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A CLASS OF PERVERSE SHEAVES ON A PARTIAL FLAG MANIFOLD

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Abstract. We introduce a class of perverse sheaves on a partial flag manifold of a connected reductive group G defined over a finite field which are equivariant for the action of the group of rational points of G. The definition of this class is similar to the definition of parabolic character sheaves.

Introduction

Let $f : X \to Y$ be a morphism of algebraic varieties over an algebraically closed field $\mathbf{k}$. The theory of perverse sheaves [BBD] associates to f a collection of invariants, namely the collection of simple perverse sheaves on Y which appear as subquotients of the l-adic perverse sheaf $\oplus_{j\in\mathbf{Z}}{}^pH^j(f_!\bar{\mathbf{Q}}_l)$ on Y. More precisely, if f is equivariant for given actions of a finite group Γ on X and Y, then we denote by $C_\Gamma(f)$ the (finite) collection of simple Γ-equivariant perverse sheaves on Y which appear as subquotients of the l-adic Γ-equivariant perverse sheaf $\oplus_{j\in\mathbf{Z}}{}^pH^j(f_!\bar{\mathbf{Q}}_l)$ on Y. (The perverse sheaves in $C_\Gamma(f)$ are not necessarily simple if the Γ-equivariant structure is disregarded.)

In this paper we try to understand the collection $C_\Gamma(f)$ in the case where:

$\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$,

G is a connected reductive algebraic group defined over $\mathbf{F}_q$ with Frobenius map $F : G \to G$,

Γ is the group of $\mathbf{F}_q$-rational points of G,

X is the variety of all pairs (B', g) where B' is a Borel subgroup of G and $g \in G$ is such that $g^{-1}F(g)$ is in the unipotent radical of B',

Y is the variety of parabolic subgroups of G of a fixed type,

f associates to $(B', g) \in X$ the unique parabolic subgroup in Y that contains B'.

(The action of Γ on X is $g_0 : (B', g) \mapsto (g_0B'g_0^{-1}, gg_0^{-1})$; the action of Γ on Y is by conjugation.)

Note that the variety X can be viewed as a family of varieties of the type considered in [DL] indexed by the full flag manifold.

In this paper we define a finite collection $\mathbb{S}(Y)$ of simple Γ-equivariant perverse sheaves on Y by two methods (see Sections 3 and 4). These methods and the proof of their equivalence are similar to those used in the theory of parabolic character sheaves [L5]. The second method (see Section 4) gives a description of these perverse

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sheaves in terms of some explicit local systems on some pieces of a finite partition of Y. This partition, introduced by the author in 1977, and further studied in [BE], reduces in the case where Y is the full flag manifold to the partition introduced in [DL].

In 7.6 we show that $C_f(Y) \subset \mathbb{S}(Y)$.

In Section 6 we construct an explicit basis for the space of intertwining operators between certain cohomologically induced representations of Γ, extending an idea of [L4]. As a bi-product we obtain a disjointness theorem for the objects of $\mathbb{S}(Y)$ which in the special case where $Y = \{G\}$ reduces to the disjointness theorem [DL, 6.2, 6.3] (but the present proof is quite different from that of [DL]).

In Section 7 we study the variety X (see above). In particular we show (using results in Section 6) that X is connected if G is simply connected.

In Section 8 we give a conjecture (based on results in Section 6 and some combinatorial results in Section 5) which should explain in an intrinsic way the "Jordan decomposition" [L1] for irreducible representations of Γ.

Contents

1. Preliminaries

1.1. Let $\mathbf{k}$ be an algebraically closed field. In this paper all algebraic varieties are over $\mathbf{k}$. Let G be a connected reductive algebraic group. Let $\mathcal{B}$ be the variety of Borel subgroups of G. We fix $B \in \mathcal{B}$ and a maximal torus T of B. Let $N(T)$ be the normalizer of T in G. Let $W = N(T)/T$. Note that W acts on T by conjugation; we use this action to identify W with a subgroup of $\mathrm{Aut}(T)$. For any $(B', B'') \in \mathcal{B} \times \mathcal{B}$ there is a unique $w \in W$ such that $gB'g^{-1} = B, gB''g^{-1} = \tilde{w}B\tilde{w}^{-1}$ for some $g \in G$ and some representative $\tilde{w}$ of w in $N(T)$. We then write $\mathrm{pos}(B', B'') = w$. Let $l : W \to \mathbf{N}$ be the length function: $l(w) = \dim(B'/(B' \cap B''))$ where $(B', B'') \in \mathcal{B} \times \mathcal{B}$, $\mathrm{pos}(B', B'') = w$. Let $\mathbf{I} = \{w \in W; l(w) = 1\}$.

If H is a group acting on a set X, we denote by X^H the fixed point set of H on X.

1.2. Let $\leq$ be the standard partial order on W regarded as a Coxeter group with generators $\mathbf{I}$. If X is a subset of W and $w \in X$ we write $w = \min X$ if $l(w) < l(w')$ for all $w' \in W - \{w\}$. For $J \subset \mathbf{I}$ let W_J be the subgroup of W generated by J. For $J, J' \subset W$ let ${}^JW = \{w \in W; w = \min(W_J w)\}$, $W^{J'} = \{w \in W; w = \min(wW_{J'})\}$, ${}^JW^{J'} = {}^JW \cap W^{J'}$.

Let $w_{\mathbf{I}}$ be the unique element of maximal length in W.

1.3. Let $\mathcal{P}$ be the variety of parabolic subgroups of G. For any $P\in\mathcal{P}$ let U_P be the unipotent radical of P. We set $U=U_B$. For $J\subset\mathbf{I}$ let $P_J\in\mathcal{P}$ be the subgroup of G generated by B and by representatives in $N(T)$ of the various elements of J. Let L_J be the unique Levi subgroup of P that contains T. For $s\in\mathbf{I}$ we write P_s instead of $P_{\{s\}}$. For $J\subset\mathbf{I}$ let $\mathcal{P}_J$ be the G-conjugacy class of parabolic subgroups of G that contains P_J. For $B'\in\mathcal{B}$ let $P_{B',J}$ be the unique subgroup in $\mathcal{P}_J$ that contains B'. For $P\in\mathcal{P}_J$, $Q\in\mathcal{P}_{J'}$ the element $\mathrm{pos}(P,Q):=\min\{w\in W; w=\mathrm{pos}(B',B'')$ for some $B'\subset P, B''\subset Q\}$ is well defined and $\mathrm{pos}(P,Q)\in {}^JW^{J'}$. We set

(a) $P^Q=(P\cap Q)U_P\in\mathcal{P}_{J\cap uJ'u^{-1}}$ where $u=\mathrm{pos}(P,Q)$.

For any $g\in G$ we define $k(g)\in N(T)$ by $g\in Uk(g)U$.

1.4. Let $R\subset\mathrm{Hom}(T,\mathbf{k}^*)$ be the set of roots. Let $\check R\subset\mathrm{Hom}(\mathbf{k}^*,T)$ be the set of coroots. Let $\check\alpha\leftrightarrow\alpha$ be the standard bijection $\check R\leftrightarrow R$. For $\alpha\in R$ let U_α be the one-dimensional root subgroup (normalized by T) corresponding to α. Let $R^+=\{a\in R; U_\alpha\subset B\}$, $R^-=R-R^+$. For $s\in\mathbf{I}$ let α_s be the unique root such that $U_{\alpha_s}\subset P_s$. We write $\check\alpha_s$ instead of $(\alpha_s)^\vee$. The natural action of W on T induces an action of W on R.

1.5. For any $s\in\mathbf{I}$ we fix a homomorphism $h_s:SL_2(\mathbf{k})\to G$ such that

$h_s\left(\begin{smallmatrix} b & 0\\ 0 & b^{-1}\end{smallmatrix}\right)=\check\alpha_s(b)$ for all $b\in\mathbf{k}^*$,

$a\mapsto h_s\left(\begin{smallmatrix} 1 & a\\ 0 & 1\end{smallmatrix}\right)$ is an isomorphism $x_s:\mathbf{k}\xrightarrow{\sim}U_{\alpha_s}$,

$a\mapsto h_s\left(\begin{smallmatrix} 1 & 0\\ a & 1\end{smallmatrix}\right)$ is an isomorphism $y_s:\mathbf{k}\xrightarrow{\sim}U_{\alpha_s^{-1}}$.

We say that $\{B,T,h_s(s\in\mathbf{I})\}$ is an *épinglage* of G. Let $\dot s=h_s\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)\in N(T)$. For any $w\in W$ we set $\dot w=\dot s_1\dot s_2\dots\dot s_n\in N(T)$ where $s_1,s_2,\dots,s_n\in\mathbf{I}$ are chosen so that $w=s_1s_2\dots s_n$, $l(w)=n$. (This is independent of the choice.) In particular, $\dot 1=1$.

For any sequence $\mathbf{w}=(w_1,w_2,\dots,w_r)$ in W we set $[\mathbf{w}]=w_1w_2\dots w_r\in W$ and $[\mathbf{w}]^\bullet=\dot w_1\dot w_2\dots\dot w_r\in N(T)$.

1.6. Equivariant structures. If X is an algebraic variety we write $\mathcal{D}(X)$ for the derived category of bounded constructible $\bar{\mathbf{Q}}_l$-sheaves on X. If $K\in\mathcal{D}(X)$, let $\mathfrak{D}(K)\in\mathcal{D}(X)$ be the Verdier dual of X. Let $\mathcal{M}(X)$ be the subcategory of $\mathcal{D}(X)$ whose objects are perverse sheaves. If $\mathcal{E}$ is a local system on X we denote by $\check{\mathcal{E}}$ the dual local system. For $K\in\mathcal{D}(X)$ we write ${}^pH^\cdot(K)$ instead of $\oplus_j{}^pH^j(K)$. If $f:X\to Y$ is a smooth morphism between algebraic varieties with connected fibres of dimension δ, we set $f^\bigstar A=f^*A[\delta]$ for any $A\in\mathcal{D}(Y)$.

Let $m:H\times Y\to Y$ be an action of an algebraic group H on an algebraic variety Y. For any connected component C of H define $m_C:C\times Y\to Y$ by $(g,y)\mapsto gy$ and $\pi_C:C\times Y\to Y$ by $(g,y)\mapsto y$. Let $K\in\mathcal{M}(Y)$. An H-equivariant structure on K is a collection of isomorphisms $\phi_C:\pi_C^\bigstar K\xrightarrow{\sim}m_C^\bigstar K$ (one for each C) such that for any two connected components C,C' of H,

$$(\tilde m\boxtimes 1)^\bigstar(\phi_{CC'}):(\tilde m\boxtimes 1)^\bigstar(\pi_{CC'}^\bigstar K)\to(\tilde m\boxtimes 1)^\bigstar(m_{CC'}^\bigstar K)$$

is equal to the composition

$$(\tilde m\boxtimes 1)^\bigstar(\pi_{CC'}^\bigstar K)\xrightarrow{\overset{(1)}{=}}\tilde\pi^\bigstar(\pi_{C'}^\bigstar K)\xrightarrow{\beta}\tilde\pi^\bigstar(m_{C'}^\bigstar K)$$
$$\xrightarrow{\overset{(3)}{=}}(1\times m_{C'})^\bigstar(\pi_C^\bigstar K)\xrightarrow{\gamma}(1\times m_{C'})^\bigstar(m_C^\bigstar K)\xrightarrow{\overset{(2)}{=}}(\tilde m\boxtimes 1)^\bigstar(m_{CC'}^\bigstar K)$$

where

$\tilde{m} : C \times C' \to CC'$ is $(g, g') \mapsto gg'$, $\tilde{\pi} : C \times C' \times Y \to C' \times Y$ is $(g, g', y) \mapsto (g', y)$;
the equality (1) comes from $\pi_{C'}\tilde{\pi} = \pi_{CC'}(\tilde{m} \times 1) : C \times C' \times Y \to Y$;
the equality (2) comes from $m_{CC'}(1 \times m_{C'}) = m_{CC'}(\tilde{m} \times 1) : C \times C' \times Y \to Y$;
the equality (3) comes from $m_{C'}\tilde{\pi} = \pi_C(1 \times m_{C'}) : C \times C' \times Y \to Y$;
$\beta = \tilde{\pi}^\bigstar \phi_{C'}$, $\gamma = (1 \times m_{C'})^\bigstar \phi_C$.

The notion of H-equivariant structure on a local system on Y is defined in the same way (but replace $()^\bigstar$ by $()^*$).

We denote by $\mathcal{M}_H(Y)$ the (abelian) category of perverse sheaves on Y endowed with an H-equivariant structure; the morphisms are morphisms of perverse sheaves which are compatible with the equivariant structures. Note that any object of $\mathcal{M}_H(Y)$ has finite length. If $K \in \mathcal{M}_H(Y)$ is semisimple as an object of $\mathcal{M}(Y)$ and H is finite, then K is semisimple as an object of $\mathcal{M}_H(Y)$.

Let $f : X \to Y$ be a morphism compatible with H-actions on X and Y. If $K \in \mathcal{M}_H(Y)$, then ${}^pH^i(f^*K)$ is naturally an object of $\mathcal{M}_H(X)$. If $K' \in \mathcal{M}_H(X)$, then ${}^pH^i(f_!K')$ is naturally an object of $\mathcal{M}_H(Y)$. Similarly, if $\mathcal{E}$ is a local system on Y with a given H-equivariant structure, then the local system $f^*\mathcal{E}$ on X has an induced H-equivariant structure.

If Y_0 is a locally closed smooth H-stable subvariety of pure dimension of Y and $\mathcal{F}$ is an H-equivariant local system on Y_0 we denote by $\mathcal{F}^\sharp$ the complex $IC(\bar{Y}_0, \mathcal{F})$ (where $\bar{Y}_0$ is the closure of Y_0 in Y) extended by 0 on $Y - \bar{Y}_0$. Then $\mathcal{F}^\sharp[\dim Y_0]$ is an object of $\mathcal{M}_H(Y)$.

If K is a simple object of $\mathcal{M}_H(Y)$, then there exists $Y_0, \mathcal{F}$ as above so that the connected components of Y_0 are permuted transitively by the H-action and $K = \mathcal{F}^\sharp[\dim Y_0]$. Note that K is not necessarily simple as an object of $\mathcal{M}(Y)$. If K, K' are objects of $\mathcal{M}_H(Y)$ we write $K \dashv_\Gamma K'$ if K is isomorphic to a subquotient of K'.

If $K \in \mathcal{M}_H(Y)$, then $\mathfrak{D}(K)$ is naturally an object of $\mathcal{M}_H(Y)$.

If H is connected, then K has at most one H-equivariant structure.

Now assume that H is finite. Let $m_g : Y \to Y, y \mapsto gy$. An H-equivariant structure on K is a collection of isomorphisms $\phi_g : K \to m_g^*K$ (one for each $g \in H$) such that for any g, g' in H, $K \xrightarrow{\phi_{gg'}} {}^* K$ is equal to the composition $K \xrightarrow{\phi_{g'}} m_{g'}^*K \xrightarrow{m_{g'}^*\phi_g} m_{gg'}^*K$. The same definition applies to H-equivariant structures on local systems on Y.

Assume now that A, A' are two simple objects of $\mathcal{M}_H(Y)$. Let $S = \text{supp}(A)$, $S' = \text{supp}(A')$. Note that the irreducible components of S (resp. S') are permuted transitively by H hence S (resp. S') has pure dimmension d (resp. d'). We show:

(a) *$H_c^0(Y, A \otimes A')^H = \bar{\mathbf{Q}}_l(-d)$ if $A' = \mathfrak{D}(A)$ in $\mathcal{M}_H(Y)$ and $H_c^0(Y, A \otimes A')^H = 0$ if $A' \not\cong \mathfrak{D}(A)$ in $\mathcal{M}_H(Y)$.*

Note that there exists an open dense smooth H-stable subset S_0 (resp. S'_0) of S (resp. S') and an H-equivariant local system $\mathcal{E}$ (resp. $\mathcal{E}'$) on S_0 (resp. S'_0) such that $A = \mathcal{E}^\sharp[d]$, $\mathfrak{D}(A) = \check{\mathcal{E}}^\sharp[d]$, $A' = \mathcal{E}'^\sharp[d']$. By an argument as in [L2, II, 7.4] we see that $H_c^0(Y, A \otimes A') = 0$ if $S \neq S'$. In the rest of the proof we assume that $S = S'$. Then we can also assume that $S_0 = S'_0$. Again by the argument in [L2, II, 7.4] we see that $H_c^0(Y, A \otimes A') = H_c^{2d}(S_0, \mathcal{E} \otimes \mathcal{E}')$. By Poincaré duality this can be identified with $H^0(S_0, \check{\mathcal{E}} \otimes \check{\mathcal{E}}')^*(-d)$. It remains to describe $H^0(S_0, \check{\mathcal{E}} \otimes \check{\mathcal{E}}')^H$. This is the vector space of homomorphisms of local systems $\mathcal{E}' \to \check{\mathcal{E}}$ which are compatible with the H-equivariant structures. This is the same as the space of morphisms from A' to $\mathfrak{D}(A)$ in $\mathcal{M}_H(Y)$ which is $\bar{\mathbf{Q}}_l$ if $A' = \mathfrak{D}(A)$ and is 0 if $A' \not\cong \mathfrak{D}(A)$.

Let E be a finite dimensional $\bar{\mathbf{Q}}_l$-vector space and let $r : H \to GL(E)$ be a homomorphism. We regard E as an H-equivariant local system over a point in an obvious way. If X is an algebraic variety with H-action, we denote by $\epsilon : X \to$ point the obvious map and we set $E_X = \epsilon^* E$; this is naturally an H-equivariant local system on X (since ϵ is compatible with the H-action on X and the trivial H-action on the point).

1.7. For any torus T' let $\mathcal{S}(T')$ be the category whose objects are the local systems of rank 1 on T' that are equivariant for the transitive T'-action $z : t \mapsto z^n t$ on T' for some $n \in \mathbf{Z}_{>0}$ invertible in $\mathbf{k}$.

1.8. Let $f : T' \to T''$ be a morphism of tori and let $\mathcal{L} \in \mathcal{S}(T'')$. We show that the following two conditions are equivalent:

(i) $f^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$;

(ii) $\mathcal{L}$ is equivariant for the T'-action $t' : t'' \mapsto f(t')t''$ on T''.

We can find $\kappa \in \mathrm{Hom}(T'', \mathbf{k}^*)$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^*)$ such that $\mathcal{L} \cong \kappa^*(\mathcal{E})$. If the result holds for $\kappa f, \mathcal{E}$ instead of $f, \mathcal{L}$, then it also holds for $f, \mathcal{L}$. Thus we may assume that $T'' = \mathbf{k}^*$. We can assume that $T' \neq \{1\}$. Let T_0 be a codimension 1 subtorus of T' contained in $\ker f$. Then f induces a homomorphism $f' : T'/T_0 \to T''$. If the result holds for $f', \mathcal{L}$ instead of $f, \mathcal{L}$, then it also holds for $f, \mathcal{L}$. Thus we may assume that $T' = T'' = \mathbf{k}^*$. In this case the result is immediate.

1.9. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$. Let $F' : T \to T$ be the Frobenius map for some $\mathbf{F}_q$-rational structure on the torus T. Let $T^{F'} = \{t \in T; F'(t) = t\}$. The following three sets coincide:

(i) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $\mathcal{L}$ is T-equivariant for the T-action $t_0 : t \mapsto t_0 t F'(t_0)^{-1}$ on T;

(ii) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $F'^*\mathcal{L} \cong \mathcal{L}$;

(iii) the set of $\mathcal{L} \in \mathcal{S}(T)$ (up to isomorphism) such that $\mathcal{L}$ is a direct summand of $L_!\bar{\mathbf{Q}}_l$ where $L : T \to T$ is $t \mapsto tF'(t^{-1})$;

moreover, they are in a natural bijection with

(iv) the set of characters $\mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)$.

For $\mathcal{L} \in \mathcal{S}(T)$ we have $L^*\mathcal{L} \cong \mathcal{L} \otimes F'^*(\check{\mathcal{L}})$. Hence $\mathcal{L}$ satisfies (ii) if and only if it satisfies $L^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$. The last condition is clearly equivalent to the condition in (iii) and by 1.8 it is also equivalent to the condition in (i).

If L is as in (iii), $T^{F'}$ acts in an obvious way on $L_!\bar{\mathbf{Q}}_l$ and we have $L_!\bar{\mathbf{Q}}_l = \oplus_{\chi \in \mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)} L_!^{\chi}\bar{\mathbf{Q}}_l$ where $L_!^{\chi}\bar{\mathbf{Q}}_l$ is the subsheaf on which $T^{F'}$ according to χ. It is clear that $\chi \mapsto L_!^{\chi}\bar{\mathbf{Q}}_l$ defines a bijection between the sets (iv) and (iii). The inverse of this bijection can be described as follows. Let $\mathcal{L}$ be as in (i). By restriction of the equivariant T-structure we obtain an equivariant $T^{F'}$-structure on $\mathcal{L}$. Since $T^{F'}$ acts trivially on T, it acts naturally on the stalk of $\mathcal{L}$ at 1; this action is via a character $\chi_{\mathcal{L}} : T'^F \to \bar{\mathbf{Q}}_l^*$. Now $\mathcal{L} \mapsto \chi_{\mathcal{L}}$ is the inverse of the bijection above.

1.10. Let $\mathcal{L} \in \mathcal{S}(T)$. Let $R_{\mathcal{L}} = \{\alpha \in R; \check{\alpha}^*\mathcal{L} \cong \bar{\mathbf{Q}}_l\}$. Then $R_{\mathcal{L}}$ is a root system and $R_{\mathcal{L}}^+ = R_{\mathcal{L}} \cap R^+$ is a set of positive roots for $R_{\mathcal{L}}$. Let $\Pi_{\mathcal{L}}$ be the unique set of simple roots of $R_{\mathcal{L}}$ such that $\Pi_{\mathcal{L}} \subset R_{\mathcal{L}}^+$. Let $W_{\mathcal{L}}$ be the subgroup of W generated by the reflections with respect to the roots in $R_{\mathcal{L}}$. Let $\mathbf{I}_{\mathcal{L}}$ be the set of reflections with respect to the roots in $\Pi_{\mathcal{L}}$. Then $(W_{\mathcal{L}}, \mathbf{I}_{\mathcal{L}})$ is a Coxeter group. Let $\check{R}_{\mathcal{L}} = \{\check{\alpha}; \alpha \in R_{\mathcal{L}}\}$.

2. The variety $\mathcal{Z}^{\mathbf{s}}$ and the local system $\overline{\mathcal{L}}$

2.1. In this and the next subsection we assume that $\mathbf{I}$ consists of a single element s. Let $a = \check{\alpha}_s : \mathbf{k}^* \to T$. To any $\mathcal{L} \in \mathcal{S}(T)$ such that

(a) $a^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$

we will associate a local system $\underline{\mathcal{L}}$ of rank 1 on G.

Case 1. a is an imbedding. We have a diagram $T \xrightarrow{c} T/a(\mathbf{k}^*) \xrightarrow{d} G/G_{der} \xleftarrow{e} G$ where c, e are the obvious maps and d is induced by the inclusion $T \subset G$; note that d is an isomorphism. Now $\mathbf{k}^*$ acts on T by $x : t \mapsto a(x)t$ and on $T/a(\mathbf{k}^*)$ trivially; c is compatible with the $\mathbf{k}^*$-actions. From (a) we see that $\mathcal{L}$ is $\mathbf{k}^*$-equivariant. Since $\mathbf{k}^*$ acts freely on T there is a well defined local system $\mathcal{L}_1$ of rank 1 on $T/a(\mathbf{k}^*)$ such that $\mathcal{L} = c^*\mathcal{L}_1$. We set $\underline{\mathcal{L}} = e^*(d^{-1})^*\mathcal{L}_1$.

Case 2. a is not an imbedding. Then the centre $\mathcal{Z}$ of G is connected, the obvious homomorphism $G_{der} \times \mathcal{Z} \to G$ is an isomorphism and we can identify $G_{der} = PGL_2(\mathbf{k})$ compatibly with the standard épinglages. Thus we can identify $G = PGL_2(\mathbf{k}) \times \mathcal{Z}$. Let $G' = GL_2(\mathbf{k}) \times \mathcal{Z}$ and let $\pi : G' \to G$ be the obvious homomorphism. Let $K = \ker \pi$, a one-dimensional torus. Let $T' = \pi^{-1}(T)$. Let $\pi_0 : T' \to T$ be the restriction of π. Let $a' : \mathbf{k}^* \to T'$ be the coroot of G' such that $\pi_0 a' = a$. Let $\mathcal{L}' = \pi_0^*\mathcal{L}$. Note that $a'^*\mathcal{L}' \cong \bar{\mathbf{Q}}_l$. Applying the construction in Case 1 to $G', T', \mathcal{L}'$ instead of $G, T, \mathcal{L}$ we obtain a local system $\underline{\mathcal{L}}'$ on G'. Now K acts on $T', G', G'/G'_{der}, T'/a'(\mathbf{k}^*)$ by translation, the analogues of c, d, e for G' are compatible with the K-action and $\mathcal{L}'$ is K-equivariant. Hence $\underline{\mathcal{L}}'$ is K-equivariant. Since K acts freely on G' there is a well defined local system $\underline{\mathcal{L}}$ of rank 1 on G such that $\pi^*\underline{\mathcal{L}} = \underline{\mathcal{L}}'$.

2.2. Define $f : G - B \to T$ by $f(y) = k(y)\dot{s}^{-1}$ and $f^1 : B \to T$ by $f^1(y) = k(y)$. We show:

(a) *we have canonically* $\underline{\mathcal{L}}|_{G-B} = f^*\mathcal{L}$ *and* $\underline{\mathcal{L}}|_B = f^{1*}\mathcal{L}$.

In the setup of 2.1, assume first that we are in Case 1. Let $j : G - B \to G$, $h : B \to G$ be the inclusions. We must show that

$$j^*e^*(d^{-1})^*\mathcal{L}_1 = f^*c^*\mathcal{L}_1, h^*e^*(d^{-1})^*\mathcal{L}_1 = f^{1*}c^*\mathcal{L}_1.$$

It is enough to show that $d^{-1}ej = cf$, $d^{-1}eh = cf^1$ or that $ej = dcf$ (resp. $eh = dcf^1$). Both maps take $ut\dot{s}u'$ (resp. ut), where $u, u' \in U, t \in T$, to the image of t in G/G_{der}. (We use that $u, u', \dot{s} \in G_{der}$.)

Next we assume that we are in Case 2. Define $\dot{s}' \in G'$ in terms of the unique épinglage of G' compatible under π with that of G in the same way that $\dot{s} \in G$ is defined in terms of the épinglage of G. Let $B' = \pi^{-1}(B)$. Define $f' : G' - B' \to T'$, $f'^1 : B' \to T'$ in terms of $G', B', T', \dot{s}'$ in the same way that f, f^1 are defined in terms of $G, B, T, \dot{s}$. Let $\pi_s : G' - B' \to G - B$, $\pi_1 : B' \to B$ be the restrictions of π. It is enough to show that

$$\pi_s^*(\underline{\mathcal{L}}|_{G-B}) = \pi_s^*f^*\mathcal{L}, \quad \pi_1^*(\underline{\mathcal{L}}|_B) = \pi_1^*f^{1*}\mathcal{L}.$$

We have

$$\pi_s^*(\underline{\mathcal{L}}|_{G-B}) = \underline{\mathcal{L}}'|_{G'-B'}, \pi_s^*f^*\mathcal{L} = f'^*\mathcal{L}', \pi_1^*(\underline{\mathcal{L}}|_B) = \underline{\mathcal{L}}'|_{B'}, \pi_1^*f^{1*}\mathcal{L} = f'^{1*}\mathcal{L}'.$$

Hence it is enough to show that $\underline{\mathcal{L}}'|_{G'-B'} = f'^*\mathcal{L}'$, $\underline{\mathcal{L}}'|_{B'} = f'^{1*}\mathcal{L}'$. But these are known from Case 1 applied to $G', \mathcal{L}'$ instead of $G, \mathcal{L}$. This proves (a).

2.3. We return to the general case. Let $s \in \mathbf{I}$. Let $P = P_s$. Let $\pi_P : P \to P/U_P$ be the obvious map. Note that P/U_P inherits an épinglage from G and that T, identified with its image under π_P is a maximal torus of P/U_P. To any $\mathcal{L} \in \mathcal{S}(T)$ such that $\check{\alpha}_s^*\mathcal{L} \cong \bar{\mathbf{Q}}_l$ we associate a local system of rank 1 on P, namely the inverse image of the local system $\underline{\mathcal{L}}$ on P/U_P (see 2.1) under π_P; this local system on P is denoted again by $\underline{\mathcal{L}}$.

Define $f_s : P - B \to T$ by $f_s(y) = k(y)\dot{s}^{-1}$ and $f_s^1 : B \to T$ by $f_s^1(y) = k(y)$. From 2.2(a) we deduce by taking inverse image under π_P:

(a) *we have canonically* $\underline{\mathcal{L}}|_{P-B} = f_s^*\mathcal{L}$ *and* $\underline{\mathcal{L}}|_B = f_s^{1*}\mathcal{L}$, (as local systems over subsets of P).

2.4. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$. Let $\mathcal{L} \in \mathcal{S}(T)$. Let

$$\mathcal{I}_{\mathbf{s}} = \{i \in [1, r]; s_1 s_2 \dots s_i \dots s_2 s_1 \in W_{\mathcal{L}}\}.$$

Let

$$\mathcal{Y} = \{(y_i) \in G^{[1,r]}; y_i \in P_{s_i}(i \in \mathcal{I}_{\mathbf{s}}), y_i \in P_{s_i} - B(i \in [1, r] - \mathcal{I}_{\mathbf{s}})\}.$$

For $i \in [1, r]$ we define $f_{s_i} : P_{s_i} - B \to T$ by $f_{s_i}(y) = k(y)\dot{s}_i^{-1}$ and $f_{s_i}^1 : B \to T$ by $f_{s_i}^1(y) = k(y)$. We have obvious projections $p_i : \mathcal{Y} \to P_{s_i}(i \in \mathcal{I}_{\mathbf{s}})$, $p_i : \mathcal{Y} \to P_{s_i} - B(i \in [1, r] - \mathcal{I}_{\mathbf{s}})$. Let

$$\underline{\underline{\mathcal{L}}} = \otimes_{i \in [1,r]} \mathcal{F}_i$$

with

$$\mathcal{F}_i = p_i^* \underline{s_{i-1}^* \dots s_2^* s_1^* \mathcal{L}} \text{ for } i \in \mathcal{I}_{\mathbf{s}}, \mathcal{F}_i = p_i^* f_{s_i}^* s_{i-1}^* \dots s_2^* s_1^* \mathcal{L} \text{ for } i \in [1, r] - \mathcal{I}_{\mathbf{s}}.$$

Here $\mathcal{F}_i, \underline{\underline{\mathcal{L}}}$ are local systems on $\mathcal{Y}$. Note that if $i \in \mathcal{I}_{\mathbf{s}}$, then the local system $\underline{s_{i-1}^* \dots s_2^* s_1^* \mathcal{L}}$ on P_{s_i} is well defined (see 2.1) since $\check{\alpha}_{s_i}^*(s_{i-1}^* \dots s_2^* s_1^* \mathcal{L}) \cong \bar{\mathbf{Q}}_l$.

For any $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$, let

$$\mathcal{Y}^{\mathcal{J}} = \{(y_i) \in G^{[1,r]}; y_i \in P_{s_i} - B(i \in [1, r] - \mathcal{J}), y_i \in B(i \in \mathcal{J})\}$$

and let $\mathbf{s}_{\mathcal{J}} = (s'_1, s'_2, \dots, s'_r)$ where $s'_i = s_i$ if $i \in [1, r] - \mathcal{J}$, $s'_i = 1$ if $i \in \mathcal{J}$. Define $f^{\mathcal{J}} : \mathcal{Y}^{\mathcal{J}} \to T$ by $(y_i) \mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet -1}$. We show:

(a) *We have canonically* $\underline{\underline{\mathcal{L}}}|_{\mathcal{Y}^{\mathcal{J}}} = (f^{\mathcal{J}})^*\mathcal{L}$.

We have obvious projections $p'_i : \mathcal{Y}^{\mathcal{J}} \to P_{s_i} - B(i \in [1, r] - \mathcal{J})$, $p'_i : \mathcal{Y}^{\mathcal{J}} \to B(i \in \mathcal{J})$. Using 2.3(a) we have canonically $\underline{\underline{\mathcal{L}}}|_{\mathcal{Y}^{\mathcal{J}}} = \otimes_{i \in [1,r]} \mathcal{F}'_i$ where

$$\mathcal{F}'_i = p_i'^* f_{s_i}^* s_{i-1}^* \dots s_2^* s_1^* \mathcal{L} \text{ for } i \in [1, r] - \mathcal{J}, \mathcal{F}'_i = p_i'^* f_{s_i}^{1*} s_{i-1}^* \dots s_2^* s_1^* \mathcal{L} \text{ for } i \in \mathcal{J}.$$

We define $\tilde{f} : \mathcal{Y}^{\mathcal{J}} \to T^{[1,r]}$ by $\tilde{f} = (\tilde{f}_i)$ where for $i \in [1, r]$, $\tilde{f}_i : \mathcal{Y}^{\mathcal{J}} \to T$ is given by

$$\tilde{f}_i = s_1 s_2 \dots s_{i-1} f_{s_i} p'_i \text{ for } i \in [1, r] - \mathcal{J}, \tilde{f}_i = s_1 s_2 \dots s_{i-1} f_{s_i}^1 p'_i \text{ for } i \in \mathcal{J}.$$

Then $\mathcal{F}'_i = \tilde{f}_i^* \mathcal{L}$ for $i \in [1, r]$ and

$$\otimes_{i \in [1,r]} \mathcal{F}'_i = \tilde{f}^*(\mathcal{L} \boxtimes \mathcal{L} \boxtimes \dots \boxtimes \mathcal{L}).$$

For $y \in \mathcal{Y}^{\mathcal{J}}$ we have $f^{\mathcal{J}}(y) = \tilde{f}_1(y)\tilde{f}_2(y)\dots\tilde{f}_r(y) = m\tilde{f}(y)$ where $m : T^{[1,r]} \to T$ is multiplication. Hence $(f^{\mathcal{J}})^*\mathcal{L} = \tilde{f}^* m^* \mathcal{L}$. It is then enough to show that $m^*\mathcal{L} = \mathcal{L} \boxtimes \mathcal{L} \boxtimes \dots \boxtimes \mathcal{L}$; this is a known property of any local system in $\mathcal{S}(T)$. This proves (a).

Let

$$Y = \{(y_i) \in G^{[1,r]}; y_i \in P_{s_i}(i \in [1,r])\}.$$

Note that $\mathcal{Y}$ is an open dense subset of Y. Hence $IC(Y, \underline{\underline{\mathcal{L}}})$ is well defined. We show:

(b) $$IC(Y, \underline{\underline{\mathcal{L}}})|_{Y-\mathcal{Y}} = 0.$$

For any $j \in [1,r] - \mathcal{I}_{\mathbf{s}}$ let

$$\Delta_j = \{\{(y_i) \in G^{[1,r]}; y_i \in P_{s_i}(i \in [1,r] - \{j\}), y_j \in B\}.$$

Clearly, $\{\Delta_j, j \in [1,r] - \mathcal{I}_{\mathbf{s}}\}$ are smooth divisors with normal crossings in the smooth variety Y. Using [L2, I, 1.6] we see that it suffices to prove the following statement.

(c) *For $j \in [1,r] - \mathcal{I}_{\mathbf{s}}$, the monodromy of $\underline{\underline{\mathcal{L}}}$ around the divisor Δ_j is non-trivial.*

We define a cross-section $\xi : \mathbf{k} \to Y$ to Δ_j in Y by

$$\xi(a) = (\dot{s}_1, \dots, \dot{s}_{j-1}, y_{s_j}(-a), \dot{s}_{j+1}, \dots, \dot{s}_r).$$

We have $\xi(0) \in \Delta_j$, $\xi(a) \in \mathcal{Y}$ for $a \in \mathbf{k}^*$. Let $\xi' : \mathbf{k}^* \to \mathcal{Y}$ be the restriction of ξ. It is enough to show that $\xi'^* \underline{\underline{\mathcal{L}}} \ncong \bar{\mathbf{Q}}_l$ or, with notation in 2.4, that

$$\xi'^* p_j^* f_{s_j}^* s_{j-1}^* \dots s_2^* s_1^* \mathcal{L} \ncong \bar{\mathbf{Q}}_l$$

or that

$$(s_1 s_2 \dots s_{j-1} f_{s_j} p_j \xi')^* \mathcal{L} \ncong \bar{\mathbf{Q}}_l, (s_1 s_2 \dots s_{j-1} \check{\alpha}_{s_j})^* \mathcal{L} \ncong \bar{\mathbf{Q}}_l.$$

(We have $f_{s_j} p_j \xi'(a) = k(y_{s_j}(-a)) \dot{s}_j^{-1} = \check{\alpha}_{s_j}(a)$.) This follows from the fact that $j \notin \mathcal{I}_{\mathbf{s}}$. This proves (c) and hence (b).

(A similar result with a similar proof appears in [L6, VI, 28.10(b)].)

In the remainder of this paper we assume that $\mathbf{k}$ *is an algebraic closure of* $\mathbf{F}_q$*, a finite field with* q *elements and that we are given a fixed* $\mathbf{F}_q$*-rational structure on* G *such that* B *and* T *are defined over* $\mathbf{F}_q$. Let $F : G \to G$ be the corresponding Frobenius map. We set

$$\Gamma = \{g \in G; F(g) = g\}.$$

Now $F : G \to G$ induces an isomorphism $F : T \to T$. For $w \in W$ we write $\mathcal{L} \in \mathcal{S}(T)^{wF}$ instead of "$\mathcal{L} \in \mathcal{S}(T)$ and $(wF)^* \mathcal{L} \cong \mathcal{L}$".

Define $F_0 : T \to T$ by $t \mapsto t^q$. For any $t \in T$ we have $F(t) = F_0(\mathbf{c}(t)) = \mathbf{c}(F_0(t))$ where $\mathbf{c} : T \to T$ is a well defined automorphism. Then $w \mapsto \mathbf{c} w \mathbf{c}^{-1}$ is an automorphism of W denoted also by $w \mapsto \mathbf{c}(w)$. This restricts to a bijection $\mathbf{I} \xrightarrow{\sim} \mathbf{I}$. Let

$$\underline{B} := \{(b_i) \in B^{[0,r]}; k(b_r^{-1} F(b_0)) = 1\}.$$

We show:

(d) *If $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$, then the local system $\underline{\underline{\mathcal{L}}}$ is equivariant for the $\underline{B}$-action*

$$(b_0, b_1, \dots, b_r) : (y_1, y_2, \dots y_r) \mapsto (b_0 y_1 b_1^{-1}, b_1 y_2 b_2^{-1}, \dots, b_{r-1} y_r b_r^{-1})$$

on $\mathcal{Y}$.

By (a), the restriction of $\underline{\underline{\mathcal{L}}}$ to the $\underline{B}$-stable open dense subset $\mathcal{Y}^\emptyset$ of $\mathcal{Y}$ is $(f^\emptyset)^* \mathcal{L}$. Since $\mathcal{Y}$ is smooth, it is enough to show that the local system $(f^\emptyset)^* \mathcal{L}$ on $\mathcal{Y}^\emptyset$ is $\underline{B}$-equivariant. Now $\underline{B}$ acts on T by $(b_0, b_1, \dots, b_r) : t \mapsto k(b_0)^{-1} t([\mathbf{s}] F(b_0))$ and $f^\emptyset : \mathcal{Y}^\emptyset \to T$ is compatible with the $\underline{B}$-actions. Hence it is enough to show that $\mathcal{L}$ is $\underline{B}$-equivariant. An equivalent statement is that $\mathcal{L}$ is T-equivariant for the T-action $t_0 : t \mapsto t_0^{-1} t([\mathbf{s}] F(t_0))$. This follows from our assumption on $\mathcal{L}$; (d) is proved.

2.5. Let $\mathbf{w} = (w_1, w_2, \dots, w_r)$ be a sequence in W. Let

$$Z^{\mathbf{w}} = \{(B_i) \in \mathcal{B}^{[0,r]}; \mathrm{pos}(B_{i-1}, B_i) = w_i (i \in [1,r]), B_r = F(B_0)\},$$
$$\dot{Z} = \{(g_iU) \in (G/U)^{[0,r]}; k(g_{i-1}^{-1}g_i) = \dot{w}_i (i \in [1,r]), g_r^{-1}F(g_0) \in U\}.$$

Let

$$\mathfrak{T} = \{(t_i) \in T^{[0,r]}; t_i = w_i^{-1}(t_{i-1})(i \in [1,r]), t_r = F(t_0)\},$$

a finite subgroup of $T^{[0,r]}$ which may be identified via $(t_i) \mapsto t_0$ with $T^{F'}$ where $F' : T \to T$ is $t \mapsto [\mathbf{w}]F(t)$. The free $\mathfrak{T}$-action $(t_i) : (g_iU) \mapsto (g_it_i^{-1}U)$ on $\dot{Z}$ makes $\dot{Z}$ into a principal $\mathfrak{T}$-bundle over $Z^{\mathbf{w}}$ via the map $f : \dot{Z} \to Z^{\mathbf{w}}$, $(g_iU) \mapsto (g_iBg_i^{-1})$. Now $f_!\bar{\mathbf{Q}}_l$ is a local system on $Z^{\mathbf{w}}$ with a free action of $\mathfrak{T} = T^{F'}$ on each stalk. We have $f_!\bar{\mathbf{Q}}_l = \oplus_{\chi \in \mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)} f_!^{\chi}\bar{\mathbf{Q}}_l$ where $f_!^{\chi}\bar{\mathbf{Q}}_l$ is the subsheaf of $f_!\bar{\mathbf{Q}}_l$ on which $T^{F'}$ acts according to χ.

Now Γ acts on $Z^{\mathbf{w}}$ by $g : (B_i) \mapsto (gB_ig^{-1})$, and on $\dot{Z}$ by $g : (g_iU) \mapsto (gg_iU)$. This last action commutes with the $\mathfrak{T}$-action. Hence $f_!\bar{\mathbf{Q}}_l$ has a natural Γ-equivariant structure and each $f_!^{\chi}\bar{\mathbf{Q}}_l$ inherits a Γ-equivariant structure from $f_!\bar{\mathbf{Q}}_l$.

We now give an alternative construction of the local systems $f_!^{\chi}\bar{\mathbf{Q}}_l$. Let

$$\underline{Z} = \{(g_iU) \in (G/U)^{[0,r]}; g_{i-1}^{-1}g_i \in B\dot{w}_iB (i \in [1,r]), g_r^{-1}F(g_0) \in U\}.$$

Define $\gamma : \underline{Z} \to Z^{\mathbf{w}}$ by $(g_iU) \mapsto (g_iBg_i^{-1})$. Define $\pi_{\mathbf{w}} : \underline{Z} \to T$ by

$$(g_iU) \mapsto k(g_0^{-1}g_1)k(g_1^{-1}g_2)\dots k(g_{r-1}^{-1}g_r)[\mathbf{w}]^{\bullet -1}.$$

The torus $\underline{T} := \{(t_i) \in T^{[0,r]}; t_r = F(t_0)\}$ acts on $\underline{Z}$ by $(t_i) : (g_iU) \mapsto (g_it_i^{-1}U)$ and on T by $(t_i) : t \mapsto t_0t([\mathbf{w}]F(t_0^{-1}))$. These actions are compatible with $\pi_{\mathbf{w}}$. Let $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$. By 1.8, $\mathcal{L}$ is equivariant for the $\underline{T}$-action on T. Hence $\pi_{\mathbf{w}}^*\mathcal{L}$ is equivariant for the (free) $\underline{T}$-action on $\underline{Z}$. Hence $\pi_{\mathbf{w}}^*\mathcal{L} = \gamma^*\mathcal{L}_{\mathbf{w}}$ for a well defined local system $\mathcal{L}_{\mathbf{w}}$ on $Z^{\mathbf{w}}$.

Now Γ acts on $\underline{Z}$ by $g : (g_iU) \mapsto (gg_iU)$ and on T trivially. Also, $\mathcal{L}$ has a natural Γ-equivariant structure in which Γ acts trivially on each stalk of $\mathcal{L}$. Since $\pi_{\mathbf{w}}$ is compatible with the Γ-actions it follows that $\pi_{\mathbf{w}}^*\mathcal{L}$ has a natural Γ-equivariant structure. Since γ is compatible with the Γ-actions it follows that $\mathcal{L}_{\mathbf{w}}$ has a natural Γ-equivariant structure.

Now assume that $\mathcal{L}$ and $\chi \in \mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)$ correspond to each other as in 1.9. Thus we assume that $\mathcal{L} = L_!^{\chi}\bar{\mathbf{Q}}_l$ where $L : T \to T$ is as in 1.9. We show that

(a) $$\mathcal{L}_{\mathbf{w}} = f_!^{\chi}\bar{\mathbf{Q}}_l.$$

Since γ is smooth with connected fibres it is enough to show that $\pi_{\mathbf{w}}^*\mathcal{L} = \gamma^* f_!^{\chi}\bar{\mathbf{Q}}_l$. Let

$$\mathfrak{P} = \{(g_iU, \tau_i) \in (G/U \times T)^{[0,r]}; k(\tau_{i-1}g_{i-1}^{-1}g_i\tau_i^{-1}) = \dot{w}_i (i \in [1,r]),$$
$$\tau_r = F(\tau_0), g_r^{-1}F(g_0) \in U\}.$$

Define $f' : \mathfrak{P} \to \underline{Z}$ by $(g_iU, \tau_i) \mapsto (g_iU)$ and $\gamma' : \mathfrak{P} \to \dot{Z}$ by $(g_iU, \tau_i) \mapsto (g_i\tau_i^{-1}U)$. Define $\pi' : \mathfrak{P} \to T$ by $(g_iU, \tau_i) \mapsto \tau_0$. Now $\mathfrak{T}$ acts on $\mathfrak{P}$ by $(t_i) : (g_iU, \tau_i) \mapsto (g_iU, t_i\tau_i)$, making f' into a principal $\mathfrak{T}$-bundle. We have a cartesian diagram of principal $\mathfrak{T} = T^{F'}$ bundles:

$$\begin{array}{ccccc} \dot{Z} & \xleftarrow{\gamma'} & \mathfrak{P} & \xrightarrow{\pi'} & T \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle L}\downarrow \\ Z^{\mathbf{w}} & \xleftarrow{\gamma} & \underline{Z} & \xrightarrow{\pi_{\mathbf{w}}} & T \end{array}$$

It follows that $\gamma^*(f_!\bar{\mathbf{Q}}_l) = f'_!\bar{\mathbf{Q}}_l = \pi^*_{\mathbf{w}}(L_!\bar{\mathbf{Q}}_l)$, and taking χ-eigenspaces: $\gamma^*(f^{\chi}_!\bar{\mathbf{Q}}_l) = f'^{\chi}_!\bar{\mathbf{Q}}_l = \pi^*_{\mathbf{w}}(L^{\chi}_!\bar{\mathbf{Q}}_l)$. Thus, $\gamma^*(f^{\chi}_!\bar{\mathbf{Q}}_l) = \pi^*_{\mathbf{w}}\mathcal{L}$, as required.

From the definitions we see that (a) is compatible with the Γ-equivariant structures.

When $\mathbf{w} = (w)$ is a one term sequence with $w \in W$ we can identify $Z^{\mathbf{w}}$ with

$$\mathcal{B}_w = \{B' \in \mathcal{B}; \mathrm{pos}(B', F(B')) = w\}$$

via $B' \leftrightarrow (B', F(B'))$. Note that $\mathcal{B}_w$ is stable under conjugation by Γ. For $\mathcal{L} \in \mathcal{S}(T)^{wF}$, the local system $\mathcal{L}_{(w)}$ on $Z^{\mathbf{w}}$ can be then identified with a local system $\mathcal{L}_w$ on $\mathcal{B}_w$. The subvarieties $\mathcal{B}_w$ of $\mathcal{B}$ and the local systems $\mathcal{L}_w$ were introduced in [DL].

2.6. In the remainder of this section we fix a sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Let $\mathcal{I}_{\mathbf{s}}$ be as in 2.4. Let

$$\bar{Z}^{\mathbf{s}} = \{(B_i) \in \mathcal{B}^{[0,r]}; \mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}(i \in [1,r]), B_r = F(B_0)\},$$

$$\begin{aligned} \mathcal{Z}^{\mathbf{s}} = \{&(B_i) \in \mathcal{B}^{[0,r]}; \mathrm{pos}(B_{i-1}, B_i) \in \{1, s_i\}(i \in \mathcal{I}_{\mathbf{s}}) \\ &\mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1,r] - \mathcal{I}_{\mathbf{s}}), B_r = F(B_0)\}. \end{aligned}$$

The variety $\bar{Z}^{\mathbf{s}}$ was introduced in [DL] (in the case where $l([\mathbf{s}]) = r$). For $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ the variety $Z^{\mathbf{s}_{\mathcal{J}}}$ (as in 2.5) can be also described as

$$\begin{aligned} Z^{\mathbf{s}_{\mathcal{J}}} &= \{(B_i) \in \mathcal{B}^{[0,r]}; B_{i-1} = B_i(i \in \mathcal{J}), \mathrm{pos}(B_{i-1}, B_i) = s_i(i \in [1,r] - \mathcal{J}), \\ &\quad B_r = F(B_0)\} \subset \mathcal{Z}^{\mathbf{s}}. \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccccccc} \bar{Z} & \xleftarrow{\delta_0} & \bar{Z}_0 & \xleftarrow{\delta_1} & \bar{Z}_1 & \xrightarrow{\delta_2} & \bar{Z}_2 \\ {\scriptstyle \epsilon}\uparrow & & {\scriptstyle \epsilon_0}\uparrow & & {\scriptstyle \epsilon_1}\uparrow & & {\scriptstyle \epsilon_2}\uparrow \\ Z & \xleftarrow{d_0} & Z_0 & \xleftarrow{d_1} & Z_1 & \xrightarrow{d_2} & Z_2 \\ {\scriptstyle e}\uparrow & & {\scriptstyle e_0}\uparrow & & {\scriptstyle e_1}\uparrow & & {\scriptstyle e_2}\uparrow \\ Z^{\mathcal{J}} & \xleftarrow{d'_0} & Z_0^{\mathcal{J}} & \xleftarrow{d'_1} & Z_1^{\mathcal{J}} & \xrightarrow{d'_2} & Z_2^{\mathcal{J}} \\ & & {\scriptstyle f_0^{\mathcal{J}}}\downarrow & & {\scriptstyle f_1^{\mathcal{J}}}\downarrow & & {\scriptstyle f_2^{\mathcal{J}}}\downarrow \\ & & T & \xleftarrow{=} & T & \xrightarrow{=} & T \end{array}$$

where the following notation is used.

$\bar{Z} = \bar{Z}^{\mathbf{s}}, Z = \mathcal{Z}^{\mathbf{s}}, Z^{\mathcal{J}} = Z^{\mathbf{s}_{\mathcal{J}}}$.

$\bar{Z}_0$ is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i}$ for $i \in [1,r]$ and $g_r^{-1}F(g_0) \in U$.

Z_0 is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i}$ for $i \in \mathcal{I}_{\mathbf{s}}$, $g_{i-1}^{-1}g_i \in P_{s_i} - B$ for $i \in [1,r] - \mathcal{I}_{\mathbf{s}}$ and $g_r^{-1}F(g_0) \in U$.

$Z_0^{\mathcal{J}}$ is the set of all $(g_0U, g_1U, \dots, g_rU) \in (G/U)^{[0,r]}$ such that $g_{i-1}^{-1}g_i \in P_{s_i} - B$ for $i \in [1,r] - \mathcal{J}$, $g_{i-1}^{-1}g_i \in B$ for $i \in \mathcal{J}$ and $g_r^{-1}F(g_0) \in U$.

d_0, δ_0 are given by $(g_0U, g_1U, \dots, g_rU) \mapsto (g_0Bg_0^{-1}, g_1Bg_1^{-1}, \dots, g_rBg_r^{-1})$.

$f_0^{\mathcal{J}}$ is given by $(g_0U, g_1U, \dots, g_rU) \mapsto k(g_0^{-1}g_1)\dots k(g_{r-1}^{-1}g_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet-1}$.

$\bar{Z}_1$ is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i}(i \in [1,r])$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

Z_1 is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i}(i \in \mathcal{I}_{\mathbf{s}})$, $y_i \in P_{s_i} - B(i \in [1,r] - \mathcal{I}_{\mathbf{s}})$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

$Z_1^{\mathcal{J}}$ is the set of all $(y_0, y_1, \dots, y_r) \in G^{[0,r]}$ such that $y_i \in P_{s_i} - B(i \in [1,r] - \mathcal{J})$, $y_i \in B(i \in \mathcal{J})$, $y_0^{-1}F(y_0) \in y_1y_2\dots y_rU$.

d_1, δ_1 are given by $(y_0, y_1, \dots, y_r) \mapsto (y_0U, y_0y_1U, \dots, y_0y_1\dots y_rU)$.

$f_1^{\mathcal{J}}$ is $(y_0, y_1, \dots, y_r) \mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet-1}$.

$\bar{Z}_2 = Y, Z_2 = \mathcal{Y}, Z_2^{\mathcal{J}} = \mathcal{Y}^{\mathcal{J}}$. (See 2.4.)

d_2, δ_2 are given by $(y_0, y_1, \dots, y_r) \mapsto (y_1, \dots, y_r)$.

$f_2^{\mathcal{J}}$ is $(y_1, \dots, y_r) \mapsto k(y_1)k(y_2)\dots k(y_r)[\mathbf{s}_{\mathcal{J}}]^{\bullet-1}$.

The maps $e, e_i, \epsilon_i (i \in [0,2])$ are the obvious imbeddings. For $i \in [0,2]$ the map d_i' is the restriction of d_i. From the definitions we have:

(a) *In our commutative diagram, all squares that do not involve T are cartesian.*

2.7. Γ acts:

on Z and $\bar{Z}$ by $g : (B_0, B_1, \dots, B_r) \mapsto (gB_0g^{-1}, gB_1g^{-1}, \dots, gB_rg^{-1})$;

on Z_0 and $\bar{Z}_0$ by $g : (g_0U, g_1U, \dots, g_rU) \mapsto (gg_0U, gg_1U, \dots, gg_rU)$;

on Z_1 and $\bar{Z}_1$ by $g : (y_0, y_1, \dots, y_r) \mapsto (gy_0, y_1, y_2, \dots, y_r)$;

on Z_2, $\bar{Z}_2$ trivially.

The subsets $Z^{\mathcal{J}}, Z_i^{\mathcal{J}}$ of $Z, Z_i (i \in [0,2])$ are stable under the Γ-action. The maps $d_i, d_i', \delta_i, f_i^{\mathcal{J}}$ are compatible with the Γ-actions.

$\underline{B}$ (see 2.4) acts:

on Z trivially;

on Z_0 by $(b_0, b_1, \dots, b_r) : (g_0U, g_1U, \dots, g_rU) \mapsto (g_0b_0^{-1}U, g_1b_1^{-1}U, \dots, g_rb_r^{-1}U)$;

on Z_1 by $(b_0, b_1, \dots, b_r) : (y_0, y_1, \dots, y_r) \mapsto (y_0b_0^{-1}, b_0y_1b_1^{-1}, b_1y_2b_2^{-1}, \dots, b_{r-1}y_rb_r^{-1})$;

on Z_2 by $(b_0, b_1, \dots, b_r) : (y_1, \dots, y_r) \mapsto (b_0y_1b_1^{-1}, b_1y_2b_2^{-1}, \dots, b_{r-1}y_rb_r^{-1})$.

The maps d_i are compatible with the $\underline{B}$-actions.

2.8. Now

(a) d_0, δ_0, d_0' are principal $\underline{B}/U^{[0,r]}$-bundles.

(b) d_1, δ_1, d_1' are principal $U^{[0,r]}$-bundles. (The action of $U^{[0,r]}$ on $\bar{Z}_1, Z_1, Z_1^{\mathcal{J}}$ is by restriction of the $\underline{B}$-action.)

(c) Each of d_2, δ_2, d_2' is a composition of a principal Γ-bundle with a principal U-bundle.

2.9. We show for $i \in [0,2]$ that

(a) *Z_i is smooth of pure dimension say $\mathfrak{d}_i$ and it is open dense in $\bar{Z}_i$.*

Let P_i be the property expressed by (a). It is obvious that P_2 holds. Using P_2, 2.6(a) and 2.8(c) we see that P_1 holds. Using P_1, 2.6(a) and 2.8(b) we see that P_0 holds. Thus (a) holds. Using P_0, 2.6(a) and 2.8(a) we see that

(b) *Z is smooth of pure dimension say $\mathfrak{d}$ and it is open dense in $\bar{Z}$.*

We show:

(c) $\mathfrak{d} = r$.

From the definitions we see that $\mathfrak{d}_2 = r(\dim B + 1)$. From the arguments above we see successively that $\mathfrak{d}_1 = r(\dim B + 1) + \dim U$, $\mathfrak{d}_0 = \mathfrak{d}_1 - (r+1)\dim U$ $\mathfrak{d} = \mathfrak{d}_0 - r\dim T$; (c) follows.

We show:

(d) *The natural* Γ*-action on the set of connected components of* Z_i *(*$i \in [0,2]$*) or of* Z *is transitive.*

For Z_2 this is clear since Z_2 is connected. This also implies the result for Z_1 (see 2.8(c)). Using 2.8(b),(a) we deduce that the result also holds for Z_0 and for Z.

2.10. Let $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$. For $i \in [0,2]$ we set $\mathcal{L}_i^{\mathcal{J}} = (f_i^{\mathcal{J}})^*\mathcal{L}$, a local system of rank 1 on $Z_i^{\mathcal{J}}$. Since $\mathcal{L}$ has a natural Γ-equivariant structure (with Γ acting trivially on each stalk) and $f_i^{\mathcal{J}}$ is compatible with the Γ-actions we see that $\mathcal{L}_i^{\mathcal{J}}$ has a natural Γ-equivariant structure. From the definitions we have isomorphisms compatible with the Γ-equivariant structures as follows:

(a) $$d_1'^{\,*}\mathcal{L}_1^{\mathcal{J}} \cong \mathcal{L}_0{}^{\mathcal{J}}; d_2'^{\,*}\mathcal{L}_2^{\mathcal{J}} \cong \mathcal{L}_1^{\mathcal{J}}.$$

From the definitions we see that

(b) Γ *acts trivially on any stalk of* $\mathcal{L}_2^{\mathcal{J}}$.

Let $\mathcal{L}^{\mathcal{J}}$ be the local system on $Z^{\mathcal{J}} = Z^{\mathbf{s}_{\mathcal{J}}}$ denoted in 2.5 by $\mathcal{L}_{\mathbf{w}}$ where $\mathbf{w} = \mathbf{s}_{\mathcal{J}}$. This is well defined since for $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ we have $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}_{\mathcal{J}}]F}$. (We use that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ and $(s_1 s_2 \dots s_j \dots s_2 s_1)^*\mathcal{L} \cong \mathcal{L}$ for any $j \in \mathcal{J}$.) As in 2.5, $\mathcal{L}^{\mathcal{J}}$ has a natural Γ-equivariant structure. From the definitions we have $d_0'^{\,*}\mathcal{L}^{\mathcal{J}} \cong \mathcal{L}_0^{\mathcal{J}}$ compatibly with the Γ-equivariant structures.

2.11. For $i \in [1,2]$ we define a local system $\bar{\mathcal{L}}_i$ on Z_i by $\bar{\mathcal{L}}_2 = \underline{\underline{\mathcal{L}}}$, $\bar{\mathcal{L}}_1 = d_2^*\bar{\mathcal{L}}_2$ where $\underline{\underline{\mathcal{L}}}$ is as in 2.4. From 2.4(b) and the results in 2.7 we see that $\bar{\mathcal{L}}_i$ is $\underline{B}$-equivariant. Since $d_0 d_1 : Z_1 \to Z$ is a principal $\underline{B}$-bundle, we see that there is a well defined local system $\bar{\mathcal{L}}$ on Z such that $(d_0 d_1)^*\bar{\mathcal{L}} = \bar{\mathcal{L}}_1$. Let $\bar{\mathcal{L}}_0 = d_0^*\bar{\mathcal{L}}$. Then $\bar{\mathcal{L}}_1 = d_1^*\bar{\mathcal{L}}_0$. We regard $\bar{\mathcal{L}}_2$ as a Γ-equivariant local system on Z_2 with Γ acting trivially on each stalk. Since each d_i is compatible with the Γ-actions we see that $\bar{\mathcal{L}}_i$ ($i \in [0,2]$) and $\bar{\mathcal{L}}$ have natural Γ-equivariant structures which are compatible with d_i^*.

2.12. We show:

(a) *For any* $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ *we have* $\bar{\mathcal{L}}|_{Z^{\mathcal{J}}} \cong \mathcal{L}^{\mathcal{J}}$ *compatibly with the* Γ*-equivariant structures.*

(b) *For any* $i \in [0,2]$ *and* $\mathcal{J} \subset \mathcal{I}_{\mathbf{s}}$ *we have* $\bar{\mathcal{L}}_i|_{Z_i^{\mathcal{J}}} \cong \mathcal{L}_i^{\mathcal{J}}$ *compatibly with the* Γ*-equivariant structures.*

Note that (b) holds for $i = 2$ by 2.4(a) (the compatibility with the Γ-equivariant structures is automatic since Γ acts trivially on each stalk of the local systems involved). From this we get (using 2.10, 2.11) that (b) holds for $i = 1$, then for $i = 0$, and then that (a) holds.

2.13. We show:

(a) *We have* $IC(\bar{Z}, \bar{\mathcal{L}})|_{\bar{Z}-Z} = 0$.

(b) *For* $i \in [0,2]$ *we have* $IC(\bar{Z}_i, \bar{\mathcal{L}}_i)|_{\bar{Z}_i - Z_i} = 0$.

Note that the IC complexes in (a),(b) are well defined by 2.9(a),(b). Now (b) holds for $i = 2$ by 2.4(b). From this we get (using 2.8, 2.11) that (b) holds for $i = 1$, then for $i = 0$, and then that (a) holds.

2.14. Assume that $r \geq 2, h \in [2, r] \cap \mathcal{I}_{\mathbf{s}}, s_{h-1} = s_h$. We set

$$\mathbf{s}' := (s_1, \dots, s_{h-1}, s_{h+1}, \dots, s_r).$$

Then $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}']F}$ so that $Z' := \mathcal{Z}^{\mathbf{s}'}$ is defined as in 2.6. We have a commutative diagram

$$\begin{array}{ccccccc}
Z & \xleftarrow{d_0} & Z_0 & \xleftarrow{d_1} & Z_1 & \xrightarrow{d_2} & Z_2 \\
\beta\downarrow & & \beta_0\downarrow & & \beta_1\downarrow & & \beta_2\downarrow \\
Z' & \xleftarrow{d'_0} & Z'_0 & \xleftarrow{d'_1} & Z'_1 & \xrightarrow{d'_2} & Z'_2
\end{array}$$

where:

the upper row is as in 2.6,

the lower row is defined analogously in terms of $\mathbf{s}', \mathcal{L}$ instead of $\mathbf{s}, \mathcal{L}$,

β is $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r)$,

β_0 is $(g_0U, g_1U, \dots, g_rU) \mapsto (g_0U, g_1U, \dots, g_{h-2}U, g_hU, g_rU)$,

β_1 is $(y_0, y_1, \dots, y_r) \mapsto (y_0, y_1, \dots, y_{h-2}, y_{h-1}y_h, y_{h+1}, \dots, y_r)$,

β_2 is $(y_1, \dots, y_r) \mapsto (y_1, \dots, y_{h-2}, y_{h-1}y_h, y_{h+1}, \dots, y_r)$.

Let $\bar{\mathcal{L}}, \bar{\mathcal{L}}_i (i \in [0, 2])$ be the local systems on Z, Z_i defined in 2.11; let $\bar{\mathcal{L}}', \bar{\mathcal{L}}'_i$ be the analogous local systems on Z', Z'_i. We show:

(a) $$\bar{\mathcal{L}} \cong \beta^*(\bar{\mathcal{L}}').$$

It is enough to show that $\delta_1^*\delta_0^*\bar{\mathcal{L}} \cong \delta_1^*\delta_0^*\beta^*(\bar{\mathcal{L}}')$ or equivalently that $\bar{\mathcal{L}}_1 \cong \beta_1^*\bar{\mathcal{L}}'_1$. Hence it is enough to show that $\delta_2^*\bar{\mathcal{L}}_2 \cong \beta_1^*\delta_2'^*\bar{\mathcal{L}}'_2$ or equivalently that $\delta_2^*\bar{\mathcal{L}}_2 \cong \delta_2^*\beta_2{}^*\bar{\mathcal{L}}'_2$. It is enough to show that $\bar{\mathcal{L}}_2 \cong \beta_2{}^*\bar{\mathcal{L}}'_2$. From the definition of $\underline{\underline{\mathcal{L}}}$ in 2.4 and with the notation in 2.4 we see that it is enough to show that $m^*{}'\underline{\underline{\mathcal{L}}} \cong {}'\underline{\mathcal{L}} \boxtimes {}'\underline{\mathcal{L}}$ (local systems on P_{s_h}) where ${}'\mathcal{L} = s_{h-1}^* \dots s_2^*s_1^*\mathcal{L} \cong s_h^*s_{h-1}^* \dots s_2^*s_1^*\mathcal{L}$ and $m : P_{s_h} \times P_{s_h} \to P_{s_h}$ is multiplication. It is enough to show that, in the setup of 2.1 we have $m'^*\underline{\mathcal{L}} \cong \underline{\mathcal{L}} \boxtimes \underline{\mathcal{L}}$ (local systems on G) where $m' : G \times G \to G$ is multiplication. This follows from the definitions in 2.1 using the isomorphism $m_1^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ (local systems on T) where $m_1 : T \times T \to T$ is multiplication.

2.15. Assume that $r \geq 2, h \in [2, r], h \notin \mathcal{I}_{\mathbf{s}}, s_{h-1} = s_h$. Let Z^1 be the open subset of Z defined by the condition $\mathrm{pos}(B_{h-2}, B_h) = s_i$. Define $\beta : Z^1 \to \mathcal{B}^r$ by $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r)$. We show:

(a) $$\beta_!(\bar{\mathcal{L}}|_{Z^1}) = 0.$$

Let $p = (B_0, B_1, \dots, B_{h-2}, B_h, \dots, B_r) \in \mathcal{B}^r$ be such that $\Phi := \beta^{-1}(p) \neq \emptyset$. Then

$$\Phi = \{(B_0, B_1, \dots, B_{h-2}, \tilde{B}, B_h, \dots, B_r); \tilde{B} \in \mathcal{B}, \mathrm{pos}(B_{h-2}, \tilde{B}) = \mathrm{pos}(\tilde{B}, B_h) = s_h\}.$$

It is enough to show that $H_c^*(\Phi, \bar{\mathcal{L}}) = 0$. Let

$$\begin{aligned}\Phi' =& \{(B_0, B_1, \dots, B_{h-2}, \tilde{B}, B_h, \dots, B_r); \tilde{B} \in \mathcal{B}, \mathrm{pos}(B_{h-2}, \tilde{B}) = s_h, \\ & \mathrm{pos}(\tilde{B}, B_h) \in \{1, s_h\}\}.\end{aligned}$$

Then Φ' is an affine line which is a cross section in $\bar{Z}$ to the divisor Δ_h (see 2.4) and $\Phi' \cap \Delta_h$ is the point $p' = (B_0, B_1, \dots, B_{h-2}, B_h, B_h, \dots, B_r)$. Moreover, $\Phi = \Phi' - \{p'\}$. The vanishing $H_c^*(\Phi, \bar{\mathcal{L}}) = 0$ follows from 2.4(c).

Now let $Z^2 = Z - Z^1$, that is the closed subset of Z defined by the condition $B_{h-2} = B_h$. Let $\mathbf{s}' := (s_1, \dots, s_{h-2}, s_{h+1}, \dots, s_r)$. We have $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}']F}$ so that $Z' := \mathcal{Z}^{\mathbf{s}'}$ is defined as in 2.6. Define $\beta' : Z^2 \to Z'$ by $(B_0, B_1, \dots, B_r) \mapsto$

$(B_0, B_1, \dots, B_{h-2}, B_{h+1}, \dots, B_r)$, an affine line bundle. Let $\bar{\mathcal{L}}$ be the local systems on Z defined in 2.11; let $\bar{\mathcal{L}}'$ be the analogous local system on Z'. From the definitions we have

(b) $$\bar{\mathcal{L}}|_{Z^2} = \beta'^*(\bar{\mathcal{L}}').$$

3. The class $\mathbb{S}'(\mathcal{P}_J)$ of simple objects in $\mathcal{M}_\Gamma(\mathcal{P}_J)$

3.1. Let $J \subset \mathbf{I}$. We view $\mathcal{P}_J$ as a variety with Γ-action (conjugation). Hence $\mathcal{M}_\Gamma(\mathcal{P}_J)$ is well defined.

3.2. Let $\mathcal{L} \in \mathcal{S}(T)$. If $\mathbf{w}$ is as in 2.5 and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$, then the local system $\mathcal{L}_{\mathbf{w}}$ on $Z^{\mathbf{w}}$ has a natural Γ-equivariant structure (see 2.5). The map $\Pi^{\mathbf{w}} : Z^{\mathbf{w}} \to \mathcal{P}_J$, $(B_0, B_1, \dots, B_r) \mapsto P_{B_0,J}$, commutes with the Γ-actions. Hence for any $j \in \mathbf{Z}$, ${}^pH^j(\Pi^{\mathbf{w}}_!\mathcal{L}_{\mathbf{w}})$ is an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

If $\mathbf{s}$ is as in 2.6 and $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$, then the local system $\bar{\mathcal{L}}$ on $\mathcal{Z}^{\mathbf{s}}$ has a natural Γ-equivariant structure (see 2.6, 2.11). Hence $\bar{\mathcal{L}}^\sharp = IC(\bar{Z}^{\mathbf{s}}, \bar{\mathcal{L}})$ (see 2.6, 2.11, 2.13) has a natural Γ-equivariant structure. Define $\Upsilon^{\mathbf{s}} : \mathcal{Z}^{\mathbf{s}} \to \mathcal{P}_J$ and $\bar{\Upsilon}^{\mathbf{s}} : \bar{Z}^{\mathbf{s}} \to \mathcal{P}_J$ by $(B_0, B_1, \dots, B_r) \mapsto P_{B_0,J}$. These maps commute with the Γ-actions. Hence for any $j \in \mathbf{Z}$,

(a) $${}^pH^j(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}) = {}^pH^j(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$$

is an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. (The equality in (a) follows from by 2.13.)

In 3.3–3.7 we will show that the following conditions for a simple object K in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ are equivalent:

(i) $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{w}}_!\mathcal{L}_{\mathbf{w}})$ for some one term sequence $\mathbf{w}$ in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(ii) $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{w}}_!\mathcal{L}_{\mathbf{w}})$ for some sequence $\mathbf{w}$ in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(iii) $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{w}}_!\mathcal{L}_{\mathbf{w}})$ for some sequence $\mathbf{w}$ in $\mathbf{I} \cup \{1\}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$.

(iv) $K \dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}}_!\mathcal{L}_{\mathbf{s}})$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

(v) $K \dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

(vi) $K \dashv_\Gamma {}^pH^\cdot(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$.

3.3. Let $\mathbf{w} = (w_1, \dots, w_r)$ be a sequence in W such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{w}]F}$. Assume that for some $i \in [1, r]$, $w'_i, w''_i \in W$ satisfy $w_i = w'_i w''_i$ and $l(w_i) = l(w'_i) + l(w''_i)$. Let $\mathbf{w}' = (w_1, \dots, w_{i-1}, w'_i, w''_i, w_{i+1}, \dots, w_r)$. Define an isomorphism $Z^{\mathbf{w}'} \xrightarrow{\sim} Z^{\mathbf{w}}$ by

$$(B_0, B_1, \dots, B_{r+1}) \mapsto (B_0, B_1, \dots, B_{i-1}, B_{i+1}, \dots, B_{r+1}).$$

This isomorphism is compatible with the Γ-actions, with the maps $\Pi^{\mathbf{w}'}, \Pi^{\mathbf{w}}$ and with the local systems $\mathcal{L}_{\mathbf{w}'}, \mathcal{L}_{\mathbf{w}}$. Hence for any j we have

(a) $${}^pH^j(\Pi^{\mathbf{w}}_!\mathcal{L}_{\mathbf{w}}) = {}^pH^j(\Pi^{\mathbf{w}'}_!\mathcal{L}_{\mathbf{w}'})$$

(as objects of $\mathcal{M}_\Gamma(\mathcal{P}_J)$). Applying (a) repeatedly we see that conditions 3.2(ii), 3.2(iii), 3.2(iv) are equivalent.

3.4. We prove the equivalence of conditions 3.2(iii), 3.2(v).

Let $\mathbf{s}=(s_1,\dots,s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L}\in\mathcal{S}(T)^{[\mathbf{s}]F}$. Define a sequence ${}^0\mathcal{Z}\supset{}^1\mathcal{Z}\supset\dots$ of closed subsets of $\mathcal{Z}^{\mathbf{s}}$ by ${}^i\mathcal{Z}=\cup_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|\ge i}Z^{\mathbf{s}_{\mathcal{J}}}$ (notation of 2.6). Let $f^i:{}^i\mathcal{Z}\to\mathcal{Z}^{\mathbf{s}}$, $f'^i:{}^i\mathcal{Z}-{}^{i+1}\mathcal{Z}\to\mathcal{Z}^{\mathbf{s}}$ be the inclusions. The natural distinguished triangle

$$(\Upsilon^{\mathbf{s}}_! f'^i_! f'^{i*}\bar{\mathcal{L}},\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}},\Upsilon^{\mathbf{s}}_! f^{i+1}_!(f^{i+1*}\bar{\mathcal{L}}))$$

gives rise for any $i\ge 0$ to a long exact sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$:

$$\dots\to{}^pH^{j-1}(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})\to\oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}{}^pH^j(\Pi^{\mathbf{s}_{\mathcal{J}}}_!\mathcal{L}_{\mathbf{s}_{\mathcal{J}}})\to{}^pH^j(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$$

(a)

$$\to{}^pH^j(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})\to\oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}{}^pH^{j+1}(\Pi^{\mathbf{s}_{\mathcal{J}}}_!\mathcal{L}_{\mathbf{s}_{\mathcal{J}}})\to\dots.$$

Here we have used the equality

$$\Upsilon^{\mathbf{s}}_! f'^i_! f'^{i*}\bar{\mathcal{L}}=\oplus_{\mathcal{J}\subset\mathcal{I}_{\mathbf{s}};|\mathcal{J}|=i}\Pi^{\mathbf{s}_{\mathcal{J}}}_!\mathcal{L}_{\mathbf{s}_{\mathcal{J}}}.$$

(See 2.12(a).) Note that $\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}}=\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}$ and $\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}}=0$ for i large.

If K does not satisfy 3.2(iii), then from (b) we see that for any $i\ge 0$ we have $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ if and only if $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})$. Since $K\not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ with large i, it follows that $K\not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}})$; that is, $K\not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Thus, K does not satisfy 3.2(v).

Assume now that K satisfies 3.2(iii). We may assume that $K\dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}}_!\mathcal{L}_{\mathbf{s}})$ where $\mathbf{s}$ as in 3.2(iii) has a minimum possible number of terms in $\mathbf{I}$. By the equivalence of 3.2(ii), 3.2(iii) we see that we may assume that all terms of $\mathbf{s}$ are in $\mathbf{I}$ and that $K\not\dashv_\Gamma {}^pH^\cdot(\Pi^{\mathbf{s}_{\mathcal{J}}}_!\mathcal{L}_{\mathbf{s}_{\mathcal{J}}})$ for and any $\mathcal{J}$ such that $|\mathcal{J}|>0$. Then from (b) we see that for any $i>0$, $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ if and only if $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^{i+1}_! f^{i+1*}\bar{\mathcal{L}})$. Since $K\not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^i_! f^{i*}\bar{\mathcal{L}})$ with large i it follows that $K\not\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^1_! f^{1*}\bar{\mathcal{L}})$. Using again (b) (with $i=0$) we see that $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_! f^0_! f^{0*}\bar{\mathcal{L}})$ hence $K\dashv_\Gamma {}^pH^\cdot(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Thus, K satisfies 3.2(v). The equivalence of 3.2(iii), 3.2(v) is proved.

3.5. Let $\mathbf{s}=(s_1,s_2,\dots,s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L}\in\mathcal{S}(T)^{[\mathbf{s}]F}$. Assume that $r\ge 2,h\in[2,r]\cap\mathcal{I}_{\mathbf{s}}$, $s_{h-1}=s_h$. Let $\mathbf{s}',\bar{\mathcal{L}}',\beta$ be as in 2.14. We have $\Upsilon^{\mathbf{s}}=\Upsilon^{\mathbf{s}'}\beta$ and using 2.14(a) we have $\Upsilon^{\mathbf{s}}_!(\bar{\mathcal{L}})=\Upsilon^{\mathbf{s}'}_!\beta_!\beta^*\bar{\mathcal{L}}'$. Since β is a projective line bundle we have an exact sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$:

(a) $$\dots\to{}^pH^{j-2}(\Upsilon^{\mathbf{s}'}_!\bar{\mathcal{L}}')(-1)\to{}^pH^j(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})\to{}^pH^j(\Upsilon^{\mathbf{s}'}_!\bar{\mathcal{L}}')\to\dots$$

3.6. Let $\mathbf{s}=(s_1,s_2,\dots,s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L}\in\mathcal{S}(T)^{[\mathbf{s}]F}$. Assume that $r\ge 2,h\in[2,r],h\notin\mathcal{I}_{\mathbf{s}}$, $s_{h-1}=s_h$. Let $Z^1,Z^2,\beta,\beta',\mathbf{s}',\bar{\mathcal{L}}'$ be as in 2.15; let $f_1:Z^1\to\mathcal{Z}^{\mathbf{s}},f_2:Z^2\to\mathcal{Z}^{\mathbf{s}}$ be the inclusions. We have a distinguished triangle

$$(\Upsilon^{\mathbf{s}}_! f_{1!}f_1^*\bar{\mathcal{L}},\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}},\Upsilon^{\mathbf{s}}_! f_{2!}f_2^*\bar{\mathcal{L}}).$$

We have $\Upsilon^{\mathbf{s}}_! f_{1!}=e_!\beta_!$ where $e:\mathcal{B}^r\to\mathcal{P}_J$ is $(B_0,B_1,\dots,B_{h-2},B_h,\dots,B_r)\mapsto P_{B_0,J}$. Using 2.15(a) we have $\Upsilon^{\mathbf{s}}_! f_{1!}f_1^*\bar{\mathcal{L}}=e_!\beta_!(\bar{\mathcal{L}}|_{Z^1})=0$. Hence the distinguished triangle above yields $\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}=\Upsilon^{\mathbf{s}}_! f_{2!}f_2^*\bar{\mathcal{L}}$. We have $\Upsilon^{\mathbf{s}}f_2=\Upsilon^{\mathbf{s}'}\beta'$. Using 2.15(b) we have

$$\Upsilon^{\mathbf{s}}_! f_{2!}f_2^*\bar{\mathcal{L}}=\Upsilon^{\mathbf{s}'}_!\beta'_!(\bar{\mathcal{L}}|_{Z^2})=\Upsilon^{\mathbf{s}'}_!\beta'_!\beta'^*\bar{\mathcal{L}}'=\Upsilon^{\mathbf{s}'}_!(\bar{\mathcal{L}}'\otimes\beta'_!\beta'^*\bar{\mathbf{Q}}_l).$$

We see that

(a) $$\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}=\Upsilon^{\mathbf{s}'}_!\bar{\mathcal{L}}'[-2](-1).$$

Hence for any j we have

(b) $$ {}^pH^j(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}) = {}^pH^{j-2}(\Upsilon_!^{\mathbf{s}'}\bar{\mathcal{L}}')(-1)$$

in $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

3.7. Assume that 3.2(iv) holds. We show that 3.2(i) holds.

We may assume that $K \dashv_\Gamma {}^pH^{\cdot}(\Pi_!^{\mathbf{s}}\mathcal{L}_{\mathbf{s}})$ for some sequence $\mathbf{s} = (s_1, s_2, \dots, s_r)$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ and that r is minimum possible. From the proof in 3.4 we see that $K \dashv_\Gamma {}^pH^{\cdot}(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}})$ and that r is also minimal for this property.

Assume first that $l(s_1s_2\dots s_r) < r$. We can find $h \in [2, r]$ such that

$$l(s_hs_{h+1}\dots s_r) = r - h + 1, l(s_{h-1}s_h\dots s_r) < r - h + 2.$$

We can find $s'_h, s'_{h+1}, \dots, s'_r$ in $\mathbf{I}$ such that

$$s'_hs'_{h+1}\dots s'_r = s_hs_{h+1}\dots s_r = y$$

and $s'_h = s_{h-1}$. Let

$$\mathbf{u}' = (s_1, s_2, \dots, s_{h-1}, s'_h, s'_{h+1}, \dots, s'_r), \mathbf{u}'' = (s_1, s_2, \dots, s_{h-1}, y).$$

From 3.3(a) we see that $\Pi_!^{\mathbf{s}}\mathcal{L}_{\mathbf{s}} = \Pi_!^{\mathbf{u}'}\mathcal{L}_{\mathbf{u}'} = \Pi_!^{\mathbf{u}''}\mathcal{L}_{\mathbf{u}''}$. Hence we may assume that $s_h = s_{h-1}$.

If $h \in \mathcal{I}_{\mathbf{s}}$, then using 3.5(a) we see that $K \dashv_\Gamma {}^pH^{\cdot}(\Upsilon_!^{\mathbf{s}'}\bar{\mathcal{L}}')$ (notation of 3.5); since $\mathbf{s}'$ has $r-1$ terms this is a contradiction. If $h \notin \mathcal{I}_{\mathbf{s}}$, then using 3.6 we see that $K \dashv_\Gamma {}^pH^{\cdot}(\Upsilon_!^{\mathbf{s}'}\bar{\mathcal{L}}')$ (notation of 3.6); since $\mathbf{s}'$ has $r-2$ terms this is a contradiction.

We see that $l(s_1s_2\dots s_r) = r$. Using 3.3(a) repeatedly we see that ${}^pH^j(\Pi_!^{\mathbf{s}}\mathcal{L}_{\mathbf{s}}) = {}^pH^j(\Pi_!^{\mathbf{w}}\mathcal{L}_{\mathbf{w}})$ where $\mathbf{w} = (w_1)$, $w_1 = s_1s_2\dots s_r$. Thus, 3.2(i) holds.

Since the implication 3.2(i) $\Longrightarrow$ 3.2(ii) is obvious and the equivalence of 3.2(v), 3.2(vi) follows from 3.2(a) we see that the equivalence of 3.2(i)–3.2(vi) is established.

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ we write $A \in \mathbb{S}'(\mathcal{P}_J)$ instead of "A satisfies the equivalent conditions of 3.2(i)–3.2(vi) for some $\mathcal{L} \in \mathcal{S}(T)$".

3.8. The results in this and the next subsection are not used in the subsequent sections.

Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$, $s_1 \in J$. Let $\mathbf{s}' = (s'_1, s'_2, \dots, s'_r)$ where $s'_i = s_{i+1}$ for $i \in [1, r-1]$ and $s'_r = \mathbf{c}(s_1)$ where $\mathbf{c} : W \to W$ is as in 2.4. Let $\mathcal{L}' = s_1^*\mathcal{L}$. We have $\mathcal{L}' \in \mathcal{S}(T)^{[\mathbf{s}']F}$. Let $\bar{\mathcal{L}}$ be the local system on $\mathcal{Z}^{\mathbf{s}}$ defined in 2.11 and let $\bar{\mathcal{L}}'$ be the analogous local system on $\mathcal{Z}^{\mathbf{s}}$ defined in terms of $\mathcal{L}'$. We show:

(a) $$\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}} \cong \Upsilon_!^{\mathbf{s}'}\bar{\mathcal{L}}'.$$

Define $\mathcal{I}'_{\mathbf{s}'}$ in terms of $\mathbf{s}', \mathcal{L}'$ in the same way as $\mathcal{I}_{\mathbf{s}}$ was defined in 2.4 in terms of $\mathbf{s}, \mathcal{L}$. If $i \in [2, r]$ we have $i \in \mathcal{I}_{\mathbf{s}}$ if and only if $i - 1 \in \mathcal{I}'_{\mathbf{s}'}$. Moreover, we have $1 \in \mathcal{I}_{\mathbf{s}}$ if and only if $r \in \mathcal{I}_{\mathbf{s}'}$. It follows that $f : \mathcal{Z}^{\mathbf{s}} \to \mathcal{Z}^{\mathbf{s}'}$, $(B_0, B_1, \dots, B_r) \mapsto (B_1, B_2, \dots, B_r, F(B_1))$, is well defined. From the definitions we see that $f^*\bar{\mathcal{L}}' \cong \bar{\mathcal{L}}$. Hence $\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}} = \Upsilon_!^{\mathbf{s}}f^*\bar{\mathcal{L}}'$. It remains to show that $\Upsilon^{\mathbf{s}}f = \Upsilon^{\mathbf{s}'}$. The first (resp. second) map takes $(B_0, B_1, \dots, B_r)$ to $P_{B_1,J}$ (resp. $P_{B_0,J}$). It is enough to show that $P_{B_1,J} = P_{B_0,J}$. This follows from the fact that $\mathrm{pos}(B_0, B_1) \in J$.

3.9. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$ be a sequence in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Let $s \in \mathbf{I}$ be such that $s \notin W_{\mathcal{L}}$. Let $\mathbf{u} = (s, s_1, s_2, \dots, s_r, \mathbf{c}(s))$, $\mathcal{L}' = s^*\mathcal{L}$. Let $\mathbf{v} = (s, s, s_1, s_2, \dots, s_r)$. We have $\mathcal{L}' \in \mathcal{S}(T)^{[\mathbf{u}]F}$, $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{v}]F}$. Let $\bar{\mathcal{L}}$ be as in 2.11; let $\bar{\mathcal{L}}', \bar{\mathcal{L}}''$ be the analogous local systems on $\mathcal{Z}^{\mathbf{u}}$, $\mathcal{Z}^{\mathbf{v}}$ defined in terms of $\mathcal{L}', \mathcal{L}$. We show that

(a) $$\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}[-2](-1) = \Upsilon_!^{\mathbf{u}}\bar{\mathcal{L}}'.$$

From 3.8 we have $\Upsilon_!^{\mathbf{u}}\bar{\mathcal{L}}' = \Upsilon_!^{\mathbf{v}}\bar{\mathcal{L}}''$. From 3.6(a) we have $\Upsilon_!^{\mathbf{v}}\bar{\mathcal{L}}'' = \Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}}[-2](-1)$ and (a) follows. We see that

(b) $${}^pH^j(\Upsilon_!^{\mathbf{u}}\bar{\mathcal{L}}') = {}^pH^{j-2}(\Upsilon_!^{\mathbf{s}}\bar{\mathcal{L}})(-1).$$

4. The class $\mathbb{S}(\mathcal{P}_J)$ of simple objects in $\mathcal{M}_\Gamma(\mathcal{P}_J)$

4.1. In this section we fix $J \subset \mathbf{I}$.

In 1977 the author generalized the partition $(\mathcal{B}_w)_{w\in W}$ of $\mathcal{B}$ (see 2.5) by defining a partition of $\mathcal{P}_J$ into finitely many pieces stable under conjugation by Γ, as follows. To any $P \in \mathcal{P}_J$ we associate a sequence $P^0 \supset P^1 \supset P^2 \supset \dots$ in $\mathcal{P}$ by

$$P^0 = P, \quad P^n = (P^{n-1})^{F(P^{n-1})} \text{ for } n \geq 1,$$

a sequence $J_0 \supset J_1 \supset J_2 \supset \dots$ of subsets of $\mathbf{I}$ by $P^n \in \mathcal{P}_{J_n}$ and a sequence $w_0, w_1, w_2, \dots$ in W by

$$w_n = \mathrm{pos}(P^n, F(P^n)).$$

We have

(a) $J_0 = J$,

(b) $J_n = J_{n-1} \cap w_{n-1}\mathbf{c}(J_{n-1})w_{n-1}^{-1}$ for $n \geq 1$. (see 1.3(a)),

(c) $w_n \in {}^{J_n}W^{\mathbf{c}(J_n)}$ for $n \geq 0$.

Clearly, for $n \geq |\mathbf{I}|$ we have $P^n = P^{n+1} = \dots$ hence

(d) $w_n = w_{n+1} = \dots$ and $J_n = J_{n+1} = \dots$.

We set $P^\infty = P^n$ for $n \geq |\mathbf{I}|$, $w_\infty = w_n$ for $n \geq |\mathbf{I}|$, $J_\infty = J_n$ for $n \geq |\mathbf{I}|$.

For any $\mathbf{t} = (J_n, w_n)_{n\geq 0}$ where $J_0 \supset J_1 \supset J_2 \supset \dots$ are subsets of J satisfying (a) and $w_0, w_1, w_2, \dots$ are elements of W satisfying (b),(c) let $\mathcal{P}_J^{\mathbf{t}}$ be the set of all $P \in \mathcal{P}_J$ which give rise to $\mathbf{t}$ by the procedure above. Let $\mathcal{T}'(J, \mathbf{c})$ be the set of all sequences $\mathbf{t}$ as above such that $\mathcal{P}_J^{\mathbf{t}} \neq \emptyset$. From (d) we see that $\mathcal{T}'(J, \mathbf{c})$ is a finite set. From (a),(b) we see that for $(J_n, w_n)_{n\geq 0} \in \mathcal{T}'(J, \mathbf{c})$, the J_n are uniquely determined by the w_n. The locally closed subvarieties $\mathcal{P}_J^{\mathbf{t}}$, $\mathbf{t} \in \mathcal{T}'(J, \mathbf{c})$, form the desired partition of $\mathcal{P}_J$. (See [L5, I, 1.3, 1.4] for some examples in the classical groups.)

4.2. In this subsection we review some results in R. Bédard's Ph.D. Thesis (M.I.T. 1983); see also [BE].

(a) *$\mathcal{T}'(J, \mathbf{c})$ is precisely the set of all $(J_n, w_n)_{n\geq 0}$ with $J_n \subset \mathbf{I}$, $w_n \in W$ such that 4.1(a),(b),(c) hold and $w_n \in W_{J_n} w_{n-1} W_{\mathbf{c}(J_{n-1})}$ for $n \geq 1$.*

(With notation in [L5, I, 2.2] we have $\mathcal{T}'(J, \mathbf{c}) = \mathcal{T}(\mathbf{c}(J), \mathbf{c}^{-1})$.)

(b) *The assignment $(J_n, w_n)_{n\geq 0} \mapsto w_\infty$ defines a bijection $\mathcal{T}'(J, \mathbf{c}) \xrightarrow{\sim} {}^JW$.*

(c) *Let $z \in {}^JW^{\mathbf{c}(J)}$, $J_1 = J \cap z\mathbf{c}(J)z^{-1}$. Let $V = \{P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = z$, $V' = \{Q \in \mathcal{P}_{J_1}; \mathrm{pos}(Q, F(Q)) \in zW_{\mathbf{c}(J)}\}$. Then $f : V \to V'$, $P \mapsto P^1 := P^{F(P)}$ is an isomorphism.*

Define $V' \to \mathcal{P}_J$ by $Q \mapsto P$ where P is the unique parabolic in $\mathcal{P}_J$ such that $Q \subset P$. We have automatically $P \in V$ hence $Q \mapsto P$ is a map $f' : V' \to V$. Clearly

$f'f = 1$. We show $ff' = 1$. It is enough to show that, if Q, P are as above, then $P^{F(P)} = Q$. We have $\mathrm{pos}(Q, F(Q)) = zu$ where $u \in W_{\mathbf{c}(J)}$. We can find $B_0, B_1 \in \mathcal{B}$ such that $B_0 \subset Q$, $B_1 \subset F(Q)$, $\mathrm{pos}(B_0, B_1) = zu$. Since $l(zu) = l(z) + l(u)$ we can find $B_2 \in \mathcal{B}$ such that $\mathrm{pos}(B_0, B_2) = z, \mathrm{pos}(B_2, B_1) = u$. Since $u \in W_{\mathbf{c}(J)}$ and $B_1 \subset F(P)$ we have $B_2 \subset F(P)$. Since $B_0 \subset P, B_2 \subset F(P)$, $\mathrm{pos}(B_0, B_2) = z$, we have $B_0 \subset P^{F(P)}$. Since $Q, P^{F(P)}$ are in $\mathcal{P}_{J_1}$ and both contain B_0 we have $Q = P^{F(P)}$. This proves (c).

Let $\mathbf{t} = (J_n, w_n)_{n \ge 0} \in \mathcal{T}'(J, \mathbf{c})$. For $m \ge 0$ we set $\mathbf{t}_m = (J'_n, w'_n)_{n\ge 0}$ where $J'_n = J_{n+m}, w'_n = w_{n+m}$. We have $\mathbf{t}_m \in \mathcal{T}'(J_m, \mathbf{c})$. We set $\mathbf{t}_\infty = (J'_n, w'_n)_{n \ge 0}$ where $J'_n = J_\infty, w'_n = w_\infty$. We have $\mathbf{t}_\infty \in \mathcal{T}'(J_\infty, \mathbf{c})$. Clearly, $P \mapsto P^1$ is a map

$$\vartheta : \mathcal{P}_J^{\mathbf{t}} \to \mathcal{P}_{J_1}^{\mathbf{t}_1}.$$

(d) *The map* $P \mapsto P^1$ *is an isomorphism* $\mathcal{P}_J^{\mathbf{t}} \xrightarrow{\sim} \mathcal{P}_{J_1}^{\mathbf{t}_1}$. *The map* $P \mapsto P^\infty$ *is an isomorphism* $\mathcal{P}_J^{\mathbf{t}} \xrightarrow{\sim} \mathcal{P}_{J_\infty}^{\mathbf{t}_\infty}$.

The first assertion of (d) follows from (c). The second assertion follows using the first assertion repeatedly.

(e) *Let* $\mathbf{t} = (J_n, w_n)_{n \ge 0} \in \mathcal{T}'(J, \mathbf{c})$ *be such that* $J_n = J$ *and* $w_n = w$ *for all* $n \ge 0$ *where* $w \in W$. *We have* $\mathbf{c}(J) = w^{-1}Jw$, $w \in {}^J W^{\mathbf{c}(J)}$. *If* $P \in \mathcal{P}_J^{\mathbf{t}}$, *then* $P^n = P$ *for* $n \ge 0$ *and* $\mathrm{pos}(P, F(P)) = w$. *From* $P = P^{F(P)}$ *we see that* $P, F(P)$ *have a common Levi. We have* $\mathcal{P}_J^{\mathbf{t}} = \{P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w\}$.

(f) *Let* $(J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. *For* $n \ge 0$ *we have* $w_n = \min(W_J w_\infty W_{\mathbf{c}(J_n)})$.

4.3. In the setup of 4.2(e) we show:

(a) $$\dot{w}^{-1} L_J \dot{w} = L_{\mathbf{c}(J)} = F(L_J).$$

From $\mathrm{pos}(P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}) = w$ we see that $P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}$ have a common Levi subgroup containing T which must be L_J and also $\dot{w}L_{\mathbf{c}(J)}\dot{w}^{-1}$.

Let

$$\tilde{\mathcal{P}}_J^{\mathbf{t}} = \{gU_{P_J} \in G/U_{P_J}; g^{-1}F(g) \in U_{P_J}\dot{w}U_{P_{\mathbf{c}(J)}}\}.$$

Define $F' : L_J \to L_J$ by $g \mapsto \dot{w}F(g)\dot{w}^{-1}$. This is the Frobenius map for an $\mathbf{F}_q$-rational structure on L_J. We set

$$L_J^{F'} = \{l \in L_J; F'(l) = l\}.$$

The finite group $L_J^{F'}$ acts freely on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ by $l : gU_{P_J} \mapsto gl^{-1}U_{P_J}$ and the map $f : \tilde{\mathcal{P}}_J^{\mathbf{t}} \to \mathcal{P}_J^{\mathbf{t}}$, $gU_{P_J} \mapsto gP_Jg^{-1}$ is constant on the orbits of this action. We show:

(b) f *is a principal* $L_J^{F'}$*-bundle.*

We only show this at the level of sets. If $P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w$, we have $P = gP_Jg^{-1}$ where $g \in G$ satisfies

$$\begin{aligned} w &= \mathrm{pos}(gP_Jg^{-1}, F(g)F(P_J)F(g^{-1})) \\ &= \mathrm{pos}(P_J, g^{-1}F(g)P_{\mathbf{c}(J)}F(g^{-1})g) = \mathrm{pos}(P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}). \end{aligned}$$

Hence there exists $y \in P_J$ such that $g^{-1}F(g)P_{\mathbf{c}(J)}F(g^{-1})g = y\dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}y^{-1}$ hence $g^{-1}F(g) \in P_J\dot{w}P_{\mathbf{c}(J)}$; that is, $g^{-1}F(g) \in l'U_{P_J}\dot{w}U_{P_{\mathbf{c}(J)}}$ for some $l' \in L_J$. (We use (a).) By Lang's theorem for F' we can find $l \in L_J$ such that $l^{-1}F'(l) = l'$. Then $gl^{-1}U_{P_J} \in \tilde{\mathcal{P}}_J^{\mathbf{t}}$. We see that f is surjective.

Assume that $gU_{P_J}, g'U_{P_J}$ in $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ have the same image under f that is $gP_Jg^{-1} = g'P_Jg'^{-1}$. Then $g' = gp^{-1}$ where $p \in P_J$. We may assume that $g' = gl^{-1}$, $l \in L_J$. We have $g^{-1}F(g) \in P_J\dot{w}P_{\mathbf{c}(J)}$ and $(gl^{-1})^{-1}F(gl^{-1}) \in U_{P_J}\dot{w}U_{P_{\mathbf{c}(J)}}$ that is

$$g^{-1}F(g) = U_{P_J}l^{-1}\dot{w}F(l)U_{P_{\mathbf{c}(J)}} \text{ and } U_{P_J}l^{-1}F'(l)\dot{w}U_{P_{\mathbf{c}(J)}} = U_{P_J}\dot{w}U_{P_{\mathbf{c}(J)}}.$$

Using [L3, 3.2] we deduce $l^{-1}F'(l)\dot{w} = \dot{w}$ hence $l \in L_J^{F'}$.

Let

$${}'\tilde{\mathcal{P}}_J^{\mathbf{t}} = \{g(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w})) \in G/(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w})); g^{-1}F(g) \in U_{P_J}\dot{w}\}.$$

We show:

(c) *The map* $g(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w})) \mapsto gU_{P_J}$ *is an isomorphism* $\gamma : {}'\tilde{\mathcal{P}}_J^{\mathbf{t}} \xrightarrow{\sim} \tilde{\mathcal{P}}_J^{\mathbf{t}}$.

We only show this at the level of sets. Let $gU_{P_J} \in \tilde{\mathcal{P}}_J^{\mathbf{t}}$. We have $g^{-1}F(g) = u\dot{w}F(u')$ for some $u \in U_{P_J}, u' \in U_{P_J}$. Then $(gu'^{-1})^{-1}F(gu'^{-1}) = u'u\dot{w}$ so that $\gamma(gu'^{-1}(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w}))) = gU_{P_J}$. We see that γ is surjective. The injectivity is immediate.

Next we show:

(d) $\mathcal{P}_J^{\mathbf{t}}$ *is a smooth variety of pure dimension equal to* $\dim U_{P_J} - \dim(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w}))$ *and its connected components are permuted transitively by the* Γ-*action on* $\mathcal{P}_J^{\mathbf{t}}$.

By (b),(c) it is enough to show that ${}'\tilde{\mathcal{P}}_J^{\mathbf{t}}$ is smooth, of pure dimension equal to $\dim U_{P_J} - \dim(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w}))$ and its connected components are permuted transitively by the Γ-action

$$g_0 : g(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w})) \mapsto g_0g(U_{P_J} \cap F^{-1}(\dot{w}^{-1}U_{P_J}\dot{w}))$$

on ${}'\tilde{\mathcal{P}}_J^{\mathbf{t}}$. This follows from the fact that $\{g \in G; g^{-1}F(g) \in U_{P_J}\dot{w}\}$ is smooth of dimension $\dim U_{P_J}$ and $U_{P_J}\dot{w}$ is connected.

4.4. We now consider a general $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $\mathcal{P}_J^{\mathbf{t}} \xrightarrow{\sim} \mathcal{P}_{J_\infty}^{\mathbf{t}_\infty}$ be the isomorphism in 4.2(d). By 4.3 for $\mathbf{t}_\infty$ instead of $\mathbf{t}$, $\tilde{\mathcal{P}}_{J_\infty}^{\mathbf{t}_\infty}$ is defined. Let

$$\tilde{\mathcal{P}}_J^{\mathbf{t}} := \tilde{\mathcal{P}}_{J_\infty}^{\mathbf{t}_\infty} = \{gU_{P_{J_\infty}} \in G/U_{P_{J_\infty}}; g^{-1}F(g) \in U_{P_{J_\infty}}\dot{w}U_{P_{\mathbf{c}(J_\infty)}}\}.$$

Define $F' : L_{J_\infty} \to L_{J_\infty}$ by $l \mapsto \dot{w}_\infty F(l)\dot{w}_\infty^{-1}$. (We have $\dot{w}_\infty^{-1}L_{J_\infty}\dot{w}_\infty = L_{\mathbf{c}(J_\infty)} = F(L_{J_\infty})$; see 4.3(a).) Let $\Lambda = L_{J_\infty}^{F'}$. The finite group $\Gamma \times \Lambda$ acts on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ by

$$(g_0, l) : gU_{P_{J_\infty}} \mapsto g_0gl^{-1}U_{P_{J_\infty}}$$

and on $\mathcal{P}_J^{\mathbf{t}}$ by $(g_0, l) : P \mapsto g_0Pg_0^{-1}$. From 4.3(b) we see that:

(a) *The map* $f : \tilde{\mathcal{P}}_J^{\mathbf{t}} \to \mathcal{P}_J^{\mathbf{t}}$, $gU_{P_{J_\infty}} \mapsto gP_Jg^{-1}$ *(which is compatible with the* $\Gamma \times \Lambda$-*actions) is a principal* Λ-*bundle.*

From 4.3(d) we see that:

(b) $\mathcal{P}_J^{\mathbf{t}}$ *is a smooth variety of pure dimension equal to* $\dim U_{P_{J_\infty}} - \dim(U_{P_{J_\infty}} \cap F^{-1}(\dot{w}_\infty^{-1}U_{P_{J_\infty}}\dot{w}_\infty))$ *and its connected components are permuted transitively by the* Γ-*action on* $\mathcal{P}_J^{\mathbf{t}}$.

Let M be a finite dimensional Λ-irreducible module over $\bar{\mathbf{Q}}_l$. We view M as a $\Gamma \times \Lambda$-module with Γ acting trivially and we form the $\Gamma \times \Lambda$-equivariant local system $M_{\tilde{\mathcal{P}}_J^{\mathbf{t}}}$ on $\tilde{\mathcal{P}}_J^{\mathbf{t}}$ as in 1.6. Using (a) we see that there is a well defined $\Gamma \times \Lambda$-equivariant local system $\underline{M}$ on $\mathcal{P}_J^{\mathbf{t}}$ with trivial action of Λ such that $f^*\underline{M} = M_{\tilde{\mathcal{P}}_J^{\mathbf{t}}}$ as $\Gamma \times \Lambda$-equivariant local systems. We will regard $\underline{M}$ as a Γ-equivariant local system.

Let $d = \dim \mathcal{P}_J^{\mathbf{t}}$ (see (b)). We show:

(c) $\underline{M}[d]$ *(an object of* $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ *by (b)) is simple.*

Let $r : \tilde{C} \to C$ be a finite principal covering with finite group H. Assume that C is connected. There is an obvious functor $E \mapsto E'$ from H-modules of finite dimension over $\bar{\mathbf{Q}}_l$ to local systems on C which are direct summands of $r_! \bar{\mathbf{Q}}_l$. If E is irreducible, then E' is irreducible as a local system.

We apply this statement in the case where C is a connected component of $\mathcal{P}_J^{\mathbf{t}}$, $\tilde{C} = f^{-1}(C)$ (f as in (a)), r is the restriction of f, $H = \Lambda$ and $E = M$. Note that $E' = \underline{M}|_C$. We see that the local system $\underline{M}|_C$ is irreducible. It remains to use the transitivity statement in (b).

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ we write $A \in \mathbb{S}(\mathcal{P}_J^{\mathbf{t}})$ instead of "A is isomorphic to $\underline{M}[d]$ for some M as above."

From (c) we see that:

(d) $\underline{M}^\sharp[d]$ *is a simple object of* $\mathcal{M}_\Gamma(\mathcal{P}_J)$.

For an object A of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ we write $A \in \mathbb{S}(\mathcal{P}_J)$ instead of "A is isomorphic to $\underline{M}^\sharp[d] \in \mathcal{M}_\Gamma(\mathcal{P}_J)$ for some $\mathbf{t} \in \mathcal{T}'(J, \mathbf{c})$ and some M as above."

4.5. For $w \in W$ we identify $\mathcal{B}_w$ with $Z^{(w)}$ as in 2.5. We show:

(a) *Let* $a_1, a_2, b \in W$. *Let* $\mathcal{L} \in \mathcal{S}(T)^{a_1 a_2 F}$. *Let* V *be a locally closed* Γ*-stable subvariety of* $\mathcal{B}_b$. *Let* $X_{a_1,a_2} = \{(B_0, B_1, B_2) \in Z^{(a_1,a_2)}; B_1 \in V\}$, *(see 2.5). Define* $\kappa : X_{a_1,a_2} \to V$ *by* $(B_0, B_1, B_2) \mapsto B_1$. *Let* $\mathcal{E} = \mathcal{L}_{(a_1,a_2)}|_{X_{a_1,a_2}}$. *Let* V' *be an algebraic variety with a* Γ*-action and let* $m : V \to V'$ *be a morphism compatible with the* Γ*-actions. Let* A' *be a simple object of* $\mathcal{M}_\Gamma(V')$ *such that* $A' \dashv_\Gamma {}^pH^\cdot((m\kappa)_! \mathcal{E})$. *Then there exists* $e \in W$ *such that* $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ *and* $A' \dashv_\Gamma {}^pH^\cdot(m_!(e^*\mathcal{L})_b|_V)$.

We argue by induction on $l(a_1)$. If $l(a_1) = 0$, then $a_2 = b$, κ is an isomorphism and the result is obvious (with $e = 1$). Assume now that $l(a_1) > 0$. We can find $s \in \mathbf{I}$ such that $l(a_1) > l(sa_1)$. Let $\mathcal{E}_1 = (s^*\mathcal{L})_{(sa_1, a_2\mathbf{c}(s))}|_{X_{sa_1,a_2\mathbf{c}(s)}}$.

Assume first that $l(a_2\mathbf{c}(s)) = l(a_2) + 1$. We have an isomorphism $\iota : X_{a_1,a_2} \to X_{sa_1,a_2\mathbf{c}(s)}$, $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$ where $B'_0 \in \mathcal{B}$ is defined by

(b) $$\mathrm{pos}(B_0, B'_0) = s, \mathrm{pos}(B'_0, B_1) = sa_1.$$

Define $\kappa' : X_{sa_1,a_2\mathbf{c}(s)} \to V$ by $(B_0, B_1, B_2) \mapsto B_1$. We have $\kappa = \kappa'\iota$, $\iota^*\mathcal{E}_1 = \mathcal{E}$ hence $\iota_!\mathcal{E} = \mathcal{E}_1$. Thus $(m\kappa)_!\mathcal{E} = (m\kappa')_!\mathcal{E}_1$ and $A' \dashv_\Gamma {}^pH^\cdot((m\kappa')_!\mathcal{E}_1)$. By the induction hypothesis there exists $e' \in W$ such that $(bF)^*(e'^*s^*\mathcal{L}) \cong e'^*s^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e'^*s^*\mathcal{L})_b|_V)$. The result follows with $e = se'$.

Assume next that $l(a_2\mathbf{c}(s)) = l(a_2) - 1$. We have a partition $X_{a_1,a_2} = X' \cup X''$ where X' (resp. X'') is the open (resp. closed) subset of X_{a_1,a_2} defined by $\mathrm{pos}(B_1, F(B'_0)) = a_2$ (resp. $\mathrm{pos}(B_1, F(B'_0)) = a_2\mathbf{c}(s)$). Let $j' = \kappa|_{X'}$, $j'' = \kappa|_{X''}$. By general principles we have either

(c) $A' \dashv_\Gamma {}^pH^\cdot((mj')_!(\mathcal{E}|_{X'}))$ or

(d) $A' \dashv_\Gamma {}^pH^\cdot((mj'')_!(\mathcal{E}|_{X''}))$.

Assume that (d) holds. We have $j'' = \kappa''\iota''$ where $\kappa'' : X_{sa_1,a_2\mathbf{c}(s)} \to V$ is given by $(B_0, B_1, B_2) \mapsto B_1$ and $\iota'' : X'' \to X_{sa_1,a_2\mathbf{c}(s)}$ is $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$ with B'_0 as in (b). We have $\mathcal{E}|_{X''} = \iota''^*\mathcal{E}_1$. Now ι'' is an affine line bundle hence $\iota''_!(\mathcal{E}|_{X''}) = \mathcal{E}_1[-2](-1)$. Hence $A' \dashv_\Gamma {}^pH^\cdot((\mapsto \kappa'')_!\mathcal{E}_1)$. By the induction hypothesis there exists $e' \in W$ such that $(bF)^*(e'^*s^*\mathcal{L}) \cong e'^*s^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e'^*s^*\mathcal{L})_b|_V)$. The result follows with $e = se'$.

Assume now that (c) holds. We have $j' = \kappa'\iota'$ where $\kappa' : X_{sa_1,a_2} \to V$ is $(B_0, B_1, B_2) \mapsto B_1$ and $\iota' : X' \to X_{sa_1,a_2}$ is $(B_0, B_1, B_2) \mapsto (B'_0, B_1, F(B'_0))$

with B'_0 as in (b). Note that ι' makes X' into the complement of a section of an affine line bundle over X_{sa_1,a_2}. If $s \notin W_{\mathcal{L}}$, then by an argument as in the proof of 2.15 we see that $\iota'_!(\mathcal{E}|_{X'}) = 0$ contradicting (c). Thus we may assume that $s \in W_{\mathcal{L}}$. Then $\mathcal{E}_2 = \mathcal{L}_{(sa_1,a_2)}|_{X_{sa_1,a_2}}$ is defined and $\mathcal{E}|_{X'} = \iota'^*\mathcal{E}_2$. Hence we have a distinguished triangle $(\iota'_!\mathcal{E}_{X'}, \mathcal{E}_2, \mathcal{E}_2[-2](-1))$ hence a distinguished triangle $(m_! j'_!(\mathcal{E}|_{X'}, m_!\kappa'_!\mathcal{E}_2, m_!\kappa'_!\mathcal{E}_2[-2](-1))$. It follows that $A' \dashv_\Gamma {}^pH^\cdot(m_!\kappa'_!\mathcal{E}_2)$. By the induction hypothesis there exists $e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $A' \dashv_\Gamma {}^pH^\cdot(m_!(e^*\mathcal{L})_b|_V)$. This completes the proof of (a).

4.6. Let $\mathbf{t} = (J_n, w_n)_{n\ge 0} \in \mathcal{T}'(J,\mathbf{c})$. Let $\mathcal{B}_{\mathbf{t}} = \{B' \in \mathcal{B}; P_{B',J} \in \mathcal{P}_J^{\mathbf{t}}\}$. For $a \in W$ let $\mathcal{B}_{\mathbf{t},a} = \mathcal{B}_{\mathbf{t}} \cap \mathcal{B}_a$. Define $\xi_{\mathbf{t},a} : \mathcal{B}_{\mathbf{t},a} \to \mathcal{P}_J^{\mathbf{t}}$ by $B' \mapsto P_{B',J}$. We show:

(a) *Let $\mathcal{L} \in \mathcal{S}(T)^{aF}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J^{\mathbf{t}})$ such that $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}))$. Then there exist $b, e \in W$ such that $b^*e^*\mathcal{L} \cong e^*\mathcal{L}$ and $A \dashv_\Gamma {}^pH^\cdot(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$.*

Since $\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}) \ne 0$ we have $\mathcal{B}_{\mathbf{t},a} \ne \emptyset$. Thus there exists $B' \in \mathcal{B}$ such that $\mathrm{pos}(B', F(B')) = a$, $P_{B',J} \in \mathcal{P}_J^{\mathbf{t}}$. We have $\mathrm{pos}(P_{B',J}, F(P_{B',J})) = w_0$. Since $B' \subset P_{B',J}$, $F(B') \subset F(P_{B',J})$, it follows that $a \in W_J w_0 W_{\mathbf{c}(J)}$ and $w_0 = \min(W_J a W_{\mathbf{c}(J)})$.

Define $\phi : \mathcal{B}_{\mathbf{t},a} \to \mathcal{B}_{\mathbf{t}_1}$ by $\phi(B') = (P_{B',J})^{F(B')}$. (For $B' \in \mathcal{B}_{\mathbf{t}}$ we have $(P_{B',J})^{F(B')} \in \mathcal{B}_{\mathbf{t}_1}$ since $(P_{B',J})^{F(B')} \subset (P_{B',J})^{F(P_{B',J})} \in \mathcal{P}_{J_1}^{\mathbf{t}_1}$.) We have a partition $\mathcal{B}_{\mathbf{t}_1} = \sqcup_{b\in W}\mathcal{B}_{\mathbf{t}_1,b}$. Setting $\mathcal{B}_{\mathbf{t},a,b} = \phi^{-1}(\mathcal{B}_{\mathbf{t}_1,b})$ we get a partition $\mathcal{B}_{\mathbf{t},a} = \sqcup_{b\in W}\mathcal{B}_{\mathbf{t},a,b}$. Let $\xi_{\mathbf{t},a,b} : \mathcal{B}_{\mathbf{t},a,b} \to \mathcal{P}_J^{\mathbf{t}}$ be the restriction of $\xi_{\mathbf{t},a}$. By general principles we have $A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a,b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}}))$ for some $b \in W$. Let $\phi_b : \mathcal{B}_{\mathbf{t},a,b} \to \mathcal{B}_{\mathbf{t}_1,b}$ be the restriction of ϕ. We have $\vartheta\xi_{\mathbf{t},a,b} = \xi_{\mathbf{t}_1,b}\phi_b$ (both compositions carry B' to $(P_{B',J})^{F(P_{B',J})}$.) Hence $\xi_{\mathbf{t},a,b} = \vartheta^{-1}\xi_{\mathbf{t}_1,b}\phi_b$. Thus, $A \dashv_\Gamma {}^pH^\cdot((\vartheta^{-1})_!\xi_{\mathbf{t}_1,b!}\phi_{b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}}))$ and

(b) $$\vartheta_! A \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t}_1,b!}\phi_{b!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a,b}})).$$

We can write uniquely $a = a_1a_2$ where $a_1 \in W_J, a_2 \in {}^JW$. We show that for any $\tilde{B} \in \mathcal{B}_{\mathbf{t}_1,b}$ we have

(c) $$\phi_b^{-1}(\tilde{B}) = \{B' \in \mathcal{B}; \mathrm{pos}(B', \tilde{B}) = a_1, \mathrm{pos}(\tilde{B}, F(B')) = a_2\}.$$

Assume first that $B' \in \phi_b^{-1}(\tilde{B})$. We know that $\mathrm{pos}(B', F(B')) = a$. We have $\mathrm{pos}(B', (P_{B',J})^{F(B')}) \in W_J$ since $B', (P_{B',J})^{F(B')}$ are two Borel subgroups of $P_{B',J}$. From the definitions we have $\mathrm{pos}((P_{B',J})^{F(B')}, F(B')) \in {}^JW$. We have automatically $\mathrm{pos}(B', (P_{B',J})^{F(B')}) = a_1$, $\mathrm{pos}((P_{B',J})^{F(B')}, F(B')) = a_2$ that is $\mathrm{pos}(B', \tilde{B}) = a_1$, $\mathrm{pos}(\tilde{B}, F(B')) = a_2$.

Conversely, assume that B' belongs to the right hand side of (c). We have $l(a_1a_2) = l(a_1) + l(a_2)$ hence $\mathrm{pos}(B', F(B')) = a_1a_2 = a$. Since the properties $\mathrm{pos}(B', \tilde{B}) \in W_J$, $\mathrm{pos}(\tilde{B}, F(B')) \in {}^JW$ characterize $\tilde{B}$ and $(P_{B',J})^{F(B')}$ has the same properties, it follows that $P^{F(B')} = \tilde{B}$ where $P = P_{B',J}$. Since $B' \subset P, F(B') \subset F(P)$ we have

$$\mathrm{pos}(P, F(P)) = \min(W_J \mathrm{pos}(B', F(B')) W_{\mathbf{c}(J)}) = \min(W_J a W_{\mathbf{c}(J)}) = w_0.$$

It follows that $P^{F(P)} \in \mathcal{P}_{J\cap w_0\mathbf{c}(J)w_0^{-1}} = J_1$. Clearly, $\tilde{B} = P^{F(B')} \subset P^{F(P)}$. Hence $P^{F(P)} = P_{\tilde{B},J_1} \in \mathcal{P}_{J_1}^{\mathbf{t}_1}$. It follows that $P \in \mathcal{P}_J^{\mathbf{t}}$. Thus, $B' \in \mathcal{B}_{\mathbf{t},a}$ and $\phi(B') = \tilde{B}$. Since $\tilde{B} \in \mathcal{B}_{\mathbf{t}_1,b}$, we see that $B' \in \phi_b^{-1}(\tilde{B})$. This proves (c).

From (c) we see that $(B_0, B_1, B_2) \mapsto B_0$ is an isomorphism $X_{a_1,a_2} \to \mathcal{B}_{\mathbf{t},a,b}$ where X_{a_1,a_2} is defined as in 4.5(a) in terms of $V = \mathcal{B}_{\mathbf{t}_1,b}$. Under this isomorphism, ϕ_b

corresponds to $X_{a_1,a_2} \to \mathcal{B}_{\mathbf{t}_1,b}$, $(B_0, B_1, B_2) \mapsto B_1$. Applying 4.5(a) with $V' = \mathcal{P}^{\mathbf{t}_1}_{J_1}$, $m = \xi_{\mathbf{t}_1,b}$, $A' = \vartheta_! A$ we see that there exists $e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $\vartheta_! A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. (The assumption of 4.5(a) is verified by (b).) Since ϑ is an isomorphism, it follows that $A \dashv_\Gamma {}^pH^{\cdot}(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. This proves (a).

4.7. Let $\mathbf{t} = (J_n, w_n)_{n\geq 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $d = \dim \mathcal{P}^{\mathbf{t}}_J$. Let $a \in W$. We show:

(a) *Let $\mathcal{L} \in \mathcal{S}(T)^{a\bar{F}}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ such that $A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}))$. Then $A \in \mathbb{S}(\mathcal{P}^{\mathbf{t}}_J)$.*

More generally, we show that (a) holds when $J, \mathbf{t}$ are replaced by $J_n, \mathbf{t}_n$, $n \geq 0$. First we show:

(b) If the result holds for $n = 1$, then it holds for $n = 0$.

Let A be as in (a). By 4.6(a) there exist $b, e \in W$ such that $(bF)^*(e^*\mathcal{L}) \cong e^*\mathcal{L}$ and $A \dashv_\Gamma {}^pH^{\cdot}(\vartheta^*\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. Since ϑ is an isomorphism, there exists a simple object A' of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}_1}_{J_1})$ such that $A = \vartheta^* A'$. From our assumption we have $A' \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}_1,b!}((e^*\mathcal{L})_b|_{\mathcal{B}_{\mathbf{t}_1,b}}))$. Since (b) holds for $n = 1$ we have $A' \cong \underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M. Hence A is of the same form. Thus (b) holds.

Similarly, if the result holds for some $n \geq 1$, then it holds for $n-1$. In this way we see that it suffices to prove the result for n large. Thus in the remainder of this proof we assume, as we may, that $J_0 = J_1 = \cdots = J$ and $w_0 = w_1 = \cdots = w$. We can write uniquely $a = a_1 a_2$ where $a_1 \in W_J, a_2 \in {}^JW$. Since $\xi_{\mathbf{t},a!}(\mathcal{L}_a|_{\mathcal{B}_{\mathbf{t},a}}) \neq 0$, we have $\mathcal{B}_{\mathbf{t},a} \neq \emptyset$. From this we deduce as in the proof of 4.6(a) that $a \in W_J w W_{\mathbf{c}(J)}$. Since $w\mathbf{c}(J)w^{-1} = J$ we must have $wW_{\mathbf{c}(J)} = W_J w$ so that $a \in W_J w$. Since $w \in {}^JW$ (see 4.2(b)) it follows that $w = a_2$. In particular, we have $a_2 \in {}^JW^{\mathbf{c}(J)}$. Hence if $B'' \in \mathcal{B}_a$, then $\mathrm{pos}(P_{B'',J}, F(P_{B'',J})) = a_2 = w$. Since in our case $\mathcal{P}^{\mathbf{t}}_J = \{P \in \mathcal{P}_J; \mathrm{pos}(P, F(P)) = w\}$ (see 4.2(e)), we see that $P_{B'',J} \in \mathcal{P}^{\mathbf{t}}_J$. Thus we have $\mathcal{B}_a = \mathcal{B}_{\mathbf{t},a}$ and $A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t},a!}(\mathcal{L}_a))$. Let $\tilde{\mathcal{B}}_a = \{gU \in G/U; g^{-1}F(g) \in U\dot{a}U\}$. Recall from 4.3(b) that $f_a : \tilde{\mathcal{B}}_a \to \mathcal{B}_a$, $gU \mapsto gBg^{-1}$ is a finite principal covering with group $\mathfrak{T} = \{t \in T; \dot{a}F(t)\dot{a}^{-1} = t\}$. From 2.5 we see that $\mathcal{L}_a$ is a direct summand of $f_{a!}\bar{\mathbf{Q}}_l$. Thus we have $A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$.

By 4.3(a), $L_J = \dot{w}L_{\mathbf{c}(J)}\dot{w}^{-1}$ is a common Levi subgroup of $P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}$. Let $F' : L_J \to L_J$ be as in 4.3. Let $\Lambda = L^{F'}_J$. Let $\mathcal{B}'$ be the variety of Borel subgroups of L_J. For β, β' in $\mathcal{B}'$ we have $\mathrm{pos}(\beta U_{P_J}, \beta' U_{P_J}) = y$ for a unique $y \in W_J$; we then also write $\mathrm{pos}'(\beta, \beta') = y$. Let $Y' = \{\beta \in \mathcal{B}'; \mathrm{pos}'(\beta, F'(\beta)) = a_1\}$. Now Λ acts on Y' by conjugation. Let $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ be as in 4.3. Note that Λ acts (freely) on $\tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y'$ by $l : (gU_{P_J}, \beta) \mapsto (gl^{-1}U_{P_J}, l\beta l^{-1})$ and we can form the orbit space $\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y')$.

We define $\psi : \tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y' \to \mathcal{B}_a$ by $(gU_{P_J}, \beta) \mapsto g\beta U_{P_J} g^{-1}$. We show that ψ is well defined; that is, $\mathrm{pos}(g\beta U_{P_J} g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})) = a$ for $gU_{P_J} \in \tilde{\mathcal{P}}^{\mathbf{t}}_J$. Since $a = a_1 w$, $l(a) = l(a_1) + l(w)$, it is enough to show that for $(gU_{P_J}, \beta) \in \tilde{\mathcal{P}}^{\mathbf{t}}_J \times Y'$ we have

(c) $\mathrm{pos}(g\beta U_{P_J} g^{-1}, gF'(\beta)U_{P_J} g^{-1}) = a_1$,

(d) $\mathrm{pos}(gF'(\beta)U_{P_J} g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})) = w$.

Now the left hand side of (c) is equal to $\mathrm{pos}(\beta U_{P_J}, F'(\beta)U_{P_J}) = \mathrm{pos}'(\beta, F'(\beta)) = a_1$, proving (c). We have $g^{-1}F(g) = u\dot{w}u'$ where $u \in U_{P_J}, u' \in U_{P_{\mathbf{c}(J)}}$. The left hand side of (d) is equal to

$$\begin{aligned}
&\mathrm{pos}(F'(\beta)U_{P_J}, g^{-1}F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1})g)\\
&= \mathrm{pos}(F'(\beta)U_{P_J}, u\dot{w}u'F(\beta)U_{P_{\mathbf{c}(J)}}u'^{-1}\dot{w}^{-1}u^{-1})\\
&= \mathrm{pos}(F'(\beta)U_{P_J}, \dot{w}F(\beta)U_{P_{\mathbf{c}(J)}}\dot{w}^{-1})\\
&= \mathrm{pos}(F'(\beta)U_{P_J}, F'(\beta)U_{\dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}}) = \mathrm{pos}(P_J, \dot{w}P_{\mathbf{c}(J)}\dot{w}^{-1}).
\end{aligned}$$

(The last equality follows from [L5], [II, 8.3].) This equals w.

We see that ψ is well defined. We show:

(e) *The map* $\bar{\psi}:\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times Y')\to\mathcal{B}_a$ *induced by* ψ *is an isomorphism.*

Let $B'\in\mathcal{B}_a$. We can find uniquely $\tilde{B}\in\mathcal{B}$ such that $\mathrm{pos}(B',\tilde{B})=a_1$, $\mathrm{pos}(\tilde{B},F(B'))=w$. Then $P_{B',J}=P_{\tilde{B},J}$ and $\mathrm{pos}(P_{\tilde{B},J},P_{F(B'),\mathbf{c}(J)})=w$. Hence $\mathrm{pos}(P_{B',J},F(P_{B',J}))=w$; that is, $P_{B',J}\in\mathcal{P}^{\mathbf{t}}_J$. Since $\tilde{\mathcal{P}}^{\mathbf{t}}_J\to\mathcal{P}^{\mathbf{t}}_J$ is surjective (see 4.3) we can find $gU_{P_J}\in\tilde{\mathcal{P}}^{\mathbf{t}}_J$ such that $gP_Jg^{-1}=P_{B',J}$; note that gU_{P_J} is unique up to the action of Λ. Since $B'\subset P_{B',J}$ we have $g^{-1}B'g\subset P_J$ hence there is a unique $\beta\in\mathcal{B}'$ such that $g^{-1}B'g=\beta U_{P_J}$. As above we see that

$$\mathrm{pos}(gF'(\beta)U_{P_J}g^{-1}, F(g)F(\beta)U_{P_{\mathbf{c}(J)}}F(g^{-1}))=w;$$

that is, $\mathrm{pos}(gF'(\beta)U_{P_J}g^{-1},F(B'))=w$. Thus $gF'(\beta)U_{P_J}g^{-1}$ is a Borel subgroup of $P_{B',J}$ whose relative position with $F(B')$ is w. But there is only one such Borel subgroup. Therefore $gF'(\beta)U_{P_J}g^{-1}=\tilde{B}$. Since $\mathrm{pos}(B',\tilde{B})=a_1$ we have $\mathrm{pos}(g\beta U_{P_J}g^{-1},gF'(\beta)U_{P_J}g^{-1})=a_1$ hence $\mathrm{pos}(\beta U_{P_J},F'(\beta)U_{P_J})=a_1$; that is, $\mathrm{pos}'(\beta,F'(\beta))=a_1$. Thus to $B'\in\mathcal{B}_a$ we have associated $(gU_{P_J},\beta)\in\tilde{\mathcal{P}}^{\mathbf{t}}_J\times Y'$; its Λ-orbit is well defined. Thus we have a well defined map $\mathcal{B}_a\to\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times Y')$. Clearly, this is the inverse of $\bar{\psi}$. This proves (e).

Now let $\beta'=B\cap L_J$, and let U' be the unipotent radical of β'. Let $U''=U'\cap F'^{-1}(\dot{a}_1^{-1}U'\dot{a}_1)$. Let $\tilde{Y}'=\{lU''\in L_J/U''; l^{-1}F'(l)\in U'\dot{a}_1\}$. As in 4.3(b),(c) the map $\rho:\tilde{Y}'\to Y'$, $lU''\mapsto l\beta'l^{-1}$ is a finite principal covering with group $\{t\in T;\dot{a}_1F'(t)\dot{a}_1^{-1}=t\}=\mathfrak{T}$. Now Λ acts freely on $\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}'$ by

$$l_0:(gU_{P_J},lU'')\mapsto(gl_0^{-1}U_{P_J},l_0lU'')$$

and we can form the orbit space $\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}')$.

The map $\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}'\to\tilde{\mathcal{B}}_a$, $(gU_{P_J},lU'')\mapsto glU$ induces a map $\tilde{\psi}:\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}')\to\tilde{\mathcal{B}}_a$ which is easily seen to be an isomorphism. We have a commutative diagram

$$\begin{array}{ccc}
\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}') & \xrightarrow{\tilde{\psi}} & \tilde{\mathcal{B}}_a\\
{\scriptstyle\xi}\big\downarrow & & {\scriptstyle f_a}\big\downarrow\\
\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times Y') & \xrightarrow{\bar{\psi}} & \mathcal{B}_a
\end{array}$$

where ξ is induced by $(gU_{P_J},lU'')\mapsto(gU_{P_J},l\beta'l^{-1})$ (a principal $\mathfrak{T}$-bundle). Note that the horizontal maps in this diagram are isomorphisms.

Under the isomorphism (e), the map $\xi_{\mathbf{t},a}$ becomes the map $\xi':\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times Y')\to\mathcal{P}^{\mathbf{t}}_J$ induced by $(gU_{P_J},\beta)\mapsto gP_Jg^{-1}$. It follows that $\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l=(\xi'\xi)_!\bar{\mathbf{Q}}_l$ and $A\dashv_\Gamma {}^pH^\cdot((\xi'\xi)_!\bar{\mathbf{Q}}_l)$. We extend the natural Γ-actions on $\mathcal{P}^{\mathbf{t}}_J$ and on $\Lambda\backslash(\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}')$ to $\Gamma\times\Lambda$-actions with Λ acting trivially. Then $A,{}^pH^j((\xi'\xi)_!\bar{\mathbf{Q}}_l)$ are naturally objects of $\mathcal{M}_{\Gamma\times\Lambda}(\mathcal{P}^{\mathbf{t}}_J)$ and $A\dashv_{\Gamma\times\Lambda}{}^pH^\cdot((\xi'\xi)_!\bar{\mathbf{Q}}_l)$. Let $f:\tilde{\mathcal{P}}^{\mathbf{t}}_J\to\mathcal{P}^{\mathbf{t}}_J$ be as in 4.3. Now $\Gamma\times\Lambda$ acts on $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ as in 4.4 (compatibly with f). Moreover, f^*A, $f^*({}^pH^j((\xi'\xi)_!\bar{\mathbf{Q}}_l))=$

${}^pH^j(f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l)$ are objects of $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}^{\mathbf{t}}_J)$ and $f^*A \dashv_{\Gamma\times\Lambda} {}^pH^{\cdot}(f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l)$. Let $p_1:\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}'\to\tilde{\mathcal{P}}^{\mathbf{t}}_J$ be the first projection. Now $\Gamma\times\Lambda$ acts on $\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}'$ by $(g_0,l_0):(gU_J,lU'')\mapsto(g_0gl_0^{-1}U_J,l_0lU'')$ and p_1 is compatible with the $\Gamma\times\Lambda$-actions. We have $f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l=p_{1!}\bar{\mathbf{Q}}_l$. We see that $f^*A\dashv_{\Gamma\times\Lambda}{}^pH^{\cdot}(p_{1!}\bar{\mathbf{Q}}_l)$. Let $p_2:\tilde{\mathcal{P}}^{\mathbf{t}}_J\times\tilde{Y}'\to\tilde{Y}'$ be the second projection. Now $\Gamma\times\Lambda$ acts on $\tilde{Y}'$ by $(g_0,l_0):lU''\mapsto l_0lU''$ and p_2 is compatible with the $\Gamma\times\Lambda$-actions. The obvious maps $\tilde{\mathcal{P}}^{\mathbf{t}}_J\xrightarrow{e}\text{point}\xleftarrow{e'}\tilde{Y}'$ are again compatible with the $\Gamma\times\Lambda$-actions (the action on the point is trivial). We have $p_{1!}\bar{\mathbf{Q}}_l=e^*e'_!\bar{\mathbf{Q}}_l$ hence $f^*A\dashv_{\Gamma\times\Lambda}{}^pH^{\cdot}(e^*e'_!\bar{\mathbf{Q}}_l)$. We have a spectral sequence in $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}^{\mathbf{t}}_J)$ with $E_2={}^pH^{\cdot}(e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l))$ and E_∞ is an associated graded of ${}^pH^{\cdot}(e^*e'_!\bar{\mathbf{Q}}_l)$. We have $f^*A\dashv_{\Gamma\times\Lambda}E_\infty$ hence $f^*A\dashv_{\Gamma\times\Lambda}E_2$. Now ${}^pH^{\cdot}(e'_!\bar{\mathbf{Q}}_l)$ is just a $\Gamma\times\Lambda$-module M with trivial action of Γ and $e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l)$ is the local system $f^*\underline{M}$ (notation of 4.4). Since $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ is smooth of pure dimension d, ${}^pH^j(e^{*p}H^{\cdot}(e'_!\bar{\mathbf{Q}}_l))$ is 0 if $j\ne d$ and is $f^*\underline{M}[d]$ if $j=d$. Thus $E_2=f^*\underline{M}[d]$. We see that $f^*A\dashv_{\Gamma\times\Lambda}f^*(\underline{M})[d]$. It follows that $A\dashv_\Gamma\underline{M}[d]$. This implies (a).

4.8. Let $\mathbf{t}=(J_n,w_n)_{n\ge0}\in\mathcal{T}'(J,\mathbf{c})$. Let $d=\dim\mathcal{P}^{\mathbf{t}}_J$. Let $\mathcal{L}\in\mathcal{S}(T)$ and $\mathbf{s}$ be as in 2.6. Let $\bar{\mathcal{L}}$ be the local system on $\mathcal{Z}^{\mathbf{s}}$ as in 2.11. Let $\Upsilon^{\mathbf{s}}:\mathcal{Z}^{\mathbf{s}}\to\mathcal{P}_J$, $\bar{\Upsilon}^{\mathbf{s}}:\bar{Z}^{\mathbf{s}}\to\mathcal{P}_J$ be as in 3.2. We show:

(a) *Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A\dashv_\Gamma{}^pH^{\cdot}(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. Let $h:\mathcal{P}^{\mathbf{t}}_J\to\mathcal{P}_J$ be the inclusion. Let A' be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ such that $A'\dashv_\Gamma{}^pH^{\cdot}(h^*A)$. Then $A'\cong\underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M.*

Let $K=\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp$. Using 2.13(a) we see that $\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}}=K$. Hence $A\dashv_\Gamma{}^pH^{\cdot}(K)$. We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ with $E_2={}^pH^{\cdot}(h^*({}^pH^{\cdot}K))$ and E_∞ is an associated graded of ${}^pH^{\cdot}(h^*K)$. Since $\bar{\Upsilon}^{\mathbf{s}}$ is proper, we see from the decomposition theorem [BBD] that $K\cong\oplus_i{}^pH^i(K)[-i]$ and that each ${}^pH^i(K)$ is semisimple as an object of $\mathcal{M}(\mathcal{P}_J)$ hence also as an object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. It also follows that $h^*K\cong\oplus_ih^*({}^pH^i(K))[-i]$ hence ${}^pH^j(h^*K)\cong\oplus_i{}^pH^{j-i}(h^*({}^pH^i(K)))$. This shows that $E_2\cong E_\infty$ as objects of $\mathcal{M}(\mathcal{P}^{\mathbf{t}}_J)$. Thus the spectral sequence above is degenerate when regarded in $\mathcal{M}(\mathcal{P}^{\mathbf{t}}_J)$. But then it is also degenerate in $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$. Using $A'\dashv_\Gamma{}^pH^{\cdot}(h^*A)$ and the fact that A is a direct summand of ${}^pH^{\cdot}(K)$ (in $\mathcal{M}_\Gamma(\mathcal{P}_J)$) we see that $A'\dashv_\Gamma E_2$. It follows that $A'\dashv_\Gamma E_\infty$; that is, $A'\dashv_\Gamma{}^pH^{\cdot}(h^*\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$. As in the proof of the implication 3.2(v) $\implies$ 3.2(i) we deduce that $A'\dashv_\Gamma{}^pH^{\cdot}(h^*\Pi^{(a)}_!\mathcal{L}_{(a)})$ ($\Pi^{(a)}$ as in 3.2) for some $a\in W$. Now (a) follows from 4.7(a).

4.9. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A\dashv_\Gamma{}^pH^{\cdot}(\Upsilon^{\mathbf{s}}_!\bar{\mathcal{L}})$ with $\mathcal{L},\mathbf{s}$ as in 2.6. We show:

(a) *There exists $\mathbf{t}=(J_n,w_n)_{n\ge0}\in\mathcal{T}'(J,\mathbf{c})$ and an irreducible $L^{F'}_{J_\infty}$-module M such that $A\cong\underline{M}^\sharp[d]$ where $\underline{M}^\sharp$ is as in 4.4 and $d=\dim\mathcal{P}^{\mathbf{t}}_J$.*

Since $\mathcal{P}_J=\cup_{\mathbf{t}\in\mathcal{T}'(J,\mathbf{c})}\mathcal{P}^{\mathbf{t}}_J$, we can find $\mathbf{t}\in\mathcal{T}'(J,\mathbf{c})$ such that $\text{supp}(A)\cap\mathcal{P}^{\mathbf{t}}_J$ is open dense in $\text{supp}(A)$. Then, denoting by $h:\mathcal{P}^{\mathbf{t}}_J\to\mathcal{P}_J$ the inclusion, we see that h^*A is a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$. As in 4.8(a), we have $h^*A\cong\underline{M}[d]$ for some irreducible $L^{F'}_{J_\infty}$-module M. It follows that A is of the required form.

4.10. Let $\mathbf{t}=(J_n,w_n)_{n\ge0}\in\mathcal{T}'(J,\mathbf{c})$. We set $w=w_\infty\in{}^JW$. We show:

(a) *for any $b\in W_{J_\infty}$, $B'\mapsto P_{B',J}$ is a well defined map $\mathcal{B}_{bw}\to\mathcal{P}^{\mathbf{t}}_J$;*

(b) *for $b=1$, the map $\mathcal{B}_w\to\mathcal{P}^{\mathbf{t}}_J$ in (a) is surjective.*

We prove (a). Let $B'\in\mathcal{B}_{bw}$. Let $P=P_{B',J}$. By 4.2(f) we have $\text{pos}(P,F(P))=\min(W_JwW_{\mathbf{c}(J)})=w_0$. Define P^n in terms of P as in 4.1. We have $P^1=P^{F(P)}\in$

$\mathcal{P}_{J_1}$. As in the proof of 4.6(c) (with $a_1 = b, a_2 = w$) we see that $b = \mathrm{pos}(B', P^{F(B')})$, $w = \mathrm{pos}(P^{F(B')}, F(B'))$. Since $b \in W_{J_1}$ we have $\mathrm{pos}(B', P^{F(B')}) \in W_{J_1}$ hence $B' \subset P^1$. From the definitions we have $w_1 = \min(W_{J_1} w W_{\mathbf{c}(J_1)})$ hence

$$\mathrm{pos}(P^1, F(P^1)) = \min(W_{J_1}\mathrm{pos}(B', F(B'))W_{\mathbf{c}(J_1)}) = \min(W_{J_1} bw W_{\mathbf{c}(J_1)}) = w_1.$$

By the same argument applied to $B', P^1, \mathbf{t}_1$ instead of $B', P, \mathbf{t}$ we see that $P^2 \in \mathcal{P}_{J_2}$ and $\mathrm{pos}(P^2, F(P^2)) = w_2$. (We have $w \in {}^{J_1}W$ since $J_1 \subset J$.) Continuing in this way we see that $P^n \in \mathcal{P}_{J_n}$ and $\mathrm{pos}(P^n, F(P^n)) = w_n$ for all $n \geq 0$. Thus $P \in \mathcal{P}^{\mathbf{t}}_J$. This proves (a).

We prove (b). Let $P \in \mathcal{P}^{\mathbf{t}}_J$. Define P^∞ in terms of P as in 4.1. We have $\mathrm{pos}(P^\infty, F(P^\infty)) = w$. Hence

$$\begin{aligned} pr_2 :&\{(B', B'') \in \mathcal{B} \times \mathcal{B}; B' \subset P^\infty, B'' \subset F(P^\infty), \mathrm{pos}(B', B'') = w\} \\ &\to \{B'' \in \mathcal{B}; B'' \subset F(P^\infty)\} \end{aligned}$$

is a bijection with inverse $B'' \mapsto ((P^\infty)^{B''}, B'')$. The condition that (B', B'') in the domain of pr_2 satisfies $B'' = F(B')$ is that B'' is a fixed point of the map $B'' \mapsto F((P^\infty)^{B''})$ of the flag manifold of $F(P^\infty)$ into itself. This map may be identified with the map induced by $F' : L_J \to L_J$ (see 4.3) on the flag manifold of L_J hence it has at least one fixed point. Thus there exist $(B', B'') \in \mathcal{B} \times \mathcal{B}$ such that $B'' = F(B')$, $B' \subset P^\infty$, $\mathrm{pos}(B', B'') = w$. Then $B' \in \mathcal{B}_a$ and $B' \subset P$ (since $P^\infty \subset P$). This proves (b).

4.11. Let $\mathbf{t} = (J_n, w_n)_{n \geq 0} \in \mathcal{T}'(J, \mathbf{c})$. We set $w = w_\infty \in {}^J W$. We show:

(a) *Let $b \in W_{J_\infty}$. Let $\mathcal{L} \in \mathcal{S}(T)^{bwF}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}_1}_J)$ such that $A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}_1, w\mathbf{c}(b)!}(b^*\mathcal{L})_{w\mathbf{c}(b)})$. Then $\vartheta^* A \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}, bw!}\mathcal{L}_{bw})$.*

The result makes sense since $\mathcal{B}_{\mathbf{t}, bw} = \mathcal{B}_{bw}$ by 4.10(a) and $\mathcal{B}_{\mathbf{t}_1, w\mathbf{c}(b)} = \mathcal{B}_{w\mathbf{c}(b)}$ (this follows from 4.10(a) applied to $\mathbf{t}_1, w, w\mathbf{c}(b)w^{-1}$ instead of $\mathbf{t}, w, b$; note that $w\mathbf{c}(J_\infty)w^{-1} = J_\infty$ by 4.2(e) hence $w\mathbf{c}(b)w^{-1} \in W_{J_\infty}$). This shows also that $l(w\mathbf{c}(b)) = l(w) + l(\mathbf{c}(b))$. (Since $w \in {}^J W$ we have $l((w\mathbf{c}(b)w^{-1})w) = l(w\mathbf{c}(b)w^{-1}) + l(w)$.) We define $h : \mathcal{B}_{bw} \to \mathcal{B}_{w\mathbf{c}(b)}$ by $B' \mapsto B''$ with B'' defined by $\mathrm{pos}(B', B'') = b, \mathrm{pos}(B'', F(B')) = w$. This is an isomorphism whose inverse $\mathcal{B}_{w\mathbf{c}(b)} \to \mathcal{B}_{bw}$ is given by $B'' \mapsto B'$ with B' defined by $\mathrm{pos}(B', B'') = b, \mathrm{pos}(B'', F(B')) = w$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{bw} & \xrightarrow{h} & \mathcal{B}_{w\mathbf{c}(b)} \\ {\scriptstyle \xi_{\mathbf{t}, bw}}\downarrow & & {\scriptstyle \xi_{\mathbf{t}_1, w\mathbf{c}(b)}}\downarrow \\ \mathcal{P}^{\mathbf{t}}_J & \xrightarrow{\vartheta} & \mathcal{P}^{\mathbf{t}_1}_{J_1} \end{array}$$

where the horizontal maps are isomorphism. From the definitions we see that $h^*((b^*\mathcal{L})_{w\mathbf{c}(b)}) = \mathcal{L}_{bw}$. The result follows.

4.12. Let $\mathbf{t} = (J_n, w_n)_{n \geq 0} \in \mathcal{T}'(J, \mathbf{c})$. We set $w = w_\infty \in {}^J W$. Let $d = \dim \mathcal{P}^{\mathbf{t}}_J$. Let $\Lambda = L^{F'}_{J_\infty}$. Let M be a finite dimensional irreducible Λ-module over $\bar{\mathbf{Q}}_l$. Let $\underline{M}[d] \in \mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ be as in 4.4(c). We show:

(a) *there exists $b \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{bwF}$ such that $\underline{M}[d] \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}, bw!}\mathcal{L}_{bw})$;*

(b) *there exists $b \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{bwF}$ such that $\underline{M}^\sharp[d] \dashv_\Gamma {}^pH^{\cdot}(\Pi^{(bw)}_!\mathcal{L}_{(bw)})$.*

We prove (a). More generally, we show that for any $n \geq 0$:

(c) *there exists $b_n \in W_{J_\infty}$ and $\mathcal{L} \in \mathcal{S}(T)^{b_n wF}$ such that $\underline{M}[d]$ (regarded as an object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}_n}_{J_n})$) satisfies $\underline{M}[d] \dashv_\Gamma {}^pH^{\cdot}(\xi_{\mathbf{t}_n, b_n w!}\mathcal{L}_{b_n w})$.*

If (c) holds for $n = 1$, then, by 4.11(a) it holds for $n = 0$. Similarly, if (c) holds for some $n \geq 1$, then it holds for $n - 1$. Hence it is enough to prove (c) for large n. Thus we may assume that $J_0 = J_1 = \cdots = J$ and $w_0 = w_1 = \cdots = w$. By [DL, 7.7] applied to Λ, there exists $a_1 \in W_{J_\infty}$ such that $M \dashv_\Lambda {}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l)$ (in $\mathcal{M}_\Lambda($ point$)$). (The reference to [DL] could be replaced by a self-contained proof, see 7.9.) By definition, $f^*\underline{M}[d] = e^*M[d]$ with f as in 4.3, e as in 4.7. We have $e^*M[d] \dashv_{\Gamma\times\Lambda} e^*({}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l))[d]$ hence $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} e^*({}^pH^\cdot(e'_!\bar{\mathbf{Q}}_l))[d]$ in $\mathcal{M}_{\Gamma\times\Lambda}(\tilde{\mathcal{P}}^{\mathbf{t}}_J)$. Since $\tilde{\mathcal{P}}^{\mathbf{t}}_J$ is smooth of pure dimension d we have $e^*({}^pH^i(e'_!\bar{\mathbf{Q}}_l))[d] = {}^pH^{i+d}(e^*e'_!\bar{\mathbf{Q}}_l)$. Since

$$e^*e'_!\bar{\mathbf{Q}}_l = p_{1!}\bar{\mathbf{Q}}_l = f^*(\xi'\xi)_!\bar{\mathbf{Q}}_l = f^*\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l$$

(notation of 4.7 with $a = a_1w$) we have $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} {}^pH^\cdot(f^*\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$. Since f is a finite principal covering we also have $f^*\underline{M}[d] \dashv_{\Gamma\times\Lambda} f^*({}^pH^\cdot(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l))$. It follows that $\underline{M}[d] \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}f_{a!}\bar{\mathbf{Q}}_l)$. Now $f_{a!}\bar{\mathbf{Q}}_l = \oplus_{\mathcal{L}}\mathcal{L}_a$ where $\mathcal{L}$ runs over the local systems in $\mathcal{S}(T)^{aF}$ (up to isomorphism). Hence for some such $\mathcal{L}$ we have $\underline{M}[d] \dashv_\Gamma {}^pH^\cdot(\xi_{\mathbf{t},a!}\mathcal{L}_a)$. This proves (a).

We prove (b). Let $b, \mathcal{L}$ be such that (a) holds. Let $K = \Pi_!^{(bw)}\mathcal{L}_{(bw)}$, $K' = \xi_{\mathbf{t},bw!}\mathcal{L}_{bw}$. We have $\underline{M}[d] \dashv_\Gamma {}^pH^\cdot(K')$. Let $\kappa : \mathcal{P}^{\mathbf{t}}_J \to \mathcal{P}_J$ be the inclusion. We have $K = \kappa_!K'$. Let C be the closure of $\mathcal{P}^{\mathbf{t}}_J$ in $\mathcal{P}_J$. Let $\kappa' : \mathcal{P}^{\mathbf{t}}_J \to C$, $\kappa'' : C \to \mathcal{P}_J$ be the inclusions. Let $\tilde{M} = IC(C, \underline{M})[d]$. Let $K_1 = \kappa'_!K$. We show:

(d) $$\tilde{M} \dashv_\Gamma {}^pH^\cdot(K_1).$$

We have $\kappa'^*K_1 = K'$ and $\kappa'^*({}^pH^j(K_1)) = {}^pH^j(K')$. Let $0 = X_0 \subset X_1 \subset \ldots \subset X_m = {}^pH^j(K_1)$ be a composition series of ${}^pH^j(K_1)$ in $\mathcal{M}_\Gamma(C)$. Aplying κ'^* to the exact sequence $0 \to X_{i-1} \to X_i \to X_i/X_{i-1} \to 0$ (where $1 \leq i \leq m$) we get an exact sequence $0 \to \kappa'^*(X_{i-1}) \to \kappa'^*(X_i) \to \kappa'^*(X_i/X_{i-1}) \to 0$ in $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_\Gamma)$. Since X_i/X_{i-1} is simple and $\mathcal{P}^{\mathbf{t}}_\Gamma$ is open dense in C we see that $\kappa'^*(X_i/X_{i-1})$ is either simple or 0. It follows that any composition factor of $\kappa'^*(X_m)$ is isomorphic to $\kappa'^*(X_i/X_{i-1})$ for some i. In particular, $\underline{M}[d] \cong \kappa'^*(X_i/X_{i-1})$ for some i. It follows that $\tilde{M} \cong X_i/X_{i-1}$ for some i. Thus (d) holds.

Applying $\kappa''_!$ to (d) we obtain

$$\underline{M}^\sharp[d] = \kappa''_!\tilde{M} \dashv_\Gamma \kappa''^p_!H^\cdot(K_1) = {}^pH^\cdot(\kappa''_!K_1) = {}^pH^\cdot(K).$$

This proves (b).

Theorem 4.13. *Let K be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. Then $K \in \mathbb{S}'(\mathcal{P}_J)$ (see 3.7) if and only if $K \in \mathbb{S}(\mathcal{P}_J)$.*

If $K \in \mathbb{S}(\mathcal{P}_J)$, then by 4.12(b) it satisfies 3.2(i). If K satisfies 3.2(v), then by 4.9(a), $K \in \mathbb{S}(\mathcal{P}_J)$. This completes the proof.

Theorem 4.14. *Let $\mathbf{t} = (J_n, w_n)_{n\geq 0} \in \mathcal{T}'(J, \mathbf{c})$. Let $h : \mathcal{P}^{\mathbf{t}}_J \to \mathcal{P}_J$ be the inclusion. Let $A \in \mathbb{S}(\mathcal{P}_J)$. Let A' be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}^{\mathbf{t}}_J)$ such that $A' \dashv_\Gamma {}^pH^\cdot(h^*A)$. Then $A' \in \mathbb{S}(\mathcal{P}^{\mathbf{t}}_J)$.*

By assumption, A is as in 4.8(a). The result now follows from 4.8(a).

5. Some computations in the Weyl group

5.1. Let $\mathcal{L} \in \mathcal{S}(T)$. Define $\tilde{l} : W \to \mathbf{N}$ by $\tilde{l}(w) = |(\alpha \in R^+_{\mathcal{L}}; w(\alpha) \in R^-\}|$.

Let $\mathbf{s} = (s_1, \ldots s_r)$ be a sequence in $\mathbf{I} \cup \{1\}$. Let

$$\mathcal{I}_{\mathbf{s}} = \{i \in [1, r]; s_i \neq 1, s_1s_2 \ldots s_i \ldots s_2s_1 \in W_{\mathcal{L}}\}.$$

This agrees with the definition in 2.4.

Lemma 5.2. *We have* $|\mathcal{I}_{\mathbf{s}}| \ge \tilde{l}(s_r \dots s_1)$ *with equality if* $l(s_r \dots s_1) = l(s_1) + \dots + l(s_r)$.

Let

$$X = \{\alpha \in R^+_{\mathcal{L}}; (s_r \dots s_1)(\alpha) \in R^-\},$$
$$X' = \{\alpha \in R^+_{\mathcal{L}}; \alpha = s_1 s_2 \dots s_{i-1}(\alpha_{s_i}) \text{ for some } i \in [1,r] \text{ such that } s_i \ne 1\}.$$

We have $X \subset X'$. We have $|X| = \tilde{l}(s_r \dots s_1)$ hence $\tilde{l}(s_r \dots s_1) \le |X'|$. Define $f : \mathcal{I}_{\mathbf{s}} \to R_{\mathcal{L}}$ by $f(i) = s_1 s_2 \dots s_{i-1}(\alpha_{s_i})$; then $X' = f(\mathcal{I}_{\mathbf{s}}) \cap R^+_{\mathcal{L}}$. Hence $|X'| \le |f(\mathcal{I}_{\mathbf{s}})| \le |\mathcal{I}_{\mathbf{s}}|$ and the desired inequality is proved. Assume now that $l(s_r \dots s_1) = l(s_1) + \dots + l(s_r)$. Then $s_1 s_2 \dots s_{i-1}(\alpha_{s_i}) (i \in [1,r], s_i \ne 1)$ are distinct in R^+. Hence for $i \in \mathcal{I}_{\mathbf{s}}$, $s_1 s_2 \dots s_{i-1}(\alpha_{s_i})$ are distinct elements of X. Thus $|\mathcal{I}_{\mathbf{s}}| \le |X|$. It follows that $|\mathcal{I}_{\mathbf{s}}| = |X|$.

Lemma 5.3. *Let* $j \in \mathcal{I}_{\mathbf{s}}$. *Let* $\mathbf{s}(j) = (s'_1, s'_2, \dots, s'_r)$ *with* $s'_i = s_i$ *for* $i \ne j$, $s'_j = 1$. *We have* $\mathcal{I}_{\mathbf{s}(j)} = \mathcal{I}_{\mathbf{s}} - \{j\}$.

Let $h \in \mathcal{I}_{\mathbf{s}} - \{j\}$. We have $s_1 s_2 \dots s_h \dots s_2 s_1 \in W_{\mathcal{L}}$. Hence if $j < h$, we have

$$s_1 s_2 \dots \hat{s}_j \dots s_h \dots \hat{s}_j \dots s_2 s_1$$
$$= (s_1 s_2 \dots s_j \dots s_2 s_1)(s_1 s_2 \dots s_h \dots s_2 s_1)(s_1 s_2 \dots s_j \dots s_2 s_1) \in W_{\mathcal{L}},$$

so that $h \in \mathcal{I}_{\mathbf{s}(j)}$. ($\hat{}$ denotes an omitted symbol.) If $j > h$, then $h \in \mathcal{I}_{\mathbf{s}(j)}$ is obvious.

Conversely, assume that $h \in \mathcal{I}_{\mathbf{s}(j)}$. Clearly, $h \ne j$. Assume first that $j < h$. We have

$$s_1 s_2 \dots s_h \dots s_2 s_1$$
$$= (s_1 s_2 \dots s_j \dots s_2 s_1)(s_1 s_2 \dots \hat{s}_j \dots s_h \dots \hat{s}_j \dots s_2 s_1)(s_1 s_2 \dots s_j \dots s_2 s_1) \in W_{\mathcal{L}}.$$

Hence $h \in \mathcal{I}_{\mathbf{s}}$. If $j > h$, then it is clear that $h \in \mathcal{I}_{\mathbf{s}}$. The lemma is proved.

Lemma 5.4. *Let* h *be the smallest element of* $\mathcal{I}_{\mathbf{s}}$. *Then* $s_1 s_2 \dots s_h \dots s_2 s_1 \in \mathbf{I}_{\mathcal{L}}$.

Let $\mathbf{s}' = (s_1, s_2, \dots, s_h, \dots, s_2, s_1)$. We have
$s_1 s_2 \dots s_h \dots s_2 s_1 \in W_{\mathcal{L}}$,
$s_1 s_2 \dots s_{h-1} \dots s_2 s_1 \notin W_{\mathcal{L}}$ or $s_{h-1} = 1$,
$s_1 s_2 \dots s_{h-2} \dots s_2 s_1 \notin W_{\mathcal{L}}$ or $s_{h-2} = 1$, ...

Hence the middle term in $\mathbf{s}'$ has index in $\mathcal{I}_{\mathbf{s}'}$ but all terms preceding it have index not in $\mathcal{I}_{\mathbf{s}'}$. We show that the term in $\mathbf{s}'$ immediately following the middle term has index not in $\mathcal{I}_{\mathbf{s}'}$. Otherwise it would be $\ne 1$ and

$$s_1 s_2 \dots s_{h-1} s_h s_{h-1} s_h s_{h-1} \dots s_2 s_1 \notin W_{\mathcal{L}}.$$

Multiplying on the left and right by $s_1 s_2 \dots s_h \dots s_2 s_1$ we find

$$s_1 s_2 \dots s_{h-1} \dots s_2 s_1 \in W_{\mathcal{L}},$$

a contradiction. Similarly we see that all terms in $\mathbf{s}'$ following the middle term have index not in $\mathcal{I}_{\mathbf{s}'}$. Thus $|\mathcal{I}_{\mathbf{s}'}| = 1$. By Lemma 5.2 we have

$$\tilde{l}(s_1 s_2 \dots s_h \dots s_2 s_1) \le 1.$$

Since $s_1 s_2 \dots s_h \dots s_2 s_1 \in W_{\mathcal{L}} - \{1\}$, it follows that $s_1 s_2 \dots s_h \dots s_2 s_1 \in \mathbf{I}_{\mathcal{L}}$. The lemma is proved.

5.5. We write the elements of $\mathcal{I}_{\mathbf{s}}$ in ascending order: $i_1 < i_2 < \dots < i_b$. Define a sequence $\mathbf{S} = (S_1, S_2, \dots, S_b)$ in W by

$S_1 = s_1 s_2 \dots s_{i_1} \dots s_2 s_1$,
$S_2 = s_1 s_2 \dots \hat{s}_{i_1} \dots s_{i_2} \dots \hat{s}_{i_1} \dots s_2 s_1, \dots$,
$S_b = s_1 s_2 \dots \hat{s}_{i_1} \dots \hat{s}_{i_2} \dots \hat{s}_{i_{b-1}} \dots s_{i_b} \dots \hat{s}_{i_{b-1}} \dots \hat{s}_{i_2} \dots \hat{s}_{i_1} \dots s_2 s_1$,
$\omega = s_1 \dots \hat{s}_{i_1} \dots \hat{s}_{i_2} \dots \hat{s}_{i_b} \dots s_r$.

Lemma 5.6. *(a) $S_1, S_2, \dots, S_b$ belong to $\mathbf{I}_{\mathcal{L}}$.*
(b) $\tilde{l}(\omega^{-1}) = 0$.
(c) $s_1 s_2 \dots s_r = S_1 S_2 \dots S_b \omega$.

We use induction on b. Assume first that $b = 0$. By 5.2 we have $\tilde{l}(s_r \dots s_1) = 0$ that is $\tilde{l}(\omega^{-1}) = 0$ and the lemma is clear. Assume now that $b \geq 1$ and that the lemma holds for $b - 1$. Let $h = i_1$. By 5.3 we have $\mathcal{I}_{\mathbf{s}(h)} = \mathcal{I}_{\mathbf{s}} - \{h\}$. By the induction hypothesis, $S_2, S_3, \dots, S_b$ belong to $\mathbf{I}_{\mathcal{L}}$, $s_1 \dots \hat{s}_{i_1} \dots s_r = S_2 S_3 \dots S_b \omega$, $\tilde{l}(\omega^{-1}) = 0$. By 5.4 we have $S_1 \in \mathbf{I}_{\mathcal{L}}$. It follows that

$$s_1 s_2 \dots s_r = (s_1 \dots s_{i_1} \dots s_1)(s_1 \dots \hat{s}_{i_1} \dots s_r) = S_1 S_2 \dots S_b \omega.$$

The lemma is proved.

5.7. Let $W'_{\mathcal{L}} = \{w \in W; w^* \mathcal{L} \cong \mathcal{L}\}$, a subgroup of W. Let $W'^0_{\mathcal{L}} = \{w \in W'_{\mathcal{L}}; w(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}\}$, a subgroup of $W'_{\mathcal{L}}$. Note that $W'_{\mathcal{L}} = W'^0_{\mathcal{L}} W_{\mathcal{L}}$ (semidirect product with $W_{\mathcal{L}}$ normal).

In the remainder of this section we fix an automorphism c of finite order of T that induces a permutation of R, one of $\check{R}$ and one of R^+.

Lemma 5.8. *Assume that $F_0^*(s_1 s_2 \dots s_r c)^* \mathcal{L} \cong \mathcal{L}$. Let $c' = \omega c$ so that $s_1 s_2 \dots s_r c = S_1 S_2 \dots S_b c'$ and $F_0^*(S_1 S_2 \dots S_b c')^* \mathcal{L} \cong \mathcal{L}$. Then c' is an automorphism of finite order of T that induces a permutation of $R_{\mathcal{L}}$, one of $\check{R}_{\mathcal{L}}$ and one of $R^+_{\mathcal{L}}$.*

We first show:

(i) $$c'(R_{\mathcal{L}}) = R_{\mathcal{L}}.$$

It is enough to show that for any coroot $\kappa : \mathbf{k}^* \to T$ such that $\kappa^*(\mathcal{L}) \cong \bar{\mathbf{Q}}_l$ we have $(c'\kappa)^* \mathcal{L} \cong \bar{\mathbf{Q}}_l$. From $F_0^*(S_1 S_2 \dots S_b c')^* \mathcal{L} \cong \mathcal{L}$ and $S_1 S_2 \dots S_r \in W_{\mathcal{L}}$ we deduce $F_0^* c'^* \mathcal{L} \cong \mathcal{L}$. Hence $\kappa^* F_0^* c'^* \mathcal{L} \cong \kappa^* \mathcal{L} \cong \bar{\mathbf{Q}}_l$. Now $F_0 \kappa = \kappa F'_0$ where $F'_0 : \mathbf{k}^* \to \mathbf{k}^*$ is $x \mapsto x^q$. Hence $F'^*_0 \kappa^* c'^* \mathcal{L} = \bar{\mathbf{Q}}_l = F'^*_0 \bar{\mathbf{Q}}_l$ and $\kappa^* c'^* \mathcal{L} = \bar{\mathbf{Q}}_l$. This proves (i).

By 5.6(b) we have $\omega^{-1}(R^+_{\mathcal{L}}) \subset R^+$. Moreover, $c^{-1}(R^+) = R^+$ hence $c^{-1}\omega^{-1}(R^+_{\mathcal{L}}) \subset R^+$. Using (i) we see that $c'^{-1}(R^+_{\mathcal{L}}) \subset R^+_{\mathcal{L}}$ hence $c'(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. The lemma is proved.

5.9. Let $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{\tilde{r}})$ be a second sequence in $\mathbf{I}$. Then the subset $\mathcal{I}_{\tilde{\mathbf{s}}}$ is defined in terms of $\tilde{\mathbf{s}}, \mathcal{L}$ in the same way as $\mathcal{I}_{\mathbf{s}}$ is defined in terms of $\mathbf{s}, \mathcal{L}$.

We write the elements of $\mathcal{I}_{\tilde{\mathbf{s}}}$ in ascending order: $j_1 < j_2 < \dots < j_{\tilde{b}}$. Define $\tilde{\mathbf{S}} = (\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_{\tilde{b}}), \tilde{\omega}$ in terms of $\tilde{\mathbf{s}}, \mathcal{L}$ in the same way that $\mathbf{S}, \omega$ are defined in 5.5 in terms of $\mathbf{s}, \mathcal{L}$. As in 5.6 we have $\tilde{S}_i \in \mathbf{I}_{\mathcal{L}}$ for $i \in [1, \tilde{b}]$ and $\tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_{\tilde{r}} = \tilde{S}_1 \tilde{S}_2 \dots \tilde{S}_{\tilde{b}} \tilde{\omega}$. We assume that

$$F_0^*([\mathbf{s}]c)^* \mathcal{L} \cong \mathcal{L}, F_0^*([\tilde{\mathbf{s}}]c)^* \mathcal{L} \cong \mathcal{L}.$$

From 5.8 we see that ωc and $\tilde{\omega} c$ are automorphisms of finite order of T that induce a permutation of $R_{\mathcal{L}}$, one of $\check{R}_{\mathcal{L}}$ and one of $R^+_{\mathcal{L}}$. Moreover, we have

$$F_0^*(S_1 S_2 \dots S_b \omega c)^* \mathcal{L} \cong \mathcal{L}, F_0^*(\tilde{S}_1 \tilde{S}_2 \dots \tilde{S}_{\tilde{b}} \tilde{\omega} c)^* \mathcal{L} \cong \mathcal{L},$$

or equivalently $F_0^*(\omega c)^*\mathcal{L} \cong \mathcal{L}, F_0^*(\tilde{\omega}c)^*\mathcal{L} \cong \mathcal{L}$. Let

$$\mathfrak{F} = \{f \in W'^0_{\mathcal{L}}; f^{-1}\omega c f = \tilde{\omega}c\}.$$

Now for $f \in \mathfrak{F}$ we have $f\mathbf{I}_{\mathcal{L}}f^{-1} = \mathbf{I}_{\mathcal{L}}$; we set

$${}^f\tilde{\mathbf{S}} = (f\tilde{S}_1f^{-1}, f\tilde{S}_2f^{-1}, \dots f\tilde{S}_{\tilde{b}}f^{-1}),$$

a sequence in $\mathbf{I}_{\mathcal{L}}$. Since $F_0^*(\omega c)^*\mathcal{L} \cong \mathcal{L}$, we have

$$F_0^*((f\tilde{S}_1f^{-1})(f\tilde{S}_2f^{-1})\dots(f\tilde{S}_{\tilde{b}}f^{-1})\omega c)^*\mathcal{L} \cong \mathcal{L}.$$

We set $\rho = r + \tilde{r}$. Let $\mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$ be the set of all sequences $(a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1}a_j \in \{1, \tilde{s}_j\}$ for $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$;

$a_{j-1}^{-1}a_j = \tilde{s}_j$ for $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$;

$a_{\tilde{r}+i}a_{\tilde{r}+i-1}^{-1} \in \{1, s_i\}$ for $i \in \mathcal{I}_{\mathbf{s}}$;

$a_{\tilde{r}+i}a_{\tilde{r}+i-1}^{-1} = s_i$ for $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$;

$a_\rho = ca_0c^{-1}$;

$a_0^*\mathcal{L} \cong \mathcal{L}$.

Replacing here $W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}$ by $W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}$, we obtain for any $f \in \mathfrak{F}$ a set $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. From the definition, $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$ consists of all sequences $(A_0, A_1, \dots, A_{b+\tilde{b}})$ in $W_{\mathcal{L}}$ such that

$A_{j-1}^{-1}A_j \in \{1, f\tilde{S}_jf^{-1}\}$ for $j \in [1, \tilde{b}]$;

$A_{\tilde{b}+i}A_{\tilde{b}+i-1}^{-1} \in \{1, S_i\}$ for $i \in [1, b]$;

$A_{b+\tilde{b}} = (\omega c)A_0(\omega c)^{-1}$.

We now state the following result.

Proposition 5.10. *There is a canonical bijection*

$$\Psi : \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}) \xrightarrow{\sim} \sqcup_{f\in\mathfrak{F}}\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}).$$

Let $(a_0, a_1, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$. Consider the product

$$a_{\tilde{r}}(a_{\tilde{r}+i_1}^{-1}a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} = (a_{\tilde{r}}a_{\tilde{r}+1}^{-1})(a_{\tilde{r}+1}\alpha_{\tilde{r}+2}^{-1})\dots(a_{\tilde{r}+i_1-2}a_{\tilde{r}+i_1-1}^{-1})$$
$$\times (a_{\tilde{r}+i_1-1}a_{\tilde{r}+i_1}^{-1})(a_{\tilde{r}+i_1-1}a_{\tilde{r}+i_1-2}^{-1})\dots(a_{\tilde{r}+2}a_{\tilde{r}+1}^{-1})(a_{\tilde{r}+1}a_{\tilde{r}}^{-1}).$$

The right hand side is equal to $s_1s_2\dots s_{i_1-1}xs_{i_1-1}\dots s_2s_1$ where x is either s_{i_1} or 1. Hence it is equal to $s_1s_2\dots s_{i_1-1}s_{i_1}s_{i_1-1}\dots s_2s_1$ or to 1. Thus we have

$$a_{\tilde{r}}(a_{\tilde{r}+i_1}^{-1}a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} \in \{S_1, 1\}.$$

Similarly,

$$a_{\tilde{r}}(a_{\tilde{r}+i_1-1}^{-1}a_{\tilde{r}+i_1})(a_{\tilde{r}+i_2-1}^{-1}a_{\tilde{r}+i_2})\dots(a_{\tilde{r}+i_{e-1}-1}^{-1}a_{\tilde{r}+i_{e-1}})(a_{\tilde{r}+i_e}^{-1}a_{\tilde{r}+i_e-1})$$
$$\times (a_{\tilde{r}+i_{e-1}}^{-1}a_{\tilde{r}+i_{e-1}-1})\dots(a_{\tilde{r}+i_2}^{-1}a_{\tilde{r}+i_2-1})(a_{\tilde{r}+i_1}^{-1}a_{\tilde{r}+i_1-1})a_{\tilde{r}}^{-1} \in \{S_e, 1\}$$

for $e \in [1, b]$,

$$a_0^{-1}(a_{j_1-1}a_{j_1}^{-1})(a_{j_2-1}a_{j_2}^{-1})\dots(a_{j_{e-1}-1}a_{j_{e-1}}^{-1})(a_{j_e}a_{j_e-1}^{-1})$$
$$(a_{j_{e-1}}a_{j_{e-1}-1}^{-1})\dots(a_{j_2}a_{j_2-1}^{-1})(a_{j_1}a_{j_1-1}^{-1}) \in \{\tilde{S}_e, 1\}$$

for $e \in [1, \tilde{b}]$,

$$\omega = a_{\tilde{r}}(a_{\tilde{r}+i_1-1}^{-1}a_{\tilde{r}+i_1})(a_{\tilde{r}+i_2-1}^{-1}a_{\tilde{r}+i_2})\dots(a_{\tilde{r}+i_b-1}^{-1}a_{\tilde{r}+i_b})a_\rho^{-1},$$

$$\tilde{\omega} = a_0^{-1}(a_{j_1-1}a_{j_1}^{-1})(a_{j_2-1}a_{j_2}^{-1})\dots(a_{j_{\tilde{b}-1}}a_{j_{\tilde{b}}}^{-1})a_{\tilde{r}}.$$

Setting

(a) $\hat{a}_e = a_{j_e} a_{j_e-1}^{-1}$ for $e \in [1, \tilde{b}]$, $\hat{a}_{\tilde{b}+e} = a_{\tilde{r}} a_{\tilde{r}+i_e}^{-1} a_{\tilde{r}+i_e-1} a_{\tilde{r}}^{-1}$ for $e \in [1, b]$

we see that

(b) $\hat{a}_{\tilde{b}+1}^{-1} \hat{a}_{\tilde{b}+2}^{-1} \dots \hat{a}_{\tilde{b}+e-1}^{-1} \hat{a}_{\tilde{b}+e} \hat{a}_{\tilde{b}+e-1} \dots \hat{a}_{\tilde{b}+2} \hat{a}_{\tilde{b}+1} \in \{S_e, 1\}$ for $e \in [1, b]$,

(c) $a_0^{-1} \hat{a}_1^{-1} \hat{a}_2^{-1} \dots \hat{a}_{e-1}^{-1} \hat{a}_e \hat{a}_{e-1} \dots \hat{a}_2 \hat{a}_1 a_0 \in \{\tilde{S}_e, 1\}$ for $e \in [1, \tilde{b}]$,

(d) $$\omega = \hat{a}_{\tilde{b}+1}^{-1} \hat{a}_{\tilde{b}+2}^{-1} \dots \hat{a}_{\tilde{b}+b}^{-1} a_{\tilde{r}} a_\rho^{-1},$$

(e) $$\tilde{\omega} = a_0^{-1} \hat{a}_1^{-1} \hat{a}_2^{-1} \dots \hat{a}_{\tilde{b}}^{-1} a_{\tilde{r}}.$$

Since $S_e = S_e^{-1} \in W_{\mathcal{L}}$, $\tilde{S}_e = \tilde{S}_e^{-1} \in W_{\mathcal{L}}$, it follows by induction on e that

(f) $$\hat{a}_{\tilde{b}+e} = \hat{a}_{\tilde{b}+e}^{-1} \in W_{\mathcal{L}} \text{ for } e \in [1, b],$$

$$a_0^{-1} \hat{a}_e a_0 = a_0^{-1} \hat{a}_e^{-1} a_0 \in W_{\mathcal{L}} \text{ for } e \in [1, \tilde{b}].$$

Since a_0 normalizes $W_{\mathcal{L}}$ it follows that

(g) $$\hat{a}_e = \hat{a}_e^{-1} \in W_{\mathcal{L}} \text{ for } e \in [1, \tilde{b}].$$

Since $a_0^* \mathcal{L} \cong \mathcal{L}$ we can write uniquely

(h) $$a_0 = A_0 f \text{ with } A_0 \in W_{\mathcal{L}}, f \in W'^0_{\mathcal{L}}.$$

We set

(i) $$A_e = \hat{a}_e \dots \hat{a}_2 \hat{a}_1 a_0 f^{-1} \quad (e \in [1, \tilde{b}]),$$

(j) $$A_{\tilde{b}+e} = \hat{a}_{\tilde{b}+1} \hat{a}_{\tilde{b}+2} \dots \hat{a}_{\tilde{b}+e} \hat{a}_{\tilde{b}} \dots \hat{a}_2 \hat{a}_1 a_0 f^{-1} \quad (e \in [1, b]).$$

From (f),(g),(h) we see that $A_e \in W_{\mathcal{L}}$ for $e \in [0, b + \tilde{b}]$. From the definitions we have

$$A_{e-1}^{-1} A_e = f S_e f^{-1} \quad (e \in [1, \tilde{b}]),$$
$$A_{\tilde{b}+e-1} A_{\tilde{b}+e}^{-1} = S_e \quad (e \in [1, b]).$$

We have

$$A_{\tilde{b}+b} = \hat{a}_{\tilde{b}+1} \hat{a}_{\tilde{b}+2} \dots \hat{a}_{\tilde{b}+b} \hat{a}_{\tilde{b}} \dots \hat{a}_2 \hat{a}_1 a_0 f^{-1} = \omega a_\rho \tilde{\omega}^{-1} f^{-1}$$
$$= (\omega c)(c^{-1} a_\rho c)(\tilde{\omega} c)^{-1} f^{-1} = (\omega c) a_0 (\tilde{\omega} c)^{-1} f^{-1} = (\omega c) A_0 f (\tilde{\omega} c)^{-1} f^{-1}.$$

In particular, $(\omega c) w f (\tilde{\omega} c)^{-1} f^{-1} = w'$ for some $w \in W_{\mathcal{L}}$. We have $(\omega c) w = w_1 (\omega c)$ for some $w_1 \in W_{\mathcal{L}}$ hence $(\omega c) f (\tilde{\omega} c)^{-1} f^{-1} = w_1^{-1} w'$ belongs to $W_{\mathcal{L}} \cap W'^0_{\mathcal{L}} = \{1\}$. Thus $(\omega c) f (\tilde{\omega} c)^{-1} f^{-1} = 1$; that is, $f \in \mathfrak{F}$. We also see that

$$A_{\tilde{b}+b} = (\omega c) A_0 f (\tilde{\omega} c)^{-1} f^{-1} = (\omega c) A_0 (\omega c)^{-1}.$$

Thus to each $(a_0, a_1, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$ we have associated $f \in \mathfrak{F}$ and an element $(A_0, A_1, \dots, A_{b+\tilde{b}})$ in $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. This defines the map Ψ in the proposition.

Conversely, assume that we are given $f \in \mathfrak{F}$ and an element $(A_0, A_1, \dots, A_{b+\tilde{b}})$ in $\mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}})$. We will construct a sequence $(a_0, a_1, \dots, a_\rho)$ in W as follows.

We set $a_0 = A_0 f$. We define $\hat{a}_1, \hat{a}_2, \dots, \hat{a}_{\tilde{b}}$ inductively so that (i) holds. We define $\hat{a}_{\tilde{b}+1}, \hat{a}_{\tilde{b}+2}, \dots, \hat{a}_{\tilde{b}+b}$ inductively so that (j) holds. We define the elements $a_0, a_1, a_2, \dots, a_{\tilde{r}}$ by induction as follows. Note that a_0 is already defined. Assume that $a_0, a_1, \dots, a_{u-1}$ are already defined for some $u \in [1, \tilde{r}]$. If $u = j_e$ for some $e \in [1, \tilde{b}]$, we set $a_u = \hat{a}_e a_{u-1}$. If $u \notin \mathcal{I}_{\tilde{\mathbf{s}}}$, we set $a_u = a_{u-1} \tilde{s}_u$. This completes the definition of $a_0, a_1, a_2, \dots, a_{\tilde{r}}$. We define the elements $a_{\tilde{r}}, a_{\tilde{r}+1}, \dots, a_{\tilde{r}+r}$ by induction as follows. Note that $a_{\tilde{r}}$ is already defined. Assume that $a_{\tilde{r}}, a_{\tilde{r}+1}, \dots, a_{\tilde{r}+u-1}$ are already defined for some $u \in [1, r]$. If $u = i_e$ for some $e \in [1, b]$ we set $a_{\tilde{r}+u} = a_{\tilde{r}+u-1} a_{\tilde{r}}^{-1} \hat{a}_{\tilde{b}+e} a_{\tilde{r}}$. If $u \notin \mathcal{I}_{\mathbf{s}}$, we set $a_{\tilde{r}+u} = s_u a_{\tilde{r}+u-1}$. This completes the definition of $a_{\tilde{r}}, a_{\tilde{r}+1}, \dots, a_{\tilde{r}+r}$.

From the definitions we see that $(a_0, a_1, a_2, \dots, a_\rho) \in \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}})$. We have thus constructed a map

$$\sqcup_{f \in \mathfrak{F}} \mathcal{A}(W_{\mathcal{L}}, \omega c, \bar{\mathbf{Q}}_l, \mathbf{S}, {}^f\tilde{\mathbf{S}}) \to \mathcal{A}(W, c, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}).$$

From the definitions we see that this is an inverse to Ψ. The proposition is proved.

6. A basis for a space of intertwining operators

6.1. Let $\mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{S}(T)$. Let $\mathbf{s} = (s_1, s_2, \dots, s_r)$, $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{\tilde{r}})$ be two sequences in $\mathbf{I}$ such that $([\mathbf{s}]F)^* \mathcal{L} \cong \mathcal{L}, ([\tilde{\mathbf{s}}]F)^* \tilde{\mathcal{L}} \cong \tilde{\mathcal{L}}$. Define $\mathcal{I}_{\mathbf{s}} \subset [1, r]$ in terms of $\mathbf{s}, \mathcal{L}$ as in 2.4 or 5.1; define $\mathcal{I}_{\tilde{\mathbf{s}}} \subset [1, \tilde{r}]$ similarly in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}$.

We set $\rho = r + \tilde{r}$. Until the end of 6.13 we fix a sequence $\mathbf{a} = (a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1} a_j \in \{1, \tilde{s}_j\}$ for $j \in [1, \tilde{r}]$;

$a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} \in \{1, s_i\}$ for $i \in [1, r]$;

if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{j-1}\tilde{s}_j > a_{j-1}$, then $a_{j-1}^{-1} a_j = \tilde{s}_j$;

if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$, $s_i a_{\tilde{r}+i-1} > a_{\tilde{r}+i-1}$, then $a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} = s_i$;

$a_\rho = \mathbf{c}(a_0)$.

We define representatives $\ddot{a}_i$ for a_i in $N(T)$ ($i \in [0, \rho]$) as follows. We set $\ddot{a}_0 = \dot{a}_0$. For $j \in [1, \tilde{r}]$ we define $\ddot{a}_j$ inductively by

$\ddot{a}_j = \ddot{a}_{j-1}$ if $a_j = a_{j-1}$,

$\ddot{a}_j = \ddot{a}_{j-1} \dot{\tilde{s}}_j$ if $a_j = a_{j-1} \tilde{s}_j$.

For $i \in [1, r]$ we define $\ddot{a}_{\tilde{r}+i}$ inductively by

$\ddot{a}_{\tilde{r}+i} = \ddot{a}_{\tilde{r}+i-1}$ if $a_{\tilde{r}+i-1} = a_{\tilde{r}+i}$,

$\ddot{a}_{\tilde{r}+i} = \dot{s}_i^{-1} \ddot{a}_{\tilde{r}+i-1}$ if $a_{\tilde{r}+i} = s_i a_{\tilde{r}+i-1}$.

Consider the commutative diagrams

$$\begin{array}{ccccccccc}
Z \times \tilde{Z} & \xleftarrow{d_0 \times \tilde{d}_0} & Z_0 \times \tilde{Z}_0 & \xleftarrow{d_1 \times \tilde{d}_1} & Z_1 \times \tilde{Z}_1 & \xrightarrow{d_2 \times \tilde{d}_2} & Z_2 \times \tilde{Z}_2 & \xrightarrow{=} & Z_2 \times \tilde{Z}_2 \\
{\scriptstyle e}\uparrow & & {\scriptstyle e_0}\uparrow & & {\scriptstyle e_1}\uparrow & & {\scriptstyle e_2}\uparrow & & {\scriptstyle e_3}\uparrow \\
X & \xleftarrow{b_0} & X_0 & \xleftarrow{b_1} & X_1 & \xrightarrow{b_2} & X_2 & \xrightarrow{b_3} & X_3
\end{array}$$

$$\begin{array}{ccccccccc}
Z_2 \times \tilde{Z}_2 & \xrightarrow{=} & Z_2 \times \tilde{Z}_2 & \xrightarrow{=} & Z_2 \times \tilde{Z}_2 & \xleftarrow{=} & Z_2 \times \tilde{Z}_2 & & X_8 \\
{\scriptstyle e_3}\uparrow & & {\scriptstyle e_4}\uparrow & & {\scriptstyle e_5}\uparrow & & {\scriptstyle e_6}\uparrow & & {\scriptstyle b_8}\uparrow \\
X_3 & \xleftarrow{b_4} & X_4 & \xrightarrow{b_5} & X_5 & \xleftarrow{b_6} & X_6 & \xrightarrow{b_7} & X_7
\end{array}$$

where the following notation is used.

Z, Z_i, d_i are as in 2.6, $\tilde{Z}, \tilde{Z}_i, \tilde{d}_i$ are the analogous objects defined in terms of $\tilde{\mathbf{s}}, \tilde{\mathcal{L}}$ instead of $\mathbf{s}, \mathcal{L}$.

X is the set of all $((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in \mathcal{B}^{[0,r]} \times \mathcal{B}^{[0,\tilde{r}]}$ such that $(A0)$–$(A5)$ below hold:

$(A0)$ $\text{pos}(B_0, \tilde{B}_j) = a_j (j \in [0, \tilde{r}])$, $\text{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [0, r])$,

$(A1)$ $\text{pos}(B_{i-1}, B_i) = s_i$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(A2)$ $\text{pos}(B_{i-1}, B_i) \in \{1, s_i\}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(A3)$ $\text{pos}(\tilde{B}_{j-1}, \tilde{B}_j) = \tilde{s}_j$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(A4)$ $\text{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(A5)$ $F(B_0) = B_r, F(\tilde{B}_0) = \tilde{B}_{\tilde{r}}$.

X_0 is the set of all

$$((g_0U, g_1U, \dots, g_rU), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)) \in (G/U)^{[0,r]} \times (G/U)^{[0,\tilde{r}]}$$

such that $(B0)$–$(B5)$ below hold:
$(B0)$ $k(g_0^{-1}\tilde{g}_j) = \ddot{a}_j (j \in [0, \tilde{r}]), k(g_i^{-1}\tilde{g}_{\tilde{r}}) = \ddot{a}_{\tilde{r}+i} (i \in [0, r])$,
$(B1)$ $g_{i-1}^{-1}g_i \in P_{s_i} - B$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(B2)$ $g_{i-1}^{-1}g_i \in P_{s_i}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(B3)$ $\tilde{g}_{j-1}^{-1}\tilde{g}_j \in P_{\tilde{s}_j} - B$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(B4)$ $\tilde{g}_{j-1}^{-1}\tilde{g}_j \in P_{\tilde{s}_j}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(B5)$ $g_r^{-1}F(g_0) \in U$, $\tilde{g}_{\tilde{r}}^{-1}F(\tilde{g}_0) \in U$.

b_0 is

$$((g_0U, g_1U, \dots, g_rU), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)) \mapsto$$
$$((g_0Bg_0^{-1}, g_1Bg_1^{-1}, \dots, g_rBg_r^{-1}), (\tilde{g}_0B\tilde{g}_0^{-1}, \tilde{g}_1B\tilde{g}_1^{-1}, \dots, \tilde{g}_{\tilde{r}}B\tilde{g}_{\tilde{r}}^{-1})).$$

X_1 is the set of all $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in G^{[0,r]} \times G^{[0,\tilde{r}]}$ such that $(B0)$–$(B5)$ hold.

b_1 is

$$((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto ((g_0U, g_1U, \dots, g_rU)), (\tilde{g}_0U, \tilde{g}_1U, \dots, \tilde{g}_{\tilde{r}}U)).$$

X_2 is the set of all $(x, x', u, u', y_0, y_1, \dots, y_\rho) \in G \times G \times U \times U \times G^{[0,\rho]}$ such that $(C0)$–$(C'5)$ below hold:
$(C0)$ $k(y_z) = \ddot{a}_z (z \in [0, \rho])$,
$(C1)$ $y_{\tilde{r}+i-1}y_{\tilde{r}+i}^{-1} \in P_{s_i} - B$ if $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$,
$(C2)$ $y_{\tilde{r}+i-1}y_{\tilde{r}+i}^{-1} \in P_{s_i}$ if $i \in \mathcal{I}_{\mathbf{s}}$,
$(C3)$ $y_{j-1}^{-1}y_j \in P_{\tilde{s}_j} - B$ if $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(C4)$ $y_{j-1}^{-1}y_j \in P_{\tilde{s}_j}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$(C5)$ $uF(y_0) = y_\rho u'$,
$(C'5)$ $y_0x' = xF(y_0)$, $y_\rho y_{\tilde{r}}^{-1}x = u$.

b_2 is $((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \mapsto (x, x', u, u', y_0, y_1, \dots, y_\rho)$ where

$$x = g_0^{-1}F(g_0), x' = \tilde{g}_0^{-1}F(\tilde{g}_0), u = g_r^{-1}F(g_0), u' = \tilde{g}_{\tilde{r}}^{-1}F(\tilde{g}_0),$$
$$y_j = g_0^{-1}\tilde{g}_j (j \in [0, \tilde{r}]), y_{\tilde{r}+i} = g_i^{-1}\tilde{g}_{\tilde{r}} (i \in [0, r]).$$

X_3 is the set of all $(u, u', y_0, y_1, \dots, y_\rho) \in U^2 \times G^{[0,\rho]}$ such that $(C0)$–$(C5)$ hold.

b_3 is $(x, x', u, u', y_0, y_1, \dots, y_\rho) \mapsto (u, u', y_0, y_1, \dots, y_\rho)$.

X_4 is the set of all $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \in U^6 \times G^{[0,\rho]}$ such that $(C0)$–$(C4)$ hold and

$$y_0 = v\ddot{a}_0v', y_\rho = \tilde{v}F(\ddot{a}_0)\tilde{v}', uF(v)F(\ddot{a}_0)F(v') = \tilde{v}F(\ddot{a}_0)\tilde{v}'u'.$$

b_4 is $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \mapsto (u, u', y_0, y_1, \dots, y_\rho)$.

X_5 is the set of all $(v, v', y_0, y_1, \dots, y_\rho) \in U^2 \times G^{[0,\rho]}$ such that $(C0)$–$(C4)$ hold and $y_0 = v\ddot{a}_0v'$.

b_5 is $(u, u', v, v', \tilde{v}, \tilde{v}', y_0, y_1, \dots, y_\rho) \mapsto (v, v', y_0, y_1, \dots, y_\rho)$.

X_6 is the set of all $((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\in G^{[0,r]}\times G^{[0,\tilde r]}$ such that $(B0)-(B4)$ hold and $g_0\in U,\tilde g_0\in \ddot a_0U$.

b_6 is $((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto (v,v',y_0,y_1,\dots,y_\rho)$ where

$$v=g_0^{-1},v'=\ddot a_0^{-1}\tilde g_0,y_j=g_0^{-1}\tilde g_j(j\in[0,\tilde r]),y_{\tilde r+i}=g_i^{-1}\tilde g_{\tilde r}(i\in[0,r]).$$

X_7 is the set of all

$$((g_0U,g_1U,\dots,g_rU),(\tilde g_0U,\tilde g_1U,\dots,\tilde g_{\tilde r}U))\in (G/U)^{[0,r]}\times (G/U)^{[0,\tilde r]}$$

such that $(B0)$–$(B4)$ hold and $g_0\in U,\tilde g_0\in\ddot a_0U$.

b_7 is given by the same formula as b_1.

X_8 is the set of all $((B_0,B_1,\dots,B_r),(\tilde B_0,\tilde B_1,\dots,\tilde B_{\tilde r}))\in\mathcal{B}^{[0,r]}\times\mathcal{B}^{[0,\tilde r]}$ such that $(A0)$–$(A4)$ hold and $B_0=B,\tilde B_0=\ddot a_0B\ddot a_0^{-1}$.

b_8 is given by the same formula as b_0.

e,e_0 are the obvious imbeddings.

e_1 is

$$((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\tilde g_{\tilde r}))\mapsto((g_0,g_0g_1^{-1},\dots,g_{r-1}^{-1}g_r),(\tilde g_0,\tilde g_0\tilde g_1^{-1},\dots,\tilde g_{\tilde r-1}^{-1}\tilde g_{\tilde r})).$$

e_2 is $(x,x',u,u',y_0,y_1,\dots,y_\rho)\mapsto\lambda(y_0,y_1,\dots,y_\rho)$ where

$$\lambda(y_0,y_1,\dots,y_\rho)=((y_{\tilde r}y_{\tilde r+1}^{-1},y_{\tilde r+1}y_{\tilde r+2}^{-1},\dots,y_{\tilde r+r-1}y_{\tilde r+r}^{-1}),(y_0^{-1}y_1,y_1^{-1}y_2,\dots,y_{\tilde r-1}^{-1}y_{\tilde r})).$$

e_3 is $(u,u',y_0,y_1,\dots,y_\rho)\mapsto\lambda(y_0,y_1,\dots,y_\rho)$.

e_4 is $(u,u',v,v',\tilde v,\tilde v',y_0,y_1,\dots,y_\rho)\mapsto\lambda(y_0,y_1,\dots,y_\rho)$.

e_5 is $(v,v',y_0,y_1,\dots,y_\rho)\mapsto\lambda(y_0,y_1,\dots,y_\rho)$.

e_6 is

$$((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$$
$$((g_0^{-1}g_1,g_1^{-1}g_2,\dots,g_{r-1}^{-1}g_r),(\tilde g_0^{-1}\tilde g_1,\tilde g_1^{-1}\tilde g_2,\dots,\tilde g_{r-1}^{-1}\tilde g_r)).$$

6.2. Let $\underline{B},\bar{\mathcal{L}},\bar{\mathcal{L}}_i$ be as in 2.4, 2.11. Define $\underline{\tilde B},\bar{\tilde{\bar{\mathcal{L}}}},\bar{\tilde{\bar{\mathcal{L}}}}_i$ similarly, in terms of $\tilde{\mathbf{s}},\tilde{\mathcal{L}}$ instead of $\mathbf{s},\mathcal{L}$. Note that $\Gamma,\underline{B}$ act on Z and $Z_i(i\in[0,2])$ as in 2.7. Similarly, $\Gamma,\underline{\tilde B}$ act on $\tilde Z$ and $\tilde Z_i(i\in[0,2])$. These actions give rise to (commuting) actions of $\Gamma\times\Gamma,\underline{B}\times\underline{\tilde B}$ on $Z\times\tilde Z$ and $Z_i\times\tilde Z_i(i\in[0,2])$ hence to actions of $\Gamma\times\Gamma\times\underline{B}\times\underline{\tilde B}$. By 2.11, $\bar{\mathcal{L}}$ and $\bar{\mathcal{L}}_i$ are $\underline{B}$-equivariant local systems with natural Γ-equivariant structures. Similarly, $\bar{\tilde{\bar{\mathcal{L}}}}$ and $\bar{\tilde{\bar{\mathcal{L}}}}_i$ are $\underline{\tilde B}$-equivariant local systems with natural Γ-equivariant structures. Hence $\bar{\mathcal{L}}\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}$ and $\bar{\mathcal{L}}_i\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_i$ are $\underline{B}\times\underline{\tilde B}$-equivariant local systems on $Z\times\tilde Z$ and $Z_i\times\tilde Z_i$ with natural $\Gamma\times\Gamma$-equivariant structures. These structures give rise to $\Gamma\times\Gamma\times\underline{B}\times\underline{\tilde B}$-equivariant structures on $\bar{\mathcal{L}}\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}$ and $\bar{\mathcal{L}}_i\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_i$. From the definitions we have

$$(d_0\times\tilde d_0)^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\bar{\mathcal{L}}}})=\bar{\mathcal{L}}_0\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_0,(d_1\times\tilde d_1)^*(\bar{\mathcal{L}}_0\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_0)=\bar{\mathcal{L}}_1\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_1,$$
$$(d_2\times\tilde d_2)^*(\bar{\mathcal{L}}_2\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_2)=\bar{\mathcal{L}}_1\boxtimes\bar{\tilde{\bar{\mathcal{L}}}}_1,$$

compatibly with the $\Gamma\times\Gamma\times\underline{B}\times\underline{\tilde B}$-equivariant structures.

6.3. Let

$$\mathcal{T}=\{((t_0,t_1,\dots,t_r),(\tilde t_0,\tilde t_1,\dots,\tilde t_{\tilde r}))\in T^{[0,r]}\times T^{[0,\tilde r]};$$
$$t_r=F(t_0),\tilde t_{\tilde r}=F(\tilde t_0),\tilde t_j=a_j^{-1}(t_0)\text{ for }j\in[0,\tilde r],t_i=a_{\tilde r+i}(\tilde t_{\tilde r})\text{ for }i\in[0,r].$$

This is a subgroup of $T^{[0,r]}\times T^{[0,\tilde r]}$ isomorphic to the finite subgroup

$$\mathcal{T}_0=\{t_0\in T;a_{\tilde r}^{-1}(t_0)=F(a_0^{-1}(t_0))\}$$

of T under $((t_i),(\tilde t_j))\mapsto t_0$. Hence $\mathcal{T}$ is finite.

$\Gamma\times\mathcal{T}$ acts on X by

$$(g,(t_i),(\tilde t_j)):((B_0,B_1,\dots,B_r),(\tilde B_0,\tilde B_1,\dots,\tilde B_{\tilde r}))\mapsto$$
$$((gB_0g^{-1},gB_1g^{-1},\dots,gB_rg^{-1}),(g\tilde B_0g^{-1},g\tilde B_1g^{-1},\dots,g\tilde B_{\tilde r}g^{-1}));$$

on X_0 by

$$(g,(t_i),(\tilde t_j)):((g_0U,g_1U,\dots,g_rU),(\tilde g_0U,\tilde g_1U,\dots,\tilde g_{\tilde r}U))\mapsto$$
$$((gg_0t_0^{-1}U,gg_1t_1^{-1}U,\dots,gg_rt_r^{-1}U),(g\tilde g_0\tilde t_0^{-1}U,g\tilde g_1\tilde t_1^{-1}U,\dots,g\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}U));$$

on X_1 by

$$(g,(t_i),(\tilde t_j)):((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$$
$$((gg_0t_0^{-1},gg_1t_1^{-1},\dots,gg_rt_r^{-1}),(g\tilde g_0\tilde t_0^{-1},g\tilde g_1\tilde t_1^{-1},\dots,g\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}));$$

on X_2 by

$$(g,(t_i),(\tilde t_j)):(x,x',u,u',y_0,y_1,\dots,y_\rho)\mapsto(t_0xF(t_0)^{-1},\tilde t_0x'F(\tilde t_0)^{-1},t_rut_r^{-1},$$
$$\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_3 by

$$(g,(t_i),(\tilde t_j)):(u,u',y_0,y_1,\dots,y_\rho)\mapsto$$
$$(t_rut_r^{-1},\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_4 by

$$(g,(t_i),(\tilde t_j)):(u,u',v,v',\tilde v,\tilde v',y_0,y_1,\dots,y_\rho)\mapsto(t_rut_r^{-1},\tilde t_{\tilde r}u'\tilde t_{\tilde r}^{-1},$$
$$t_0vt_0^{-1},\tilde t_0v'\tilde t_0^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_5 by

$$(g,(t_i),(\tilde t_j)):(v,v',\tilde v,\tilde v',y_0,y_1,\dots,y_\rho)\mapsto$$
$$(t_0vt_0^{-1},\tilde t_0v'\tilde t_0^{-1},t_0y_0\tilde t_0^{-1},t_0y_1\tilde t_1^{-1},\dots,t_0y_{\tilde r}\tilde t_{\tilde r}^{-1},t_1y_{\tilde r+1}\tilde t_{\tilde r}^{-1},\dots,t_ry_\rho\tilde t_{\tilde r}^{-1});$$

on X_6 by

$$(g,(t_i),(\tilde t_j)):((g_0,g_1,\dots,g_r),(\tilde g_0,\tilde g_1,\dots,\tilde g_{\tilde r}))\mapsto$$
$$((t_0g_0t_0^{-1},t_0g_1t_1^{-1},\dots,t_0g_rt_r^{-1}),(t_0\tilde g_0\tilde t_0^{-1},t_0\tilde g_1\tilde t_1^{-1},\dots,t_0\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}));$$

on X_7 by

$$(g,(t_i),(\tilde t_j)):((g_0U,g_1U,\dots,g_rU),(\tilde g_0U,\tilde g_1U,\dots,\tilde g_{\tilde r}U))\mapsto$$
$$((t_0g_0t_0^{-1}U,t_0g_1t_1^{-1}U,\dots,t_0g_rt_r^{-1}U),(t_0\tilde g_0\tilde t_0^{-1}U,t_0\tilde g_1\tilde t_1^{-1}U,\dots,t_0\tilde g_{\tilde r}\tilde t_{\tilde r}^{-1}U));$$

on X_8 by

$$(g,(t_i),(\tilde t_j)):((B_0,B_1,\dots,B_r),(\tilde B_0,\tilde B_1,\dots,\tilde B_{\tilde r}))\mapsto$$
$$((t_0B_0t_0^{-1},t_0B_1t_0^{-1},\dots,t_0B_rt_0^{-1}),(t_0\tilde B_0t_0^{-1},t_0\tilde B_1t_0^{-1},\dots,t_0\tilde B_{\tilde r}t_0^{-1})).$$

The maps b_i are compatible with the $\Gamma\times\mathcal{T}$-actions. Let

$$\mathcal{G}=\{((b_0,b_1,\dots,b_r),(\tilde b_0,\tilde b_1,\dots,\tilde b_{\tilde r}))\in\underline{B}\times\underline{\tilde B};$$
$$((k(b_0),k(b_1),\dots,k(b_r)),(k(\tilde b_0),k(\tilde b_1),\dots,k(\tilde b_{\tilde r})))\in\mathcal{T}\},$$

a subgroup of $\underline{B} \times \tilde{\underline{B}}$. The $\Gamma \times \mathcal{T}$-action on X_6 extends to a $\Gamma \times \mathcal{G}$-action on X_6:

$(g,(b_i),(\tilde{b}_j)) : ((g_0,g_1,\dots,g_r),(\tilde{g}_0,\tilde{g}_1,\dots,\tilde{g}_{\tilde{r}})) \mapsto$

$((k(b_0)g_0b_0^{-1},k(b_0)g_1b_1^{-1},\dots,k(b_0)g_rb_r^{-1}),(k(b_0)\tilde{g}_0\tilde{b}_0^{-1},k(b_0)\tilde{g}_1\tilde{b}_1^{-1},\dots,k(b_0)\tilde{g}_{\tilde{r}}\tilde{b}_{\tilde{r}}^{-1}))$.

6.4. Now

(a) b_0 is a principal $\mathcal{T}$-bundle

($\mathcal{T}$ acts on X_0 by restriction of the $\Gamma \times \mathcal{T}$-action) and induces an isomorphism $\mathcal{T}\backslash X_0 \xrightarrow{\sim} X$. (See [L4, 3.4].)

(b) b_1 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle.

(c) b_2 is a principal Γ-bundle

(Γ acts on X_1 by restriction of the $\Gamma \times \mathcal{T}$-action) and induces an isomorphism $\Gamma\backslash X_1 \xrightarrow{\sim} X_2$. (See [L4, 3.8].)

(d) b_3 is an isomorphism.

(e) b_4 is a quasi-vector bundle (see [L4, 3.2]) with fibres of dimension $2(\dim U - l(a_0))$.

(f) b_5 is a quasi-vector bundle with fibres of dimension $2(\dim U - l(a_0))$.

(g) b_6 is an isomorphism.

(h) b_7 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle.

(i) b_8 is an isomorphism. (See [L4, 3.24].)

6.5. Now $\mathcal{T}$ is naturally a subgroup of $\underline{B} \times \tilde{\underline{B}}$ and Γ is a subgroup of $\Gamma \times \Gamma$ (the diagonal) hence $\Gamma \times \mathcal{T}$ is a subgroup of $\Gamma \times \Gamma \times \underline{B} \times \tilde{\underline{B}}$. Hence the actions in 6.2 give by restriction actions of $\Gamma \times \mathcal{T}$ on $Z \times \tilde{Z}$ and $Z_i \times \tilde{Z}_i (i \in [0,2])$ and the equivariant structures on $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$ and $\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i$ in 6.2 restrict to $\Gamma \times \mathcal{T}$-equivariant structures on these local systems. Since e and $e_i(i \in [0,6])$ are compatible with the $\Gamma \times \mathcal{T}$-actions, we see that the local systems $\mathcal{E} = e^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})$, $\mathcal{E}_i = e_i^*(\bar{\mathcal{L}}_i \boxtimes \bar{\tilde{\mathcal{L}}}_i)$, $(i \in [0,2])$ and $\mathcal{E}_i = e_i^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2)$, $(i \in [3,6])$ have natural $\Gamma \times \mathcal{T}$-equivariant structures. Moreover, the $\Gamma \times \mathcal{T}$-equivariant structure on $\mathcal{E}_6$ extends to a $\Gamma \times \mathcal{G}$-equivariant structure since e_6 is compatible with the $\Gamma \times \mathcal{G}$ actions (see 6.3). Since the restriction of the $\Gamma \times \mathcal{G}$ action on X_6 to the subgroup $U^{[0,r]} \times U^{[0,\tilde{r}]}$ is the free action which makes X_6 a principal bundle over X_7 (see 6.4(h)), it follows that there is a well defined local system $\mathcal{E}_7$ on X_7 with a natural $\Gamma \times \mathcal{T}$-equivariant structure such that $b_7^*\mathcal{E}_7 = \mathcal{E}_6$. Since b_8 is an isomorphism, there is a well defined local system $\mathcal{E}_8$ on X_7 with a natural $\Gamma \times \mathcal{T}$-equivariant structure such that $b_8^*\mathcal{E}_8 = \mathcal{E}_7$. We have

$$b_0^*\mathcal{E} = \mathcal{E}_0; \quad b_1^*\mathcal{E}_1 = \mathcal{E}_0; \quad b_2^*\mathcal{E}_2 = \mathcal{E}_1; \quad b_3^*\mathcal{E}_3 = \mathcal{E}_2; \quad b_4^*\mathcal{E}_3 = \mathcal{E}_4; \quad b_5^*\mathcal{E}_5 = \mathcal{E}_4;$$

(a) $$b_6^*\mathcal{E}_5 = \mathcal{E}_6; \quad b_7^*\mathcal{E}_7 = \mathcal{E}_6; \quad b_8^*\mathcal{E}_8 = \mathcal{E}_7;$$

compatibly with the $\Gamma \times \mathcal{T}$-equivariant structures. We show:

(b) $\mathcal{T}$ *acts trivially on any stalk of* $\mathcal{E}$.

(c) Γ *acts trivially on any stalk of* $\mathcal{E}_i$ *(if* $i \in [2,8]$*).*

(b) follows from the fact that $\underline{B} \times \tilde{\underline{B}}$ acts trivially on $Z \times \tilde{Z}$ and, being connected, it acts trivially on any stalk of $\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}$. Now (c) follows from the fact that $\Gamma \times \Gamma$ acts trivially on $Z_2 \times \tilde{Z}_2$ and on any stalk of $\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2$.

6.6. For any $h \in [0,\tilde{r}]$ let $\mathfrak{Y}_h$ be the set of all $(\tilde{B}_0,\tilde{B}_1,\dots,\tilde{B}_h) \in \mathcal{B}^{[0,h]}$ such that

$\operatorname{pos}(B,\tilde{B}_j) = a_j (j \in [0,h])$,

$\operatorname{pos}(\tilde{B}_{j-1},\tilde{B}_j) = \tilde{s}_j$ if $j \in [1,h], j \notin \mathcal{I}_{\tilde{\mathbf{s}}}$,

$\text{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in [1, h] \cap \mathcal{I}_{\tilde{\mathbf{s}}}$,
$\tilde{B}_0 = \ddot{a}_0 B \ddot{a}_0^{-1}$.

Note that $\mathfrak{Y}_0$ is a point. Moreover, if $h \in [1, \tilde{r}]$, then we have an obvious map $\mathfrak{Y}_h \to \mathfrak{Y}_{h-1}$ which is
-an isomorphism if $a_h = a_{h-1}\tilde{s}_h < a_{h-1}$;
-an isomorphism if $h \in \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h > a_{h-1} = a_h$;
-a line bundle minus the zero section if $h \notin \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h < a_{h-1} = a_h$;
-a line bundle if $a_h = a_{h-1}\tilde{s}_h > a_{h-1}$;
-a line bundle if $h \in \mathcal{I}_{\tilde{\mathbf{s}}}$, $a_{h-1}\tilde{s}_h < a_{h-1} = a_h$.

For any $h \in [0, r]$ let $\mathfrak{Y}_{\tilde{r}+h}$ be the set of all

$$((B_0, B_1, \ldots, B_h), (\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_{\tilde{r}})) \in \mathcal{B}^{[0,h]} \times \mathcal{B}^{[0,\tilde{r}]}$$

such that
$\text{pos}(B_0, \tilde{B}_j) = a_j (j \in [0, \tilde{r}])$, $\text{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [0, h])$,
$\text{pos}(B_{i-1}, B_i) = s_i$ if $i \in [1, h], j \notin \mathcal{I}_{\mathbf{s}}$,
$\text{pos}(B_{i-1}, B_i) \in \{1, s_i\}$ if $i \in [1, h] \cap \mathcal{I}_{\mathbf{s}}$,
$\text{pos}(\tilde{B}_{j-1}, \tilde{B}_j) = \tilde{s}_j$ if $j \in [1, \tilde{r}], j \notin \mathcal{I}_{\tilde{\mathbf{s}}}$,
$\text{pos}(\tilde{B}_{j-1}, \tilde{B}_j) \in \{1, \tilde{s}_j\}$ if $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$,
$B_0 = B, \tilde{B}_0 = \ddot{a}_0 B \ddot{a}_0^{-1}$.

Note that $\mathfrak{Y}_{\tilde{r}}$ in the last definition may be identified with $\mathfrak{Y}_{\tilde{r}}$ in the earlier one. Moreover, if $h \in [1, r]$, then we have an obvious map $\mathfrak{Y}_{\tilde{r}+h} \to \mathfrak{Y}_{\tilde{r}+h-1}$ which is
-an isomorphism if $a_{\tilde{r}+h} = s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1}$;
-an isomorphism if $h \in \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} > a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$;
-a line bundle minus the zero section if $h \notin \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$;
-a line bundle if $a_{\tilde{r}+h} = s_h a_{\tilde{r}+h-1} > a_{\tilde{r}+h-1}$;
-a line bundle if $h \in \mathcal{I}_{\mathbf{s}}$, $s_h a_{\tilde{r}+h-1} < a_{\tilde{r}+h-1} = a_{\tilde{r}+h}$.

6.7. Assume that for some $h \in [1, r] - \mathcal{I}_{\mathbf{s}}$ we have $a := a_{\tilde{r}+h} = a_{\tilde{r}+h-1}$, $s := s_h$, $sa < a$. We show that

(a) $H_c^*(X_8, \mathcal{E}_8) = 0$.

We have an obvious map $\phi : X_8 \to \mathfrak{Y}_{\tilde{r}+h-1}$. (See 6.6.) It is enough to show that for any $p = ((B_0, \ldots, B_{h-1}), (\tilde{B}_0, \tilde{B}_1, \ldots, t B_{\tilde{r}})) \in \mathfrak{Y}_{\tilde{r}+h-1}$ we have $H_c^*(\mathfrak{Y}', \mathcal{E}_8) = 0$ where $\mathfrak{Y}' = \phi^{-1}(p)$. We may identify $\mathfrak{Y}'$ with the set of all $(B_h, B_{h+1}, \ldots, B_r) \in \mathcal{B}^{[h,r]}$ such that
$\text{pos}(B_i, \tilde{B}_{\tilde{r}}) = a_{\tilde{r}+i} (i \in [h, r])$,
$\text{pos}(B_{i-1}, B_i) \in \{s_i, 1\}$ if $i \in [h+1, r] \cap \mathcal{I}_{\mathbf{s}}$.
$\text{pos}(B_{i-1}, B_i) = s_i$ if $i \in [h, r], i \notin \mathcal{I}_{\mathbf{s}}$.

Since $sa < a$, there is a unique $D \in \mathcal{B}$ such that $\text{pos}(B_{h-1}, D) = s$, $\text{pos}(D, \tilde{B}_{\tilde{r}}) = sa$. Pick $C \in \mathcal{B}$ such that $\text{pos}(\tilde{B}_{\tilde{r}}, C) = a^{-1} w_{\mathbf{I}}$. Since $\text{pos}(D, C) = s w_{\mathbf{I}}$, $V := U_D \cap U_C$ is a one-dimensional connected unipotent group. Now $\tilde{B}_{\tilde{r}}$ is the unique Borel such that $\text{pos}(D, \tilde{B}_{\tilde{r}}) = sa$, $\text{pos}(\tilde{B}_{\tilde{r}}, C) = a^{-1} w_{\mathbf{I}}$. Hence it is normalized by V and $V \subset U_{\tilde{B}_{\tilde{r}}}$. Since $\text{pos}(B_{h-1}, C) = w_{\mathbf{I}}$, we have $U_{B_{h-1}} \cap U_C = \{1\}$ hence $U_{B_{h-1}} \cap V = \{1\}$. Let

$$\begin{aligned}\Xi &= \{E \in \mathcal{B}; \text{pos}(B_{h-1}, E) = s, \text{pos}(E, \tilde{B}_{\tilde{r}}) = a\} \\ &= \{E \in \mathcal{B}; \text{pos}(B_{h-1}, E) = s, \text{pos}(E, D) = s\}.\end{aligned}$$

Since $V \subset U_D$, $V \cap U_{B_{h-1}} = \{1\}$, V acts simply transitively (by conjugation) on the affine line $\{E' \in \mathcal{B}; \text{pos}(E', D) = s\} = \Xi \sqcup \{B_{h-1}\}$. Pick $B_h \in \xi$. Define

$v_0 \in V - \{1\}$ by $v_0 B_h v_0^{-1} = B_{h-1}$. Let

$$\mathfrak{Y}'' = \{(B_{h+1}, \dots, B_r) \in \mathcal{B}^{[h+1,r]}; (B_h, B_{h+1}, \dots, B_r) \in \mathfrak{Y}'\}.$$

The map $\zeta : (V - \{v_0\}) \times \mathfrak{Y}'' \to \mathfrak{Y}'$,

$$(v, (B_{h+1}, \dots, B_r)) \mapsto (vB_h v^{-1}, vB_{h+1}v^{-1}, \dots, vB_r v^{-1})$$

is an isomorphism. Hence it is enough to show that

$$H_c^*((V - \{v_0\}) \times \mathfrak{Y}'', \zeta^*\mathcal{E}_8) = 0.$$

Let $\pi'' : (V - \{v_0\}) \times \mathfrak{Y}'' \to \mathfrak{Y}''$ be the projection. It is enough to show that for any $p' = (B_{h+1}, \dots, B_r) \in \mathfrak{Y}''$ we have $H_c^*(\pi''^{-1}(p'), \zeta^*\mathcal{E}_8) = 0$ or equivalently that $H_c^*(V - \{v_0\}, \zeta'^*\mathcal{E}_8) = 0$ where $\zeta' : V - \{v_0\} \to X_8$ is

$$v \mapsto ((B_0, B_1, \dots, B_{h-1}, vB_h v^{-1}, vB_{h+1}v^{-1}, \dots, vB_r v^{-1}), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})).$$

Since $\zeta'^*\mathcal{E}_8$ is a local system of rank 1 on $V - \{v_0\} \cong \mathbf{k}^*$ with monodromy of finite order invertible in $\mathbf{k}$, it is enough to show that $\zeta'^*\mathcal{E}_8 \not\cong \bar{\mathbf{Q}}_l$. We can find $\epsilon = ((g_0, g_1, \dots, g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})) \in X_6$ such that

$$b_8(b_7(\epsilon)) = ((B_0, B_1, \dots, B_{h-1}, B_h, B_{h+1}, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})),$$

$g_{h-1}^{-1}\tilde{g}_{\tilde{r}} = \ddot{a}$ and $V = g_{h-1} y_{s_h}(\mathbf{k}) g_{h-1}^{-1}$. Define $\lambda_0 \in \mathbf{k}$ by $y_{s_h}(\lambda_0) g_{h-1}^{-1} g_h \in B$. Define $\tilde{\zeta} : \mathbf{k} - \{\lambda_0\} \to X_6$ by

$$\lambda \mapsto ((g_0, g_1, \dots, g_{h-1}, g_{h-1}y_{s_h}(\lambda)g_{h-1}^{-1}g_h, \dots, g_{h-1}y_{s_h}(\lambda)g_{h-1}^{-1}g_r), (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_{\tilde{r}})).$$

We have $\zeta'^*\mathcal{E}_8 = \tilde{\zeta}^*\mathcal{E}_6$. It is enough to prove that $\tilde{\zeta}^*\mathcal{E}_6 \not\cong \bar{\mathbf{Q}}_l$ or that $\tilde{\zeta}^* e_6^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2) \not\cong \bar{\mathbf{Q}}_l$. Note that $e_6\tilde{\zeta} : \mathbf{k} - \{\lambda_0\} \to Z_2 \times \tilde{Z}_2$ is

$$\lambda \mapsto ((y_1, y_2, y_{h-1}, y_{s_h}(\lambda)y_h, y_{h+1}, \dots, y_r), (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\tilde{r}}))$$

where

$$((y_1, y_2, y_{h-1}, y_h, y_{h+1}, \dots, y_r), (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\tilde{r}})) = e_6(\epsilon).$$

From this and the definition of $\bar{\mathcal{L}}_2 = \underline{\underline{\mathcal{L}}}$ (see 2.4) we see that $(e_6\tilde{\zeta})^*(\bar{\mathcal{L}}_2 \boxtimes \bar{\tilde{\mathcal{L}}}_2)$ is isomorphic to the inverse image of $\mathcal{L}$ under a map $\mathbf{k} - \{\lambda_0\} \to T$ of the form

$$\lambda \mapsto t'\dot{s}_1 \dots \dot{s}_{h-1} k(y_{s_h}(\lambda - \lambda_0))\dot{s}_h^{-1} \dots \dot{s}_1^{-1}$$

where t' is a fixed element of T. This is also of the form $\lambda \mapsto \beta(\lambda - \lambda_0)t'$ where $\beta : \mathbf{k}^* \to T$ is one of the two coroots with associated reflection $s_1 s_2 \dots s_h \dots s_2 s_1$. The desired result follows from the fact that $\beta^*(\mathcal{L}) \not\cong \bar{\mathbf{Q}}_l$. (Recall that $h \notin \mathcal{I}_{\mathbf{s}}$).

6.8. Assume that for some $h \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$ we have $a := a_h = a_{h-1}$, $s := \tilde{s}_h$, $as < a$. We have

(a) $H_c^*(X_8, \mathcal{E}_8) = 0$.

The proof is entirely similar to that of 6.7(a).

6.9. Let

$N_{\mathbf{a}}=|\{h\in[1,\tilde r];a_{h-1}\le a_h\ge a_{h-1}\tilde s_h\}|+|\{h\in[1,r];a_{\tilde r+h-1}\le a_{\tilde r+h}\ge s_h a_{\tilde r+h-1}\}|.$

Consider the following condition on $\mathbf{a}$:

(a) *for any* $h\in[1,r]-\mathcal{I}_{\mathbf{s}}$ *we have* $a_{\tilde r+h}=s_h a_{\tilde r+h-1}$*; for any* $h\in[1,\tilde r]-\mathcal{I}_{\tilde{\mathbf{s}}}$ *we have* $a_h=a_{h-1}\tilde s_h$.

If $\mathbf{a}$ satisfies (a), we have the following results:

(b) X_8 *is isomorphic to an affine space of dimension* $N_{\mathbf{a}}$.

(c) X *has pure dimension* $N_{\mathbf{a}}$.

Indeed, from the results in 6.6 we see by induction on $h\in[0,\rho]$ that $\mathfrak{Y}_h$ is an affine space. (In this case each of the maps $\mathfrak{Y}_h\to\mathfrak{Y}_{h-1}$, ($h\in[1,\rho]$) in 6.6 is either an isomorphism or an affine line bundle.) The same argument yields the dimension of each $\mathfrak{Y}_h$. This yields (b). Now (c) follows from (b) and the results in 6.4.

6.10. We assume that $\mathbf{a}$ satisfies 6.9(a). Define $x=((g_i),\tilde g_j))\in G^{[0,r]}\times G^{[0,\tilde r]}$ by $g_i=\ddot a_{\tilde r}\ddot a_{\tilde r+i}^{-1}$, $\tilde g_j=\ddot a_j$. Our assumption on $\mathbf{a}$ implies that $x\in X_6$. From the definitions we see that x is a fixed point of the $\mathcal{T}$-action 6.3 on X_6. Hence, in the $\Gamma\times\mathcal{T}$-structure of $\mathcal{E}_6$, $\mathcal{T}$ acts on the stalk $\mathcal{E}_{6,x}$ of $\mathcal{E}_6$ at x. We show:

(a) *the* $\mathcal{T}$*-action on* $\mathcal{E}_{6,x}$ *is trivial if and only if* $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}\cong\bar{\mathbf{Q}}_l$.

Let

$$\mathcal{J}=\{i\in[1,r];a_{\tilde r+i-1}=a_{\tilde r+i}\},\tilde{\mathcal{J}}=\{j\in[1,\tilde r];a_{j-1}=a_j\}.$$

From our assumption on $\mathbf{a}$ we see that $\mathcal{J}\subset\mathcal{I}_{\mathbf{s}}$, $\tilde{\mathcal{J}}\subset\mathcal{I}_{\tilde{\mathbf{s}}}$. Let $f^{\mathcal{J}}:\mathcal{Y}^{\mathcal{J}}\to T$ be as in 2.4; let $\tilde f^{\tilde{\mathcal{J}}}:\tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}\to T$ be the analogous map defined in terms of $\tilde{\mathbf{s}},\tilde{\mathcal{L}},\tilde{\mathcal{J}}$ instead of $\mathbf{s},\mathcal{L},\mathcal{J}$. Define sequences $\mathbf{s}_{\mathcal{J}}=(s'_1,s'_2,\dots,s'_r)$, $\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}}=(\tilde s'_1,\tilde s'_2,\dots,\tilde s'_{\tilde r})$ by

$s'_i=1$ if $i\in\mathcal{J}$, $s'_i=s_i$ if $i\in[1,r]-\mathcal{J}$, $\tilde s'_j=1$ if $j\in\tilde{\mathcal{J}}$, $\tilde s'_j=\tilde s_j$ if $j\in[1,\tilde r]-\tilde{\mathcal{J}}$.

We have $e_6(x)=((y_1,\dots,y_r),(\tilde y_1,\dots,\tilde y_{\tilde r}))$ where $y_i=\ddot a_{\tilde r+i-1}\ddot a_{\tilde r+i}^{-1}$ for $i\in[1,r]$ and $\tilde y_j=\ddot a_{j-1}^{-1}\ddot a_j$ for $j\in[1,\tilde r]$. Equivalently, $y_i=\dot s'_i$ for $i\in[1,r]$ and $\tilde y_j=\dot{\tilde s}'_j$ for $j\in[1,\tilde r]$. Thus we have $e_6(x)\in\mathcal{Y}^{\mathcal{J}}\times\tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}$. Moreover, we have $(f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}})e_6(x)=(1,1)\in T\times T$. It is enough to show that the $\mathcal{T}$-action on the stalk of $\bar{\mathcal{L}}_2\boxtimes\bar{\tilde{\mathcal{L}}}_2$ at $e_6(x)$ is trivial if and only if $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}\cong\bar{\mathbf{Q}}_l$. (The $\mathcal{T}$-equivariant structure on $\bar{\mathcal{L}}_2\boxtimes\bar{\tilde{\mathcal{L}}}_2$ is obtained by restricting the $\underline{B}\times\underline{\tilde B}$-equivariant structure in 6.2.) By 2.4(a) we have canonically

$$(\bar{\mathcal{L}}_2\boxtimes\bar{\tilde{\mathcal{L}}}_2)|_{\mathcal{Y}^{\mathcal{J}}\times\tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}}=(f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}})^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}}).$$

Now $\mathcal{Y}^{\mathcal{J}}\times\tilde{\mathcal{Y}}^{\tilde{\mathcal{J}}}$ is stable under the $\underline{B}\times\underline{\tilde B}$-action on $Z_2\times\tilde Z_2$ and the previous equality shows that $(f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}})^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}})$ has a $\underline{B}\times\underline{\tilde B}$-equivariant structure; this structure is unique since $\underline{B}\times\underline{\tilde B}$ is connected. By restriction to the subgroup $\mathcal{T}$ of $\underline{B}\times\underline{\tilde B}$ we obtain a $\mathcal{T}$-equivariant structure on $(f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}})^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}})$ and it is enough to show that the $\mathcal{T}$-action on the stalk of $(f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}})^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}})$ at $e_6(x)$ is trivial if and only if $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}\cong\bar{\mathbf{Q}}_l$. We define a $\underline{B}\times\underline{\tilde B}$-action on $T\times T$ by

$$((b_i),(\tilde b_j)):(t,\tilde t)\mapsto(k(b_0)t([\mathbf{s}_{\mathcal{J}}]Fk(b_0^{-1})),k(\tilde b_0)\tilde t([\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}}]Fk(\tilde b_0^{-1}))).$$

From the definitions we see that $f^{\mathcal{J}}\times\tilde f^{\tilde{\mathcal{J}}}:Z_2\times\tilde Z_2\to T\times T$ is compatible with the $\underline{B}\times\underline{\tilde B}$-actions. Moreover, $\mathcal{L}\boxtimes\tilde{\mathcal{L}}$ is equivariant for the $\underline{B}\times\underline{\tilde B}$-action on $T\times T$ as above. (In fact, it is equivariant for the action of the bigger group $B^{[0,r]}\times B^{[0,\tilde r]}$ given by the same formula; this follows from 1.8 using that $\mathcal{L}\in\mathcal{S}(T)^{[\mathbf{s}_{\mathcal{J}}]F}$ and $\mathcal{L}\in\mathcal{S}(T)^{[\tilde{\mathbf{s}}_{\tilde{\mathcal{J}}}]F}$.) By restriction to the subgroup $\mathcal{T}$ of $\underline{B}\times\underline{\tilde B}$ we obtain a $\mathcal{T}$-equivariant structure on

$\mathcal{L}\boxtimes\tilde{\mathcal{L}}$. We may identify the stalk of $(f^{\mathcal{J}}\times\tilde{f}^{\tilde{\mathcal{J}}})^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}})$ at $e_6(x)$ with the stalk of $\mathcal{L}\boxtimes\tilde{\mathcal{L}}$ at $(1,1)$ as $\mathcal{T}$-modules. Let $\chi:\mathcal{T}\to\bar{\mathbf{Q}}_l^*$ be the character by which $\mathcal{T}$ acts on the stalk of $\mathcal{L}\boxtimes\tilde{\mathcal{L}}$ at $(1,1)$. It is enough to show that $\chi=1$ if and only if $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}\cong\bar{\mathbf{Q}}_l$.

Now let T' be the subgroup of $B^{[0,r]}\times B^{[0,\tilde{r}]}$ consisting of all elements $((t_0,t_1,\dots,t_r),(\tilde{t}_0,\tilde{t}_1,\dots,\tilde{t}_{\tilde{r}}))$ with coordinates in T such that $\tilde{t}_j=a_j^{-1}(t_0)$ for $j\in[0,\tilde{r}]$, $t_i=a_{\tilde{r}+i}(\tilde{t}_{\tilde{r}})$ for $i\in[0,r]$. We have $\mathcal{T}\subset T'$ and $((t_i),(\tilde{t}_j))\mapsto t_0$ is an isomorphism $T'\xrightarrow{\sim}T$. Moreover, $\mathcal{L}\boxtimes\tilde{\mathcal{L}}$ is T'-equivariant where T' acts on $T\times T$ by restriction of the $B^{[0,r]}\times B^{[0,\tilde{r}]}$ action. Now T' acts on T by $t_0:t\mapsto t_0tF'(t_0^{-1})$ where $F':T\to T$, $F'(t_0)=(a_{\tilde{r}}a_\rho^{-1}F)(t_0)$ is the Frobenius map for an $\mathbf{F}_q$-rational structure on T. The map $m:T\to T\times T$, $t\mapsto(t,a_0^{-1}(t))$ is compatible with the T'-actions. Hence $m^*(\mathcal{L}\boxtimes\tilde{\mathcal{L}})=\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}$ is a T'-equivariant local system on T and the natural action of $\mathcal{T}$ at any stalk of this local system is via χ. If we identify T' with T as above, $\mathcal{T}$ becomes the subgroup $T^{F'}$ of T. It remains to use the bijection (i)–(iv) in 1.9, with $\mathcal{L}$ replaced by $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}$.

6.11. We assume that $\mathbf{a}$ satisfies 6.9(a). We show:

(a) $\mathcal{E}_6\cong\bar{\mathbf{Q}}_l$.

We write $\bar{\mathcal{L}}_2=\underline{\underline{\mathcal{L}}}=\otimes_{i\in[1,r]}\mathcal{F}_i$ as in 2.4. Similarly $\bar{\tilde{\mathcal{L}}}_2=\otimes_{j\in[1,\tilde{r}]}\tilde{\mathcal{F}}_j$ where $\tilde{\mathcal{F}}_j$ are local systems on $\tilde{Z}_2$. From the definitions we have

$$\mathcal{E}_6=\otimes_{i\in[1,r]}e_6^*(\mathcal{F}_i\boxtimes\bar{\mathbf{Q}}_l)\otimes\otimes_{j\in[1,\tilde{r}]}e_6^*(\bar{\mathbf{Q}}_l\boxtimes\tilde{\mathcal{F}}_j).$$

It is enough to show:

(b) $e_6^*(\mathcal{F}_i\boxtimes\bar{\mathbf{Q}}_l)\cong\bar{\mathbf{Q}}_l$ for any $i\in[1,r]$;

(c) $e_6^*(\bar{\mathbf{Q}}_l\boxtimes\tilde{\mathcal{F}}_j)\cong\bar{\mathbf{Q}}_l$ for any $j\in[1,\tilde{r}]$.

We prove (b). The general case can be reduced to the case where G has simply connected derived group, which we now assume. Let $\psi:Z_2\times\tilde{Z}_2\to Z_2$ be the projection. Let p_i,f_{s_i} be as in 2.4. Let $\tilde{f}_i$ be the obvious map from P_{s_i} to the quotient of $P_{s_i}/U_{P_{s_i}}$ by its derived subgroup. Then $e_6^*(\mathcal{F}_i\otimes\bar{\mathbf{Q}}_l)$ is the inverse image of a local system of rank 1 under $\tilde{f}_ip_i\psi e_6$ (if $i\in\mathcal{I}_{\mathbf{s}}$), or $f_{s_i}p_i\psi e_6$ (if $i\in[1,r]-\mathcal{I}_{\mathbf{s}}$).

It is enough to show that the image of $\tilde{f}_ip_i\psi e_6$ (if $i\in\mathcal{I}_{\mathbf{s}}$) or of $f_{s_i}p_i\psi e_6$ (if $i\in[1,r]-\mathcal{I}_{\mathbf{s}}$) is a point.

Let $\xi=((g_0,g_1,\dots,g_r),(\tilde{g}_0,\tilde{g}_1,\dots,\tilde{g}_{\tilde{r}}))\in X_6$. We have $p_i\psi e_6(\xi)=g_{i-1}^{-1}g_i$. We have $g_i^{-1}\tilde{g}_{\tilde{r}}\in U\ddot{a}_{\tilde{r}+i}U$, $g_{i-1}^{-1}\tilde{g}_{\tilde{r}}\in U\ddot{a}_{\tilde{r}+i-1}U$, hence $g_{i-1}^{-1}g_i\in u_1\ddot{a}_{\tilde{r}+i-1}u_2\ddot{a}_{\tilde{r}+i}^{-1}u_3$ with u_1,u_2,u_3 in U. Moreover, $g_{i-1}^{-1}g_i\in P_{s_i}$.

Case 1. Assume that $i\in\mathcal{I}_{\mathbf{s}}$ and $a_{\tilde{r}+i-1}=a_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i-1}=\ddot{a}_{\tilde{r}+i}^{-1}$ and $\ddot{a}_{\tilde{r}+i-1}u_2\ddot{a}_{\tilde{r}+i}^{-1}=\ddot{a}_{\tilde{r}+i-1}u_2\ddot{a}_{\tilde{r}+i-1}^{-1}\in P_{s_i}$ is unipotent. Hence $g_{i-1}^{-1}g_i$ is a product of three unipotent elements of P_{s_i} so that $\tilde{f}_i(g_{i-1}^{-1}g_i)=1$.

Case 2. Assume that $i\in\mathcal{I}_{\mathbf{s}}$ and $a_{\tilde{r}+i-1}=s_ia_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i}=\dot{s}_i^{-1}\ddot{a}_{\tilde{r}+i-1}^{-1}$ and

$$\ddot{a}_{\tilde{r}+i-1}u_2\ddot{a}_{\tilde{r}+i}^{-1}=\ddot{a}_{\tilde{r}+i-1}u_2\ddot{a}_{\tilde{r}+i-1}^{-1}\dot{s}_i=u_2'\dot{s}_i$$

where $u_2'\in G$ is unipotent. Thus $g_{i-1}^{-1}g_i=u_1u_2'\dot{s}_iu_3$. Since $u_1u_2'\dot{s}_iu_3,u_1,\dot{s}_i,u_3$ belong to P_{s_i} we see that u_2' is a unipotent element of P_{s_i}. Note also that $\dot{s}_i$ belongs to the derived subgroup of P_{s_i}. We see that $\tilde{f}_i(g_{i-1}^{-1}g_i)=1$.

Case 3. Assume that $i\notin\mathcal{I}_{\mathbf{s}}$. By our assumption we have $a_{\tilde{r}+i-1}=s_ia_{\tilde{r}+i}$. Then $\ddot{a}_{\tilde{r}+i}=\dot{s}_i^{-1}\ddot{a}_{\tilde{r}+i-1}^{-1}$ and as before we have $g_{i-1}^{-1}g_i=u_1u_2'\dot{s}_iu_3$ where u_2' belongs to the unipotent group $P_{s_i}\cap\ddot{a}_{\tilde{r}+i-1}U\ddot{a}_{\tilde{r}+i-1}^{-1}$. This unipotent group is normalized by

T. By [Bo, 14.4] we have $P_{s_i} \cap \ddot{a}_{\tilde{r}+i-1} U \ddot{a}_{\tilde{r}+i-1}^{-1} = U_1 U_2 \dots U_n$ where $U_1, U_2, \dots, U_n$ are the connected one-dimensional unipotent subgroups of $P_{s_i} \cap \ddot{a}_{\tilde{r}+i-1} U \ddot{a}_{\tilde{r}+i-1}^{-1}$ normalized by T, in any order. Thus any of $U_1, U_2, \dots, U_n$ is either $y_{s_i}(\mathbf{k})$ or else is contained in U; moreover, we can assume that $U_1, U_2, \dots, U_{n-1}$ are contained in U and $U_n = y_{s_i}(\mathbf{k})$. Thus $u'_2 \in U y_{s_i}(\mathbf{k})$ and

$$g_{i-1}^{-1} g_i \in U y_{s_i}(\mathbf{k}) \dot{s}_i U = U \dot{s}_i x_{s_i}(\mathbf{k}) U = U \dot{s}_i U.$$

Note that $f_{s_i}(g_{i-1}^{-1} g_i) = 1$.

We see that in Cases 1 and 2 we have $\tilde{f}_i p_i \psi e_6(\xi) = 1$ for any $\xi \in X_6$. In Case 3 we have $f_{s_i} p_i \psi e_6(\xi) = 1$ for any $\xi \in X_6$. This completes the proof of (b).

The proof of (c) is entirely similar.

6.12. We set $X_{-1} = X$, $\mathcal{E}_{-1} = \mathcal{E}$. Let $\delta_i = \dim X_i (i \in [-1, 8])$. For $i \in [-1, 8]$ we show:

(a) *Assume that* $\mathbf{a}$ *satisfies 6.9(a); if* $n \neq 2\delta_i$ *or if* $\tilde{\mathcal{L}} \not\cong a_0^* \check{\mathcal{L}}$, *then* $H_c^n(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}} = 0$*; if* $\tilde{\mathcal{L}} = w^* \check{\mathcal{L}}$ *where* $w \in W$ *and* $w^* \mathcal{L} \cong a_0^* \mathcal{L}$, *then* $H_c^{2\delta_i}(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}}(\delta_i) = \bar{\mathbf{Q}}_l$ *canonically. Assume that* $\mathbf{a}$ *does not satisfy 6.9(a); then* $H_c^n(X_i, \mathcal{E}_i)^{\Gamma \times \mathcal{T}} = 0$ *for all* n.

Here (δ_i) is a Tate twist, The upper index denotes $\Gamma \times \mathcal{T}$-invariants (the action of $\Gamma \times \mathcal{T}$ comes from the $\Gamma \times \mathcal{T}$-equivariant structure of $\mathcal{E}_i$).

Let P_i be the statement of (a). From 6.4 we see that the statements P_0, P_1 are equivalent and that the statements $P_2, P_3, \dots, P_8$ are equivalent. From 6.4 we see also that $H_c^n(X_2, \mathcal{E}_2) = H_c^n(X_1, \mathcal{E}_1)^\Gamma$ and that $H_c^n(X_{-1}, \mathcal{E}_{-1}) = H_c^n(X_0, \mathcal{E}_0)^{\mathcal{T}}$ for any n. Since Γ acts trivially on any stalk of $\mathcal{E}_2$ (see 6.5(c)) and $\mathcal{T}$ acts trivially on any stalk of $\mathcal{E}_{-1}$ (see 6.5(b)), it follows that $H_c^n(X_2, \mathcal{E}_2)^\Gamma = H_c^n(X_1, \mathcal{E}_1)^\Gamma$ and that $H_c^n(X_{-1}, \mathcal{E}_{-1})^{\mathcal{T}} = H_c^n(X_0, \mathcal{E}_0)^{\mathcal{T}}$ for any n. This shows that the statements P_1, P_2 are equivalent and the statements P_{-1}, P_0 are equivalent. We see that the statements $P_{-1}, P_0, \dots, P_8$ are all equivalent. Thus it is enough to show that P_8 holds.

Assume first that $\mathbf{a}$ does not satisfy 6.9(a). Then the result follows from 6.7(a) or 6.8(a).

Next we assume that $\mathbf{a}$ satisfies 6.9(a). Let $\bar{x} = b_8(b_7(x)) \in X_8$ where $x \in X_6$ is as in 6.10. We can write 6.11(a) in the form $b_7^* b_8^* \mathcal{E}_8 \cong \bar{\mathbf{Q}}_l$. Since b_8 is an isomorphism and b_7 is a principal $U^{[0,r]} \times U^{[0,\tilde{r}]}$-bundle, it follows that $\mathcal{E}_8 \cong \bar{\mathbf{Q}}_l$. Using this and 6.9(b) we see that

$$H_c^n(X_8, \mathcal{E}_8) = 0 \text{ for } i \neq 2\delta_8, H_c^{2\delta_8}(X_8, \mathcal{E}_8)(\delta_8) = \mathcal{E}_{8,\bar{x}},$$

where $\mathcal{E}_{8,\bar{x}}$ is the stalk of $\mathcal{E}_8$ at $\bar{x}$. Moreover, the last equality is compatible with the natural $\mathcal{T}$-actions (coming from the $\mathcal{T}$-equivariant structure of $\mathcal{E}_8$). Now $\mathcal{E}_{8,\bar{x}}$ may be canonically identified with the stalk $\mathcal{E}_{6,x}$ of $\mathcal{E}_6$ at x. Thus we have $H_c^{2\delta_8}(X_8, \mathcal{E}_8)(\delta_8) = \mathcal{E}_{6,x}$, compatibly with the $\mathcal{T}$-actions. Using 6.10(a) we see that the $\mathcal{T}$-action on $H_c^{2\delta_8}(X_8, \mathcal{E}_8)$ is trivial if and only if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$. Taking $\mathcal{T}$-invariants we see that

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^{\mathcal{T}}(\delta_8) = \mathcal{E}_{6,x}$ if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \cong \bar{\mathbf{Q}}_l$;

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^{\mathcal{T}} = 0$ if $\mathcal{L} \otimes (a_0^{-1})^* \tilde{\mathcal{L}} \not\cong \bar{\mathbf{Q}}_l$.

By 6.5(c), Γ acts trivially on $H_c^{2\delta_8}(X_8, \mathcal{E}_8)$ hence

$H_c^{2\delta_8}(X_8, \mathcal{E}_8)^{\Gamma \times \mathcal{T}} = H_c^{2\delta_8}(X_8, \mathcal{E}_8)^{\mathcal{T}}$.

We see that the first assertion in (a) for $i = 8$ holds. It remains to prove that, if $\tilde{\mathcal{L}} = w^* \check{\mathcal{L}}$ where $w \in W$ and $w^* \mathcal{L} \cong a_0^* \mathcal{L}$, then $\mathcal{E}_{6,x} = \bar{\mathbf{Q}}_l$ canonically. By the proof of 6.10(a), $\mathcal{E}_{6,x}$ is canonically isomorphic to the stalk of $\mathcal{L}\otimes(a_0^{-1})^*\tilde{\mathcal{L}}$ at 1; that is, to the stalk of $\mathcal{L}\otimes(wa_0^{-1})^*\check{\mathcal{L}}$ at 1. The stalk of $(wa_0^{-1})^*\check{\mathcal{L}}$ at 1 is the same as the stalk of $\check{\mathcal{L}}$ at $wa_0(1)=1$. Thus $\mathcal{E}_{6,x}$ is canonically isomorphic to the stalk of $\mathcal{L}\otimes\check{\mathcal{L}}=\bar{\mathbf{Q}}_l$ at 1. Thus P_8 holds. We see that (a) holds for $i\in[-1,8]$.

6.13. We now write $X_{\mathbf{a}},\mathcal{E}_{\mathbf{a}},e_{\mathbf{a}}$ instead of $X,\mathcal{E},e$ in 6.1. We identify $X_{\mathbf{a}}$ with its image under the imbedding $e_{\mathbf{a}}$. Let $\bar{X}_{\mathbf{a}}$ be the closure of $X_{\mathbf{a}}$ in $Z\times\tilde{Z}$. Recall that $\mathcal{E}_{\mathbf{a}}=e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})$. Let $\bar{e}_{\mathbf{a}}:\bar{X}_{\mathbf{a}}\to Z\times\tilde{Z}$ by the inclusion. By 6.9(c), $X_{\mathbf{a}}$ (hence also $\bar{X}_{\mathbf{a}}$) has pure dimension $N_{\mathbf{a}}$ and $\dim(\bar{X}_{\mathbf{a}}-X_{\mathbf{a}})<N_{\mathbf{a}}$. Hence the natural map

(a) $$H_c^{2N_{\mathbf{a}}}(X_{\mathbf{a}},e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))\to H_c^{2N_{\mathbf{a}}}(\bar{X}_{\mathbf{a}},\bar{e}_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))$$

(induced by the open imbedding $X_{\mathbf{a}}\subset\bar{X}_{\mathbf{a}}$) is an isomorphism. Let

$$\xi'_{\mathbf{a}}:H_c^{2N_{\mathbf{a}}}(Z\times\tilde{Z},(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))\to H_c^{2N_{\mathbf{a}}}(X_{\mathbf{a}},e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))$$

be the linear map obtained by composing the linear map

$$H_c^{2N_{\mathbf{a}}}(Z\times\tilde{Z},(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))\to H_c^{2N_{\mathbf{a}}}(\bar{X}_{\mathbf{a}},\bar{e}_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))$$

with the inverse of (a). By taking Γ-invariants and applying a Tate twist we obtain from $\xi'_{\mathbf{a}}$ a linear map

(b) $$\xi_{\mathbf{a}}:H_c^{2N_{\mathbf{a}}}(Z\times\tilde{Z},(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))^\Gamma(N_{\mathbf{a}})\to H_c^{2N_{\mathbf{a}}}(X_{\mathbf{a}},e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))^\Gamma(N_{\mathbf{a}}).$$

6.14. Let $\mathcal{A}$ be the set of all $\mathbf{a}$ as in 6.1. Note that the subvarieties $X_{\mathbf{a}}$ ($\mathbf{a}\in\mathcal{A}$) form a partition of $Z\times\tilde{Z}$.

(a) *For any $n\in\mathbf{Z}$, the linear map*

$$\xi_n:H_c^{2n}(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma(n)\to\oplus_{\mathbf{a}\in\mathcal{A};N_{\mathbf{a}}=n}H_c^{2n}(X_{\mathbf{a}},e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))^\Gamma(n)$$

whose components are the maps $\xi_{\mathbf{a}}$ ($\mathbf{a}\in\mathcal{A},N_{\mathbf{a}}=n$), is an isomorphism. Moreover, $H_c^{n'}(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma=0$ for any odd n'.

The proof is almost identical to that of [L4, 2.7]. (We use the fact that

$$H_c^m(X_{\mathbf{a}},e_{\mathbf{a}}^*(\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}}))^\Gamma=0$$

for any $\mathbf{a}\in\mathcal{A}$ and any $m\neq 2N_{\mathbf{a}}$; this follows from 6.12(a) for $i=-1$; note that, in view of 6.5(b), 6.12(a) for $i=-1$ remains valid if $\Gamma\times\mathcal{T}$-invariants are replaced by Γ-invariants.)

From (a) and 6.12(a) for $i=-1$ we see that the following holds.

(b) *If for any $w\in W$ we have $\tilde{\mathcal{L}}\not\cong w^*\check{\mathcal{L}}$, then $H_c^m(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma=0$ for any $m\in\mathbf{Z}$.*

Now assume that $\tilde{\mathcal{L}}=w^*\check{\mathcal{L}}$ for some $w\in W$. Let

$\mathcal{A}_w=\{\mathbf{a}=(a_0,a_1,\dots,a_\rho)\in\mathcal{A};\mathbf{a}$ satisfies 6.9(a), $a_0^*\mathcal{L}\cong w^*\mathcal{L}.\}$.

Note that for $w=1$ we have $\mathcal{A}_1=\mathcal{A}(W,\mathbf{c},\mathcal{L},\mathbf{s},\tilde{\mathbf{s}})$; see 5.9.

By 6.12(a) for $i=-1$, the summand in the target of ξ_n corresponding to $\mathbf{a}$ is canonically $\bar{\mathbf{Q}}_l$ if $\mathbf{a}\in\mathcal{A}_w$ and is 0 if $\mathbf{a}\notin\mathcal{A}_w$. Hence for any $\mathbf{a}\in\mathcal{A}_w$ with $N_{\mathbf{a}}=n$ there is a unique element $b_{\mathbf{a}}^w\in H_c^{2n}(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma(n)$ such that $\xi_n(b_{\mathbf{a}}^w)$ is contained in the summand corresponding to $\mathbf{a}$ and, as an element of that summand, it corresponds to $1\in\bar{\mathbf{Q}}_l$. Moreover,

(c) *$\{b_{\mathbf{a}}^w;\mathbf{a}\in\mathcal{A}_w,N_{\mathbf{a}}=n\}$ is a $\bar{\mathbf{Q}}_l$-basis of $H_c^{2n}(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma(n)$.*

Taking $w=1$ and taking direct sum over n we obtain

(d) *$\{b_{\mathbf{a}}^1;\mathbf{a}\in\mathcal{A}(W,\mathbf{c},\mathcal{L},\mathbf{s},\tilde{\mathbf{s}})\}$ is a $\bar{\mathbf{Q}}_l$-basis of $\oplus_nH_c^{2n}(Z\times\tilde{Z},\bar{\mathcal{L}}\boxtimes\bar{\tilde{\mathcal{L}}})^\Gamma(n)$.*

6.15. More generally, if J is as in 3.1, we set $\mathcal{A}_J = \{\mathbf{a} = (a_0, a_1, \dots, \alpha_\rho) \in \mathcal{A}; a_0 \in W_J\}$. Let

$$(Z \times \tilde{Z})_J = \{((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \in Z \times \tilde{Z}; P_{B_0,J} = P_{\tilde{B}_0,J}\}.$$

Note that the subvarieties $X_\mathbf{a}$ ($\mathbf{a} \in \mathcal{A}_J$) form a partition of $(Z \times \tilde{Z})_J$. As in 6.14 we see that for any $n \in \mathbf{Z}$ we have an isomorphism

(a) $$H_c^{2n}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma(n) \xrightarrow{\sim} \oplus_{\mathbf{a} \in \mathcal{A}_J; N_\mathbf{a} = n} H_c^{2n}(X_\mathbf{a}, e_\mathbf{a}^*(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}))^\Gamma(n).$$

Moreover,

(b) $$H_c^{n'}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = 0 \text{ for any odd } n'.$$

As in 6.14 we see that the following holds:

(c) *If for any $w \in W_J$ we have $\tilde{\mathcal{L}} \not\cong w^* \check{\mathcal{L}}$, then $H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = 0$ for any $m \in \mathbf{Z}$.*

Define $h : (Z \times \tilde{Z}) \to \mathcal{P}_J$ by $((B_0, B_1, \dots, B_r), (\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_{\tilde{r}})) \mapsto P_{B_0,J} = P_{\tilde{B}_0,J}$. We have

$$H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = H_c^m(\mathcal{P}_J, h_!(\bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})) = H_c^m(\mathcal{P}_J, (\Upsilon_!^\mathbf{s} \bar{\mathcal{L}}) \otimes (\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}})).$$

(Notation of 3.2.) From 2.13 we see that

(d) $$\Upsilon_!^\mathbf{s} \bar{\mathcal{L}} = \bar{\Upsilon}_!^\mathbf{s} \bar{\mathcal{L}}^\sharp$$

(notation of 3.2). By the decomposition theorem [BBD], $\bar{\Upsilon}_!^\mathbf{s} \bar{\mathcal{L}}^\sharp \cong \oplus_j {}^pH^j(\bar{\Upsilon}_!^\mathbf{s} \bar{\mathcal{L}}^\sharp)[-j]$. Hence $\Upsilon_!^\mathbf{s} \bar{\mathcal{L}} \cong \oplus_j {}^pH^j(\Upsilon_!^\mathbf{s} \bar{\mathcal{L}})[-j]$. Similarly, $\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}} \cong \oplus_{\tilde{j}} {}^pH^{\tilde{j}}(\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}})[-\tilde{j}]$. We see that

(e) $$H_c^m((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = \oplus_{j,j'} H_c^{m-j-\tilde{j}}(\mathcal{P}_J, {}^pH^j(\Upsilon_!^\mathbf{s} \bar{\mathcal{L}}) \otimes {}^pH^{\tilde{j}}(\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}}))$$

(compatibly with the Γ-actions).

Proposition 6.16. *Let $J \subset \mathbf{I}$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$. Assume that $A \dashv_\Gamma {}^pH^j(\Upsilon_!^\mathbf{s} \bar{\mathcal{L}})$ and $A \dashv_\Gamma {}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}})$ where $j, j' \in \mathbf{Z}$. Then $\tilde{\mathcal{L}} \cong w^* \check{\mathcal{L}}$ for some $w \in W_J$ and $j = j' \mod 2$.*

Using 6.15(d) and the fact that $\bar{\Upsilon}^\mathbf{s}$ is proper we see that

$$\mathfrak{D}(\Upsilon_!^\mathbf{s} \bar{\mathcal{L}}) = \bar{\Upsilon}_!^\mathbf{s} \mathfrak{D}(IC(\bar{Z}^\mathbf{s}, \bar{\mathcal{L}})) = \bar{\Upsilon}_!^\mathbf{s}(IC(\bar{Z}^\mathbf{s}, \bar{\check{\mathcal{L}}}))[2r] = \Upsilon_!^\mathbf{s} \bar{\check{\mathcal{L}}}[2r].$$

Hence $\mathfrak{D}({}^pH^j(\Upsilon_!^\mathbf{s} \bar{\mathcal{L}})) = {}^pH^{-j+2r}(\Upsilon_!^\mathbf{s} \bar{\check{\mathcal{L}}})$. We see that $\mathfrak{D}(A) \dashv_\Gamma {}^pH^{-j+2r}(\Upsilon_!^\mathbf{s} \bar{\check{\mathcal{L}}})$. By 1.6(a) we have $\dim H_c^0(\mathcal{P}_J, \mathfrak{D}(A) \otimes A)^\Gamma = 1$. Since $\mathfrak{D}(A)$ (resp. A) is a direct summand of ${}^pH^{-j+2r}(\Upsilon_!^\mathbf{s} \bar{\check{\mathcal{L}}})$ (resp. ${}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}})$) we see that

$$H_c^0(\mathcal{P}_J, {}^pH^{-j+2r}(\Upsilon_!^\mathbf{s} \bar{\check{\mathcal{L}}}) \otimes {}^pH^{j'}(\Upsilon_!^{\tilde{\mathbf{s}}} \bar{\tilde{\mathcal{L}}}))^\Gamma \neq 0.$$

Using 6.15(e) we see that $H_c^{-j+2r+j'}((Z \times \tilde{Z})_J, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma \neq 0$. Using 6.15(b),(c) we see that the proposition holds.

6.17. In the case where $J = \mathbf{I}$, the first assertion of 6.16 reduces to the disjointness theorem [DL, 6.2, 6.3] (we use also the equivalence of 3.2(i), 3.2(vi)).

6.18. In this subsection we assume that $\tilde{\mathcal{L}} = \check{\mathcal{L}}$ and

(a) any $s \in \mathbf{I}$ is in the $\mathbf{c}$-orbit of some $s_i (i \in [1, r])$.

Since $\dim(Z \times \tilde{Z}) = \rho$, we have $H_c^i(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}}) = 0$ for $i < 2\rho$. We show:

(b) $\dim H_c^{2\rho}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = n_{\mathcal{L}}$ *where* $n_{\mathcal{L}}$ *is* 1 *if* $R = R_{\mathcal{L}}$ *and* 0 *if* $R \neq R_{\mathcal{L}}$.

By 6.14(c) we have $\dim H_c^{2\rho}(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \bar{\tilde{\mathcal{L}}})^\Gamma = |\mathfrak{A}|$ where $\mathfrak{A} = \{\mathbf{a} \in \mathcal{A}_1; N_{\mathbf{a}} = \rho\}$. Note that $\mathfrak{A}$ is the set of all sequences $\mathbf{a} = (a_0, a_1, \dots, a_\rho)$ in W such that

$a_{j-1}^{-1} a_j \in \{1, \tilde{s}_j\}$ for $j \in \mathcal{I}_{\tilde{\mathbf{s}}}$;
$a_{j-1}^{-1} a_j = \tilde{s}_j$ for $j \in [1, \tilde{r}] - \mathcal{I}_{\tilde{\mathbf{s}}}$;
$a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} \in \{1, s_i\}$ for $i \in \mathcal{I}_{\mathbf{s}}$;
$a_{\tilde{r}+i} a_{\tilde{r}+i-1}^{-1} = s_i$ for $i \in [1, r] - \mathcal{I}_{\mathbf{s}}$;
if $h \in [1, \tilde{r}]$, then $a_{h-1} \le a_h \ge a_{h-1} \tilde{s}_h$;
if $h \in [1, r]$, then $a_{\tilde{r}+h-1} \le a_{\tilde{r}+h} \ge s_h a_{\tilde{r}+h-1}$;
$a_\rho = \mathbf{c}(a_0)$;
$a_0^* \mathcal{L} \cong \mathcal{L}$.

If $\mathbf{a} \in \mathfrak{A}$, then $l(a_0) \le l(a_1) \le \dots \le l(a_\rho) = l(a_0)$ hence $l(a_0) = l(a_1) = \dots = l(a_\rho)$ and since $a_0 \le a_1 \le \dots \le a_\rho$, we have $a_0 = a_1 = \dots = a_\rho$. This forces $\mathcal{I}_{\tilde{\mathbf{s}}} = [1, \tilde{r}]$. Thus, $s_1, s_1 s_2 s_1, \dots, s_1 s_2 \dots s_r \dots s_2 s_1$ are in $W_{\mathcal{L}}$ hence $s_i \in W_{\mathcal{L}}$ for $i \in [1, r]$. Using $([\mathbf{s}]F)^* \mathcal{L} \cong \mathcal{L}$ and $[\mathbf{s}] \in W_{\mathcal{L}}$ we deduce $F^* \mathcal{L} \cong \mathcal{L}$. Hence if $i \in [1, r]$, then $\mathbf{c}^n(s_i) \in W_{\mathcal{L}}$ for any $n \ge 1$. Using (a) we deduce that $\mathbf{I} \subset W_{\mathcal{L}}$ hence $W = W_{\mathcal{L}}$ and $R = R_{\mathcal{L}}$.

If $R = R_{\mathcal{L}}$, then $\mathfrak{A}$ is in bijection with the set

$$\mathfrak{A}' = \{a_0 \in W; a_0 = \mathbf{c}(a_0), a_0 \tilde{s}_h < a_0 \text{ for } h \in [1, \tilde{r}], s_h a_0 < a_0 \text{ for } h \in [1, r]\}.$$

We set $I' = \{s \in \mathbf{I}; s = s_h \text{ for some } h \in [1, r]\}$, $I'' = \{s \in \mathbf{I}; s = \tilde{s}_h \text{ for some } h \in [1, \tilde{r}]\}$. We see that

$$\mathfrak{A}' = \{a_0 \in W; I' \subset L_{a_0}, I'' \subset R_{a_0}, a_0 = \mathbf{c}(a_0)\}$$

where for $w \in W$ we set $L_w = \{s \in \mathbf{I}; sw < w\}, R_w = \{s \in \mathbf{I}; ws < w\}$.

For a_0 such that $a_0 = \mathbf{c}(a_0)$, the set L_{a_0} is $\mathbf{c}$-stable; hence the condition $I' \subset L_{a_0}$ is equivalent to $I' \cup \mathbf{c}(I') \cup \mathbf{c}^2(I') \cup \dots \subset L_{a_0}$; that is, (by (a)) to $\mathbf{I} = L_{a_0}$. We see that $\mathfrak{A}'$ has exactly one element: $w_{\mathbf{I}}$. This proves (b).

7. The variety X

7.1. In this section we study the variety

$$X = \{(B', g) \in \mathcal{B} \times G; g^{-1} F(g) \in U_{B'}\}.$$

Note that X is an étale covering of the smooth connected variety $X' = \{(B', u) \in \mathcal{B} \times G; u \in U_{B'}\}$ of dimension via the map $\rho_1 : X \to X'$, $(B', g) \mapsto (B', g^{-1} F(g))$. Since $\dim X' = 2d$ where $d = \dim \mathcal{B}$, we see that X is smooth of pure dimension $2d$. Here is one of the main results of this section.

Proposition 7.2. *If G is simply connected, then X is connected.*

The proof is given in 7.16. Note that X is not necessarily connected without the assumption that G is simply connected.

7.3. Let $w \in W$. Let $i_w : \mathcal{B}_w \to \mathcal{B}$ be the inclusion. Let $A' \in \mathbb{S}(\mathcal{B}_w)$. Let A be a simple object of $\mathcal{M}_\Gamma(\mathcal{B})$ such that $A \dashv_\Gamma {}^pH^\cdot(i_{w!}A')$. We show:

(a) $$A \in \mathbb{S}(\mathcal{B}).$$

We argue by induction on $l(w)$. If $l(w) = 0$, then $i_{w!}A' \in \mathbb{S}(\mathcal{B})$ and the result is clear. We now assume that $l(w) > 0$. We have $A = \mathcal{L}_w[l(w)]$ where $\mathcal{L} \in \mathcal{S}(T)^{wF}$. Let $K = \mathcal{L}_w^\sharp[l(w)]$. We have $K \in \mathbb{S}(\mathcal{B})$. Assume first that $A \not\dashv_\Gamma {}^pH^\cdot(i_{w'!}i_{w'}^*K)$ for any $w' \in W$ such that $w' < w$. Then $A \not\dashv_\Gamma {}^pH^\cdot(u_!u^*K)$ where $u : \cup_{w';w'<w}\mathcal{B}_{w'} \to \mathcal{B}$ is the inclusion. Let $u' : \cup_{w';w'\le w}\mathcal{B}_{w'} \to \mathcal{B}$ be the inclusion. Since $i_{w!}i_w^*K = A'$ we see that $A \dashv_\Gamma {}^pH^\cdot(u'_!u'^*K)$. Thus $A \dashv_\Gamma K$ hence $A \cong K$ and $A \in \mathbb{S}(\mathcal{B})$.

Next we assume that $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}i_{w'}^*K)$ for some $w' \in W$ such that $w' < w$. It follows that there exists j' such that $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}({}^pH^{j'}(i_{w'}^*K)))$. Hence there exists a simple object A'' of $\mathcal{M}_\Gamma(\mathcal{B}_{w'})$ such that $A'' \dashv {}^pH^{j'}(i_{w'}^*K)$ and $A \dashv_\Gamma {}^pH^\cdot(i_{w'!}A'')$. From 4.14 we see that $A'' \in \mathbb{S}(\mathcal{B}_{w'})$. From the induction hypothesis we see that $A \in \mathbb{S}(\mathcal{B})$. This proves (a).

7.4. Let $w \in W$. Let $f_w : \tilde{\mathcal{B}}_w \to \mathcal{B}_w$ be as in 4.7, a finite principal covering. Let $U_w = \{u \in U; \dot{w}F(u)\dot{w}^{-1} \in U\}$. Let $\hat{\mathcal{B}}_w = \{z \in G; z^{-1}F(z) \in U\dot{w}\}$. Define $a' : \hat{\mathcal{B}}_w \to \tilde{\mathcal{B}}_w$ by $z \mapsto zU$, a principal U_w-bundle. Let

$X_w = \{(B', g) \in \mathcal{B} \times G; g^{-1}F(g) \in U_{B'}, \mathrm{pos}(B', F(B')) = w\}$,
$\tilde{X}_w = \{(zU_w, g) \in G/U_w \times G; z^{-1}F(z) \in U\dot{w}, g^{-1}F(g) \in zUz^{-1}\}$,
$\tilde{X}'_w = \{(z, y) \in G \times G; z^{-1}F(z) \in U\dot{w}, y^{-1}F(y) \in U\dot{w}\}$,
$\tilde{X}''_w = U_w \backslash \tilde{X}'_w$,

where U_w acts (freely) by $u : (z, y) \mapsto (zu^{-1}, yu^{-1})$. Define $\pi_w : X_w \to \mathcal{B}_w$ by $(B', g) \mapsto B'$. Define $\pi' : \tilde{X}_w \to \tilde{\mathcal{B}}_w$ by $(zU_w, g) \mapsto zU$. Under the identification $\tilde{X}_w = \tilde{X}''_w$, $(zU_w, g) \leftrightarrow (z, gz)U_w$, π' becomes $\pi'' : \tilde{X}''_w \to \tilde{\mathcal{B}}_w$, $(z, y)U_w \mapsto zU$. Define $a : \tilde{X}'_w \to \tilde{X}''_w$ by $(z, y) \mapsto (z, y)U_w$. Define $h : \tilde{X}'_w \to \hat{\mathcal{B}}_w$ by $(z, y) \mapsto z$. Define $\gamma : \tilde{X}_w \to X_w$ by $(zU_w, g) \mapsto (zBz^{-1}, g)$. Now Γ acts on $\tilde{X}''_w$ by $g_0 : (z, y) \mapsto (g_0z, y)$. We show:

(i) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(h_!\bar{\mathbf{Q}}_l)$, then $A \cong \bar{\mathbf{Q}}_l[\dim U]$;
(ii) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(a'_!h_!\bar{\mathbf{Q}}_l)$, then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(iii) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi''_!a_!\bar{\mathbf{Q}}_l)$, then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(iv) if $A \in \mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi'_!\bar{\mathbf{Q}}_l)$, then $A \cong \bar{\mathbf{Q}}_l[l(w)]$;
(v) if $A \in \mathcal{M}_\Gamma(\mathcal{B}_w)$ is simple and $A \dashv_\Gamma {}^pH^\cdot(\pi_{w!}\bar{\mathbf{Q}}_l))$, then $A \in \mathbb{S}(\mathcal{B}_w)$;
(vi) if $A \in \mathcal{M}_\Gamma(\mathcal{B})$ is simple and $A \dashv_\Gamma {}^pH^\cdot(i_{w!}\pi_{w!}\bar{\mathbf{Q}}_l)$, then $A \in \mathbb{S}(\mathcal{B})$.

Now (i) is obvious; (ii) follows from (i) using $a'_!\bar{\mathbf{Q}}_l = \bar{\mathbf{Q}}_l[-2\delta](-\delta)$ where $\delta = \dim U_w$; (iii) follows from (ii) using $a'h = \pi''a$; (iv) follows from (iii) using $a_!\bar{\mathbf{Q}}_l = \bar{\mathbf{Q}}_l[-2\delta](-\delta)$ and the identification $\pi' = \pi''$. Since the diagram formed by π', f_w, γ, π_w is cartesian we have $\pi'_!\bar{\mathbf{Q}}_l = \pi'_!\gamma^*\bar{\mathbf{Q}}_l = f_w^*\pi_{w!}\bar{\mathbf{Q}}_l$.

We prove (v). Let A be as in (v). Let A' be a simple object of $\mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$ such that $A' \dashv_\Gamma f_w^*A$. We have

$$A' \dashv_\Gamma f_w^*({}^pH^\cdot(\pi_{w!}\bar{\mathbf{Q}}_l)) = {}^pH^\cdot(f_w^*\pi_{w!}\bar{\mathbf{Q}}_l) = {}^pH^\cdot(\pi'_!\bar{\mathbf{Q}}_l)$$

and using (iv) we see that $A' \cong \bar{\mathbf{Q}}_l[l(w)]$. Then there exists a nonzero morphism $f_w^*A \to \bar{\mathbf{Q}}_l[l(w)]$ in $\mathcal{M}_\Gamma(\tilde{\mathcal{B}}_w)$. Hence there exists a nonzero morphism $A \to f_{w!}\bar{\mathbf{Q}}_l[l(w)]$ in $\mathcal{M}_\Gamma(\mathcal{B}_w)$. Now $f_{w!}\bar{\mathbf{Q}}_l[l(w)] = \oplus_\mathcal{L}\mathcal{L}_w[l(w)]$ where $\mathcal{L}$ runs over the local systems in $\mathcal{S}(T)^{wF}$ (up to isomorphism). We must have $A \cong \mathcal{L}_w[l(w)]$ for some $\mathcal{L}$ as above. This proves (v).

We prove (vi). We have $A \dashv_\Gamma {}^pH^\cdot(i_{w!}({}^pH^{j'}(\pi_{w!}\bar{\mathbf{Q}}_l)))$ for some j'. Hence there exists a simple object A_1 of $\mathcal{M}_\Gamma(\mathcal{B}_w)$ such that $A_1 \dashv_\Gamma {}^pH^{j'}(\pi_{w!}\bar{\mathbf{Q}}_l)$ and $A \dashv_\Gamma {}^pH^\cdot(i_{w!}A_1)$. From (v) we see that $A_1 \in \mathbb{S}(\mathcal{B}_w)$. From 7.3 we see that $A \in \mathbb{S}(\mathcal{B})$.

7.5. Let $J \subset \mathbf{I}$. Define $\pi_J : X \to \mathcal{P}_J$ by $(B', g) \mapsto P_{B',J}$. This is compatible with the Γ-actions where Γ acts on X by $g_0 : (B', g) \mapsto (g_0 B' g_0^{-1}, g g_0^{-1})$. Hence ${}^pH^j(\pi_{J!}\bar{\mathbf{Q}}_l)$ has a Γ-equivariant structure. We show:

Theorem 7.6. *(a) If A is a simple object of $\mathcal{M}_\Gamma(\mathcal{P}_J)$ such that $A \dashv_\Gamma {}^pH^\cdot(\pi_{J!}\bar{\mathbf{Q}}_l)$, then $A \in \mathbb{S}(\mathcal{P}_J)$.*

Note that π_J is a composition $\pi'\pi''$ where $\pi'' : X \to \mathcal{B}$ is $(B', g) \mapsto B'$ and $\pi'' : \mathcal{B} \to \mathcal{P}_J$ is $B' \mapsto P_{B',J}$. We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ with $E_2 = {}^pH^\cdot(\pi'_!({}^pH^\cdot(\pi''_!\bar{\mathbf{Q}}_l)))$ and E_∞ is an associated graded of ${}^pH^\cdot(\pi'_!\pi''_!\bar{\mathbf{Q}}_l)$. We have $A \dashv_\Gamma E_\infty$ hence $A \dashv_\Gamma E_2$. Hence we can find a simple object $A_1 \in \mathcal{M}_\Gamma(\mathcal{B})$ such that $A_1 \dashv_\Gamma {}^pH^\cdot(\pi''_!\bar{\mathbf{Q}}_l)$ and $A \dashv_\Gamma {}^pH^\cdot(\pi'_!A_1)$. Using the partition $\mathcal{B} = \sqcup_w \mathcal{B}_w$, we see that $A_1 \dashv_\Gamma {}^pH^\cdot(i_{w!}i_w^*\pi''_!\bar{\mathbf{Q}}_l)$ for some $w \in W$ (with i_w as in 7.3). Since $i_w^*\pi''_!\bar{\mathbf{Q}}_l = \pi_{w!}\bar{\mathbf{Q}}_l$ (with π_w as in 7.4), we see that $A_1 \dashv_\Gamma {}^pH^\cdot(i_{w!}\pi_{w!}\bar{\mathbf{Q}}_l)$. Using 7.4(vi) we see that $A_1 \in \mathbb{S}(\mathcal{B}_w)$. By 4.13 (for $\mathcal{B}$ instead of $\mathcal{P}_J$) we see that $A_1 \dashv_\Gamma {}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$ for some sequence $\mathbf{s}$ in $\mathbf{I}$ and some $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$ (here ${}^\emptyset\bar{\Upsilon}^{\mathbf{s}}$ is $\bar{\Upsilon}^{\mathbf{s}}$ of 3.2 with J replaced by $\emptyset$). We have a spectral sequence in $\mathcal{M}_\Gamma(\mathcal{P}_J)$ with $E_2 = {}^pH^\cdot(\pi'_!({}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)))$ and E_∞ is an associated graded of

$$ {}^pH^\cdot(\pi'_!{}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp) = {}^pH^\cdot(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp) $$

(with $\bar{\Upsilon}^{\mathbf{s}}$ as in 3.2). Now A_1 is a direct summand of ${}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$. Hence ${}^pH^\cdot(\pi'_!A_1)$ is a direct summand of ${}^pH^\cdot(\pi'_!({}^pH^\cdot({}^\emptyset\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)))$. Hence $A \dashv_\Gamma E_2$. Our spectral sequence is degenerate (by an argument as in 4.8). Hence $A \dashv_\Gamma E_\infty$ and $A \dashv_\Gamma {}^pH^\cdot(\bar{\Upsilon}^{\mathbf{s}}_!\bar{\mathcal{L}}^\sharp)$. Using 4.13 we deduce that $A \in \mathbb{S}(\mathcal{P}_J)$. The theorem is proved.

7.7. We have a Γ-action $g_1 : (B', g) \mapsto (B', g_1 g)$ on X (this is different from the Γ-action on X in 7.5). This induces a Γ-module structure on $H_c^{2d}(X, \bar{\mathbf{Q}}_l)$ (d as in 7.1).

Proposition 7.8. *The Γ-module $H_c^{2d}(X, \bar{\mathbf{Q}}_l)$ contains a copy of Reg, the left regular representation of Γ.*

Let $\rho_1 : X \to X'$ be as in 7.1. Let $X'' = \{u \in G; u \text{ unipotent}\}$. Define $\rho_2 : X' \to X''$ by $(B', u) \mapsto u$. Let $\rho = \rho_2\rho_1 : X \to X''$. It is well known that ρ_2 is a semismall morphism. Recall that ρ_1 is a finite étale covering. Hence ρ is a semismall morphism. Using this, we see that $\rho_!\bar{\mathbf{Q}}_l[2d]$ is a perverse sheaf on X' (recall that X is smooth of pure dimension $2d$ and ρ is proper). By the decomposition theorem [BBD], ${}^pH^\cdot(\rho_!\bar{\mathbf{Q}}_l)$ is a semisimple perverse sheaf. Hence $\rho_!\bar{\mathbf{Q}}_l[2d]$ is a semisimple perverse sheaf. Now Γ acts on X'' trivially and ρ is compatible with the Γ-actions. Since $\bar{\mathbf{Q}}_l$ is naturally a Γ-equivariant local system on X we see that we have naturally $\rho_!\bar{\mathbf{Q}}_l[2d] \in \mathcal{M}_\Gamma(X'')$. Let $j : \{1\} \to X''$ be the inclusion of the unit element into X''. If E is an irreducible Γ-module over $\bar{\mathbf{Q}}_l$ we can regard E as a Γ-equivariant local system on $\{1\}$. Then $j_!E$ is a simple object of $\mathcal{M}_\Gamma(X'')$. Let n_E be the number of times $j_!E$ appears in a direct sum decomposition of $\rho_!\bar{\mathbf{Q}}_l[2d]$ (a semisimple object of $\mathcal{M}_\Gamma(X'')$) into a direct sum of simple objects. Since $\rho_!\bar{\mathbf{Q}}_l[2d]$

is selfdual, we have using 1.6(a):

$$n_E = \dim H_c^0(X'', j_! E \otimes \rho_! \bar{\mathbf{Q}}_l[2d])^\Gamma = \dim H_c^{2d}(\{1\}, E \otimes j^* \rho_! \bar{\mathbf{Q}}_l)^\Gamma$$
$$= \dim(H_c^{2d}(\mathcal{B} \times \Gamma, \bar{\mathbf{Q}}_l) \otimes E)^\Gamma = \dim(\mathrm{Reg}(-d) \otimes E)^\Gamma = \dim E.$$

We see that $\rho_! \bar{\mathbf{Q}}_l[2d]$ contains as a direct summand the perverse sheaf $j_!\mathrm{Reg}$ where *Reg* is regarded as an object of $\mathcal{M}_\Gamma(\{1\})$. It follows that the Γ-module $H_c^0(\rho_! \bar{\mathbf{Q}}_l[2d])$ contains as a direct summand the Γ-module $H_c^0(j_!\mathrm{Reg}) = \mathrm{Reg}$. Equivalently the Γ-module $H^{2d}(X, \bar{\mathbf{Q}}_l)$ contains Reg as a direct summand. The proposition is proved.

Corollary 7.9. *Let E be an irreducible Γ-module over $\bar{\mathbf{Q}}_l$. There exists $w \in W$ such that E appears in the Γ-module $\oplus_i H_c^i(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$. ($\Gamma$ acts on $\tilde{\mathcal{B}}_w$ by $g_1 : zU \to g_1 zU$.)*

From 7.8 we see that E appears in the Γ-module $H_c^{2d}(X, \bar{\mathbf{Q}}_l)$. Using this and the partition $X = \sqcup_{w \in W} X_w$ (with X_w as in 7.4, Γ-stable) we see that there exists $w \in W$ such that E appears in the Γ-module $H_c^{2d}(X_w, \bar{\mathbf{Q}}_l)$.

Now Γ acts:
on $\tilde{X}_w$ by $g_1 : (zU_w, g) \mapsto (zU_w, g_1 g)$,
on $\tilde{X}'_w$ by $g_1 : (z, y) \mapsto (z, g_1 y)$,
on $\tilde{X}''_w$ by $g_1 : (z, y)U_w \mapsto (z, g_1 y)U_w$,
on $\hat{\mathcal{B}}_w$ by $g_1 : z \mapsto g_1 z$.

(Notation of 7.4.) Moreover, the maps γ, a, a' in 7.4 are compatible with the Γ-actions. Since γ is a finite principal covering we see that E appears in the Γ-module $H_c^{2d}(\tilde{X}_w, \bar{\mathbf{Q}}_l)$. We identify $\tilde{X}_w = \tilde{X}''_w$ as in 7.4. We see that E appears in the Γ-module $H_c^{2d}(\tilde{X}''_w, \bar{\mathbf{Q}}_l)$. Since a is an affine space bundle we see that E appears in the Γ-module $H_c^{4d-2l(w)}(\tilde{X}'_w, \bar{\mathbf{Q}}_l)$. We have $\tilde{X}'_w = \hat{\mathcal{B}}_w \times \hat{\mathcal{B}}_w$ and $\oplus_i H_c^i(\tilde{X}'_w, \bar{\mathbf{Q}}_l) = \oplus_{i,i'} H_c^i(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l) \otimes H_c^{i'}(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$ with Γ acting only on the seond factor. It follows that E appears in the Γ-module $\oplus_i H_c^i(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)$. Since a' is an affine space bundle we see that the corollary holds.

7.10. For $w \in W$ let $\mathfrak{T}_w = \{t \in T; \dot{w}F(t)\dot{w}^{-1} = t\}$. Now $\Gamma \times \Gamma \times \mathfrak{T}_w$ acts on X_w by $(g_0, g_1, t) : (B', g) \mapsto (g_0 B' g_0^{-1}, g_1 g g_0^{-1})$, on $\tilde{X}_w$ by $(g_0, g_1, t) : (zU_w, g) \mapsto (g_0 z t^{-1} U_w, g_1 g g_0^{-1})$, on $\tilde{X}'_w$ by $(g_0, g_1, t) : (z, y) \mapsto (g_0 z t^{-1}, g_1 y t^{-1})$, on $\tilde{X}''_w$ by $(g_0, g_1, t) : (z, y)U_w \mapsto (g_0 z t^{-1}, g_1 y t^{-1})U_w$. Moreover, the maps γ, a in 7.4 are compatible with the $\Gamma \times \Gamma \times \mathfrak{T}_w$-actions. Also $\Gamma \times \mathfrak{T}_w$ acts on $\hat{\mathcal{B}}_w$ by $(g_1, t) : z \mapsto g_1 z t^{-1}$, on $\tilde{\mathcal{B}}_w$ by $(g_1, t) : zU \mapsto g_1 z t^{-1} U$; the map a' in 7.4 is compatible with the $\Gamma \times \mathfrak{T}_w$-actions.

For any $\theta \in \hat{\mathfrak{T}}_w := \mathrm{Hom}(\mathfrak{T}_w, \bar{\mathbf{Q}}_l^*)$ let $H_c^i(\tilde{\mathcal{B}}_w)_\theta$ be the subspace of $H_c^i(\tilde{\mathcal{B}}_w)$ on which $\mathfrak{T}_w$ acts via θ; this is naturally a $\Gamma \times \mathfrak{T}_w$-module. Now $\Gamma \times \Gamma$ acts on X by $(g_0, g_1) : (B', g) \mapsto (g_0 B' g_0^{-1}, g_1 g g_0^{-1})$.

Let $\mathcal{G}(\Gamma)$ (resp. $\mathcal{G}(\Gamma \times \Gamma)$ or $\mathcal{G}(\Gamma \times \Gamma \times \mathfrak{T}_w)$) be the Grothendieck group of Γ-modules (resp $\Gamma \times \Gamma$-modules or $\Gamma \times \Gamma \times \mathfrak{T}_w$-modules) of finite dimension over $\bar{\mathbf{Q}}_l$. Let $\Pi : \mathcal{G}(\Gamma \times \Gamma \times \mathfrak{T}_w) \to \mathcal{G}(\Gamma \times \Gamma \times \mathfrak{T}_w)$ be the homomorphism which takes an irreducible $\Gamma \times \Gamma \times \mathfrak{T}_w$-module to the space of $\mathfrak{T}_w$-invariants (an irreducible $\Gamma \times \Gamma \times \mathfrak{T}_w$-module or 0). In the setup of 7.9 we show:

Proposition 7.11.

(a) $$\sum_i (-1)^i H_c^i(X, \bar{\mathbf{Q}}_l) = \sum_{\substack{w \in W \\ \theta \in \hat{\mathfrak{T}}_w \\ i,i'}} (-1)^{i+i'} H_c^i(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_{\theta^{-1}} \otimes H_c^{i'}(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_\theta.$$

equality in $\mathcal{G}(\Gamma \times \Gamma)$.

By the arguments in the proof of 7.9 we have

$$\sum_i (-1)^i H_c^i(X, \bar{\mathbf{Q}}_l) = \sum_{w \in W} \sum_i (-1)^i H_c^i(X_w, \bar{\mathbf{Q}}_l),$$

equality in $\mathcal{G}(\Gamma \times \Gamma)$ and

$$\sum_i (-1)^i H_c^i(X_w, \bar{\mathbf{Q}}_l) = \Pi(\sum_i (-1)^i H_c^i(\tilde{X}_w, \bar{\mathbf{Q}}_l)) = \Pi(\sum_i (-1)^i H_c^i(\tilde{X}''_w, \bar{\mathbf{Q}}_l))$$
$$= \Pi(\sum_i (-1)^i H_c^i(\tilde{X}'_w, \bar{\mathbf{Q}}_l)) = \Pi(\sum_{i,i'} (-1)^{i+i'} H_c^i(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l) \otimes H_c^{i'}(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l))$$
$$= \sum_{\theta \in \hat{\mathfrak{T}}_w} \sum_{i,i'} (-1)^{i+i'} H_c^i(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_{\theta^{-1}} \otimes H_c^{i'}(\hat{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_\theta$$
$$= \sum_{\theta \in \hat{\mathfrak{T}}_w} \sum_{i,i'} (-1)^{i+i'} H_c^i(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_{\theta^{-1}} \otimes H_c^{i'}(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_\theta,$$

equalities in $\mathcal{G}(\Gamma \times \Gamma \times \mathfrak{T}_w)$. This proves (a).

7.12. Restricting the $\Gamma \times \Gamma$-action on the modules in 7.11(a) to Γ by $g_1 \mapsto (1, g_1)$ gives

$$\sum_i (-1)^i H_c^i(X, \bar{\mathbf{Q}}_l) = \sum_{\substack{w \in W \\ \theta \in \hat{\mathfrak{T}}_w \\ i,i'}} (-1)^{i+i'} \dim(H_c^i(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_{\theta^{-1}}) H_c^{i'}(\tilde{\mathcal{B}}_w, \bar{\mathbf{Q}}_l)_\theta.$$

equality in $\mathcal{G}(\Gamma)$. From [DL, 7.1, (7.6.3)] we see that the right hand side is equal to $|W|\text{Reg}$. Thus we have

Proposition 7.13.

(a)
$$\sum_i (-1)^i H_c^i(X, \bar{\mathbf{Q}}_l) = |W| \textit{Reg in } \mathcal{G}(\Gamma).$$

Lemma 7.14. *Assume that G is simply connected. Let $\mathbf{s} = (s_1, \dots, s_r), \mathcal{L}$ be as in 2.6 and that 6.18(a) holds. Let $Z^{\mathbf{s}}$ be as in 2.5. Let $\dot{Z}$ be as in 2.5 (with $\mathbf{w} = \mathbf{s}$). Let $Z = \mathcal{Z}^{\mathbf{s}}$ be as in 2.6. Then:*
(a) $\dim H_c^{2r}(Z, \bar{\mathcal{L}}) = n_{\mathcal{L}}$*;*
(b) $\dim H_c^{2r}(Z^{\mathbf{s}}, \mathcal{L}_{\mathbf{s}}) = n_{\mathcal{L}}$*;*
(c) $\dim H_c^{2r}(\dot{Z}, \bar{\mathbf{Q}}_l) = 1$.

We prove (a). We shall use 6.18(b) assuming that $\tilde{\mathbf{s}} = \mathbf{s}$. Then $Z = Z'$ and $\dim Z = r$ hence $H_c^i(Z, \bar{\mathcal{L}}) = 0$ for $i < 2r$ and $H_c^i(Z, \bar{\bar{\mathcal{L}}}) = 0$ for $i < 2r$. Hence $H_c^{4r}(Z \times Z, \bar{\mathcal{L}} \boxtimes \bar{\bar{\mathcal{L}}}) = H_c^{2r}(Z, \bar{\mathcal{L}}) \otimes H_c^{2r}(Z, \bar{\bar{\mathcal{L}}})$. Since in this case $\rho = 2r$ we see from 6.18(b) that $\dim(H_c^{2r}(Z, \bar{\mathcal{L}}) \otimes H_c^{2r}(Z, \bar{\bar{\mathcal{L}}}))^\Gamma = n_{\mathcal{L}}$. Using Poincaré duality on the smooth variety Z of pure dimension r (see 2.9) we deduce that $\dim(H^0(Z, \bar{\bar{\mathcal{L}}}) \otimes H^0(Z, \bar{\mathcal{L}}))^\Gamma = n_{\mathcal{L}}$. Note that the Γ-modules $H^0(Z, \bar{\bar{\mathcal{L}}})$, $H^0(Z, \bar{\mathcal{L}})$ are dual to each other. Hence the previous equality can be written as $\dim \text{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) = n_{\mathcal{L}}$ where $\text{End}_\Gamma()$ is the space of Γ-module endomorphisms. If $H^0(Z, \bar{\mathcal{L}}) = 0$, it follows that $n_\lambda = 0$; in this case we have also $H^0(Z, \bar{\bar{\mathcal{L}}}) = 0$ and by Poincaré duality, $H_c^{2r}(Z, \bar{\mathcal{L}}) = 0$ and (a) holds. Thus we may assume that $H^0(Z, \bar{\mathcal{L}}) \neq 0$. Then $\dim \text{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) \geq 1$ hence $\dim \text{End}_\Gamma(H^0(Z, \bar{\mathcal{L}})) = n_{\mathcal{L}} = 1$. Then $R = R_{\mathcal{L}}$.

Since G is simply connected, it follows that $\mathcal{L} \cong \bar{\mathbf{Q}}_l$. Then $H^0(Z, \bar{\mathcal{L}}) = H^0(Z, \bar{\mathbf{Q}}_l)$ may be identified with the permutation representation V of Γ on the set C of connected components of Z. Let Γ' be the isotropy group in Γ of some connected component of Z. Since Γ acts transitively on C (see 2.9) we see that $\dim V = |\Gamma/\Gamma'|$, $1 = \dim \mathrm{End}_G(V) = |\Gamma' \backslash \Gamma/\Gamma'|$. It follows that $\Gamma = \Gamma'$ hence $\dim V = 1$. Then we have $\dim H^0(Z, \tilde{\bar{\mathcal{L}}}) = 1$ and, by Poincaré duality, $\dim H_c^{2r}(Z, \bar{\mathcal{L}}) = 1$. This proves (a).

We prove (b). Note that $Z^{\mathbf{s}}$ is an open dense subset of Z (using the commutative diagram in 2.6 with $\mathcal{J} = \emptyset$ and 2.8, this statement is reduced to the statement that $Z_2^{\emptyset}$ in 2.6 is open dense in Z_2 which is clear). Note also that $\bar{\mathcal{L}}|_{Z^{\mathbf{s}}} = \mathcal{L}_{\mathbf{s}}$ (see 2.12) and that Z has pure dimension r (see 2.9). We see that (b) follows from (a).

We prove (c). Let $f_!^{\chi}\bar{\mathbf{Q}}_l$ be as in 2.5 with $\mathbf{w} = \mathbf{s}$. From the definitions we have $H_c^{2r}(\dot{Z}, \bar{\mathbf{Q}}_l) = \oplus_{\chi} H_c^{2r}(Z^{\mathbf{s}}, f_!^{\chi}\bar{\mathbf{Q}}_l)$ where χ runs over $\mathrm{Hom}(T^{F'}, \bar{\mathbf{Q}}_l^*)$ (as in 2.5). Using (b) and 2.5(a) we see that $\dim H_c^{2r}(Z^{\mathbf{s}}, f_!^{\chi}\bar{\mathbf{Q}}_l)$ is 1 if $\chi = 1$ and is 0 if $\chi \neq 1$. The result follows.

7.15. Assume that G is simply connected. Let d be as in 7.1. Let $w = w_{\mathbf{I}}$. We show:

(a) $\tilde{\mathcal{B}}_w$ *is connected;*

(b) X_w *is connected (notation of 7.4);*

Assume that $\mathbf{s}$ is a reduced expression for w. The associated variety $\dot{Z}$ (see 2.5) is connected by 7.14(c) (note that $\dot{Z}$ has pure dimension $d = l(w)$). But $\dot{Z}$ may be identified with $\tilde{\mathcal{B}}_w$. Hence (a) holds. Since $a' : \hat{\mathcal{B}}_w \to \tilde{\mathcal{B}}_w$ is a principal U_w-bundle (as in 7.4) we see that $\hat{\mathcal{B}}_w$ is connected. Since $\tilde{X}'_w \cong \hat{\mathcal{B}}_w \times \hat{\mathcal{B}}_w$ (as in 7.4) we see that $\tilde{X}'_w$ is connected. Since $a : \tilde{X}'_w \to \tilde{X}''_w$ (as in 7.4) is a principal U_w-bundle we see that $\tilde{X}''_w$ is connected. Since $\tilde{X}_w = \tilde{X}''_w$ (as in 7.4) we see that $\tilde{X}_w$ is connected. Since $\gamma : \tilde{X}_w \to X_w$ (as in 7.4) is surjective, we see that X_w is connected. Hence (b) holds.

Note that (a),(b) above do not necessarily hold without the assumption that G is simply connected.

7.16.

Proof of Proposition 7.2. Note that $X = \cup_{w' \in W} X_{w'}$, that X is of pure dimension $2d$, that X_w is an open subset of X and that for any $w' \in W - \{w\}$, $X_{w'}$ has pure dimension equal to $d + l(w') < 2d$. It follows that X is connected if and only if X_w is connected. Hence 7.2 is a consequence of 7.15(b). □

8. A conjecture

8.1. In this section we assume that G has connected centre. Let $\mathcal{L} \in \mathcal{S}(T)$ be such that $(wF)^*\mathcal{L} \cong \mathcal{L}$ for some $w \in W$. Our assumption on G guarantees that

(a) $w \in W'_{\mathcal{L}} \implies w \in W_{\mathcal{L}}$.

(Notation of 5.7.) Let $\mathcal{X}_{\mathcal{L}}$ be the set of all sequences $\mathbf{s} = (s_1, \ldots, s_r)$ in $\mathbf{I}$ such that $\mathcal{L} \in \mathcal{S}(T)^{[\mathbf{s}]F}$. Note that $\mathcal{X}_{\mathcal{L}} \neq \emptyset$.

To any $\mathbf{s} \in \mathcal{X}_{\mathcal{L}}$ we associate an element $\omega \in W$ as in 5.5. We show that ω is independent of the choice of $\mathbf{s}$. We must show that if $\tilde{\mathbf{s}}$ is another element of $\mathcal{X}_{\mathcal{L}}$ and $\tilde{\omega} \in W$ is associated to $\tilde{\mathbf{s}}$ in the same way as ω is associated to $\mathbf{s}$, then $\tilde{\omega} = \omega$. Using 5.6 we see that $F^*\omega^*\mathcal{L} \cong \mathcal{L}$, $F^*\tilde{\omega}^*\mathcal{L} \cong \mathcal{L}$. Hence $\omega^*\mathcal{L} \cong \tilde{\omega}^*\mathcal{L}$ so that $\tilde{\omega}\omega^{-1} \in W'_{\mathcal{L}}$ and, using (a), $\tilde{\omega}\omega^{-1} \in W_{\mathcal{L}}$. By the proof of 5.8 (with $c = \mathbf{c}$) we have $\omega\mathbf{c}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$ and similarly $\tilde{\omega}\mathbf{c}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. Thus $\omega^{-1}(R^+_{\mathcal{L}}) = \tilde{\omega}^{-1}(R^+_{\mathcal{L}})$ and $\tilde{\omega}\omega^{-1}(R^+_{\mathcal{L}}) = R^+_{\mathcal{L}}$. This together with $\tilde{\omega}\omega^{-1} \in W_{\mathcal{L}}$ yields $\tilde{\omega}^{-1}\omega = 1$, as desired.

Using the previous paragraph and (a) we see that for any $\mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{X}_{\mathcal{L}}$, the set $\mathfrak{F}$ defined as in 5.9 (with $c = \mathbf{c}$) is equal to $\{1\}$.

8.2. For $\mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{X}_{\mathcal{L}}$ let $Z, \tilde{Z}, r, \tilde{r}$ be as in 6.1. Let $\bar{\mathcal{L}}$ be as in 6.2 and let $\bar{\tilde{\mathcal{L}}}$ be the analogous local system on $\tilde{Z}$ defined in terms of $\mathcal{L}$. Then $\check{\bar{\tilde{\mathcal{L}}}}$ (dual of $\bar{\tilde{\mathcal{L}}}$) is the analogous local system on $\tilde{Z}$ defined in terms of $\check{\mathcal{L}}$. Let

$$V_{\tilde{\mathbf{s}},\mathbf{s}} = \mathrm{Hom}_{\Gamma}(\oplus_i H^i_c(\tilde{Z}, \bar{\tilde{\mathcal{L}}})(i/2), \oplus_{i'} H^{i'}_c(Z, \bar{\mathcal{L}})(i'/2)).$$

where Hom_{Γ} is the space of homomorphisms of Γ-modules. Using 2.13(a) we see that Poincaré duality holds on $\tilde{Z}$ in the form

$$\mathrm{Hom}(H^i_c(\tilde{Z}, \bar{\tilde{\mathcal{L}}})(i/2), \bar{\mathbf{Q}}_l) = H^{2\tilde{r}-i}_c(\tilde{Z}, \check{\bar{\tilde{\mathcal{L}}}})(\tilde{r} - i/2).$$

Hence

$$\mathrm{Hom}(\oplus_i H^i_c(\tilde{Z}, \bar{\tilde{\mathcal{L}}})(i/2), \bar{\mathbf{Q}}_l) = \oplus_i H^i_c(\tilde{Z}, \check{\bar{\tilde{\mathcal{L}}}})(i/2),$$

$$V_{\mathbf{s},\tilde{\mathbf{s}}} = ((\oplus_{i'} H^{i'}_c(Z, \bar{\mathcal{L}})(i'/2)) \otimes (\oplus_i H^i_c(\tilde{Z}, \check{\bar{\tilde{\mathcal{L}}}})(i/2)))^{\Gamma},$$

$$V_{\mathbf{s},\tilde{\mathbf{s}}} = (\oplus_n H^n_c(Z \times \tilde{Z}, \bar{\mathcal{L}} \boxtimes \check{\bar{\tilde{\mathcal{L}}}})(n/2))^{\Gamma}.$$

By 6.14(d) and 6.15(b), the last vector space has a distinguished basis $\{b^1_{\mathbf{a}}; \mathbf{a} \in \mathcal{A}_1\}$, with $\mathcal{A}_1$ as in 6.14.

Let $C_{\mathcal{L}}$ be the category whose objects are the elements of $\mathcal{X}_{\mathcal{L}}$ and in which the set of morphisms from $\tilde{\mathbf{s}}$ to $\mathbf{s}$ is the vector space $V_{\tilde{\mathbf{s}},\mathbf{s}}$. The composition of morphisms is given by composing linear maps.

8.3. We will view T as a maximally $\mathbf{F}_q$-split torus of a second connected reductive algebraic group G' over $\mathbf{F}_q$ in such a way that $R_{\mathcal{L}}$ is the set of roots of G' with respect to T' and $R^+_{\mathcal{L}}$ is the set of positive roots of G' with respect to T' and a Borel subgroup B' of G' which is defined over $\mathbf{F}_q$ and contains T'.

Replacing $G, T, B, \mathcal{L}$ by $G', T, B', \bar{\mathbf{Q}}_l$ in the definition of the set $\mathcal{X}_{\mathcal{L}}$ and of the category $C_{\mathcal{L}}$ in 8.2 we obtain a set $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$ and a category $C'_{\bar{\mathbf{Q}}_l}$. Note that the objects of $C'_{\bar{\mathbf{Q}}_l}$ are the elements of $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$; that is, the sequences $\mathbf{S} = (S_1, S_2, \dots, S_b)$ in $W_{\mathcal{L}}$. For $\tilde{\mathbf{S}}, \mathbf{S}$ in $\mathcal{X}'_{\bar{\mathbf{Q}}_l}$ we denote by $V'_{\tilde{\mathbf{S}},\mathbf{S}}$ the vector space of morphisms from $\tilde{\mathbf{S}}$ to $\mathbf{S}$ in $C'_{\bar{\mathbf{Q}}_l}$.

The following is conjecturally a functor $\Phi : C_{\mathcal{L}} \to C'_{\bar{\mathbf{Q}}_l}$. To an object $\mathbf{s}$ of $C_{\mathcal{L}}$, Φ associates the object $\mathbf{S}$ of $C'_{\bar{\mathbf{Q}}_l}$ defined as in 5.5. Given two objects $\tilde{\mathbf{s}}, \mathbf{s}$ of $C_{\mathcal{L}}$ we set $\tilde{\mathbf{S}} = \Phi(\tilde{\mathbf{s}}), \mathbf{S} = \Phi(\mathbf{s})$ and we define a linear map $\Phi : V_{\tilde{\mathbf{s}},\mathbf{s}} \to V'_{\tilde{\mathbf{S}},\mathbf{S}}$ to be the isomorphism which maps the distinguished basis of $V_{\tilde{\mathbf{s}},\mathbf{s}}$ onto the analogous distinguished basis of $V'_{\tilde{\mathbf{S}},\mathbf{S}}$ according to the bijection $\mathcal{A}(W, \mathbf{c}, \mathcal{L}, \mathbf{s}, \tilde{\mathbf{s}}) \xrightarrow{\sim} \mathcal{A}(W_{\mathcal{L}}, \omega\mathbf{c}, \bar{\mathbf{Q}}_l, \mathbf{S}, \tilde{\mathbf{S}})$ described in 5.10. (As pointed out in 8.1, the set $\mathfrak{F}$ which appears in 5.10 is in our case equal to $\{1\}$.) We expect that Φ is a functor and that moreover it is an equivalence of categories.

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