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HOLOMORPHIC CONTINUATION OF GENERALIZED JACQUET INTEGRALS FOR DEGENERATE PRINCIPAL SERIES

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ABSTRACT. This paper introduces a class of parabolic subgroups of real reductive groups (called "very nice"). For these parabolic subgroups we study the generalized Whittaker vectors for their degenerate principal series. It is shown that there is a holomorphic continuation of the Jacquet integrals associated with generic characters of their unipotent radicals. Also, in this context an analogue of the "multiplicity one" theorem is proved. Included is a complete classification of these parabolic subgroups (due to K. Baur and the author). These parabolic subgroups include all known examples of such continuations and multiplicity theorems.

1. Introduction

The theory of Whittaker vectors and Jacquet integrals was initiated in the thesis of Jacquet [J] (this is the reason for the name of the integrals) under the direction of Godement. The importance of this theory derives from the fact that the generic Fourier coefficients (suitably interpreted) of an automorphic form at a cusp can be expressed in terms of Whittaker vectors on the spaces of smooth vectors of admissible representations. Since a significant part of the application of the theory of automorphic forms to number theory involves these Fourier coefficients the theory of Whittaker vectors has a major role in the theory.

In this paper we give a definitive treatment of Whittaker vectors for generic characters of unipotent radicals of a class of parabolic subgroups (called very nice and defined in the next section) and prove a holomorphic continuation of Jacquet integrals for the induced representations from finite dimensional representations of the parabolic subgroup. In addition, an analogue of the multiplicity one theorem is proved. The precise forms of these theorems can be found in the next section with a discussion of earlier related work. We also observe that the class of very nice parabolic subgroups is a part of those for which the vanishing theorem of Lynch is not vacuous (these parabolic subgroups are called nice in [BW], see Theorem 3.1). In the case when all of the simple factors are of type A_n, then every nice parabolic subgroup is very nice. In Section 6 we give a condition that depends only on the complexification of P that implies that P is very nice. In that section we will also give the complete classification (proofs will appear in [BW2]) of parabolic subgroups (over the complexes) satisfying this criterion. We note that the class is

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a substantial part of the set of nice parabolic subgroups over $\mathbb{C}$ (for example, in the case of E_8 and E_6, all of the nice parabolic subgroups are very nice and for E_7 all but 3 of the nice are very nice). The set of real forms of this class contains all minimal parabolic subgroups, parabolic subgroups for groups over $\mathbb{R}$ or $\mathbb{C}$ whose unipotent radical is abelian or if the group has no factors of type C_n of Heisenberg type over the corresponding field (these results are in Section 6). It also contains all parabolic subgroups that are defined by the "h" part of a standard $\mathfrak{sl}_2$ triple corresponding to an even nilpotent element (see Corollary 2.3, this case contains all of the pertinent Heisenberg examples). In fact, the results in [BW2] imply that all non-simply laced examples are given in this way.

As mentioned above, in [BW2] we give a complete classification of the parabolic subgroups that satisfy the conditions in Theorem 6.2 in this paper which gives a manageable sufficient condition for very niceness. Thus to see if the results in this paper apply one need only check to see if the complexification of the parabolic subgroup in question is on the list that is repeated here. The experts in the more delicate aspects of the structure of Richardson elements and the connectivity properties of stabilizers of nilpotent elements would have no problem proving these conditions (using the results in [He]). However, this paper is not aimed at experts in the structure of nilpotent elements. Rather, it is aimed at people who only wish to know that the Jacquet integrals in question have the properties that they need. Furthermore, in many cases the exact forms of the results needed are not explicitly stated in the literature.

This paper is a continuation of the author's earlier work in this subject [Wa1]. In that paper the conclusions 1, 2, and 3 in the next section were proved under very complicated hypotheses. Also, there was a conjecture which had as a consequence the corollary in the next section. This conjecture is false (but not the consequence) and the first counterexample appears in [Wa3] for split G_2 and thereby for almost all cases of a parabolic with a unipotent radical isomorphic with a Heisenberg group. Since that was exactly the case of interest in [Wa3], a new approach to the Bruhat theory was necessary in order to prove the needed results. The work of Kolk and Varadarajan [KV] gave precisely the extension needed. In this paper, the most general result in this context is given and with similar use of [KV].

2. Notation and main results

Let G be a real reductive group of inner type with compact center and let P be a parabolic subgroup of G. Let N be the unipotent radical of P and let K be a maximal compact subgroup of G with corresponding Cartan involution θ. Let $P = MAN$ be a Langlands decomposition of P. Let $P_o = M_oA_oN_o$ be a minimal parabolic subgroup of G such that $M_o \subset M$, $A \subset A_o$, $N \subset N_o$. Set $\overline{N} = \theta(N)$. We will use lower case fraktur letters for Lie algebras so $\mathfrak{a} = Lie(A)$, $\mathfrak{n} = Lie(N)$. Let (σ, H_σ) be a smooth Fréchet representation of M of moderate growth (for the purposes of this paper one can assume that H_σ is continuous and finite dimensional since then it is automatically a smooth Fréchet representation of moderate growth) and let $\nu \in \mathfrak{a}^*_{\mathbb{C}}$; then we can form the smooth induced representation

$$I^\infty_{P.\sigma,\nu} = \{f \in C^\infty(G; H_\sigma) | f(namg) = a^{\nu+\rho}\sigma(m)f(g),$$
$$n \in N, a \in A, m \in M, g \in G\}$$

with the C^∞ topology and the action of G by right translation

$$\pi_{P,\sigma,\nu}(g)f(x) = f(xg).$$

We note that since $G = PK$ if $f \in I^\infty_{P.\sigma,\nu}$, then $f_{|K} \in I^\infty_{\sigma_{|M\cap K}}$ (the smooth induced representation of $\sigma_{|K\cap M}$ from $K \cap M$ to K) and we have an inverse map. That is, if $f \in I^\infty_{\sigma_{|M\cap K}}$, then we set $f_{P,\sigma,\nu}(namk) = a^{\nu+\rho}\sigma(m)f(k)$, $n \in N$, $a \in A$, $m \in M$, $k \in K$. Finally, let $\chi : \overline{N} \to S^1$ be a unitary character of $\overline{N}$. If $f \in I^\infty_{\sigma_{|M\cap K}}$, then we define (after having chosen a Haar measure on $\overline{N}$)

$$J^\chi_{P.\sigma,\nu}(f) = \int_{\overline{N}} f_{P,\sigma,\nu}(\overline{n})\chi(\overline{n})^{-1}d\overline{n}.$$

Fix B, a G-invariant real-valued bilinear form on $\mathfrak{g}$ such that $B(\theta X, X) < 0$ for all $X \in \mathfrak{g}$. Then $B_{|\mathfrak{a}}$ is positive definite. We write $(\ldots,\ldots)$ for the dual form on $\mathfrak{a}^*$ and for the complex bilinear extension to the complexification. It is not hard to show that there exists a constant C_σ (depending on σ) such that in the set of all ν satisfying

$$\mathrm{Re}(\nu, \alpha) > C_\sigma, \alpha \in \Phi(P, A)$$

the integral defining $J^\chi_{P,\sigma,\nu}(f)$ converges absolutely and uniformly on compacta. Thus defining for each ν satisfying the above inequality a continuous functional on $I^\infty_{\sigma|_{M\cap K}}$.

One desires a meromorphic continuation of these integrals to all of $\mathfrak{a}^*_{\mathbb{C}}$. To be precise here is our "wish list" for "generic characters" χ:

1. The map $\nu \mapsto J^\chi_{P.\sigma,\nu}(f)$ extends to a holomorphic map of $\mathfrak{a}^*_{\mathbb{C}}$ to H_σ for all $f \in I^\infty_{\sigma_{|P\cap K}}$.

2. Set

$$Wh_\chi(I^\infty_{P,\sigma,\nu}) = \{T \in \left(I^\infty_{P,\sigma,\nu}\right)' | T(\pi_{P,\sigma,\nu}(\overline{n})f) = \chi(\overline{n})^{-1}T(f), \overline{n} \in \overline{N}, f \in I^\infty_{P,\sigma,\nu}\}.$$

Here the upper prime means continuous functionals. Then if $T \in Wh_\chi(I^\infty_{P,\sigma,\nu})$, then there exists $\lambda \in (H_\sigma)'$ such that $T = \lambda \circ J^\chi_{P.\sigma,\nu}$.

3. Set $M_\chi = \{m \in M | \chi(m\overline{n}m^{-1}) = \chi(\overline{n})$ for all $\overline{n} \in \overline{N}$. Then we can define an action of M_χ on $Wh_\chi(I^\infty_{P,\sigma,\nu})$ via $\pi'_{P,\sigma.\nu}(m)T = T \circ \pi_{P,\sigma.\nu}(m^{-1})$. The map $(H_\sigma)' \to Wh_\chi(I^\infty_{P,\sigma,\nu})$ given by $\lambda \mapsto \lambda \circ J^\chi_{P.\sigma,\nu}$ is an equivalence of representations of M_χ.

Let $\chi : \overline{N} \to S^1$ be a unitary character. Then there exists $x \in \mathfrak{n}(= Lie(N))$ such that $d\chi(Y) = iB(x, Y)$ for $Y \in \theta\mathfrak{n}$.

Definition 2.1. A unitary character, χ, of $\overline{N}$ is said to be non-degenerate if there exists $x \in \mathfrak{n}$ such that $Ad(P)x$ is open in $\mathfrak{n}$ and $d\chi(Y) = iB(x, Y)$ for $Y \in \theta\mathfrak{n}$.

In other words, the orbit of x is a Richardson element in $\mathfrak{n}_{\mathbb{C}}$. We note that there exist parabolic groups such that there are no such non-degenerate characters of $\overline{N}$. Here is an example that was communicated to us by McGovern. Let $G = SL(5, \mathbb{R})$ and let P be the parabolic subgroup consisting of matrices of the form

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}.$$

One can show by a direct calculation that of $x \in \mathfrak{n}$ and if $B(x, \theta[\mathfrak{n}, \mathfrak{n}]) = 0$, then $[\mathfrak{p}, x] \neq \mathfrak{n}$. We will say that a parabolic subgroup of G is "nice" if there exists $x \in \mathfrak{n}$ such that $B(x, \theta[\mathfrak{n}, \mathfrak{n}]) = 0$ and $[\mathfrak{p}, x] = \mathfrak{n}$. If P is nice, then our condition of non-degenerate is generic (i.e., defines an open dense subset of the characters). We also note that this notion of non-degenerate is the same as that in [Wa1] (see Lemma 3.3 below) so all of the examples in Section 4 of [Wa1] are examples of nice parabolic subgroups. In Section 6 we will study the condition of niceness more thoroughly. We will in fact need a stronger condition that we will call *strongly non-degenerate*. We will now describe the additional condition.

Let $\overline{N}_o$ be the nilradical of the opposite parabolic subgroup to P_o corresponding to the above choice of Langlands' decomposition. Let $W(A_o) = W$ be the Weyl group of A_o and let Φ^+ denote the roots of A_o in $\mathrm{Lie}(N_o)$. Let Φ_M^- denote the set of roots in $-\Phi^+$ such that the corresponding root space is contained in M. The additional condition is that if $s \in W$ is such that $s\Phi^+ \supset \Phi_M^-$, if s^* is a representative of s in K and if there exists $m \in \overline{N}_o \cap M$ such that $B(Ad(s^*)^{-1}Lie(N_o), Ad(m)x) = 0$, then s is the longest element of the Weyl group of MA acting on A_o.

We will say that a standard parabolic subgroup is nice if the standard opposite parabolic subgroup admits a non-degenerate character and it will be said to be very nice if every non-degenerate character is strongly non-degenerate. Thus if a parabolic subgroup is very nice, then all non-degenerate characters are strongly non-degenerate.

The simplest examples of parabolic subgroups that is nice but not very nice are the Heisenberg parabolic subgroups of $Sp_{2n}(\mathbb{R})$ (this example is due to the referee).

In Section 6 we will prove that if G is semi-simple with $Lie(G) = \mathfrak{g}$ and if $G_{\mathbb{C}}$ is the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, the complexification of $Lie(G)$, and $P_{\mathbb{C}}$ is the subgroup corresponding to $Lie(P)_{\mathbb{C}}$, then if P is nice and the moment map from $T^*(G_{\mathbb{C}}/P_{\mathbb{C}})$ to $\mathfrak{g}_{\mathbb{C}}^*$ is birational to its image, then P is very nice. This condition plays a key role in the work of Yamashita [Y1, Y2]. One can also show using the methods of Section 6 that a parabolic subgroup is very nice if and only if it is nice and the condition on the moment map is satisfied.

We have not resolved the question: If the unipotent radical of P has one strongly non-degenerate character then is every non-degenerate character strongly non-degenerate?

The main result of this paper is

Theorem 2.2. *If χ is a strongly non-degenerate character of $\overline{N}$ and if* $\dim H_\sigma < \infty$*, then 1, 2, and 3 above are true.*

The strong non-degeneracy condition is not necessarily easy to verify. We will now describe a corollary to the main theorem that is more applicable and is the essence of a conjecture in [Wa1] (more will be said about it after the statement). We assume that $y \in \theta\mathfrak{n}$ has the property that $B(y, [\mathfrak{n}, \mathfrak{n}]) = 0$. We define $\chi(\exp X) = e^{iB(y,X)}$ for $X \in \mathfrak{n}$. We assume that there exists $x \in \mathfrak{n}$ such that $[x, y] = h \in \mathfrak{n}$, $[h, x] = 2x, [h, y] = -2y$ and that the space $\{X \in \mathfrak{n} | [h, X] = 2X\}$ generates $\mathfrak{n}$ as a Lie algebra. Then, if we choose $k \in (\exp(\pi(x - y)/2)P) \cap K$, we have

$$kPk^{-1} = \theta(P).$$

We write

$$J^{\chi}_{P,\sigma,\nu}(f) = \int_N f_{P,\sigma,\nu}(k^{-1}n)\chi(n)^{-1}dn.$$

Notice that we have apparently used the same notation for two different objects, however, this is not so since that type of characters is different. Hopefully, there will be no confusion. We have

Corollary 2.3. *Assume the discussion above and replacing $\overline{N}$ by N. If $\dim H_\sigma < \infty$, then 1, 2, and 3 are true.*

This result has been proved in many special cases and our notion of generic agrees with all of the previous definitions. If P is minimal and G split over $\mathbb{R}$, Jacquet [J] has proved 1, 2 and 3 and for complex groups he has proved 1 for f a K-finite function and the essence of 3 which his multiplicity one theorem. If G is of rank one over $\mathbb{R}$, this was proved by Schiffmann [S]. For general real groups and minimal parabolic subgroups, the papers [Ha1, Ha2] show special cases, there is the announcement of Varadarajan for Harish-Chandra in 1983 and there is also the independent work of the author which appears in [Wa1]. This paper goes beyond the case of minimal parabolic subgroup. The basic ideas in that paper will be used to prove the main theorem. [Wa1] contains many examples of the main theorem including most cases when the unipotent radical is abelian. The proofs in that paper used, as hypotheses, a set of conditions on certain double cosets in G and then involved standard Bruhat theory, a vanishing theorem for Lie algebra cohomology essentially due to Lynch, and an explicit method of construction of Whittaker vectors in tensor products with finite dimensional representations. In [Wa3] this result was proved in the special case of the Heisenberg parabolic for the quaternionic real form of type A, B, D, E, F, G. The Bruhat theoretic method in [Wa1] does not directly apply to these cases. It is here that the result of [KV] came to the rescue. The above corollary applies directly to all of the examples in [Wa1], Section 4. Beyond the work in [Wa1] there are the two papers of H. Yamashita, [Y1], [Y2] that have a substantial overlap with this paper and, in particular, contain a different multiplicity one theorem than that of [Wa1].

3. The category $\mathcal{W}_\psi$

Let G and $P = MAN$ be as in the introduction (we will maintain all of the notation therein). Let $x \in \mathfrak{n}$ be such that $B(x, [\theta\mathfrak{n}, \theta\mathfrak{n}]) = 0$. Then we can define a Lie algebra homomorphism of $\theta\mathfrak{n}$ to $i\mathbb{R}$ by $\psi(Y) = iB(x, Y)$. Let $\mathcal{W}_\psi$ denote the category of all $\mathfrak{g}$-modules, M, such that if $m \in M$, $Y \in \theta\mathfrak{n}$, there exits k such that $(Y - \psi(Y))^k m = 0$. The following is a reformulation of Theorem 4.3 in [L] (cf. Theorem 2.2 of [Wa1]).

Theorem 3.1. *Assume that $Ad(P)x$ is open in $\mathfrak{n}$. Then $H^i(\theta\mathfrak{n}, M \otimes \mathbb{C}_{-\psi}) = 0$ for all $i > 0$.*

To prove this we need only show that x satisfies the conditions (1), (2) in the beginning of Section 2 in [Wa1]. (Here we note that the roles of $\mathfrak{n}$ and $\theta\mathfrak{n}$ have been reversed). We must therefore show that

(1) There is an element $h \in \mathfrak{a}$ such that $ad(h)$ has eigenvalues included in $\{0, \pm 2, \pm 4, \ldots\}$ such that if $\mathfrak{g}^j$ is the eigenspace for eigenvalue j of $ad(h)$, then $x \in \mathfrak{g}^2$ and $\mathfrak{g}^2$ generates $\mathfrak{n}$ as a Lie algebra.

(2) $ad(x) : \theta\mathfrak{n} \to \mathfrak{g}$ is injective.

To prove the first condition we show that there exists an h with the appropriate properties. We will show that there exists $H \in \mathfrak{a}_o$ such that the eigenvalues of

$ad(H)$ are integers and the eigenspace with eigenvalue 1 generates $\mathfrak{n}$. Let $\Phi(P_o, A_o)$ be the set of roots for A_o on $\mathfrak{n}_o$. Then the weights of A_o on $\mathfrak{n}$ are the elements of $\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote the simple roots of $\Phi(P_o, A_o)$ and (after possibly relabelling) $\Delta_M = \{\alpha_{r+1}, \ldots, \alpha_l\}$ the simple roots of $\Phi(P_o \cap (MA), A_o)$. We prove

Lemma 3.2. *If $\alpha = \sum m_i \alpha_i \in \Phi(P_o, A_o)$ and $\sum_{i \leq r} m_i \geq 2$, then $\alpha = \gamma + \delta$ with $\gamma, \delta \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$.*

Before we prove this lemma let us show how it implies the existence of the element H. Denote by $\mathfrak{g}^\alpha$ the root space for $\alpha \in \Phi(P_o, A_o)$. We note that if $\alpha.\beta \in \Phi(P_o, A_o)$, then $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$ if $\alpha + \beta \in \Phi(P_o, A_o)$ and 0 otherwise. So the lemma plus this observation imply that $\mathfrak{g}^\alpha \subset [\mathfrak{n}, \mathfrak{n}]$ if and only if $\alpha = \sum m_i \alpha_i$ with $\sum_{i \leq r} m_i \geq 2$. This implies that $\mathfrak{n}$ is generated by the sum of the $\mathfrak{g}^\alpha$ with $\alpha = \sum m_i \alpha_i$ and $\sum_{i \leq r} m_i = 1$. Let $h_1, \ldots, h_l \in \mathfrak{a}_o$ be defined by $\alpha_i(h_j) = \delta_{ij}$. Set $H = h_1 + \cdots + h_r$. Then $H \in \mathfrak{a}$ has the desired properties.

We will now prove the lemma. We prove it by induction on $|\alpha| = \sum_{i \leq l} m_i$. If $|\alpha| = 2$, then $\alpha = \alpha_i + \alpha_j$ with $i, j \leq r$ as desired. We now assume the result for $2 \leq |\alpha| \leq s$ and consider the case when $|\alpha| = s + 1$. There exists i such that $(\alpha, \alpha_i) > 0$ (since otherwise $(\alpha, \alpha) \leq 0$). Thus since $\sum_{i \leq r} m_i \geq 2$, $\alpha - \alpha_i$ is in $\Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$. If $i \leq r$, then we take $\gamma = \alpha - \alpha_i$ and $\delta = \alpha_i$. Otherwise, $|\alpha - \alpha_i| = s$ and the sum of the coefficients of the α_i with $i \leq r$ hasn't changed. Hence the inductive hypothesis implies that $\alpha - \alpha_i = \gamma + \delta$ with $\gamma, \delta \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$. We therefore have $[\mathfrak{g}^\gamma, \mathfrak{g}^\delta] = \mathfrak{g}^{\alpha - \alpha_i}$. Also, $[\mathfrak{g}^{\alpha - \alpha_i}, \mathfrak{g}^{\alpha_i}] = \mathfrak{g}^\alpha$. So applying the Jacobi identity we have

$$\mathfrak{g}^\alpha = [[\mathfrak{g}^\gamma, \mathfrak{g}^\delta], \mathfrak{g}^{\alpha_i}] = [[\mathfrak{g}^\gamma, \mathfrak{g}^{\alpha_i}]\mathfrak{g}^\delta] + [\mathfrak{g}^\gamma, [\mathfrak{g}^\delta, \mathfrak{g}^{\alpha_i}]].$$

Hence, $\gamma + \alpha_i \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$ or $\delta + \alpha_i \in \Phi(P_o, A_o) - \Phi(P_o \cap (MA), A_o)$. Assume the latter, then $\alpha = \gamma + (\delta + \alpha_i)$ which is the desired form.

Let $h = 2H$. Then the space $\{X \in \mathfrak{g} | [h, X] = 2X\}$ generates $\mathfrak{n}$. We set $\mathfrak{g}^j = \{X \in \mathfrak{g} | [h, X] = jX\}$. Then we see that $[\theta\mathfrak{n}, \theta\mathfrak{n}] = \sum_{j > 1} \mathfrak{g}^{-2j}$. Thus since we are assuming that $B(x, [\theta\mathfrak{n}, \theta\mathfrak{n}]) = 0$ we have $[h, x] = 2x$. Thus (1) above is satisfied. As for (2) we have

Lemma 3.3. *If (1) above is satisfied, then (2) is equivalent to the condition $[\mathfrak{p}, x] = \mathfrak{n}$.*

Proof. We note that

$$(\theta\mathfrak{n})^x = \ker ad(x)_{|\theta\mathfrak{n}} = \{y \in \theta\mathfrak{n} | B(y, [x, \mathfrak{g}]) = 0\}.$$

Thus if $[\mathfrak{p}, x] = \mathfrak{n}$, then since the pairing between $\mathfrak{n}$ and $\theta\mathfrak{n}$ is perfect we see that $(\theta\mathfrak{n})^x = 0$. Assume that $(\theta\mathfrak{n})^x = 0$. Noting that $[x, \mathfrak{g}]$ is $ad(h)$-invariant and that $[x, \theta\mathfrak{n}] \subset \mathfrak{m} \oplus \theta\mathfrak{n}$ we see that the displayed formula implies that $[\mathfrak{p}, x] \supset \mathfrak{n}$. Since $[\mathfrak{p}, x] \subset \mathfrak{n}$, the lemma follows. □

We can now give a reformulation of Theorem 3.4 of [Wa1]. We will use the notation

$$(M \otimes \mathbb{C}_{-\psi})^{\theta\mathfrak{n}} = \{m \in M | (Y - \psi(Y))m = 0, Y \in \theta\mathfrak{n}\}.$$

Theorem 3.4. *Assume that $Ad(P)x$ is open in $\mathfrak{n}$. Let F be a finite dimensional representation of $\mathfrak{g}$. Then there exists an element $\Gamma \in U(\mathfrak{g}_{\mathbb{C}}) \otimes End(F)$ depending only on x and F such that if $M \in \mathcal{W}_\psi$, then $\Gamma : (M \otimes \mathbb{C}_{-\psi})^{\theta\mathfrak{n}} \otimes F \to ((M \otimes F) \otimes$*

$\mathbb{C}_{-\psi})^{\theta\mathfrak{n}}$ *is a linear bijection. (Here* $U(\mathfrak{g}_{\mathbb{C}})$ *is acting by the tensor product action on* $M \otimes F$*and* $End(F)$ *only on* F.*)*

4. An application of a theorem of Kolk and Varadarajan

We maintain the notation of the previous sections. We set $W_G(A_o)$ equal to the Weyl group of G with respect to A_o. For each $w \in W_G(A_o)$ we fix $w^* \in K$, normalizing $\mathfrak{a}_o$, such that $w = Ad(w^*)_{|\mathfrak{a}_o}$. Also let w_M be the element of $W_{MA}(A_o)$ such that $w_M(\Phi((P_o \cap M)A, A_o)) = -\Phi((P_o \cap M)A, A_o)$. We can choose $w_M^* \in K \cap M$.

We will now give an application of the results of [KV]. We note that $P\overline{N}$ is open in G. We will denote by $U_{P,\sigma,\nu}$ the space of all $f \in I^\infty_{P,\sigma,\nu}$ such that $\operatorname{supp}(f) \subset P\overline{N}$ is compact modulo P. The key result (which has a substantial overlap with results in [Y1] and [Y2]) is

Theorem 4.1. *Let* χ *be a strongly non-degenerate character of* $\overline{N}$ *with* $d\chi(Y) = iB(y, Y)$. *Let* (σ, H_σ) *be an irreducible finite dimensional representation of* M. *If* $\lambda \in Wh_\chi(I^\infty_{P,\sigma,\nu})$ *and if* $\lambda_{|U_{P,\sigma,\nu}} = 0$, *then* $\lambda = 0$.

Proof. We will be applying Theorem 3.15, p. 82 of [KV], case iii). However, we will be reversing the roles of right and left in their result. We consider the group $H = P \times \overline{N_o}$ acting on G by $(p, y) \cdot x = pxy^{-1}$. Then if

$$\Sigma = \{w \in W_G(A_o) | w\Phi(P_o, A_o) \supset -\Phi(P_o \cap (MA), A_o),$$

then G is the disjoint union $\bigcup_{w\in\Sigma} Pw^*\overline{N_o}$. We consider the subgroup $H' = P \times \overline{N}$ acting on G by the restriction of the action of H. This group has one open orbit which is the same as the unique open orbit of H, that is, $H \cdot 1 = P\overline{N_o} = P\overline{N} = H' \cdot 1$. The other orbits of H' are given as $H' \cdot w^*m$ with $m \in \overline{N_o} \cap M$. Assume that $w \neq w_M$ and $m \in \overline{N_o} \cap M$. We will now calculate the stabilizer of w^*m in H'. We are looking at the set of all p, y with $p \in P$ and $y \in \overline{N}$ such that $pw^*my^{-1} = w^*m$. That is, $m^{-1}(w^*)^{-1}pw^*m = y$. We are therefore looking at pairs (p, y) with $y \in \overline{N}$ and $mym^{-1} = (w^*)^{-1}pw^*$ with $p \in P$. Observe that the action of $Ad(w^*)$ stabilizes $\mathfrak{a}_o$ and permutes the root spaces. By the definition of Σ we see that $mym^{-1} \in (w^*)^{-1}N_ow^* \cap \overline{N}$. Thus if $\mathfrak{u}$ is the Lie algebra of the projection onto the second factor of the stabilizer of w^*m, then $Ad(m)\mathfrak{u} = (Ad(w^*)^{-1}\mathfrak{n}_o) \cap \overline{\mathfrak{n}}$. Hence

$$B(\mathfrak{u}, y) = B((Ad(w^*)^{-1}\mathfrak{n}_o) \cap \overline{\mathfrak{n}}, Ad(m)^{-1}y).$$

Since χ is non-degenerate this implies (since $w \neq w_M$) that $B(\mathfrak{u}, y) \neq \{0\}$. The elements of this group act unipotently under σ and in the adjoint representation, so it follows that equation (3.27) in [KV], page 82 is satisfied (with right and left interchanged). To complete the proof starting with the conclusion of Theorem 3.15 in [KV] one uses standard Bruhat theoretic arguments (now using the finite number of orbits of H on G) . We will now sketch what is necessary to implement the method.

Let $\lambda \in Wh_\chi(I^\infty_{P,\sigma,\nu})$ and assume that $\lambda_{|U_{P,\sigma,\nu}} = 0$. We label the elements of Σ as $\{w_1, \ldots, w_r\}$ such that $X_j = \cup_{i\geq j} H \cdot w_i^*$ is closed in G. We define T, a distribution on G with values in H_σ as follows: Let f be a smooth compactly supported function with values in H_σ. Set

$$f^\nu(g) = \int_{M\times A\times N} f(mang)\sigma(m)^{-1}a^{-\nu-\rho}dmdadn$$

here we have made a choice of bi-invariant measure for each of the indicated groups. Then $f^\nu \in I^\infty_{P,\sigma,\nu}$. We set $T(f) = \lambda(f^\nu)$. Our assumption on λ says that supp$T \subset X_2$. Suppose that we have shown that supp$T \subset X_j$, $j \geq 2$. Now, $H \cdot w_j^*$ is open in X_j hence there exists U open in G such that U is H-invariant and $U \cap X_j = H \cdot w_j^*$. Our assumption now implies that supp$T_{|U} \subset H \cdot w_j^*$. We can now apply Theorem 3.15 in [KV] to see that $T_{|U} = 0$. Thus supp$T \subset X_{j+1}$. Since $X_{r+1} = \emptyset$ the theorem follows. □

5. The proof of the main theorem

We will prove Theorem 2 in this section. We first need a lemma which is essentially the same as Corollary 2 of [Wa3]. We use the notation in the introduction, in particular, the action of M_χ on $Wh_\chi(I^\infty_{P,\sigma,\nu})$.

Lemma 5.1. *Assume that χ is strongly non-degenerate and that (σ, H_σ) is finite dimensional. Then $Wh_\chi(I^\infty_{P,\sigma,\nu})$ is equivalent with a subrepresentation of the representation contragradient to $(\sigma_{|M_\psi}, H_\sigma)$ as an M_χ-module.*

Proof. We first note that if $T \in (C_c^\infty(\overline{N}))'$ (continuous linear functionals in the usual topology) is such that $T \circ R_x = \chi(x)^{-1}T(R_x f(y) = f(yx))$, then

$$T(f) = c(T)\int_{\overline{N}} f(\overline{n})\chi(\overline{n})d\overline{n}.$$

This can be seen as follows. Let $\{X_j\}$ be a basis of $Lie(\overline{N})$ (thought of as left invariant vector fields). Then we have

$$X_jT = -d\chi(X_j)T$$

in the sense of distributions. Thus

$$(\sum_j X_j^2 - \sum_j (d\chi(X_j)^2)T = 0.$$

The elliptic regularity theorem implies that T is given by integration against real analytic $\dim \overline{N}$-form on $\overline{N}$. The transformation law now easily implies that the form is $\chi(\overline{n})\omega$ with ω invariant.

If $f \in C_c^\infty(\overline{N})$ and $v \in H_\sigma$, then we define $S(f \otimes v)(p\overline{n}) = \sigma_\nu(p)f(\overline{n})v$ for $p \in P$, $\overline{n} \in \overline{N}$. If $\lambda \in Wh_\chi(I^\infty_{P,\sigma,\nu})$, then we have $T_{\lambda,v}(f) = \lambda(S(f \otimes v))$ defines a distribution on $\overline{N}$ and $T_{\lambda,v}(R_x f) = \chi(x)T_{\lambda,v}(f)$ for all $x \in \overline{N}$. Thus we have a $\mathbb{C}$-bilinear pairing defined on $Wh_\chi(I^\infty_{P,\sigma,\nu}) \times H_\sigma$ defined by

$$T_{\lambda,v}(f) = \langle \lambda, v\rangle \int_{\overline{N}} f(\overline{n})\chi(\overline{n})d\overline{n}.$$

A direct calculation shows that

$$\langle m \cdot \lambda, v\rangle = \left\langle \lambda, \sigma(m)^{-1}v\right\rangle.$$

These observations combined with Theorem 4.1 complete the proof. □

We are now ready to begin in earnest to prove the main theorem. We first observe the following

Proposition 5.2. *Let (σ, H_σ) be a smooth Fréchet representation of M of moderate growth, then there exits a constant C_σ such that if $\Omega_\sigma = \{\nu \in \mathfrak{a}^*_{\mathbb{C}} | \operatorname{Re}(\nu, \alpha) > C_\sigma,$ $\alpha \in \Phi(P, A)\}$, then if χ is a unitary character of $\overline{N}$, then*

$$J^\chi_{P.\sigma,\nu}(f) = \int_{\overline{N}} f_{P,\sigma,\nu}(\overline{n})\chi(\overline{n})^{-1} d\overline{n}$$

converges absolutely and uniformly in compacta of Ω_σ.

This result is well known; see, for example, Proposition 7.1 in [Wa1] (or putting in appropriate seminorms it is an easy consequence of Lemma 4.A.2.3 in [Wa2]).

Using this result and Lemma 5.1 it is easy to prove

Proposition 5.3. *Let χ be a strongly non-degenerate character of $\overline{N}$ and $\nu \in \Omega_\sigma$ and assume that* $\dim H_\sigma < \infty$. *Then:*

1. $Wh_\chi(I^\infty_{P.\sigma,\nu}) = \{\lambda \circ J^\chi_{P,\sigma,\nu} | \lambda \in (H_\sigma)^*\}$.
2. *As an M_χ-module, $Wh_\chi(I^\infty_{P.\sigma,\nu})$ is equivalent to the contragradient of $(\sigma_{|M_\chi}, H_\sigma)$.*

With all of this at hand the proof of Theorem 2 is essentially the same as the proof of Theorem 7.2 in [Wa1]. Since the notation in that paper is different, there are several misprints in that proof and perhaps too many details were left to the reader, we will give the argument for the sake of clarity. Set $\psi = d\chi$. Let x be such that $d\chi(Y) = iB(x, Y)$ for $Y \in \theta\mathfrak{n}$. Let $H \in \mathfrak{a}$ be such that $\mathfrak{n}_1 = \{X \in \mathfrak{n} | [H, X] = X\}$ generates $\mathfrak{n}$. Then $x \in \mathfrak{n}_1$. Since $Ad(P)x$ is open in $\mathfrak{n}$, $[\mathfrak{p}, x] = \mathfrak{n}$ hence $[Lie(MA), x] = \mathfrak{n}_1$. Thus χ satisfies the hypothesis of the previous lemma.

If (π, V) is a smooth Fréchet representation of G, then we set

$$V'[\psi] = \{T \in V' \,|\, \text{if } y \in \theta\mathfrak{n} \text{ there exists } k \text{ such that } (y - \psi(y))^k T = 0\}.$$

Then $V'[\psi] \in \mathcal{W}_\psi$. If F is a finite dimensional representation of G, then since $\theta\mathfrak{n}$ acts nilpotently on F, it is clear that

$$(V \otimes F)'[\psi] = V'[\psi] \otimes F.$$

Let (μ, F) be a finite dimensional irreducible representation of G such that:

a) Setting $F^{\theta\mathfrak{n}} = \{v \in F | yv = 0, y \in \theta\mathfrak{n}\}$, then M acts trivially on $F^{\theta\mathfrak{n}}$.

This implies that $\dim F^{\theta\mathfrak{n}} = 1$. This one-dimensional space is invariant under $\mathfrak{a}$ which therefore acts by a linear functional which we will denote $-\Lambda$. We also assume:

b) If $\alpha \in \Phi(P, A)$, then $(\Lambda, \alpha) > 0$.

Such a representation always exists. Indeed, if $d = \dim \mathfrak{n}$, then we can take for F the span of $\bigwedge^d Ad(G) \cdot \bigwedge^d \theta\mathfrak{n}$ in $\bigwedge^d \mathfrak{g}$.

We set $\sigma_\nu(man) = a^{\nu+\rho}\sigma(m)$. Then (σ_ν, H_σ) is a finite dimensional representation of P. If (ξ, H_ξ) is a finite dimensional continuous (hence smooth) representation of P, then we set $I^\infty_{P,\xi}$ equal to the space of all smooth f from G to H_ξ such that $f(pg) = \xi(p)f(g)$, $g \in G$, $p \in P$. G acts on $I^\infty_{P,\xi}$ by the right regular action. We endow the space $I^\infty_{P,\xi}$ with the C^∞ topology. We also note that the map $f \longmapsto f_{|K}$ defines an isomorphism of topological vector spaces between $I^\infty_{P,\xi}$ and the space $I^\infty_{\xi_{|K\cap P}}$ consisting of the smooth maps from K to H_ξ such that $f(mk) = \xi(m)f(k)$ for $m \in K \cap P$ and $k \in K$ with the C^∞ topology. Fix a norm on H_ξ, $\|\ldots\|$. Since K is compact, we see that this topology can be defined using the seminorms $p_x(f) = \sup_k \|xf(k)\|$ for $x \in U(Lie(K))$.

We now observe that if we put on $I^\infty_{P,\xi}\otimes F$, the Fréchet space structure obtained by choosing a basis of F and looking at $I^\infty_{P,\xi}\otimes F$ as a direct sum of $\dim F$ copies of $I^\infty_{P,\xi}$, then the map $T(f\otimes v)(g)=f(g)\otimes\mu(g)v$ defines a continuous isomorphism between the smooth Fréchet representations $I^\infty_{P,\xi}\otimes F$ and $I^\infty_{P,\xi\otimes\mu_{|P}}$. Let

$$H_\xi\otimes F=W_0\supset W_1\supset\cdots\supset W_r\supset W_{r+1}=\{0\}$$

be a Jordan-Hölder series. Then W_i/W_{i+1} is equivalent to $((\sigma_i)_{\nu_i},H_{\sigma_i})$ for $i=0,\dots,r$. This leads to the composition series

$$I^\infty_{P,\xi}\otimes F=V_0\supset V_1\supset\cdots\supset V_r\supset V_{r+1}=\{0\}$$

with V_i a closed smooth Fréchet subrepresentation of V_{i-1} and V_i/V_{i-1} is topologically equivalent to $I^\infty_{P,\sigma_i,\nu_i}$. We will now apply these observations to the case when $\xi=\sigma_\nu$. We note that in this case we can assume that $\sigma_0=\sigma$ and $\nu_0=\nu-\Lambda$. We assume that ν is such that

$$\dim Wh_\chi(I^\infty_{P.\sigma,\nu})=\dim H_\sigma.$$

Then Theorem 3.4 implies that

$$\dim((I^\infty_{P.\sigma,\nu}[\psi]\otimes F)'\otimes\mathbb{C}_{-\psi})^{\theta\mathfrak{n}}=\dim H_\sigma\dim F.$$

We have

$$((I^\infty_{P.\sigma,\nu}[\psi]\otimes F)'\otimes\mathbb{C}_{-\psi})^{\theta\mathfrak{n}}{}_{|V_r}\subset V_r'[\psi]^{\theta\mathfrak{n}}$$

and since V_r is topologically isomorphic to I_{P,σ_r,ν_r} Lemma 9 implies that $\dim V_r'[\psi]^{\theta\mathfrak{n}}\le\dim H_{\sigma_r}$. We put

$$Z^j=\{\lambda\in((I^\infty_{P.\sigma,\nu}[\psi]\otimes F)'\otimes\mathbb{C}_{-\psi})^{\theta\mathfrak{n}}|\lambda_{|V_j}=0\}.$$

We therefore have $\dim Z^r\ge\dim H_\sigma\dim F-\dim H_{\sigma_r}$. Now V_{r-1}/V_r is equivalent to $I^\infty_{P,\sigma_{r-1},\nu_{r-1}}$ hence we see that

$$\dim Z_{r-1}\ge\dim H_\sigma\dim F-\dim H_{\sigma_r}-\dim H_{\sigma_{r-1}}.$$

Continuing in this way we find that

$$\dim Z_1\ge\dim H_\sigma\dim F-\sum_{i\ge1}\dim H_{\sigma_i}=\dim H_\sigma.$$

We note that Z_1 injects into $(V_0/V_1)'[\psi]^{\theta\mathfrak{n}}$ and, since $\dim(V_0/V_1)'[\psi]^{\theta\mathfrak{n}}=\dim Wh_\chi(I^\infty_{P.\sigma,\nu-\Lambda})$, we have shown that $\dim Wh_\chi(I^\infty_{P.\sigma,\nu-\Lambda})\ge\dim H_\sigma$. Since Lemma 5.1 asserts the reverse inequality we have proved:

I) If $\dim Wh_\chi(I^\infty_{P.\sigma,\nu})=\dim H_\sigma$, then $Wh_\chi(I^\infty_{P.\sigma,\nu-\Lambda})=\dim H_\sigma$.

Our hypothesis on Λ implies that if ν is given, then there exists $k>0$ with $k\in\mathbb{Z}$ such that $\nu+k\Lambda\in\Omega_\sigma$. Hence I) implies

II) $\dim Wh_\chi(I^\infty_{P.\sigma,\nu})=\dim H_\sigma$ for all irreducible finite dimensional representations (σ,H_σ) of M and all $\nu\in\mathfrak{a}^*_{\mathbb{C}}$.

We note that at this point we have proved part 3 of Theorem 2. We will now complete the proof of parts 1 and 2. We define $\delta:I^\infty_{\sigma_{|P\cap K}}\otimes F\to I^\infty_{(\sigma\otimes\mu)_{|P\cap K}}$ by $\delta(f\otimes v)(k)=f(k)\otimes\mu(k)v$. Let q denote the projection of F onto $F/\mathfrak{n}F$, then we can form $T_\Lambda=(I\otimes q)\circ\delta$. Then $T_\Lambda:I^\infty_{\sigma_{|P\cap K}}\otimes F\to I^\infty_{\sigma_{|P\cap K}}$ is surjective continuous and

$$(*)\qquad T_\Lambda\circ(\pi_{P,\sigma,\nu}(g)\otimes\mu(g))=\pi_{P,\sigma,\nu-\Lambda}(g)\circ T_\Lambda.$$

Note that T_Λ only depends on σ and F and not on ν. Now suppose that we have proved:

III) The map $\nu \mapsto J^{\chi}_{P.\sigma,\nu}(f)$ extends to a holomorphic map of $U_t = \{\nu \in \mathfrak{a}^*_{\mathbb{C}} | (\alpha, \nu) > t, \alpha \in \Phi(P, A)\}$ to H_σ, and

IV) if $\nu \in U_t$ and $\eta \in Wh_\chi(I^\infty_{P,\sigma,\nu})$, then there exists $\lambda \in H^*_\sigma$ such that $\eta = \lambda \circ J^{\chi}_{P.\sigma,\nu}$.

We set $\pi_\nu = \pi_{P,\sigma,\nu}$, $J_\nu = J_{P,\sigma\,\nu} \ldots$. Fix $\lambda_1, \ldots, \lambda_m$ as a basis of H^*_σ and let $\xi_1, \ldots, \xi_d$ be a basis of F^*. If $\nu \in U_t$, then set $\gamma_i(\nu) = \lambda_i \circ J_\nu$. Then Theorem 3.4 says that there exist $p^k_{ij} \in U(\mathfrak{g})$ and $A^k_{ij} \in End(F^*)$ with $1 \le i \le m, 1 \le j \le d, 1 \le k \le dm$ such that the elements

$$\zeta_k(\nu) = \sum_{i,j} p^k_{i,j}(\gamma_i(\nu) \otimes A^k_{ij}\xi_j)$$

form a basis of $(((I^\infty_\nu)' \otimes F^*) \otimes \mathbb{C}_{-\psi})^{\theta\mathfrak{n}}$. Fix $\nu_o \in U_t$ and set $Z_\Lambda = \ker T_\Lambda$ (not that Z_Λ depends only on σ and F but not on ν). Then the argument proving II) above implies that

$$\dim \left(\sum_{i \le dm} \mathbb{C}\zeta_i(\nu_o) \right)_{|Z_\Lambda} = dm - m.$$

We may assume, after relabelling, that $\zeta_{m+1}(\nu_o)_{|Z_\Lambda}, \ldots, \zeta_{dm}(\nu_o)_{|Z_\Lambda}$ are linearly independent. Thus, there is an open neighborhood, W, of ν_o such that $\zeta_{m+1}(\nu)_{|Z_\Lambda}, \ldots, \zeta_{dm}(\nu)_{|Z_\Lambda}$ are linearly independent for $\nu \in W$. We therefore see that there exist holomorphic functions a_{ij} $1 \le i \le m, m+1 \le j \le dm$ on W such that

$$\zeta_i(\nu)_{|Z_\Lambda} = \sum_{j>m} a_{ij}(\nu)\zeta_j(\nu)_{|Z_\Lambda}, \nu \in W.$$

If we set

$$\phi_i(\nu) = \zeta_i(\nu) - \sum_{j>m} a_{ij}(\nu)\zeta_j(\nu), 1 \le i \le m, \nu \in W,$$

then we note that $\phi_1(\nu), \ldots, \phi_m(\nu)$ are linearly independent and vanish on Z_Λ. Now

$$T_\Lambda : (I^\infty_{\sigma_{|K\cap P}} \otimes F)/Z_\Lambda \to I^\infty_{\sigma_{|P\cap K}}$$

is an isomorphism of topological vector spaces. Thus $(*)$ above implies that $\phi_1(\nu), \ldots, \phi_m(\nu)$ are holomorphic on W with values in $(I^\infty_{\sigma|K\cap P})'$ and

$$\phi_i(\nu)(\pi_{\nu-\Lambda}(\overline{n})f) = \chi(\overline{n})^{-1}\phi_i(\nu)(f), \overline{n} \in \overline{N}, f \in I^\infty_{\sigma_{|K\cap P}}.$$

If $f \in C^\infty_c(\overline{N})$ and $h \in H_\sigma$, then set

$$f_{\nu,h}(man\overline{n}) = f(\overline{n})a^{\nu+\rho}\sigma(m)h.$$

Then the $f_{\nu,h}$ span the space $U_{\sigma,\nu}$. We argue as in the proof of Lemma 5.1 to see that if we set $\Phi(f) = \int_{\overline{N}} \chi(\overline{n})^{-1} f(\overline{n}) d\overline{n}$, then

$$\phi_i(\nu)(f_{v,h}) = \alpha_i(\nu)(h)\Phi(f)$$

with $\alpha_1(\nu), \ldots, \alpha_m(\nu)$ holomorphic on W with values in H^*_σ and for each $\nu \in W$ giving a basis of H^*_σ. Thus there exist holomorphic functions $b_{ij}(\nu)$ on W such that

$$\mu_i = \sum_j b_{ji}(\nu)\alpha_j(\nu), i = 1, \ldots, m.$$

Hence if

$$\omega_i(\nu) = \sum_j b_{ji}(\nu)\phi_j(\nu), i = 1, \ldots, m,$$

then

$$\omega_i(\nu)(f) = \mu_i \left(\int\limits_{\overline{N}} \chi(\overline{n})^{-1} f(\overline{n}) d\overline{n} \right)$$

for $f \in U_{\sigma.\nu-\Lambda}$. At this point we have proved that there exist $\omega_i(\nu)$ holomorphic on U_t with values in $I^{\infty}_{\sigma_{|K\cap P}}$ such that

V) $\omega_i(\nu) \in Wh_{\chi}(I^{\infty}_{P,\sigma,\nu-\Lambda}), i = 1, \ldots, m$ and are linearly independent.

VI) $\omega_i(\nu)(f) = \omega_i(\nu)(f) = \mu_i \left(\int_{\overline{N}} \chi(\overline{n})^{-1} f(\overline{n}) d\overline{n} \right)$ for $f \in U_{\sigma,\nu-\Lambda}$.

The uniqueness statement in Theorem 3.4 implies that if $\nu - \Lambda \in U_t$, then $\omega_i(\nu) = \mu_i \circ J_{P,\sigma,\nu-\Lambda}$. Set $q = \min_{\alpha\in\Phi(P,A)}(\Lambda, \alpha)$. Then $q > 0$. We have proved that III) and IV) are true with t replaced by $t - q$. Since $U_{t-rq} \subset U_{t-(r+1)q}$ and $\bigcup_{r>0} U_{t-rq} = \mathfrak{a}^*_{\mathbb{C}}$, we have finally completed the proof of Theorem 2.

6. Nice and very nice parabolic subgroups

Let G be a real reductive group, K a maximal compact subgroup and θ the corresponding Cartan involution. Let P be a parabolic subgroup of G with unipotent radical N and set $M = P \cap \theta(P)$, then $P = MN$. Then P is nice if and only if there exists $x \in Lie(N)$ such that $B(\theta(x), [Lie(N), Lie(N)]) = 0$ with $Ad(P)x$ is open in $Lie(N)$. We will now show that this condition is a property of the complexification of $Lie(P)$.

If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ (the complexification of $Lie(G)$), then they are said to be opposite if $\mathfrak{q}_1 \cap \mathfrak{q}_2$ is a Levi factor of both $\mathfrak{q}_1$ and $\mathfrak{q}_2$.

Proposition 6.1. *Let P be a parabolic subgroup of G with complexified Lie algebra $\mathfrak{p}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$, the complexification of $Lie(G)$, and let $\mathfrak{n}_{\mathbb{C}}$ be the complexification of the Lie algebra of the unipotent radical of P. Then P is nice if and only if there exists a parabolic subalgebra $\mathfrak{q}$ opposite to $\mathfrak{p}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ and an element $x \in \mathfrak{n}_{\mathbb{C}}$ such that*

1) $[\mathfrak{p}_{\mathbb{C}}, x] = \mathfrak{n}_{\mathbb{C}}$, *and*
2) *if $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$, then* $B(x, [\mathfrak{u}, \mathfrak{u}]) = \{0\}$.

Proof. To prove this it is enough to show that if there exists an element x satisfying 1) and 2) for one choice of opposite parabolic subalgebra, $\mathfrak{q}$, and if $\mathfrak{q}_1$ is another choice of opposite parabolic subalgebra, then there exists an element $y \in \mathfrak{n}_{\mathbb{C}}$ with the corresponding two properties. So let $\mathfrak{q}_1$ be another choice of opposite parabolic subalgebra then there is a Cartan subalgebra, $\mathfrak{h}$ (resp. $\mathfrak{h}_1$), of $\mathfrak{q} \cap \mathfrak{p}_{\mathfrak{C}}$ (resp. $\mathfrak{q}_1 \cap \mathfrak{p}_{\mathfrak{C}}$) such that the roots of $\mathfrak{h}$ (resp. $\mathfrak{h}_1$) in the nilradical of $\mathfrak{q}$ (resp.$\mathfrak{q}_1$) are precisely the negatives of those of $\mathfrak{h}$ (resp. $\mathfrak{h}_1$) in $\mathfrak{n}_{\mathbb{C}}$. Since all Cartan subalgebras of $\mathfrak{p}_{\mathbb{C}}$ are conjugate by an inner automorphism of $\mathfrak{p}_{\mathbb{C}}$ (cf. [B], Corollary 11.3, p. 143) this implies that if p is the inner automorphism taking $\mathfrak{h}$ to $\mathfrak{h}_2$, then the element $px \in \mathfrak{n}_{\mathbb{C}}$ has the desired properties for $\mathfrak{q}_1$. □

If $\mathfrak{p}$ is a parabolic subalgebra of a reductive Lie algebra over $\mathbb{C}$, $\mathfrak{g}$ such that the analogues of conditions 1) and 2) of the above theorem hold, then we will say that $\mathfrak{p}$ is nice. We will now give a condition that only depends on the complexification of $Lie(P)$ that implies that if P is nice, then it is very nice. We note first of all that G is very nice if and only if the adjoint group of its Lie algebra is very nice. So we will assume that G is connected and semi-simple.

Theorem 6.2. *We will use the notation of the previous proposition. Let G be connected and semisimple and let P be a parabolic subgroup of G. Let $G_{\mathbb{C}}$ be the adjoint group of $\mathfrak{g}_{\mathbb{C}}$, let $P_{\mathbb{C}}$ be the subgroup corresponding to $\mathfrak{p}_{\mathbb{C}}$ and let $x \in \mathfrak{n}_{\mathbb{C}}$ be a Richardson element. If P is nice and if $\{g \in G_{\mathbb{C}} | Ad(g)x = x\} = \{p \in P_{\mathbb{C}} | Ad(p)x = x\}$, then P is very nice.*

Proof. Let $P = MN$ be a Levi decomposition as in the beginning of this section. Let A be a split component of M. We also fix a minimal parabolic subgroup of G, P_o such that $P_o \subset P$, so we can write $P_o = M_oN_o$ with $M_o \subset M$ and $N \subset N_o$. We can choose a split component of M_o, A_o such that $A \subset A_o$. We assume that P is nice, that $h \in Lie(A)$ defines a grade of $\mathfrak{g} = Lie(G)$ (as in (1), Section 3), that is, if $\mathfrak{g}^j = \{X \in \mathfrak{g} | ad(h)X = j\}$, then $\mathfrak{g} = \bigoplus_j \mathfrak{g}^j$, $Lie(P) = \bigoplus_{j \geq 0} \mathfrak{g}^{2j}$ and $Lie(N)$ is generated by $\mathfrak{g}^2$. We will be using some notation from Section 2. Let $x \in \mathfrak{g}^2$ be a Richardson element, we will show that if $w \in W(A_o)$ and $w\Phi(P_o, A_o) \supset -\Phi(M \cap P_o, A_o)$ is such that

$$B(Ad(w^*)^{-1}Lie(N_o), x) = \{0\},$$

then $w = w_M$. We note that this condition is the same as

$$B(Lie(N_o), Ad(w^*)x) = \{0\}.$$

If $x \in Lie(N)$, then we can write $x = \sum_{\alpha \in \Phi(P_o, A_o)} x_\alpha$ with x_α in the α rootspace. We define $\mathrm{supp}(x) = \{a \in \Phi(P_o, A_o) | x_\alpha \neq 0\}$. Then the condition above says that $w\,\mathrm{supp}(x) \subset \Phi(P_o, A_o)$. But our assumption on w implies that

$$w\Phi(P_o, A_o) \cap \Phi(P_o, A_o) \subset \Phi(P_o, A_o) - \Phi(M \cap P_o, A_o).$$

This implies that $Ad(w^*)x \in Lie(N)$. But this implies that it is a Richardson element on $Lie(N)$. Hence there exists $p \in P_{\mathbb{C}}$ such that $Ad(p)Ad(w^*)x = x$. This implies by our assumption that $pw^* \in P_{\mathbb{C}}$ and hence $Ad(w^*)Lie(N) = Lie(N)$. Our assumption on w now implies that $w = w_M$. This completes the proof. □

The condition of equality of stabilizers in the previous theorem is equivalent to the statement that the moment map from $T^*(G_{\mathbb{C}}/P_{\mathbb{C}})$ to $\mathfrak{g}_{\mathbb{C}}^*$ is birational onto its image. The list of all nice parabolic subalgebras satisfying this condition is given in [BW2]. We will develop a few generalities in this paper and leave the most delicate results to [BW2], and without proof the results therein. All of the conditions are part of the folklore of the subject. So we have decided, for the sake of completeness to give precise statements and proofs. All of the proofs are easy except for the first lemma which is also used in [Y1] and [Y2].

If G is a connected, affine algebraic, reductive group over $\mathbb{C}$, then we will say that a parabolic subgroup, P, satisfies condition (C) if the stabilizer in G of a Richardson element in the Lie algebra of the unipotent radical is contained in P. We note that if P_1 satisfies condition (C) in G_1 and P_2 satisfies condition (C) in G_2, then $P_1 \times P_2$ satisfies condition (C) in $G_1 \times G_2$.

Lemma 6.3. *If P has abelian unipotent radical, then it satisfies condition (C).*

Proof. Let $B \subset P$ be a Borel subgroup and $H \subset B$ a Cartan subgroup. Let $\mathfrak{n}$ be the Lie algebra of the unipotent radical of P. Let Σ denote the set of roots of $\mathfrak{h} = Lie(H)$ on $\mathfrak{n}$ and let Φ^+ denote the roots that have root spaces contained in $Lie(B)$. Then there exists a set $\gamma_1, \dots, \gamma_r$ of elements of Σ with the following properties:

a) $\gamma_i \pm \gamma_j$ is not a root.

b) If $\mathfrak{h}^-$ is the complex linear span of the coroots of the γ_i and if $\alpha \in \Sigma$, then $\alpha_{|_{\mathfrak{h}^-}}$ is of one of the forms: $\gamma_i, \frac{1}{2}(\gamma_i + \gamma_j)$ with $i < j$ or possibly $\frac{1}{2}\gamma_i$ for some i. Furthermore, if $\alpha_{|_{\mathfrak{h}^-}} = \gamma_i$, then $\alpha = \gamma_i$.

c) If $\alpha \in \Phi^+ - \Sigma$, then $\alpha_{|_{\mathfrak{h}^-}}$ is of one of the forms $\frac{1}{2}(\gamma_i - \gamma_j)$ with $i < j$ or possibly $\frac{1}{2}\gamma_i$.

d) Choose a non-zero root vector, X_α, for each root α. Then $X_{\gamma_1} + \cdots + X_{\gamma_r}$ is a Richardson element in $\mathfrak{n}_{\mathbb{C}}$.

These properties follow from the theory of Hermitian symmetric spaces if we observe that if M is the Levi factor of P that contains H, then there exists an involutive automorphism of $Lie(G)$ with fixed point algebra $Lie(M)$.

Let $x = X_{\gamma_1} + \cdots + X_{\gamma_r}$ as above and assume that the X_α have been chosen so that $[X_\alpha, X_{-\alpha}]$ is the α coroot, $\check{\alpha}$. Let $y = X_{-\gamma_1} + \cdots + X_{-\gamma_r}$, then $[x, y] = h = \check{\gamma}_1 + \cdots + \check{\gamma}_r$ and h, x, y is a standard basis for a three-dimensional simple Lie subalgebra (TDS for short). We assume that $g \in G$ and $Ad(g)x = x$. Then $x, h' = Ad(g)h, y' = Ad(g)y$ is another TDS. We note that by b) above $ad(h)$ is injective on $\mathfrak{n}$. We also observe that since x is a Richardson element, $ad(x)$ is injective on $\bar{\mathfrak{n}}$ (the nilradical of the opposite parabolic corresponding to the choice of H). Write $h' = h'_- + h'_0 + h'_+$ with $h'_- \in \bar{\mathfrak{n}}$, $h'_o \in \mathfrak{m}$ and $h'_+ \in \mathfrak{n}$. Here $\mathfrak{m}$ is the standard Levi factor of $\mathfrak{p} = Lie(P)$ corresponding to the choice of H. Now $[h', x] = 2x$ implies that $[h'_-, x] = 0$. Thus our observation above implies that $h'_- = 0$. Also, $y' = y'_- + y'_0 + y'_+$ with $y'_- \in \bar{\mathfrak{n}}$, $y'_o \in \mathfrak{m}$ and $y'_+ \in \mathfrak{n}$. Now $[x, y'] = h'$ implies that $[x, y'_-] = h'_0$ and $[x, y'_0] = h'_+$. We note that $[h', x] = [h'_0, x] = 2x$ and since $[x, y'] = h'$ we see that $[x, y'_-] = h'_0$. Also, since the component of $[h', y'_-]$ in $\bar{\mathfrak{n}}$ is $[h'_0, y'_-]$, we also have $[h'_0, y'_-] = -2y'_-$. Hence x, h'_0, y'_- is another TDS. We write $y'_- = \sum c_i X_{-\gamma_i} + \sum_{i<j} Y_{ij} + \sum_i \tilde{Y}_i$ with Y_{ij} in the $\mathfrak{h}^-$ weight space for $-\frac{1}{2}(\gamma_i + \gamma_j)$ and $\tilde{Y}_i$ is in the weight space $-\frac{1}{2}\gamma_i$. We therefore have (taking into account a), b), and c))

$$[x.y'_-] = \sum c_i \check{\gamma}_i + \sum_{i<j} \left([X_{\gamma_i}, Y_{ij}] + [X_{\gamma_j}, Y_{ij}]\right) + \sum_i [X_{\gamma_i}, \tilde{Y}_i].$$

Now $[X_{\gamma_i}, Y_{ij}]$ is in the $\frac{1}{2}(\gamma_i - \gamma_j)$ weight space and $[X_{\gamma_i}, \tilde{Y}_i]$ is in the $\frac{1}{2}\gamma_\iota$ weight space. Since we must have $[[x, y'_-], x] = 2x$, this can only happen if all of the $c_i = 1$ and $h'_0 = h$. If $z \in \mathfrak{n}$, then $e^{adz}h' = h' + [z, h'_0]$. Thus since h'_0 is invertible on $\mathfrak{n}$ we see that there exists $p \in P$ so that $Ad(p)x = x$ and $Ad(p)h' = h$. We therefore have $Ad(pg)x = x$ and $Ad(pg)h = h$. Let $\mathfrak{q}$ be the sum of the eigenspaces for the non-negative eigenvalues of $ad(h)$. Then b) and c) imply that $\mathfrak{q} \subset \mathfrak{p}$. We also note that we have $Ad(pg)\mathfrak{q} \subset \mathfrak{q}$. The subgroup of G that normalizes $\mathfrak{q}$ is a subgroup of P since G is assumed to be connected. This implies $g \in P$. $\square$

This result implies

Corollary 6.4. *If G is a connected real reductive group and if P is a parabolic subgroup of G with commutative unipotent radical, then P is very nice.*

Proof. The previous result implies that the complexified Lie algebra of P satisfies condition (C). Since P is nice, Theorem 6.2 implies that P is very nice. $\square$

We will now begin the proof of the main theorems in this direction. The first is an easy result that is usually given without proof. In [He] a proof is given of the stronger result that for $GL(n,\mathbb{C})$ the stabilizer of a nilpotent element is connected. We only need the condition (C) which is much easier and included below.

Proposition 6.5. *Let P be a parabolic subgroup of $GL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$, then P satisfies condition (C).*

Proof. It is enough to prove this result for $G = GL(n,\mathbb{C})$. We identify, as usual, $Lie(G)$ with the $n\times n$ matrices, $M_n(\mathbb{C})$. We may assume that P contains the upper triangular Borel subgroup and hence that P consists of matrices of a fixed upper triangular block form corresponding to diagonal blocks of sizes $n_1 \times n_1, \ldots, n_r \times n_r$ with $n_1 + \cdots + n_r = n$. If $\mathfrak{p} = Lie(P)$, then this clearly implies that $P = G \cap \mathfrak{p}$. If $x \in \mathfrak{n}$, the nilradical of $\mathfrak{p}$, is a Richardson element, we have

$$\{z \in M_n(\mathbb{C}) | [z,x] = 0\} \subset \mathfrak{p}.$$

Now, if $g \in G$ and $Ad(g)x = x$, then $[g,x] = 0$. Thus if $g \in G$ is such that $Ad(g)x = x$, then $g \in G \cap \mathfrak{p} = P$. This completes the proof. □

We next look at the main class of nice parabolic subalgebras. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$. Let G denote the adjoint group of $\mathfrak{g}$ and let P denote the normalizer of $\mathfrak{p}$ in $\mathfrak{g}$, then we will say that $\mathfrak{p}$ satisfies the condition (C) if P does in G. Let $x \in \mathfrak{g}$ be a nilpotent element, $x \neq 0$, then we will say that x is even if for some (hence for any) TDS in $\mathfrak{g}$, x, y, h, all of the eigenvalues of $ad(h)$ are even. We will say that the parabolic subalgebra that is the span of the non-negative eigenvalues for h as above *comes from an even TDS*.

The main theorem in this direction follows. It is stated in this generality in [He]. Our proof as indicated above is easy, and we include it for the sake of completeness.

Theorem 6.6. *Let $\mathfrak{p}$ be a parabolic subalgebra of the reductive Lie algebra $\mathfrak{g}$ that comes from an even TDS, then $\mathfrak{p}$ is nice and satisfies condition (C).*

Proof. Let $\mathfrak{g}_j$ be the eigenspace for adh with eigenvalue j, then $\sum_{j\leq 0} \mathfrak{g}_j$ is an opposite parabolic subalgebra to $\mathfrak{p}$ with nilradical $\overline{\mathfrak{n}} = \sum_{j<0} \mathfrak{g}_j$. Since the eigenvalues of adh on $\ker adx$ are all non-negative this implies that adx is injective on $\overline{\mathfrak{n}}$. Lemma 3.3 implies that $\mathfrak{p}$ is nice.

We will now prove that $\mathfrak{p}$ satisfies condition (C). We may assume that $\mathfrak{g}$ is semisimple. Then we have $ad : \mathfrak{g} \to End(\mathfrak{g})$ is an injective Lie algebra homomorphism and we will use the notation Ad for the injection of the adjoint group of $\mathfrak{g}$, G, into $GL(\mathfrak{g})$. Let x, y, h be an even TDS defining $\mathfrak{p}$. We set $X = adx, Y = ady, H = adh$. Then the eigenvalues of adH are $j - k$ with j and k eigenvalues of adh thus X, Y, H is an even TDS in $End(\mathfrak{g})$. Let $\mathfrak{q}$ denote the corresponding parabolic subalgebra of $End(\mathfrak{g})$ and let $Q = \mathfrak{q} \cap GL(\mathfrak{g})$ be the corresponding parabolic subgroup. In light of the first part of the proof we see that $\mathfrak{q}$ is nice and hence X is a Richardson element in the nilradical of $\mathfrak{q}$.

Let $\mathfrak{g}_j$ denote the eigenspace for adh with eigenvalue j. We note that if P is the parabolic subgroup of G corresponding to $\mathfrak{p}$, then $P \subset \{g \in G | Ad(g) \in Q\}$. Indeed, we first observe that $End(\mathfrak{g}) = \bigoplus_{j,k} Hom(\mathfrak{g}_j, \mathfrak{g}_k)$ with adH acting on $Hom(\mathfrak{g}_j, \mathfrak{g}_k)$ by $(k-j)I$. Thus $\mathfrak{q} = \bigoplus_{j\leq k} Hom(\mathfrak{g}_j, \mathfrak{g}_k)$. Hence if $q \in Q$, then $q\mathfrak{g}_j \subset \bigoplus_{k\geq j} \mathfrak{g}_k$. This implies that if $q \in Q$, then $q\mathfrak{p} \subset \mathfrak{p}$. Thus, if $g \in G$ and $Ad(g) \in Q$, then $g \in P$ as asserted. We can now complete the proof. Let $g \in G$ be such that

$Ad(g)x = x$. Then $ad(Ad(g)x) = ad(x)$. Hence $Ad(g)XAd(g)^{-1} = X$ and since X is a Richardson element in the nilradical of $\mathfrak{q}$, the previous lemma implies that $Ad(g) \in Q$. But then $g \in P$ by the above observation. □

We will now prove the other assertions in the introduction and in Section 2.

Corollary 6.7. *Let G be a connected absolutely simple Lie group over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let P be a parabolic subgroup over $\mathbb{F}$ with unipotent radical of Heisenberg type over $\mathbb{F}$. If G is not of type C_n with $n \geq 2$, then P is very nice.*

Proof. Under this condition the Lie algebra of P is defined by an even TDS in $Lie(G)$. Thus the preceding theorem combined with Theorem 6.2 implies that P is very nice. □

We will now prove Corollary 2.3. We will use the notation before the statement. Let k be as in the statement. Then if we define $\eta(\overline{n}) = \chi(k^{-1}\overline{n}k)$, we have

$$J^{\chi}_{P,\sigma,\nu}(f) = J^{\eta}_{P,\sigma,\nu}(\pi_{P,\sigma,\nu}(k)f).$$

Here the difference between the "J" on the left-hand side and on the right, is that they involve characters for different unipotent groups. Thus to prove the corollary we must show that η is strongly non-degenerate. The representation theory of $\mathfrak{sl}_2$ implies that η is non-degenerate. Since P comes from an even TDS the Theorem 6.6 combined with Theorem 6.2 implies that P is very nice and so Theorem 2.2 implies the result. We will now describe the results in [BW2]. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. If it is classical of type B_n, C_n or D_n, respectively, we take it to be a subalgebra of the $N \times N$ matrices $N = 2n+1, 2n, 2n$, respectively, given by a form such that its intersection with the upper triangular matrices is a Borel subalgebra (we will call these parabolic subalgebras standard). That is, if we set L_M to be the $M \times M$ matrix with ones on the skew diagonal, then for type B_n and D_n the form defining the Lie algebra will be taken to have matrix L_N and in the case of C_n the form will be taken to have matrix

$$\begin{bmatrix} 0 & L_n \\ -L_n & 0 \end{bmatrix}.$$

In this form the parabolic subalgebras are described (up to conjugacy) by the block form of the block diagonal Levi factor of the corresponding parabolic subalgebra of the $N \times N$ matrices. We will call the vector of block sizes along the main diagonal the block form of the parabolic subalgebra. Here is the result in the case of the classical groups. (In the case of D_n there are pairs of parabolic subalgebras that are related by an outer automorphism; the list below only has one of each such pair.)

Theorem 6.8. *The Lie algebras of very nice standard parabolic subalgebras of the classical groups are described as follows:*

a) *In the case $G = SL(n, \mathbb{C})$ they are the parabolic subalgebras that have block form that is unimodal and palendromic.*

b) *In the case when G is of type B_n, they are the standard parabolic subalgebras with an odd number of blocks that are unimodal (and palendromic).*

c) *In the case when G is of type C_n, they are the standard parabolic subalgebras with an odd number of even blocks that are unimodal (and palendromic).*

d) *In the case D_n, they are the standard parabolic subalgebras that are unimodal (and palendromic) and if its block form has an even number of blocks, then there is at most one block size that is odd and if the block sizes are given as*

$$d_1 \leq d_2 \leq \cdots \leq d_r = d_r \geq \cdots \geq d_1,$$

then the odd block must have index $i < r$ and $d_i < d_r - 3$.

For the exceptional Lie algebras we will use the Bourbaki way of labeling the Dynkin diagram and parametrize the parabolic subalgebras up to conjugacy by an n-tuple of 0's and 1's; the nodes with a 1 correspond to the index of a simple root whose root space is in the unipotent radical of the parabolic. Here are the tables taken from [BW2]:

G_2	F_4
$(1,1)$	$(1,1,1,1)$
$(1,0)$	$(1,1,0,1)$
$(0,0)$	$(1,1,0,0)$
	$(1,0,0,1)$
	$(0,1,0,1)$
	$(0,1,0,0)$
	$(0,0,0,1)$
	$(0,0,0,0)$

	E_6	E_7	E_8
1	$(1,1,1,1,1,1)$	$(1,1,1,1,1,1,1)$	$(1,1,1,1,1,1,1,1)$
2	$(1,1,1,0,1,1)$	$(1,1,1,0,1,1,1)$	$(1,1,1,0,1,1,1,1)$
3	$(1,1,1,0,1,0)$	$(1,1,1,0,1,0,1)$	$(1,1,1,0,1,0,1,1)$
4	$(1,1,0,1,0,1)$	$(1,1,0,0,1,0,1)$	$(1,0,0,1,0,1,1,1)$
5	$(1,1,0,0,1,0)$		$(1,0,0,1,0,1,0,1)$
6	$(1,1,0,0,0,1)$	$(1,0,1,1,0,1,0)$	$(1,0,0,1,0,0,1,1)$
7	$(1,1,0,0,0,0)$	$(1,0,1,0,0,1,0)$	$(1,0,0,1,0,0,1,0)$
8	$(1,0,1,1,0,1)$	$(1,0,1,0,0,0,0)$	$(1,0,0,0,1,0,0,1)$
9	$(1,0,1,0,0,1)$	$(1,0,0,1,0,1,1)$	$(1,0,0,0,0,1,1,1)$
10	$(1,0,1,0,0,0)$	$(1,0,0,1,0,1,0)$	$(1,0,0,0,0,1,0,1)$
11	$(1,0,0,1,1,1)$	$(1,0,0,1,0,0,1)$	$(1,0,0,0,0,1,0,0)$
12	$(1,0,0,1,0,1)$	$(1,0,0,0,1,0,0)$	$(1,0,0,0,0,0,1,1)$
13	$(1,0,0,1,0,0)$	$(1,0,0,0,0,1,1)$	$(1,0,0,0,0,0,1,0)$
14	$(1,0,0,0,1,1)$	$(1,0,0,0,0,1,0)$	$(1,0,0,0,0,0,0,1)$
15	$(1,0,0,0,1,0)$	$(1,0,0,0,0,0,1)$	$(1,0,0,0,0,0,0,0)$
16	$(1,0,0,0,0,1)$	$(1,0,0,0,0,0,0)$	$(0,1,0,0,0,0,0,1)$
17	$(1,0,0,0,0,0)$	$(0,1,1,0,0,1,1)$	$(0,1,0,0,0,0,0,0)$
18	$(0,1,1,0,1,1)$	$(0,1,0,0,0,0,0)$	$(0,0,1,0,0,0,1,0)$
19	$(0,1,1,0,0,1)$	$(0,0,1,0,0,1,0)$	$(0,0,0,1,0,0,1,1)$
20	$(0,1,0,1,0,0)$		$(0,0,0,1,0,0,1,0)$
21	$(0,1,0,0,0,1)$	$(0,0,1,0,0,0,0)$	$(0,0,0,1,0,0,0,1)$
22	$(0,1,0,0,0,0)$	$(0,0,0,1,0,1,0)$	$(0,0,0,0,1,0,0,1)$
23	$(0,0,1,0,0,1)$	$(0,0,0,1,0,0,1)$	$(0,0,0,0,1,0,0,0)$
24	$(0,0,1,0,0,0)$	$(0,0,0,1,0,0,0)$	$(0,0,0,0,0,1,0,0)$
25	$(0,0,0,1,0,1)$		$(0,0,0,0,0,0,1,1)$
26	$(0,0,0,1,0,0)$	$(0,0,0,0,1,0,0)$	$(0,0,0,0,0,0,1,0)$
27	$(0,0,0,0,1,1)$	$(0,0,0,0,0,1,0)$	$(0,0,0,0,0,0,0,1)$
28	$(0,0,0,0,1,0)$	$(0,0,0,0,0,0,1)$	$(0,0,0,0,0,0,0,0)$
29	$(0,0,0,0,0,1)$	$(0,0,0,0,0,0,0)$	
30	$(0,0,0,0,0,0)$		

These tables and Theorem 6.8 combine to have the following remarkable consequence.

Theorem 6.9. *If G is simple and not simply laced (i.e., not of type ADE), then a parabolic subgroup of G is very nice if and only if it comes from an even TDS.*

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References

[BW] Karin Baur and Nolan Wallach, *Nice parabolic subalgebras of reductive Lie algebras*, Represent. Theory **9** (2005), 1-29. MR2123123 (2005k:17021)

[BW2] Karin Baur and Nolan Wallach, to appear.

[B] Armand Borel, *Linear Algebraic Groups*, Second Enlarged Edition, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[CM] David H. Collingwood and William M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold Mathematics Series, New York, 1993. MR1251060 (94j:17001)

[EK] A.G. Elashvili and V.G. Kac, *Good gradings of semisimple Lie algebras*, to appear.

[Ha1] M. Hashizume, *Whittaker models for real reductive groups*, Japan J. Math. (N.S.) **5** (1979), 349-401. MR0614828 (82g:10048)

[Ha2] M. Hashizume, *Whittaker functions on semisimple Lie groups*, Hiroshima Math. J. **12** (1982), 259-293. MR0665496 (84d:22018)

[He] W. Hesselink, *Polarizations in the classical groups*, Math. Z. **160** (1978), 217-234. MR0480765 (58:916)

[J] H. Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France **95** (1967), 243-309. MR0271275 (42:6158)

[KV] Johan A.C. Kolk and V.S.Varadarajan, *On the transversal symbol of vectorial distributions and some applications to harmonic analysis*, Indag. Mathem. (N.S.) **7** (1996), 67-96. MR1621372 (99d:22025)

[L] Thomas E. Lynch, *Generalized Whittaker vectors and representation theory*, Thesis, MIT, 1979.

[Mc] William M. McGovern, *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia of Mathematical Sciences, Volume 131, Springer-Verlag, Berlin, 2002, 159-238. MR1925828 (2003c:14002)

[R] R.W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*, Bull. London Math. Soc. **6** (1974), 21-24. MR0330311 (48:8648)

[S] G. Schiffmann, *Intégrales d'entrelacement et fonctions de Whittaker*, Bull. Soc. Math. France **99** (1971), 3-72. MR0311838 (47:400)

[Wa1] Nolan R. Wallach, *Lie algebra cohomology and holomorphic continuation of generalized Jacquet integrals*, Representations of Lie groups, Kyoto, Hiroshima, 1986, 123–151, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA, 1988. MR1039836 (91d:22014)

[Wa2] Nolan R. Wallach, *Real reductive groups.* I. Pure and Applied Mathematics, 132, Academic Press, Inc., Boston, MA, 1988. MR0929683 (89i:22029)

[Wa3] Nolan R. Wallach, *Generalized Whittaker Vectors for Holomorphic and Quaternionic Representations*, Comment. Math. Helv. **78** (2003), 266-307. MR1988198 (2004e:22018)

[Y1] Hiroshi Yamashita, *On Whittaker vectors for generalized Gel′fand-Graev representations of semisimple Lie groups*, J. Math. Kyoto Univ. **26** (1986), 263-298. MR0849220 (88a:22028)

[Y2] Hiroshi Yamashita, *Multiplicity one theorems for generalized Gel′fand-Graev representations of semisimple Lie groups and Whittaker models for discrete series*, Adv. Stud. Pure Math. 14, Academic Press Boston, 1988, 31-121. MR1039835 (91e:22023)

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