frac{1}{f_E} [E : [\Gamma_1]] = 1.$$

Consequently, all the inequalities must be equalities and so  $[\Gamma] \cong [\Gamma_1]$ . □

Note that the proof of Lemma 5.6 which is given in [25, §3], uses the fact that every left cell representation has a unique “special” simple component with multiplicity 1; see the discussion in [24, 5.25]. This argument is no longer available in the case of unequal parameters: there are families which do not contain any reasonably defined “special” simple module (see, for example, [12, Remark 4.11]).

**5.7. Cuspidal families.** Let  $\mathcal{F}$  be a family of  $\text{Irr}(W)$ . Then  $\mathcal{F} \otimes \text{sgn} = \{E \otimes \text{sgn} \mid E \in \mathcal{F}\}$  also is a family. (This is clear by the definitions in (4.1).) As in [24, 8.1], we say that  $\mathcal{F}$  is *cuspidal* if neither  $\mathcal{F}$  nor  $\mathcal{F} \otimes \text{sgn}$  satisfies the hypothesis of Lemma 5.6 for any proper subset  $I \subset S$  and any family  $\mathcal{F}'$  of  $\text{Irr}(W_I)$ .

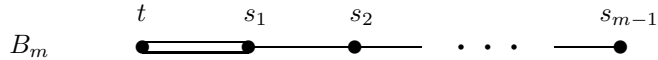
Suppose now we want to show that  $[\Gamma]$  is constructible for every left cell  $\Gamma$  of  $W$ . Proceeding by induction on  $|S|$ , we may assume that  $[\Gamma']$  is constructible for every left cell  $\Gamma'$  of a proper parabolic subgroup of  $W$ . Then, as explained in [25], it will be enough to consider only those left cells  $\Gamma$  for which  $[\Gamma]$  belongs to a cuspidal family of  $\text{Irr}(W)$ .

Indeed, let  $\Gamma$  be a left cell such that all simple components of  $[\Gamma]$  belong to a family  $\mathcal{F}$  which is not cuspidal. Then either  $\mathcal{F}$  itself or  $\mathcal{F} \otimes \text{sgn}$  satisfies the hypothesis of Lemma 5.6 for some proper subset  $I \subset S$  and some family  $\mathcal{F}'$  of  $\text{Irr}(W_I)$ . Assume first that  $\mathcal{F}$  itself satisfies that hypothesis. Then there exists a left cell  $\Gamma'$  of  $W_I$  such that  $[\Gamma] \cong J_I^S([\Gamma'])$ . By induction,  $[\Gamma']$  is constructible, hence  $[\Gamma]$  is constructible. Now assume that  $\mathcal{F} \otimes \text{sgn}$  satisfies that hypothesis. Then consider the left cell  $\Gamma w_0$ . By Lemma 4.5, all simple components of  $[\Gamma w_0] \cong [\Gamma] \otimes \text{sgn}$  belong to  $\mathcal{F} \otimes \text{sgn}$ . Hence, as before, we may conclude that  $[\Gamma w_0]$  is constructible. Consequently,  $[\Gamma]$  also is constructible.

## 6. TYPES $B_m$ AND $D_m$

We are now going to prove Conjecture 2.1 for groups of type  $B_m$  and  $D_m$ , under the hypothesis that (P1)–(P15) hold for  $W, L$ . A special case in type  $B_m$  has been already considered in Example 4.10. (For type  $A_{m-1}$ , see Example 4.9.) As far as type  $D_m$  and  $B_m$  with equal parameters are concerned, our proof is different from the one given by Lusztig [25].

**6.1. A decomposition matrix in type  $B_m$ .** Let  $W$  be of type  $B_m$  ( $m \geq 2$ ), with generators and relations given by the following diagram:



Let  $L$  be a weight function such that

$$L(t) = r \geq 0 \quad \text{and} \quad L(s_1) = L(s_2) = \dots = L(s_{m-1}) = 1.$$

Here, we explicitly allow the case where  $r = 0$ , which is related to type  $D_m$ . In fact, this case can be interpreted as indicated in Remark 2.5; see (6.5) and Lemma 6.6 below. Let us assume that (P1)–(P15) hold for  $W, L$ . It is known that this is the case if  $r = 1$  (equal parameters); see [27, Chap. 15]. In the set-up of Remark 2.5, this also holds if  $r = 0$ .

Recall that we have a natural parametrisation of  $\text{Irr}(W)$  by the set  $\mathcal{P}_m$  of all pairs of partitions  $(\alpha, \beta)$  such that  $|\alpha| + |\beta| = m$ . Let us write

$$\text{Irr}(W) = \{E^{(\alpha, \beta)} \mid (\alpha, \beta) \in \mathcal{P}_m\}.$$

By [27, Prop. 22.14],  $f_E$  is a power of 2 for all  $E \in \text{Irr}(W)$ . So let us consider the subring  $R = A_2 \subseteq \mathbb{C}(v)$  defined in Example 3.6(c), where  $p = 2$ . The ring  $R$  is local, with maximal ideal  $2R$ , residue field  $k = R/2R = \mathbb{F}_2(v)$  and field of fractions  $F = \mathbb{Q}(v)$ . As explained in (3.9), the canonical map  $R \rightarrow k$  induces a decomposition map

$$d_{r,2}: R_0(H_F) \rightarrow R_0(H_k).$$

Note that  $H_F$  and  $H_k$  are split and  $H_F$  is semisimple; see Dipper, James and Murphy [8, §4, §6]. Let  $D_{r,2}$  be the corresponding decomposition matrix. Since  $R$  is  $\mathcal{C}$ -adapted for every two-sided cell  $\mathcal{C}$  of  $W$ , we have

$$D_{r,2} = \left( [E : [\Gamma_V]] \right)_{E \in \text{Irr}(W), V \in \text{Irr}(H_k)};$$

see (3.9)(3). Note that, by (3.9)(2), the columns of  $D_{r,2}$  are linearly independent. In fact, by Dipper, James and Murphy [8, 6.5], something stronger holds: the rows and columns may be ordered so that  $D_{r,2}$  has a lower triangular shape, with 1 on the diagonal.

The discussion in (3.9) also showed that, given any left cell  $\Gamma$ , there exists a unique  $V \in \text{Irr}(H_k)$  such that  $[\Gamma] \cong [\Gamma_V]$ .

Now consider the constructible representations of  $W$  with respect to a weight function  $L$  as above. The set  $\text{Con}(W)$  and the corresponding “decomposition matrix” matrix  $\mathbf{D}_r$  are determined by Lusztig [27, 22.24, 22.25 and 22.26] in a purely combinatorial way. We shall need the following result.

**Proposition 6.2.** *Let  $W$  be of type  $B_m$  ( $m \geq 2$ ) and  $L$  be a weight function as in (6.1). Then there exist rational numbers  $n_P$  ( $P \in \text{Con}(W)$ ) such that the following identity holds:*

$$\dim E = \sum_{P \in \text{Con}(W)} n_P [E : P] \quad \text{for any } E \in \text{Irr}(W).$$

The following proof is due to B. Leclerc and reproduced here with his kind permission. It requires some standard results about highest weight modules for the Lie algebra  $\mathfrak{sl}_{n+1}(\mathbb{C})$ ; see [20] and the references there. (This proof replaces my earlier proof which also used results from [20] but which was much less elementary.)

*Proof.* First note that the above identity simply means that the regular representation of  $W$  can be written as a rational linear combination of constructible representations. To be precise, we have to work here in the appropriate Grothendieck group and identify a  $\mathbb{C}[W]$ -module  $\tilde{E}$  with

$$\sum_{E \in \text{Irr}(W)} [E : \tilde{E}] E \in \mathbb{Q}[\text{Irr}(W)].$$

To prove the above assertion, we place ourselves in the setting of Leclerc and Miyachi [20], where the constructible representations of  $W$  are interpreted in terms of canonical bases. Choose large positive integers  $k, n$ . (For example, any  $k \geq m$  and  $n \geq m - 1 + k + r$  will do; see [20, 4.1.1].) We consider the simple Lie algebra  $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ , with Chevalley generators  $e_j, h_j, f_j$  ( $1 \leq j \leq n$ ). Let  $V(\Lambda)$  be the highest weight module of  $\mathfrak{g}$  with highest weight  $\Lambda = \Lambda_k + \Lambda_{k+r}$ , where

$\Lambda_1, \dots, \Lambda_n$  are the fundamental weights<sup>1</sup>. There is a canonical embedding of  $\mathfrak{g}$ -modules  $\Phi: V(\Lambda) \rightarrow F(\Lambda)$  where  $F(\Lambda) = V(\Lambda_{k+r}) \otimes V(\Lambda_k)$ . The module  $F(\Lambda)$  has a standard basis

$$\mathcal{S}(\Lambda) = \{u_S \mid S \in \text{Sy}(n, k, r)\}$$

where  $\text{Sy}(n, k, r)$  is the set of all symbols of the form

$$(1) \quad S = \begin{pmatrix} 1 \leq \beta_1 < \dots < \beta_{k+r} \leq n+1 \\ 1 \leq \gamma_1 < \dots < \gamma_k \leq n+1 \end{pmatrix};$$

see [20, 2.2]. The action of the Chevalley generator  $f_j$  on a basis element  $u_S$  is explicitly given by the formulas in [20, 2.1]. One notices that these formulas constitute a refinement of the ‘‘branching rule’’ (see [14, 6.1.9]), that is, we have

$$(2) \quad \text{ind}(u_S) = f_1(u_S) + f_2(u_S) + \dots + f_n(u_S),$$

where  $\text{ind}(u_S)$  denotes the sum of all  $u_{S'}$  where  $S'$  is a symbol as in (1) which can be obtained from  $S$  by increasing exactly one entry by 1. (This means: we may replace some  $\beta_i$  by  $\beta_i + 1$  if the new list of coefficients still satisfies the conditions in (1); or we may replace some  $\gamma_i$  by  $\gamma_i + 1$  if the new list of coefficients still satisfies the conditions in (1).)

Let us now turn to the module  $V(\Lambda)$ . There is a well-defined symbol  $S_0$  such that  $u_{S_0} = \Phi(v_\Lambda)$  where  $v_\Lambda$  is a highest weight vector in  $V(\Lambda)$ ; see [20, 2.2]. Thus, we have

$$\Phi(V(\Lambda)) = \langle f_{i_1} \circ \dots \circ f_{i_s}(u_{S_0}) \mid s \geq 0, 1 \leq i_1, \dots, i_s \leq n \rangle_{\mathbb{C}} \subseteq F(\Lambda).$$

A symbol  $S$  as in (1) is called *standard* if  $\beta_i \leq \gamma_i$  for  $1 \leq i \leq k$ . Let  $\text{SSy}(n, k, r)$  be the set of all standard symbols in  $\text{Sy}(n, k, r)$ . By [20, 2.3], this set will label the *canonical basis* (or *lower global basis*)  $\mathcal{B}(\Lambda)$  of  $V(\Lambda)$ ; see, for example, Jantzen [18] for the general theory of canonical bases. In the present situation, we can just accept the conditions in [20, 2.4] as a definition, and then [20, Theorem 3] provides an explicit construction of  $\mathcal{B}(\Lambda) = \{b_S \mid S \in \text{SSy}(n, k, r)\}$ . That construction proceeds by induction on the ‘‘principal degree’’  $d(S)$  of a symbol  $S$ , which is defined as

$$(3) \quad d(S) = \sum_i \beta_i + \sum_j \gamma_j - \binom{k+1}{2} - \binom{k+r+1}{2}.$$

Given  $d \geq 0$ , let  $\text{Sy}(n, k, r, d)$  be the set of symbols  $S$  as in (1) such that  $d(S) = d$ . We also write

$$\text{SSy}(n, k, r, d) := \text{Sy}(n, k, r, d) \cap \text{SSy}(n, k, r).$$

The Leclerc–Miyachi construction shows that, for any  $S \in \text{SSy}(n, k, r, d)$ , there exists an explicitly known subset  $\mathcal{C}(S) \subseteq \text{Sy}(n, k, r, d)$  such that

$$(4) \quad \Phi(b_S) = \sum_{\Sigma \in \mathcal{C}(S)} u_\Sigma.$$

Let us set  $W_d = W(B_d)$  for  $d = 0, 1, 2, \dots$ . We bring now into play the representations of  $W_d$ . By [20, 4.1.1], we may naturally label the irreducible representations of  $W_d$  by the set of symbols in  $\text{Sy}(n, k, r, d)$ . Thus, we write

$$\text{Irr}(W_d) = \{E_S \mid S \in \text{Sy}(n, k, r, d)\}.$$

<sup>1</sup>Note that, in [20], the quantized versions of these modules are considered. For our purposes, it is enough to work with the specialisations at 1.

Identifying  $u_S \leftrightarrow E_S$  for any  $S$ , we may just consider  $\text{Irr}(W_d)$  as a subset of  $F(\Lambda)$  for any  $d \geq 0$ . With this convention, [20, Theorem 10] states that

$$(5) \quad \text{Con}(W_d) = \{\Phi(b_S) \mid S \in \text{SSy}(n, k, r, d)\} \quad \text{for } d = 0, 1, 2, \dots$$

Now we have natural embeddings  $\{1\} = W_0 \subset W_1 \subset W_2 \subset \dots$ . Using our identification  $u_S \leftrightarrow E_S$ , we see that (2) does yield the “branching rule” for the induction of representations from  $W_d$  to  $W_{d+1}$ , that is, we have

$$\text{Ind}_{W_d}^{W_{d+1}}(E_S) = f_1(E_S) \oplus \dots \oplus f_n(E_S) \quad \text{for any } S \in \text{Sy}(n, k, r, d).$$

Hence, starting with  $\text{Irr}(W_0) = \{\mathbf{1}\}$ , we obtain

$$(6) \quad (f_1 + \dots + f_n)^d(\mathbf{1}) = \text{Ind}_{W_0}^{W_d}(\mathbf{1}) = \text{regular representation of } W_d.$$

Under the identification  $u_S \leftrightarrow E_S$ , the unit representation  $\mathbf{1}$  of  $W_0$  corresponds to  $u_{S_0} = \Phi(v_\Lambda)$ . Now, the vector  $(f_1 + \dots + f_n)^d(v_\Lambda)$  lies in the principal degree  $d$  component of  $V(\Lambda)$  and, hence, can be expressed as a rational linear combination of basis elements  $b_S$  with  $d(S) = d$ . Applying  $\Phi$  and using (5) and (6), we can now conclude that the regular representation of  $W_d$  is a rational linear combination of  $\text{Con}(W_d)$ , as desired. Taking  $d = m$  (which is possible since  $n, k$  were chosen sufficiently large) yields the desired assertion.  $\square$

Note that, by Remark 4.2(b), the numbers  $n_P$  are actually seen to be non-negative integers, once we have shown that Conjecture 2.1 holds.

**Proposition 6.3.** *Let  $W$  be of type  $B_m$  ( $m \geq 2$ ), with generators and relations given by the diagram in Example 4.10. Let  $L$  be a weight function such that  $L(t) = b > 0$  and  $L(s_i) = a > 0$  for  $1 \leq i \leq m - 1$ . Assume that  $b = ra$  for some integer  $r \geq 1$  and that (P1)–(P15) hold for  $W, L$ . Then  $[\Gamma]$  is constructible for every left cell  $\Gamma$  and all constructible representations arise in this way. Thus, Conjecture 2.1 holds for  $W, L$ .*

*Proof.* It is easily shown (see, for example, [12, Remark 2.15]) that the left cell representations and the constructible representations only depend on  $r$ . Thus, we may assume without loss of generality that  $a = 1$  and  $b = r \geq 1$ . Now recall the description of the decomposition matrix  $D_{2,r}$  in (6.1). The representations carried by the left cells of  $W$  are completely determined by  $D_{r,2}$ .

Now consider the constructible representations of  $W$ . Let  $P \in \text{Con}(W)$ . By [27, Lemma 22.2], there exists some left cell  $[\Gamma]$  such that  $P \cong [\Gamma]$  as  $\mathbb{C}[W]$ -modules. So we have  $P \cong [\Gamma] \cong [\Gamma_V]$  for a unique  $V \in \text{Irr}(H_k)$ . Thus, there exists a subset  $\mathcal{C}' \subseteq \text{Irr}(H_k)$  such that

$$(1) \quad \text{Con}(W) = \{[\Gamma_V] \mid V \in \mathcal{C}'\}.$$

Now, by (3.9), the modules  $[\Gamma_V]_R$  ( $V \in \text{Irr}(H_k)$ ) are precisely the PIM’s of  $H_R$  (up to isomorphism). So we have an isomorphism of  $H_R$ -modules

$$H_R \cong \bigoplus_{V \in \text{Irr}(H_k)} (\dim V) [\Gamma_V]_R$$

where  $H_R$  is regarded as an  $H_R$ -module via left multiplication. Extending scalars from  $R$  to  $K$  and considering the multiplicities of simple modules, we find

$$\begin{aligned} \dim E &= [E : \mathbb{C}[W]] = [E_v : H_K] = \sum_{V \in \text{Irr}(H_k)} (\dim V) [E_v : [\Gamma_V]_K] \\ &= \sum_{V \in \text{Irr}(H_k)} (\dim V) [E : [\Gamma]] \quad \text{for any } E \in \text{Irr}(W). \end{aligned}$$

Using now (1) and Proposition 6.2, we obtain the identity

$$(2) \quad \sum_{V \in \text{Irr}(H_k)} (\dim V) [E : [\Gamma_V]] = \sum_{V \in \mathcal{C}'} n_V [E : [\Gamma_V]] \quad \text{for any } E \in \text{Irr}(W),$$

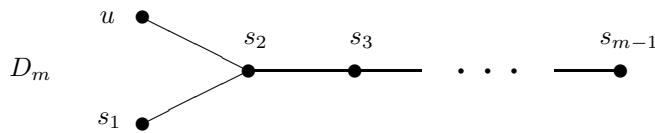
where  $n_V$  are certain rational numbers. By (6.1), the coefficients  $[E : [\Gamma_V]]$  occurring in the above identity are the entries of  $D_{r,2}$ . Now, by (3.9)(2), the columns of  $D_{r,2}$  are linearly independent over  $\mathbb{Q}$ . Thus, we can compare coefficients in (2). In particular, every summand on the left-hand side must also occur on the right-hand side and so  $\mathcal{C}' = \text{Irr}(H_k)$ . This shows that all representations carried by the left cells are constructible, as desired.  $\square$

*Remark 6.4.* Let  $W$  be of type  $B_m$  and  $L$  be a weight function as in Proposition 6.3, that is, we have  $b = ra$  for some integer  $r \geq 1$ .

(a) The above result shows that the decomposition matrix  $D_{r,2}$  in (6.1) coincides (up to a permutation of the columns) with the “decomposition matrix”  $\mathbf{D}_r$  giving the expansion of the constructible representations in terms of the irreducible ones. Thus, we obtain a new proof of [20, Theorem 16], assuming that (P1)–(P15) hold. The proof in [*loc. cit.*] uses the deep results of Ariki [3].

(b) Table II in [21, p. 35] shows that  $L$  arises (in the sense explained by Lusztig [27, Chap. 0]) in the representation theory of finite classical groups. Hence, there is some hope that the geometric realization of  $H$  in [27, Chap. 27] will lead to a proof of (P1)–(P15) in this case.

**6.5. Type  $D_m$ .** Let  $W_1$  be a Coxeter group of type  $D_m$  ( $m \geq 3$ ) with generators and relations given by the following diagram:



Let  $\omega: W_1 \rightarrow W_1$  be the automorphism such that  $\omega(u) = s_1$ ,  $\omega(s_1) = u$  and  $\omega(s_i) = s_i$  for  $i > 1$ . Then the semidirect product  $W = W_1 \rtimes \langle \omega \rangle$  can be naturally identified with a Coxeter group of type  $B_m$ , with generators  $\{\omega, s_1, s_2, \dots, s_{m-1}\}$ . Regarding  $\omega$  as an element of  $W$ , we have the relation  $\omega w = \omega(w)\omega$  for all  $w \in W_1$ . In particular, we have  $u = \omega s_1 \omega$ .

Let  $L: W \rightarrow \mathbb{Z}$  be the weight function such that

$$L(\omega) = 0 \quad \text{and} \quad L(s_1) = L(s_2) = \dots = L(s_{m-1}) = 1.$$

Let  $L_1$  be the restriction of  $L$  to  $W_1$ . Then  $L_1$  is just the usual length function on  $W_1$ ; see, for example, [14, Lemma 1.4.12]. Thus, we are precisely in the setting of Remark 2.5, where  $\tilde{l} = L$ .

Given any  $\mathbb{C}[W_1]$ -module  $E$ , we can define a new  $\mathbb{C}[W_1]$ -module structure on  $E$  by composing the original action with the automorphism  $\omega$ . We denote that new  $\mathbb{C}[W_1]$ -module by  ${}^\omega E$ .

Recall that  $\text{Irr}(W) = \{E^{(\alpha,\beta)} \mid (\alpha,\beta) \in \mathcal{P}_m\}$  see (6.1). For  $(\alpha,\beta) \in \mathcal{P}_m$ , we denote by  $E^{[\alpha,\beta]}$  the restriction of  $E^{(\alpha,\beta)}$  to  $W_1$ . Then we have

$$\begin{aligned} E^{[\alpha,\beta]} &\cong E^{[\beta,\alpha]} \in \text{Irr}(W_1) && \text{if } \alpha \neq \beta, \\ E^{[\alpha,\alpha]} &\cong E^{[\alpha,+]} \oplus E^{[\alpha,-]} && \text{if } \alpha = \beta, \end{aligned}$$

where  $E^{[\alpha,\pm]} \in \text{Irr}(W_1)$  and  $E^{[\alpha,+]} \not\cong E^{[\alpha,-]} \cong {}^\omega E^{[\alpha,+]}$ . This yields (see [14, Chap. 5] for more details):

$$\text{Irr}(W_1) = \{E^{[\alpha,\beta]} \mid \alpha \neq \beta\} \cup \{E^{[\alpha,\pm]} \mid 2|\alpha| = m\}.$$

Let  $H$  be the Iwahori–Hecke algebra associated with  $W, L$ . Let  $H_1$  be the  $A$ -subspace of  $H$  spanned by all  $T_{w_1}$  with  $w_1 \in W_1$ . Then  $H_1$  is nothing but the Iwahori–Hecke algebra associated with  $W_1, L_1$ . Note  $T_\omega^2 = T_1$  since  $L(\omega) = 0$ . We have

$$T_{w_1}T_\omega = T_{w_1\omega} \quad \text{and} \quad T_\omega T_{w_1} = T_{\omega(w_1)}T_\omega \quad \text{for any } w_1 \in W_1.$$

In the following discussion, it will be understood that a left cell of  $W_1$  is defined with respect to  $L_1$  and a left cell of  $W$  is defined with respect to  $L$ .

**Lemma 6.6.** *In the setting of (6.5), (P1)–(P15) hold for  $W, L$ . Let  $\Gamma_1$  be a left cell of  $W_1$ . Then  $\omega(\Gamma_1)$  also is a left cell of  $W_1$  and*

$$\Gamma := \Gamma_1 \cup \omega(\Gamma_1)\omega$$

*is a left cell of  $W$ . Furthermore, we have isomorphisms of  $\mathbb{C}[W]$ -modules*

$$\text{Res}_{W_1}^W([\Gamma]) \cong [\Gamma_1] \oplus [\omega(\Gamma_1)] \quad \text{and} \quad [\omega(\Gamma_1)] \cong {}^\omega[\Gamma_1].$$

*Proof.* We consider the Kazhdan–Lusztig basis  $\{c_w \mid w \in W\}$  of  $H$  (defined with respect to  $W, L$ ). Note that the construction of that basis in [27, Chap. 5] works for weight functions with arbitrary integer values (even negative ones). We have  $c_w = T_w + \sum_{y < w} p_{y,w} T_y$ , where  $p_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$ . Let  $y, w \in W$  and write  $y = y_1\omega^i$ ,  $w = w_1\omega^j$  where  $y_1, w_1 \in W_1$  and  $i, j \in \{0, 1\}$ . Then we have  $p_{y,w} = 0$  if  $i \neq j$  and  $p_{y,w} = p_{y_1,w_1}$  if  $i = j$ , where  $p_{y_1,w_1}$  is the polynomial defined with respect to  $W_1, L_1$ ; see the remarks in [26, §3] and [9, §2]. Thus,  $\{c_{w_1} \mid w_1 \in W_1\}$  is the Kazhdan–Lusztig basis of  $H_1$  (with respect to  $L_1$ ). We have  $c_\omega = T_\omega$  and

$$(1) \quad c_{w_1}c_\omega = c_{w_1\omega} \quad \text{and} \quad c_\omega c_{w_1} = c_{\omega(w_1)}c_\omega \quad \text{for any } w_1 \in W_1.$$

Furthermore, let  $\mathbf{a}(z)$  ( $z \in W$ ) be defined with respect to  $L$ , and let  $\mathbf{a}_1(z_1)$  ( $z_1 \in W_1$ ) be defined with respect to  $L_1$ . Then we have

$$\mathbf{a}(w_1\omega) = \mathbf{a}(w_1) = \mathbf{a}_1(w_1) \quad \text{for any } w_1 \in W_1.$$

Now (P1)–(P15) hold for  $W_1, L_1$ ; see [27, Chap. 15]. The above relations imply that (P1)–(P15) also hold for  $W, L$ ; see [26, §3] where this is worked out explicitly. The above relations also show that

$$\mathcal{D} = \mathcal{D}_1 = \{z \in W \mid \mathbf{a}(z) = \Delta(z)\} \subseteq W_1.$$

Let  $J$  be the  $\mathbb{Z}$ -algebra with basis  $\{t_w \mid w \in W\}$  defined with respect to  $W, L$ . Then we have  $t_\omega^2 = t_1$  and

$$(2) \quad t_{w_1}t_\omega = t_{w_1\omega} \quad \text{and} \quad t_\omega t_{w_1} = t_{\omega(w_1)}t_\omega \quad \text{for all } w_1 \in W_1.$$

Let  $J_1$  be the  $\mathbb{Z}$ -submodule of  $J$  spanned by all elements  $t_{w_1}$  for  $w_1 \in W_1$ . Then  $J_1$  is nothing but the  $\mathbb{Z}$ -algebra defined with respect to  $W_1, L_1$ . Furthermore, the  $A$ -algebra homomorphism  $\phi: H \rightarrow J_A$  restricts to the homomorphism  $\phi_1: H_1 \rightarrow (J_1)_A$  defined with respect to  $W_1, L_1$ . These remarks allow us to switch back and forth between left cells of  $W_1$  and left cells of  $W$ . For example, the characterisation of  $\sim_{\mathcal{L}}$  in Remark 2.4 shows that

- (3a) every left cell of  $W_1$  is contained in a left cell of  $W$ ;  
 (3b) if  $\Gamma_1$  is a left cell of  $W_1$ , then so is  $\omega(\Gamma_1)$ .

Now let  $\Gamma_1$  be a left cell of  $W_1$ . By (3b), we know that  $\omega(\Gamma_1)$  is a left cell of  $W_1$ . By (3a), we have  $\Gamma_1 \subseteq \Gamma$  where  $\Gamma$  is a left cell of  $W$ . First we claim that  $\Gamma \cap W_1 = \Gamma_1$ . To see this, let us fix an element  $z \in \Gamma_1$ . Now let  $y \in \Gamma \cap W_1$ . Since  $z, y \in \Gamma$ , we have  $t_y t_{z^{-1}} \neq 0$  inside  $J$ , by Remark 2.4. Since  $z, y \in W_1$ , we also have  $t_y t_{z^{-1}} \neq 0$  inside  $J_1$  and so  $y \in \Gamma_1$  (again using Remark 2.4). Thus, the above claim is proved. Now let  $y \in \Gamma$  and assume that  $y \notin W_1$ . Let us write  $y = y_1 \omega$  where  $y_1 \in W_1$ . Since  $y \in \Gamma$ , we have  $t_z t_{y^{-1}} \neq 0$  inside  $J$ . Using (2), we also have  $t_z t_{\omega(y_1)^{-1}} \neq 0$  inside  $J_1$  and so  $\omega(y_1) \in \Gamma_1$ . Thus, we have shown that  $\Gamma = \Gamma_1 \cup \omega(\Gamma_1)\omega$ , as desired.

Finally, consider the statement concerning the left cell representations. The decomposition  $\Gamma = \Gamma_1 \cup \omega(\Gamma_1)\omega$  yields that we have an isomorphism of  $\mathbb{C}[W_1]$ -modules

$$[\Gamma] \cong [\Gamma_1] \oplus [\omega(\Gamma_1)].$$

Furthermore, using (1), we see that for any  $w \in W_1$ , the action of  $c_w$  on  $[\omega(\Gamma_1)]$  will be the same as the action of  $c_{\omega(w)} = c_w c_w c_w$  on  $[\Gamma_1]$ , as required.  $\square$

The above result shows, in particular, that every left cell of  $W$  has an even number of elements. This certainly is in contrast to the more familiar picture of equal parameters.

**Proposition 6.7.** *Let  $W_1$  be of type  $D_m$ , as in (6.5), and let  $L_1$  be the weight function on  $W_1$  given by the length. By [27, Chap. 15], (P1)–(P15) hold for  $W_1, L_1$ . Then  $[\Gamma_1]$  is constructible for every left cell  $[\Gamma_1]$ , and all constructible representations arise in this way. Thus, Conjecture 2.1 holds for  $W_1, L_1$ .*

*Proof.* By [27, Lemma 22.2], we already know that every constructible representation is of the form  $[\Gamma_1]$  for some left cell  $\Gamma_1$ . Conversely, let  $\Gamma_1$  be a left cell of  $W_1$ . We must show that  $[\Gamma_1]$  is constructible. For this purpose, we place ourselves in the setting of Lemma 6.6.

Assume first that  $[\Gamma_1]$  has a simple component  $E_1$  with  $f_{E_1} = 1$ . Then Corollary 4.8 shows that  $[\Gamma_1]$  is irreducible and constructible. Now assume that  $[\Gamma_1]$  has no simple component  $E_1$  with  $f_{E_1} = 1$ . Using the formulas for  $f_E$  in [27, 22.14], we see that all simple components of  $[\Gamma_1]$  must be of the form  $E^{[\alpha, \beta]}$  with  $\alpha \neq \beta$ . Thus, by Lemma 6.6, we have  $[\omega(\Gamma_1)] \cong \omega[\Gamma_1] \cong [\Gamma_1]$  and so

$$\text{Res}_{W_1}^W([\Gamma]) \cong 2[\Gamma_1].$$

Now, the argument in the proof of Proposition 6.3 also works in the present situation, where  $L(\omega) = 0$  and  $L(s_i) = 1$  for all  $i$ . Hence,  $[\Gamma]$  is a constructible representation of  $W$ . Now [27, 22.26] (see also Leclerc and Miyachi [20, Theorem 13]) shows that  $\text{Res}_{W_1}^W([\Gamma])$  is twice a constructible representation of  $W_1$ . Hence  $[\Gamma_1]$  is constructible as desired.  $\square$



## 7. EXCEPTIONAL TYPES

We are now going to indicate proofs of Conjecture 2.1 for groups of exceptional type. If  $W$  is of type  $E_6$ ,  $E_7$ ,  $E_8$  and  $F_4$  with equal parameters, Lusztig's proof [25] requires some sophisticated results from the representation theory of reductive groups over finite field, see [24, Chap. 12]. We will show here that these arguments can be replaced by more elementary ones, involving some explicit computations with the character tables in [14].

**7.1. Type  $I_2(m)$ .** Let  $W$  be of type  $I_2(m)$  ( $m \geq 3$ ), that is, we have  $W = \langle s_1, s_2 \rangle$  where  $s_1^2 = s_2^2 = (s_1 s_2)^m = 1$ . A weight function  $L$  is specified by  $L(s_1) = a > 0$  and  $L(s_2) = b > 0$  where  $a = b$  if  $m$  is odd.

The irreducible representations of  $W$  are given as follows. We have the unit representation  $1_W$  and  $\text{sgn}$ . If  $m$  is even, there are two further 1-dimensional representations, which we denote by  $\text{sgn}_1$  and  $\text{sgn}_2$ . They are characterised by the condition that  $s_1$  acts as  $-1$  in  $\text{sgn}_2$  and  $s_2$  acts as  $-1$  in  $\text{sgn}_1$ . All other irreducible representations have dimension 2; see [14, 5.3.4] for an explicit description of these representations. We denote by  $\tau$  the direct sum of all the 2-dimensional representations.

For any  $k \geq 0$ , write  $1_k = s_1 s_2 s_1 \cdots$  ( $k$  factors) and  $2_k = s_2 s_1 s_2 \cdots$  ( $k$  factors). Then the following hold:

(i) If  $m$  is odd, then the left cells are

$$\{1_0\}, \quad \{2_m\}, \quad \{2_1, 1_2, 2_3, \dots, 1_{m-1}\}, \quad \{1_1, 2_2, 1_3, \dots, 2_{m-1}\}.$$

The left cell representations are  $1_W$ ,  $\text{sgn}$ ,  $\tau$ ,  $\tau$ , respectively.

(ii) If  $m$  is even and  $a = b$ , then the left cells are

$$\{1_0\}, \quad \{2_m\}, \quad \{2_1, 1_2, 2_3, \dots, 2_{m-1}\}, \quad \{1_1, 2_2, 1_3, \dots, 1_{m-1}\}.$$

The representations are  $1_W$ ,  $\text{sgn}$ ,  $\text{sgn}_1 \oplus \tau$ ,  $\text{sgn}_2 \oplus \tau$ , respectively.

(iii) If  $m$  is even and  $b > a$ , then the left cells are

$$\begin{aligned} &\{1_0\}, \quad \{1_1\}, \quad \{2_{m-1}\}, \quad \{2_m\}, \\ &\{2_1, 1_2, 2_3, \dots, 2_{m-2}\}, \quad \{2_2, 1_3, 2_4, \dots, 1_{m-1}\}. \end{aligned}$$

The representations are  $1_W$ ,  $\text{sgn}_2$ ,  $\text{sgn}_1$ ,  $\text{sgn}$ ,  $\tau$ ,  $\tau$ , respectively.

The left cells in all of the above cases are determined in [27, Chap. 8] (see also [14, Exc. 11.4] for the case  $a \neq b$ ). These computations do not require any of the conditions (P1)–(P15). The representations carried by the left cells are easily determined by an explicit computation; see [10, §6] for case (iii). The constructible representations are listed in [23, §12] ( $a = b$ ) and [10, §6] ( $b > a$ ). In each case, the representations carried by the left cells are precisely the constructible ones. Thus, Conjecture 2.1 holds for  $W$  and any weight function  $L$ .

**Example 7.2.** Let  $W$  be of type  $H_3$ , with generators and relations given by the following diagram:

$$H_3 \quad \begin{array}{c} s_1 \quad 5 \quad s_2 \quad s_3 \\ \bullet \text{---} \bullet \text{---} \bullet \end{array}$$

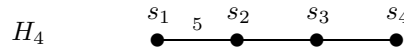
All generators of  $W$  are conjugate so every weight function on  $W$  is of the form  $L = al$  for some  $a > 0$ . We have  $|W| = 120$ . Using CHEVIE [13], one can explicitly compute the basis  $\{c_w\}$  and all polynomials  $h_{x,y,z}$ . By inspection, one sees that

$$p_{y,w} \in \mathbb{N}[v^{-1}] \quad \text{and} \quad h_{x,y,z} \in \mathbb{N}[v, v^{-1}]$$

for all  $x, y, z, w \in W$ . Thus, the arguments in [27, Chap. 15] show that (P1)–(P15) hold for  $W, L$ .

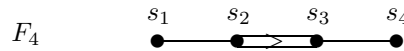
By [27, Lemma 22.2], we already know that each constructible representation is of the form  $[\Gamma]$  where  $\Gamma$  is a left cell of  $W$ . The constructible representations are listed in [23, §12]. This yields the partition of  $\text{Irr}(W)$  into families. Using the tables for the J-induction in [14, Table D.1], it is readily checked that there are no cuspidal families. We can now use the argument in (5.7) to conclude that all left cell representations of  $W$  are constructible. Thus, Conjecture 2.1 holds for  $W$ . Of course, having computed all  $p_{y,w}$  and all  $h_{x,y,z}$ , one could also directly compute the left cells and the corresponding representations. The left cells of  $W$  and the corresponding representations are given (without proof) in [22, §5]; they are also implicit in Alvis [1] which deals with type  $H_4$ .

**7.3. Type  $H_4$ .** Let  $W$  be of type  $H_4$ , with generators and relations given by the following diagram:



All generators of  $W$  are conjugate so every weight function on  $W$  is of the form  $L = al$  for some  $a > 0$ . The constructible representations are determined by Alvis and Lusztig [2]. On the other hand, the left cells and the corresponding representations have been explicitly computed by Alvis [1]. By inspection, one sees that the representations carried by the left cells are precisely the constructible ones; see [1, Prop. 3.5]. Thus, Conjecture 2.1 holds for  $W$ .

**7.4. Type  $F_4$ .** Let  $W$  be of type  $F_4$ , with generators and relations given by the following diagram:



A weight function  $L$  on  $W$  is specified by two positive integers  $a := L(s_1) = L(s_2) > 0$  and  $b := L(s_3) = L(s_4) > 0$ ; we shall write  $L = L_{a,b}$ . In [12], the partition of  $W$  into left cells has been determined for all values of  $a, b$ . Let  $L = L_{a,b}$  and  $L' = L_{a',b'}$  be two weight functions on  $W$  such that  $b \geq a > 0$  and  $b' \geq a' > 0$ . Then  $L, L'$  define the same partition of  $W$  into left cells if and only if  $L, L' \in \mathcal{L}_i$  for some  $i \in \{0, 1, 2, 3\}$ , where  $\mathcal{L}_i$  are defined as follows:

$$\begin{aligned} \mathcal{L}_0 &= \{(c, c, c, c) \mid c > 0\}, \\ \mathcal{L}_1 &= \{(c, c, 2c, 2c) \mid c > 0\}, \\ \mathcal{L}_2 &= \{(c, c, d, d) \mid 2c > d > c > 0\}, \\ \mathcal{L}_3 &= \{(c, c, d, d) \mid d > 2c > 0\}. \end{aligned}$$

Furthermore, if  $L, L' \in \mathcal{L}_i$ , then the left cells give rise to the same representations of  $W$ . On the other hand, the constructible representations are determined in [27, 22.27]<sup>2</sup>. By inspection, one sees that the left cell representations are precisely the constructible ones. Thus, Conjecture 2.1 holds for  $W$  and all weight functions  $L$ .

Note that, by Table II in [21, p. 35], only the weight functions  $L$  such that  $b = a$ ,  $b = 2a$  or  $b = 4a$  arise in the representation theory of finite reductive groups.

<sup>2</sup>As pointed out in [12, Remark 4.10], there is an error in the list of constructible representations in [27, 22.27, Case 1], where  $b = 2a > 0$ : the representations denoted  $1_3 \oplus 2_1$  and  $1_2 \oplus 2_2$  have to be removed from that list.

For the case of equal parameters, let us indicate an argument which does not require the explicit computation of all left cells. This will also be a model for the discussion of groups of type  $E_6$ ,  $E_7$  and  $E_8$ . So let us assume that  $L = al$  for some  $a > 0$ . By [27, Chap. 15], the properties (P1)–(P15) hold for  $W, L$ . Hence, by [27, Lemma 22.2], we already know that every constructible representation is carried by a left cell of  $W$ . To prove the converse, it will now be enough to consider only those left cells which belong to a cuspidal family of  $\text{Irr}(W)$ ; see (5.7).

By [24, 8.1], there is a unique cuspidal family  $\mathcal{F}_0$  of  $\text{Irr}(W)$ , the one containing the representation  $12_1$ , where we use the notation of [24, 4.10] or [14, Table C.3]. To deal with this family, we consider the parabolic subgroup  $W_I$  of type  $B_3$ , where  $I = S \setminus \{s_4\}$ . By Proposition 6.3, the left cell representations of  $W_I$  are precisely the constructible ones, and these are explicitly given in [27, 22.24].

TABLE 1. The cuspidal family in type  $F_4$ 

$E$	$f_E$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$
$1_2$	8	1	.	.	.	.
$1_3$	8	.	1	.	.	.
$4_1$	8	.	.	1	.	.
$4_3$	4	1	.	.	1	.
$4_4$	4	.	1	.	.	1
$6_1$	3	.	.	.	1	1
$6_2$	12	1	1	1	.	.
$9_2$	8	2	.	1	1	.
$9_3$	8	.	2	1	.	1
$12_1$	24	1	1	1	1	1
$16_1$	4	1	1	2	1	1

Now let  $\Gamma$  be a left cell of  $W$  belonging to  $\mathcal{F}_0$ . Then, as in the proof of Lemma 5.2, there exists a unique left cell  $\Gamma'$  of  $W_I$  such that  $\Gamma \subseteq X_I \Gamma'$ . Then  $[\Gamma]$  is a direct summand of  $\text{Ind}_I^S([\Gamma'])$ . Let us write

$$\text{Ind}_I^S([\Gamma']) \cong \bigoplus_{E \in \mathcal{F}_0} m_E E \oplus \text{combination of } E' \in \text{Irr}(W) \setminus \{\mathcal{F}_0\},$$

where  $m_E$  are non-negative integers. Thus, we can conclude that

$$[\Gamma] \cong \bigoplus_{E \in \mathcal{F}_0} n_E E \quad \text{where } 0 \leq n_E \leq m_E \text{ for all } E \in \mathcal{F}_0.$$

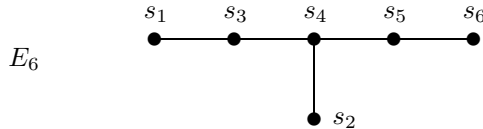
Now the idea is to look for arithmetical conditions on the numbers  $n_E$  so that the only remaining possibilities satisfying these conditions correspond to the expansion of the constructible representations in  $\mathcal{F}_0$ . One such condition is given by Lemma 4.6: the numbers  $n_E$  must satisfy

$$(\diamond) \quad \sum_{E \in \mathcal{F}_0} \frac{1}{f_E} n_E = 1.$$

Inducing all constructible representations from  $W_I$  to  $W$ , we can explicitly determine (using CHEVIE [13]) all possible non-zero vectors  $(m_E)_{E \in \mathcal{F}_0}$  as above. They are given by the columns labelled by  $P_1, P_2, P_3, P_4, P_5$  in Table 1. Now, if the vector

$(m_E)_{E \in \mathcal{F}_0}$  is given by one of the above columns, we see that  $\sum_E m_E / f_E = 1$ . Thus,  $(\diamond)$  shows that we must have  $n_E = m_E$  for all  $E$ . So  $[\Gamma]$  is given by one of the above five columns. Comparison with the table of constructible representations in [23, p. 223] shows that  $[\Gamma]$  is constructible.

**7.5. Type  $E_6$ .** Let  $W$  be of type  $E_6$ , with generators and relations given by the following diagram:



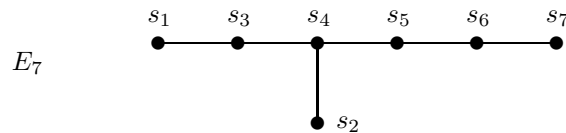
All generators of  $W$  are conjugate so every weight function on  $W$  is of the form  $L = al$  for some  $a > 0$ . By [27, Chap. 15], the properties (P1)–(P15) hold for  $W, L$ . Hence, by [27, Lemma 22.2], we already know that every constructible representation is carried by a left cell of  $W$ . To prove the converse, it will now be enough to consider only those left cells which belong to a cuspidal family of  $\text{Irr}(W)$ ; see (5.7). We shall argue as in (7.4).

By [24, 8.1], there is a unique cuspidal family  $\mathcal{F}_0$  of  $\text{Irr}(W)$ , the one containing the representation  $80_s$ , where we use the notation of [24, 4.11] or [14, Table C.4]. To deal with this family, we consider the parabolic subgroup  $W_I$  of type  $D_5$ , where  $I = S \setminus \{s_6\}$ . By Proposition 6.7, the left cell representations of  $W_I$  are precisely the constructible ones, and these are explicitly given in [27, 22.26]. The possibilities for all non-zero vectors  $(m_E)_{E \in \mathcal{F}_0}$  obtained (as above) by inducing all constructible representations of  $W_I$  are given by the following table:

$E$	$f_E$	$P_1$	$P_2$	$P_3$
$10_s$	3	1	.	.
$20_s$	6	.	1	.
$60_s$	2	1	.	1
$80_s$	6	1	1	1
$90_s$	3	.	2	1

Checking  $(\diamond)$ , we find that  $(n_E)_{E \in \mathcal{F}_0}$  must be equal to one of the vectors  $(m_E)_{E \in \mathcal{F}_0}$ . So  $[\Gamma]$  is given by one of the above three columns. Comparison with the table of constructible representations in [23, p. 223] shows that  $[\Gamma]$  is constructible. Thus, Conjecture 2.1 holds for  $W$ .

**7.6. Type  $E_7$ .** Let  $W$  be of type  $E_7$ , with generators and relations given by the following diagram:



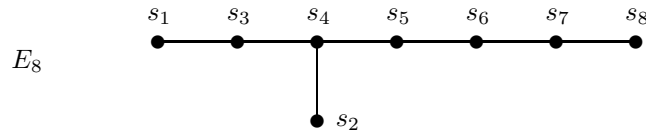
In order to prove Conjecture 2.1, we argue as in (7.5). It remains to consider the cuspidal families. By [24, 8.1], there is a unique cuspidal family  $\mathcal{F}_0$  of  $\text{Irr}(W)$ , the one containing the representation  $512'_a$ , where we use the notation of [24, 4.12] or [14, Table C.5]. To deal with this family, we consider the parabolic subgroup  $W_I$  of type  $E_6$ . The possibilities for all non-zero vectors  $(m_E)_{E \in \mathcal{F}_0}$  obtained by inducing

all constructible representations from  $W_I$  are given by the following table:

$E$	$f_E$	$P_1$	$P_2$	$P_3$
$512_a$	2	1	2	3
$512'_a$	2	1	2	3

Now let  $\Gamma$  be a left cell belonging to  $\mathcal{F}_0$ . Checking  $(\diamond)$ , we find the following possibilities for the expansion of  $[\Gamma]$ :  $512_a \oplus 512'_a$ ,  $2 \cdot 512_a$  or  $2 \cdot 512'_a$ . The first is constructible by the table in [23, p. 223]. We must show that the other two possibilities do not occur. This can be seen as follows. By Theorem 4.1, the family  $\mathcal{F}_0$  corresponds to a two-sided cell  $\mathfrak{c}$  of  $W$ . Since  $f_E = 2$  for all  $E \in \text{Irr}(\mathcal{F}_0)$ , we consider the ring  $A_2$  as in Example 3.6. In fact, since we do not yet know if  $H$  is split over the residue field of  $A_2$  (which is just  $\mathbb{F}_2(v)$ ), we work with a ring  $R \supseteq A_2$  as in (3.10). It is readily checked that  $R$  is  $\mathfrak{c}$ -adapted. So Theorem 3.7 shows that  $e_{\mathfrak{c}} = \phi_R^{-1}(t_{\mathfrak{c}})$  is a primitive idempotent in the center of  $H_R$ . Consider the decomposition matrix  $D_{\mathfrak{c}}$  of the block  $H_R e_{\mathfrak{c}}$ . That matrix has only two rows, corresponding to  $512_a$  and  $512'_a$ . The columns correspond to the projective indecomposable modules in that block. Now we already know that  $512_a \oplus 512'_a$  is constructible and, hence, is carried by some left cell contained in  $\mathfrak{c}$ . Hence  $512_a \oplus 512'_a$  gives one column of  $D_{\mathfrak{c}}$ ; see Corollary 3.8. If  $[\Gamma] \cong 2 \cdot 512_a$  or  $2 \cdot 512'_a$ , then this would give another column of  $D_{\mathfrak{c}}$ . Since  $512'_a \cong 512_a \otimes \text{sgn}$ , we would conclude that actually both  $2 \cdot 512_a$  and  $2 \cdot 512'_a$  correspond to columns of  $D_{\mathfrak{c}}$ . Thus, we would deduce that  $D_{\mathfrak{c}}$  has at least three columns and, hence, does not have full rank. Consequently,  $D_R$  (which is a block diagonal matrix where one block is given by  $D_{\mathfrak{c}}$ ) would not have full rank either, contradicting (3.9)(2). This completes the proof of Conjecture 2.1 for  $W$ .

**7.7. Type  $E_8$ .** Let  $W$  be of type  $E_8$ , with generators and relations given by the following diagram:



In order to prove Conjecture 2.1, we argue as in (7.5). It remains to consider the cuspidal families. By [24, 8.1], there is a unique cuspidal family  $\mathcal{F}_0$  of  $\text{Irr}(W)$ , the one containing the representation  $4480_y$ , where we use the notation of [24, 4.13] or [14, Table C.6]. To deal with this family, we use the parabolic subgroup  $W_I$  of type  $E_7$ . The possibilities for all non-zero vectors  $(m_E)_{E \in \mathcal{F}_0}$  obtained by inducing all constructible representations from  $W_I$  are given by the columns labelled  $P_1, \dots, P_8$  in Table 2.

Checking  $(\diamond)$  yields that  $P_1, \dots, P_7$  already give the expansion of a left cell representation in terms of the irreducible ones; furthermore, the table in [23, p. 224] shows that all these columns give constructible representations of  $W$ .

Hence, it remains to consider the case where  $\Gamma$  is a left cell such that  $[\Gamma]$  is a direct summand of the column labelled  $P_8$  in Table 2. We must show that the vector  $(n_E)_{E \in \mathcal{F}_0}$  giving the expansion of  $[\Gamma]$  is obtained by dividing all coefficients in that column by 2. Now there are many more possibilities for  $(n_E)_{E \in \mathcal{F}_0}$  satisfying  $(\diamond)$ . So we have to look for further arithmetical conditions on these numbers. Another such condition is given by (4.12)(b), where we work over the ring  $A_2$  as in Example 3.6. The remaining possibilities for the vector  $(n_E)_{E \in \mathcal{F}_0}$  are listed in the

TABLE 2. The cuspidal family in type  $E_8$

$E$	$(x, \sigma)$	$f_E$	$P_1$	$P_2$	$P_3$	$P_4$	$P_5$	$P_6$	$P_7$	$P_8$	splitting $P_8$			
$70_y$	$1, \lambda^3$	30	1	.	.	.	.	.	.	.	.	.	.	.
$168_y$	$g'_2, \varepsilon'$	8	.	1	.	.	.	.	.	.	.	.	.	.
$420_y$	$g_5, 1$	5	.	.	1	.	.	.	.	.	.	.	.	.
$448_w$	$g_2, \varepsilon$	12	1	.	.	1	.	.	.	.	.	.	.	.
$1134_y$	$g_3, \varepsilon$	6	.	.	.	1	1	.	.	.	.	.	.	.
$1344_w$	$g_4, 1$	4	.	1	1	.	1	.	.	.	.	.	.	.
$1400_y$	$1, \nu'$	24	2	1	.	.	.	1	.	.	.	.	.	.
$1680_y$	$1, \lambda^2$	20	3	.	.	1	.	1	.	.	.	.	.	.
$2688_y$	$g'_2, \varepsilon''$	8	.	.	.	.	.	1	1	2	.	2	.	2
$3150_y$	$g_3, 1$	6	.	.	1	1	1	.	1	2	2	.	2	.
$4200_y$	$g'_2, 1$	8	.	2	1	.	1	1	1	2	.	2	.	2
$4480_y$	$1, 1$	120	1	1	1	1	1	1	1	2	2	.	2	.
$4536_y$	$1, \nu$	24	3	1	.	1	.	2	1	2	2	.	2	.
$5670_y$	$1, \lambda^1$	30	3	1	.	2	1	2	1	2	2	.	2	.
$2016_w$	$g_6, 1$	6	.	.	1	.	.	.	1	2	2	.	.	2
$5600_w$	$g_2, r$	6	2	1	.	2	1	2	1	2	.	2	2	.
$7168_w$	$g_2, 1$	12	1	1	1	1	1	2	2	4	2	2	2	4

The notation in the first two columns is taken from Lusztig [24, p. 369].

The second column gives the identification with Gyoja's tables [17, §2.3].

6 rightmost columns of Table 2. (The central characters in (4.12) can be computed using the programs contained in the file `hecblloc.g` in the contributions directory on the CHEVIE homepage [13].)

Now we can argue as follows. Consider a ring  $R \supseteq A_2$  as in (3.10). Then, by (3.9)(2), the columns of the decomposition matrix of  $H_R$  are linearly independent. Assume now, if possible, that  $[\Gamma]$  is given by one of the 6 rightmost columns in Table 2. The  $H_R$ -module  $[\Gamma]_R$  is projective (see Corollary 3.8), hence it can be written as a direct sum of projective indecomposable  $H_R$ -modules. (Note that  $R$  is not  $\mathfrak{c}$ -adapted, where  $\mathfrak{c}$  is the two-sided cell containing  $\Gamma$ ; so we don't know if  $[\Gamma]_R$  is indecomposable.) Now we already know that there is a left cell  $\Gamma_7$  whose expansion in terms of irreducible representations is given by  $P_7$ , and  $[\Gamma_7]_R$  is a projective  $H_R$ -module (Corollary 3.8). Since  $P_8 = 2P_7$ , we conclude (using the linear independence of the columns of the decomposition matrix of  $H_R$ ) that  $[\Gamma]_R$  must be a sum of projective indecomposable  $H_R$ -modules which occur in the decomposition of  $[\Gamma_7]_R$  as a sum of projective indecomposable  $H_R$ -modules. Now we just compute all possible decompositions of  $[\Gamma_7]_R$  which satisfy the conditions in (4.12)(b). It turns out that the possibilities are precisely those given by the table with heading  $(\mathfrak{S}_5, \mathfrak{S}_3 \times \mathfrak{S}_2)$  ( $p = 2$ ) of Gyoja [17, p. 321]. (These tables can also be found in [28, 3.20].) Thus, at least one of the 6 rightmost columns in Table 2 should be expressible as a sum of the columns in Gyoja's table. One easily checks that this is impossible. This contradiction shows that Conjecture 2.1 holds for  $W$ .

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