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FINITE DIMENSIONAL REPRESENTATIONS OF SYMPLECTIC REFLECTION ALGEBRAS ASSOCIATED TO WREATH PRODUCTS

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Abstract. Using deformation theory of representations of algebras, we construct families of finite dimensional representations of symplectic reflection algebras associated to wreath products.

1. Introduction

In this paper we study finite dimensional representations of the wreath product symplectic reflection algebra $H_{1,k,c}(\mathbf{\Gamma}_N)$ of rank N attached to the group $\mathbf{\Gamma}_N = S_N \ltimes \Gamma^N$ ([EG], page 6), where $\Gamma \subset SL(2, \mathbf{C})$ is a finite subgroup, and $(k, c) \in C(\mathcal{S})$, where $C(\mathcal{S})$ is the space of (complex valued) class functions on the set $\mathcal{S}$ of symplectic reflections of $\mathbf{\Gamma}_N$.

In the rank 1 case, there is no parameter k, and finite dimensional representations of the wreath product algebra have been classified in [CBH] by Crawley-Boevey and Holland, by establishing a Morita equivalence between the algebra $H_{1,c}(\Gamma)$ and the deformed preprojective algebra $\Pi_{\lambda(c)}(Q)$ attached to the (extended Dynkin) quiver Q associated to Γ via the McKay correspondence.

We consider the higher rank case. When $k=0$, then $H_{1,k,c}(\mathbf{\Gamma}_N)=S_N\sharp H_{1,c}(\Gamma)^{\otimes N}$, so the finite dimensional representations of $H_{1,k,c}(\mathbf{\Gamma}_N)$ are known. Using a cohomological approach, we investigate the possibility of deforming some of these representations to values of the parameters with $k \neq 0$. This allows us to produce the first nontrivial examples of finite dimensional representations of $H_{1,k,c}(\mathbf{\Gamma}_N)$ for noncyclic Γ and $k \neq 0$.

Specifically, we show that if W is an irreducible representation of S_N whose Young diagram is a rectangle, and Y an irreducible finite dimensional representation of $H_{1,c}(\Gamma)$, then the representation $M = W \otimes Y^{\otimes N}$ of $H_{1,0,c}(\mathbf{\Gamma}_N)$ can be deformed along a hyperplane in $C(\mathcal{S})$. On the other hand, if $\dim Y = 1$ and the Young diagram of W is not a rectangle, such a deformation does not exist.

2. Preliminaries

2.1. **The wreath product construction.** In this subsection we will briefly recall the wreath product construction. Let L be a 2-dimensional complex vector space with a symplectic form ω_L, and consider the space $V = L^{\oplus N}$, endowed with the

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induced symplectic form $\omega_V = {\omega_L}^{\oplus N}$. Let Γ be a finite subgroup of $Sp(L)$, and let S_N be the symmetric group acting on V by permuting the factors. The group $\mathbf{\Gamma}_N := S_N \ltimes \Gamma^N \subset Sp(V)$ acts naturally on V. In the sequel we will write $\gamma_i \in \mathbf{\Gamma}_N$ for any element $\gamma \in \Gamma$ seen as an element in the i-th factor Γ of $\mathbf{\Gamma}_N$. The symplectic reflections in $\mathbf{\Gamma}_N$ are the elements s such that $\mathrm{rk}(Id - s)|_V = 2$. $\mathbf{\Gamma}_N$ acts by conjugation on the set $\mathcal{S}$ of its symplectic reflections. It is easy to see that there are symplectic reflections of two types in $\mathbf{\Gamma}_N$:

(S) the elements $s_{ij}\gamma_i{\gamma_j}^{-1}$ where $i, j \in [1, N]$, s_{ij} is the transposition $(ij) \in S_N$, and $\gamma \in \Gamma$,

(Γ) the elements γ_i, for $i \in [1, N]$ and $\gamma \in \Gamma\backslash\{1\}$.

Elements of type (S) are all in the same conjugacy class, while elements of type (Γ) form one conjugacy class for any conjugacy class of γ in Γ. Thus elements $f \in C[\mathcal{S}]$ can be written as pairs (k, c), where k is a number (the value of f on elements of type (S)), and c is a conjugation invariant function on $\Gamma \setminus \{1\}$ (encoding the values of f on elements of type (Γ)).

For any $s \in \mathcal{S}$ we write ω_s for the bilinear form on V that coincides with ω_V on $\mathrm{Im}(Id - s)$ and has $\mathrm{Ker}(Id - s)$ as radical. Denote by TV the tensor algebra of V.

Definition 2.1. For any $t \in \mathbf{C}$ and $f = (k, c) \in C[\mathcal{S}]$, the symplectic reflection algebra $H_{t,k,c}(\mathbf{\Gamma}_N)$ is the quotient

$$(\mathbf{\Gamma}_N \sharp TV) / \left\langle [u, v] - \kappa(u, v) \right\rangle_{u,v \in V}$$

where

$$\kappa : V \otimes V \longrightarrow \mathbf{C}[\mathbf{\Gamma}_N] : (u, v) \mapsto t \cdot \omega(u, v) \cdot 1 + \sum_{s \in \mathcal{S}} f_s \cdot \omega_s(u, v) \cdot s$$

with $f_s = f(s)$, and $\langle \dots \rangle$ is the two-sided ideal in the smash product $\mathbf{\Gamma}_N \sharp TV$ generated by the elements $[u, v] - \kappa(u, v)$ for $u, v \in V$.

We will be interested in the case $t \neq 0$, and it will be enough to consider the case $t = 1$ since $H_{t,k,c}(\mathbf{\Gamma}_N) \cong H_{1,k/t,c/t}(\mathbf{\Gamma}_N)$ for any $t \neq 0$ (cf. [EG], page 14). We recall that the case $t = 0$ is remarkably different and the corresponding representation theory has been studied in [EG], Section 3 and in [GS].

It is clear that choosing a symplectic basis x, y for L we can consider Γ as a subgroup of $SL(2, \mathbf{C})$. We will denote by x_i, y_i the corresponding vectors in the i-th L-factor of V. Following [GG] we will now give a more explicit representation of the algebra $H_{1,k,c}(\mathbf{\Gamma}_N)$.

Lemma 2.2 ([GG])**.** *The algebra $H_{1,k,c}(\mathbf{\Gamma}_N)$ is the quotient of $\mathbf{\Gamma}_N \sharp TV$ by the following relations:*

(R1) For any $i \in [1, N]$:

$$[x_i, y_i] = 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} s_{ij}\gamma_i\gamma_j^{-1} + \sum_{\gamma \in \Gamma\backslash\{1\}} c_\gamma \gamma_i \,.$$

(R2) For any $u, v \in L$ and $i \neq j$:

$$[u_i, v_j] = -\frac{k}{2} \sum_{\gamma \in \Gamma} \omega_L(\gamma u, v) s_{ij}\gamma_i\gamma_j^{-1} \,.$$

□

In the case $N = 1$, there is no parameter k (there are no symplectic reflections of type (S)) and

$$H_{1,c}(\Gamma) = \mathbf{C}\Gamma\sharp\mathbf{C}\langle x, y\rangle /(xy - yx = \lambda)$$

where $\lambda = \lambda(c) = 1+\sum_{\gamma\in\Gamma\setminus\{1\}} c_\gamma\gamma \in Z(\mathbf{C}[\Gamma])$ is the central element corresponding to c.

We end this subsection by recalling an important result that we will need later. It is stated in [EG] (Theorem 1.3) and is called the *Poincaré-Birkhoff-Witt (PBW-) property* for $H_{1,k,c}(\mathbf{\Gamma}_N)$. Consider the increasing filtration on $TV\sharp\mathbf{\Gamma}_N$ obtained by assigning degree zero to the elements of the group algebra $\mathbf{C}[\mathbf{\Gamma}_N]$ and degree one to the vectors in V. This filtration induces a filtration on $H_{1,k,c}(\mathbf{\Gamma}_N)$. The following theorem holds:

Theorem 2.3 (PBW). *The associated graded algebra to $H_{1,k,c}(\mathbf{\Gamma}_N)$ with respect to the above increasing filtration is $\mathbf{\Gamma}_N\sharp SV$, where SV is the symmetric algebra of V.* □

2.2. **Representations of S_N with a rectangular Young diagram.** We will use the following standard results from representation theory of the symmetric group. The proofs are well known, but we recall them for the reader's convenience. Denote by $\mathfrak{h}$ the reflection representation of S_N. For a Young diagram μ we denote by π_μ the corresponding irreducible representation of S_N.

Lemma 2.4. (i) $\mathrm{Hom}_{S_N}(\mathfrak{h}\otimes\pi_\mu, \pi_\mu) = \mathbf{C}^{m-1}$*, where m is the number of corners of the Young diagram μ. In particular,* $\mathrm{Hom}_{S_N}(\mathfrak{h}\otimes\pi_\mu, \pi_\mu) = 0$ *if and only if μ is a rectangle.*

(ii) *The element $C = s_{12} + s_{13} + \cdots + s_{1N}$ acts by a scalar in π_μ if and only if μ is a rectangle.*

Proof. Let $S_{N-1} \subset S_N$ be the subgroup of permutations fixing the index 1. It is well known that $\pi_\mu|_{S_{N-1}} = \sum \pi_{\mu-j}$, where the sum is taken over the corners of μ and $\mu - j$ is the Young diagram obtained from μ by cutting off the corner j. Since $\mathfrak{h}\oplus\mathbf{C} = \mathrm{Ind}_{S_{N-1}}^{S_N}\mathbf{C}$, the assertion (i) follows from the Frobenius reciprocity. To prove (ii), observe that C commutes with S_{N-1}, so it acts by a scalar on each $\pi_{\mu-j}$. Thus, if μ is a rectangle, C acts as a scalar (as we have only one summand), and the "if" part of the statement is proved. To prove the "only if" part, let Z_N be the sum of all transpositions in S_N. Z_N is a central element in the group algebra, and it is known to act in π_μ by the scalar $\mathbf{c}(\mu)$, where $\mathbf{c}(\mu)$ is the content of μ, i.e. the sum over all cells of the signed distances from these cells to the diagonal. Now, $C = Z_N - Z_{N-1}$, so it acts on $\pi_{\lambda-j}$ by the scalar $\mathbf{c}(j)$, the signed distance from the cell j to the diagonal. The numbers $\mathbf{c}(j)$ are clearly different for all corners j, so if there are 2 or more corners, then C cannot act by a scalar. This finishes the proof of (ii). □

3. The main theorem

Let Y be a finite dimensional irreducible representation of the algebra $H_{1,c}(\Gamma)$ for some[1] c. Let W be an irreducible representation of S_N. Since the algebra $H_{1,0,c}(\mathbf{\Gamma}_N)$ is naturally isomorphic to $S_N\sharp H_{1,c}(\Gamma)^{\otimes N}$, there is a natural action of $H_{1,0,c}(\mathbf{\Gamma}_N)$ on the vector space $M := W\otimes Y^{\otimes N}$. Namely, each copy of $H_{1,c}(\Gamma)$ acts

[1]Such representations exist only for special c; as for generic c, the algebra $H_{1,c}(\Gamma)$ is simple; see [CBH],Theorem 0.3, Corollary 7.6.

in the corresponding copy of Y, while S_N acts in W and simultaneously permutes the factors in the product $Y^{\otimes N}$. We will denote this representation by M_c. The main theorem tells us when such a representation can be deformed to nonzero values of k.

Assume that the Young diagram of W is a rectangle of height l and width $m = N/l$ (the trivial representation corresponds to the horizontal strip of height 1).

Let $\mathcal{H}_{Y,m,l}$ be the hyperplane in $C(\mathcal{S})$ consisting of all pairs (k, c) satisfying the equation

$$\dim Y + \frac{k}{2}|\Gamma|(m-l) + \sum_{\gamma\in\Gamma\setminus\{1\}} c_\gamma \chi_Y(\gamma) = 0\,, \tag{1}$$

where χ_Y is the character of Y.

Let $X = X(Y, m, l)$ be the moduli space of irreducible representations of $H_{1,k,c}(\mathbf{\Gamma}_N)$ isomorphic to M as $\mathbf{\Gamma}_N$-modules (where (k, c) are allowed to vary). This is a quasi-affine algebraic variety: it is the quotient of the quasi-affine variety $\widetilde{X}(Y, m, l)$ of extensions of the $\mathbf{\Gamma}_N$-module M to an irreducible $H_{1,k,c}(\mathbf{\Gamma}_N)$-module by a free action of the reductive group G of basis changes in M compatible with $\mathbf{\Gamma}_N$ modulo scalars. Let $f : X \to C(\mathcal{S})$ be the morphism which sends a representation to the corresponding values of (k, c).

The main result of this paper is the following theorem.

Theorem 3.1. (i) *For any c_0 the representation M_{c_0} of $H_{1,0,c_0}(\mathbf{\Gamma}_N)$ can be formally deformed to a representation of $H_{1,k,c}(\mathbf{\Gamma}_N)$ along the hyperplane $\mathcal{H}_{Y,m,l}$, but not in other directions. This deformation is unique.*

(ii) *The morphism f maps X to $\mathcal{H}_{Y,m,l}$ and is étale at M_{c_0} for all c_0. Its restriction to the formal neighborhood of M_{c_0} is the deformation from (i).*

(iii) *There exists a nonempty Zariski open subset $\mathcal{U}$ of $\mathcal{H}_{Y,m,l}$ such that for $(k, c) \in \mathcal{U}$, the algebra $H_{1,k,c}(\mathbf{\Gamma}_N)$ admits a finite dimensional irreducible representation isomorphic to M as a $\mathbf{\Gamma}_N$-module.*

The proof of this theorem occupies the remaining sections of the paper.

Remark. In the case of cyclic Γ and trivial W Theorem 3.1 was proved in [CE]. In this case, the deformation of the representation M can be constructed explicitly.

We expect that the condition that the Young diagram of W is a rectangle is essential to obtain the deformation of Theorem 3.1 (i). For example, this is the case if Y is 1-dimensional. This follows from the following more general statement.

Proposition 3.2. *Let W be an irreducible S_N-module. If W extends to a representation of $H_{1,k,c}(\mathbf{\Gamma}_N)$ for some (k, c) with $k \neq 0$, then the Young diagram of W is a rectangle.*

Proof. Suppose that W extends to a representation of $H_{1,k,c}(\mathbf{\Gamma}_N)$. Such an extension is, first of all, an extension of W to a representation of the wreath product group $S_N \ltimes \Gamma^N$. This can only be done by making Γ^N act by an S_N-invariant character ξ, i.e., $\xi(\gamma_1, \ldots, \gamma_N) = \chi(\gamma_1)\ldots\chi(\gamma_N)$, where $\chi : \Gamma \to \mathbf{C}^*$ is a character. But in this case Γ^N acts trivially on $\mathrm{End}_{\mathbf{C}}(W)$, and hence x_i, y_i must act by 0 on W for each $i = 1, \ldots, N$. So, denoting by ρ the possible extended representation, we obtain from relation *(R1)* for $i = 1$,

$$\rho\,(s_{12} + \cdot + s_{1N}) = -2\,\frac{1 + \sum_{\gamma\in\Gamma\setminus\{1\}} c_\gamma \chi(\gamma)}{k\,|\Gamma|},$$

i.e., $C = s_{12} + \cdots + s_{1N}$ acts by a constant on W. Now applying Lemma 2.4, part *(ii)*, we get that W must correspond to a rectangular Young diagram. $\square$

4. Proof of Theorem 3.1

4.1. **Deformation theory.** In this section we recall deformation theory of representations of algebras. This theory is well known, but we give the details for the reader's convenience.

Let A be an associative algebra over $\mathbf{C}$. In what follows, for each A-bimodule E, we write $H^n(A, E)$ for the n-th Hochschild cohomology group of A with coefficients in E. We recall that $H^n(A, E)$ is defined to be the i-th cohomology group of the Hochschild complex:

$$0 \longrightarrow C^0(A, E) \xrightarrow{d} \cdots \xrightarrow{d} C^n(A, E) \xrightarrow{d} C^{n+1}(A, E) \xrightarrow{d} \cdots$$

where $C^n(A, E) = \mathrm{Hom}_{\mathbf{C}}(A^{\otimes n}, E)$ is the space of n-linear maps from $A^{\otimes n}$ to E, and the differential d is defined as follows:

$$\begin{aligned}(d\varphi)(a_1, \ldots, a_{n+1}) : \;&=\; a_1\varphi(a_2, \ldots, a_{n+1})\\ &+\; \sum_{i=1}^{n}(-1)^i\varphi(a_1, \ldots, a_{i-1}, a_i\,a_{i+1}, a_{i+2}, \ldots, a_{n+1})\\ &-\; (-1)^n\varphi(a_1, \ldots, a_n)\,a_{n+1}.\end{aligned}$$

We remark that $H^i(A, E)$ coincides with the vector space $Ext^i_{A\otimes A^o}(A, E)$, where A^o is the opposite algebra of A.

Let A_U be a flat formal deformation of A over the formal neighborhood of zero in a finite dimensional vector space U with coordinates $t_1, \ldots, t_n$. This means that A_U is an algebra over $\mathbf{C}[[U]] = \mathbf{C}[[t_1, \ldots, t_n]]$ which is *topologically free* as a $\mathbf{C}[[U]]$-module (i.e., A_U is isomorphic as a $\mathbf{C}[[U]]$-module to $A[[U]]$), together with a fixed isomorphism of algebras $A_U/JA_U \cong A$, where J is the maximal ideal in $\mathbf{C}[[U]]$. Given such a deformation, we have a natural linear map $\phi : U \longrightarrow H^2(A, A)$ defined as follows.

We can think of A_U as $A[[t_1, \ldots, t_n]]$ equipped with a new $\mathbf{C}[[t_1, \ldots, t_n]]$-linear (and continuous) associative product defined by

$$a * b = \sum_{p_1,\ldots,p_n} c_{p_1,\ldots,p_n}(a, b) \prod_j t_j^{p_j} \qquad a, b \in A$$

where $c_{p_1,\ldots,p_n} : A \times A \longrightarrow A$ are $\mathbf{C}$-bilinear functions and $c_{0,\ldots,0}(a, b) = ab$, for any $a, b \in A$.

Imposing the associativity condition on $*$, one obtains that $c_{0,\ldots,1_j,\ldots,0}$ must be Hochschild 2-cocycles for each j. The map ϕ is given by the assignment $(t_1, ..., t_N) \to \sum_j t_j\,[c_{0,\ldots,1_j,\ldots,0}]$ for any $(t_1, \ldots, t_n) \in U$, where $[C]$ stands for the cohomology class of a cocycle C.

Now let M be a representation of A. In general it does not deform to a representation of A_U. However, we have the following standard proposition. Let $\eta : U \to H^2(A, \mathrm{End}\, M)$ be the composition of ϕ with the natural map $\psi : H^2(A, A) \longrightarrow H^2(A, \mathrm{End}\, M)$.

Proposition 4.1. *Assume that η is surjective with kernel K, and $H^1(A, \operatorname{End} M) = 0$. Then:*

(i) There exists a unique smooth formal subscheme S of the formal neighborhood of the origin in U, with tangent space K at the origin, such that M deforms to a representation of the algebra $A_S := A_U \hat{\otimes}_{\mathbf{C}[[U]]} \mathbf{C}[S]$ (where $\hat{\otimes}$ is the completed tensor product).

(ii) The deformation of M over S is unique.

Proof. Let us realize A_U explicitly as $A[[t_1, \dots, t_n]]$ equipped with a product $*$ as above. We may assume that K is the space of all vectors $(t_1, \dots, t_n)$ such that $t_{m+1} = \dots = t_n = 0$.

Let D be the formal neighborhood of the origin in K, with coordinates $h_1 = t_1, \dots, h_m = t_m$. Let $\theta : D \to U$ be a map given by the formula $\theta(h_1, \dots, h_m) = (t_1, \dots, t_n)$, where $t_i = h_i$ for $i \leq m$, and

$$t_k = \sum_{p_1, \dots, p_m} t_{k,p_1,\dots,p_m} h_1^{p_1} \dots h_m^{p_m}, k > m,$$

where $t_{k,p_1,\dots,p_m} \in \mathbf{C}$. More briefly, we can write $t_k = \sum_P t_{kP} h^P$, where P is a multi-index. We will use the notation $|P|$ for the sum of indices in a multi-index P. For brevity we also let e_j to be the multi-index $(0, \dots, 1_j, \dots, 0)$.

We claim that there exist unique formal functions $t_k = t_k(h)$, $k > m$, for which we can deform M over D. Indeed, such a deformation would be defined by a series

$$\widetilde{r}(a) = \sum_P r_P(a) h^P,$$

where $r_0(a) = r(a)$, and r is the homomorphism giving the representation M. The condition that $\widetilde{r}$ is a representation gives, for each P,

$$d\, r_P = \sum_j t_{jP} r(c_{e_j}) + C_P, \tag{2}$$

where for $j \leq m$, $t_{jP} = 1$ if $P = e_j$ and zero otherwise, and C_P is a 2-cocycle whose expression may involve r_Q and t_{kQ} **only** with $|Q| < |P|$. Since the map η is surjective, there are (unique) $t_{d+1,P}, \dots, t_{nP}$ for which the right-hand side is a coboundary. For such $t_{d+1,P}, \dots, t_{nP}$ (and only for them), we can solve (2) for r_P.

This shows the existence of the functions $t_j(h)$, $j > m$, such that the deformation of M over D is possible. To show the uniqueness of these functions, let t_j and t'_j be two sets of functions for which the deformation exists. Let r_P, r'_P be the coefficients of the corresponding representations $\widetilde{r}, \widetilde{r}'$. Let N be the maximal number such that $t_{jP} = t'_{jP}$ for $|P| < N$. Since $H^1(A, \operatorname{End} M) = 0$, the solution r_P of (2) is unique up to adding a coboundary. Thus we can use changes of basis in M to modify $\widetilde{r}$ so that $r_P = r'_P$ for $|P| < N$ (note that this does not affect t_j). Then for any Q with $|Q| = N$, $C_Q(\widetilde{r}) = C_Q(\widetilde{r'})$, and hence $t_{jQ} = t'_{jQ}$. This contradicts the maximality of N.

Thus, we have shown that the functions t_j exist and are unique; they define a parametrization of the desired subscheme S by D. Our proof also implies that the deformation of M over S is unique, so we are done. □

We end this section by recalling the following fact from algebraic geometry that will guarantee that the representations we found in Theorem 3.1 are actually irreducible.

Let X be an affine irreducible algebraic variety over $\mathbf{C}$, $R = \mathbf{C}[X]$. Let A be an algebra over R and M an A-module, such that A and M are free as R-modules and M is of finite rank. For $x \in X$, let A_x, M_x be the fibers of A, M at x; so A_x is a $\mathbf{C}$-algebra and M_x a finite dimensional module over A_x.

Proposition 4.2. *The set of x for which M_x is irreducible is open in X.*

Proof. Let x be a point of X where M_x is irreducible. Then the map $f_x : A \to \operatorname{End} M_x$ is surjective. This means that there exist elements $a_1, \dots, a_{N^2}$ in A, $N = \dim_R M$, such that $f_x(a_i)$ is a basis of $\operatorname{End} M_x$. The set U of points z of X such that $f_z(a_i)$ are a basis of $\operatorname{End} M_z$ is open and contains x. We found a neighborhood U of x such that, for all z in U, M_z is an irreducible A_z-module, as desired. □

4.2. **Homological properties of the algebra $H_{1,c}(\Gamma)$.** We recall the following definition (see [VB1, VB2, EO]):

Definition 4.3. An algebra A is defined to be in the class $VB(d)$ if it is of finite Hochschild dimension (i.e. there exists $n \in \mathbf{N}$ s.t. $H^i(A, E) = 0$ for any $i > n$ and any A-bimodule E) and $H^*(A, A \otimes A^o)$ is concentrated in degree d, where it equals A as an A-bimodule.

The meaning of this definition is clarified by the following result by van den Bergh ([VB1, VB2]).

Theorem 4.4. *If $A \in VB(d)$, then for any A-bimodule E, the Hochschild homology $H_i(A, E)$ is naturally isomorphic to the Hochschild cohomology $H^{d-i}(A, E)$.* □

Now let $B = H_{1,c}(\Gamma)$.

Proposition 4.5. *The algebra B belongs to the class $VB(2)$.*

Proof. If $\Gamma = \{1\}$, the statement is well known ([VB1, VB2]; see also [EO]). Let us consider the case $\Gamma \neq \{1\}$. We have to show that B has finite Hochschild dimension and that

$$H^i(B, B \otimes B^o) = 0 \qquad \text{for } i \neq 2$$
$$H^2(B, B \otimes B^o) \cong B \qquad \text{as } B\text{-bimodules.}$$

The algebra $\mathbf{C}\Gamma \sharp \mathbf{C}\langle x, y\rangle$ has a natural increasing filtration obtained by putting x, y in degree 1 and the elements of Γ in degree 0. This filtration clearly induces a filtration on B: $B = \bigcup_{n \geq 0} F^n B$, and the associated graded algebra is $B_0 = grB = \mathbf{C}\Gamma \sharp \mathbf{C}[x, y]$ (by the PBW theorem), which has Hochschild dimension 2. So by a deformation argument we have that B has finite Hochschild dimension (equal to 2) and $H^i(B, B \otimes B^o) = 0$ for $i \neq 2$, as this is true for B_0 (which is easily checked since B_0 is a semidirect product of a finite group with a polynomial algebra).

It remains to show the B-bimodule $E := H^2(B, B \otimes B^o)$ is isomorphic to B. Using again a deformation argument (cf. [VB1]), we can see that E is invertible and free as a right and left B-module, because this is true for B_0. So $E = B\phi$ where ϕ is an automorphism of B such that $gr\phi = 1$. We will now show that $\phi = 1$, which will conclude the proof.

Define a linear map $\xi : B_0 \to B_0$ as follows: if $z \in B_0$ is a homogeneous element of degree n, and $\widetilde{z}$ is its lifting to B, then $\xi(z)$ is defined to be the projection of the element $\phi(\widetilde{z}) - \widetilde{z}$ (which has filtration degree $n - 1$) to $(grB)_{n-1}$. It is easy to check that ξ is well defined (i.e., independent on the choice of the lifting), and is a derivation of B_0 of degree -1.

Our job is to show that $\xi = 0$. This would imply that $\phi = 1$, since B is generated by $F^1 B$.

It is clear that any homogeneous inner derivation of B_0 has nonnegative degree. Hence, it suffices to show that the degree -1 part of $H^1(B_0, B_0)$ is zero. But it is easy to compute using Koszul complexes that $H^1(B_0, B_0) = \mathrm{Vect}(L)^\Gamma$, the space of Γ-invariant vector fields on L. In particular, vector fields of degree -1 are those with constant coefficients. But such a vector field cannot be Γ-invariant unless it is zero, since the space L has no nonzero vectors fixed by Γ. Thus, $\xi = 0$ and we are done. □

Corollary 4.6. $H^2(B, \mathrm{End}\, Y) = H_0(B, \mathrm{End}\, Y) = \mathbf{C}$.

Proof. We apply Theorem 4.4 to obtain the first identity. Furthermore, $H_0(B, \mathrm{End}\, Y) = \mathrm{End}\, Y/[B, \mathrm{End}\, Y] = \mathbf{C}$ as Y is irreducible, so the second identity follows. □

Proposition 4.7. $H^1(B, \mathrm{End}\, Y) = 0$.

Proof. We have $H^1(B, \mathrm{End}\, Y) = \mathrm{Ext}^1_{B\otimes B^o}(B, \mathrm{End}\, Y) = \mathrm{Ext}^1_B(Y, Y)$. But it is known ([CBH], Corollary 7.6) that B contains only one minimal ideal J among all the nonzero ideals, and $\mathrm{Ext}^1_{B/J}(Y', Y') = 0$ for any irreducible module Y' over the (finite dimensional) quotient algebra B/J. Since any finite dimensional B-module must factor through B/J, we get $\mathrm{Ext}^1_B(Y, Y) = 0$, as desired. □

4.3. **Homological properties of** $A = H_{1,0,c}(\mathbf{\Gamma}_N)$**.** We now let A denote the algebra $H_{1,0,c_0}(\mathbf{\Gamma}_N)$. The algebra A has a flat deformation over $U = C(\mathcal{S})$, which is given by the algebra $H_{1,k,c_0+c'}(\mathbf{\Gamma}_N)$. The fact that this deformation is flat follows from Theorem 2.3.

Proposition 4.8. *If the Young diagram of W is a rectangle, then*

$$H^2(A, \mathrm{End}\, M) = H^2(B, \mathrm{End}\, Y) = \mathbf{C}.$$

Proof. The second equality follows from Corollory 4.6. Let us prove the first equality. We have

$$\begin{aligned} H^*(A, \mathrm{End}\, M) &= \mathrm{Ext}^*_{A\otimes A^o}(A, \mathrm{End}\, M) \\ &= \mathrm{Ext}^*_{S_N\sharp B^{\otimes N}\otimes S_N\sharp B^{o\otimes N}}(S_N\sharp B^{\otimes N}, \mathrm{End}\, W \otimes \mathrm{End}\, Y^{\otimes N}) \\ &= \mathrm{Ext}^*_{S_N\times S_N\sharp(B^{\otimes N}\otimes B^{o\otimes N})}(S_N\sharp B^{\otimes N}, \mathrm{End}\, W \otimes \mathrm{End}\, Y^{\otimes N}). \end{aligned}$$

Now, the $S_N \times S_N\sharp(B^{\otimes N} \otimes B^{o\otimes N})$-module $S_N\sharp B^{\otimes N}$ is induced from the module $B^{\otimes N}$ over the subalgebra $S_N\sharp B^{\otimes N} \otimes B^{o\otimes N}$, in which S_N acts simultaneously permuting the factors of $B^{\otimes N}$ and $B^{o\otimes N}$ (note that $S_N\sharp(B^{\otimes N} \otimes B^{o\otimes N})$ is indeed a subalgebra of $S_N \times S_N\sharp(B^{\otimes N} \otimes B^{o\otimes N})$ as it can be identified with the subalgebra $D\sharp(B^{\otimes N} \otimes B^{o\otimes N})$ where $D = \{(\sigma, \sigma),\ \sigma \in S_N\} \subset S_N \times S_N$). Applying the Shapiro Lemma, we get

$$\begin{aligned} &\mathrm{Ext}^*_{S_N\times S_N\sharp(B^{\otimes N}\otimes B^{o\otimes N})}(S_N\sharp B^{\otimes N}, \mathrm{End}\, W \otimes \mathrm{End}\, Y^{\otimes N}) \\ &= \mathrm{Ext}^*_{S_N\sharp(B^{\otimes N}\otimes B^{o\otimes N})}(B^{\otimes N}, \mathrm{End}\, W \otimes \mathrm{End}\, Y^{\otimes N}) \\ &= \left(\mathrm{Ext}^*_{B^{\otimes N}\otimes B^{o\otimes N}}(B^{\otimes N}, \mathrm{End}\, W \otimes \mathrm{End}\, Y^{\otimes N})\right)^{S_N}. \end{aligned}$$

But since $B^{\otimes N} \otimes B^{o\otimes N}$ does not act on $\operatorname{End} W$, the latter module equals

$$\left(\operatorname{Ext}^*_{B^{\otimes N}\otimes B^{o\otimes N}}(B^{\otimes N}, \operatorname{End} Y^{\otimes N}) \otimes \operatorname{End} W\right)^{S_N}.$$

Using Proposition 4.7 and the Künneth formula in degree 2, we get that as an S_N-module, $\operatorname{Ext}^2_{B^{\otimes N}\otimes B^{o\otimes N}}(B^{\otimes N}, \operatorname{End} Y^{\otimes N}) = \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y) \otimes \mathbf{C}^N$ where S_N acts only on $\mathbf{C}^N$ permuting the factors. But as an S_N-module, $\mathbf{C}^N = \mathbf{C} \oplus \mathfrak{h}$, where $\mathbf{C}$ is the trivial representation. As a result we get

$$\begin{aligned}\operatorname{Ext}^2_{A\otimes A^o}(A, \operatorname{End} M) &= \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y) \otimes \left(\mathbf{C}^N \otimes \operatorname{End} W\right)^{S_N} \\ &= \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y) \otimes (\mathbf{C} \otimes \operatorname{End} W \oplus \mathfrak{h} \otimes \operatorname{End} W)^{S_N} \\ &= \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y) \otimes (\operatorname{Hom}_{S_N}(W, W) \oplus \operatorname{Hom}_{S_N}(\mathfrak{h} \otimes W, W)) \\ &= \operatorname{Ext}^2_{B\otimes B^o}(B, \operatorname{End} Y)\end{aligned}$$

as $\operatorname{Hom}_{S_N}(\mathfrak{h} \otimes W, W) = 0$ by Lemma 2.4 part (i). □

Corollary 4.9. *The map $\eta : U \longrightarrow H^2(A, \operatorname{End} M)$ is surjective.*

Proof. Let $U_0 \subset U$ be the subspace of vectors $(0, c')$. It is sufficient to show that the restriction of η to U_0 is surjective. But this restriction is a composition of three natural maps:

$$U_0 \to H^2(B, B) \to H^2(A, A) \to H^2(A, \operatorname{End} M).$$

Here the first map $\eta_0 : U_0 \to H^2(B, B)$ is induced by the deformation of B along U_0, the second map $\xi : H^2(B, B) \to H^2(A, A)$ comes from the Künneth formula, and the third map $\psi : H^2(A, A) \to H^2(A, \operatorname{End} M)$ is induced by the homomorphism $A \to \operatorname{End} M$.

Now, by Proposition 4.8, the map $\psi \circ \xi$ coincides with the map $\psi_0 : H^2(B, B) \to H^2(B, \operatorname{End} Y)$ induced by the homomorphism $B \to \operatorname{End} Y$. We claim that this map is surjective. Indeed, since by Proposition 4.5, B is in $VB(2)$, by Theorem 4.4 there is a natural identification of $H^2(B, E)$ with $H_0(B, E)$ for any B-bimodule E; hence ψ_0 can be viewed as the natural map $\psi_0 : H_0(B, B) \to H_0(B, \operatorname{End} Y)$. But $H_0(B, E) = E/[B, E]$ for any B-bimodule E. Hence, ψ_0 can be viewed as the natural map

$$\psi_0 : B/[B, B] \longrightarrow \operatorname{End} Y/[B, \operatorname{End} Y].$$

This map is clearly nonzero: the representation Y is irreducible, and hence the map $B \to \operatorname{End} Y$ is surjective. Thus ψ_0 is surjective, as claimed (as the space $\operatorname{End} Y/[B, \operatorname{End} Y]$ is 1-dimensional).

Let K be the kernel of ψ_0. It remains to show that the map η_0 does not land in K. To show this, recall that by Proposition 4.1, the representation Y of B can be deformed along K. Thus it remains to show that Y does not admit a first order deformation along the entire U_0. But this follows easily by computing the trace of both sides of the commutation relation $xy - yx = \lambda$ in a deformation of Y. We are done. □

Proposition 4.10. *$H^1(A, \operatorname{End} M) = 0$.*

Proof. Arguing as in the proof of Proposition 4.8, we get that $H^1(A, \operatorname{End} M) = H^1(B, \operatorname{End} Y)$, which is zero. This proves the proposition. □

We have thus proved the following result.

Proposition 4.11. *If the Young diagram corresponding to W is a rectangle, then there exists a unique smooth codimension one formal subscheme S of the formal neighborhood of the origin in U such that the representation $M = W \otimes Y^{\otimes N}$ of $H_{1,0,c_0}(\mathbf{\Gamma}_N)$ formally deforms to a representation of $H_{1,k,c_0+c'}(\mathbf{\Gamma}_N)$ along S (i.e., abusing the language, for $(k, c') \in S$). Furthermore, the deformation of M over S is unique.*

Proof. Corollary 4.9 and Proposition 4.10 show that our case satisfies all the hypotheses of Proposition 4.1. Moreover, from $H^2(A, \operatorname{End} M) = \mathbf{C}$ we deduce $\dim \operatorname{Ker} \eta = \dim U - 1$, and the proposition follows. □

4.4. **The trace condition and the proof of Theorem 3.1.** Now we would like to find the subscheme S of Proposition 4.11. For this we take the trace in M of the commutation relation *(R1)*, and obtain a necessary condition on the parameters (k, c) for the algebra $H_{1,k,c}(\mathbf{\Gamma}_N)$ to admit a representation isomorphic to M as a Γ_N-module:

(TR) For any $i \in [1, n]$,

$$\dim M + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in \Gamma} \operatorname{tr}|_M (s_{ij} \gamma_i \gamma_j^{-1}) + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \operatorname{tr}|_M (\gamma_i) = 0\,.$$

This relation can be easily rewritten in terms of the characters χ_Y of Y as a representation of Γ and ψ_W of W as a representation of S_N. Indeed, one can check:

$$\operatorname{tr}|_M (\gamma_i) = \dim W \dim Y^{N-1} \chi_Y(\gamma), \tag{3}$$

$$\operatorname{tr}|_M (s_{ij}\, \gamma_i\, {\gamma_j}^{-1}) = \psi_W(s_{ij}) \dim Y^{N-1}. \tag{4}$$

Namely, (3) is an easy consequence of the fact that the group $\Gamma^{\times N} \subset \mathbf{\Gamma}_N$ acts only on $Y^{\otimes N}$ with character $\chi_Y^{\otimes N}$, and γ_i is by definition the element $(1, \ldots, \overset{i}{\gamma}, \ldots, 1) \in \Gamma^{\times N}$. To obtain (4), we observe that $s_{ij}\, \gamma_i\, {\gamma_j}^{-1}$ is conjugate in $\mathbf{\Gamma}_N$ to s_{ij} and that the character of S_N on M is simply the product of the characters on W and $Y^{\otimes N}$. An easy computation gives $\operatorname{tr}|_{Y^{\otimes N}} s_{ij} = \dim Y^{N-1}$, hence the formula.

We now recall that, for any transposition $\sigma \in S_N$, $\psi|_W(\sigma) = \frac{\dim W}{N\,(N-1)/2} \mathbf{c}(\mu)$, where $\mathbf{c}(\mu)$ is the content of the Young diagram μ attached to W. In particular, if μ is a rectangular diagram of size $l \times m$ with $lm = N$, it can be easily computed that

$$\mathbf{c}(\mu) = \frac{N\,(m-l)}{2},$$

so we have

$$\operatorname{tr}|_M (s_{ij} \gamma_i\, {\gamma_j}^{-1}) = \frac{(m-l) \dim W}{N-1} \dim Y^{N-1}. \tag{5}$$

Finally, substituting (3), (5) in *(TR)* and dividing the relation by $\dim Y^{N-1} \dim W$, we obtain

(TR') If the Young diagram of W is of size $l \times m$, then

$$\dim Y + \frac{k}{2}\, |\Gamma| (m-l) + \sum_{\gamma \in \Gamma \setminus \{1\}} c_\gamma \chi_Y(\gamma) = 0\,.$$

The condition *(TR')* defines exactly the hyperplane $\mathcal{H}_{Y,m,l}$.

Thus we have shown that $(0, c_0) + S \subset \mathcal{H}_{Y,m,l}$. But S and $\mathcal{H}_{Y,m,l}$ have the same dimension, which implies that S is the formal neighborhood of zero in $\mathcal{H}_{Y,m,l} - (0, c_0)$. This proves part (i) of Theorem 3.1.

We now conclude the proof of Theorem 3.1. Let X' be the formal neighborhood of M_{c_0} in X. We have shown that the morphism $f : X \to U$ lands in $\mathcal{H}_{Y,m,l}$, and that $f|_{X'} : X' \to (0, c_0) + S$ is an isomorphism. This implies that the map $f : X \to \mathcal{H}_{Y,m,l}$ is étale at M_{c_0}. This proves part (ii) of Theorem 3.1, and, together with Proposition 4.2, also implies (iii), since a map which is étale at one point is dominant.

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