

## CHARACTER SHEAVES ON DISCONNECTED GROUPS, IX

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ABSTRACT. We associate a two-sided cell to any (parabolic) character sheaf. We study the interaction between the duality operator for character sheaves and the operation of “twisted induction”.

### INTRODUCTION

Throughout this paper,  $G$  denotes a fixed, not necessarily connected, reductive algebraic group over an algebraically closed field  $\mathbf{k}$ . This paper is a part of a series [L9] which attempts to develop a theory of character sheaves on  $G$ .

One of the main constructions in [L3] (going back to [L14]) was a procedure which to any character sheaf on  $G^0$  associates a certain two-sided cell in an (extended) Coxeter group. A variant of this construction (restricted to “unipotent” character sheaves) was later given by Grojnowski [Gr]. Here we give a construction which generalizes that in [L3] (and takes into account the approach in [Gr]) which to any (parabolic) character sheaf on  $Z_{J,D}$  associates a certain type of two-sided cell.

The paper is organized as follows. In Section 40 we study certain equivariant sheaves on  $G^0/U^* \times G^0/U^*$  (where  $U^*$  is the unipotent radical of a Borel in  $G^0$ ) under the convolution operation. Some results in this section are implicit in [L14, Ch.1]. In Section 41 we study the character sheaves on  $Z_{\emptyset,D}$  (where  $D$  is a connected component of  $G$ ) by connecting them with sheaves on  $G^0/U^* \times G^0/U^*$ . We use this study to attach a two-sided cell to any character sheaf on  $Z_{J,D}$ . (See 41.4.) In Section 42 we study the interaction between the duality operation  $\mathbf{d}$  (see 38.10, 38.11) and the functor  $f_{\emptyset,\mathbf{I}}$  (see 36.4). The main result in this section is Proposition 42.9 which contains [L3, III, Cor. 15.8(b)] as a special case (with  $G = G^0, v = 1$ ).

*Notation.* We fix a 1-dimensional  $\bar{\mathbf{Q}}_l$ -vector space  $V$  with a given isomorphism  $V^{\otimes 2} \xrightarrow{\sim} \bar{\mathbf{Q}}_l(1)$  (Tate twist of  $\bar{\mathbf{Q}}_l$ ). For  $n \in \mathbf{N}$  we set  $\bar{\mathbf{Q}}_l(n/2) = V^{\otimes n}$ . For  $n \in \mathbf{Z}, n < 0$  let  $\bar{\mathbf{Q}}_l(n/2)$  be the dual space of  $\bar{\mathbf{Q}}_l(-n/2)$ . If  $X$  is an algebraic variety and  $A \in \mathcal{D}(X), n \in \mathbf{Z}$ , we write  $A(n/2)$  instead of  $A \otimes \bar{\mathbf{Q}}_l(n/2)$  and  $A[[n/2]]$  instead of  $A[n](n/2)$ . (When  $n$  is even, this agrees with the notation in [L9, II, p. 73].)

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40. SHEAVES ON  $G^0/U^* \times G^0/U^*$ 

**40.1.** Let  $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ . Let  $\hat{H}$  (resp.  $H$ ) be the  $\mathcal{A}$ -module consisting of all formal (resp. finite) linear combinations  $\sum_{w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}} a_{w, \lambda} \tilde{T}_w 1_\lambda$  with  $a_{w, \lambda} \in \mathcal{A}$ . Note that  $H$  is naturally an  $\mathcal{A}$ -submodule of  $\hat{H}$  with  $\mathcal{A}$ -basis  $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}\}$ . For any  $n \in \mathbf{N}_k^*$ , the  $\mathcal{A}$ -submodule of  $H$  spanned by  $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n\}$  may be naturally identified with  $H_n$  (see 31.2(a)). There is a unique  $\mathcal{A}$ -algebra structure on  $\hat{H}$  in which the product of two elements

$$h = \sum_{w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}} a_{w, \lambda} \tilde{T}_w 1_\lambda, \quad h' = \sum_{w' \in \mathbf{W}, \lambda' \in \underline{\mathfrak{s}}} a'_{w', \lambda'} \tilde{T}_{w'} 1_{\lambda'}$$

as above is  $hh' = \sum_{y \in \mathbf{W}, \nu \in \underline{\mathfrak{s}}} b_{y, \nu} \tilde{T}_y 1_\nu$  where for any  $\nu \in \underline{\mathfrak{s}}$ ,

$$\sum_{w, w' \in \mathbf{W}} a_{w, w'^{-1}\nu} a'_{w', \nu} \tilde{T}_w \tilde{T}_{w'} 1_\nu = \sum_{y \in \mathbf{W}} b_{y, \nu} \tilde{T}_y 1_\nu$$

is computed in the algebra structure of  $H_n$  for any  $n$  such that  $\nu \in \underline{\mathfrak{s}}_n$ . Thus  $\hat{H}$  becomes an associative algebra with 1;  $H$  is a subalgebra (without 1) and, for  $n \in \mathbf{N}_k^*$ ,  $H_n$  is a subalgebra (with a different 1) with the algebra structure as in 31.2.

Now in the definition of  $\hat{H}$  given above, although  $\tilde{T}_w 1_\lambda$  is defined, the elements  $\tilde{T}_w, 1_\lambda$  are not defined separately (as was the case in  $H_n$ ). To remedy this we set  $\tilde{T}_w = \sum_{\lambda \in \underline{\mathfrak{s}}} \tilde{T}_w 1_\lambda \in \hat{H}$  (for  $w \in \mathbf{W}$ ) and  $1_\lambda = \tilde{T}_1 1_\lambda \in H$  (for  $\lambda \in \underline{\mathfrak{s}}$ ). Then  $\tilde{T}_w 1_\lambda$  is the product of  $\tilde{T}_w, 1_\lambda$  in the algebra  $\hat{H}$ . Note that  $\tilde{T}_1$  is the unit element of  $\hat{H}$  and the following equalities hold in  $\hat{H}$ :

$$\begin{aligned} 1_\lambda 1_{\lambda'} &= 1_\lambda \text{ for } \lambda \in \underline{\mathfrak{s}}, 1_\lambda 1_{\lambda'} = 0 \text{ for } \lambda \neq \lambda' \text{ in } \underline{\mathfrak{s}}; \\ \tilde{T}_w \tilde{T}_{w'} &= \tilde{T}_{ww'} \text{ for } w, w' \in \mathbf{W} \text{ such that } l(ww') = l(w) + l(w'); \\ \tilde{T}_w 1_\lambda &= 1_w \lambda \tilde{T}_w \text{ for } w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}; \\ \tilde{T}_s^2 &= \tilde{T}_1 + (v - v^{-1}) \sum_{\lambda \in \underline{\mathfrak{s}}, s \in \mathbf{W}_\lambda} \tilde{T}_s 1_\lambda \text{ for } s \in \mathbf{I}. \end{aligned}$$

By a standard argument we see that

(a)  $H$  is exactly the  $\mathcal{A}$ -algebra defined by the generators  $\tilde{T}_w 1_l$  ( $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$ ) and the relations:

$$\begin{aligned} (\tilde{T}_w 1_\lambda)(\tilde{T}_{w'} 1_{\lambda'}) &= 0 \text{ if } w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}, w' \lambda' \neq \lambda, \\ (\tilde{T}_w 1_{w' \lambda'})(\tilde{T}_{w'} 1_{\lambda'}) &= \tilde{T}_{ww'} 1_{\lambda'} \text{ if } w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}, l(ww') = l(w) + l(w'), \\ (\tilde{T}_s 1_{s \lambda'})(\tilde{T}_s 1_{\lambda'}) &= \tilde{T}_1 1_{\lambda'} + (v - v^{-1}) c \tilde{T}_s 1_{\lambda'} \text{ if } s \in \mathbf{I}, \lambda' \in \underline{\mathfrak{s}} \text{ where } c = 1 \text{ for } s \in \mathbf{W}_{\lambda'} \\ &\text{and } c = 0 \text{ for } s \notin \mathbf{W}_{\lambda'}. \end{aligned}$$

**40.2.** Let  $R, R^+$  be as in 28.3. The following result is well known:

(a) *If  $w \in \mathbf{W}, \alpha \in R^+$  and  $s_\alpha$  is as in 28.3, then we have  $l(ws_\alpha) > l(w)$  if and only if  $w(\alpha) \in R^+$ .*

Let  $\lambda \in \underline{\mathfrak{s}}$ . Let  $R_\lambda, R_\lambda^+, \mathbf{W}_\lambda, H_\lambda$  be as in 34.2. We write  $\vee_\lambda$  instead of  $\vee_\lambda^D$  (as in 34.4 with  $D = G^0$ ). We show that

(b) *if  $w \in \mathbf{W}$ , then  $w\mathbf{W}_\lambda$  contains a unique element  $w_1$  of minimal length; it is characterized by the property  $w_1(R_\lambda^+) \subset R^+$ .*

Let  $w_1$  be an element of minimal length in  $w\mathbf{W}_\lambda$ . Let  $\alpha \in R_\lambda^+$ . Then  $l(w_1 s_\alpha) \geq l(w_1)$ . Since  $l(w_1 s_\alpha) = l(w_1) + 1 \pmod{2}$  we see that  $l(w_1 s_\alpha) > l(w_1)$ . By (a) we have  $w_1(\alpha) \in R^+$ . Thus,  $w_1(R_\lambda^+) \subset R^+$ . Now let  $u \in \mathbf{W}_\lambda - \{1\}$ . We pick  $\alpha \in R_\lambda^+$  such that  $u(\alpha)^{-1} \in R_\lambda^+$ ; then  $w_1 u(\alpha)^{-1} \in R^+$ . If  $w_1 u$  has minimal length in  $w\mathbf{W}_\lambda$ ,

then by an earlier part of the argument applied to  $w_1u$  instead of  $w_1$  we have  $w_1u(\alpha) \in R^+$ , a contradiction. We see that  $w_1$  is the unique element of minimal length in  $w\mathbf{W}_\lambda$ . It remains to show that if  $u \in \mathbf{W}_\lambda$  satisfies  $w_1u(R_\lambda^+) \subset R^+$ , then  $u = 1$ . If  $u \neq 1$ , then by an earlier part of the argument we have  $w_1u(\alpha)^{-1} \in R^+$  for some  $\alpha \in R_\lambda^+$ , a contradiction. This proves (b).

We show that

(c) if  $s \in \mathbf{I}$  and  $w \in \mathbf{W}$  has minimal length in  $w\mathbf{W}_\lambda$ , then either (i)  $sw$  has minimal length in  $sw\mathbf{W}_\lambda$  or (ii)  $w^{-1}sw \in \mathbf{W}_\lambda$ .

There is a unique  $\beta \in R^+$  such that  $s(\beta)^{-1} \in R^+$ . Assume that (i) does not hold. By (b) there exists  $\alpha \in R_\lambda^+$  such that  $sw(\alpha)^{-1} \in R^+$ ; moreover,  $w(\alpha) \in R^+$ . Hence  $w(\alpha) = \beta$ . We have  $w^{-1}(\beta) = \alpha \in R_\lambda$ , hence  $w^{-1}sw \in \mathbf{W}_\lambda$  and (ii) holds. This proves (c).

For  $z \in \mathbf{W}_\lambda$  let  $\tilde{T}_z^\lambda, c_z^\lambda \in H_\lambda$  be as in 34.2. Then  $c_z^\lambda = \sum_{z' \in \mathbf{W}_\lambda} p_{z',z}^\lambda \tilde{T}_{z'}^\lambda$  where  $p_{z',z}^\lambda \in \mathbf{Z}[v^{-1}]$  are uniquely defined.

For any  $w \in \mathbf{W}$ ,  $\lambda \in \underline{\mathfrak{g}}$  there is a unique element of  $H$  which is equal to  $c_{w,\lambda} \in H_n$  (see 34.4) for any  $n$  such that  $\lambda \in \underline{\mathfrak{g}}_n$ ; we denote this element again by  $c_{w,\lambda}$ . We have  $c_{w,\lambda} = \sum_{w' \in \mathbf{W}} \pi_{w',w,\lambda} \tilde{T}_{w'}^\lambda$  where  $\pi_{w',w,\lambda} \in \mathbf{Z}[v^{-1}]$  are uniquely defined. Note that  $\{c_{w,\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{g}}\}$  is an  $\mathcal{A}$ -basis of  $H$ .

Now:

(d) Let  $w, w' \in \mathbf{W}$ . We write  $w = w_1z, w' = w'_1z'$  where  $w_1$  has minimal length in  $w\mathbf{W}_\lambda$ ,  $w'_1$  has minimal length in  $w'\mathbf{W}_\lambda$  and  $z, z' \in \mathbf{W}_\lambda$ . If  $w_1 \neq w'_1$ , then  $\pi_{w',w,\lambda} = 0$ . If  $w_1 = w'_1$ , then  $\pi_{w',w,\lambda} = p_{z',z}^\lambda$ .

From the definitions we see that if  $w\lambda \neq w'\lambda$ , then  $\pi_{w',w,\lambda} = 0$ . Thus we may assume that  $w\lambda = w'\lambda$ . We choose a sequence  $s_1, s_2, \dots, s_r$  in  $\mathbf{I}$  such that  $w\lambda = w'\lambda = s_r s_{r-1} \dots s_1 \lambda \neq s_{r-1} \dots s_1 \lambda \neq \dots \neq s_1 \lambda \neq \lambda$ .

We show that for  $k \in [0, r]$ ,  $s_k s_{k-1} \dots s_1$  has minimal length in  $s_k s_{k-1} \dots s_1 \mathbf{W}_\lambda$ . We argue by induction. For  $k = 0$  the result is obvious. Assume now that  $k \in [1, r]$ . Since  $s_{k-1} \dots s_1$  has minimal length in  $s_{k-1} \dots s_1 \mathbf{W}_\lambda$  and  $s_k s_{k-1} \dots s_1 \lambda \neq s_{k-1} \dots s_1 \lambda$ , we see from (c) that  $s_k s_{k-1} \dots s_1$  has minimal length in  $s_k s_{k-1} \dots s_1 \mathbf{W}_\lambda$  as required.

In particular,  $s_r s_{r-1} \dots s_1$  has minimal length in  $s_r s_{r-1} \dots s_1 \mathbf{W}_\lambda$ . Since  $w\lambda = s_r s_{r-1} \dots s_1 \lambda$  we have  $w = s_r s_{r-1} \dots s_1 h_1 h_2$  where  $h_1 \in \vee_\lambda, h_2 \in \mathbf{W}_\lambda$ . Then both  $w_1$  and  $s_r s_{r-1} \dots s_1 h_1$  have minimal length in  $s_r s_{r-1} \dots s_1 h_1 \mathbf{W}_\lambda = w\mathbf{W}_\lambda = w_1 \mathbf{W}_\lambda$ ; using (b) we deduce that  $s_r s_{r-1} \dots s_1 h_1 = w_1$ . Hence  $s_1 \dots s_r w = s_1 \dots s_r w_1 z = h_1 z$ . Similarly,  $s_1 \dots s_r w' = h'_1 z'$  where  $h'_1 \in \vee_\lambda$ .

From the results in 34.7–34.10 we see that  $\pi_{w',w,\lambda} = p_{s_1 \dots s_r w', s_1 \dots s_r w}^\lambda = p_{h'_1 z', h_1 z}^\lambda$ . Using  $h_1, h'_1 \in \vee_\lambda$  and the definitions (34.2) we see that  $p_{h'_1 z', h_1 z}^\lambda = 0$  if  $h_1 \neq h'_1$  and  $p_{h'_1 z', h_1 z}^\lambda = p_{z', z}^\lambda$  if  $h_1 = h'_1$ .

It remains to show that we have  $w_1 = w'_1$  if and only if  $h_1 = h'_1$ . We have  $s_r s_{r-1} \dots s_1 = h_1^{-1} w_1$  and similarly  $s_r s_{r-1} \dots s_1 = (h'_1)^{-1} w'_1$ . Hence  $h_1^{-1} w_1 = (h'_1)^{-1} w'_1$ . We see that  $w_1 = w'_1$  if and only if  $h_1 = h'_1$ . This proves (d).

For  $w' \leq w$  in  $\mathbf{W}$ ,  $\lambda \in \underline{\mathfrak{g}}$  and  $i \in \mathbf{Z}$  we define  $N_{i,w',w,\lambda} \in \mathbf{Z}$  by

(e)  $\pi_{w',w,\lambda} = v^{l(w')-l(w)} \sum_{i \in \mathbf{Z}} N_{i,w',w,\lambda} v^i$ , that is,

$$p_{z',z}^\lambda = v^{l(w')-l(w)} \sum_{i \in \mathbf{Z}} N_{i,w',w,\lambda} v^i$$

if  $w'\mathbf{W}_\lambda = w\mathbf{W}_\lambda$  and  $z, z'$  are as in (d), then

$$N_{i,w',w,\lambda} = 0 \text{ if } w'\mathbf{W}_\lambda \neq w\mathbf{W}_\lambda.$$

Note that  $N_{i,w',w,\lambda}$  is 0 unless  $i$  is even.

**40.3.** Let  $B^* \in \mathcal{B}$ . Let  $U^* = U_{B^*}$  and let  $T$  be a maximal torus of  $B^*$ . Let  $\mathfrak{r} = \dim \mathbf{T}$ . Let  $W_T = N_{G^0}T/T$ . We identify  $T = \mathbf{T}, W_T = \mathbf{W}$  as in 28.5. For any  $w \in \mathbf{W}$  we denote by  $\dot{w}$  a representative of  $w$  in  $N_{G^0}T$ .

Let  $C = G^0/U^* \times G^0/U^*$ . We have a partition  $C = \bigcup_{w \in \mathbf{W}} C_w$  where

$$C_w = \{(hU^*, h'U^*) \in C; h^{-1}h' \in B^*\dot{w}B^*\}.$$

For  $w \in \mathbf{W}$  let  $d_w = \dim C_w$  and let

$$\bar{C}_w = \{(hU^*, h'U^*) \in C; h^{-1}h' \in \overline{B^*\dot{w}B^*}\}$$

(closure in  $G^0$ ). Now  $\bar{C}_w$  is an irreducible variety and we have a partition  $\bar{C}_w = \bigcup_{w'; w' \leq w} C_{w'}$  with  $C_w$  smooth, open dense in  $\bar{C}_w$ .

Define  $\gamma_{\dot{w}} : B^*\dot{w}B^* \rightarrow T$  by  $\gamma_{\dot{w}}(g) = t$  where  $g \in U^*\dot{w}tU^*$  with  $t \in T$ . Define  $\psi : C_w \rightarrow T$  by  $\psi(hU^*, h'U^*) = \gamma_{\dot{w}}(h^{-1}h')$ .

For  $\mathcal{L} \in \mathfrak{s}$  we set  $\mathcal{L}_w = \psi^*\mathcal{L}$ , a local system on  $C_w$ . (Using 28.1(c) we see that the isomorphism class of  $\psi^*\mathcal{L}$  is independent of the choice of  $\dot{w}$ .) Let  $\mathcal{L}_w^\sharp = IC(\bar{C}_w, \mathcal{L}_w) \in \mathcal{D}(\bar{C}_w)$ .

**40.4.** For  $w \in \mathbf{W}, \mathcal{L} \in \mathfrak{s}$  let  $\underline{\mathcal{L}}_w = j_{w!}\mathcal{L}_w, \underline{\mathcal{L}}_w^\sharp = \bar{j}_{w!}\mathcal{L}_w^\sharp$  where  $j_w : C_w \rightarrow C, \bar{j}_w : \bar{C}_w \rightarrow C$  are the inclusions. Let  $\hat{C}$  be the full subcategory of  $\mathcal{D}(C)$  whose objects are the simple perverse sheaves on  $C$  which are equivariant for the  $G^0 \times T \times T$  action

$$(a) (x, t, t') : (hU^*, h'U^*) \mapsto (xht^nU^*, xh't'^nU^*)$$

on  $C$  (for some  $n \in \mathbf{N}_k^*$ ) or equivalently, are isomorphic to  $\underline{\mathcal{L}}_w^\sharp[d_w]$  for some  $\mathcal{L} \in \mathfrak{s}$  and some  $w \in \mathbf{W}$ . Let  $\mathcal{D}^{cs}(C)$  be the subcategory of  $\mathcal{D}(C)$  whose objects are those  $K \in \mathcal{D}(C)$  such that for any  $j$ , any simple subquotient of  ${}^pH^jK$  is in  $\hat{C}$ .

If  $w, \mathcal{L}$  are as above, then  $\underline{\mathcal{L}}_w \in \mathcal{D}^{cs}(C)$ . Indeed this constructible sheaf is equivariant for the action (a) (for some  $n$ ), hence so is each  ${}^pH^j(\underline{\mathcal{L}}_w)$ .

We have a diagram  $C \times C \xleftarrow{r} (G^0/U^*)^3 \xrightarrow{s} C$  where

$$\begin{aligned} r(h_1U^*, h_2U^*, h_3U^*) &= ((h_1U^*, h_2U^*), (h_2U^*, h_3U^*)), \\ s(h_1U^*, h_2U^*, h_3U^*) &= (h_1U^*, h_3U^*). \end{aligned}$$

We define a bi-functor  $\mathcal{D}(C) \times \mathcal{D}(C) \rightarrow \mathcal{D}(C)$  by  $A, A' \mapsto A * A' = s_!r^*(A \boxtimes A')$ . The “product”  $A * A'$  is associative in an obvious sense. We show that

(b)  $A, A' \mapsto A * A'$  restricts to a bi-functor  $\mathcal{D}^{cs}(C) \times \mathcal{D}^{cs}(C) \rightarrow \mathcal{D}^{cs}(C)$ .

Let  $A, A' \in \mathcal{D}^{cs}(C)$ . To show that  $A * A' \in \mathcal{D}^{cs}(C)$  we may assume that  $A, A' \in \hat{C}$ . Then each  ${}^pH^j(A * A')$  is equivariant for the action (a) (for some  $n$ ). This proves (b).

**40.5.** For  $w' \leq w$  in  $\mathbf{W}, \lambda \in \underline{\mathfrak{s}}, \mathcal{L} \in \lambda$  and  $i \in \mathbf{Z}$  we show that

$$(a) \mathcal{H}^i(\mathcal{L}_w^\sharp)|_{C_{w'}} \cong (\mathcal{L}_{w'}(-i/2))^{\oplus N_{i,w',w,\lambda}}.$$

(Both sides are 0 unless  $i$  is even.)

Let

$$\begin{aligned}\tilde{C}_w &= \{(h, h') \in G^0 \times G^0; h^{-1}h' \in B^* \dot{w} B^*\} \times \mathbf{k}^*, \\ \bar{C}_w &= \{(h, h') \in G^0 \times G^0; h^{-1}h' \in \overline{B^* \dot{w} B^*}\} \times \mathbf{k}^*.\end{aligned}$$

Now  $\bar{C}_w$  is an irreducible variety and we have a partition  $\bar{C}_w = \bigcup_{w'; w' \leq w} \bar{C}_{w'}$  with  $\tilde{C}_w$  smooth, open dense in  $\bar{C}_w$ . Define  $\bar{d}: \bar{C}_w \rightarrow \bar{C}_w$ ,  $d: \tilde{C}_w \rightarrow \bar{C}_w$  by  $(h, h', z) \mapsto (hU^*, h'U^*)$ . Let  $\tilde{\mathcal{L}}_w = d^* \mathcal{L}_w$ , a local system on  $\tilde{C}_w$ . Let  $\tilde{\mathcal{L}}_w^\sharp = IC(\tilde{C}_w, \tilde{\mathcal{L}}_w)$ . Since  $d, \bar{d}$  are principal  $U^* \times \mathbf{k}^*$ -bundles, it is enough to show that

$$(b) \mathcal{H}^i(\tilde{\mathcal{L}}_w^\sharp)|_{\tilde{C}_{w'}} \cong (\tilde{\mathcal{L}}_{w'}(-i/2))^{\oplus N_{i, w', w, \lambda}}.$$

(Both sides are 0 unless  $i$  is even.)

We choose  $\kappa \in \text{Hom}(T, \mathbf{k}^*)$ ,  $\mathcal{E} \in \mathfrak{s}(\mathbf{k}^*)$  such that  $\mathcal{L} \cong \kappa^* \mathcal{E}$ ; see 28.1(c).

Now  $B^*$  acts on  $(B^* \dot{w} B^*) \times \mathbf{k}^*$  and on  $(\overline{B^* \dot{w} B^*}) \times \mathbf{k}^*$  by

$$t_1 u : (g, z) \mapsto (g(t_1 u)^{-1}, \kappa(t_1) z)$$

where  $t_1 \in T$ ,  $u \in U^*$ . Let  $\bar{\mathbf{P}}_w^\kappa = ((\overline{B^* \dot{w} B^*}) \times \mathbf{k}^*)/B^*$ ,  $PP_w^\kappa = ((B^* \dot{w} B^*) \times \mathbf{k}^*)/B^*$ . Now  $\mathbf{P}_w^\kappa$  is a smooth open dense subvariety of the irreducible variety  $\bar{\mathbf{P}}_w^\kappa$  and  $\bar{\mathbf{P}}_w^\kappa = \bigcup_{w'; w' \leq w} \mathbf{P}_{w'}^\kappa$  is a partition. The morphism  $(B^* \dot{w} B^*) \times \mathbf{k}^* \rightarrow \mathbf{k}^*$  given by  $(g, z) \mapsto \kappa(\gamma_{\dot{w}}(g))z$  factors through a morphism  $\phi: \mathbf{P}_w^\kappa \rightarrow \mathbf{k}^*$ . Let  $\mathcal{E}_w^\kappa = \phi^* \mathcal{E}$ , a local system of rank 1 on  $\mathbf{P}_w^\kappa$ . Let  $\mathcal{E}_w^{\kappa^\sharp} = IC(\bar{\mathbf{P}}_w^\kappa, \mathcal{E}_w^\kappa) \in \mathcal{D}(\bar{\mathbf{P}}_w^\kappa)$ . From [L14, 1.24] we see that

$$(c) \mathcal{H}^i(\mathcal{E}_w^{\kappa^\sharp})|_{\mathbf{P}_{w'}^\kappa} \cong (\mathcal{E}_{w'}^\kappa(-i/2))^{\oplus N_{i, w', w, \lambda}}.$$

(Both sides are 0 unless  $i$  is even.)

We can find  $n \in \mathbf{N}_{\mathbf{k}^*}$  such that  $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$ . Define  $\bar{c}: \bar{C}_w \rightarrow \bar{\mathbf{P}}_w$ ,  $c: \tilde{C}_w \rightarrow \bar{\mathbf{P}}_w$  by  $(h, h', z) \mapsto B^*$ -orbit of  $(h^{-1}h', z^n)$ . Now  $\bar{c}, c$  are locally trivial fibrations with smooth fibres of pure dimension. Hence (b) follows from (c) provided that we can show that  $c^* \mathcal{E}_{w'}^\kappa = \tilde{\mathcal{L}}_{w'}$  for  $w' \leq w$ . We may assume that  $w = w'$ . We have a commutative diagram

$$\begin{array}{ccccc} \mathbf{P}_w^\kappa & \xleftarrow{c} & \tilde{C}_w \times \mathbf{k}^* & \xrightarrow{d} & C_w \\ \phi \downarrow & & \phi' \downarrow & & \kappa \psi \downarrow \\ \mathbf{k}^* & \xleftarrow{c'} & \mathbf{k}^* \times \mathbf{k}^* & \xrightarrow{d'} & \mathbf{k}^* \end{array}$$

with  $\phi, \psi, c, d$  as above,  $\phi'(h, h', z) = (\kappa(\gamma_{\dot{w}}(h^{-1}h')), z)$ ,  $c'(z', z) = z'z^n$ ,  $d'(z', z) = z'$ . Using this and the definitions we have  $\tilde{\mathcal{L}}_w = \phi'^* d'^* \mathcal{E}$ ,  $c^* \mathcal{E}_w = \phi'^* c'^* \mathcal{E}$ . It remains to show that  $d'^* \mathcal{E} = c'^* \mathcal{E}$ . This expresses the fact that  $\mathcal{E}$  is equivariant for the  $\mathbf{k}^*$ -action  $z_1: z \mapsto z_1^n z$  on  $\mathbf{k}^*$  which follows from  $\mathcal{E} \in \mathfrak{s}_n(\mathbf{k}^*)$ . This proves (b), hence (a).

**40.6.** Let  $w, w' \in \mathbf{W}$ ,  $\mathcal{L}, \mathcal{L}' \in \mathfrak{s}$ . We set  $L = \underline{\mathcal{L}}_w * \underline{\mathcal{L}}_{w'} \in \mathcal{D}^{cs}(C)$ . Let

$$\begin{aligned}X &= \{(h_1 U^*, h_2 U^*, h_3 U^*) \in (G^0/U^*)^3; h_1^{-1} h_2 \in B^* \dot{w} B^*, h_2^{-1} h_3 \in B^* \dot{w}' B^*\}, \\ \bar{X} &= \{(h_1 U^*, h_2 B^*, h_3 U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*; \\ &\quad h_1^{-1} h_2 \in B^* \dot{w} B^*, h_2^{-1} h_3 \in B^* \dot{w}' B^*\}.\end{aligned}$$

We have a commutative diagram with a cartesian square

$$\begin{array}{ccccc} X & \xrightarrow{f} & \bar{X} & \xrightarrow{\bar{\sigma}} & C \\ \tau \downarrow & & \bar{\tau} \downarrow & & \\ T \times T & \xrightarrow{f'} & T & & \end{array}$$

where  $f$  is given by  $(h_1U^*, h_2U^*, h_3U^*) \mapsto (h_1U^*, h_2B^*, h_3U^*)$ ,

$f'$  is  $(t, t') \mapsto \text{Ad}(\dot{w}')^{-1}(t)t'$ ,

$\tau$  is  $(h_1U^*, h_2U^*, h_3U^*) \mapsto (t, t')$  with  $h_1^{-1}h_2 \in U^*\dot{w}tU^*$ ,  $h_2^{-1}h_3 \in U^*\dot{w}'t'U^*$ ,

$\bar{\tau}$  is  $(h_1U^*, h_2B^*, h_3U^*) \mapsto \text{Ad}(\dot{w}')^{-1}(t)t'$  with  $t, t'$  as in the definition of  $\tau$ ,

$\bar{\sigma}$  is  $(h_1U^*, h_2B^*, h_3U^*) \mapsto (h_1U^*, h_3U^*)$ .

From the definitions we have  $L = \bar{\sigma}_! f_! \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$ . Using the diagram above, we have  $L = \bar{\sigma}_! \bar{\tau}^* f'_!(\mathcal{L} \boxtimes \mathcal{L}')$ . From the definitions we see that either (i) or (ii) below holds:

- (i)  $\mathcal{L} \not\cong (\text{Ad}(\dot{w}')^{-1})^* \mathcal{L}'$  and  $f'_!(\mathcal{L} \boxtimes \mathcal{L}') = 0$ ;
- (ii)  $\mathcal{L} \cong (\text{Ad}(\dot{w}')^{-1})^* \mathcal{L}'$  and  $\mathcal{L} \boxtimes \mathcal{L}' = f'^* \mathcal{L}'$ .

If (i) holds, then  $K = 0$ . If (ii) holds, then, as in 32.16, we have

$$\begin{aligned} f'_!(\mathcal{L} \boxtimes \mathcal{L}') &= f'_! f'^* \mathcal{L}' = \mathcal{L}' \otimes f'_! \bar{\mathbf{Q}}_l \simeq \{\mathcal{L}' \otimes \mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\}, \\ \mathcal{L}' \otimes \mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l)[-e] &\simeq \{\mathcal{L}'(\mathbf{r} - e), \dots, \mathcal{L}'(\mathbf{r} - e), \left(\binom{\mathbf{r}}{2\mathbf{r} - e}\right) \text{ copies}\}. \end{aligned}$$

Setting  $\bar{L} = \bar{\sigma}_! \bar{\tau}^*(\mathcal{L}')$ , it follows that

$$L \simeq \{\bar{L}(\mathbf{r} - e)[-e], \dots, \bar{L}(\mathbf{r} - e)[-e], \left(\binom{\mathbf{r}}{2\mathbf{r} - e}\right) \text{ copies}, e \in \mathbf{Z}\}.$$

We now consider  $\bar{L}$  for certain choices of  $w, w'$ .

If  $w, w'$  satisfy  $l(ww') = l(w) + l(w')$ , then  $\bar{\sigma}$  restricts to an isomorphism  $\bar{X} \rightarrow C_{ww'}$  and  $\bar{L} = \underline{\mathcal{L}}'_{ww'}$ .

Now assume that  $\alpha, \check{\alpha}, s_\alpha$  are as in 28.3 and that  $w = w' = s_\alpha \in \mathbf{I}$ . We have

$$\bar{L} \simeq \{j_{u!} \bar{L}_u; u \in \mathbf{W}\}$$

where  $j_u : C_u \rightarrow C$  is the inclusion and  $\bar{L}_u = j_u^* \bar{L}$ . Let  $\bar{X}_u = \bar{\sigma}^{-1}(C_u)$ . Then  $\bar{L}_u = \bar{\sigma}_{u!} \bar{\tau}_u^*(\mathcal{L}')$  where  $\bar{\sigma}_u : \bar{X}_u \rightarrow C_u$ ,  $\bar{\tau}_u : \bar{X}_u \rightarrow T$  are the restrictions of  $\bar{\sigma}, \bar{\tau}$ .

If  $u \notin \{1, s_\alpha\}$ , then  $\bar{X}_u = \emptyset$  and  $\bar{L}_u = 0$ . If  $u = 1$ , then  $\bar{\sigma}_u : \bar{X}_u \rightarrow C_u$  is an affine line bundle and  $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^* \mathcal{L}'_u$ ; hence  $\bar{\sigma}_{u!} \bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_{u!} \bar{\sigma}_u^* \mathcal{L}'_u = \mathcal{L}'_u[[-1]]$ . If  $u = s_\alpha$ , then  $\bar{\sigma}_u : \bar{X}_u \rightarrow C_u$  is a principal  $\mathbf{k}^*$ -bundle and either (iii) or (iv) below holds:

- (iii)  $\check{\alpha}^* \mathcal{L}' \not\cong \bar{\mathbf{Q}}_l$  and  $\bar{\sigma}_{u!} \bar{\tau}_u^*(\mathcal{L}') = 0$ ,
- (iv)  $\check{\alpha}^* \mathcal{L}' \cong \bar{\mathbf{Q}}_l$  and  $\bar{\tau}_u^*(\mathcal{L}') = \bar{\sigma}_u^* \mathcal{L}'_u$ .

If (iv) holds, then, as in case (ii) above, we have

$$\begin{aligned} \bar{\sigma}_{u!} \bar{\tau}_u^*(\mathcal{L}') &= \bar{\sigma}_{u!} \bar{\sigma}_u^* \mathcal{L}'_u = \mathcal{L}'_u \otimes \bar{\sigma}_{u!} \bar{\mathbf{Q}}_l \simeq \{\mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!} \bar{\mathbf{Q}}_l)[-e], e \in \mathbf{Z}\}, \\ \mathcal{L}'_u \otimes \mathcal{H}^e(\bar{\sigma}_{u!} \bar{\mathbf{Q}}_l)[-e] &\simeq \{\mathcal{L}'_u(1 - e), \dots, \mathcal{L}'_u(1 - e), \left(\binom{1}{2 - e}\right) \text{ copies}\}. \end{aligned}$$

**40.7.** In this subsection we assume that  $\mathbf{k}$  is an algebraic closure of a finite field. Now the  $\mathcal{A}$ -module  $\mathfrak{K}(C)$  is defined as in 36.8 (the character sheaves on  $C$  are taken to be the objects in  $\hat{C}$ ).

For  $(w, \lambda) \in \mathbf{W} \times \underline{\mathfrak{s}}$ , let  $[w; \lambda]$  be the basis element of  $\mathfrak{K}(C)$  given by  $\underline{\mathcal{L}}_w^\sharp[[d_w/2]]$ ; we choose  $\mathcal{L} \in \lambda$  and we regard  $\underline{\mathcal{L}}_w, \underline{\mathcal{L}}_w^\sharp$  as mixed complexes on  $C$  whose restriction to  $C_w$  is pure of weight 0; then  $gr(\underline{\mathcal{L}}_w), gr(\underline{\mathcal{L}}_w^\sharp)$  are defined in  $\mathfrak{K}(C)$  as in 36.8. We denote these elements of  $\mathfrak{K}(C)$  by  $[w; \lambda]', [w; \lambda]^\sharp$  respectively. From 40.5(a) we see that

$$(a) \quad (-v)^{d_w} [w; \lambda] = [w; \lambda]^\sharp = \sum_{w' \in \mathbf{W}} \sum_{i \in 2\mathbf{Z}} N_{i, w', w, \lambda} v^i [w'; \lambda]' \text{ in } \mathfrak{K}(C)$$

where  $N_{i, w', w, \lambda}$  is as in 40.2(e).

Let  $r, s$  be as in 40.4. By 40.4(b),  $s!r^* : \mathcal{D}(C \times C) \rightarrow \mathcal{D}(C)$  restricts to a functor  $\mathcal{D}^{cs}(C \times C) \rightarrow \mathcal{D}^{cs}(C)$  where the character sheaves on  $C \times C$  are by definition complexes of the form  $A \boxtimes A'$  with  $A \in \hat{C}, A' \in \hat{C}$ . Hence the  $\mathcal{A}$ -linear map  $gr(s!r^*) : \mathfrak{K}(C \times C) \rightarrow \mathfrak{K}(C)$  or equivalently  $\mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C) \rightarrow \mathfrak{K}(C)$  is well defined. (We have canonically  $\mathfrak{K}(C \times C) = \mathfrak{K}(C) \otimes_{\mathcal{A}} \mathfrak{K}(C)$ .) We write  $\xi * \xi'$  instead of  $gr(s!r^*)(\xi \boxtimes \xi')$  where  $\xi, \xi' \in \mathfrak{K}(C)$ . Note that  $\xi, \xi' \mapsto \xi * \xi'$  defines an associative  $\mathcal{A}$ -algebra structure on  $\mathfrak{K}(C)$ .

Let  $w, w' \in \mathbf{W}, \lambda, \lambda' \in \underline{\mathfrak{s}}$ . From 40.6 we see that

if  $w' \lambda' \neq \lambda$ , then  $[w; \lambda]' * [w'; \lambda']' = 0$  in  $\mathfrak{K}(C)$ ;

if  $w' \lambda' = \lambda$  and  $l(ww') = l(w) + l(w')$ , then  $[w; \lambda]' * [w'; \lambda']' = (v^2 - 1)^r [ww'; \lambda]'$  in  $\mathfrak{K}(C)$ ;

if  $s \in \mathbf{I}$  and  $s\lambda' = \lambda$ , then  $[s; \lambda]' * [s; \lambda']' = (v^2 - 1)^r (v^2 [1; \lambda']' + (v^2 - 1)c[s; \lambda']')$

where  $c = 1$  for  $s \in \mathbf{W}_{\lambda'}$  and  $c = 0$  for  $s \notin \mathbf{W}_{\lambda'}$ .

Using this and (a), 40.1(a), 40.2(e), we see that

(b) the unique  $\mathcal{A}$ -linear isomorphism  $\omega : \mathfrak{K}(C) \rightarrow H$  ( $H$  as in 40.1) given by  $[w; \lambda]' \mapsto v^{l(w)} \tilde{T}_w 1_\lambda$  for  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$ , satisfies  $\omega([w; \lambda]) = (-v)^{-d_w} v^{l(w)} c_{w, \lambda}$  for  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$  and  $\omega(x * x') = (v^2 - 1)^r \omega(x) \omega(x')$  for any  $x, x' \in \mathfrak{K}(C)$ .

**40.8.** For  $w, w' \in \mathbf{W}$  and  $\lambda, \lambda' \in \underline{\mathfrak{s}}$  we have  $c_{w, \lambda} c_{w', \lambda'} = \sum_{y \in \mathbf{W}, \nu \in \underline{\mathfrak{s}}} \gamma_{y, \nu}^{w, \lambda; w', \lambda'} c_{y, \nu}$  in the algebra  $H$ . Here  $\gamma_{y, \nu}^{w, \lambda; w', \lambda'} \in \mathcal{A}$ . We have:

$$(a) \quad \gamma_{y, \nu}^{w, \lambda; w', \lambda'} \in \mathbf{N}[v, v^{-1}].$$

By the arguments in 34.4–34.10 (with  $D = G^0$ ) this is reduced to the analogous (well-known) statement for the structure constants of the algebra  $H_\lambda^D$  with its basis  $(c_w^\lambda)$  (see 34.2).

**40.9.** For any  $J \subset \mathbf{I}$  let  $H_J$  be the  $\mathcal{A}$ -submodule of  $H$  spanned by  $\{c_{w, \lambda}; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$  or equivalently by  $\{\tilde{T}_w 1_\lambda; w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}\}$ . From the definitions we see that  $H_J$  is a subalgebra of  $H$ . For any  $J \subset \mathbf{I}, J' \subset \mathbf{I}$  we define a relation  $\preceq_{J, J'}$  on  $\mathbf{W} \times \underline{\mathfrak{s}}$  as follows. We say that  $(y, \nu) \preceq_{J, J'} (w, \lambda)$  if there exist  $w_1 \in \mathbf{W}_J, w_2 \in \mathbf{W}_{J'}, \lambda_1, \lambda_2 \in \underline{\mathfrak{s}}$  such that in the expansion (in the algebra  $H$ )

$$c_{w_1, \lambda_1} c_{w, \lambda} c_{w_2, \lambda_2} = \sum_{y' \in \mathbf{W}, \nu' \in \underline{\mathfrak{s}}} a_{y', \nu'} c_{y', \nu'}$$

(with  $a_{y', \nu'} \in \mathcal{A}$ ) we have  $a_{y, \nu} \neq 0$ .

Using the associativity of the product in  $H$ , the fact that  $H_J, H_{J'}$  are subalgebras of  $H$  and 40.8(a), we see that  $\preceq_{J, J'}$  is transitive. Using the formula  $c_{1, w\lambda} c_{w, \lambda} c_{1, \lambda} = c_{w, \lambda}$  we see that it is reflexive. Thus, it is a preorder. Let  $\sim_{J, J'}$  be the equivalence relation attached to  $\preceq_{J, J'}$ ; thus,  $(y, \nu) \sim_{J, J'} (w, \lambda)$  if

$(y, \nu) \preceq_{J, J'} (w, \lambda)$  and  $(w, \lambda) \preceq_{J, J'} (y, \nu)$ . The equivalence classes for  $\sim_{J, J'}$  are called  $(J, J')$ -two-sided cells. The  $(\mathbf{I}, \mathbf{I})$ -two-sided cells in  $\mathbf{W} \times \underline{\mathfrak{s}}$  are also called two-sided cells.

**40.10.** Let  $w, w', w'' \in \mathbf{W}$ ,  $\mathcal{L}, \mathcal{L}', \mathcal{L}'' \in \mathfrak{s}$ . We set  $K = \underline{\mathcal{L}}_w * \underline{\mathcal{L}}_{w'}^\sharp * \underline{\mathcal{L}}_{w''} \in \mathcal{D}^{cs}(C)$ . Let

$$X = \{(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \in (G^0/U^*)^4; \\ h_1^{-1}h_2 \in B^*\dot{w}B^*, h_2^{-1}h_3 \in \overline{B^*\dot{w}'B^*}, h_3^{-1}h_4 \in B^*\dot{w}''B^*\},$$

an irreducible variety. Let  $X_0$  be the smooth open dense subset of  $X$  defined by the condition  $h_2^{-1}h_3 \in B^*\dot{w}'B^*$ . Define  $\sigma : X \rightarrow C$  by

$$(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (h_1U^*, h_4U^*).$$

Define  $\tau : X_0 \rightarrow T \times T \times T$  by

$$(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (t, t', t'')$$

with  $h_1^{-1}h_2 \in U^*\dot{w}tU^*$ ,  $h_2^{-1}h_3 \in U^*\dot{w}'t'U^*$ ,  $h_3^{-1}h_4 \in U^*\dot{w}''t''U^*$ .

Let  $\mathcal{F} = \tau^*(\mathcal{L} \boxtimes \mathcal{L}' \boxtimes \mathcal{L}'')$ , a local system on  $X_0$ . Then  $\mathcal{F}^\sharp := IC(X, \mathcal{F}) \in \mathcal{D}(X)$  is defined and we have  $K = \sigma_*\mathcal{F}^\sharp$ .

Let  $\bar{X}$  (resp.  $\bar{X}_0$ ) be the the variety of all  $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in G^0/U^* \times G^0/B^* \times G^0/B^* \times G^0/U^*$  that satisfy the same equations as those defining  $X$  (resp.  $X_0$ ). Note that  $\bar{X}$  is irreducible and  $\bar{X}_0$  is an open dense smooth subset of  $\bar{X}$ . We have a cartesian diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & \bar{X} & \xrightarrow{\bar{\sigma}} & C \\ \uparrow & & \uparrow & & \\ X_0 & \xrightarrow{f_0} & \bar{X}_0 & & \\ \tau \downarrow & & \bar{\tau} \downarrow & & \\ T \times T \times T & \xrightarrow{f'} & T & & \end{array}$$

where  $X_0 \rightarrow X$ ,  $\bar{X}_0 \rightarrow \bar{X}$  are the obvious imbeddings,

$f, f_0$  are given by  $(h_1U^*, h_2U^*, h_3U^*, h_4U^*) \mapsto (h_1U^*, h_2B^*, h_3B^*, h_4U^*)$ ,

$f'$  is  $(t, t', t'') \mapsto \text{Ad}(\dot{w}'\dot{w}'')^{-1}(t)\text{Ad}(\dot{w}'')^{-1}(t')t''$ ,

$\bar{\tau}$  is  $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto \text{Ad}(\dot{w}'\dot{w}'')^{-1}(t)\text{Ad}(\dot{w}'')^{-1}(t')t''$  with  $t, t', t''$  as in the definition of  $\tau$ ,  $\bar{\sigma}$  is  $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$ . Assume that  $\mathcal{L} \cong (\text{Ad}(\dot{w}')^{-1})^*\mathcal{L}'$  and  $\mathcal{L}' \cong (\text{Ad}(\dot{w}'')^{-1})^*\mathcal{L}''$ . Then  $\mathcal{L} \boxtimes \mathcal{L}' \boxtimes \mathcal{L}'' = f'^*\mathcal{L}''$ . We have  $\mathcal{F} = \tau^*f'^*\mathcal{L}'' = f_0^*\bar{\tau}^*\mathcal{L}''$ . Since  $f$  is a principal  $T \times T$ -bundle and  $X_0 = f^{-1}(\bar{X}_0)$  it follows that  $\mathcal{F}^\sharp = f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'')$ . Note that  $f_!\bar{\mathbf{Q}}_l \simeq \{\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l)[-e], 2\mathbf{r} \leq e \leq 4\mathbf{r}\}$ ,

$$\mathcal{H}^e(f_!\bar{\mathbf{Q}}_l) \simeq \{\bar{\mathbf{Q}}_l(2\mathbf{r} - e), \dots, \bar{\mathbf{Q}}_l(2\mathbf{r} - e), \left(\binom{2\mathbf{r}}{4\mathbf{r} - e} \text{ copies}\right)\}.$$

Hence setting  $\bar{K} = \bar{\sigma}_!(IC(\bar{X}, \bar{\tau}^*\mathcal{L}''))$  we have

$$K = \sigma_*f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') = \bar{\sigma}_!f_!f^*IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') = \bar{\sigma}_!(IC(\bar{X}, \bar{\tau}^*\mathcal{L}'') \otimes f_!\bar{\mathbf{Q}}_l),$$

$$(a) \quad K \simeq \{\bar{K}(2\mathbf{r} - e)[-e], \dots, \bar{K}(2\mathbf{r} - e)[-e], \left(\binom{2\mathbf{r}}{4\mathbf{r} - e} \text{ copies}\right), 2\mathbf{r} \leq e \leq 4\mathbf{r}\}.$$

We now show that

$$(b) \quad \text{if } A \in \hat{C} \text{ is such that } A \dashv \bar{K}, \text{ then } A \dashv K.$$



We may regard  $\mathcal{L}, \mathcal{L}', \mathcal{L}''$  as mixed local systems (with respect to a rational structure over a sufficiently large finite subfield of  $\mathbf{k}$ ) which are pure of weight 0. Then  $K, \bar{K}$  are naturally mixed complexes and (a) is compatible with the mixed structures. For any mixed perverse sheaf  $P$ , let  $P_h$  be the subquotient of  $P$  of pure weight  $h$ . We can find  $h \in \mathbf{Z}$  such that  $A \dashv {}^p H^j(\bar{K})_h$  for some  $j \in \mathbf{Z}$ ; moreover, we may assume that  $h$  is maximum possible. Note that  $A \dashv {}^p H^{j+4\mathbf{r}}(\bar{K}[-4\mathbf{r}](-2\mathbf{r}))_{h+2\mathbf{r}}$  and  $A \dashv {}^p H^{j'}(\bar{K}[-e](2\mathbf{r}-e))_{h+2\mathbf{r}}$  for  $2\mathbf{r} \leq e < 4\mathbf{r}$  and any  $j'$ ; hence from (a) we see that  $A \dashv {}^p H^{j+4\mathbf{r}}(K)_{h+2\mathbf{r}}$ . In particular,  $A \dashv K$ , and (b) is proved.

**40.11.** Let  $w, w', \mathcal{L}, \mathcal{L}', X, \bar{X}, \tau$  be as in 40.6. We set  $\mathbf{L} = \underline{\mathcal{L}}_w^\# * \underline{\mathcal{L}}_{w'}^\# \in \mathcal{D}^{cs}(C)$ . Let  $A = \underline{\mathcal{L}}_{w''}^\#[d_{w''}]$ . We show that

(a) if  $A \dashv \mathbf{L}$ , then  $[w'', \lambda'']$  appears with nonzero coefficient in the expansion of the product  $[w, \lambda] * [w', \lambda']$  in terms of the basis  $([y, \nu])$  of  $\mathfrak{K}(C)$ .

Let

$$\begin{aligned} \mathbf{X} &= \{(h_1 U^*, h_2 U^*, h_3 U^*) \in (G^0/U^*)^3; h_1^{-1} h_2 \in \overline{B^* \dot{w} B^*}, h_2^{-1} h_3 \in \overline{B^* \dot{w}' B^*}\}, \\ \bar{\mathbf{X}} &= \{(h_1 U^*, h_2 B^*, h_3 U^*) \in G^0/U^* \times G^0/B^* \times G^0/U^*; \\ &\quad h_1^{-1} h_2 \in \overline{B^* \dot{w} B^*}, h_2^{-1} h_3 \in \overline{B^* \dot{w}' B^*}\}. \end{aligned}$$

Note that  $X$  (resp.  $\bar{X}$ ) is naturally an open dense subset of  $\mathbf{X}$  (resp.  $\bar{\mathbf{X}}$ ). Define  $\sigma' : \mathbf{X} \rightarrow C$  by  $(h_1 U^*, h_2 U^*, h_3 U^*) \mapsto (h_1 U^*, h_3 U^*)$ . Define  $\bar{\sigma}' : \bar{\mathbf{X}} \rightarrow C$  by  $(h_1 U^*, h_2 B^*, h_3 U^*) \mapsto (h_1 U^*, h_3 U^*)$ . Let  $\mathcal{F} = \tau^*(\mathcal{L} \boxtimes \mathcal{L}')$ , a local system on  $X$ . Then  $\mathcal{F}^\# := IC(\mathbf{X}, \mathcal{F}) \in \mathcal{D}(\mathbf{X})$  is defined and we have  $\mathbf{L} = \sigma'_! \mathcal{F}^\#$ . We have a cartesian diagram

$$\begin{array}{ccccc} \mathbf{X} & \xrightarrow{\tilde{f}} & \bar{\mathbf{X}} & \xrightarrow{\bar{\sigma}'} & C \\ \uparrow & & \uparrow & & \\ X & \xrightarrow{f} & \bar{X} & & \\ \tau \downarrow & & \bar{\tau} \downarrow & & \\ T \times T & \xrightarrow{f'} & T & & \end{array}$$

where  $X \rightarrow \mathbf{X}, \bar{X} \rightarrow \bar{\mathbf{X}}$  are the obvious imbeddings,  $f, f', \bar{\tau}$  are as in 40.6 and  $\tilde{f}$  is the obvious map.

Assume first that 40.6(i) holds. Let  $m' : T \times \mathbf{X} \rightarrow \mathbf{X}$  be the free  $T$ -action  $t_1 : (h_1 U^*, h_2 U^*, h_3 U^*) \mapsto (h_1 U^*, h_2 t_1^{-1} U^*, h_3 U^*)$ . This restricts to a free  $T$ -action  $m : T \times X \rightarrow X$ . Define a free  $T$  action  $m_0 : T \times (T \times T) \rightarrow T \times T$  by  $t_1 : (t, t') \mapsto (t_1^{-1} t, \text{Ad}(\dot{w}')^{-1}(t_1) t')$ . Then  $m, m_0$  are compatible with  $\tau$ . By our assumption we have  $m_0^*(\mathcal{L} \boxtimes \mathcal{L}') = \mathcal{L}_0 \boxtimes \mathcal{L} \boxtimes \mathcal{L}'$  where  $\mathcal{L}_0 \in \mathfrak{s}(T)$ ,  $\mathcal{L}_0 \not\cong \mathbf{Q}_l$ . It follows that  $m^*(\mathcal{F}) \cong \mathcal{L}_0 \boxtimes \mathcal{F}$ . From the properties of intersection cohomology we then have  $m'^*(\mathcal{F}^\#) \cong \mathcal{L}_0 \boxtimes \mathcal{F}^\#$ . Let  $r : T \times \mathbf{X} \rightarrow \mathbf{X}$  be the second projection. Since  $\mathcal{L}_0 \in \mathfrak{s}(T)$ ,  $\mathcal{L}_0 \not\cong \mathbf{Q}_l$ , we have  $r_!(\mathcal{L}_0 \boxtimes \mathcal{F}^\#) = 0$ . Hence  $r_! m'^*(\mathcal{F}^\#) = 0$ . Since  $m', f', r, f'$  form a cartesian diagram we must have  $f'^* f'(\mathcal{F}^\#) = 0$ . Since  $f'$  is a principal  $T$ -bundle we deduce that  $f'_!(\mathcal{F}^\#) = 0$ . We have  $\mathbf{L} = \bar{\sigma}'_! f'_!(\mathcal{F}^\#)$  hence  $\mathbf{L} = 0$ . In this case (a) is clear.

Assume next that 40.6(ii) holds. Then  $\mathcal{L} \boxtimes \mathcal{L}' = f'^* \mathcal{L}'$  and  $\mathcal{F} = \tau^* f'^* \mathcal{L}' = f^* \bar{\tau}^* \mathcal{L}'$ . Since  $f'$  is a principal  $T$ -bundle and  $X = f'^{-1}(\bar{X})$  it follows that  $\mathcal{F}^\# =$

$f'^*IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}')$ . Note that  $f'_! \bar{\mathbf{Q}}_l \simeq \{\mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l)[-e], \mathbf{r} \leq e \leq 2\mathbf{r}\}$ ,

$$\mathcal{H}^e(f'_! \bar{\mathbf{Q}}_l) \simeq \{\bar{\mathbf{Q}}_l(\mathbf{r} - e), \dots, \bar{\mathbf{Q}}_l(\mathbf{r} - e), \left(\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies}\right)\}.$$

Hence setting  $\bar{\mathbf{L}} = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}'))$  we have

$$\begin{aligned} \mathbf{L} &= \sigma'_! f'^* IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}') = \bar{\sigma}'_! f'_! f'^* IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}') = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}') \otimes f'_! \bar{\mathbf{Q}}_l), \\ \mathbf{L} &\simeq \{\bar{\mathbf{L}}(\mathbf{r} - e)[-e], \dots, \bar{\mathbf{L}}(\mathbf{r} - e)[-e], \left(\binom{\mathbf{r}}{2\mathbf{r} - e} \text{ copies}\right), \mathbf{r} \leq e \leq 2\mathbf{r}\}. \end{aligned}$$

Since  $A \dashv \mathbf{L}$ , this shows that  $A \dashv \bar{\mathbf{L}}$ . We regard  $\mathcal{L}'$  as a pure local system of weight 0. Then  $\bar{\mathbf{L}} = \bar{\sigma}'_!(IC(\bar{\mathbf{X}}, \bar{\tau}^* \mathcal{L}'))$  is again pure of weight 0, since  $\bar{\sigma}'$  is proper (see [BBD]). Hence the coefficient with which  $A$  appears in the expansion of  $gr(\bar{\mathbf{L}})$  is a polynomial in  $-v$  with coefficients given by the multiplicities of  $A$  in the various  ${}^p H^j(\bar{\mathbf{L}})$ ; in particular,  $A$  appears with coefficient  $\neq 0$  in  $gr(\bar{\mathbf{L}})$ . On the other hand, the arguments above show that  $[w, \lambda] * [w', \lambda'] = (v^2 - 1)^{\mathbf{r}} gr(\bar{\mathbf{L}})$ . It follows that  $A$  appears with coefficient  $\neq 0$  in  $[w, \lambda] * [w', \lambda']$ . This proves (a).

#### 41. CHARACTER SHEAVES AND TWO-SIDED CELLS

**41.1.** In this section we preserve the notation of 40.3. We fix a connected component  $D$  of  $G$  and we pick  $\delta \in N_D B^* \cap N_D T$ . We write  $\epsilon$  instead of  $\epsilon_D : \mathbf{W} \rightarrow \mathbf{W}$ . For  $w \in \mathbf{W}$  we set

$$Z_{\emptyset, D}^w = \{(B, B', xU_B) \in Z_{\emptyset, D}; \text{pos}(B, B') = w\}.$$

(This is the same as  ${}^{w^{-1}}Z_{\emptyset, D}$  in 36.2.) Define  $\xi_D : C \rightarrow Z_{\emptyset, D}$  by  $(hU^*, h'U^*) \mapsto (hB^*h^{-1}, h'B^*h'^{-1}, h'\delta h^{-1}U_{hB^*h^{-1}})$ , a principal  $T$ -bundle for the free  $T$ -action on  $C$  given by  $t : (hU^*, h'U^*) \rightarrow (htU^*, h'(\delta t \delta^{-1})U^*)$ .

Since  $\xi_D^{-1}(Z_{\emptyset, D}^w) = C_w$ ,  $\xi_D$  restricts to a principal  $T$ -bundle  $\xi_{D, w} : C_w \rightarrow Z_{\emptyset, D}^w$ . We have a commutative diagram

$$\begin{array}{ccccc} T & \xleftarrow{\psi} & C_w & \xrightarrow{=} & C_w \\ \zeta \downarrow & & j' \uparrow & & \xi_{D, w} \downarrow \\ \mathbf{d} & \xleftarrow{pr_2} & G^0/(U^* \cap \dot{w}U^* \dot{w}^{-1}) \times \mathbf{d} & \xrightarrow{j} & Z_{\emptyset, D}^w \end{array}$$

where  $\psi$  is as in 40.3,

$$\begin{aligned} \mathbf{d} &= \dot{w}\delta T, \\ j(f(U^* \cap \dot{w}U^* \dot{w}^{-1}), s) &= (fB^*f^{-1}, f\dot{w}B^*\dot{w}^{-1}f^{-1}, fsf^{-1}U_{fB^*f^{-1}}), \\ j'(f(U^* \cap \dot{w}U^* \dot{w}^{-1}), s) &= (fU^*, fs\delta^{-1}U^*), \\ \zeta(t) &= \dot{w}\delta(\delta^{-1}t\delta). \end{aligned}$$

Note that the lower row in the diagram is as in 36.2(a).

Define  $\iota : \mathbf{d} \rightarrow T$  by  $\iota(\dot{w}\delta t) = t$  where  $t \in T$ . If  $\mathcal{L} \in \mathfrak{s}$  is such that  $\text{Ad}((\dot{w}d)^{-1})^* \mathcal{L} \cong \mathcal{L}$ , then  $pr_2^* \iota^*(\mathcal{L})$  is a local system on  $G^0/(U^* \cap \dot{w}U^* \dot{w}^{-1}) \times \mathbf{d}$ , equivariant for the  $T$ -action  $t_0 : (f(U^* \cap \dot{w}U^* \dot{w}^{-1}), s) = (ft_0^{-1}(U^* \cap \dot{w}U^* \dot{w}^{-1}), t_0 s t_0^{-1})$  on  $G^0/(U^* \cap \dot{w}U^* \dot{w}^{-1}) \times \mathbf{d}$ , which makes  $j$  a principal  $T$ -bundle. It follows that there is a well-defined local system  $\dot{\mathcal{L}}_w$  (of rank 1) on  $Z_{\emptyset, D}^w$  such that  $j^* \dot{\mathcal{L}}_w = pr_2^* \iota^*(\mathcal{L})$ . We show that

$$(a) \quad \xi_{D, w}^*(\dot{\mathcal{L}}_w) = (\text{Ad}(\delta^{-1})^* \mathcal{L})_w.$$

Since  $j'$  is an isomorphism, it is enough to show that  $j'^*\xi_{D,w}^*(\dot{\mathcal{L}}_w) = j'^*(\text{Ad}(\delta^{-1})^*\mathcal{L})_w$  or that  $j^*\dot{\mathcal{L}}_w = j'^*(\text{Ad}(\delta^{-1})^*\mathcal{L})_w$  or that  $pr_2^*\iota^*\mathcal{L} = j'^*\psi^*(\text{Ad}(\delta^{-1})^*\mathcal{L})$  or that  $j'^*\psi^*\zeta^*\iota^*\mathcal{L} = j'^*\psi^*(\text{Ad}(\delta^{-1})^*\mathcal{L})$ . It is enough to show that  $\zeta^*\iota^*\mathcal{L} = \text{Ad}(\delta^{-1})^*\mathcal{L}$ . This follows from  $\text{Ad}(\delta^{-1}) = \iota\zeta : T \rightarrow T$ .

Let  $h_w : Z_{\emptyset,D}^w \rightarrow Z_{\emptyset,D}$ ,  $\bar{h}_w : \bar{Z}_{\emptyset,D}^w \rightarrow Z_{\emptyset,D}$  be the inclusions ( $\bar{Z}_{\emptyset,D}^w = \bigcup_{w'; w' \leq w} Z_{\emptyset,D}^{w'}$  is the closure of  $Z_{\emptyset,D}^w$  in  $Z_{\emptyset,D}$ ). Let  $\dot{\mathcal{L}}_w = h_{w!}\dot{\mathcal{L}}_w$ ,  $\dot{\mathcal{L}}_w^\sharp = \bar{h}_{w!}\dot{\mathcal{L}}_w^\sharp$ . Using (a) and the fact that  $\xi_D$  is a principal  $T$ -bundle we deduce

$$\begin{aligned} \text{(b)} \quad \xi_D^*(\dot{\mathcal{L}}_w) &= \underline{(\text{Ad}(\delta^{-1})^*\mathcal{L})_w}, \\ \text{(c)} \quad \xi_D^*(\dot{\mathcal{L}}_w^\sharp) &= \underline{(\text{Ad}(\delta^{-1})^*\mathcal{L})_w^\sharp}. \end{aligned}$$

Now let  $D'$  be another connected component of  $G$ . We pick  $\delta' \in N_{D'}B^* \cap N_{D'}T$ . We have a commutative diagram with a cartesian right square

$$\begin{array}{ccccc} C \times C & \xleftarrow{r} & (G^0/U^*)^3 & \xrightarrow{s} & C \\ \xi_D \times \xi_{D'} \downarrow & & \xi_0 \downarrow & & \xi_{D'} \downarrow \\ Z_{\emptyset,D} \times Z_{\emptyset,D'} & \xleftarrow{b_1} & Z_0 & \xrightarrow{b_2} & Z_{\emptyset,D'D} \end{array}$$

where  $r, s$  are as in 40.4,  $Z_0, b_1, b_2$  are as in 32.5 (with  $J = \emptyset$ ) and

$$\begin{aligned} &\xi_0(h_1U^*, h_2U^*, h_3U^*) \\ &= (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_2\delta h_1^{-1}U_{h_1B^*h_1^{-1}}, h_3\delta' h_2^{-1}U_{h_2B^*h_2^{-1}}). \end{aligned}$$

Hence, if  $A \in \mathcal{D}(Z_{\emptyset,D})$ ,  $A' \in \mathcal{D}(Z_{\emptyset,D'})$ , then  $\xi_{D'D}^*b_2!b_1^*(A \boxtimes A') = s_!r^*(\xi_D^*A \boxtimes \xi_{D'}^*A')$ , or equivalently

$$\text{(d)} \quad \xi_{D'D}^*(A * A') = (\xi_D^*A) * (\xi_{D'}^*A').$$

**41.2.** Let  $u \in \mathbf{W}$ . Let

$$\begin{aligned} \Upsilon_u &= \{(B, B', g(U_B \cap U_{B'})); \\ &B \in \mathcal{B}, B' \in \mathcal{B}, g(U_B \cap U_{B'}) \in D/(U_B \cap U_{B'}), \text{pos}(B, B') = u\} \end{aligned}$$

and let  $\Phi_u : \mathcal{D}(Z_{\emptyset,D}) \rightarrow \mathcal{D}(Z_{\emptyset,D})$  be the composition  $\mathfrak{h}!j^*$  where  $j : \Upsilon_u \rightarrow Z_{\emptyset,D}$  is  $(B, B', g(U_B \cap U_{B'})) \mapsto (B, gBg^{-1}, gU_B)$  and  $\mathfrak{h} : \Upsilon_u \rightarrow Z_{\emptyset,D}$  is

$$(B, B', g(U_B \cap U_{B'})) \mapsto (B', gB'g^{-1}, gU_{B'}).$$

(A special case of definitions in 37.1.) Let

$$\begin{aligned} \Upsilon' &= \{(B', B, \tilde{B}, \tilde{B}', gU_{B'}); B' \in \mathcal{B}, B \in \mathcal{B}, \tilde{B} \in \mathcal{B}, \tilde{B}' \in \mathcal{B}, \\ &gU_{B'} \in D/U_{B'}, \text{pos}(B', B) = u^{-1}, \text{pos}(\tilde{B}, \tilde{B}') = \epsilon(u), gB'g^{-1} = \tilde{B}'\}, \\ s : \Upsilon_u &\rightarrow \Upsilon', (B, B', g(U_B \cap U_{B'})) \mapsto (B', B, gBg^{-1}, gB'g^{-1}, gU_{B'}). \end{aligned}$$

Note that  $s$  is an isomorphism. (We show this only at the level of sets. Define  $s' : \Upsilon' \rightarrow \Upsilon_u$  by  $(B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, B', x(U_B \cap U_{B'}))$  where  $x \in D$  is such that  $xBx^{-1} = \tilde{B}$ ,  $xU_{B'} = gU_{B'}$ . This is well defined and clearly an inverse of  $s$ .) It follows that  $\mathfrak{h}!j^* = \mathfrak{h}'!j'^*$  where

$$\begin{aligned} \mathfrak{h}' &= \mathfrak{h}s' : \Upsilon' \rightarrow Z_{\emptyset,D} \text{ is } (B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B', \tilde{B}', gU_{B'}), \\ j' &= js' : \Upsilon' \rightarrow Z_{\emptyset,D} \text{ is } (B', B, \tilde{B}, \tilde{B}', gU_{B'}) \mapsto (B, \tilde{B}, xU_B) \end{aligned}$$

and  $x \in D$  is such that  $xBx^{-1} = \tilde{B}$ ,  $xU_{B'} = gU_{B'}$  (then  $x(U_B \cap U_{B'})$  is well defined). We have a commutative diagram with a cartesian right square

$$\begin{array}{ccccc} C & \xleftarrow{\tilde{j}} & \tilde{C} & \xrightarrow{\tilde{h}} & C \\ \xi_D \downarrow & & \xi' \downarrow & & \xi_D \downarrow \\ Z_{\emptyset, D} & \xleftarrow{j'} & \Upsilon'_u & \xrightarrow{b'} & Z_{\emptyset, D} \end{array}$$

where  $\xi_D$  is as in 41.1,

$$\begin{aligned} \tilde{C} = & \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in (G^0/U^*)^4; \\ & h_1^{-1}h_2 \in B^*\dot{u}^{-1}B^*, h_3^{-1}h_4 \in B^*\delta\dot{u}\delta^{-1}B^*\}, \end{aligned}$$

$\tilde{h}$  is  $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_1U^*, h_4U^*)$ ,  $\xi'$  is

$$\begin{aligned} & (h_1U^*, h_2B^*, h_3B^*, h_4U^*) \\ & \mapsto (h_1B^*h_1^{-1}, h_2B^*h_2^{-1}, h_3B^*h_3^{-1}, h_4B^*h_4^{-1}, h_4\delta h_1^{-1}U_{h_1B^*h_1^{-1}}), \end{aligned}$$

$\tilde{j}$  is  $(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \mapsto (h_2t^{-1}U^*, h_3\tilde{t}U^*)$  where  $t, \tilde{t} \in T$  are given by  $h_1^{-1}h_2 \in U^*\dot{u}^{-1}tU^*$ ,  $h_3^{-1}h_4 \in U^*\tilde{t}\delta\dot{u}\delta^{-1}U^*$ .

We see that for  $A \in \mathcal{D}(Z_{\emptyset, D})$  we have

$$\xi_D^* \Phi_u(A) = \xi_D \mathfrak{h}_j^* A = \xi_D^* \mathfrak{h}'_j{}^* A = \tilde{h}_1 \xi'^*{}^* j'^* A = \tilde{h}_1 \tilde{j}^* \xi_D^* A.$$

Taking here  $A = \dot{\underline{\mathcal{L}}}_w^\sharp$  (with  $w \in \mathbf{W}$ ,  $\lambda \in \mathfrak{s}$ ,  $\mathcal{L} \in \lambda$  with  $wD\lambda = \lambda$ ) and using 41.1(c) we obtain  $\xi_D^* \Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \tilde{h}_1 \tilde{j}^* ((\text{Ad}(\delta^{-1})^* \mathcal{L})_w^\sharp)$  or equivalently  $\xi_D^* \Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar{\sigma}_1 \tilde{j}'^* ((\text{Ad}(\delta^{-1})^* \mathcal{L})_w^\sharp)$  where

$$\bar{X} = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde{C}; h_2^{-1}h_3 \in \overline{B^*\dot{w}B^*}\}$$

and  $\tilde{j}' : \bar{X} \rightarrow \bar{C}_w$ ,  $\bar{\sigma} : \bar{X} \rightarrow C$  are the restrictions of  $\tilde{j}, \tilde{h}$ . Let

$$\bar{X}_0 = \{(h_1U^*, h_2B^*, h_3B^*, h_4U^*) \in \tilde{C}; h_2^{-1}h_3 \in B^*\dot{w}B^*\}$$

and let  $\tilde{j}'_0 : \bar{X}_0 \rightarrow C_w$  be the restriction of  $\tilde{j}$ . Let  $\mathcal{F}_0 = \tilde{j}'_0^* (\text{Ad}(\delta^{-1})^* \mathcal{L})$ , a local system on  $\bar{X}_0$ . Since  $\tilde{j}'$  is a fibration with smooth connected fibres, we have  $\tilde{j}'^* ((\text{Ad}(\delta^{-1})^* \mathcal{L})_w^\sharp) = IC(\bar{X}, \mathcal{F}_0)$ . Thus,  $\xi_D^* \Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar{\sigma}_1(IC(\bar{X}, \mathcal{F}_0))$ . From the definitions we see that  $\mathcal{F}_0 = \bar{\tau}^* \mathcal{L}''$ , hence  $\bar{\sigma}_1(IC(\bar{X}, \mathcal{F}_0)) = \bar{K}$  and

$$(a) \quad \xi_D^* \Phi_u(\dot{\underline{\mathcal{L}}}_w^\sharp) = \bar{K}$$

where  $\bar{\tau}^* \mathcal{L}'$ ,  $\bar{K}$  are given as in 40.10 in terms of

$$(u^{-1}, \mathcal{L}), (w, \text{Ad}(\delta^{-1})^* \mathcal{L}), (\epsilon(u), \text{Ad}(\delta\dot{u}\delta^{-1})^* \text{Ad}(\delta^{-1})^* \mathcal{L})$$

instead of  $(w, \mathcal{L}), (w', \mathcal{L}'), (w'', \mathcal{L}'')$ .

**41.3.** For  $J \subset \mathbf{I}$  let  $\mathcal{D}_J^{\text{cs}}(C)$  be the subcategory of  $\mathcal{D}^{\text{cs}}(C)$  whose objects are those  $K \in \mathcal{D}(C)$  such that for any  $j$ , any simple subquotient of  ${}^p H^j K$  is isomorphic to  $\dot{\underline{\mathcal{L}}}_w^\sharp$  for some  $\mathcal{L} \in \mathfrak{s}$  and some  $w \in \mathbf{W}_J$ .

Let  $J, J' \subset \mathbf{I}$ . Let  $K \in \mathcal{D}_J^{\text{cs}}(C)$ ,  $K' \in \mathcal{D}_{J'}^{\text{cs}}(C)$ , and let  $w', w'' \in \mathbf{W}$ ,  $\lambda', \lambda'' \in \mathfrak{s}$ ,  $\mathcal{L}' \in \lambda', \mathcal{L}'' \in \lambda''$ . Let  $A = \dot{\underline{\mathcal{L}}}_{w''}^\sharp[d_{w''}]$ . We show that

(a) if (i)  $A \dashv K * \dot{\underline{\mathcal{L}}}_{w'}^\sharp[d_{w'}]$  or (ii)  $A \dashv \dot{\underline{\mathcal{L}}}_{w'}^\sharp[d_{w'}] * K'$  or (iii)  $A \dashv K * \dot{\underline{\mathcal{L}}}_{w'}^\sharp[d_{w'}] * K'$ , then  $(w'', \lambda'') \preceq_{J, J'} (w', \lambda')$ .

For the proof we may assume that  $\mathbf{k}$  is an algebraic closure of a finite field. Then the results in 40.7 are applicable. We first consider the case (i). In this case we can find  $\mathcal{L} \in \mathfrak{s}, w \in \mathbf{W}_J$  such that  $A \dashv \underline{\mathcal{L}}_w^\sharp[d_w] * \underline{\mathcal{L}}_{w'}^\sharp[d_{w'}]$ . By 40.11(a),  $[w'', \lambda'']$  appears with nonzero coefficient in the expansion of the product  $[w, \lambda] * [w', \lambda']$  in terms of the basis  $([y, \nu])$  of  $\mathfrak{K}(C)$ . Applying  $\omega$  (see 40.7(b)) we see that  $c_{w'', \lambda''}$  appears with nonzero coefficient in the expansion of the product  $c_{w, \lambda} c_{w', \lambda'}$  in terms of the basis  $(c_{y, \nu})$  of  $H$  and the desired result follows. Case (ii) is treated in an entirely similar way. We now consider case (iii). In this case we must have  $A \dashv A' * K'$  for some simple perverse sheaf  $A'$  such that  $A' \dashv K * \underline{\mathcal{L}}_{w'}^\sharp[d_{w'}]$ . We have  $A' = \underline{\mathcal{M}}_y^\sharp[d_y]$  where  $y \in \mathbf{W}, \mathcal{M} \in \mathfrak{s}$ . Let  $\nu$  be the isomorphism class of  $\mathcal{M}$ . From case (ii) applied to  $A \dashv A' * K'$  we see that  $(w'', \lambda'') \preceq_{J, J'} (y, \nu)$ . From case (i) applied to  $A' \dashv K * \underline{\mathcal{L}}_{w'}^\sharp[d_{w'}]$  we see that  $(y, \nu) \preceq_{J, J'} (w', \lambda')$ . Combining these two inequalities we obtain  $(w'', \lambda'') \preceq_{J, J'} (w', \lambda')$ , as desired.

**41.4.** Let  $J \subset \mathbf{I}$ . In the remainder of this section we write  $\mathfrak{f}, \mathfrak{e}$  instead of  $\mathfrak{f}_{\emptyset, J} : \mathcal{D}(Z_{\emptyset, D} \rightarrow \mathcal{D}(Z_{J, D}), \mathfrak{e}_{\emptyset, J} : \mathcal{D}(Z_{J, D} \rightarrow \mathcal{D}(Z_{\emptyset, D}))$ . We note that

(a) if  $A \in \mathcal{D}(Z_{J, D})$ , then  $\mathfrak{f}\mathfrak{e}(A) \cong A[m] \oplus A'$  for some  $m \in \mathbf{Z}$  and some  $A' \in \mathcal{D}(Z_{J, D})$ .

See [G], [MV] for the special case  $D = G^0, J = \mathbf{I}$  and [L10, 6.6] for the general case. Now:

(b) Let  $A$  be a simple perverse sheaf on  $Z_{J, D}$ . Then  $A \dashv \mathfrak{f}({}^p H^j(\mathfrak{e}(A)))$  for some  $j \in \mathbf{Z}$ .

Assume that this is not true. As in [BBD, p. 142], for any  $n \in \mathbf{Z}$  we have a distinguished triangle  $({}^p \tau_{\leq n-1} \mathfrak{e}A, {}^p \tau_{\leq n} \mathfrak{e}A, {}^p H^n(\mathfrak{e}A)[-n])$ , hence a distinguished triangle

$$(\mathfrak{f}({}^p \tau_{\leq n-1} \mathfrak{e}A), \mathfrak{f}({}^p \tau_{\leq n} \mathfrak{e}A), \mathfrak{f}({}^p H^n(\mathfrak{e}A))[-n]).$$

Using our assumption, we see that  $A \dashv \mathfrak{f}({}^p \tau_{\leq n-1} \mathfrak{e}A)$  if and only if  $A \dashv \mathfrak{f}({}^p \tau_{\leq n} \mathfrak{e}A)$ . Thus we have  $A \dashv \mathfrak{f}({}^p \tau_{\leq n} \mathfrak{e}A)$  for some  $n$  if and only if  $A \dashv \mathfrak{f}({}^p \tau_{\leq n} \mathfrak{e}A)$  for any  $n$ . Since  ${}^p \tau_{\leq n} \mathfrak{e}A = 0$  for some  $n$ , we see that  $A \dashv \mathfrak{f}({}^p \tau_{\leq n} \mathfrak{e}A)$  for any  $n$ . Since  ${}^p \tau_{\leq n} \mathfrak{e}A = \mathfrak{e}A$  for some  $n$ , we deduce that  $A \dashv \mathfrak{f}\mathfrak{e}A$ . This contradicts (a); (b) is proved.

We show that

(c) if  $A$  is a simple perverse sheaf on  $Z_{J, D}$ , then there exists a simple perverse sheaf  $A'$  on  $Z_{\emptyset, D}$  such that  $A \dashv \mathfrak{f}(A'), A' \dashv \mathfrak{e}(A)$ .

By (b) we can find  $i, j \in \mathbf{Z}$  such that  $A \dashv {}^p H^i(\mathfrak{f}(P))$  where  $P = {}^p H^j(\mathfrak{e}(A))$ .

Assume that  $A \dashv {}^p H^i(\mathfrak{f}(A'))$  for any simple subquotient  $A'$  of  $P$ . We claim that  $A \dashv {}^p H^i(\mathfrak{f}(P'))$  for any subobject  $P'$  of  $P$ . We argue by induction on the length of  $P'$ . If  $P'$  has length 1, the claim holds by assumption. If  $P'$  has length  $\geq 2$ , we can find a simple subobject  $P''$  of  $P'$ . We have a distinguished triangle  $(\mathfrak{f}(P''), \mathfrak{f}(P'), \mathfrak{f}(P'/P''))$ . Hence we have an exact sequence  ${}^p H^i(\mathfrak{f}(P'')) \rightarrow {}^p H^i(\mathfrak{f}(P')) \rightarrow {}^p H^i(\mathfrak{f}(P'/P''))$ . By the induction hypothesis, we have  $A \dashv {}^p H^i(\mathfrak{f}(P''))$ ,  $A \dashv {}^p H^i(\mathfrak{f}(P'/P''))$ . Hence  $A \dashv {}^p H^i(\mathfrak{f}(P'))$ . This proves the claim. In particular,  $A \dashv {}^p H^i(\mathfrak{f}(P))$ , contradicting the definition of  $i, P$ .

We see that there exists a simple subquotient  $A'$  of  $P$  such that  $A \dashv {}^p H^i(\mathfrak{f}(A'))$ . Then  $A'$  is as required by (c).

Let  $\bar{d}_w = \dim Z_{\emptyset, D}^w$ . Let

(d)  $A' = \underline{\mathcal{L}}_w^\sharp[\bar{d}_w], A'' = \underline{\mathcal{M}}_y^\sharp[\bar{d}_y] \in \hat{Z}_{\emptyset, D}, \mathcal{L} \in \lambda, \mathcal{M} \in \nu$ .

Here  $w\underline{D}\lambda = \lambda, y\underline{D}\nu = \nu$ . Now:

(e) *Let  $A$  be a character sheaf on  $Z_{J,D}$  such that  $A \dashv \mathfrak{f}(A'), A'' \dashv \mathfrak{e}(A)$ . Then  $(y, \underline{D}\nu) \preceq_{J,J'} (w, \underline{D}\lambda)$ .*

Since  $\mathfrak{f}$  is proper,  $\mathfrak{f}(A')$  is a semisimple complex (see [BBD]). Hence  $\mathfrak{f}(A') \cong A[m] \oplus A_1$  for some  $m \in \mathbf{Z}$ ,  $A' \in \mathcal{D}(Z_{J,D})$  and  $\mathfrak{e}\mathfrak{f}(A') \cong \mathfrak{e}(A)[m] \oplus \mathfrak{e}(A_1)$ . Hence from  $A'' \dashv \mathfrak{e}(A)$  we can deduce  $A'' \dashv \mathfrak{e}\mathfrak{f}(A')$ . By 37.2 we have  $\mathfrak{e}\mathfrak{f}(A') \simeq \{\Phi_u(A')[[-m_u]]; u \in \mathbf{W}_J\}$  where  $m_u$  are certain integers. Hence for some  $u \in \mathbf{W}_J$  we have  $A'' \dashv \Phi_u(A')[[-m_u]]$ , that is,  $A'' \dashv \Phi_u(A')$  and  $\xi_D^* A''[\mathbf{r}] \dashv \xi_D^* \Phi_u(A')[\mathbf{r}]$ . Hence using 41.2(a) we have  $\xi_D^* A''[\mathbf{r}] \dashv \bar{K}$  where  $\bar{K}$  is as in the end of 41.2. Thus,  $\underline{\mathcal{M}}_y^\sharp[d_y] \dashv \bar{K}$ . Using 40.10(b) we deduce that

$$\underline{\mathcal{M}}_y^\sharp[d_y] \dashv \underline{\text{Ad}(w)^{-1}}^* \underline{\text{Ad}(\delta^{-1})}^* \mathcal{L}_{u^{-1}} * (\underline{\text{Ad}(\delta^{-1})}^* \mathcal{L}_w^\sharp * \underline{\text{Ad}(\delta u \delta^{-1})}^* \underline{\text{Ad}(\delta^{-1})}^* \mathcal{L}_{\epsilon(u)}).$$

Using this and 41.3(a) we see that (e) holds.

Now:

(f) *Let  $A$  be a character sheaf on  $Z_{J,D}$ . In the setup of (d) assume that  $A \dashv \mathfrak{f}(A'), A' \dashv \mathfrak{e}(A), A \dashv \mathfrak{f}(A''), A'' \dashv \mathfrak{e}(A)$ . Then  $(y, \underline{D}\nu) \sim_{J,J'} (w, \underline{D}\lambda)$ .*

Applying (e) to  $A', A''$  we see that  $(y, \underline{D}\nu) \preceq_{J,J'} (w, \underline{D}\lambda)$ . Applying (e) to  $A'', A'$  (instead of  $A', A''$ ) we see that  $(w, \underline{D}\lambda) \preceq_{J,J'} (y, \underline{D}\nu)$ . Hence (f) holds.

From (c),(f) we see that there is a well-defined map  $A \mapsto \mathbf{c}_A$  from the set of character sheaves on  $Z_{J,D}$  (up to isomorphism) to the set of  $(J, J')$ -two-sided cells in  $\mathbf{W} \times \underline{\mathcal{F}}$  where  $\mathbf{c}_A$  is the unique  $(J, J')$ -two-sided cell that contains

$$\{(w, \underline{D}\lambda) \in \mathbf{W} \times \underline{\mathfrak{g}}; w\underline{D}\lambda = \lambda, A \dashv \mathfrak{f}(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w]), \underline{\mathcal{L}}_w^\sharp[\bar{d}_w] \dashv A\}$$

(a nonempty set); here  $\mathcal{L} \in \lambda$ .

**41.5.** In the setup of 41.4, let  $A$  be a character sheaf on  $Z_{J,D}$ . We show that:

(a) *There exists  $(w, \underline{D}\lambda) \in \mathbf{c}_A$  such that  $w\underline{D}\lambda = \lambda, A \dashv \mathfrak{f}(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w])$ . If  $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{g}}$  is such that  $w'\underline{D}\lambda' = \lambda', A \dashv \mathfrak{f}(\underline{\mathcal{L}}_{w'}^\sharp[\bar{d}_{w'}])$ , then  $(w, \underline{D}\lambda) \preceq_{J,J'} (w', \underline{D}\lambda')$ . Here  $\mathcal{L} \in \lambda, \mathcal{L}' \in \lambda'$ .*

(b) *There exists  $(w, \underline{D}\lambda) \in \mathbf{c}_A$  such that  $w\underline{D}\lambda = \lambda, \underline{\mathcal{L}}_w^\sharp[\bar{d}_w] \dashv \mathfrak{e}(A)$ . If  $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{g}}$  is such that  $w'\underline{D}\lambda' = \lambda', \underline{\mathcal{L}}_{w'}^\sharp[\bar{d}_{w'}] \dashv \mathfrak{e}(A)$ , then  $(w', \underline{D}\lambda') \preceq_{J,J'} (w, \underline{D}\lambda)$ . Here  $\mathcal{L} \in \lambda, \mathcal{L}' \in \lambda'$ .*

Note that (a) follows immediately from 41.4(c),(e) and the definition of  $\mathbf{c}_A$ . Similarly, (b) follows from 41.4(c),(e) and the definition of  $\mathbf{c}_A$ .

**41.6.** In this subsection we assume that  $J = \mathbf{I}$ . The  $\mathcal{A}$  linear map  $H \rightarrow H$  given by

$$(a) \tilde{T}_w 1_\lambda \mapsto \tilde{T}_{\epsilon(w)} 1_{\underline{D}\lambda} \text{ for } w \in \mathbf{W}, \lambda \in \underline{\mathfrak{g}}$$

is an  $\mathcal{A}$ -algebra isomorphism. It carries  $c_{w,\lambda}$  to  $c_{\epsilon(w),\underline{D}\lambda}$  for any  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{g}}$ . It induces a bijection  $\mathbf{c} \mapsto \mathbf{c}'$  from the set of two-sided cells in  $\mathbf{W} \times \underline{\mathfrak{g}}$  onto itself. We show that

(b) *if  $A$  is a character sheaf on  $D$ , then  $(\mathbf{c}_A)' = \mathbf{c}_A$ .*

Consider the automorphism  $\text{Ad}(\delta) : D \rightarrow D$ . From the definitions we see that for  $(w, \lambda) \in \mathbf{W} \times \underline{\mathfrak{g}}$  such that  $w\underline{D}\lambda = \lambda$  we have  $A \dashv \mathfrak{f}(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w])$  if and only if

$\text{Ad}(\delta^{-1})^*A \dashv \text{f}(\underline{\mathcal{D}}^{-1})^* \mathcal{L}_{\epsilon(w)}^\sharp[\bar{d}_w]$ . Using this and 41.5(a) we see that

$$\mathbf{c}_{\text{Ad}(\delta^{-1})^*A} = (\mathbf{c}_A)'$$

It is then enough to show that  $\text{Ad}(\delta^{-1})^*A \cong A$ . By the  $G^0$ -equivariance of  $A$  we have  $m^*A \cong q^*A$  where  $m : G^0 \times D \rightarrow D$  is  $(x, g) \mapsto xgx^{-1}$  and  $q : G^0 \times D \rightarrow D$  is  $(x, g) \mapsto g$ . Define  $r : D \rightarrow G^0 \times D$  by  $r(g) = (\delta g^{-1}, g)$ . Then  $r^*m^*A \cong r^*q^*A$  that is,  $(mr)^*A \cong (qr)^*A$ . We have  $mr = \text{Ad}(\delta)$ ,  $qr = 1$ , hence  $\text{Ad}(\delta)^*A \cong A$  and  $\text{Ad}(\delta^{-1})^*A \cong A$ , as required.

Note also that for  $(w, \lambda)$  as above we have

$$(c) \text{f}(\underline{\mathcal{D}}^{-1})^* \mathcal{L}_{\epsilon(w)}^\sharp[\bar{d}_w] \cong \text{f}(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w]).$$

Indeed, let  $K = \text{f}(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w])$ . Clearly, we have  $m^*K \cong q^*K$  with  $m, q$  as above. Then as in the proof of (b) we see that  $\text{Ad}(\delta)^*K \cong K$ . From the definitions we see that  $\text{f}(\underline{\mathcal{D}}^{-1})^* \mathcal{L}_{\epsilon(w)}^\sharp[\bar{d}_w] = \text{Ad}(\delta^{-1})^*K$ . Since  $\text{Ad}(\delta^{-1})^*K \cong K$ , (c) follows.

**41.7.** In this and the next subsection we assume that  $\mathbf{k}$  is an algebraic closure of a finite field. From 41.1(c) we see that  $\xi_D^* : \mathcal{D}(Z_{\emptyset, D}) \rightarrow \mathcal{D}(C)$  restricts to a functor  $\mathcal{D}^{cs}(Z_{\emptyset, D}) \rightarrow \mathcal{D}^{cs}(C)$ , hence, as in 36.8, the  $\mathcal{A}$ -linear map  $gr(\xi_D^*) : \mathfrak{K}(Z_{\emptyset, D}) \rightarrow \mathfrak{K}(C)$  is well defined; from 41.1(c) we see also that

$$(a) \text{gr}(\xi_D^*)(\underline{\mathcal{L}}_w^\sharp[\bar{d}_w]) = (-v)^r[w; \underline{D}\lambda]$$

for  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}$  such that  $w\underline{D}\lambda = \lambda$  and  $\mathcal{L} \in \lambda$ . From (a) we see that  $gr(\xi_D^*)$  is injective with image equal to  $\mathfrak{K}(C)^D$ , the  $\mathcal{A}$ -submodule of  $\mathfrak{K}(C)$  spanned by  $\{[w; \underline{D}\lambda]; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$  or equivalently by  $\{[w; \underline{D}\lambda]'; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$ . Thus,  $gr(\xi_D^*)$  defines an isomorphism  $\eta' : \mathfrak{K}(Z_{\emptyset, D}) \xrightarrow{\sim} \mathfrak{K}(C)^D$ . Let  $\eta = \eta'^{-1}$ .

Let  $n \in \mathbf{N}_{\mathbf{k}}^*$ . Let  $\mathfrak{K}(C)_n^D$  be the  $\mathcal{A}$ -submodule of  $\mathfrak{K}(C)$  spanned by  $\{[w; \underline{D}\lambda]; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$  or equivalently by  $\{[w; \underline{D}\lambda]'; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$ .

Let  $u, w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$  be such that  $w\underline{D}\lambda = \lambda$  and let  $\mathcal{L} \in \lambda$ . From 37.3(c) we see that the  $\mathcal{A}$ -linear map  $gr(\Phi_u) : \mathfrak{K}(Z_{\emptyset, D}) \rightarrow \mathfrak{K}(Z_{\emptyset, D})$  is well defined; we denote it again by  $\Phi_u$ . From 40.10(a), 41.2(a) we have

$$[u^{-1}; \lambda]' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]' = (v^2 - 1)^{2r} \eta' \Phi_u \eta([w; \underline{D}\lambda]'^\sharp),$$

equality in  $\mathfrak{K}(C)$ . If  $\lambda' \in \underline{\mathfrak{s}}_n, \lambda' \neq \lambda$ , we have (from 40.7) that  $[u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)']' = 0$ . It follows that

$$(v^2 - 1)^{2r} \eta' \Phi_u \eta([w; \underline{D}\lambda]'^\sharp) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * [w; \underline{D}\lambda]'^\sharp * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)']'.$$

Using this and the definition of  $\mathfrak{K}(C)_n^D$  we see that

$$(v^2 - 1)^{2r} \eta' \Phi_u \eta(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} [u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)']'$$

for any  $x \in \mathfrak{K}(C)_n^D$ . Applying  $\eta$  to both sides we obtain

$$(b) \quad (v^2 - 1)^{2r} \Phi_u \eta'(x) = \sum_{\lambda' \in \underline{\mathfrak{s}}_n} \eta([u^{-1}; \lambda']' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)']')$$

for any  $x \in \mathfrak{K}(C)_n^D$ .

**41.8.** In the setup of 41.4, let  $A$  be a character sheaf on  $Z_{J,D}$ . From 36.9(b) we see that the condition that, if  $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{g}}$  is such that  $w' \underline{D}\lambda' = \lambda'$ , then we have  $A \dashv \mathfrak{f}(\dot{\mathcal{L}}_{w'}^\#[\bar{d}_{w'}])$  if and only if  $A$  appears with coefficient  $\neq 0$  in the expansion of  $\mathfrak{f}(\dot{\mathcal{L}}_{w'}^\#[\bar{d}_{w'}]) \in \mathfrak{K}(Z_{J,D})$  as a linear combination of the canonical basis of  $\mathfrak{K}(Z_{J,D})$ . Hence from 41.5(a) we deduce:

(a) *There exists  $(w, \underline{D}\lambda) \in \mathbf{c}_A$  such that  $w \underline{D}\lambda = \lambda$  and  $A$  appears with nonzero coefficient in  $\mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w]) \in \mathfrak{K}(Z_{J,D})$ . If  $(w', \underline{D}\lambda') \in \mathbf{W} \times \underline{\mathfrak{g}}$  is such that  $w' \underline{D}\lambda' = \lambda'$  and  $A$  appears with nonzero coefficient in  $\mathfrak{f}(\dot{\mathcal{L}}_{w'}^\#[\bar{d}_{w'}]) \in \mathfrak{K}(Z_{J,D})$ , then  $(w, \underline{D}\lambda) \preceq_{J,J'} (w', \underline{D}\lambda')$ . Here  $\mathcal{L} \in \lambda, \mathcal{L}' \in \lambda'$ .*

Clearly, property (a) characterizes  $\mathbf{c}_A$ .

**41.9.** Let  $J \subset J' \subset \mathbf{I}$  and let  $D'$  be another connected component of  $G$ . Let  $A_0 \in \mathcal{D}(Z_{J,D})$ ,  $A' \in \mathcal{D}(Z_{\epsilon_D(J'), D'})$ . We show that

$$(a) \mathfrak{f}_{J,J'}(A_0) * A' \cong \mathfrak{f}_{J,J'}(A_0 * \mathfrak{e}_{\epsilon_D(J), \epsilon_D(J')} A') \text{ in } \mathcal{D}(Z_{J', D'D}).$$

Indeed, from the definitions we see that both sides of (a) can be identified with  $b_! c^*(A_0 \boxtimes A')$  where  $b, c$  are as in the diagram

$$Z_{J,D} \times Z_{\epsilon_D(J'), D'} \xleftarrow{c} Y \xrightarrow{b} Z_{J', D'D}$$

where

$$Y = \{(P, R, R', gU_R, g'U_{R'}); P \in \mathcal{P}_J, R \in \mathcal{P}_{J'}, R' \in \mathcal{P}_{\epsilon_D(J')}, \\ gU_R \in D/U_R, g'U_{R'} \in D'/U_{R'}, gRg^{-1} = R', P \subset R\},$$

$c$  is  $(P, R, R', gU_R, g'U_{R'}) \mapsto ((P, gU_P), (R', g'U_{R'}))$ ,  $b$  is  $(P, R, R', gU_R, g'U_{R'}) \mapsto (R, g'gU_R)$ .

An entirely similar proof shows that, if  $A \in \mathcal{D}(Z_{J', D})$ ,  $A'_0 \in \mathcal{D}(Z_{\epsilon_D(J), D'})$ , then

$$(b) A * \mathfrak{f}_{\epsilon_D(J), \epsilon_D(J')} (A'_0) \cong \mathfrak{f}_{J,J'} (\mathfrak{e}_{J,J'} A * A'_0) \text{ in } \mathcal{D}(Z_{J', D'D}).$$

**41.10.** Let  $\mathbf{c}$  be a two-sided cell in  $\mathbf{W} \times \underline{\mathfrak{g}}$ . Let  $\bar{\mathbf{c}}$  be the set of all  $(w, \lambda) \in \mathbf{W} \times \underline{\mathfrak{g}}$  such that  $(w, \lambda) \preceq_{\mathbf{I}, \mathbf{I}} (y, \nu)$  for some/any  $(y, \nu) \in \mathbf{c}$ .

If  $K \in \mathcal{D}(Z_{\emptyset, D})$ , we say that  $K \in \mathcal{D}_{\bar{\mathbf{c}}}^{cs}(Z_{\emptyset, D})$  if for any  $j \in \mathbf{Z}$  and simple subquotient  $A$  of  ${}^p H^j(K)$  satisfies  $\mathbf{c}_A \subset \bar{\mathbf{c}}$ .

Let  $D'$  be another connected component of  $G$ . We show that

$$(a) \text{ if } K \in \mathcal{D}_{\bar{\mathbf{c}}}^{cs}(Z_{\emptyset, D}), K' \in \mathcal{D}^{cs}(Z_{\epsilon_D(J'), D'}), \text{ then } K * K' \in \mathcal{D}_{\bar{\mathbf{c}}}^{cs}(Z_{\emptyset, D'D}).$$

We may assume that  $\mathbf{k}$  is an algebraic closure of a finite field. We may assume that  $K \in \hat{Z}_{\emptyset, D}$  and  $\mathbf{c}_K \subset \bar{\mathbf{c}}$ . Then there exists  $(w, \underline{D}\lambda) \in \mathbf{c}_K$  such that  $w \underline{D}\lambda = \lambda$ ,  $K \dashv \mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w])$ ,  $\mathcal{L} \in \lambda$ . It is enough to show that, if  $\tilde{A} \in \hat{Z}_{\emptyset, D'D}$  is such that  $\tilde{A} \dashv K * K'$ , then  $\mathbf{c}_{\tilde{A}} \subset \bar{\mathbf{c}}$ . Since  $\mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w])$  is a semisimple complex (see the line after 41.4(e)) we have  $\mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w]) \cong K[m] \oplus \tilde{K}$  for some  $m \in \mathbf{Z}$ ,  $\tilde{K} \in \mathcal{D}(Z_{\emptyset, D'D})$ . It follows that  $\mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w]) * K' \cong K * K'[m] \oplus \tilde{K} * K'$  hence  $\tilde{A} \dashv \mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w]) * K'$ . By 41.9(a) we have  $\mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w]) * K' \cong \mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w] * \mathfrak{e}(K'))$  hence  $\tilde{A} \dashv \mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w] * \mathfrak{e}(K'))$ . We deduce that there exists  $K'_0 \in \hat{Z}_{\emptyset, D'}$  such that  $\tilde{A} \dashv \mathfrak{f}(\dot{\mathcal{L}}_w^\#[\bar{d}_w] * K'_0)$  and  $K''_0 \in \hat{Z}_{\emptyset, D'D}$  such that  $K''_0 \dashv \dot{\mathcal{L}}_w^\#[\bar{d}_w] * K'_0$ ,  $\tilde{A} \dashv \mathfrak{f}(K''_0)$ . We then have  $\xi_{D'D}^* K''_0[\mathbf{r}] \dashv \xi_{D'D}^*(\dot{\mathcal{L}}_w^\#[\bar{d}_w] * K'_0)[\mathbf{r}]$ , hence, using 41.1(d),  $\xi_{D'D}^* K''_0[\mathbf{r}] \dashv (\xi_D^* \dot{\mathcal{L}}_w^\#[\bar{d}_w]) * \xi_{D'} K'_0$ . Setting  $gr(\xi_{D'D}^* K''_0[\mathbf{r}]) = [w_1, \underline{D}' \underline{D}\lambda_1] \in \mathfrak{K}(C)$  with  $(w_1, \lambda_1) \in \mathbf{W} \times \underline{\mathfrak{g}}$  we see, using 41.3(a) that  $(w_1, \underline{D}' \underline{D}\lambda_1) \preceq_{\mathbf{I}, \mathbf{I}} (w, \underline{D}\lambda)$ . From  $\tilde{A} \dashv \mathfrak{f}(K''_0)$  we see using 41.5(a) that



$\mathbf{c}_{\bar{A}} \preceq_{\mathbf{I}, \mathbf{I}} (w_1, \underline{D}' \underline{D} \lambda_1)$  (that is, some/any element of  $\mathbf{c}_{\bar{A}}$  is  $\preceq_{\mathbf{I}, \mathbf{I}} (w_1, \underline{D}' \underline{D} \lambda_1)$ ). Using the transitivity of  $\preceq_{\mathbf{I}, \mathbf{I}}$  we see that  $\mathbf{c}_{\bar{A}} \preceq_{\mathbf{I}, \mathbf{I}} (w, \underline{D} \lambda)$ . This proves (a).

An entirely similar argument shows that

(b) if  $K \in \mathcal{D}^{cs}(Z_{\emptyset, D})$ ,  $K' \in \mathcal{D}_{\bar{c}}^{cs}(Z_{\epsilon_D(J'), D'})$ , then  $K * K' \in \mathcal{D}_{\bar{c}}^{cs}(Z_{\emptyset, D'})$ .

#### 42. DUALITY AND THE FUNCTOR $\mathfrak{f}_{\emptyset, \mathbf{I}}$

**42.1.** In this section we fix a connected component  $D$  of  $G$ . We write  $\epsilon$  instead of  $\epsilon_D : \mathbf{W} \rightarrow \mathbf{W}$ . We write  $\mathfrak{f}$  instead of  $\mathfrak{f}_{\emptyset, \mathbf{I}} : \mathcal{D}(Z_{\emptyset, D}) \rightarrow \mathcal{D}(Z_{\mathbf{I}, D})$ . We assume that  $\mathbf{k}$  is an algebraic closure of a finite field.

Let  $J \subset \mathbf{I}$  be such that  $\epsilon(J) = J$ . Recall from 30.3 that  $V_{J, D} = \{(P, gU_P); P \in \mathcal{P}_J, gU_P \in N_D P/U_P\}$ . As in 30.4 (with  $J' = \mathbf{I}$ ) we consider the diagram  $V_{J, D} \xleftarrow{c} V_{J, \mathbf{I}, D} \xrightarrow{d} D$  where  $V_{J, \mathbf{I}, D} = \{(P, g); P \in \mathcal{P}_J, g \in N_D P\}$ ,  $c$  is  $(P, g) \mapsto (P, gU_P)$  and  $d$  is  $(P, g) \mapsto g$ . Define  $\tilde{f}_J : \mathcal{D}(V_{J, D}) \rightarrow \mathcal{D}(D)$ ,  $\tilde{e}_J : \mathcal{D}(D) \rightarrow \mathcal{D}(V_{J, D})$  by  $\tilde{f}_J A = d_! c^* A$ ,  $\tilde{e}_J A' = c_! d^* A'$ . (In the notation of 30.4 we have  $\tilde{f}_J = \tilde{f}_{J, \mathbf{I}}$ ,  $\tilde{e}_J = \tilde{e}_{J, \mathbf{I}}$ .) Define  $f_J : \mathcal{D}(V_{J, D}) \rightarrow \mathcal{D}(D)$ ,  $e_J : \mathcal{D}(D) \rightarrow \mathcal{D}(V_{J, D})$  by  $f_J A = \tilde{f}_J A[[\alpha_J/2]]$ ,  $e_J A' = \tilde{e}_J A'[[\alpha_J/2]]$  where  $\alpha_J = \dim \mathcal{P}_J$ . (In the notation of 30.4 we have  $f_J A = f_{J, \mathbf{I}} A(\alpha_J/2)$ ,  $e_J A' = e_{J, \mathbf{I}} A'(-\alpha_J/2)$ ). Thus,  $f_J, e_J$  are the same, up to a twist, as  $f_{J, \mathbf{I}}, e_{J, \mathbf{I}}$ .

From 30.5 (with  $J' = \mathbf{I}$ ) we see that for  $A \in \mathcal{D}(V_{J, D})$ ,  $A' \in \mathcal{D}(D)$  we have canonically

(a)  $\mathrm{Hom}_{\mathcal{D}(V_{J, D})}(e_J A', A) = \mathrm{Hom}_{\mathcal{D}(D)}(A', f_J A)$ .

Let  $CS(V_{J, D}), CS(D)$  be as in 38.1. From 38.2, 38.3 we see that

(b)  $f_J, e_J$  restrict to functors  $CS(V_{J, D}) \rightarrow CS(D)$ ,  $CS(D) \rightarrow CS(V_{J, D})$  denoted again by  $f_J, e_J$ .

We show that

(c) if  $A \in CS(V_{J, D})$  comes from a pure complex of weight 0 with respect to a rational structure over a finite subfield of  $\mathbf{k}$ , then  $f_J A$  (naturally regarded as a mixed complex) is pure of weight 0.

Indeed, the functor  $c^*$  preserves pure complexes of weight 0 since  $c$  is smooth with connected fibres; the functor  $d_!$  preserves pure complexes of weight 0 since  $d$  is proper (see [De, 6.2.6]) and  $[[\alpha_J/2]]$  also preserves pure complexes of weight 0.

We show that

(d) if  $A' \in CS(D)$  comes from a pure complex of weight 0 with respect to a rational structure over a finite subfield of  $\mathbf{k}$ , then  $e_J A'$  (naturally regarded as a mixed complex) is pure of weight 0.

Using (b), it is enough to show that for any simple  $A$  as in (c), the natural action of Frobenius on the vector space  $\mathrm{Hom}_{\mathcal{D}(V_{J, D})}(e_J A', A)$  has weight 0. Using (a) we see that it is enough to show that the natural action of Frobenius on the vector space  $\mathrm{Hom}_{\mathcal{D}(D)}(A', f_J A)$  has weight 0. This follows from (c) using (b).

Define an imbedding  $s : V_{J, D} \rightarrow Z_{J, D}$  by  $(P, gU_P) \mapsto (P, P, gU_P)$ . From the definitions we see that

(e)  $\tilde{f}_J : \mathcal{D}(V_{J, D}) \rightarrow \mathcal{D}(D)$  is the composition  $\mathcal{D}(V_{J, D}) \xrightarrow{s_!} \mathcal{D}(Z_{J, D}) \xrightarrow{\tilde{f}_{J, \mathbf{I}}} \mathcal{D}(D)$ ,

(f)  $\tilde{e}_J : \mathcal{D}(D) \rightarrow \mathcal{D}(V_{J, D})$  is the composition  $\mathcal{D}(D) \xrightarrow{\tilde{e}_{J, \mathbf{I}}} \mathcal{D}(Z_{J, D}) \xrightarrow{s^*} \mathcal{D}(V_{J, D})$ .

Let  $Y = \{(B, B', gU_B) \in Z_{\emptyset, D}; \mathrm{pos}(B, B') \in \mathbf{W}_J\}$  and let  $r : Y \rightarrow Z_{\emptyset, D}$  be the inclusion. From the definitions we have

(g)  $s_! s^* \mathfrak{f}_{\emptyset, J} = \mathfrak{f}_{\emptyset, J} r_! r^* : \mathcal{D}(Z_{\emptyset, D}) \rightarrow \mathcal{D}(Z_{J, D})$ .

Note that  $V_{J,D} = {}^1Z_{J,D}$  (see 36.2); hence the “character sheaves” on  $V_{J,D} = {}^1Z_{J,D}$  are defined as in 36.8 and  $\mathcal{D}^{cs}(V_{J,D} = \mathcal{D}^{cs}({}^1Z_{J,D}))$  is defined as 36.8. In particular,  $\mathfrak{K}(V_{J,D}) = \mathfrak{K}({}^1Z_{J,D})$  is defined. Let  $\mathfrak{K}_0(V_{J,D}) = \bigoplus_A \mathbf{Z}A \subset \mathfrak{K}(V_{J,D})$  where  $A$  runs through the character sheaves on  $V_{J,D}$  (up to isomorphism).

From (b) we see that  $\tilde{f}_J, \tilde{e}_J$  restrict to functors  $\mathcal{D}^{cs}(V_{J,D}) \rightarrow \mathcal{D}^{cs}(D)$ ,  $\mathcal{D}^{cs}(D) \rightarrow \mathcal{D}^{cs}(V_{J,D})$ , hence the  $\mathcal{A}$ -linear maps  $gr(\tilde{f}_J) : \mathfrak{K}(V_{J,D}) \rightarrow \mathfrak{K}(D)$ ,  $gr(\tilde{e}_J) : \mathfrak{K}(D) \rightarrow \mathfrak{K}(V_{J,D})$  are well defined; we denote them by  $\hat{f}_J, \hat{e}_J$ . Define  $f_J : \mathfrak{K}(V_{J,D}) \rightarrow \mathfrak{K}(D)$  by  $f_J = (-v)^{-\alpha_J} \hat{f}_J$  and  $e_J : \mathfrak{K}(D) \rightarrow \mathfrak{K}(V_{J,D})$  by  $e_J = (-v)^{-\alpha_J} \hat{e}_J$ . We show that

(h)  $f_J : \mathfrak{K}(V_{J,D}) \rightarrow \mathfrak{K}(D)$ ,  $e_J : \mathfrak{K}(D) \rightarrow \mathfrak{K}(V_{J,D})$  restrict to group homomorphisms  $\mathfrak{K}_0(V_{J,D}) \rightarrow \mathfrak{K}_0(D)$ ,  $\mathfrak{K}_0(D) \rightarrow \mathfrak{K}_0(V_{J,D})$  denoted again by  $f_J, e_J$ .

It is enough to prove the following statement. If  $x$  is a canonical basis element of  $\mathfrak{K}(V_{J,D})$  (resp.  $\mathfrak{K}(D)$ ), then  $f_J(x)$  (resp.  $e_J(x)$ ) is an  $\mathbf{N}$ -linear combination of canonical basis elements of  $\mathfrak{K}(D)$  (resp.  $\mathfrak{K}(V_{J,D})$ ). This is immediate from (c), (d).

Now, one checks easily that  $r_1 r^* : \mathcal{D}(Z_{\emptyset,D}) \rightarrow \mathcal{D}(Z_{\emptyset,D})$  restricts to a functor  $\mathcal{D}^{cs}(Z_{\emptyset,D}) \rightarrow \mathcal{D}^{cs}(Z_{\emptyset,D})$ . (Note that, if  $w \in \mathbf{W}$ ,  $\lambda \in \underline{\mathfrak{g}}$ ,  $\mathcal{L} \in \lambda$  and  $w\underline{D}\lambda = \lambda$ , then  $r_1 r^*(\underline{\mathcal{L}}_w) = \underline{\mathcal{L}}_w$  for  $w \in \mathbf{W}_J$  and  $r_1 r^*(\underline{\mathcal{L}}_w) = 0$  for  $w \in \mathbf{W} - \mathbf{W}_J$ .) It follows that the  $\mathcal{A}$ -linear map  $gr(r_1 r^*) : \mathfrak{K}(Z_{\emptyset,D}) \rightarrow \mathfrak{K}(Z_{\emptyset,D})$  (denoted by  $\rho_J$ ) is well defined.

Let  $\mathfrak{K}(C)^D, \eta$  be as in 41.7. Define an  $\mathcal{A}$ -linear map  $\tilde{\rho}_J : \mathfrak{K}(C)^D \rightarrow \mathfrak{K}(C)^D$  by  $[w; \underline{D}\lambda]' \mapsto [w; \underline{D}\lambda]'$  if  $w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{g}}, w\underline{D}\lambda = \lambda$  and  $[w; \underline{D}\lambda]' \mapsto 0$  if  $w \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{g}}, w\underline{D}\lambda = \lambda$ . From the definitions we see that

(i)  $\rho_J \eta(x) = \eta \tilde{\rho}_J(x)$  for all  $x \in \mathfrak{K}(C)^D$ .

**42.2.** We define an  $\mathcal{A}$ -linear map  $\mathbf{d} : \mathfrak{K}(D) \rightarrow \mathfrak{K}(D)$  by

$$\mathbf{d}(x) = \sum_{J; J \subset \mathbf{I}; \epsilon(J)=J} (-1)^{|J_\epsilon|} f_J e_J(x)$$

where  $f_J, e_J$  are as in 42.1(h) and  $J_\epsilon$  is as in 38.1. Now:

(a) Let  $A$  be a character sheaf on  $D$ . Then  $\mathbf{d}(A) = \pm A'$  where  $A'$  is a character sheaf on  $D$ . Moreover,  $\pm$  and  $A'$  are the same as in 38.11(a).

For any  $J \subset \mathbf{I}$  such that  $\epsilon(J) = J$  let  $\mathcal{K}(V_{J,D})$  be as in 38.9. We shall identify  $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D}) = \mathcal{K}(V_{J,D})$  as abelian groups in such a way that the image of  $A_1$  (a character sheaf on  $V_{J,D}$ ) in  $\mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D})$  is identified with the basis element  $A_1$  of  $\mathcal{K}(V_{J,D})$ . From the definitions we see that the homomorphisms

$$\mathfrak{K}(D)/(v-1)\mathfrak{K}(D) \rightarrow \mathfrak{K}(V_{J,D})/(v-1)\mathfrak{K}(V_{J,D}) \rightarrow \mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$$

induced by  $e_J, f_J$  in 42.1(h) are then identified with the homomorphisms

$$e_{J,\mathbf{I}} : \mathcal{K}(D) \rightarrow \mathcal{K}(V_{J,D}), \quad f_{J,\mathbf{I}} : \mathcal{K}(V_{J,D}) \rightarrow \mathcal{K}(D)$$

in 38.2, 38.3. It follows that the endomorphism of  $\mathfrak{K}(D)/(v-1)\mathfrak{K}(D)$  induced by  $\mathbf{d} : \mathfrak{K}(D) \rightarrow \mathfrak{K}(D)$  is identified with the homomorphism  $\mathcal{K}(D) \rightarrow \mathcal{K}(D)$  denoted in 38.10(a), 38.11 again by  $\mathbf{d}$ . Hence we have  $\mathbf{d}(A) = \pm A' + (v-1)x$  (in  $\mathfrak{K}(D)$ ) where  $\pm, A'$  are as in 38.11(a) and  $x \in \mathfrak{K}(D)$ . From 42.1(h) we see that  $\mathbf{d}(A) \in \mathfrak{K}_0(D)$ . Since  $\pm A' \in \mathfrak{K}_0(D)$ , we see that  $(v-1)x \in \mathfrak{K}_0(D)$ . Since  $\mathfrak{K}_0(D) \cap (v-1)\mathfrak{K}(D) = 0$ , we have  $(v-1)x = 0$  and  $x = 0$ . This proves (a).

**42.3.** We have  $H = H_D \oplus H'_D$  where  $H_D$  (resp.  $H'_D$ ) is the  $\mathcal{A}$ -submodule of  $H_n$  spanned by  $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda = \lambda\}$  (resp. by  $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}, w\underline{D}\lambda \neq \lambda\}$ ). Equivalently,

$$H_D = \sum_{\lambda \in \underline{\mathfrak{s}}} 1_\lambda H 1_{\underline{D}\lambda} \subset H, \quad H'_D = \sum_{\lambda, \lambda' \in \underline{\mathfrak{s}}; \lambda \neq \lambda'} 1_{\lambda'} H 1_{\underline{D}\lambda} \subset H.$$

Recall that  $\omega : \mathfrak{K}(C) \xrightarrow{\sim} H$  is defined in 40.7(b). Define an  $\mathcal{A}$ -linear map  $\tilde{\omega} : H \rightarrow \mathfrak{K}(C)^D$  by

$$\begin{aligned} \tilde{\omega}(y) &= \omega^{-1}(y) \quad \text{if } y \in H_D, \\ \tilde{\omega}(y) &= 0 \quad \text{if } y \in H'_D. \end{aligned}$$

Then  $\eta\tilde{\omega}(y) \in \mathfrak{K}(Z_{\emptyset, D})$  is well defined for any  $y \in H$ . Here  $\eta$  is as in 41.7.

Let  $n \in \mathbf{N}_{\mathbf{k}}^*$ . Let  $H_{n, D} = H_D \cap H_n$ . Note that  $H_{n, D}$  is the  $\mathcal{A}$ -submodule of  $H_n$  spanned by  $\{\tilde{T}_w 1_{\underline{D}\lambda}; w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda\}$ .

For  $J \subset \mathbf{I}$  such that  $\epsilon(J) = J$  we define an  $\mathcal{A}$ -linear map  $\rho_{J, n} : H_{n, D} \rightarrow H_{n, D}$  by

$$\begin{aligned} \tilde{T}_w 1_{\underline{D}\lambda} &\mapsto \tilde{T}_w 1_{\underline{D}\lambda} \quad \text{if } w \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda, \\ \tilde{T}_w 1_{\underline{D}\lambda} &\mapsto 0 \quad \text{if } w \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n, w\underline{D}\lambda = \lambda. \end{aligned}$$

We have the following result.

**Lemma 42.4.** *For any  $y \in H_{n, D}$  we have  $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\delta(y))$  where*

$$\delta = \sum_{J \subset \mathbf{I}; \epsilon(J) = J} (-1)^{|J\epsilon|} \delta_J$$

with  $\delta_J : H_{n, D} \rightarrow H_{n, D}$  given by

$$\delta_J(y) = \rho_{J, n} \left( \sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} y \tilde{T}_{\epsilon_D(u)} \right)$$

(the sum in the right-hand side is computed in  $H_n$  but it belongs to  $H_{n, D}$ ).

Applying 37.2 with  $K, K', J$  repaced by  $\emptyset, J, \mathbf{I}$  and with  $A' \in \mathcal{D}^{cs}(Z_{\emptyset, D})$  we obtain

$$\mathbf{e}_{J, \mathbf{I}} \mathfrak{f} A' \simeq \{\mathfrak{f}_{\emptyset, J} \Phi_u A' [[-m_u]]; u \in \mathbf{W}^J\}$$

(in  $\mathcal{D}(Z_{J, D})$ , with  $\Phi_u : \mathcal{D}(Z_{\emptyset, D}) \rightarrow \mathcal{D}(Z_{\emptyset, D})$  as in 37.1 and  $m_u = \alpha_J - \lambda(u)$  where  $\alpha_J = \dim \mathcal{P}_J$ . Applying here  $s^*$  we obtain

$$s^* \mathbf{e}_{J, \mathbf{I}} \mathfrak{f} A' \simeq \{s^* \mathfrak{f}_{\emptyset, J} \Phi_u A' [[-m_u]]; u \in \mathbf{W}^J\}.$$

We replace  $s^* \mathbf{e}_{J, \mathbf{I}}$  by  $\tilde{e}_J$  (see 42.1(f)) and we apply  $\tilde{f}_J = \mathfrak{f}_{J, \mathbf{I}} s!$  (see 42.1(e)); we obtain

$$\tilde{f}_J \tilde{e}_J \mathfrak{f} A' \simeq \{\mathfrak{f}_{J, \mathbf{I}} s! s^* \mathfrak{f}_{\emptyset, J} \Phi_u A' [[-m_u]]; u \in \mathbf{W}^J\}.$$

Using now 42.1(g) we obtain

$$\tilde{f}_J \tilde{e}_J \mathfrak{f} A' \simeq \{\mathfrak{f}_{J, \mathbf{I}} \mathfrak{f}_{\emptyset, J} r! r^* \Phi_u A' [[-m_u]]; u \in \mathbf{W}^J\}.$$

Here we replace  $\mathfrak{f}_{J, \mathbf{I}} \mathfrak{f}_{\emptyset, J}$  by  $\mathfrak{f}$  (see 36.4(b)). This (or rather its mixed analogue) gives rise to the following equality in  $\mathfrak{K}(D)$ :

$$\tilde{f}_J \tilde{e}_J \mathfrak{f}(x') = \sum_{u \in \mathbf{W}^J} v^{2m_u} \mathfrak{f} \rho_J \Phi_u(x')$$

for any  $x' \in \mathfrak{K}(Z_{\emptyset, D})$ , or equivalently

$$f_J e_J \mathfrak{f}(x') = \sum_{u \in \mathbf{W}^J} v^{2m_u - 2\alpha_J} \mathfrak{f} \rho_J \Phi_u(x').$$

Taking  $x' = \eta(x)$  where  $x \in \mathfrak{K}(C)_n^D$  (see 41.7) and using 41.7(b) we obtain

$$\begin{aligned} & (v^2 - 1)^{2r} f_J e_J \mathfrak{f} \eta(x) \\ &= \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} v^{-2l(u)} \mathfrak{f} \rho_J \eta([u^{-1}; \lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]') \end{aligned}$$

and using 42.1(i),

$$\begin{aligned} & (v^2 - 1)^{2r} f_J e_J \mathfrak{f} \eta(x) \\ &= \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} v^{-2l(u)} \mathfrak{f} \eta \tilde{\rho}_J([u^{-1}; \lambda]' * x * [\epsilon_D(u); \underline{D}(u^{-1}\lambda)]') \end{aligned}$$

for any  $x \in \mathfrak{K}(C)_n^D$ . Here we replace  $x$  by  $\tilde{\omega}(y)$  where  $y \in H_{n, D}$  and  $\tilde{\rho}_J|_{\mathfrak{K}(C)_n^D}$  by  $\tilde{\omega}|_{H_{n, D}} \rho_{J, n} \omega_{\mathfrak{K}(C)_n^D}$ ; using 40.7(b) we obtain:

$$\begin{aligned} f_J e_J \mathfrak{f} \eta \tilde{\omega}(y) &= \sum_{u \in \mathbf{W}^J} \sum_{\lambda \in \underline{\mathfrak{s}}_n} \mathfrak{f} \eta \tilde{\omega} \rho_{J, n}(\tilde{T}_{u^{-1}} 1_{\lambda} y \tilde{T}_{\epsilon_D(u)} 1_{\underline{D}(u^{-1}\lambda)}) \\ &= \mathfrak{f} \eta \tilde{\omega} \rho_{J, n} \left( \sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}} y \tilde{T}_{\epsilon_D(u)} \right). \end{aligned}$$

The lemma is proved.

**42.5.** As in 34.12 let  $\mathfrak{U}$  be the subfield of  $\bar{\mathbf{Q}}_l$  generated by the roots of 1. Let  $\Phi : H_n^D \rightarrow \mathcal{A} \otimes_{\mathbf{Z}} H_n^{D, \infty}$  be as in 34.12 (a special case of a definition in 34.1) and let  $\Phi^1 : H_n^{D, 1} \rightarrow \mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D, \infty}$  be the specialization of  $\Phi$  for  $v = 1$  (see 34.12(b)). Let  $\tilde{\mathcal{A}} = \mathfrak{U}[v, v^{-1}]$ , let  $H_n^{D, \tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes_{\mathcal{A}} H_n^D$  and let  $\Phi^{\tilde{\mathcal{A}}} : H_n^{D, \tilde{\mathcal{A}}} \rightarrow \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D, \infty}$  be the homomorphism obtained from  $\Phi$  by extending the scalars from  $\mathcal{A}$  to  $\tilde{\mathcal{A}}$ .

Let  $E$  be an  $H_n^{D, 1}$ -module of finite dimension over  $\mathfrak{U}$ . Since  $\Phi^1$  is an isomorphism of  $\mathfrak{U}$ -algebras (see 34.12(b)) we may regard  $E$  as an  $\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D, \infty}$ -module  $E^\infty$  via  $(\Phi^1)^{-1}$ . By extension of scalars,  $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} E^\infty$  is naturally a module over

$$\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} (\mathfrak{U} \otimes_{\mathbf{Z}} H_n^{D, iy}) = \tilde{\mathcal{A}} \otimes_{\mathbf{Z}} H_n^{D, iy}$$

and this can be regarded as an  $H_n^{D, \tilde{\mathcal{A}}}$ -module  $E^{\tilde{\mathcal{A}}}$  via  $\Phi^{\tilde{\mathcal{A}}}$ .

Let  $J \subset \mathbf{I}$  be such that  $\epsilon(J) = J$ . Let  $H_{J, n}^D$  be the  $\mathcal{A}$ -algebra of  $H_n^D$  generated by  $1_\lambda, \lambda \in \underline{\mathfrak{s}}_n$  by  $\tilde{T}_w, w \in \mathbf{W}_J$  and by  $\tilde{T}_{\underline{D}}$ . Note that  $\{\tilde{T}_{w \underline{D}'} 1_\lambda; w \in \mathbf{W}_J, \underline{D}' = \text{power of } \underline{D}\}$  is an  $\mathcal{A}$ -basis of  $H_{J, n}^D$ . Let  $H_{J, n}^{D, 1} = \mathfrak{U} \otimes_{\mathcal{A}} H_{J, n}^D$  where  $\mathfrak{U}$  is regarded as an  $\mathcal{A}$ -algebra via  $v \mapsto 1$ . Let  $H_{J, n}^{D, \tilde{\mathcal{A}}} = \tilde{\mathcal{A}} \otimes_{\mathcal{A}} H_{J, n}^D$ . Note that  $H_{J, n}^{D, \tilde{\mathcal{A}}}$  is naturally a subalgebra of  $H_n^{D, \tilde{\mathcal{A}}}$ . Hence  $E^{\tilde{\mathcal{A}}}$  may be regarded as an  $H_{J, n}^{D, \tilde{\mathcal{A}}}$ -module  $(E^{\tilde{\mathcal{A}}})_J$ . This  $H_{J, n}^{D, \tilde{\mathcal{A}}}$ -module may be induced to an  $H_n^{D, \tilde{\mathcal{A}}}$ -module

$$\text{IND}((E^{\tilde{\mathcal{A}}})_J) := H_n^{D, \tilde{\mathcal{A}}} \otimes_{H_{J, n}^{D, \tilde{\mathcal{A}}}} E_J^{\tilde{\mathcal{A}}}.$$

Next,  $H_{J, n}^{D, 1}$  is naturally a subalgebra of  $H_n^{D, 1}$ . Hence  $E$  may be regarded as an  $H_{J, n}^{D, 1}$ -module  $E_J$ . This  $H_{J, n}^{D, 1}$ -module may be induced to an  $H_n^{D, 1}$ -module  $\text{ind}(E_J) := H_n^{D, 1} \otimes_{H_{J, n}^{D, 1}} E_J$ . Define an  $H_{J, n}^{D, \tilde{\mathcal{A}}}$ -module  $(\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  in terms of  $\text{ind}(E_J)$

in the same way as  $E^{\tilde{\mathcal{A}}}$  was defined in terms of  $E$ . By extension of scalars from  $\tilde{\mathcal{A}}$  to  $\mathfrak{U}(v)$  (the quotient field of  $\tilde{\mathcal{A}}$ ),  $\text{IND}((E^{\tilde{\mathcal{A}}})_J)$ ,  $(\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  give rise to  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -modules  $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} \text{IND}((E^{\tilde{\mathcal{A}}})_J)$ ,  $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\text{ind}(E_J))^{\tilde{\mathcal{A}}}$ . We show that

(a) *these two  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -modules are isomorphic.*

Since  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ ,  $H_n^{D,1}$  are (finite dimensional) semisimple algebras (see 34.12) it follows by standard arguments that it is enough to show that  $\text{IND}((E^{\tilde{\mathcal{A}}})_J)$ ,  $(\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  become isomorphic  $H_n^{D,1}$ -modules under the specialization  $v = 1$ . First we note that under the specialization  $v = 1$ ,  $E^{\tilde{\mathcal{A}}}$  becomes the  $H_n^{D,1}$ -module  $E$ . (This is because the specialization of  $\Phi^{\tilde{\mathcal{A}}}$  at  $v = 1$  cancels  $(\Phi_1)^{-1}$ .) In particular, the specialization of  $(\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  for  $v = 1$  is  $\text{ind}(E_J)$ . Moreover, from the definition of induction, the specialization of  $\text{IND}((E^{\tilde{\mathcal{A}}})_J)$  for  $v = 1$  is the same as  $\text{ind}(E'_J)$  where  $E'$  is the specialization of  $E^{\tilde{\mathcal{A}}}$  for  $v = 1$ , that is,  $E' = E$ . This proves (a).

**Lemma 42.6.** *We preserve the setup of 42.5. Let  $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}}$ ,  $\mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  be the  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -module obtained from  $E^{\tilde{\mathcal{A}}}$ ,  $(\text{ind}(E_J))^{\tilde{\mathcal{A}}}$  by extension of scalars from  $\tilde{\mathcal{A}}$  to  $\mathfrak{U}(v)$ . Let  $y \in H_{n,D}$ . We have*

$$\text{tr}(\delta_J(y)\tilde{T}_D, \mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} E^{\tilde{\mathcal{A}}}) = \text{tr}(y\tilde{T}_D, \mathfrak{U}(v) \otimes_{\tilde{\mathcal{A}}} (\text{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

Let  $H_{J,n}$  be the  $\mathcal{A}$ -subalgebra of  $H_n$  defined in 31.8. Define an  $\mathcal{A}$ -linear map  $p_J : H_n \rightarrow H_{J,n}$  by  $p_J(\tilde{T}_z 1_\lambda) = \tilde{T}_z 1_\lambda$  if  $z \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$ ,  $p_J(\tilde{T}_z 1_\lambda) = 0$  if  $z \in \mathbf{W} - \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$ . We show that

(a)  $p_J(\tilde{T}_u h') = \delta_{u,1} h'$  if  $u \in \mathbf{W}^J, h' \in H_{J,n}$ .

We may assume that  $h' = \tilde{T}_b 1_\lambda, b \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{s}}_n$ . Then  $p_J(\tilde{T}_u \tilde{T}_b 1_\lambda) = p_J(\tilde{T}_{ub} 1_\lambda) = \delta_{u,1} \tilde{T}_{ub} 1_\lambda = \delta_{u,1} \tilde{T}_b 1_\lambda$ , as required.

We show that

(b)  $p_J(hh') = p_J(h)h'$  for any  $h \in H_n, h' \in H_{J,n}$ .

We may assume  $h = \tilde{T}_u \tilde{T}_b 1_\nu, h' = \tilde{T}_a 1_\lambda, u \in \mathbf{W}^J, a, b \in \mathbf{W}_J, \lambda, \nu \in \underline{\mathfrak{s}}_n$ . We must show that  $p_J(\tilde{T}_u \tilde{T}_b 1_\nu \tilde{T}_a 1_\lambda) = p_J(\tilde{T}_u \tilde{T}_b 1_\nu) \tilde{T}_a 1_\lambda$ . If  $u \neq 1$ , both sides are zero by (a). If  $u = 1$ , both sides are  $\tilde{T}_b 1_\nu \tilde{T}_a 1_\lambda$ . This proves (b).

By 34.13(a) we have

(c)  $p_\emptyset(\tilde{T}_x \tilde{T}_{x'} 1_\lambda) = \delta_{xx',1}$  for  $x, x' \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n$ .

For  $u, u' \in \mathbf{W}^J, \lambda \in \underline{\mathfrak{s}}_n$  we write  $\tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda = \sum_{a \in \mathbf{W}} f_a \tilde{T}_a 1_\lambda$  where  $f_a \in \mathcal{A}$ . For  $a' \in \mathbf{W}_J$  we have

$$\tilde{T}_{a'^{-1}u^{-1}} \tilde{T}_{u'} 1_\lambda = \tilde{T}_{a'^{-1}} \tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda = \sum_{a \in \mathbf{W}} f_a \tilde{T}_{a'^{-1}} \tilde{T}_a 1_\lambda.$$

Applying  $p_\emptyset$  to this and using (c) gives  $f_{a'} = \delta_{u',ua'} = \delta_{a',1} \delta_{u,u'}$  so that

$$p_J(\tilde{T}_{u^{-1}} \tilde{T}_{u'} 1_\lambda) = \sum_{a \in \mathbf{W}_J} f_a \tilde{T}_a 1_\lambda = \delta_{u,u'} \tilde{T}_1 1_\lambda.$$

Since this holds for any  $\lambda \in \underline{\mathfrak{s}}_n$  we have

(d)  $p_J(\tilde{T}_{u^{-1}} \tilde{T}_{u'}) = \delta_{u,u'} \tilde{T}_1$ .

Let  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{s}}_n, u \in \mathbf{W}^J$ . We have

$$\tilde{T}_w 1_\lambda \tilde{T}_u = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,u,u',a,\lambda} \tilde{T}_{u'} \tilde{T}_a 1_{u^{-1}\lambda}$$

where  $c_{w,u,u',a,\lambda} \in \mathcal{A}$  are uniquely determined. It follows that

$$\tilde{T}_{u^{-1}}\tilde{T}_w1_\lambda\tilde{T}_{\epsilon(u)} = \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \tilde{T}_{u^{-1}}\tilde{T}_{u'}\tilde{T}_a1_{\epsilon(u)^{-1}\lambda}.$$

Applying  $p_J$  and using (b),(d) we obtain

$$\begin{aligned} p_J(\tilde{T}_{u^{-1}}\tilde{T}_w1_\lambda\tilde{T}_{\epsilon(u)}) &= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} p_J(\tilde{T}_{u^{-1}}\tilde{T}_{u'})\tilde{T}_a1_{\epsilon(u)^{-1}\lambda} \\ \text{(e)} \quad &= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \delta_{u,u'} \tilde{T}_a1_{\epsilon(u)^{-1}\lambda} = \sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \tilde{T}_a1_{\epsilon(u)^{-1}\lambda}. \end{aligned}$$

Let  $(e_i)_{i \in X}$  be a basis of the free  $\tilde{\mathcal{A}}$ -module  $E^{\tilde{\mathcal{A}}}$ . For  $a \in \mathbf{W}_J, \lambda \in \underline{\mathfrak{g}}_n$  we have  $\tilde{T}_a1_\lambda\tilde{T}_{\underline{D}}e_i = \sum_{i' \in X} \tilde{c}_{a,\lambda,i,i'}e_{i'}$  where  $\tilde{c}_{a,\lambda,i,i'} \in \tilde{\mathcal{A}}$ .

Since  $H_n^{D,\tilde{\mathcal{A}}}$  is a free right  $H_{J,n}^{D,\tilde{\mathcal{A}}}$ -module with basis  $\{\tilde{T}_u; u \in \mathbf{W}^J\}$ , we see that  $\{\tilde{T}_u \otimes e_i; u \in \mathbf{W}^J, i \in X\}$  is a basis of the free  $\tilde{\mathcal{A}}$ -module  $\text{ind}((E^{\tilde{\mathcal{A}}})_J)$ .

Let  $w \in \mathbf{W}, \lambda \in \underline{\mathfrak{g}}_n, u \in \mathbf{W}^J$  be such that  $w\underline{D}\lambda = \lambda$ . In  $\text{IND}((E^{\tilde{\mathcal{A}}})_J)$  we have

$$\begin{aligned} \tilde{T}_w1_\lambda\tilde{T}_{\underline{D}}(\tilde{T}_u \otimes e_i) &= (\tilde{T}_w1_\lambda\tilde{T}_{\epsilon(u)}\tilde{T}_{\underline{D}}) \otimes e_i \\ &= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} (\tilde{T}_{u'}\tilde{T}_a1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}) \otimes e_i \\ &= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u',a,\lambda} \tilde{T}_{u'} \otimes (\tilde{T}_a1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}e_i) \\ &= \sum_{u' \in \mathbf{W}^J, a \in \mathbf{W}_J, i' \in X} c_{w,\epsilon(u),u',a,\lambda} \tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i'} \tilde{T}_{u'} \otimes e_{i'}. \end{aligned}$$

Hence, using (e),

$$\begin{aligned} \text{tr}(\tilde{T}_w1_\lambda\tilde{T}_{\underline{D}}, \text{IND}((E^{\tilde{\mathcal{A}}})_J)) &= \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J, i \in X} c_{w,\epsilon(u),u,a,\lambda} \tilde{c}_{a,\epsilon(u)^{-1}\lambda,i,i} \\ &= \sum_{u \in \mathbf{W}^J, a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \text{tr}(\tilde{T}_a1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\ &= \sum_{u \in \mathbf{W}^J} \text{tr}(\sum_{a \in \mathbf{W}_J} c_{w,\epsilon(u),u,a,\lambda} \tilde{T}_a1_{\epsilon(u)^{-1}\lambda}\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\ &= \text{tr}(p_J(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}}\tilde{T}_w1_\lambda\tilde{T}_{\epsilon(u)})\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) \\ &= \text{tr}(\rho_{J,n}(\sum_{u \in \mathbf{W}^J} \tilde{T}_{u^{-1}}\tilde{T}_w1_\lambda\tilde{T}_{\epsilon(u)})\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \text{tr}(\delta_J(\tilde{T}_w1_\lambda)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}). \end{aligned}$$

Thus we have

$$\text{tr}(\delta_J(\tilde{T}_w1_\lambda)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \text{tr}(\tilde{T}_w1_\lambda\tilde{T}_{\underline{D}}, \text{IND}((E^{\tilde{\mathcal{A}}})_J)) = \text{tr}(\tilde{T}_w1_\lambda\tilde{T}_{\underline{D}}, (\text{ind}(E_J))^{\tilde{\mathcal{A}}})$$

where the second equality follows from 42.5(a). Since the elements  $\tilde{T}_w1_\lambda$  as above generate the  $\mathcal{A}$ -module  $H_{n,D}$ , the lemma follows.

**42.7.** Let  $\mathcal{V}$  be the  $\mathbf{Q}$ -vector subspace of  $\mathbf{Q} \otimes \text{Hom}(\mathbf{k}^*, \mathbf{T})$  spanned by the coroots. Let  $\mathcal{V}_{\mathbf{R}} = \mathbf{R} \otimes_{\mathbf{Q}} \mathcal{V}$ . The kernels of the roots  $\mathcal{V}_{\mathbf{R}} \rightarrow \mathbf{R}$  a hyperplane arrangement which defines a partition of  $\mathcal{V}_{\mathbf{R}}$  into facets in a standard way. Let  $\mathcal{F}$  be the set of facets. Now the orbits of  $\mathbf{W}$  on  $\mathcal{F}$  are naturally indexed by the various subsets  $J$

of  $\mathbf{I}$ . This gives a partition  $\mathcal{F} = \bigsqcup_{J \subset \mathbf{I}} \mathcal{F}_J$ . For example,  $\mathcal{F}_\emptyset$  consists of all Weyl chambers. If  $F \in \mathcal{F}_J$ , then  $F$  is homeomorphic to a real affine space of dimension  $|\mathbf{I} - J|$  hence we have  $H_c^i(F) = 0$  if  $i \neq |\mathbf{I} - J|$  and  $H_c^{|\mathbf{I} - J|}(F) = \Lambda^{|\mathbf{I} - J|}[F]$ ; here we write  $H_c^i(?)$  instead of  $H_c^i(?, \mathbf{R})$ ,  $[F]$  denotes the vector subspace of  $\mathcal{V}_{\mathbf{R}}$  in which  $F$  is open dense and  $\Lambda^{|\mathbf{I} - J|}[F]$  is the top exterior power of  $[F]$ . Note that  $[F] = \mathbf{R} \otimes_{\mathbf{Q}} ([F]_{\mathbf{Q}})$  for a well-defined  $\mathbf{Q}$ -subspace  $[F]_{\mathbf{Q}}$  of  $\mathcal{V}$ . For any  $\underline{D}$ -orbit  $\eta$  on the set of subsets of  $\mathbf{I}$  let  $\mathcal{V}_{\mathbf{R}}^\eta = \bigcup_{J \in \eta} \bigcup_{F \in \mathcal{F}_J} F \subset \mathcal{V}_{\mathbf{R}}$  and let  $r_\eta = |\mathbf{I} - J|$  for some/any  $J \in \eta$ . We have  $H_c^i(\mathcal{V}_{\mathbf{R}}^\eta) = 0$  if  $i \neq r_\eta$ ,  $H_c^{r_\eta}(\mathcal{V}_{\mathbf{R}}^\eta) = \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]$ . Note also that  $H_c^i(\mathcal{V}_{\mathbf{R}}) = 0$  if  $i \neq |\mathbf{I}|$  and  $H_c^{|\mathbf{I}|}(\mathcal{V}_{\mathbf{R}}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}}$ . The  $\mathbf{W}^D$ -action on  $\mathbf{T}$  induces a linear action of  $\mathbf{W}^D$  on  $\mathcal{V}_{\mathbf{R}}$ . This action restricts for any  $\eta$  to a  $\mathbf{W}^D$ -action on  $\mathcal{V}_{\mathbf{R}}^\eta$  and this induces a  $\mathbf{W}^D$ -action on  $H_c^{r_\eta}(\mathcal{V}_{\mathbf{R}}^\eta)$ . It also induces a natural  $\mathbf{W}^D$ -action on  $H_c^{|\mathbf{I}|}(\mathcal{V}_{\mathbf{R}}) = \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}}$ . The long cohomology exact sequences attached to the partition  $\mathcal{V}_{\mathbf{R}} = \bigcup_{\eta} \mathcal{V}_{\mathbf{R}}^\eta$  show that  $(-1)^{|\mathbf{I}|} H_c^{|\mathbf{I}|}(\mathcal{V}_{\mathbf{R}}) = \sum_{\eta} (-1)^{r_\eta} H_c^{r_\eta}(\mathcal{V}_{\mathbf{R}}^\eta)$  in the Grothendieck group of representations of  $\mathbf{W}^D$  over  $\mathbf{R}$ , that is,

$$\begin{aligned} & \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathbf{R}} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1} \bigoplus_{\text{mod } 2} \left( \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F] \right) \\ & \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}|} \bigoplus_{\text{mod } 2} \left( \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F] \right) \end{aligned}$$

as representations of  $\mathbf{W}^D$  over  $\mathbf{R}$ . All real representations in this formula come naturally from representations of  $\mathbf{W}^D$  over  $\mathbf{Q}$ . Hence the previous formula remains valid (as representations of  $\mathbf{W}^D$  over  $\mathbf{Q}$ ) if  $\mathcal{V}_{\mathbf{R}}, [F]$  are replaced by  $\mathcal{V}, [F]_{\mathbf{Q}}$  and the exterior powers are taken over  $\mathbf{Q}$ . Tensoring both sides (over  $\mathbf{Q}$ ) by  $\mathfrak{U}$  (as in 42.5) we obtain

$$\begin{aligned} & \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1} \bigoplus_{\text{mod } 2} \left( \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}} \right) \\ \text{(a)} \quad & \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}|} \bigoplus_{\text{mod } 2} \left( \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}} \right) \end{aligned}$$

as representations of  $\mathbf{W}^D$  over  $\mathfrak{U}$ ; here  $\mathcal{V}_{\mathfrak{U}} = \mathfrak{U} \otimes_{\mathbf{Q}} \mathcal{V}$ ,  $[F]_{\mathfrak{U}} = \mathfrak{U} \otimes_{\mathbf{Q}} [F]_{\mathbf{Q}}$  and the exterior powers are taken over  $\mathfrak{U}$ . We may view (a) as an isomorphism of  $H_n^{D,1}$ -modules: the  $\mathbf{W}^D$ -modules in (a) may be viewed as  $H_n^{D,1}$ -modules via the algebra homomorphism  $H_n^{D,1} \rightarrow \mathfrak{U}[\mathbf{W}^D]$  given by  $\tilde{T}_w \mapsto w$  for  $w \in \mathbf{W}^D$ ,  $1_\lambda \mapsto 0$  for  $\lambda \neq \lambda_0$ ,  $1_{\lambda_0} \mapsto 1$  (here  $\lambda_0$  is the neutral element of the abelian group  $\underline{\mathfrak{s}}_n$ ; see 28.1).

We define an  $\mathfrak{U}$ -linear map  $\Delta : H_n^{D,1} \rightarrow H_n^{D,1} \otimes H_n^{D,1}$  by  $\Delta(\tilde{T}_w) = \tilde{T}_w \otimes \tilde{T}_w$  for  $w \in \mathbf{W}^D$  and  $\Delta(1_\lambda) = \sum_{\lambda_1, \lambda_2 \in \underline{\mathfrak{s}}_n; \lambda_1 \lambda_2 = \lambda} 1_{\lambda_1} \otimes 1_{\lambda_2}$  for  $\lambda \in \underline{\mathfrak{s}}_n$ . (Here we use the abelian group structure on  $\underline{\mathfrak{s}}_n$ , a subgroup of  $\underline{\mathfrak{s}}$ ; see 28.1.) This makes  $H_n^{D,1}$  into a Hopf algebra. (Note that the analogous formulas do not make  $H_n^D$  into a Hopf algebra.) It follows that for any two  $H_n^{D,1}$ -modules  $E_1, E_2$ , the  $\mathfrak{U}$ -vector space  $E_1 \otimes E_2$  is naturally an  $H_n^{D,1}$ -module.

Now let  $E$  be an  $H_n^{D,1}$ -module of finite dimension over  $\mathfrak{U}$ . Then we can take the tensor product of each  $H_n^{D,1}$ -module in (a) with  $E$  and we obtain an isomorphism of  $H_n^{D,1}$ -modules

$$E \otimes \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1} \bigoplus_{\text{mod } 2} X_\eta \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}|} \bigoplus_{\text{mod } 2} X_\eta$$

where  $X_\eta = E \otimes \bigoplus_{J \in \eta} \bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}}$ . Applying to this the functor  $E \mapsto E^{\tilde{\mathcal{A}}}$  (see 42.5), we deduce an isomorphism of  $H_n^{D, \tilde{\mathcal{A}}}$ -modules

$$(E \otimes \Lambda^{|\mathbf{I}|} \mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}} \oplus \bigoplus_{\eta; r_\eta = |\mathbf{I}|+1 \pmod 2} X_\eta^{\tilde{\mathcal{A}}} \cong \bigoplus_{\eta; r_\eta = |\mathbf{I}| \pmod 2} X_\eta^{\tilde{\mathcal{A}}}.$$

We deduce that for  $y \in H_{n, D}$  we have

$$(b) \quad \text{tr}(y\tilde{T}_{\underline{D}}, (E \otimes \Lambda^{|\mathbf{I}|} \mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}) = \sum_{\eta} (-1)^{r_\eta + |\mathbf{I}|} \text{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}).$$

We have  $X_\eta = \bigoplus_{J \in \eta} X^J$  where  $X^J = E \otimes (\bigoplus_{F \in \mathcal{F}_J} \Lambda^{r_\eta}[F]_{\mathfrak{U}})$ .

Assume first that  $\eta$  consists of at least two subsets of  $\mathbf{I}$ . Then each  $X_J$  is stable under  $H_n^{D, 1}$  and is mapped by  $\tilde{T}_{\underline{D}}$  into  $X_{J'}$  with  $J \neq J'$ . From the definitions we have  $X_\eta^{\tilde{\mathcal{A}}} = \bigoplus_{J \in \eta} \tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_J$  as an  $\tilde{\mathcal{A}}$ -module and each summand  $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_J$  is stable under  $H_n$  and is mapped by  $\tilde{T}_{\underline{D}}$  into a summand  $\tilde{\mathcal{A}} \otimes_{\mathfrak{U}} X_{J'}$  with  $J \neq J'$ . It follows that for our  $\eta$  we have

$$(c) \quad \text{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}) = 0.$$

Next assume that  $\eta$  consists of a single subset  $J$  of  $\mathbf{I}$ . We have  $\underline{D}(J) = J$ . Let  $F_J$  be the unique facet in  $\mathcal{F}_J$  such that  $F_J$  is contained in the closure of the dominant Weyl chamber. Then  $F_J$  is stable under the the subgroup  $\mathbf{W}_J^D$  of  $\mathbf{W}^D$  generated by  $\mathbf{W}_J$  and  $\underline{D}$  and  $X_\eta$  may be identified with  $E \otimes (H_n^{D, 1} \otimes_{H_{J, n}^{D, 1}} (\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}))$ . Here  $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$  is regarded as a  $WW_J^D$ -module and then is viewed as a  $H_{J, n}^{D, 1}$ -module via the canonical algebra homomorphism  $H_{J, n}^{D, 1} \rightarrow \mathfrak{U}[\mathbf{W}_J^D]$ ; thus  $1_\lambda$  acts on it as 1 if  $\lambda = \lambda_0$  and as 0 if  $\lambda \neq \lambda_0$ . Note that in the  $\mathbf{W}_J^D$ -module  $\Lambda^{|\mathbf{I}-J|}[F_J]_{\mathfrak{U}}$ ,  $\mathbf{W}_J$  acts trivially (since  $\mathbf{W}_J$  acts trivially on  $[F_J]_{\mathfrak{U}}$ ) and  $\underline{D}$  acts as multiplication by  $(-1)^{|\mathbf{I}-J| - |(\mathbf{I}-J)^\epsilon|}$ . Let  $X'_\eta = E \otimes (H_n^{D, 1} \otimes_{H_{J, n}^{D, 1}} \mathfrak{U})$  where  $\mathfrak{U}$  is regarded as a  $H_{J, n}^{D, 1}$ -module coming from the trivial representation of  $\mathbf{W}_J^D$ . We see that we may identify  $X_\eta, X'_\eta$  in a way compatible with the  $H_n^1$ -module structures and so that the action of  $\tilde{T}_{\underline{D}}$  on  $X_\eta$  corresponds to  $(-1)^{|\mathbf{I}-J| - |(\mathbf{I}-J)^\epsilon|}$  times the action of  $\tilde{T}_{\underline{D}}$  on  $X'_\eta$ . Using the definitions we see that we may identify  $X_\eta^{\tilde{\mathcal{A}}}, X'^{\tilde{\mathcal{A}}}$  in a way compatible with the  $H_n$ -module structures and so that the action of  $\tilde{T}_{\underline{D}}$  on  $X_\eta^{\tilde{\mathcal{A}}}$  corresponds to  $(-1)^{|\mathbf{I}-J| - |(\mathbf{I}-J)^\epsilon|}$  times the action of  $\tilde{T}_{\underline{D}}$  on  $X'^{\tilde{\mathcal{A}}}$ . From the definitions we have  $X'_\eta = \text{ind}(E_J)$  (notation of 42.5). We see that for our  $\eta$  we have

$$(d) \quad \text{tr}(y\tilde{T}_{\underline{D}}, X_\eta^{\tilde{\mathcal{A}}}) = (-1)^{|\mathbf{I}-J| - |(\mathbf{I}-J)^\epsilon|} \text{tr}(y\tilde{T}_{\underline{D}}, (\text{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

From the definitions (34.4) we see that there is a unique  $\tilde{\mathcal{A}}$ -algebra homomorphism  $\vartheta : H_n^{D, \tilde{\mathcal{A}}} D \rightarrow H_n^{D, \tilde{\mathcal{A}}}$  such that

$$\begin{aligned} \vartheta(1_\lambda) &= 1_\lambda \text{ for any } \lambda \in \underline{\mathfrak{s}}_n, \\ \vartheta(\tilde{T}_w) &= (-1)^{l(w)} \tilde{T}_{w^{-1}}^{-1} \text{ for any } w \in \mathbf{W}, \\ \vartheta(\tilde{T}_{\underline{D}}) &= (-1)^{|\mathbf{I}| - |\mathbf{I}^\epsilon|} \tilde{T}_{\underline{D}}. \end{aligned}$$

We have  $\vartheta^2 = 1$ .

Using  $\vartheta$  and  $E^{\tilde{\mathcal{A}}}$  we can define a new  $H_n^{D, \tilde{\mathcal{A}}}$ -module  $E^{\tilde{\mathcal{A}}, \vartheta}$  with the same underlying  $\tilde{\mathcal{A}}$ -module as  $E^{\tilde{\mathcal{A}}}$  but with  $x \in H_n^{D, \tilde{\mathcal{A}}}$  acting on  $E^{\tilde{\mathcal{A}}, \vartheta}$  in the same way that  $\vartheta(x)$  acts on  $E^{\tilde{\mathcal{A}}}$ . We show that



(e) under extension of scalars from  $\tilde{\mathcal{A}}$  to  $\mathfrak{U}(v)$ , the  $H_n^{D,\tilde{\mathcal{A}}}$ -modules  $E^{\tilde{\mathcal{A}},\vartheta}$  and  $(E \otimes \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}})^{\tilde{\mathcal{A}}}$  become isomorphic  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -modules.

As in the proof of 42.5(a) it is enough to show that these  $H_n^{D,\tilde{\mathcal{A}}}$ -modules become isomorphic  $H_n^{D,1}$ -modules under the specialization  $v = 1$ . Thus it is enough to show that  $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1} \cong E \otimes \Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$  as  $H_n^{D,1}$ -modules. Now the underlying  $\mathfrak{U}$ -vector space of  $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$  is  $E$  but the action of  $x \in H_n^{D,1}$  on  $E^{\tilde{\mathcal{A}},\vartheta}|_{v=1}$  is the same as the action of  $\vartheta_1(x)$  on  $E$ . Here  $\vartheta_1 : H_n^{D,1} \rightarrow H_n^{D,1}$  is the specialization of  $\vartheta_1$  for  $v = 1$ . Note that  $\vartheta_1(1_\lambda) = 1_\lambda$  for any  $\lambda \in \underline{\mathfrak{g}}_n$  and  $\vartheta_1(\tilde{T}_w) = \gamma_w \tilde{T}_w$  for any  $w \in \mathbf{W}^D$ , where  $\gamma_w = \pm 1$  is the scalar by which  $w$  acts in the  $\mathbf{W}^D$ -module  $\Lambda^{|\mathbf{I}|}\mathcal{V}_{\mathfrak{U}}$ . The desired result follows.

Combining (b),(c),(d),(e) we see that for any  $y \in H_{n,D}$  we have

$$(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|} \operatorname{tr}(\vartheta(y\tilde{T}_{\underline{D}}), E^{\tilde{\mathcal{A}}}) = \sum_{J \subset \mathbf{I}; \epsilon(J)=J} (-1)^{|J_\epsilon|} \operatorname{tr}(y\tilde{T}_{\underline{D}}, (\operatorname{ind}(E_J))^{\tilde{\mathcal{A}}}).$$

Replacing here  $(-1)^{|\mathbf{I}|+|\mathbf{I}_\epsilon|} \vartheta(y\tilde{T}_{\underline{D}})$  by  $\vartheta(y)\tilde{T}_{\underline{D}}$  and using Lemma 42.6 we may rewrite this as

$$\operatorname{tr}(\vartheta(y)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \sum_{J \subset \mathbf{I}; \epsilon(J)=J} (-1)^{|J_\epsilon|} \operatorname{tr}(\delta_J(y)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}})$$

or equivalently (see 42.4)  $\operatorname{tr}(\vartheta(y)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}}) = \operatorname{tr}(\delta(y)\tilde{T}_{\underline{D}}, E^{\tilde{\mathcal{A}}})$ . Since any simple  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -module can be obtained by extension of scalars (from  $\tilde{\mathcal{A}}$  to  $\mathfrak{U}(v)$ ) from some  $E^{\tilde{\mathcal{A}}}$  as above, we deduce that

$$\operatorname{tr}((\delta(y) - \vartheta(y))\tilde{T}_{\underline{D}}, \mathbf{E}) = 0$$

for any simple  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ -module  $\mathbf{E}$ . Since  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$  is a semisimple algebra, it follows that  $(\delta(y) - \vartheta(y))\tilde{T}_{\underline{D}}$  belongs to the  $\mathfrak{U}(v)$ -subspace of  $\mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$  spanned by commutators  $xx' - x'x$  with  $x, x' \in \mathfrak{U}(v) \otimes_{\mathcal{A}} H_n^D$ . Hence we have

$$g(\delta(y) - \vartheta(y))\tilde{T}_{\underline{D}} = \sum_{i=1}^m g_i(x_i \tilde{T}_{\underline{D}}^{s_i} x'_i \tilde{T}_{\underline{D}}^{1-s_i} - x'_i \tilde{T}_{\underline{D}}^{1-s_i} x_i \tilde{T}_{\underline{D}}^{s_i})$$

with  $g \in \mathcal{A} - \{0\}$ ,  $g_i \in \mathcal{A}$ ,  $x_i \in H_n$ ,  $x'_i \in H_n$ ,  $s_i \in \mathbf{Z}$ , that is,

$$(f) \quad g(\delta(y) - vt(y)) = \sum_{i=1}^m g_i(x_i \tilde{T}_{\underline{D}}^{s_i} x'_i \tilde{T}_{\underline{D}}^{-s_i} - x'_i \tilde{T}_{\underline{D}}^{1-s_i} x_i \tilde{T}_{\underline{D}}^{s_i-1}).$$

**42.8.** We show that for any  $y, y' \in H_n$  we have

$$(a) \quad \mathfrak{f}\eta\tilde{\omega}(yy' - y'\tilde{T}_{\underline{D}}y\tilde{T}_{\underline{D}}^{-1}) = 0.$$

Let  $w \in \mathbf{W}$ ,  $\lambda \in \underline{\mathfrak{g}}_n$ . Let  $\mathcal{L} \in \lambda$ . If  $w\underline{D}\lambda = \lambda$ , using notation and results in 31.6, 31.7, we have

$$\begin{aligned} \mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w 1_{\underline{D}\lambda}) &= gr(K_{\mathbf{I},\underline{D}}^{w,\mathcal{L}}) \\ &= \sum_A \chi_v^A(K_{\mathbf{I},\underline{D}}^{w,\mathcal{L}}) = \sum_A \zeta^A(v^{l(w)}\tilde{T}_w 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}) = \sum_A \zeta^A(v^{l(w)}\tilde{T}_w 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}}) \end{aligned}$$

(the last equation comes from 31.7(e);  $A$  runs over the objects in  $\hat{D}$  up to isomorphism such that  $\zeta^A \neq 0$ .) The equation

$$\mathfrak{f}\eta\tilde{\omega}(v^{l(w)}\tilde{T}_w 1_{\underline{D}\lambda}) = \sum_A \zeta^A(v^{l(w)}\tilde{T}_w 1_{\underline{D}\lambda}\tilde{T}_{\underline{D}})$$

holds also if  $w\underline{D}\lambda \neq \lambda$  (in this case both sides are 0). It follows that

$$\mathfrak{f}\eta\tilde{\omega}(x) = \sum_A \zeta^A(x\tilde{T}_D) \text{ for any } x \in H_n.$$

We deduce

$$\mathfrak{f}\eta\tilde{\omega}(yy' - y'\tilde{T}_D y\tilde{T}_D^{-1}) = \sum_A (\zeta^A(yy'\tilde{T}_D) - \zeta^A(y\epsilon(y)\tilde{T}_D)) = 0$$

where the last equality follows from 31.8. This proves (a).

**Proposition 42.9.** *Let  $y \in H$ . We have  $\mathbf{d}(\mathfrak{f}\eta\tilde{\omega}(y)) = \mathfrak{f}\eta\tilde{\omega}(\vartheta(y)) \in \mathfrak{K}(D)$  with  $\mathbf{d} : \mathfrak{K}(D) \rightarrow \mathfrak{K}(\Delta)$  as in 4.2.2.*

If  $y \in H'_D$  (see 42.3), both sides of the desired equality are 0. (Note that  $\vartheta$  maps  $H_D$  into itself and  $H'_D$  into itself.) Hence we may assume that  $y \in H_D$ . We can assume that  $y \in H_n$  where  $n \in \mathbf{N}_k^*$ . Then  $y \in H_{n,D}$ . By 42.4 it is enough to show that  $\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$ . Let  $g, g_i, x_i, x'_i, s_i$  be as in 42.7(f). Since  $g \neq 0$ , it is enough to show that  $g\mathfrak{f}\eta\tilde{\omega}(\delta(y) - \vartheta(y)) = 0$  or that  $\mathfrak{f}\eta\tilde{\omega}(g(\delta(y) - \vartheta(y))) = 0$ . Using 42.7 it is enough to show that

$$\mathfrak{f}\eta\tilde{\omega}\left(\sum_{i=1}^m g_i(x_i\tilde{T}_D^{s_i}x'_i\tilde{T}_D^{-s_i} - x'_i\tilde{T}_D^{1-s_i}x_i\tilde{T}_D^{s_i-1})\right) = 0.$$

Hence it is enough to show that

$$\mathfrak{f}\eta\tilde{\omega}(x\tilde{T}_D^s x'\tilde{T}_D^{-s} - x'\tilde{T}_D^{1-s} x\tilde{T}_D^{s-1}) = 0$$

for any  $x, x' \in H_n$  and any  $s \in \mathbf{Z}$ . We have

$$x\tilde{T}_D^s x'\tilde{T}_D^{-s} - x'\tilde{T}_D^{1-s} x\tilde{T}_D^{s-1} = (z - \tilde{T}_D^{-s} z\tilde{T}_D^s) + (z'x' - x'\tilde{T}_D z'\tilde{T}_D^{-1})$$

where  $z = x\tilde{T}_D^s x'\tilde{T}_D^{-s} \in H_n$  and  $z' = \tilde{T}_D^{-s} x\tilde{T}_D^s \in H_n$ . Hence it is enough to show that  $\mathfrak{f}\eta\tilde{\omega}(z'x' - x'\tilde{T}_D z'\tilde{T}_D^{-1}) = 0$  (see 42.8(a)) and

$$(a) \mathfrak{f}\eta\tilde{\omega}(z - \tilde{T}_D^{-s} z\tilde{T}_D^s) = 0$$

for any  $z \in H_n$ . This follows from 41.6(c).

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