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CATEGORICAL LANGLANDS CORRESPONDENCE FOR $\mathrm{SO}_{n,1}(\mathbb{R})$

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Abstract. In the context of the local Langlands philosopy for $\mathbb{R}$, Adams, Barbasch and Vogan describe a bijection between the simple Harish-Chandra modules for a real reductive group $G(\mathbb{R})$ and the space of "complete geometric parameters"—a space of equivariant local systems on a variety on which the Langlands-dual of $G(\mathbb{R})$ acts. By a conjecture of Soergel, this bijection can be enhanced to an equivalence of categories. In this article, that conjecture is proven in the case where $G(\mathbb{R})$ is a generalized Lorentz group $\mathrm{SO}_{n,1}(\mathbb{R})$.

Contents

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1. Introduction

The Langlands philosophy for the local field $\mathbb{R}$ in the form presented in [1] states more or less the following: For any real reductive algebraic group G, there is a correspondence between the irreducible Harish-Chandra modules for $G(\mathbb{R})$ and the set of "complete geometric parameters"—a geometric space constructed from the Langlands dual group $G^\vee$. In [12], Soergel turns this space into a category and conjectures that this "geometric category" is equivalent to the category of Harish-Chandra modules for $G(\mathbb{R})$.

If $G(\mathbb{R})$ is a generalized Lorentz group $\mathrm{SO}_{m-1,1}(\mathbb{R})$, the category of Harish-Chandra modules has been realized explicitly as a category of certain modules over a quiver algebra in [9]. In this article we will realize the corresponding geometric category in the same way, thereby proving the conjecture of [12] for the case $\mathrm{SO}_{m-1,1}(\mathbb{R})$.

The reader is assumed to be familiar with [1] up to chapter 6 and with equivariant derived sheaves as presented in [3]. We will use methods of [11] and [10], so it might be helpful to know these, too; however, this is not absolutely necessary as we will repeat all the results we will use. What we are doing here can be found in more detail (but in german) in [7].

1.1. **Some notation.** Given a complex algebraic group G, we will write G instead of $G(\mathbb{C})$ for the complex-valued points most of the time.

The notation "$G \multimap X$" means "G acts on X". This notation is also suited for commutative diagrams. Instead of "G-equivariant sheaves on X", we will often write "sheaves on $G \multimap X$". By the way: All sheaves in this article will be equivariant; we will not always say this explicitly.

1.2. **The conjecture of Soergel in detail.** Suppose we are given the following data:

- a complex algebraic group G,
- an inner class F of real forms of G,
- a central character $\chi \in \operatorname{Max} \mathrm{Z}(U(\mathfrak{g}))$ of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ corresponding to G.

From these data one defines the two categories which are claimed to be equivalent. We start with the representation theoretic side:

Let ${}^{\Gamma}G = G \rtimes \Gamma$ with $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ be a *weak extended group* corresponding to the inner class F of real forms (see [1], ch. 2). The equivalence classes of those strong real forms $\delta \in {}^{\Gamma}G \setminus G$ which satisfy $\delta^2 = 1$ correspond to the elements of the non-abelian cohomology $H^1(\Gamma, G)$. Choose a set $S \subset {}^{\Gamma}G \setminus G$ of representatives of these equivalence classes.

Given such a representative $\delta \in S$, let $G(\mathbb{R}, \delta) := G(\mathbb{C})^{\delta}$ be the corresponding real algebraic group and let $\mathcal{M}(G(\mathbb{R}, \delta))$ be its category of Harish-Chandra modules. This category decomposes according to generalized infinitesimal characters. Let $\mathcal{M}(G(\mathbb{R}, \delta))_{\chi}$ be the part corresponding to our character χ. Now we are ready to define the representation theoretic category:

Definition 1.1. The *representation theoretic category* associated to (G, F, χ) is the direct sum

$$\bigoplus_{\delta \in S} \mathcal{M}(G(\mathbb{R}, \delta))_{\chi} \, .$$

For the geometric category, we first have to construct a variety X out of the data $(G,\, F,\, \chi)$. This variety comes with an action of the Langlands dual group $G^{\vee}$ of G. We will postpone the definition of X to Chapter 2. See [1] for details. (The *complete geometric parameters* of Adams, Barbasch and Vogan are the equivariant local systems on orbits of $G^{\vee} \circlearrowright X$.)

Let $\mathrm{PSh}_{G^{\vee}}(X)$ be the category of equivariant perverse sheaves on X (with respect to the middle perversity). The simple objects in this category are in bijection to the complete geometric parameters, and therefore also to the simple objects in the representation theoretic category.

Let $\mathcal{L}_{\mathrm{all}}$ be the direct sum of all simple objects in $\mathrm{PSh}_{G^{\vee}}(X)$. (To be more precise: of one representative of each isomorphism class of simple objects.) The endomorphisms $\mathrm{End}^{\bullet}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}}) := \bigoplus_{n \in \mathbb{N}} \mathrm{Hom}^{n}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}}, \mathcal{L}_{\mathrm{all}})$ in the equivariant derived category $\mathrm{D}^{+}_{G^{\vee}}(X)$ form a graded algebra, the *geometric extension algebra.*

One easily checks that there is a bijection between the simple equivariant perverse sheaves and the simple $\mathrm{End}^{\bullet}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}})$-modules. The geometric category is now defined to be a subcategory of the $\mathrm{End}^{\bullet}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}})$-modules: For any graded algebra A, we define the category A-nil of *nil-modules* by

$$A\text{-nil} := \{M \in A\text{-mod} \mid \dim M < \infty, A^n M = 0 \text{ for } n \gg 0\} \, .$$

Definition 1.2. The *geometric category* associated to (G, F, χ) is the category of nil-modules,

$$\mathrm{End}^{\bullet}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}})\text{-nil} \, .$$

Now here is Soergel's conjecture:

Conjecture 1.3 ([12])**.** *The representation theoretic category and the geometric category associated to (G, F, χ) are equivalent:*

$$\bigoplus_{\delta \in S} \mathcal{M}(G(\mathbb{R}, \delta))_{\chi} \cong \mathrm{End}^{\bullet}_{G^{\vee}}(\mathcal{L}_{\mathrm{all}})\text{-nil} \, .$$

1.3. **Our case.** In the case which will be treated in this article, the data are the following:

- The algebraic group G is SO_m with $m \geq 3$.

- The inner class F of real forms is the one containing $\mathrm{SO}_{m-1,1}(\mathbb{R})$.
- The central character χ is the trivial one (i.e., the one corresponding to the trivial representation of $\mathfrak{g}$).

In [9], only the category $\mathcal{M}(\mathrm{SO}_{m-1,1}(\mathbb{R}))_\chi$ is examined and not the complete representation theoretic category corresponding to this situation (which also includes the representations of the other real forms of G). Accordingly, we will only examine a part of the geometric category: $\mathcal{L}_{\mathrm{use}}$ will be the direct sum of some of the simple equivariant perverse sheaves in $\mathrm{PSh}_{G^\vee}(X)$ (the "useful" ones; in the whole article, we will always call those objects "useful" that correspond to the right part of the geometric category).

The goal of this article is to define $\mathcal{L}_{\mathrm{use}}$ and then to prove:

Theorem 1.4. (1) $\mathrm{End}^\bullet_{G^\vee}(\mathcal{L}_{\mathrm{use}})$ *is a direct summand of* $\mathrm{End}^\bullet_{G^\vee}(\mathcal{L}_{\mathrm{all}})$.
(2) *There is an equivalence of categories,*

$$\mathcal{M}(\mathrm{SO}_{m-1,1}(\mathbb{R}))_\chi \cong \mathrm{End}^\bullet_{G^\vee}(\mathcal{L}_{\mathrm{use}})\text{-nil}\,.$$

Part (1) will be proven in Chapter 4. In fact, we will prove more generally that blocks of representations correspond to summands of $\mathrm{End}^\bullet_{G^\vee}(\mathcal{L}_{\mathrm{all}})$ (Corollary 4.8) and we will check that our choosen summands of $\mathcal{L}_{\mathrm{use}}$ indeed consists of entire blocks.

Part (2) follows from Theorem 1.6, the proof of which will last until Chapter 6.

Remark. In the case of SO_m with m even, there are two strong real forms whose weak real forms are isomorphic to $\mathrm{SO}_{m-1,1}(\mathbb{R})$. Indeed, we will see in the proof that in this case, we get two copies of $\mathcal{L}_{\mathrm{use}}$.

1.4. **The Harish-Chandra modules for** $\mathrm{SO}_{m-1,1}(\mathbb{R})$**.** In this section we present the results of [9] about the category $\mathcal{M}(\mathrm{SO}_{m-1,1}(\mathbb{R}))_\chi$ (where χ is the trivial central character). This category is equivalent to the category A_m-nil of nil-modules of some algebra A_m. The definition of A_m depends on the parity of m:

Case $m = 2n$ ("B_{2n+1}" in [9]): A_{2n} is the direct sum of two copies of the graded algebra given by the following quiver and relations:

$$(1)\qquad \begin{tikzcd} \bullet \arrow[r, bend left, "a_1"] & \bullet \arrow[l, bend left, "b_1"] & \cdots & \bullet \arrow[r, bend left, "a_{n-1}"] & \bullet \arrow[l, bend left, "b_{n-1}"] \arrow[loop right, "c"] \end{tikzcd} \qquad \left|\; \begin{array}{c} a_{\mu+1}a_\mu = b_\mu b_{\mu+1} = 0 \\ ca_{n-1} = b_{n-1}c = 0 \end{array}\right.$$

All arrows have degree 1.

Case $m = 2n+1$ ("B_{2n}" in [9]): A_{2n+1} is the algebra corresponding to the following quiver:

$$(2)\qquad \begin{tikzcd} \bullet \arrow[r, bend left, "a_1"] & \bullet \arrow[l, bend left, "b_1"] & \cdots & \bullet \arrow[r, bend left, "a_{n-1}"] & \bullet \arrow[l, bend left, "b_{n-1}"] \arrow[dr, bend left, "a_n"] & \\ & & & & & \bullet \arrow[ul, bend left, "b_n"] \arrow[dl, bend left, "b_{-n}"] \\ \bullet \arrow[r, bend left, "a_{-1}"] & \bullet \arrow[l, bend left, "b_{-1}"] & \cdots & \bullet \arrow[r, bend left, "a_{-(n-1)}"] & \bullet \arrow[l, bend left, "b_{-(n-1)}"] \arrow[ur, bend left, "a_{-n}"] & \end{tikzcd} \qquad \left|\; \begin{array}{r} a_{\pm(\mu+1)}a_{\pm\mu} = 0 \\ b_{\pm\mu}b_{\pm(\mu+1)} = 0 \\ b_{\pm n}a_{\mp n} = 0 \end{array}\right.$$

Again, all arrows have degree 1. In fact, there is nothing special about the right dot of this quiver; one could have drawn the quiver in a straight line. However, we will see in the construction of the geometric category that in some sense, this representation is natural: The dots will correspond to simple perverse sheaves and the x-coordinate of a dot will indicate the dimension of the support of that sheaf.

Now we can formulate the main result of [9]:

Proposition 1.5 ([9], Theorem 1). *Let A_m be the graded algebra just defined. Then the category $\mathcal{M}(\mathrm{SO}_{m-1,1}(\mathbb{R}))_\chi$ is equivalent to the category A_m-nil.*

With this proposition, assertion (2) of Theorem 1.4 follows from the next theorem:

Theorem 1.6. *For our data (G, F, χ) and our $\mathcal{L}_{\mathrm{use}}$, we have an isomorphism of graded algebras:*

$$\mathrm{End}^{\bullet}_{G^\vee}(\mathcal{L}_{\mathrm{use}}) \cong A_m \,.$$

1.5. **Summary of the construction of the geometric category.** As before, let $G := \mathrm{SO}_m$, let F be the inner class of real forms containing $\mathrm{SO}_{m-1,1}(\mathbb{R})$, and χ the trivial central character.

Notation. In the entire article, n will be the rank of G, that is, $n = \lfloor \frac{m}{2} \rfloor$.

In Chapter 2, using the group G, the inner class of real forms F, and the trivial central character χ, we will construct the variety X with the action of $G^\vee$ according to [1]. This variety will consist of several components. We will see that the equivariant derived category on each component is equivalent to the equivariant derived category on $K \times B^\vee \multimap\!\!\!\rightarrow G^\vee$ for a Borel subgroup $B^\vee$ of $G^\vee$ and some other subgroup K of $G^\vee$. (Here K depends on the component.)

The useful perverse sheaves $\mathcal{L}_{\mathrm{use}}$ all live on one single component of X if m is odd and on two isomorphic components if m is even. We will choose the right component(s) at the end of Chapter 2.

Simple equivariant perverse sheaves are intermediate extensions of shifted equivariant local systems on orbits. We will call the local systems corresponding to $\mathcal{L}_{\mathrm{use}}$ "useful local systems" and the underlying orbits "useful orbits". First, we have to determine these useful orbits. This will be done in Chapter 3, using the methods of [11].

After that, we will examine the equivariant local systems on the useful orbits—still using [11]—and choose the useful ones. This is the last choice we have to make to define $\mathcal{L}_{\mathrm{use}}$.

What is left is to determine the endomorphisms of $\mathcal{L}_{\mathrm{use}}$; this is the most difficult part. To achieve this, we will first use the methods of [10] to determine "where the distinct summands of $\mathcal{L}_{\mathrm{use}}$ live". These methods involve computations in a module M over the Hecke algebra $\mathcal{H}$ corresponding to $G^\vee$. This module M has the equivariant local systems as a basis, and we will be interested in the submodule $M_{\mathrm{use}} \subset M$ generated by the useful equivariant local systems. To check that M_{use} really is a submodule of M and to determine the action of $\mathcal{H}$ on M_{use}, we will still use the methods of [11].

The results about where the summands of $\mathcal{L}_{\mathrm{use}}$ live will imply that each endomorphism of $\mathcal{L}_{\mathrm{use}}$ in some sense "lives" only on a part of $K \times B^\vee \multimap\!\!\!\rightarrow G^\vee$ which looks quite similar to an analogous situation for PGL_2. In this small situation, one can describe the equivariant derived category very explicitly. So we can determine endomorphisms there and then lift them to our $K \times B^\vee \multimap\!\!\!\rightarrow G^\vee$. This will be done in Chapter 5. In Chapter 6 we will then put everything together.

Remark. If m is even, we have to find four copies of the block depicted in (1); two for each of the two strong real forms inducing $\mathrm{SO}_{m-1,1}(\mathbb{R})$. There will be two of them on each of the two isomorphic components of X. To be precise (according to the classification of the irreducible representations by equivariant local systems), each of the two real forms has one block on each of the two components of X.

2. Construction of X according to [1]

Let G, χ and F be the same as in the previous chapter. In this chapter, we will determine the geometric parameter space X. More details on the definition can be found in [1]. Afterwards, we will choose the "useful" component of X, that is the one on which we will find the useful equivariant perverse sheaves.

2.1. **Data and notation for SO_{2n}, SO_{2n+1}, and SP_{2n}.** We will use the following notation:

Notation. Let G be one of the above groups and let T and B be a maximal torus, resp. a Borel subgroup of G. We will use the following notation:

$X^\bullet(T)$, $X_\bullet(T)$: Root lattice, co-root lattice,
$\{\varepsilon_1, \dots, \varepsilon_n\}$, $\{e_1, \dots, e_n\}$: "Bourbaki basis" of $X^\bullet(T)$, resp. $X_\bullet(T)$,
$\Pi = \{\alpha_1, \dots, \alpha_{n-1}, \alpha_\star\}$: Simple roots: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_\star = \varepsilon_n$, resp. $2\varepsilon_n$, resp. $\varepsilon_{n-1} + \varepsilon_n$ (if $G = \mathrm{SO}_{2n+1}$, resp. SP_{2n}, resp. SO_{2n}),
W, w_0: Weyl group, longest element of W,
$S = \{s_1, \dots, s_{n-1}, s_\star\}$: Simple reflections,
U, U_α: Unipotent radical of B; Root subgroup corresponding to the root α,
$\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, $\mathfrak{n}$: Lie algebras corresponding to G, T, B, U,
$\mathfrak{g}_{\mathrm{ss}}$: Semi-simple elements in $\mathfrak{g}$.

The following notation for elements of the torus will also be useful:

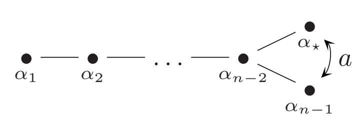
$$\lceil t_1, \dots, t_n \rfloor := e_1(t_1) \cdots e_n(t_n) \in T.$$

2.2. **The real form and the L-group.** From now on (and for the rest of this chapter) let G be SO_m with $m = 2n$ or $m = 2n+1$. The Langlands dual group of G is

$$G^\vee \cong \begin{cases} \mathrm{SO}_{2n} & \text{if } m \text{ is even,} \\ \mathrm{SP}_{2n} & \text{if } m \text{ is odd}. \end{cases}$$

Fix tori T and $T^\vee$ and Borel subgroups B and $B^\vee$ of G and $G^\vee$, respectively.

We compute the *L-group* corresponding to the inner class of $\mathrm{SO}_{m-1,1}(\mathbb{R})$: First, we need the involution a of the root datum corresponding to that inner class. a is the identity if $m \equiv 1, 2, 3 \pmod 4$. If $m \equiv 0 \pmod 4$, the two "branches" of the Dynkin diagram are exchanged:



The L-group is a semidirect product $G^\vee \rtimes \Gamma$. (Remember that $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$.) The action of $\gamma \in \Gamma$ on $G^\vee$ fixes $B^\vee$ and $T^\vee$ and acts on the root datum in the same way as a does. So we finally get

$$ {}^{\Gamma}G^{\vee} \cong \begin{cases} G^{\vee} \times \mathbb{Z}/2\mathbb{Z} & \text{if } m \equiv 1,2,3 \pmod 4, \\ \mathrm{O}_m & \text{if } m \equiv 0 \pmod 4\,. \end{cases} $$

2.3. **The variety X for the trivial character χ.** For regular integral characters, the definition of the variety X (on which the geometric category lives) is much simpler than in the general case. We follow the definition of [1] anyway:

First, one gets a $G^{\vee}$-orbit $\mathcal{O}(\chi)$ of semisimple elements $\mathfrak{g}^{\vee}$ from the central character χ by using the Harish-Chandra homomorphism, the identification $\mathfrak{h}^* \cong \mathfrak{h}^{\vee}$, and the Chevalley theorem:

$$ \begin{array}{ccccccc} \operatorname{Max} \mathrm{Z}(U(\mathfrak{g})) & \cong & W\backslash\mathfrak{h}^* & \cong & W\backslash\mathfrak{h}^{\vee} & \cong & G^{\vee}\backslash\mathfrak{g}^{\vee}_{\mathrm{ss}} \\ \chi \text{ (trivial)} & \mapsto & W\rho & \mapsto & W\rho & \mapsto & G^{\vee}\rho =: \mathcal{O}(\chi)\,. \end{array} $$

Let $\lambda := \rho$ be a representative of $\mathcal{O}(\chi)$. To define X, we first need several other objects:

$$ \begin{aligned} \mathfrak{g}^{\vee}(\lambda)_n &:= \{\mu \in \mathfrak{g}^{\vee} \mid [\lambda,\mu] = n\mu\}\,, \\ \mathfrak{n}^{\vee}(\lambda) &:= \sum_{n\in\mathbb{N}_{>0}} \mathfrak{g}^{\vee}(\lambda)_n, \\ U^{\vee}(\lambda) &:= \begin{array}{l}\text{Connected unipotent subgroup of} \\ G \text{ with Lie algebra } \mathfrak{n}^{\vee}(\lambda),\end{array} \\ L^{\vee}(\lambda) &:= \mathrm{Z}_{G^{\vee}}(\lambda), \\ P^{\vee}(\lambda) &:= L^{\vee}(\lambda)U^{\vee}(\lambda). \end{aligned} $$

One easily checks that in our case (that is, for $\lambda = \rho$) $U^{\vee}(\lambda) = U^{\vee}$ and $L^{\vee}(\lambda) = T^{\vee}$, and thus $P^{\vee}(\lambda) = B^{\vee}$.

Next, we need

$$ \begin{aligned} e(\lambda) &:= e^{2\pi i\lambda} \in G^{\vee}, \\ G^{\vee}(\lambda) &:= \mathrm{Z}_{G^{\vee}}(e(\lambda)). \end{aligned} $$

One checks that in our case

$$ e(\lambda) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ -1 & \text{if } m \text{ is odd}\,, \end{cases} $$

so $G^{\vee}(\lambda) = G^{\vee}$.

The last thing we need for the definition of X is a set of representatives of the orbits of

$$ \begin{aligned} G^{\vee} \looparrowright \mathcal{I} &:= \left\{y \in {}^{\Gamma}G^{\vee} \smallsetminus G^{\vee} \mid y^2 \in e(\mathcal{O}(\chi))\right\}, \\ &\overset{\text{in our case}}{=} \left\{y \in {}^{\Gamma}G^{\vee} \smallsetminus G^{\vee} \mid y^2 = e(\lambda)\right\}\,. \end{aligned} $$

(The action of $G^{\vee}$ is by conjugation.) There are finitely many such orbits. Fix a set Y of representatives of these orbits. X can now be defined to consist of one component for each orbit in $\mathcal{I}$, namely:

$$ \begin{aligned} X_y &:= G^{\vee} \times_{\mathrm{Z}_{G^{\vee}}(y)} G^{\vee}(\lambda)/P^{\vee}(\lambda) \quad \text{for } y \in Y, \\ X &:= \dot{\bigcup_{y\in Y}} X_y\,. \end{aligned} $$

In our case, we get

$$X_y := G^\vee \times_{Z_{G^\vee}(y)} G^\vee/B^\vee .$$

The action of $G^\vee$ on X is by multiplication from the left.

The equivariant derived category on a fixed component X_y of X is equivalent to the one on

$$Z_{G^\vee}(y) \circlearrowright G^\vee/B^\vee , \tag{3}$$

and also to the one on

$$Z_{G^\vee}(y) \times B^\vee \circlearrowright G^\vee . \tag{4}$$

Most of the time, it will be more convenient to work with (4).

2.4. **The useful component(s) X_y of X.** We are looking for the part of the geometric category which corresponds to the category of Harish-Chandra modules for $\mathrm{SO}_{m-1,1}(\mathbb{R})$. This part lives on one or two components X_y, where y is such that the (holomorphic) involution $\operatorname{Int} y \in \operatorname{Aut}(G^\vee)$ is quasi-split in the sense of [14]. We repeat the definition of quasi-split, together with a lemma needed for the definition:

Lemma 2.1. *Suppose G is an algebraic group and θ a (holomorphic) involution of G. Then there exists a Borel subgroup B and a torus T such that $\theta(B) = B$ and $\theta(T) = T$.*

Definition 2.2. Suppose G, θ, B, and T are as in Lemma 2.1. θ is called *quasi-split* if there exists an $x \in G$ such that $x^{-1}\theta(x)$ lies in $\mathrm{N}(T)$ and is a representative of the longest element w_0 of the Weyl group W (with respect to B and T).

The statement that $\operatorname{Int} y$ is a quasi-split involution can be reformulated as follows: There exists a representative $\dot{w}_0 \in \mathrm{N}(T)$ of the longest element $w_0 \in W$ such that y and $\dot{w}_0 y$ are conjugate. (If $x^{-1}yx = \dot{w}_0 y$, then $x^{-1}(\operatorname{Int} y)(x) = \dot{w}_0$.)

Using this formulation, one can easily find a $y \in {}^\Gamma G^\vee \setminus G^\vee$ with this property and satisfying $y^2 = e(\lambda)$: Just represent $G^\vee$ by matrices and find explicit matrices for $\dot{w}_0$ and y. (This is carried out in detail in [7].) This yields one desired component X_y of X. If m is even, we also have the second component we need: X_{-y}. It is isomorphic to X_y, and one checks that y and $-y$ are not conjugate and so indeed yield two different components of X.

In principle it doesn't matter which representant y of an orbit of $G^\vee \circlearrowright \mathcal{I}$ we choose. However, for simplicity we choose y such that the Borel subgroup and the torus which are fixed by $\operatorname{Int} y$ (by Lemma 2.1) are our $B^\vee$ and $T^\vee$. This can be achieved by conjugating y with a suitable element of $G^\vee$.

3. Construction of the Hecke algebra module M_{use}

3.1. **Goal of the chapter.** From this chapter on, we will only work on the side of the dual group, so we will omit all "$^\vee$" from the notation.

The current situation is the following: We have a group G (which is isomorphic to either SO_{2n} or SP_{2n}) together with a torus T and a Borel subgroup B, an extension ${}^\Gamma G$ of order two of G, and an element $y \in {}^\Gamma G \setminus G$ of order two which stabilizes T and B. We are interested in the equivariant sheaves on

$$K := Z_G(y) \circlearrowright G/B ,$$

or, equivalently, on

$$K \times B \looparrowright G\,.$$

As $y^2 = 1$, ${}^{\Gamma}G$ can be written as semidirect product $G \rtimes \{1, y\}$. From now on, let Γ be the subgroup $\{1, y\}$ of ${}^{\Gamma}G$ (in contrast to $\Gamma = \{1, \gamma\}$ in the definition of the L-group.)

As mentioned in the summary, we want to use the methods of [10] to get some information about our useful equivariant perverse sheaves. The main ingredient to these methods is a module M over the Hecke algebra $\mathcal{H}$ corresponding to G. M is a kind of Grothendieck group for the equivariant derived category on $K \times B \looparrowright G$. A $\mathbb{Z}[q, q^{-1}]$-basis of M is the set

$$\mathcal{D} = \left\{ \delta_{\mathcal{O},\gamma} \,\middle|\, \begin{array}{l} \mathcal{O} \text{ is an orbit of } K \times B \looparrowright G, \\ \gamma \text{ is an equivariant local system on } \mathcal{O}. \end{array} \right\}$$

The first goal of this chapter is to define the useful subset $\mathcal{D}_{\mathrm{use}} \subset \mathcal{D}$. The second goal is to check that $\mathcal{D}_{\mathrm{use}}$ consists of one entire block (resp. two in the case of SO_{2n}) and to describe the action of $\mathcal{H}$ on $M_{\mathrm{use}} := \langle \mathcal{D}_{\mathrm{use}} \rangle_{\mathbb{Z}[q,q^{-1}]}$ explicitly. (As $\mathcal{D}_{\mathrm{use}}$ consists of entire blocks, M_{use} is a submodule of M.)

This will be done using the methods of [11]. The main idea there is to define a map ϕ from the orbits of $K \looparrowright G/B$ to the set of involutions in the "extended Weyl group" ${}^{\Gamma}W$. This map enables us to get information about orbits by doing combinatorial computations in ${}^{\Gamma}W$.

We will explicitly specify a set of "useful involutions" in section 3.5. Then, in sections 3.6 and 3.7, we will define the useful orbits as appropriate preimages under ϕ. After that, we will analyze the equivariant local systems on the useful orbits and choose some of them (in sections 3.11 and 3.12) to be the useful ones. Finally, we will determine the action of the Hecke algebra $\mathcal{H}$.

3.2. **Involutions in ${}^{\Gamma}W \smallsetminus W$.** To analyze the orbits of $K \times B \looparrowright G$, Richardson and Springer ([11]) define a set of "twisted" involutions in the Weyl group W and a map ϕ from the orbits to these twisted involutions. Instead of using twisted involutions in W, we will use normal involutions in "${}^{\Gamma}W \smallsetminus W$"; this doesn't really change anything, but it fits better into the general context and simplifies some computations.

Definition 3.1. (1) Define the *extended normalizer* ${}^{\Gamma}\mathrm{N}(T)$ of T to be its normalizer $\mathrm{N}_{{}^{\Gamma}G}(T)$ in ${}^{\Gamma}G$ and the *extended Weyl group* ${}^{\Gamma}W$ to be the quotient ${}^{\Gamma}\mathrm{N}(T)/T$.
(2) Let $\mathcal{I} := \left\{ a \in {}^{\Gamma}W \smallsetminus W \mid a^2 = 1 \right\}$ be the set of involutions in ${}^{\Gamma}W \smallsetminus W$.

Remark. As y stabilizes the torus, it is an element of ${}^{\Gamma}\mathrm{N}(T)$, so we get

$$\begin{aligned} {}^{\Gamma}\mathrm{N}(T) &\cong \mathrm{N}(T) \rtimes \Gamma \quad \text{and} \\ {}^{\Gamma}W &\cong W \rtimes \Gamma\,. \end{aligned}$$

By abuse of notation we will write y for the image of y in ${}^{\Gamma}W$.

Remark. The action of W on T and on $X^{\bullet}(T)$ naturally extends to an action of ${}^{\Gamma}W$. In the cases where ${}^{\Gamma}G \cong G \times \mathbb{Z}/2\mathbb{Z}$, y acts trivially; in the case ${}^{\Gamma}G \cong \mathrm{O}_{2n}$ (n even), y maps ε_n to $-\varepsilon_n$.

We will need a kind of "Bruhat order" on $\mathcal{I}$.

Definition 3.2. For two elements of $\mathcal{I}$ written in the form wy and $w'y$ for $w, w' \in W$, define

$$wy < w'y :\Leftrightarrow w < w',$$

where the "<" on the right-hand side is the usual Bruhat order on W.

Remark. ${}^{\Gamma}W$ is a Coxeter group. However, our order on ${}^{\Gamma}W \smallsetminus W$ is *not* (the restriction of) a Bruhat order in this sense.

3.3. **The map ϕ from the orbits to the involutions.** To define the map $\phi\colon K\backslash G/B \to \mathcal{I}$, we first need another map κ:

Definition 3.3.
1. Define $\kappa\colon G \to {}^{\Gamma}G \smallsetminus G, x \mapsto x^{-1}yx$.
2. Suppose $\mathcal{O} \in K\backslash G/B$ and $x \in \mathcal{O}$ is a representative with $\kappa(x) \in {}^{\Gamma}\mathrm{N}(T)$. Then, define $\phi(\mathcal{O}) := \kappa(x)T \in {}^{\Gamma}W \smallsetminus W$ to be the element represented by $\kappa(x)$.

Remark. It is shown in [11] that this defines a map $\phi\colon K\backslash G/B \to \mathcal{I}$; indeed, each orbit $\mathcal{O}$ has representatives x with $\kappa(x) \in {}^{\Gamma}\mathrm{N}(T)$, all such representatives of one orbit yield the same element of ${}^{\Gamma}W \smallsetminus W$, and these elements are involutions.

Later, we will need the following lemma:

Lemma 3.4. *The preimage of $w_0y \in \mathcal{I}$ under ϕ consists exactly of the open orbit $\mathcal{O}_{\max}$ in G: $\phi^{-1}(w_0y) = \{\mathcal{O}_{\max}\}$.*

Proof. By results of [11], the preimage of $\phi(\mathcal{O}_{\max})$ is only $\mathcal{O}_{\max}$ itself and the image of ϕ is $\{a \in \mathcal{I} \mid a \le \phi(\mathcal{O}_{\max})\}$. (In particular, $\phi(\mathcal{O}_{\max})$ is the longest element in the image.) So all we have to check is that w_0y lies in the image of ϕ.

This follows from our choice of y (see section 2.4): $\mathrm{Int}\, y$ is quasi-split, that is, there is an $x \in G$ with $x^{-1}yxy^{-1}T = w_0$. For $\mathcal{O} := KxB$ this means that $\phi(\mathcal{O}) = \kappa(x)T = w_0y$. □

3.4. **Neighborhoods of orbits.** Now we take a closer look at the geometry of the orbits. A common way of doing this is the following: Fix a simple reflection $s \in S$ and let $P_s := B \cup BsB$ be the corresponding minimal parabolic subgroup. We will look at sets of the form $\mathcal{O}P_s$ for an orbit $\mathcal{O} \in K\backslash G/B$. The geometry of those sets is well known (see e. g. [11], §2, [10], and [15]) and it is using this geometry that the action of the Hecke algebra $\mathcal{H}$ on the module M is defined. In this section, we will describe the results of Richardson and Springer on how to extract information about this geometry from the involutions.

We start with a definition to simplify the language:

Definition 3.5. Suppose $s \in S$ is a simple reflection.
1. For an orbit $\mathcal{O} \in K\backslash G/B$, call $\mathcal{O}P_s \in K\backslash G/P_s$ the *s-neighborhood* of $\mathcal{O}$.
2. Two orbits $\mathcal{O}$ and $\mathcal{O}'$ are called *s-neighbors* if $\mathcal{O}P_s = \mathcal{O}'P_s$.
3. The open orbit in an s-neighborhood is called *s-large* (or simply *large* if there is no risk of confusion), the other ones are called *(s-)small*.

Now fix a simple reflection $s \in S$ and a neighborhood $\mathcal{N} \in K\backslash G/P_s$. Let U_s be unipotent radical of P_s and $T_s = \ker\alpha \subset T$ the kernel of the simple root α corresponding to s.

TABLE 1. The possible cases for $K_\mathcal{N}$

$K_\mathcal{N}$	$T_\mathcal{N}$			$\mathrm{N}(T_\mathcal{N})$		$B_\mathcal{N}$		$G_\mathcal{N}$
Orbits of $K_\mathcal{N} \circlearrowright \mathbb{P}^1\mathbb{C}$	$\{0\}$	$\mathbb{C}^\times$	$\{\infty\}$	$\{0,\infty\}$	$\mathbb{C}^\times$	$\{\infty\}$	$\mathbb{C}$	$\mathbb{P}^1\mathbb{C}$
$sa \gtrless a$	$>$	$<$	$>$	$>$	$<$	$>$	$<$	$>$
$sa \overset{=}{\neq} as$	$sa = as$			$sa = as$		$sa \neq as$		$sa = as$

The following groups can be defined:

$$\begin{aligned} G_\mathcal{N} &:= P_s/U_sT_s \cong \mathrm{PGL}_2, \\ B_\mathcal{N} &:= B/U_sT_s \subset G_\mathcal{N}, \\ T_\mathcal{N} &:= T/T_s \subset B_\mathcal{N}\,. \end{aligned}$$

In fact, these groups only depend on s and not on $\mathcal{N}$, but this notation will be more consistent with what follows.

Now choose a representative $x \in \mathcal{N}$ and define

$$K_x := x^{-1}Kx\,.$$

To $\mathcal{N}$ we associate the group

$$K_\mathcal{N} := (K_x \cap P_s)/(K_x \cap U_sT_s) \cong (K_xU_sT_s \cap P_s)/U_sT_s \subset G_\mathcal{N}\,.$$

$K_\mathcal{N}$ does depend on the choice of x, but it is well defined up to conjugacy in $G_\mathcal{N}$. We won't need more.

The main point about the geometry on neighborhoods is that there is a one-to-one correspondence between the orbits of $K \times B \circlearrowright \mathcal{N}$ and the orbits of $K_\mathcal{N} \times B_\mathcal{N} \circlearrowright G_\mathcal{N}$ which respects geometry (orbit closure). In Chapter 5 we will see that in certain cases even the equivariant derived categories on $K \times B \circlearrowright \mathcal{N}$ and on $K_\mathcal{N} \times B_\mathcal{N} \circlearrowright G_\mathcal{N}$ are quite similar.

There are only four different possibilities for the geometry on $K_\mathcal{N} \times B_\mathcal{N} \circlearrowright G_\mathcal{N}$ (and thus on $\mathcal{N}$). In each case, there is one large (open) orbit and up to two small ones. We list the orbits in each case in Table 1, together with some extra information. (To shorten the notation, $K_\mathcal{N} \times B_\mathcal{N} \circlearrowright G_\mathcal{N}$ has been written as $K_\mathcal{N} \circlearrowright G_\mathcal{N}/B_\mathcal{N} \cong \mathbb{P}^1\mathbb{C}$.)

For a given orbit $\mathcal{O}$ and a given simple reflection $s \in S$, we are interested in two things: Which of the four cases do we have for the s-neighborhood $\mathcal{N}$ of $\mathcal{O}$ and is $\mathcal{O}$ s-small or s-large? By results of [11], these questions can almost be answered only by examining the involution $a := \phi(\mathcal{O}) \in \mathcal{I}$ corresponding to our orbit $\mathcal{O}$: $a > sa$ if and only if $\mathcal{O}$ is s-large and $K_\mathcal{N} \ncong G_\mathcal{N}$. Furthermore, $sa \neq as$ if and only if $K_\mathcal{N} \cong B_\mathcal{N}$.

The next thing we want to know is: If $\mathcal{O}$ and $\mathcal{O}'$ are s-neighbors, then what is the relation between $\phi(\mathcal{O})$ and $\phi(\mathcal{O}')$?

Proposition 3.6 (compare [11], Prop. 3.3.3)**.** *Suppose $s \in S$ is a simple reflection, $\mathcal{N} \in K\backslash G/P_s$ is an s-neighborhood, and $\mathcal{O}$ and $\mathcal{O}'$ are a small and a large orbit in that neighborhood. (In particular, $K_\mathcal{N} \ncong G_\mathcal{N}$.) Then $\phi(\mathcal{O}) < \phi(\mathcal{O}')$ and we have:*

(1) *If $K_\mathcal{N} \cong B_\mathcal{N}$, then $\phi(\mathcal{O}') = s\phi(\mathcal{O})s$. (The map $\mathcal{O} \mapsto \mathcal{O}'$ is the cross-action of $s \in W$ on the set of orbits.)*

(2) *If $K_{\mathcal{N}} \cong T_{\mathcal{N}}$ or $K_{\mathcal{N}} \cong \mathrm{N}(T_{\mathcal{N}})$, then $\phi(\mathcal{O}') = s\phi(\mathcal{O})$. (The map $\mathcal{O} \mapsto \mathcal{O}'$ is the Cayley transform.)*

Finally, we define notations which will make it easy to draw neighborhood diagrams:

Notation. (1) Suppose $\mathcal{O}$ and $\mathcal{O}'$ are two orbits. $\mathcal{O} \overset{s}{\vdash\!\textcircled{T}\!\rightarrow} \mathcal{O}'$, resp. $\mathcal{O} \overset{s}{\vdash\!\textcircled{N}\!\rightarrow} \mathcal{O}'$, resp. $\mathcal{O} \overset{s}{\vdash\!\textcircled{B}\!\rightarrow} \mathcal{O}'$ means that $\mathcal{O}$ and $\mathcal{O}'$ are s-neighbors, $\mathcal{O}$ being small and $\mathcal{O}'$ being large, and that $K_{\mathcal{N}} \cong T_{\mathcal{N}}$ resp. $K_{\mathcal{N}} \cong \mathrm{N}(T_{\mathcal{N}})$ resp. $K_{\mathcal{N}} \cong B_{\mathcal{N}}$.

(2) We also define the corresponding notation on the level of the involutions. However, here we can't distinguish between the $T_{\mathcal{N}}$ and the $\mathrm{N}(T_{\mathcal{N}})$ case:

Suppose $a, a' \in \mathcal{I}$ are two involutions. $a \overset{s}{\vdash\!\textcircled{*}\!\rightarrow} a'$ means that $a' > a$ and $a' = sa = as$, and $a \overset{s}{\vdash\!\textcircled{B}\!\rightarrow} a'$ means that $a' > a$ and $a' = sas \neq a$.

3.5. **The useful involutions.** Now we are ready to list the useful involutions (the images under ϕ of what will be the useful orbits):

Definition 3.7. (1) For $1 \leq i \leq n$, let a_i be the element of $\mathcal{I}$ acting by 1 on ε_i and by -1 on ε_j for $j \neq i$.

(2) In the case $G = \mathrm{SP}_{2n}$, let additionally a_∞ be the element of $\mathcal{I}$ acting by -1 on the whole of $X^\bullet(T)$.

(3) Define the set of *useful involutions* to be $\{a_1, \ldots, a_n\}$ in the case $G = \mathrm{SO}_{2n}$ and $\{a_1, \ldots, a_n, a_\infty\}$ in the case $G = \mathrm{SP}_{2n}$.

The reader should check that there are indeed elements of $W \setminus {}^\Gamma W$ behaving like this. If $G = \mathrm{SP}_{2n}$, the Weyl group can apply any permutation and sign change to the elements ε_i, so here, this is clear. However, if $G = \mathrm{SO}_{2n}$, one has to check that the parity of the number of sign changes is the right one.

Remark. Note that we have $a_n = w_0 y$ in the case $G = \mathrm{SO}_{2n}$ and $a_\infty = w_0 y$ in the case $G = \mathrm{SP}_{2n}$.

We are especially interested in the neighborhood between useful involutions (and later between useful orbits). So let's define:

Definition 3.8. Neighborhoods consisting of two useful involutions (resp. only of useful orbits) are called *useful*. Neighborhoods consisting of one useful and one useless involution (resp. useful and useless orbits) are called *semi-useful*[1].

Figure 1 contains all useful and semi-useful neighborhoods. The vertical arrows are the useful neighborhoods and the almost horizontal arrows the semi-useful ones.

3.6. **The useful orbits in the case SO_{2n}.** Now we can define the useful orbits. In the SO_{2n}-case this is easy using the following lemma:

Lemma 3.9. *In the SO_{2n}-case, each useful involution a_i $(1 \leq i \leq n)$ has exactly one preimage under ϕ.*

Before we get to the proof, we define:

Definition 3.10. For $1 \leq i \leq n$, define $\mathcal{O}_i$ to be the preimage of a_i under ϕ. These orbits $\mathcal{O}_i$ will be called the *useful orbits*.

[1] "Annoying" might be more appropriate, as later we will have to get rid of those.

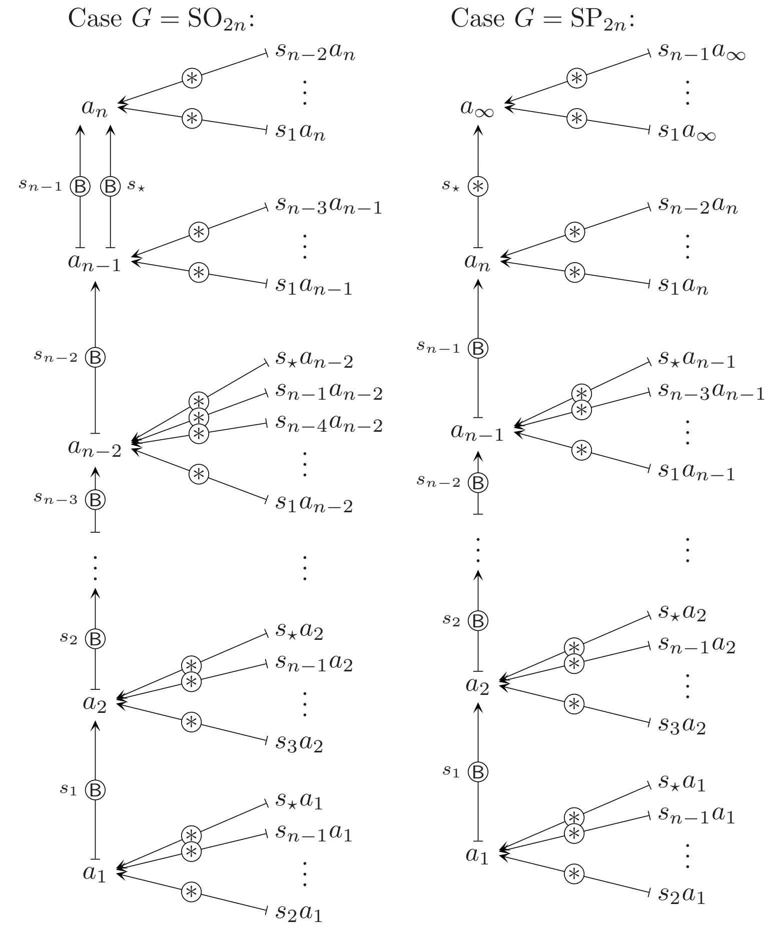


FIGURE 1. All useful and semi-useful neighbourhoods of involutions

Proof of the lemma. For $a_n = w_0y$, we know that the preimage is $\{\mathcal{O}_{\max}\}$ by Lemma 3.4.

Now suppose $1 \le i \le n-1$. The s_i-neighborhood of an orbit $\mathcal{O} \in \phi^{-1}(a_i)$ is of type $K_{\mathcal{N}} \cong B_{\mathcal{N}}$, so $\mathcal{O}$ has exactly one s_i-neighbor $\mathcal{O}'$ and we have $\phi(\mathcal{O}') = a_{i+1}$. Conversely, each orbit $\mathcal{O}' \in \phi^{-1}(a_{i+1})$ has exactly one s_i-neighbor $\mathcal{O} \in \phi^{-1}(a_i)$. So the s_i-neighborhood defines a bijection between $\phi^{-1}(a_i)$ and $\phi^{-1}(a_{i+1})$. Starting from $\phi^{-1}(a_n) = \{\mathcal{O}_{\max}\}$ and going down inductively, we find that each a_i has exactly one preimage. $\square$

In the proof, we already saw the useful neighborhoods of our orbits. Together with the information about the semi-useful neighborhoods of the involutions we get:

Proposition 3.11. *In the case* SO_{2n} *the useful neighborhoods of the orbits are*

$$\mathcal{O}_1 \xrightarrow{s_1} \mathcal{O}_2 \xrightarrow{s_2} \dots \xrightarrow{s_{n-1}} \mathcal{O}_n\,.$$

All semi-useful neighbourhoods are of the form $\mathcal{O} \xmapsto{s_j \; \text{N}} \mathcal{O}_i$ *or* $\mathcal{O} \xmapsto{s_j \; \text{T}} \mathcal{O}_i$ *.*

In other words, the useful orbits are just one orbit under the cross-action.

3.7. **The useful orbits in the case** SP_{2n}**.** In the SP_{2n}-case, there is still some information missing that we need to define the useful orbits: We know that the preimage of $a_\infty = w_0 y$ is $\{\mathcal{O}_{\max}\}$, but we don't know yet whether the $s_\star$-neighborhood of $\mathcal{O}_{\max}$ is of type $T_\mathcal{N}$ or $\mathrm{N}(T_\mathcal{N})$. This would be necessary to know the number of $s_\star$-neighbors of $\mathcal{O}_{\max}$ in $\phi^{-1}(a_n)$. (In addition, $\phi^{-1}(a_n)$ could contain even more orbits: orbits whose $s_\star$-neighborhood is of type $G_\mathcal{N}$. However, we don't care about these.)

In Lemma 3.19, we will see that the $s_\star$-neighborhood of $\mathcal{O}_{\max}$ is of type $T_\mathcal{N}$. Using this (and the same bijections between the preimages $\phi^{-1}(a_i), 1 \le i \le n$ as in the SO_{2n}-case), we can define:

Definition 3.12. (1) Let $\mathcal{O}_\infty := \mathcal{O}_{\max}$ be the only preimage of a_∞.
(2) Let $\mathcal{O}_n$ and $\mathcal{O}_{-n}$ be the two small $s_\star$-neighbors of $\mathcal{O}_\infty$.
(3) Let inductively $\mathcal{O}_i$ and $\mathcal{O}_{-i}$ be the small s_i-neighbor of $\mathcal{O}_{i+1}$ resp. $\mathcal{O}_{-i-1}$.
(4) These orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n, \mathcal{O}_{-1}, \ldots, \mathcal{O}_{-n}, \mathcal{O}_\infty$ are called the *useful orbits*.

As in the SO_{2n}-case, we now can describe the useful and semi-useful neighborhoods of our useful orbits:

Proposition 3.13. *In the case* SP_{2n} *the useful neighborhoods of the orbits are:*

$$\begin{tikzcd}
\mathcal{O}_1 \arrow[r, mapsto, "s_1"] \arrow[r, mapsto, swap, "\text{B}"] & \mathcal{O}_2 \arrow[r, mapsto, "s_2"] \arrow[r, mapsto, swap, "\text{B}"] & \ldots \arrow[r, mapsto, "s_{n-1}"] \arrow[r, mapsto, swap, "\text{B}"] & \mathcal{O}_n \arrow[dr, mapsto, "s_\star"] \arrow[dr, mapsto, swap, "\text{T}"] & \\
 & & & & \mathcal{O}_\infty . \\
\mathcal{O}_{-1} \arrow[r, mapsto, swap, "s_1"] \arrow[r, mapsto, "\text{B}"] & \mathcal{O}_{-2} \arrow[r, mapsto, swap, "s_2"] \arrow[r, mapsto, "\text{B}"] & \ldots \arrow[r, mapsto, swap, "s_{n-1}"] \arrow[r, mapsto, "\text{B}"] & \mathcal{O}_{-n} \arrow[ur, mapsto, swap, "s_\star"] \arrow[ur, mapsto, "\text{T}"] &
\end{tikzcd}$$

All semi-useful neighbourhoods are of the form $\mathcal{O} \xmapsto{s_j \; \text{N}} \mathcal{O}_i$ *or* $\mathcal{O} \xmapsto{s_j \; \text{T}} \mathcal{O}_i$ *.*

So in this case, the useful orbits consist of three orbits under the cross-action ($\{\mathcal{O}_\infty\}$ and the two "branches"), linked together by a Cayley transform.

3.8. **Equivariant local systems on** G/B**.** In this section, we will repeat how to describe the equivariant local systems on orbits using the methods of [11].

Definition 3.14. For an involution $a \in \mathcal{I}$ define:

(1) $T_a := \{t \in T \mid at = t\}$.
(2) $A_a := T_a / T_a^0$, the component group of T_a.
(3) Γ_a the set of characters of A_a.

Lemma 3.15 (cf. [11], Lemma 7.2.1)**.** *Suppose* $\mathcal{O}$ *is an orbit of* $K \times B \multimap G$ *and* $x \in \mathcal{O}$ *is a representative of* $\mathcal{O}$*. Then the isotropy group of* x *is solvable and its maximal torus is isomorphic to* $T_{\phi(\mathcal{O})}$*. In particular, there is a one-to-one correspondence between equivariant local systems on* $\mathcal{O}$ *and the characters* $\Gamma_{\phi(\mathcal{O})}$*.*

Notation. For an element $\delta_{\mathcal{O},\gamma} \in \mathcal{D}$ of the basis of the Hecke module (for the definition of $\mathcal{D}$ see section 3.1), we will also write $\delta_{\mathcal{O},\chi}$ if $\chi \in \Gamma_{\phi(\mathcal{O})}$ is the corresponding character.

To define the action of the Hecke algebra on the module M, we have to extend the definition of neighborhoods to the basis elements $\mathcal{D}$. Suppose $\bar{\gamma}$ is an equivariant local system on a whole neighborhood $\mathcal{N} \in K\backslash G/P_s$ of orbits. Then we want the restrictions of $\bar{\gamma}$ to the distinct orbits to form a neighborhood. Here is a formal definition:

Definition 3.16. Suppose s is a simple reflection, P_s is the corresponding minimal parabolic subgroup, and $\mathcal{N} \in K\backslash G/P_s$ is a neighborhood of orbits. Denote the orbits in $\mathcal{N}$ by $(\mathcal{O}_\nu)_{\nu\in I}$ and the inclusions of those orbits by $\iota_\nu\colon \mathcal{O}_\nu \hookrightarrow \mathcal{N}$. Let $\bar{\gamma}$ be an equivariant local system on $\mathcal{N}$ and for each $\nu \in I$ let $\gamma_\nu := \iota_\nu^*\bar{\gamma}$ be its restriction to $\mathcal{O}_\nu$.

(1) An *s-neighborhood* in $\mathcal{D}$ is a set of the form $\{\delta_{\mathcal{O}_\nu,\gamma_\nu} \mid \nu \in I\}$.
(2) δ is called *(s-)large* if its underlying orbit is s-large.
(3) If $\delta, \delta' \in \mathcal{D}, \delta \neq \delta'$ are s-neighbors and δ' is large, we write $\delta \overset{s}{\mapsto\!\!\text{(T)}\!\!\to} \delta'$, resp. $\delta \overset{s}{\mapsto\!\!\text{(N)}\!\!\to} \delta'$, resp. $\delta \overset{s}{\mapsto\!\!\text{(B)}\!\!\to} \delta'$ (depending on the kind of the underlying neighborhood of orbits). We will write $\delta \overset{s}{\mapsto} \delta'$ if we are not interested in the kind of neighborhood.

Remark. If these neighborhoods are represented as a graph, then the connected components correspond exactly to the blocks of representations.

[11] gives an explicit way to determine the neighborhoods between equivariant local systems when they are given as characters of the groups $A_{\phi(\mathcal{O})}$. This is the result:

Case $K_{\mathcal{N}} \cong G_{\mathcal{N}}$: Let $\mathcal{O} := \mathcal{N}$ be the only orbit in that neighborhood. Each $\delta_{\mathcal{O},\chi}$ (for $\chi \in \Gamma_{\phi(\mathcal{O})}$) is a neighborhood on its own.

Case $K_{\mathcal{N}} \cong B_{\mathcal{N}}$ (cross-action): Let $\mathcal{O}$ and $\mathcal{O}'$ be the orbits in $\mathcal{N}$, $\mathcal{O}'$ being the large one. Let $a := \phi(\mathcal{O})$ and $a' := \phi(\mathcal{O}')$ be the corresponding involutions. There is an isomorphism

$$\begin{aligned} T_a &\longrightarrow T_{a'} = T_{sas}, \\ t &\longmapsto s(t)\,, \end{aligned} \tag{5}$$

which induces an isomorphism $A_a \cong A_{a'}$ and therefore also an isomorphism of the character groups $\Gamma_{a'} \xrightarrow{\sim} \Gamma_a$. In this case, neighborhoods of equivariant local systems are sets of the form $\{\delta_{\mathcal{O},\chi}, \delta_{\mathcal{O}',\chi'}\}$, where $\chi' \in \Gamma_{a'}$ is arbitrary and χ is the image of χ' under the above isomorphism.

Cases $K_{\mathcal{N}} \cong T_{\mathcal{N}}$ and $K_{\mathcal{N}} \cong \mathrm{N}(T_{\mathcal{N}})$ (Cayley transform): Let $\mathcal{O}'$ be the large orbit in $\mathcal{N}$, and $\mathcal{O}$ resp. $\mathcal{O}_1$, $\mathcal{O}_2$ the other orbit(s). Let $a' := \phi(\mathcal{O}')$ the involution corresponding to the large orbit and a the involution corresponding to the small one(s). Let α be the simple root corresponding to the simple reflection s.

Define $T_{\text{middle}} := T_{a'} \cap \ker(\alpha)$ and $A_{\text{middle}} := T_{\text{middle}}/T^0_{\text{middle}}$. The following facts are proven in [11]:

Lemma 3.17. (1) *The induced map $\rho_1\colon A_{\text{middle}} \to A_{a'}$ is an injection of degree either 1 or 2.*
(2) *ρ_1 is an isomorphism if and only if $\alpha(T_{a'}) = 1$, if and only if $K_{\mathcal{N}} \cong T_{\mathcal{N}}$.*
(3) *T_{middle} is a subset of T_a.*
(4) *The induced map $\rho_2\colon A_{\text{middle}} \to A_a$ is a surjection of degree either 1 or 2. The kernel of ρ_2 is $\alpha^\vee(\pm 1) \cdot T^0_{\text{middle}}$.*

Each character $\chi' \in \Gamma_{a'}$ can be restricted to a character of A_{middle} using ρ_1. If χ' vanishes on $\alpha^\vee(\pm 1)$, that character factors over ρ_2 and we get a character $\chi \in \Gamma_a$. In the case $K_{\mathcal{N}} \cong \mathrm{N}(T_{\mathcal{N}})$, the neighborhoods of equivariant local systems are of the form $\{\delta_{\mathcal{O},\chi}, \delta_{\mathcal{O}',\chi'}\}$ for characters χ' that vanish on $\alpha^\vee(\pm 1)$. In the case $K_{\mathcal{N}} \cong T_{\mathcal{N}}$, they are of the form $\{\delta_{\mathcal{O}_1,\chi}, \delta_{\mathcal{O}_2,\chi}, \delta_{\mathcal{O}',\chi'}\}$.

Remark. Characters $\chi' \in \Gamma_{a'}$ which do not vanish on $\alpha^\vee(\pm 1)$ have no s-neighborhood. (They correspond to equivariant local systems which are “twisted” around the small orbit and so cannot be extended.)

Remark. In the case $K_{\mathcal{N}} \cong \mathrm{N}(T_{\mathcal{N}})$ each equivariant local system on the small orbit $\mathcal{O}$ has two s-neighborhoods (the restriction of characters $\Gamma_{a'} \to \Gamma_{\text{middle}}$ is 2-to-1); in the case $K_{\mathcal{N}} \cong T_{\mathcal{N}}$, the equivariant local systems on $\mathcal{O}_1$ and $\mathcal{O}_2$ have only one s-neighborhood.

3.9. **The action of the Hecke algebra $\mathcal{H}$.** The action of the Hecke algebra $\mathcal{H}$ on the module M can be defined either geometrically (see [10], Lemma 3.3) or in a purely combinatorial way (see e. g. [11]). Here is a definition which lies somewhere in between, using neighborhoods of equivariant local systems.

$\mathcal{H}$ is generated by the set $\{T_s \mid s \in S\}$. Instead of defining the action of these elements, we use another set of generators, namely,

$$C_s := T_s + 1\,.$$

Now here is our definition of the action of $\mathcal{H}$ on M (which is equivalent to the ones in [10] and [11]):

Definition 3.18. Suppose $s \in S$ is a simple reflection and $\delta \in \mathcal{D}$. The action of C_s on δ depends on whether δ is s-large or not.

(1) If δ is s-small: δ can lie in one or two s-neighborhoods. Let $\{\delta_{i,j} \mid 1 \le i \le l\}$ for $j = 1$ (resp. $j = 1, 2$) be this s-neighborhood (resp. those s-neighborhoods). Define

$$C_s\delta := \sum_{i,j} \delta_{i,j}\,.$$

(2) If δ is s-large: If δ has no s-neighborhood, define

$$C_s\delta := 0\,.$$

Otherwise, let $\{\delta_i \mid 1 \le i \le l\}$ be the s-neighborhood of δ. Define $C_s\delta$ so that the following formula holds:

$$\sum_i C_s\delta_i = (q+1)\sum_i \delta_i\,.$$

(This does make sense, because on the left-hand side, δ is the only large summand, and for small δ_i, $C_s\delta_i$ has already been defined in (1).)

3.10. **The characters corresponding to useful orbits.** In this section, we determine the character groups Γ_{a_i} corresponding to the useful orbits and fix some notation.

Notation. Define the following shortcuts:

$$\begin{aligned} T_i &:= T_{a_i} = \{t \in T \mid a_i(t) = t\}\,,\\ A_i &:= A_{a_i} = T_i/T_i^0,\\ \Gamma_i &:= \Gamma_{a_i} \quad \text{the set of characters of } A_i. \end{aligned}$$

For elements of the torus, we use the notation $\lceil t_1, \ldots, t_n \rfloor := e_1(t_1)\cdots e_n(t_n)$ (see section 2.1).

Using these notations, we can determine the groups T_i and A_i. In the case SO_{2n}, we get:

$$\begin{aligned} T_i &= \left\{ \lceil \pm 1, \dots, \pm 1, \underset{i}{*}, \pm 1, \dots, \pm 1 \rfloor \right\}, \\ A_i &\cong \{\pm 1\} \times \cdots \times \{\pm 1\} \times \underset{i}{\{1\}} \times \{\pm 1\} \times \cdots \times \{\pm 1\}\ . \end{aligned}$$

In the case SP_{2n}, we get the same and additionally:

$$\begin{aligned} T_\infty &= \{ \lceil \pm 1, \dots, \pm 1 \rfloor \}, \\ A_\infty &\cong \{\pm 1\} \times \cdots \times \{\pm 1\}\ . \end{aligned}$$

Here is a good opportunity to compute the neighborhood which was missing in section 3.7:

Lemma 3.19. *The $s_\star$-neighborhood of $\mathcal{O}_\infty$ is of type $T_\mathcal{N}$.*

Proof. By Lemma 3.17, we have to check $\alpha_\star(T_\infty) = \{1\}$. This is the case, because $\alpha_\star = 2\varepsilon_n$. □

Back to the characters: For $1 \le i \le n$, a character $\chi \in \Gamma_i$ is determined by the images of the elements $(1, \dots, 1, \underset{j}{-1}, 1, \dots, 1) \in A_i$ for $j \ne i$. A character of A_∞ is determined by the images of all the $(1, \dots, 1, \underset{j}{-1}, 1, \dots, 1)$.

Notation. For $1 \le i \le n$, we denote a character $\chi \in \Gamma_i$ by the tuple

$$(b_1, \dots, b_{i-1}, \bullet, b_{i+1}, \dots, b_n)$$

where $b_j = \chi(1, \dots, 1, \underset{j}{-1}, 1, \dots, 1)$ for $j \ne i$. A character $\chi \in \Gamma_\infty$ will be denoted by the full tuple $(b_1, \dots, b_n)$ of the images.

3.11. **The useful basis in the case SO_{2n}.** Now we are ready to define the useful equivariant local systems:

Definition 3.20. (1) For $1 \le i \le n$, define the following characters of Γ_i:

$$\begin{aligned} \chi_i &:= (1, -1, 1, \dots, \pm 1, \underset{i}{\bullet}, \mp 1, \dots) \in \Gamma_i, \\ \chi_{-i} &:= (-1, 1, -1, \dots, \mp 1, \underset{i}{\bullet}, \pm 1, \dots) \in \Gamma_i\ . \end{aligned}$$

Note that the signs before and after the "$\bullet$" are different.

(2) For $i \in \{1, \dots, n, -1, \dots, -n\}$ let γ_i be the equivariant local system on $\mathcal{O}_{|i|}$ corresponding to χ_i.
(3) Let $\delta_i := \delta_{\mathcal{O}_{|i|}, \gamma_i} \in \mathcal{D}$ be the corresponding element of the basis of the Hecke module.
(4) Let $\mathcal{D}_{\mathrm{use}} := \{\delta_i \mid i \in \{1, \dots, n, -1, \dots, -n\}\} \subset \mathcal{D}$ be the *useful basis*.

The choice of the χ_i may seem a bit strange at first glance, but we will see in moment that these are the ones that define the (rather small) two complete blocks of equivariant local systems that we need.

Proposition 3.21. *In the case SO_{2n}, all neighborhoods of the useful basis elements are:*

$$\begin{tikzcd} \delta_1 \arrow[r, mapsto, "s_1", "\textcircled{B}" description] & \delta_2 \arrow[r, mapsto, "s_2", "\textcircled{B}" description] & \dots \arrow[r, mapsto, "s_{n-2}", "\textcircled{B}" description] & \delta_{n-1} \arrow[r, mapsto, "{s_{n-1}, s_\star}", "\textcircled{B}" description] & \delta_n \\ \delta_{-1} \arrow[r, mapsto, "s_1"', "\textcircled{B}" description] & \delta_{-2} \arrow[r, mapsto, "s_2"', "\textcircled{B}" description] & \dots \arrow[r, mapsto, "s_{n-2}"', "\textcircled{B}" description] & \delta_{-(n-1)} \arrow[r, mapsto, "{s_{n-1}, s_\star}"', "\textcircled{B}" description] & \delta_{-n} \end{tikzcd}$$

In particular, the useful basis elements make up two complete blocks.

Proof. The s_i-neighborhood $\mathcal{O}_i, \mathcal{O}_{i+1}$ is of type $B_{\mathcal{N}}$, so the neighborhoods of equivariant local systems are given by the isomorphism (5) between the character groups Γ_i and Γ_{i+1} (see section 3.8). Applying this isomorphism yields: $\gamma \in \Gamma_i$ and $\gamma' \in \Gamma_{i+1}$ form a neighborhood if and only if they differ only in the location of the "$\bullet$". Therefore, we get the neighborhoods $\delta_{\pm i} \overset{s_i}{\mapsto \mathrm{B} \rightarrow} \delta_{\pm(i+1)}$. In the same way, we also get $\delta_{\pm(n-1)} \overset{s_\star}{\mapsto \mathrm{B} \rightarrow} \delta_{\pm n}$.

It remains to prove that there are no other neighborhoods. The semi-useful neighborhoods of the underlying orbits are of type $T_{\mathcal{N}}$ or $\mathrm{N}(T_{\mathcal{N}})$, and the useful orbit is always the large one. So by section 3.8, to show that χ_i has no s_j-neighbors we have to check that χ_i does not vanish on $\alpha^\vee(\pm 1)$, where $\alpha^\vee$ is the simple coroot corresponding to s_j. This is indeed the case because of the alternating signs of χ_i. $\square$

3.12. **The useful basis in the case SP_{2n}.** Now let's do the same for SP_{2n}.

Definition 3.22. (1) Define

$$\chi_i := \Big((-1)^{n-1}, (-1)^{n-2}, \dots, (-1)^{n-i+1}, \underset{i}{\bullet}, (-1)^{n-i}, \dots, (-1)^2, (-1)^1\Big) \in \Gamma_i$$

for $1 \le i \le n$, and

$$\chi_\infty := \big((-1)^{n-1}, (-1)^{n-2}, \dots, (-1)^1, (-1)^0\big) \in \Gamma_\infty .$$

(Here we have to be a bit more careful about the signs than in the SO_{2n}-case.)

(2) For $1 \le i \le n$ let γ_i, resp. γ_{-i}, be the equivariant local system on $\mathcal{O}_i$, resp. $\mathcal{O}_{-i}$ corresponding to χ_i. Let γ_∞ be the equivariant local system on $\mathcal{O}_\infty$ corresponding to χ_∞.
(3) Let $\delta_i := \delta_{\mathcal{O}_i, \chi_i}$, $\delta_{-i} := \delta_{\mathcal{O}_{-i}, \chi_i}$ and $\delta_\infty := \delta_{\mathcal{O}_\infty, \chi_\infty}$ be the corresponding elements of the basis of the Hecke module.
(4) Let $\mathcal{D}_{\mathrm{use}} := \{\delta_i \mid i \in \{1, \dots, n, -1, \dots, -n, \infty\}\} \subset \mathcal{D}$ be the *useful basis.*

Proposition 3.23. *In the case SP_{2n}, all neighborhoods of the useful basis elements are:*

$$\begin{array}{l} \delta_1 \overset{s_1}{\mapsto \mathrm{B} \rightarrow} \delta_2 \overset{s_2}{\mapsto \mathrm{B} \rightarrow} \dots \overset{s_{n-1}}{\mapsto \mathrm{B} \rightarrow} \delta_n \overset{s_\star}{\mapsto \mathrm{T} \searrow} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \delta_\infty . \\ \delta_{-1} \underset{s_1}{\mapsto \mathrm{B} \rightarrow} \delta_{-2} \underset{s_2}{\mapsto \mathrm{B} \rightarrow} \dots \underset{s_{n-1}}{\mapsto \mathrm{B} \rightarrow} \delta_{-n} \underset{s_\star}{\mapsto \mathrm{T} \nearrow} \end{array}$$

In particular, the useful basis elements make up one complete block.

Proof. Getting the neighborhoods $\delta_{\pm i} \overset{s_i}{\mapsto \mathrm{B} \rightarrow} \delta_{\pm(i+1)}$ $(1 \le i \le n-1)$ is analogue to the case SO_{2n}. For the $s_\star$-neighborhood of δ_∞, we use the notation of section 3.8: $T_{\mathrm{middle}} = T_\infty \cap \ker(\alpha_\star) = T_\infty$. One easily checks that applying the maps $\rho_1 \colon A_{\mathrm{middle}} \to A_\infty$ and $\rho_2 \colon A_{\mathrm{middle}} \to A_n$ to χ_∞ as in section 3.8 yields χ_n, so we get the neighborhood $\{\delta_n, \delta_{-n}, \delta_\infty\}$.

Again, we have to check that there are no semi-useful neighborhoods. For the s_j-neighborhoods with $1 \le j \le n-1$, this works exactly as in the SO_n-case. For the $s_\star$-neighborhood of δ_i with $1 \le i \le n-1$, one also checks that $\alpha_\star^\vee$ does not vanish on χ_i; this time, the reason is that the last sign in χ_i is negative. $\square$

3.13. **The action of $\mathcal{H}$ on M_{use}.** Now that we know our useful basis $\mathcal{D}_{\mathrm{use}}$ and its neighborhood relations, we can define our useful $\mathcal{H}$-submodule $M_{\mathrm{use}} \subset M$ and determine the action of $\mathcal{H}$ on M_{use}.

Definition 3.24. Let the *useful $\mathcal{H}$-module* be the $\mathbb{Z}[q,q^{-1}]$-submodule $M_{\mathrm{use}} := \langle \mathcal{D}_{\mathrm{use}} \rangle_{\mathbb{Z}[q,q^{-1}]} \subset M$ generated by the useful basis.

As the useful basis elements consist of entire blocks, M_{use} is a $\mathcal{H}$-submodule of M. To get the action of $\mathcal{H}$ on M_{use}, we now apply Definition 3.18 to our situation, using the neighborhood relations determined in the two previous sections. We use the following notation.

Notation. $C_j := C_{s_j}$ for $j \in \{1, \ldots, n-1, \star\}$.

We get in the SO_{2n}-case:

$$\begin{aligned} C_i \delta_{\pm i} &= \delta_{\pm i} + \delta_{\pm(i+1)} && \text{for } 1 \le i \le n-1, \\ C_i \delta_{\pm(i+1)} &= q(\delta_{\pm i} + \delta_{\pm(i+1)}) && \text{for } 1 \le i \le n-1, \\ C_\star \delta_{\pm(n-1)} &= \delta_{\pm(n-1)} + \delta_{\pm n}, \\ C_\star \delta_{\pm n} &= q(\delta_{\pm(n-1)} + \delta_{\pm n}) \,. \end{aligned}$$

And in the SP_{2n}-case:

$$\begin{aligned} C_i \delta_{\pm i} &= \delta_{\pm i} + \delta_{\pm(i+1)} && \text{for } 1 \le i \le n-1, \\ C_i \delta_{\pm(i+1)} &= q(\delta_{\pm i} + \delta_{\pm(i+1)}) && \text{for } 1 \le i \le n-1, \\ C_\star \delta_{\pm n} &= \delta_n + \delta_{-n} + \delta_\infty, \\ C_\star \delta_\infty &= q(\delta_n + \delta_{-n} + \delta_\infty) \,. \end{aligned}$$

In either case, all other $C_j \delta_i$ are zero.

3.14. **Length and Bruhat order.** To apply [10] to our module $\mathcal{D}_{\mathrm{use}}$, we need a length function l and a Bruhat order $\le$ on $\mathcal{D}_{\mathrm{use}}$.

Definition 3.25 (cf. [10], Def. 1.1)**.** For $\delta_{\mathcal{O},\gamma} \in \mathcal{D}$ define the *length* $l(\delta_{\mathcal{O},\gamma}) := \dim_{\mathbb{C}} \mathcal{O}$.

Set $l_0 := l(\delta_1) - 1$. The dimensions of a small and the large orbit in a neighborhood differ by one, so we get (in both, the SO_{2n} and the SP_{2n}-case): $l(\delta_{\pm i}) = l_0 + i$ for $1 \le i \le n$. In the SP_{2n}-case, we additionally have $l(\delta_\infty) = l_0 + n + 1$. That's all we need to know about the length.

Definition 3.26 ([10], Def. 1.8)**.** Let the *Bruhat order* $\le$ on $\mathcal{D}$ be the smallest order with the following properties:

(1) If $s \in S$ is a simple reflection and $\delta, \delta' \in \mathcal{D}$ are basis elements with $\delta \overset{s}{\mapsto} \delta'$, then $\delta \le \delta'$.
(2) If $s \in S$ is a simple reflection and $\delta, \delta', \gamma, \gamma' \in \mathcal{D}$ with $\delta \overset{s}{\mapsto} \delta'$, $\gamma \overset{s}{\mapsto} \gamma'$ and $\gamma \le \delta$, then $\gamma' \le \delta'$.

In the SO_{2n}-case, we get $\delta_i \le \delta_j$ and $\delta_{-i} \le \delta_{-j}$ for $1 \le i \le j \le n$. In the SP_{2n}-case, we get the same and additionally $\delta_i \le \delta_\infty$ for all i. Elements of $\mathcal{D}_{\mathrm{use}}$ and $\mathcal{D} \smallsetminus \mathcal{D}_{\mathrm{use}}$ are incomparable.

4. Construction of the intermediate extensions

Let the groups G, B, and K, the Hecke module $M = \langle \mathcal{D} \rangle$, and its submodule $M_{\text{use}} = \langle \mathcal{D}_{\text{use}} \rangle$ be as in the previous chapter. We start by defining the useful equivariant simple perverse sheaves. They are (up to a shift) the intermediate extensions of the equivariant local systems in $\mathcal{D}_{\text{use}}$:

Definition 4.1. (1) Suppose $\delta = \delta_{\mathcal{O},\gamma} \in \mathcal{D}$, and $\iota\colon \mathcal{O} \hookrightarrow G$ is the inclusion of the orbit. Define $\mathcal{L}_\delta := \iota_{!*}\gamma[\dim_{\mathbb{C}} \mathcal{O}]$ to be the simple perverse sheaf corresponding to γ.

(2) For $\delta_k \in \mathcal{D}_{\text{use}}$ ($k = \pm 1, \dots, \pm n$, resp. $k = \pm 1, \dots, \pm n, \infty$), we call $\mathcal{L}_k := \mathcal{L}_{\delta_k}$ a *useful* perverse sheaf.

(3) Define $\mathcal{L}_{\text{use}} := \bigoplus_{\delta \in \mathcal{D}_{\text{use}}} \mathcal{L}_\delta$ to be the direct sum of all useful perverse sheaves.

(4) We will call $\mathcal{L}_\delta$ and $\mathcal{L}_{\delta'}$ *s-neighbors* (for a simple reflection $s \in S$) if δ and δ' are s-neighbors.

We know the action of the Hecke algebra $\mathcal{H}$ on M_{use}. In this chapter, we will use the methods of [10] to determine the elements of M_{use} corresponding to the useful perverse sheaves. In section 4.3, we will then draw conclusions.

4.1. The dualizing map D.

Proposition 4.2 ([10], Theorem 1.10). *There is exactly one $\mathbb{Z}$-linear map $D\colon M \to M$ with the following properties:*

(1) *D is compatible to the usual duality on $\mathcal{H}$, i.e.,*

$$D(qm) = q^{-1}D(m) \qquad \text{for } m \in M \qquad \text{and} \tag{6}$$

$$D(C_s m) = q^{-1}C_s D(m) \qquad \text{for } m \in M, s \in S\,. \tag{7}$$

(2) *There are polynomials $R_{\delta',\delta} \in \mathbb{Z}[q]$ ($\delta', \delta \in \mathcal{D}$) such that*

$$D(\delta) = q^{-l(\delta)}(\delta + \sum_{\delta' < \delta} R_{\delta',\delta}\delta')\,. \tag{8}$$

The degree of the polynomial $R_{\delta',\delta}$ is at most $l(\delta) - l(\delta')$.

We determine the map D on our submodule $M_{\text{use}} \subset M$, starting with the case SO_{2n}. Recall that our $\mathbb{Z}[q, q^{-1}]$-basis of M_{use} is $\mathcal{D}_{\text{use}} = \{\delta_{\pm 1}, \dots, \delta_{\pm n}\}$ and that the length $l(\delta_{\pm k})$ is $l_0 + k$ for a fixed integer l_0.

Equation (8) applied to δ_1 yields

$$D(\delta_1) = q^{-l_0-1}\delta_1\,. \tag{9}$$

Now suppose $1 \le k \le n-1$. Then we get

$$\begin{aligned} D(\delta_k + \delta_{k+1}) &= D(C_k\delta_k) \\ &\overset{(7)}{=} q^{-1}C_k D(\delta_k) \\ &\overset{(8)}{=} q^{-1}C_k\left(q^{-l_0-k}(\delta_k + \sum_{j=1}^{k-1} R_{j,k}\delta_j)\right)\,. \end{aligned}$$

As $C_k\delta_j = 0$ for $j < k$, the whole sum disappears, and we get

$$D(\delta_k + \delta_{k+1}) = q^{-l_0-k-1}C_k\delta_k = q^{-l_0-k-1}(\delta_k + \delta_{k+1})\,. \tag{10}$$

The same computation also works for δ_{-1} and $\delta_{-k} + \delta_{-(k+1)}$.

In the case SP_{2n}, we get the same results, and additionally (with an analogous computation),

$$D(\delta_n + \delta_{-n} + \delta_\infty) = q^{-l_0-n-1}(\delta_n + \delta_{-n} + \delta_\infty) \tag{11}$$

Using these formulas, we could determine D inductively. However, the result is a bit ugly and we don't need it anyway.

4.2. The new basis $(\lambda_\delta)_{\delta\in\mathcal{D}_{\mathrm{use}}}$.

Proposition 4.3 ([10], Theorem 1.11)**.** *For each $\delta \in \mathcal{D}$ there exists exactly one*

$$\lambda_\delta = \sum_{\delta'\leq\delta} P_{\delta',\delta}\delta' \in M\,, \qquad P_{\delta',\delta} \in \mathbb{Z}[q]\,,$$

with the following properties:

1. $D(\lambda_\delta) = q^{-l(\delta)}\lambda_\delta$.
2. $P_{\delta,\delta} = 1$.
3. *If $\delta' \neq \delta$, $P_{\delta',\delta}$ is a polynomial of degree at most $\frac{1}{2}(l(\delta) - l(\delta') - 1)$.*

We will denote the useful λ_{δ_k} by λ_k.

Equations (9), (10), and (11) provide exactly those elements:

$$\lambda_{\pm 1} = \delta_{\pm 1}, \tag{12}$$

$$\lambda_{\pm k} = \delta_{\pm k} + \delta_{\pm(k-1)} \qquad \text{for } 2 \leq k \leq n \tag{13}$$

in the SO_{2n}-case, and in the SP_{2n}-case the same ones, and additionally,

$$\lambda_\infty = \delta_\infty + \delta_n + \delta_{-n}\,. \tag{14}$$

4.3. Implications for the simple perverse sheaves $\mathcal{L}_k$.

The following theorem explains what the elements λ_δ tell us about the simple perverse sheaves $\mathcal{L}_\delta$:

Proposition 4.4 ([10], Theorem 1.12)**.** *Suppose $\delta = \delta_{\mathcal{O},\gamma} \in \mathcal{D}$ and $\iota\colon \mathcal{O} \hookrightarrow G$ is the inclusion. Then the following holds:*

1. *$H^p(\iota_{!*}\gamma) = 0$ for odd p.*
2. *Suppose, additionally, $\delta' = \delta_{\mathcal{O}',\gamma'} \in \mathcal{D}$ and $\iota'\colon \mathcal{O}' \hookrightarrow G$ is the inclusion. Then*

   $$P_{\delta',\delta} = \sum_p [H^{2p}(\iota_{!*}\gamma) : \iota'_!\gamma']q^p\,.$$

   Here $[\mathcal{F} : \mathcal{G}]$ denotes the multiplicity of the simple (non-perverse) equivariant sheaf $\mathcal{G}$ in the Jordan-Hölder series of $\mathcal{F}$.

Using this and equations (12), (13), and (14), we can read off which orbits the useful simple perverse sheaves "live" on. By "$\mathcal{L}$ lives on X" we mean: If $\iota\colon G \smallsetminus X \hookrightarrow G$ is the inclusion of the complement, then $\iota^*\mathcal{L} = 0$.

Corollary 4.5. *Suppose $\mathcal{O} \in K\backslash G/B$ is a useful orbit, γ is a useful equivariant local system on $\mathcal{O}$, and $\mathcal{L}$ is the corresponding useful simple perverse sheaf. If $\mathcal{O}$ has small useful neighbors, then $\mathcal{L}$ lives on $\mathcal{O}$ and its small useful neighbors; otherwise $\mathcal{L}$ only lives on $\mathcal{O}$.*

From our equations (12), (13), and (14), we also can conclude that many homomorphism groups between useful simple perverse sheaves vanish. To do this, we need the following proposition, which is an adaption of Theorem 3.4.1 in [2] to our situation and our needs:

Proposition 4.6. *Suppose $\delta, \delta' \in \mathcal{D}$ are basis elements such that λ_δ and $\lambda_{\delta'}$ have no common summand, that is, for each $\delta'' \in \mathcal{D}$ either $P_{\delta'',\delta} = 0$ or $P_{\delta'',\delta'} = 0$. Then $R\operatorname{Hom}_K(\mathcal{L}_\delta, \mathcal{L}_{\delta'}) = 0$.*

We postpone the proof to the next section.

Corollary 4.7. *Suppose $\mathcal{L}_\delta$ and $\mathcal{L}_{\delta'}$ are two useful simple perverse sheaves with neither $\delta \overset{s}{\mapsto} \delta'$ nor $\delta' \overset{s}{\mapsto} \delta$ for any $s \in S$. Then $R\operatorname{Hom}_K(\mathcal{L}_\delta, \mathcal{L}_{\delta'}) = 0$.*

Remark. This corollary says that each endomorphism of $\mathcal{L}_{\text{use}}$ somehow "lives in a single neighborhood". This is the reason why we can determine $\operatorname{End}^\bullet_K(\mathcal{L}_{\text{use}})$ quite easily. We will see in the next chapter how to lift endomorphisms in the derived category on $K_\mathcal{N} \times B_\mathcal{N} \multimap\!\!\rightarrow G_\mathcal{N}$ to $K \times B \multimap\!\!\rightarrow \mathcal{N}$. Then the only thing left to do is to put everything together.

Another conclusion of Proposition 4.6 is the following:

Corollary 4.8. *Blocks in the $\mathcal{H}$-module M correspond to direct summands of $\operatorname{End}^\bullet_K(\mathcal{L}_{\text{all}})$. More precisely, a decomposition of M of the form*

$$M = \bigoplus_i \langle \mathcal{D}_i \rangle_{\mathbb{Z}[q,q^{-1}]} \quad \textit{where } \dot{\bigcup_i} \mathcal{D}_i = \mathcal{D}$$

provides a corresponding decomposition of the algebra $\operatorname{End}^\bullet_K(\mathcal{L}_{\text{all}})$:

$$\operatorname{End}^\bullet_K(\mathcal{L}_{\text{all}}) = \bigoplus_i \operatorname{End}^\bullet_K(\bigoplus_{\delta \in \mathcal{D}_i} \mathcal{L}_\delta)\,.$$

This proves Theorem 1.4 (1): As M_{use} is a direct summand of M, $\operatorname{End}^\bullet_K(\mathcal{L}_{\text{use}})$ is a direct summand of $\operatorname{End}^\bullet_K(\mathcal{L}_{\text{all}})$.

Remark. In the SO_{2n}-case, M_{use} itself consists of two blocks, so we have

$$\operatorname{End}^\bullet_K(\mathcal{L}_{\text{use}}) \cong \operatorname{End}^\bullet_K(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n) \oplus \operatorname{End}^\bullet_K(\mathcal{L}_{-1} \oplus \cdots \oplus \mathcal{L}_{-n})\,.$$

4.4. **Homomorphisms vanish.** In the last section of this chapter, we prove Proposition 4.6. To do this, we first have to analyze the equivariant derived category on a single orbit of $K \times B \multimap\!\!\rightarrow G$:

Lemma 4.9. *Suppose $\mathcal{O} \in K\backslash G/B$ is an orbit. Then the corresponding derived category $\mathrm{D}^+_{K\times B}(\mathcal{O})$ decomposes into a direct sum, each summand of which has exactly one of the equivariant local systems on $\mathcal{O}$ as simple object.*

Proof. Fix a representative $x \in \mathcal{O}$ of our orbit and suppose $I_x \subset K \times B$ is the isotropy group of x. By *induction equivalence*, $\mathrm{D}^+_{K\times B}(\mathcal{O}) \cong \mathrm{D}^+_{I_x}(\mathrm{pt})$.

Let T_x be a maximal torus of I_x. As I_x is solvable (by Lemma 3.15), we get $\mathrm{D}^+_{K\times B}(\mathcal{O}) \cong \mathrm{D}^+_{T_x}(\mathrm{pt})$. (The equivariant derived category does not change when one divides out a contractible normal subgroup from the acting group; this is well known and can be proved by the methods used in [3]. See also [7].)

T_x is a direct product of its component group T_x/T_x^0 (which is also the component group of I_x) and a connected group. The assertion follows. ∎

Now let's get to the proof of Proposition 4.6. Recall that we have two simple perverse sheaves $\mathcal{L}_\delta$ and $\mathcal{L}_{\delta'}$ such that λ_δ and $\lambda_{\delta'}$ have no common summand and that we want to show that $R\operatorname{Hom}_K(\mathcal{L}_\delta, \mathcal{L}_{\delta'}) = 0$.

In the proof, "$\mathcal{F}$ appears in $\mathcal{G}$" means: The simple (non-perverse) sheaf $\mathcal{F}$ appears in the Jordan-Hölder series of $H^p(\mathcal{G})$ for some $p \in \mathbb{Z}$.

Proof of Proposition 4.6. We will prove the following assertion: Suppose $X \subset G$ is a union of orbits. Suppose further that $\mathcal{L}$ and $\mathcal{L}'$ are two simple perverse sheaves on $K \times B \multimap X$ such that for each orbit $\iota \colon \mathcal{O} \hookrightarrow X$ and for each equivariant local system γ on $\mathcal{O}$, the equivariant local system $\iota_! \gamma$ does not appear in both $\mathcal{L}$ and $\mathcal{L}'$. Then $R\operatorname{Hom}_K(\mathcal{L}, \mathcal{L}') = 0$.

This obviously implies the proposition.

To prove the assertion, choose a closed orbit $F \subset X$ (closed relative to X). Let $U := X \smallsetminus F$ be its complement and let $i \colon F \hookrightarrow X$ and $j \colon U \hookrightarrow X$ the inclusions.

Applying $R\operatorname{Hom}(\cdot, \mathcal{L}')$ to the distinguished triangle

$$j_! j^* \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow i_! i^* \mathcal{L} \xrightarrow{+1}$$

yields

$$\begin{tikzcd} R\operatorname{Hom}(j_! j^*\mathcal{L}, \mathcal{L}') \arrow[d, equal, "\wr"] & R\operatorname{Hom}(\mathcal{L}, \mathcal{L}') \arrow[l] & R\operatorname{Hom}(i_! i^*\mathcal{L}, \mathcal{L}') \arrow[l] \arrow[d, equal, "\wr"] & \arrow[l, "+1"'] \\ R\operatorname{Hom}(j^*\mathcal{L}, j^*\mathcal{L}') & & R\operatorname{Hom}(i^*\mathcal{L}, i^!\mathcal{L}') . & \end{tikzcd} \tag{15}$$

So it suffices to show $R\operatorname{Hom}(j^*\mathcal{L}, j^*\mathcal{L}') = 0$ and $R\operatorname{Hom}(i^*\mathcal{L}, i^!\mathcal{L}') = 0$.

For $R\operatorname{Hom}(j^*\mathcal{L}, j^*\mathcal{L}') = 0$, we use induction on the number of orbits of $K \times B \multimap X$; $j^*\mathcal{L}$ and $j^*\mathcal{L}'$ are simple equivariant perverse sheaves on U satisfying the prerequisites of the assertion.

$R\operatorname{Hom}(i^*\mathcal{L}, i^!\mathcal{L}')$: If an equivariant local system δ appears in $i^*\mathcal{L}$, then $i_!\delta$ appears in $\mathcal{L}$. When we know the analogue for $i^!\mathcal{L}'$ we are done. Then by assumption no equivariant local system appears in both $i^*\mathcal{L}$ and $i^!\mathcal{L}'$; so by Lemma 4.9 we have $R\operatorname{Hom}(i^*\mathcal{L}, i^!\mathcal{L}') = 0$.

Fix $p \in \mathbb{Z}$. We want to examine the Jordan-Hölder series of $H^p(i^!\mathcal{L}')$. Equivariant local systems on orbits of $K \times B \multimap X$ are self-dual, so $\mathcal{L}'$ is self-dual and we have $i^!\mathcal{L}' \cong D i^* \mathcal{L}'$. As equivariant sheaves on F are locally constant (and so, in particular, $H^p(i^*\mathcal{L}')$ is locally constant), we get

$$H^p(i^!\mathcal{L}') \cong H^p(Di^*\mathcal{L}') \cong H^{p'}(i^*\mathcal{L}')^\vee := \mathcal{H}om(H^{p'}(i^*\mathcal{L}'), \mathrm{or}_F)$$

where $p' := -p - \dim_{\mathbb{R}} F$.

The Jordan-Hölder series of $H^{p'}(i^*\mathcal{L}')^\vee$ consists of the duals $\delta^\vee$ of the equivariant local systems δ appearing in $H^{p'}(i^*\mathcal{L}')$ (again using the local constantness). But the δ are self-dual, so putting everything together we get what we want: The equivariant local systems δ appearing in $i^!\mathcal{L}'$ are the same ones as in $i^*\mathcal{L}'$, and those are the ones for which $i_!\delta$ appears in $\mathcal{L}'$. $\square$

5. Homomorphisms in a single neighborhood

5.1. **Goal of this chapter.** From the combinatorial computations of the preceding chapter we know that the only endomorphisms of $\mathcal{L}_{\mathrm{use}}$ are homomorphisms between neighboring simple perverse sheaves. What is missing is how these homomorphisms look and how they compose. This is provided by the following proposition, which will be proven in this chapter.

Proposition 5.1. *Suppose $s \in S$ is a simple reflection, $\mathcal{N} \in K\backslash G/P_s$ is a useful s-neighborhood of orbits, and $\mathcal{L} \in \mathrm{PSh}_{K\times B}(G)$ is the direct sum of all simple perverse sheaves of one useful neighborhood on $\mathcal{N}$.*

Let $\iota\colon \mathcal{N} \hookrightarrow G$ be the inclusion. Then the endomorphism algebra $\mathrm{End}^\bullet_{K\times B}(\iota^*\mathcal{L})$ (in the derived category) is the following—depending on the kind of neighborhood:

(1) *Case $K_\mathcal{N} \cong B_\mathcal{N}$:*

$$\mathrm{End}^\bullet_{K\times B}(\iota^*\mathcal{L}) \cong \begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \arrow[loop right, "c"] \end{tikzcd} \qquad | \qquad ca = bc = 0\,.$$

The left dot corresponds to the small neighbor, the right dot to the large one.

(2) *Case $K_\mathcal{N} \cong T_\mathcal{N}$:*

$$\mathrm{End}^\bullet_{K\times B}(\iota^*\mathcal{L}) \cong \begin{tikzcd} \bullet \arrow[r, bend left, "a_+"] & \bullet \arrow[l, bend left, "b_+"] \arrow[r, bend right, swap, "b_-"] & \bullet \arrow[l, bend right, swap, "a_-"] \end{tikzcd} \qquad | \qquad b_+a_- = b_-a_+ = 0\,.$$

The outer dots correspond to the small neighbors, the middle dot to the large one.

All arrows are of degree 1.

We will keep the simple reflection s and the neighborhood $\mathcal{N}$ fixed throughout this chapter. We also fix the equivariant local system $\bar{\gamma}$ on $\mathcal{N}$ "generating" the neighborhood of perverse sheaves (in the sense of Definition 3.16).

For the proof, we first need a more general notation for "all perverse sheaves in one neighborhood":

Definition 5.2. Suppose K is a group acting on a space X with finitely many orbits $\mathcal{O}_\nu$, $\nu \in I$. (X will be a neighborhood of orbits in a slightly more general sense.) Denote by $i_\nu\colon \mathcal{O}_\nu \hookrightarrow X$ the inclusions of those orbits in X.

For an equivariant local system $\bar{\gamma}$ on X, define the corresponding *perverse neighborhood* by

$$\mathcal{L}(\bar{\gamma}) := \bigoplus_{\nu\in I} i_{\nu!*} i_\nu^* \bar{\gamma}[\dim_\mathbb{C} \mathcal{O}_\nu]\,,$$

i.e., the direct sum of all simple perverse sheaves in the neighborhood induced by $\bar{\gamma}$ in the sense of Definition 3.16.

The main idea of the proof is the following lemma, which we will prove in the next sections.

Lemma 5.3. *Suppose $s \in S$ is a simple reflection, $\mathcal{N} \in K\backslash G/P_s$ is a useful neighborhood of orbits, $\bar{\gamma}$ is an equivariant local system on $\mathcal{N}$, and $\bar{\gamma}_\mathcal{N}$ is the constant (one-dimensional) sheaf on $K_\mathcal{N} \times B_\mathcal{N} \multimap\!\!\!\rightarrow G_\mathcal{N}$. Then there is a fully faithful functor*

$$\mathrm{D}^+_{K_\mathcal{N}\times B_\mathcal{N}}(G_\mathcal{N}) \longrightarrow \mathrm{D}^+_{K\times B}(\mathcal{N})$$

mapping $\mathcal{L}(\bar{\gamma}_\mathcal{N})$ to $\mathcal{L}(\bar{\gamma})$.

Using this, the computation of the endomorphisms of Proposition 5.1 can be reduced to computations on PGL_2:

Proof of Proposition 5.1. Let $\mathcal{L}$ and $\iota\colon \mathcal{N} \hookrightarrow G$ be as in the proposition. One easily checks that the pull-back $\iota^*\mathcal{L}$ is a perverse neighborhood in the sense of Definition 5.2: $\iota^*\mathcal{L} = \mathcal{L}(\bar{\gamma})$. Therefore, we can apply Lemma 5.3: $\mathrm{End}^\bullet_{K\times B}(\iota^*\mathcal{L}) \cong \mathrm{End}^\bullet_{K_\mathcal{N}\times B_\mathcal{N}}(\mathcal{L}(\bar{\gamma}_\mathcal{N}))$.

So the only thing remaining to do is to determine the endomorphisms (including the higher ones) of the constant sheaf $\bar{\gamma}_\mathcal{N}$ in the derived category $\mathrm{D}^+_{K_\mathcal{N}\times B_\mathcal{N}}(G_\mathcal{N})$, where $G_\mathcal{N} \cong \mathrm{PGL}_2$, $B_\mathcal{N}$ is a Borel subgroup of PGL_2, and and $K_\mathcal{N}$ is either a torus or a Borel subgroup. This can be done "by hand" and is left to the reader. The result is the one claimed in the proposition. (Note that although this is a "small" situation, the computation is still a bit of work, especially concerning the composition of morphisms. It is carried out in detail in [7].) □

Remark. Lemma 5.3 is not true for arbitrary neighborhoods $\mathcal{N}$: For most of the orbits, the part of a torus of K acting trivially is too big. This causes the endomorphisms in $\mathrm{D}^+_{K\times B}(G)$ to get more complicated than those in $\mathrm{D}^+_{K_\mathcal{N}\times B_\mathcal{N}}(G_\mathcal{N})$.

5.2. **Lifting the endomorphisms.** In this section we prove Lemma 5.3.

Proof of lemma 5.3. The idea is to turn the known correspondence between the geometry of $K \times B \mathrel{-\!\circ\!\!\rightarrow} X$ and of $K_\mathcal{N} \times B_\mathcal{N} \mathrel{-\!\circ\!\!\rightarrow} G_\mathcal{N}$ into a functorial one.

In this and in the next section, it will be more convenient to work with the derived categories on $K \mathrel{-\!\circ\!\!\rightarrow} G/B$ and on $K_\mathcal{N} \mathrel{-\!\circ\!\!\rightarrow} G_\mathcal{N}/B_\mathcal{N}$ instead of those on $K \times B \mathrel{-\!\circ\!\!\rightarrow} G$ and on $K_\mathcal{N} \times B_\mathcal{N} \mathrel{-\!\circ\!\!\rightarrow} G_\mathcal{N}$.

We fix a representative $x \in \mathcal{N}$ of our neighborhood. (For now, it can be any representative; in section 5.3, we will put some restrictions on the choice of x to simplify computations.) Recall the definition

$$K_x = x^{-1}Kx\,.$$

To define the fully faithful functor from $D_{K_\mathcal{N}}(G_\mathcal{N}/B_\mathcal{N})$ to $D_K(X/B)$, we will proceed in several steps. Figure 2 lists all the involved spaces, together with maps between them.

$$\begin{tikzcd}
1. & K \arrow[r, "\circ" marking] & KxP_s/B = \mathcal{N}/B \\
2. & K_x \arrow[u, "\wr"] \arrow[r, "\circ" marking] & K_xP_s/B \cong K_x \times_{K_x\cap P_s} P_s/B \arrow[u, "\wr"'] \\
3. & K_x \cap P_s \arrow[u, hook] \arrow[r, "\circ" marking] \arrow[d, two heads] & P_s/B \arrow[u, hook] \arrow[d, equal] \\
4. & K^4_\mathcal{N} := (K_x\cap P_s)/(K_x\cap U_s) \arrow[r, "\circ" marking] \arrow[d, two heads] & P_s/B \arrow[d, equal] \\
5. & K^5_\mathcal{N} := (K_x\cap P_s)/(K_x\cap U_sT^0_s) \arrow[r, "\circ" marking] \arrow[d, two heads] & P_s/B \arrow[d, equal] \\
6. & K_\mathcal{N} = (K_x\cap P_s)/(K_x\cap U_sT_s) \arrow[r, "\circ" marking] & P_s/B \cong G_\mathcal{N}/B_\mathcal{N}
\end{tikzcd}$$

FIGURE 2. All spaces involved in the proof of Lemma 5.3

We will proceed from top to bottom: For each pair of adjacent lines i and $i+1$, we will specify a fully faithful functor $\mathcal{F}^i_{i+1}$ from the derived category of the lower line to the one of the upper line. The image of this functor will contain $\bar{\gamma}_i := (\mathcal{F}^{i-1}_i)^{-1} \dots (\mathcal{F}^1_2)^{-1}\bar{\gamma}$. All of these functors $\mathcal{F}^i_{i+1}$ will be "nice" in the following sense:

(a) They will commute with $\mathcal{L}(\cdot)$ (taking the perverse neighborhood corresponding to an equivariant local system). Therefore, we will get

$$(\mathcal{F}^1_2 \circ \cdots \circ \mathcal{F}^5_6)(\mathcal{L}(\bar{\gamma}_6)) = \mathcal{L}(\bar{\gamma})\,.$$

(b) All $\bar{\gamma}_i$ will be locally constant one-dimensional sheaves on the whole space.

In $\mathrm{D}^+_{K_\mathcal{N}}(P_s/B)$, the only locally constant one-dimensional sheaf is the constant one, so $\bar{\gamma}_6 \cong \bar{\gamma}_\mathcal{N}$. So once we have these functors, the proof of the lemma is complete.

Now let's get to work:

$2 \to 1$: This is only conjugation by x, so it is an equivalence of categories.

$3 \to 2$: We have an equivalence of categories by the *induction equivalence*.

$4 \to 3$: Dividing out a connected normal subgroup $(K_x \cap U_s)$ from the acting group induces an equivalence of the derived categories.

$5 \to 4$: This is the only step where actually something happens. It is also the step which does not work on non-useful orbits in general. Let $a \in \mathcal{I}$ be the involution corresponding to the small orbit(s) of our neighborhood. Recall that $T_a = \{t \in T \mid at = t\}$ and $A_a = T_a/T^0_a$. We need the following lemma, the proof of which will be postponed to the next section:

Lemma 5.4. $K^4_\mathcal{N} \cong K^5_\mathcal{N} \times A_a$.

As the action $\ker(K^4_\mathcal{N} \twoheadrightarrow K^5_\mathcal{N}) \cong A_a \multimap P_s/B$ is trivial, the derived category $\mathrm{D}^+_{K^4_\mathcal{N}}(P_s/B)$ decomposes into a direct sum of components, each of which is equivalent to $\mathrm{D}^+_{K^5_\mathcal{N}}(P_s/B)$. (The components only differ in the action of A_a.) We choose for $\mathcal{F}^4_5$ the identification of $\mathrm{D}^+_{K^5_\mathcal{N}}(P_s/B)$ with the component that contains $\bar{\gamma}_4$.

$6 \to 5$: We distinguish two cases:

If $K_\mathcal{N} \cong B_\mathcal{N}$, then

$$T_s = \{\lceil *, \dots, *, \underbrace{*, *,}_{k\text{th},\ (k+1)\text{th entry equal}} *, \dots, * \rfloor\}$$

(with k depending on s). In particular, T_s is connected and the fifth and sixth line are the same.

The case $K_\mathcal{N} \cong T_\mathcal{N}$: This only happens for the $s_\star$-neighborhood in the SP_{2n}-case. Here we have

$$T_s = \{\lceil *, \dots, *, \pm 1 \rfloor\} \qquad \text{and}$$
$$T^0_s = \{\lceil *, \dots, *, 1 \rfloor\}\,.$$

So the map $K^5_\mathcal{N} \twoheadrightarrow K_\mathcal{N}$ has a finite kernel $\{\pm 1\}$ which acts trivially on $G_\mathcal{N}/B_\mathcal{N}$. By [3], 8.7.1, it follows that the restriction

$$\mathcal{F}^5_6 := \mathrm{Res}_{K^5_\mathcal{N}, K_\mathcal{N}} \colon \mathrm{D}^+_{K_\mathcal{N}}(P_s/B) \longrightarrow \mathrm{D}^+_{K^5_\mathcal{N}}(P_s/B)$$

is an identification of $\mathrm{D}^+_{K_\mathcal{N}}(P_s/B)$ with the full subcategory of $\mathrm{D}^+_{K^5_\mathcal{N}}(P_s/B)$ consisting of those sheaves on which the kernel acts trivially.

Now the only thing left to check is that $\bar{\gamma}_5$ lies in the image of this restriction. This follows from the fact that $\bar{\gamma}_5$ lives on the whole space and therefore, in particular, on the one-point-orbit $\{B\} \in P_s/B$. As $K^5_{\mathcal{N}}$ is connected, it has to act trivially on the stalk of $\bar{\gamma}_5$ at B. In particular, the kernel $\{\pm 1\}$ acts trivially on that stalk and therefore also on the remainder of $\bar{\gamma}_5$. □

5.3. $K^4_{\mathcal{N}}$ **decomposes into a direct sum.** In this section we prove Lemma 5.4; this is the only thing remaining of the proof of Proposition 5.1.

Proof of Lemma 5.4. Suppose $\mathcal{O}$ is a small orbit in our neighborhood. Recall that $a = \phi(\mathcal{O})$ is the corresponding involution. To prove the lemma, we will assume that the representative $x \in \mathcal{N}$ of our neighborhood lies in $\mathcal{O}$ and that $\kappa(x) \in {}^{\Gamma}\mathrm{N}(T)$. (Such an x exists by the remark after Definition 3.3.) This implies two things:

First, note that $K_x = \left\{g \in G \mid \kappa(x) g \kappa(x)^{-1} = g\right\}$, so we get

$$T_a = \{t \in T \mid at = t\} = \left\{t \in T \mid \kappa(x) t \kappa(x)^{-1} = t\right\} = K_x \cap T\,.$$

Second, we easily get that the semidirect product $B \cong U \rtimes T$ can be restricted to

$$K_x \cap B \cong (K_x \cap U) \rtimes (K_x \cap T)\,. \tag{16}$$

Further restriction yields

$$K_x \cap U_s T_s^0 \cong (K_x \cap U_s) \rtimes (K_x \cap T_s^0)\,,$$

which yields a short exact sequence

$$\begin{tikzcd} K_x \cap T_s^0 \arrow[r, hook] & (K_x \cap P_s)/(K_x \cap U_s) \arrow[r, two heads] \arrow[d, equal, "\wr"] & (K_x \cap P_s)/(K_x \cap U_s T_s^0) \arrow[d, equal, "\wr"] \\ & K^4_{\mathcal{N}} & K^5_{\mathcal{N}} \end{tikzcd} \tag{17}$$

We will now construct a group homomorphism $K^4_{\mathcal{N}} \twoheadrightarrow A_a$. Then we will show that the composition

$$K_x \cap T_s^0 \hookrightarrow K^4_{\mathcal{N}} \twoheadrightarrow A_a$$

is an isomorphism. When this is done, we know that the sequence (17) splits into a direct product $K^4_{\mathcal{N}} \cong K^5_{\mathcal{N}} \times A_a$, so we are finished.

Construction of the homomorphism $K^4_{\mathcal{N}} \twoheadrightarrow A_a$: We take the following composition (explanation follows):

$$\begin{tikzcd} K^4_{\mathcal{N}} = (K_x \cap P_s)/(K_x \cap U_s) \arrow[d, equal, "(a)"'] \\ (K_x \cap B)/(K_x \cap U_s) \arrow[d, two heads, "(b)"'] \\ (K_x \cap B)/(K_x \cap B)^0 \arrow[d, equal, "(c)"', "\wr"] \\ (K_x \cap T)/(K_x \cap T)^0 = T_a/T_a^0 = A_a \end{tikzcd}$$

For the equality (a) we have to show $K_x \cap P_s = K_x \cap B$. "⊃" is clear. "⊂": The orbit of $K_{\mathcal{N}} \curvearrowright P_s/B$ corresponding to $\mathcal{O}$ is $(K_x \cap P_s)B/B$ (because we chose

$x \in \mathcal{O}$). Since we are in one of the cases $K_{\mathcal{N}} \cong T_{\mathcal{N}}$ or $K_{\mathcal{N}} \cong B_{\mathcal{N}}$, this orbit consists only of one point, namely B/B. It follows $K_x \cap P_s \subset B$, and therefore $K_x \cap P_s \subset K_x \cap B$.

To get the surjection (b), we have to check $K_x \cap U_s \subset (K_x \cap B)^0$. This is because $U_s \subset B$ and because $K_x \cap U_s$ is connected.

The existence of isomorphism (c) follows from (16) and from the fact that $K_x \cap U$ is connected.

We now have the desired homomorphism $K^4_{\mathcal{N}} \twoheadrightarrow A_a$; it remains to check that the composition $K_x \cap T^0_s \hookrightarrow K^4_{\mathcal{N}} \twoheadrightarrow A_a$ is an isomorphism, i.e., that there is exactly one element of $K_x \cap T^0_s = (K_x \cap T) \cap T^0_s = T_a \cap T^0_s$ in each connected component of T_a. This can easily be done manually (and is left to the reader), as we know T_a explicitly from section 3.10. □

6. Putting everything together

Now we have all the ingredients needed to determine the geometric extension algebra $\mathrm{End}^\bullet_K(\mathcal{L}_{\mathrm{use}})$. The result will be what was claimed in Theorem 1.6.

6.1. The case SO_{2n} $(n \geq 2)$.

We will use the following notation:

Notation. Given a family $\mu_1, \ldots, \mu_l$ of indices, we will write $\mathcal{L}_{\mu_1,\ldots,\mu_l}$ for the direct sum $\mathcal{L}_{\mu_1} \oplus \cdots \oplus \mathcal{L}_{\mu_l}$.

At the end of section 4.3 we saw that $\mathrm{End}^\bullet_K(\mathcal{L}_{\mathrm{use}})$ decomposes into a direct sum $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,n}) \oplus \mathrm{End}^\bullet_K(\mathcal{L}_{-1,\ldots,-n})$. The computation for these two blocks is exactly the same, so we will only consider $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,n})$.

We will determine $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,k})$ by induction on k ($k = 2, \ldots, n$). The idea for the induction step is: We know $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,k-1})$ by the induction hypothesis and $\mathrm{End}^\bullet_K(\mathcal{L}_{k-1,k})$ (more or less) by Chapter 5. By Corollary 4.7 these are all endomorphisms of $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,k})$.

So let's get to work. We claim:

$$\mathrm{End}^\bullet_K(\mathcal{L}_{1,\ldots,k}) = \begin{tikzcd} \underset{\mathcal{L}_1}{\bullet} \arrow[r, bend left, "a_1"] & \underset{\mathcal{L}_2}{\bullet} \arrow[l, bend left, "b_1"] & \cdots & \underset{\mathcal{L}_{k-1}}{\bullet} \arrow[r, bend left, "a_{k-1}"] & \underset{\mathcal{L}_k}{\bullet} \arrow[l, bend left, "b_{k-1}"] \arrow[loop right, "c"] \end{tikzcd} \quad \left| \quad \begin{aligned} a_{\mu+1}a_\mu &= 0 \\ b_\mu b_{\mu+1} &= 0 \\ c a_{k-1} &= 0 \\ b_{k-1} c &= 0 \end{aligned} \right. \tag{18}$$

(All arrows are of degree 1.)

For the following computation we will restrict ourselves to the closed subset $X \subset G$ consisting of $\mathcal{O}_k$ and all orbits of smaller dimension. We further decompose X into an open subset U and a closed subset F:

$$\begin{tikzcd} & G \\ & X := \Big(\bigcup\limits_{\substack{\mathcal{O} \in K\backslash G/B, \\ \dim_{\mathbb{C}} \mathcal{O} \leq \dim_{\mathbb{C}} \mathcal{O}_{k-1}}} \mathcal{O} \Big) \cup \mathcal{O}_k \arrow[u, hook, "\iota"] \\ U := \mathcal{O}_{k-1} \cup \mathcal{O}_k \arrow[ur, hook, "j"] & F := X \smallsetminus U = \Big(\bigcup\limits_{\substack{\mathcal{O} \in K\backslash G/B, \\ \dim_{\mathbb{C}} \mathcal{O} \leq \dim_{\mathbb{C}} \mathcal{O}_{k-1}}} \mathcal{O} \Big) \smallsetminus \mathcal{O}_{k-1} \arrow[u, hook, "i"] \end{tikzcd}$$

The simple perverse sheaves $\mathcal{L}_\mu$ for $1 \le \mu \le k$ all have support in X, so we have $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k+1}) = \mathrm{End}^\bullet_K(\iota^*\mathcal{L}_{1,\dots,k+1})$. In the rest of this section, we will only work with these sheaves and only on X, so by abuse of notation we will write $\mathcal{L}_{\mu,\dots,\nu}$ instead of $\iota^*\mathcal{L}_{\mu,\dots,\nu}$.

The start of induction ($k = 2$) follows directly from Proposition 5.1 (which describes the endomorphisms of $j^*\mathcal{L}_{1,2}$) and Corollary 4.5 (which says that $i^*\mathcal{L}_{1,2} = 0$). Now let's get to the induction step:

As a graded vector space, $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k})$ is the direct sum of the $\mathrm{Hom}^\bullet_K(\mathcal{L}_\mu, \mathcal{L}_\nu)$ with $1 \le \mu, \nu \le k$. We will look at these summands individually:

The summands where both $\mu \le k-1$ and $\nu \le k-1$ are already known by induction hypothesis. The remaining ones are those where (at least) one side is $\mathcal{L}_k$. As $i^*\mathcal{L}_k = 0$ (by Corollary 4.5), we get

$$\mathrm{Hom}^\bullet_K(\mathcal{L}_\mu, \mathcal{L}_\nu) \cong \mathrm{Hom}^\bullet_K(j^*\mathcal{L}_\mu, j^*\mathcal{L}_\nu) \quad \text{for } \mu = k \text{ or } \nu = k\,. \tag{19}$$

(To see this, take a look at the exact triangle (15) of section 4.4 and use $\mathcal{L}_k \cong D\mathcal{L}_k$ to check that $i^!\mathcal{L}_k = 0$.)

In particular, as $j^*\mathcal{L}_\mu = 0$ for $1 \le \mu \le k-2$ (again by Corollary 4.5) we get

$$\mathrm{Hom}^\bullet_K(\mathcal{L}_k, \mathcal{L}_\mu) = 0 \text{ and } \mathrm{Hom}^\bullet_K(\mathcal{L}_\mu, \mathcal{L}_k) = 0 \quad \text{for } \mu \le k-2\,. \tag{20}$$

((20) also follows directly from Corollary 4.7.)

So the only homomorphism spaces left to consider are $\mathrm{Hom}^\bullet_K(j^*\mathcal{L}_k, j^*\mathcal{L}_{k-1})$, $\mathrm{Hom}^\bullet_K(j^*\mathcal{L}_{k-1}, j^*\mathcal{L}_k)$, and $\mathrm{Hom}^\bullet_K(j^*\mathcal{L}_k, j^*\mathcal{L}_k)$; but these have already been determined in Proposition 5.1 (1).

Now that we know all $\mathrm{Hom}^\bullet_K(\mathcal{L}_\mu, \mathcal{L}_\nu)$ as graded vector spaces, we have to determine the composition of the morphisms. We will use the following approach: First, we will find elements of $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k})$ which generate $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k})$ as an algebra; however, there will be one element more than those mentioned in (18). (By "generators" we mean generators of the degrees > 0.) Then we check that these generators fulfill the relations postulated in (18). Finally, we get rid of the superfluous generator. As the dimensions of our homomorphism spaces agree with those postulated in (18), we know that there are no more relations and we are finished.

Search of generators: By induction we have elements a_l, b_l for $1 \le l \le k-2$ and c which generate $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k-1})$. So the only thing left is to find elements of $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k})$ which generate (a superset of)

$$\mathit{Rest} := \mathrm{Hom}^\bullet_K(\mathcal{L}_{k-1}, \mathcal{L}_k) \oplus \mathrm{Hom}^\bullet_K(\mathcal{L}_k, \mathcal{L}_{k-1}) \oplus \mathrm{End}^\bullet_K(\mathcal{L}_k)\,.$$

Let's have a look at the following morphism of algebras:

$$\rho\colon\ \mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,k}) \longrightarrow \mathrm{End}^\bullet_K(j^*\mathcal{L}_{1,\dots,k}) \cong \mathrm{End}^\bullet_K(j^*\mathcal{L}_{k-1,k})\,.$$

By Proposition 5.1 we have

$$\mathrm{End}^\bullet_K(j^*\mathcal{L}_{k-1,k}) = \begin{tikzcd} \bullet \arrow[r, bend left, "a'"] & \bullet \arrow[l, bend left, "b'"] \arrow[loop right, "c'"] \end{tikzcd} \quad \Big| \quad c'a' = b'c' = 0\,,$$

i.e., a', b', and c' generate $\mathrm{End}^\bullet_K(j^*\mathcal{L}_{k-1,k})$. Equation (19) says that $\rho|_{\mathit{Rest}}$ is bijective, so we can find preimages $a_{k-1}, b_{k-1}, c_{\mathrm{new}} \in \mathit{Rest}$ of a', b', c' and those preimages generate Rest.

Now we have enough generators. The current picture of our geometric extension algebra is the following:

$$\begin{tikzcd} \mathcal{L}_1 \arrow[r, bend left, "a_1"] & \mathcal{L}_2 \arrow[l, bend left, "b_1"] & \cdots & \mathcal{L}_{k-2} \arrow[r, bend left, "a_{k-2}"] & \mathcal{L}_{k-1} \arrow[l, bend left, "b_{k-2}"] \arrow[r, bend left, "a_{k-1}"] \arrow[loop right, "c"] & \mathcal{L}_k \arrow[l, bend left, "b_{k-1}"] \arrow[loop right, "c_{\mathrm{new}}"] \end{tikzcd}$$

Let's get to the relations. By induction we know that

$$\begin{aligned} a_{\mu+1}a_\mu = b_\mu b_{\mu+1} &= 0 \qquad \text{for } 0 \le \mu \le k-1-2 \quad \text{and} \\ ca_{k-2} = b_{k-2}c &= 0\,. \end{aligned}$$

As $c_{\mathrm{new}}a_{k-1}$ and $b_{k-1}c_{\mathrm{new}}$ lie in *Rest* and $\rho|_{Rest}$ is injective, from $\rho(c_{\mathrm{new}}a_{k-1}) = c'a' = 0$ and $\rho(b_{k-1}c_{\mathrm{new}}) = b'c' = 0$, we get

$$c_{\mathrm{new}}a_{k-1} = b_{k-1}c_{\mathrm{new}} = 0\,.$$

Finally, from $\mathrm{Hom}^\bullet_K(\mathcal{L}_k, \mathcal{L}_{k-2}) = 0$ and $\mathrm{Hom}^\bullet_K(\mathcal{L}_{k-2}, \mathcal{L}_k) = 0$ follows

$$a_{k-1}a_{k-2} = b_{k-2}b_{k-1} = 0\,.$$

Thus the relations of (18) are fulfilled. It remains to get rid of the generator c. Assume $c \notin \langle a_{k-2}b_{k-2}, b_{k-1}a_{k-1}\rangle_{\mathbb{C}}$. As $\mathrm{End}^2_K(\mathcal{L}_{k-1})$ is two-dimensional, it follows that $a_{k-2}b_{k-2}$ and $b_{k-1}a_{k-1}$ are linearly dependent. $a_{k-2}b_{k-2} \neq 0$ by induction, so there is a $z \in \mathbb{C}$ with $b_{k-1}a_{k-1} = za_{k-2}b_{k-2}$. It follows that $a_{k-1}b_{k-1}a_{k-1} = za_{k-1}a_{k-2}b_{k-2} = 0$. This is a contradiction to the fact that $a_{k-1}b_{k-1}a_{k-1}$ is a generator of $\mathrm{Hom}^3_K(\mathcal{L}_{k-1}, \mathcal{L}_k)$.

6.2. **The case** SP_{2n} $(n \geq 1)$**.** The claim of Theorem 1.6 in this case is:

$$\mathrm{End}^\bullet_K(\mathcal{L}_{\mathrm{use}}) = \begin{tikzcd} \mathcal{L}_1 \arrow[r, bend left, "a_1"] & \mathcal{L}_2 \arrow[l, bend left, "b_1"] & \cdots & \mathcal{L}_{n-1} \arrow[r, bend left, "a_{n-1}"] & \mathcal{L}_n \arrow[l, bend left, "b_{n-1}"] \arrow[dr, bend left, "a_n"] & \\ & & & & & \mathcal{L}_\infty \arrow[ul, bend left, "b_n"] \arrow[dl, bend left, "b_{-n}"] \\ \mathcal{L}_{-1} \arrow[r, bend left, "a_{-1}"] & \mathcal{L}_{-2} \arrow[l, bend left, "b_{-1}"] & \cdots & \mathcal{L}_{-(n-1)} \arrow[r, bend left, "a_{-(n-1)}"] & \mathcal{L}_{-n} \arrow[l, bend left, "b_{-(n-1)}"] \arrow[ur, bend left, "a_{-n}"] & \end{tikzcd} \quad \left| \begin{aligned} a_{\pm(\mu+1)}a_{\pm\mu} &= 0 \\ b_{\pm\mu}b_{\pm(\mu+1)} &= 0 \\ b_{\pm n}a_{\mp n} &= 0 \end{aligned} \right.$$

The case $n = 1$ follows directly from Proposition 5.1 (2) (in the same way as the start of induction in the SO_{2n}-case). So now let $n \geq 2$.

Determining $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,n})$ and $\mathrm{End}^\bullet_K(\mathcal{L}_{-1,\dots,-n})$ works exactly in the same way as determining $\mathrm{End}^\bullet_K(\mathcal{L}_{1,\dots,n})$ in the SO_{2n}-case. The result is:

(21)

$$\mathrm{End}^\bullet_K(\mathcal{L}_{\pm1,\dots,\pm n}) = \begin{tikzcd} \mathcal{L}_{\pm1} \arrow[r, bend left, "a_{\pm1}"] & \mathcal{L}_{\pm2} \arrow[l, bend left, "b_{\pm1}"] & \cdots & \mathcal{L}_{\pm(n-1)} \arrow[r, bend left, "a_{\pm(n-1)}"] & \mathcal{L}_{\pm n} \arrow[l, bend left, "b_{\pm(n-1)}"] \arrow[loop right, "c_\pm"] \end{tikzcd} \quad \left| \begin{aligned} a_{\pm(\mu+1)}a_{\pm\mu} &= 0 \\ b_{\pm\mu}b_{\pm(\mu+1)} &= 0 \\ c_\pm a_{\pm(n-1)} &= 0 \\ b_{\pm(n-1)}c_\pm &= 0 \end{aligned} \right.$$

There are no homomorphisms between $\mathcal{L}_{1,\dots,n}$ and $\mathcal{L}_{-1,\dots,-n}$ by Corollary 4.7.

The remainder of the computation is analogous to the induction step in the SO_{2n}-case: We check that the individual homomorphism spaces are the right ones as graded vector spaces, we find generators (this time two more than necessary: c_+ and c_-), we check the relations, and we get rid of $c_\pm$.

References

[1] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston Inc., Boston, MA, 1992. MR1162533 (93j:22001)

[2] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527. MR1322847 (96k:17010)

[3] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994. MR1299527 (95k:55012)

[4] Fokko du Cloux, *Combinatorics for the representation theory of real reductive groups*, Notes from the AIM workshop, Palo Alto, July 2005.

[5] Mark Goresky, Robert Kottwitz, and Robert MacPherson, *Equivariant cohomology, koszul duality, and the localization theorem*, Invent. Math. **131** (1998), 25–83. MR1489894 (99c:55009)

[6] Mark Goresky and Robert MacPherson, *Intersection homology II*, Invent. Math. **72** (1983), 77–129. MR0696691 (84i:57012)

[7] Immanuel Halupczok, *Kategorielle Langlands-Korrespondenz für die verallgemeinerten Lorentz-Gruppen*, Ph.D. thesis, University of Freiburg, Germany, 2005, `http://www.freidok.uni-freiburg.de/volltexte/1715/`.

[8] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1990. MR1074006 (92a:58132)

[9] S. M. Khoroshkin, *Irreducible representations of Lorentz groups*, Funct. Anal. Appl. **15** (1981), 114–122.

[10] George Lusztig and David A. Vogan, Jr., *Singularities of closures of k-orbits on flag manifolds*, Invent. Math. **71** (1983), 365–379. MR0689649 (84h:14060)

[11] R. W. Richardson and T. A. Springer, *Combinatorics and geometry of k-orbits on the flag manifold*, Contemporary Mathematics **153** (1993), 109–142. MR1247501 (94m:14065)

[12] Wolfgang Soergel, *Langlands' philosophy and Koszul duality*, Algebra—representation theory (Constanta, 2000), NATO Sci. Ser. II Math. Phys. Chem., vol. 28, Kluwer Acad. Publ., Dordrecht, 2001, pp. 379–414. MR1858045 (2002j:22019)

[13] T. A. Springer, *Linear algebraic groups*, Progress in Mathematics, vol. 9, Birkhäuser Boston, Mass., 1981. MR0632835 (84i:20002)

[14] ———, *The classification of involutions of simple algebraic groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 655–670. MR0927604 (89b:20096)

[15] David A. Vogan, Jr., *Irreducible characters of semisimple lie groups III*, Invent. Math. **71** (1983), 381–417. MR0689650 (84h:22036)

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