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# FORMULAS FOR PRIMITIVE IDEMPOTENTS IN FROBENIUS ALGEBRAS AND AN APPLICATION TO DECOMPOSITION MAPS

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ABSTRACT. In the first part of this paper we present explicit formulas for primitive idempotents in arbitrary Frobenius algebras using the entries of representing matrices coming from projective indecomposable modules with respect to a certain choice of basis. The proofs use a generalisation of the well-known Frobenius-Schur relations for semisimple algebras.

The second part of this paper considers $\mathcal{O}$-free $\mathcal{O}$-algebras of finite $\mathcal{O}$-rank over a discrete valuation ring $\mathcal{O}$ and their decomposition maps under modular reduction modulo the maximal ideal of $\mathcal{O}$, thereby studying the modular representation theory of such algebras.

Using the formulas from the first part we derive general criteria for such a decomposition map to be an isomorphism that preserves the classes of simple modules involving explicitly known matrix representations on projective indecomposable modules.

Finally, we show how this approach could eventually be used to attack a conjecture by Gordon James in the formulation of Meinolf Geck for Iwahori-Hecke algebras, provided the necessary matrix representations on projective indecomposable modules could be constructed explicitly.

## 1. Introduction

Primitive idempotents play a crucial rôle in the representation theory of finite groups and finite-dimensional algebras. In the semisimple case one has explicit formulas for central primitive idempotents using the irreducible characters, and for primitive idempotents using the entries of irreducible matrix representations. The crucial ingredient to prove these formulas are the well-known Frobenius-Schur relations that involve the matrix coordinate functions of the irreducible representations.

In this paper we generalise this to an arbitrary, finite-dimensional Frobenius algebra $H$ over a field. To this end, we prove generalisations of the Frobenius-Schur relations involving the matrix coordinate functions of representations coming from projective indecomposable modules. However, we have to choose their basis carefully, namely, the basis must be adjusted to the socle and the radical of the module; see Section 2 for details.

We then consider algebras over rings and study their decomposition maps under modular reduction. A general version of Brauer reciprocity shows that the dual map

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of such a decomposition map can alternatively be defined using idempotents and the corresponding projective Grothendieck groups. Thus, explicit knowledge about matrix representations on projective indecomposable modules can be translated using our formulas for primitive idempotents into knowledge about decomposition maps.

Finally, we hope that this theory could eventually be applied as part of a proof of a conjecture by Gordon James in a formulation by Meinolf Geck about Iwahori-Hecke algebras.

In Section 2 we fix our notation and briefly recall some definitions and facts. Then we present the well-known averaging operator and some of its properties including the theorem by Gaschütz and Ikeda for Frobenius algebras. In Section 3 we apply the averaging operator to linear maps, the image of which has dimension 1, and derive new proofs in modern language of the generalised Frobenius-Schur relations that already appear in [14] and [1].

We then proceed to explicitly construct primitive idempotents using the above relations in Section 4. An important feature of these formulas is that we can control the denominators in the coefficients quite explicitly in terms of the given matrix representation. This will be crucial later in applying these formulas to duals of decomposition maps.

Having these preparations at hand, we then set up the concept of decomposition maps and their duals in Section 5 using the well-known duality between the Grothendieck group $R_0(H)$ of the category of finitely generated $H$-modules and the Grothendieck group $K_0(H)$ of the category of finitely generated projective $H$-modules for a finite-dimensional algebra $H$ over a field. This involves an extension of the classical Brauer reciprocity to the non-semisimple case. In a more general setting, such an extension has already been discussed in [9].

The definitions in Section 6 then allow us to use our formulas for primitive idempotents to derive criteria for a decomposition map to be trivial, that is, being an isomorphism that preserves the classes of simple modules.

Finally, in Section 7, we show how this whole theory could eventually be applied as a part of a proof for James' conjecture. We give another argument that the statement of the conjecture can only be false for a finite number of cases. However, so far we cannot give an explicit bound. We show in detail what is needed to apply our results.

## 2. Averaging

Let $K$ be a field, $H$ a finite-dimensional associative $K$-algebra and mod-$H$ the category of finite-dimensional right $H$-modules. We assume throughout that $K$ is a splitting field for $H$.

For a given $K$-basis $(b_i)_{i=1}^n$ of a right $H$-module $M$ we denote by $(b_i^*)_{i=1}^n$ the basis of the dual space $M^* = \operatorname{Hom}_K(M, K)$ with $b_i^*(b_j) = \delta_{i,j}$ for $i, j \in \{1, \dots, n\}$.

**Definition 1.** If $H^*$ contains a linear map $\tau$, such that

$$\varphi : H \to H^*, a \mapsto (b \mapsto \tau(ab))$$

is a left $H$-module isomorphism, then the pair $(H, \tau)$ is called a **Frobenius algebra**.

By [2, Theorem 61.3] the bilinear form $(a|b) := \tau(ab)$ is non-degenerate and associative if and only if $(H, \tau)$ is a Frobenius algebra. Therefore we have for every $K$-basis $(C_w)_{w \in W}$ of a Frobenius algebra $H$ a uniquely determined $K$-basis

$(C_y^\vee)_{y\in W}$ such that $\tau(C_y^\vee C_w) = \delta_{y,w}$ for all $y, w \in W$. We call $(C_y^\vee)_{y\in W}$ the **dual basis** of $(C_y)_{y\in W}$.

In the next lemma we use the following notation. Let $\alpha : H \to H$ be an automorphism and $L$ a right $H$-module. Then the vector space $L$ is a right $H$-module with the following action: $l * h := l\alpha(h)$ for all $l \in L$ and $h \in H$. We denote the $H$-module $(L, *)$ with $L^\alpha$.

**Lemma 2** (Frobenius algebras, Nakayama automorphism)**.**

(1) *Let $(H, \tau)$ be a Frobenius algebra, then there exists exactly one automorphism $\alpha$ of $H$ such that $\tau(ab) = \tau(\alpha(b)a)$ for all $a, b \in H$. We call $\alpha$ the* **Nakayama automorphism***.*

(2) *Let $(H, \tau)$ be a Frobenius algebra, $\alpha$ the Nakayama automorphism and $P$ a projective indecomposable module of $H$ with socle $S := \mathrm{soc}(P)$ and head $V := P/\mathrm{rad}(P)$. Then $V$ and $S$ are simple and $V$ is isomorphic to $S^{\alpha^{-1}}$.*

*Proof.* (1) As the map $\varphi$ from the first definition is an isomorphism, there is for each $b \in H$ a unique element $\alpha(b)$ with $\tau(ab) = \tau(\alpha(b)a)$ for all $a \in H$. One easily checks that the map $b \mapsto \alpha(b)$ is linear and we have $\tau(\alpha(bc)a) = \tau(abc) = \tau(\alpha(c)ab) = \tau(\alpha(b)\alpha(c)a)$ for all $a \in H$. By the non-degeneracy of $\tau$ we conclude $\alpha(bc) = \alpha(b)\alpha(c)$ for $b, c \in H$ and that $\alpha$ is bijective. Thus $\alpha$ is an automorphism.

(2) The modules $V$ and $S$ are both simple by [3, Prop. (9.9).(ii)]. Let $e \in H$ be an idempotent with $eH \cong P$. If we identify $P$ and $eH$, we have $eS = S$. The module $S^{\alpha^{-1}}$ is simple, so $\mathrm{Hom}_H(eH, S^{\alpha^{-1}}) \cong S^{\alpha^{-1}}e$ is not equal to $\{0\}$ if and only if $S^{\alpha^{-1}} \cong V$. But $\tau(S^{\alpha^{-1}}e) = \tau(S\alpha^{-1}(e)) = \tau(eS) \neq \{0\}$, because $eS$ is a right ideal and the kernel of $\tau$ does not contain a non-zero right ideal. It follows that $S^{\alpha^{-1}}e \neq \{0\}$ and thus $S^{\alpha^{-1}} \cong V$. □

*Notation* 3 (Conventions for further reference). In the following we assume that $(H, \tau)$ is a Frobenius algebra, $\alpha$ the Nakayama automorphism and $P$ a projective indecomposable module with $\mathrm{soc}(P) = S$ and $P/\mathrm{rad}(P) = V$. Then $P$ is isomorphic to $eH$ for a primitive idempotent $e$. We want to give a formula to compute such primitive idempotents $e$ in $H$ with $P \cong eH$.

Let $(B_w)_{w\in W}$ be an arbitrary basis of $H$ with dual basis $(B_w^\vee)_{w\in W}$. Because of Lemma 2(2) the dimension $d$ of $V$ is equal to the dimension of $S$. Let $n$ be the dimension of $P$ and $m := n - d$. We choose a basis $(b_i)_{i=1}^n$ of $P$ in the following way:

We extend a basis $(b_i)_{i=1}^d$ of $S$ to a basis $(b_i)_{i=1}^m$ of $\mathrm{rad}(P)$. We can then extend the basis $(b_i)_{i=1}^m$ of $\mathrm{rad}(P)$ to a basis $(b_i)_{i=1}^n$ such that $b_i^*(b_j h) = b_{m+i}^*(b_{m+j}\alpha(h))$ for all $h \in H$ and $i, j \in \{1, \dots d\}$. Lemma 2(2) makes sure that there is such a basis.

We now introduce the averaging operator, which is similar to the standard proof of Maschke's Theorem. It will be crucial for our proofs.

**Theorem 4** (Averaging operator)**.** *We use the conventions in Notation* 3*. Let $M, N \in \text{mod-}H$ and $f \in \mathrm{Hom}_K(M, N)$. Then the map $I(f) : M \to N$ with*

$$I(f)(x) := \sum_{w\in W} f(xB_w)B_w^\vee \text{ for all } x \in M$$

*is a homomorphism of $H$-modules from $M$ to $N$. Moreover, $I(f)$ does not depend on the choice of basis.*

*Let $X, Y \in \text{mod-}H$ and $\pi \in \text{Hom}_H(X, M)$ and $\psi \in \text{Hom}_H(N, Y)$, then*

$$I(f \circ \pi) = I(f) \circ \pi \text{ and } I(\psi \circ f) = \psi \circ I(f).$$

*Proof.* Straightforward computation. See [2, Lemma 62.8]. □

**Theorem 5** (Gaschütz-Ikeda)**.** *A right $H$-module $L$ is projective if and only if there is a $\psi \in \text{End}_K(L)$ with $I(\psi) = \text{id}_L$.*

*Proof.* [2, Theorem 62.11]. □

**Lemma 6** (Averaging homomorphisms between simple modules and PIMs)**.**
*We use the conventions in Notation 3.*

(1) *Let $L$ be a simple right $H$-module which is not isomorphic to $S$, $f$ a linear form on $L$, $p \in P$ an element of $P$ and*

$$\psi : L \to P,\ l \mapsto f(l)p \quad \text{for all} \quad l \in L$$

*a linear map. Then $I(\psi) = 0$. It follows that*

$$\sum_{w \in W} f(lB_w)pB_w^\vee = I(\psi)(l) = 0 \quad \text{for all} \quad l \in L.$$

(2) *Let $R$ be a simple right $H$-module which is not isomorphic to $V$, $f$ a linear form on $P$, $r \in R$ an element of $R$ and*

$$\psi : P \to R,\ p \mapsto f(p)r \quad \text{for all} \quad p \in P$$

*a linear map. Then $I(\psi) = 0$. It follows that*

$$\sum_{w \in W} f(pB_w)rB_w^\vee = I(\psi)(p) = 0 \quad \text{for all} \quad p \in P.$$

*Proof.* (1) The map $I(\psi)$ is a homomorphism of right $H$-modules from $L$ to $P$. As $L$ is simple, $\ker(I(\psi)) = L$ or $I(\psi)$ is injective and thus $I(\psi)(L) \cong L$. But $S$ is the only simple submodule of $P$ and not isomorphic to $L$. It follows that $I(\psi) = 0$.

(2) The map $I(\psi)$ is an element of $\text{Hom}_H(P, R)$. As $R$ is a simple module, we have $\text{im}(I(\psi)) = 0$ or $I(\psi)$ is surjective and thus $P/\ker(I(\psi)) \cong R$. But $\text{rad}(P)$ is the only maximal submodule of $P$, so $V$ is the only simple factor module of $P$. Thus $I(\psi) = 0$. □

**Theorem 7** (Identity component of endomorphisms)**.** *For every $\psi \in \text{End}_H(P)$ there is a unique constant $c \in K$, such that*

$$\text{im}(\psi - c \cdot \text{id}_P) \subseteq \text{rad}(P).$$

*For this $c$ we have $(\psi - c \cdot \text{id}_P)(S) = 0$.*

*Proof.* As $\psi(\text{rad}(P)) \subset \text{rad}(P)$, the map $\psi$ induces an endomorphism $\bar{\psi} : V \to V$. Since $V$ is simple, we have by Schur's Lemma, that $\bar{\psi}$ is multiplication by a scalar $c$. Therefore $\text{im}(\psi - c \cdot \text{id}_P) \subseteq \text{rad}(P)$ holds. Since $P$ is finite-dimensional and $\psi - c \cdot \text{id}_P$ is not surjective, it follows that $\psi - c \cdot \text{id}_P$ has a non-trivial kernel. Since $S$ is the only simple submodule of $P$, it is contained in the kernel of $\psi - c \cdot \text{id}_P$. □

## 3. Averaging linear maps of rank 1

We keep the conventions in Notation 3 from Section 2. By a "linear map of rank 1" we mean a linear map the image of which has dimension 1. In particular, we are interested in linear maps that expressed as a matrix with respect to the basis $(b_i)_{i=1}^n$ contain only zeros except in one position where they have a one. Note that whenever we express our results about endomorphisms for the convenience of the reader in matrix terms, we use row convention in the matrices!

**Definition 8** (Constants $c(i,j,s,t)$)**.** Let $f_{s,t} \in \mathrm{End}_K(P)$ with $f_{s,t}(p) := b_s^*(p)b_t$ for $s,t \in \{1,\ldots,n\}$ and $p \in P$. Note that with respect to the basis $(b_i)_{i=1}^n$ this linear map corresponds to a matrix that contains only zeros except for a one in row $s$ and column $t$. The $f_{s,t}$ form a basis of $\mathrm{End}_K(P)$. We define

$$c(i,j,s,t) := b_j^*(I(f_{s,t})(b_i)) = \sum_{w \in W} b_s^*(b_i B_w) \cdot b_j^*(b_t B_w^\vee).$$

Note that if we express $I(f_{s,t})$ in terms of the basis $(b_i)_{i=1}^n$ as a matrix, then $c(i,j,s,t)$ is the entry in row $i$ and column $j$.

It is useful to imagine all occurring endomorphisms as matrices, as always expressed with respect to the basis $(b_i)_{i=1}^n$ using row convention. Then the elements of $\mathrm{End}_H(P)$ are lower block-triangular matrices and can be visualised as follows:

$$\left[\begin{array}{c|c|c} c \cdot E_d & 0 & 0 \\ \hline * & * & 0 \\ \hline * & * & c \cdot E_d \end{array}\right]$$

where $c$ is the constant in $K$ from Theorem 7 and $E_d$ stands for a $(d \times d)$-identity matrix. For the convenience of the reader we briefly indicate in the following results which regions and in what matrices we are discussing.

**Lemma 9** (Frobenius-Schur relations I)**.** *The following equation holds:*

$$c(i,j,s,t) = \begin{cases} 0 & \text{if } \ i \le d \text{ and } j > d, & (1) \\ 0 & \text{if } \ i \le m \text{ and } j > m, & (2) \\ \delta_{i,j} c(1,1,s,t) & \text{if } \ i,j \le d, & (3) \\ \delta_{i,j} c(1,1,s,t) & \text{if } \ i,j > m. & (4) \end{cases}$$

Note that this lemma is about the parts of the matrix of $I(f_{s,t})$ that are above the block diagonal and about the upper left and lower right corner blocks.

*Proof.* Let $c_{s,t}$ be the unique constant (see Theorem 7) such that

$$(\spadesuit) \qquad \mathrm{im}(I(f_{s,t}) - c_{s,t} \cdot \mathrm{id}_P) \subseteq \mathrm{rad}(P) \quad \text{and}$$
$$(\clubsuit) \qquad ((I(f_{s,t}) - c_{s,t} \cdot \mathrm{id}_P)(S) = 0.$$

It follows immediately from equation ($\clubsuit$) for $i \le d$ that

$$c(i,j,s,t) = b_j^*(I(f_{s,t})(b_i)) = b_j^*(c_{s,t} \cdot b_i) = c_{s,t} \cdot \delta_{i,j}.$$

As $c_{s,t}$ only depends on $s$ and $t$, we have $c_{s,t} = c(1,1,s,t) = c(i,j,s,t)$ for $i = j \leq d$. This gives the third and the first equation. We know from (♠) that

$$c(i,j,s,t) - b_j^*(c_{s,t} \cdot \mathrm{id}_P(b_i)) = b_j^*(I(f_{s,t}) - c_{s,t} \cdot \mathrm{id}_P)(b_i)) = 0$$

if $j > m$. Thus $c_{s,t} = c(i,j,s,t) = \delta_{i,j} \cdot c_{s,t}$ if $j > m$. As $c_{s,t}$ only depends on $s$ and $t$ this gives us the last and the second equation. □

**Lemma 10** (Shifting)**.** *If $s, i \leq d$, then we have $c(i,j,s,t) = c(t, s+m, j, i+m)$.*

Note that this lemma relates certain entries of the matrices of $I(f_{s,t})$ and $I(f_{j,i+m})$ to each other. Namely, it says that for $s \leq d$ the entries in the topmost block row of $I(f_{s,t})$ can be determined by looking at the $(t, s+m)$-entry of all the matrices $I(f_{j,i+m})$ for $1 \leq i \leq d$ and $1 \leq j \leq n$.

*Proof.* If $s, i \leq d$, then we have

$$\begin{aligned} c(i,j,s,t) &= \sum_{w \in W} b_s^*(b_i B_w) b_j^*(b_t B_w^\vee) = \sum_{w \in W} b_{s+m}^*(b_{i+m}\alpha(B_w)) b_j^*(b_t B_w^\vee) \\ &= \sum_{w \in W} b_j^*(b_t B_w^\vee) b_{s+m}^*(b_{i+m}\alpha(B_w)) = c(t, s+m, j, i+m), \end{aligned}$$

using in the last step the fact that $(\alpha(B_w))_{w \in W}$ is the dual basis of $(B_w^\vee)_{w \in W}$ and that the formula for $I$ is independent of the choice of basis (see Theorem 4). □

**Lemma 11** (Frobenius-Schur relations II)**.** *We have the following relations:*

$$c(i,j,s,t) = \begin{cases} 0 & \text{if } t \leq m \text{ and } (i,j \leq d \text{ or } i,j > m), \quad (5) \\ 0 & \text{if } s > d \text{ and } (i,j \leq d \text{ or } i,j > m), \quad (6) \\ \delta_{s+m,t}\delta_{i,j} c & \text{if } s \leq d \text{ and } t > m \text{ and } i,j \leq d, \quad (7) \end{cases}$$

*where we abbreviate $c(1,1,1,1+m)$ to $c$.*

*Additionally, if $S$ is not isomorphic to $V$ we have:*

$$c(i,j,s,t) = \begin{cases} 0 & \text{for} \quad t \leq d, \quad (8) \\ 0 & \text{for} \quad s > m. \quad (9) \end{cases}$$

Note that this lemma is concerned with the upper left and lower right corner blocks of the matrix of $I(f_{s,t})$ for different cases of $s$ and $t$. It shows that the result is non-zero only if $s \leq d$ and $t = s+m$, that is, if $f_{s,t}$ has its non-zero entry on the diagonal of the upper right corner block.

*Proof.* If $t \leq m$ and $i,j \leq d$ or $i,j > m$, then by Lemma 9 we have $c(i,j,s,t) = \delta_{i,j} c(1,1,s,t) = \delta_{i,j} c_{s,t}$ where the constant $c_{s,t}$ is the same as in the proof of Lemma 9. Since $\mathrm{im}(f_{s,t}) \subseteq \mathrm{rad}(P)$ and thus $\mathrm{im}(I(f_{s,t})) \subseteq \mathrm{rad}(P)$, the constant $c_{s,t} = c(1,1,s,t)$ is equal to 0, which proves (5).

If $s > d$, then $f_{s,t}(S) = 0$ and thus $I(f_{s,t})(S) = 0$. The map is not injective and therefore not surjective. So $\mathrm{im}(I(f_{s,t})) \subseteq \mathrm{rad}(P)$ and analogously $0 = c_{s,t}\delta_{i,j}$ for $i,j \leq d$ or $i,j > m$ proving equation (6).

If $s \leq d$ and $t > m$ and $i,j \leq d$, then we have:

$$\begin{aligned} c(i,j,s,t) &= \delta_{i,j} c(1,1,s,t) = \delta_{i,j} c(t, s+m, 1, 1+m) \\ &= \delta_{i,j}\delta_{s+m,t} c(1,1,1,1+m) \end{aligned}$$

by Lemmas 9 and 10. This proves the statement in equation (7).

If $V$ is not isomorphic to $S$ the set $\{\psi \in \mathrm{End}_H(P) | \mathrm{im}(\psi) \subseteq S\} = \{0\}$. But $\mathrm{im}(I(f_{s,t})) \subseteq S$ for $t \leq d$ and therefore $I(f_{s,t}) = 0$. This gives equation (8).

Furthermore, if $s > m$, then $f_{s,t}$ and thus $I(f_{s,t})$ contain $\mathrm{rad}(P)$ in their kernel. Therefore the image of $I(f_{s,t})$ is either isomorphic to the simple module $V$ or is equal to zero. Since the socle of $P$ is the simple module $S \ncong V$, it follows that $I(f_{s,t}) = 0$ and thus $c(i, j, s, t) = 0$ in that case proving equation (9). □

**Lemma 12.** *The constant $c(1, 1, 1, 1 + m)$ is not equal to zero.*

*Proof.* By Theorem 5, we know that there is a linear map $\psi : P \to P$ such that $I(\psi) = \mathrm{id}_P$. It follows that $b_i^*(I(\psi)(b_i)) = 1$ for all $i \in \{1, \ldots, n\}$. The map $\psi$ can be written as a linear combination of the elements $(f_{s,t})_{s,t=1}^n$ with coefficients $(d_{s,t})_{s,t=1}^n$. If we choose $i \leq d$, we get $1 = \sum_{t=1+m}^n d_{t-m,t} c(1, 1, 1, 1 + m)$ using Lemma 11. This gives $c(1, 1, 1, 1 + m) \neq 0$. □

*Remark* 13. If we just change the basis $(b_i)_{i=1}^n$ in the way that we multiply all basis vectors for $i > m$ with the same non-zero constant, then $c(1, 1, 1, 1+m)$ also changes by the same constant. At the same time this change does not violate our hypotheses in Notation 3. This argument shows that even if the constants $c(1, 1, 1, 1 + m)$ (for the various projective indecomposable modules $P$) look like the Schur elements in the semisimple case, they are by no means canonical or invariants of the algebra, but depend on the actual choice of the bases in the projective indecomposable modules. For those projective indecomposable modules that are simple this constant is the Schur element.

## 4. Primitive idempotents

We have now finished all preparations to derive explicit formulas for primitive idempotents. We continue to use our notation from Sections 2 to 3.

**Theorem 14** (Primitive idempotents)**.**

(1) *Let $c := c(1, 1, 1, 1 + m)$. Then by Lemma 12 we know that $c \neq 0$ and we can set:*

$$\tilde{e}_i := c^{-1} \sum_{w \in W} b_i^*(b_{i+m} B_w) B_w^\vee \tag{4.1}$$

*for $i \in \{1, \ldots, d\}$. Then there is a polynomial $f \in \mathbb{Z}[X]$ such that for $e_i := f(\tilde{e}_i)$ the following is true:*

$$e_i^2 = e_i, \qquad e_i H \cong P \quad \text{and} \quad e_i e_{i'} \in \mathrm{rad}(H)$$

*for $i, i' \in \{1, \ldots, d\}$ and $i \neq i'$. The polynomial $f$ is given explicitly in the proof.*

(2) *We set*

$$\tilde{E}_i := c^{-1} \sum_{w \in W} b_i^*(b_{i+m} B_w^\vee) B_w$$

*for $i \in \{1, \ldots, d\}$. Let $\bar{P}$ be a projective indecomposable module, the head of which is isomorphic to $S$. Then for the $E_i := f(\tilde{E}_i)$ the following is true:*

$$E_i^2 = E_i, \qquad E_i H \cong \bar{P} \quad \text{and} \quad E_i E_{i'} \in \mathrm{rad}(H)$$

*for $i, i' \in \{1, \ldots, d\}$ and $i \neq i'$.*

Note that the entries of the representing matrices of the $B_w$ we need are those in the lower left block corner!

*Proof.* (1) For $c \neq 0$ see Lemma 12. In order to determine the action of $\tilde{e}_i$ on $V$, we choose $s, j > m$ and consider

$$\begin{aligned} b_s^*\big(b_j\tilde{e}_i\big) &= c^{-1}b_s^*\left(b_j\left(\sum_{w\in W} b_i^*\big(b_{i+m}B_w\big)B_w^\vee\right)\right) \\ &= c^{-1}\sum_{w\in W} b_i^*\big(b_{i+m}B_w\big)b_s^*\big(b_jB_w^\vee\big) \\ &= c^{-1}c(i+m,s,i,j) = c^{-1}c(j-m,i,s-m,i+m) \\ &= \delta_{i,j-m}\delta_{s,i+m}, \end{aligned}$$

using Lemma 10 read backwards and equation (7) of Lemma 11 in the last step. This means that $\tilde{e}_i$ acts on $V$ as the projection onto $\langle b_{i+m} + \mathrm{rad}(P)\rangle_K$ and thus $V\tilde{e}_i$ is one-dimensional. This means that the representing matrix of $\tilde{e}_i$ on $V$ with respect to the basis

$$(b_{i+m} + \mathrm{rad}(P))_{1\leq i\leq d}$$

is a matrix containing a single 1 on the diagonal and apart from that only zeros. So the product $\tilde{e}_i\tilde{e}_{i'}$ annihilates $V$ for $i \neq i'$.

Let $R$ be a simple module that is not isomorphic to $V$ and $r \in R$ an arbitrary element, then we know by Lemma 6(2) with $f := b_i^*$ that

$$r\tilde{e}_i = c^{-1}\sum_{w\in W} b_i^*\big(b_{i+m}B_w\big)rB_w^\vee = 0.$$

Thus $\tilde{e}_i$ annihilates every simple right module which is not isomorphic to $V$. Since the elements $\tilde{e}_i^2 - \tilde{e}_i$ for $i \in \{1,\dots,d\}$ and $\tilde{e}_i\tilde{e}_{i'}$ for $i, i' \in \{1,\dots,d\}$ and $i \neq i'$ annihilate every simple right $H$-module, they are contained in the radical of $H$.

The associative algebra $H$ is finite-dimensional and thus Artinian. So the radical of $H$ is nilpotent and we can choose an integer $a$ so that $(\mathrm{rad}(H))^a = 0$. Set

$$f := \sum_{s=0}^{a}\binom{2a}{s}X^{2a-s}(1-X)^s \in \mathbb{Z}[X],$$

where $X$ is an indeterminate. Then with [3, (6.7)] we get the following properties of $f$:

(i) $f \in \mathbb{Z}[X]$,
(ii) $f^2 - f \equiv 0 \pmod{X^a(1-X)^a}$,
(iii) $f \equiv X \pmod{X(1-X)}$.

Because of (iii) we know that $e_i := f(\tilde{e}_i)$ acts on every simple right $H$-module exactly like $\tilde{e}_i$. Therefore $e_i \neq 0$ and $e_ie_{i'} \in \mathrm{rad}(H)$ for $i \neq i'$. With the choice of $a$ we have

$$\tilde{e}_i^a(1-\tilde{e}_i^a) = (\tilde{e}_i - \tilde{e}_i^2)^a = 0$$

because polynomials in $e_i$ commute. Therefore and because of (ii) the $e_i$ are idempotents for $i \in \{1,\dots,d\}$. This shows that $e_iH$ is a projective $H$-module. Proposition [3, (6.6)] states that the isomorphism type of a projective module is determined by the isomorphism type of its head. So $e_iH \cong P$ if and only if $e_iH/\mathrm{rad}(e_iH) \cong V$. Let $L$ be an arbitrary simple right $H$-module, then $\mathrm{Hom}_H(e_iH, L) \cong Le_i$. As $K$ is a splitting field and $e_iH$ is projective, the $K$-dimension of $\mathrm{Hom}_H(e_iH, L)$ is equal to the multiplicity of $L$ in the head of $e_iH$. We have already shown that $Le_i$ is one-dimensional for $L \cong V$ and equal to 0 otherwise. Thus $e_iH$ is indecomposable and isomorphic to $P$.

(2) We first determine the action of the $\tilde{E}_i$ on $S$. Therefore we choose $j, s \leq d$ and consider:

$$\begin{aligned} b_s^*\big(b_j\tilde{E}_i\big) &= c^{-1}b_s^*\left(b_j\left(\sum_{w\in W} b_i^*\big(b_{i+m}B_w^\vee\big)B_w\right)\right) \\ &= c^{-1}\sum_{w\in W} b_i^*\big(b_{i+m}B_w^\vee\big)b_s^*\big(b_jB_w\big) \\ &= c^{-1}c(j,i,s,i+m) = \delta_{j,i}\delta_{s,i}, \end{aligned}$$

using Lemma 11 in the last step. So $S\tilde{E}_i$ is one-dimensional.

Let $L$ be a simple right $H$-module that is not isomorphic to $S$ and $l \in L$ an arbitrary element. Let $f$ be an arbitrary linear form on $L$, then

$$\begin{aligned} f(l\tilde{E}_i) &= c^{-1}\sum_{w\in W} b_i^*\big(b_{i+m}B_w^\vee\big)f(lB_w) \\ &= c^{-1}b_i^*\left(\sum_{w\in W} f(lB_w)b_{i+m}B_w^\vee\right) = 0, \end{aligned}$$

using in the last step that $\sum_{w\in W} f(lB_w)b_{i+m}B_w^\vee = 0$ by Lemma 6(1) with $p := b_{i+m}$. Since $f$ is an arbitrary linear form we can deduce that $l\tilde{E}_i = 0$ for all $i \in \{1,\ldots,d\}$. Thus $\tilde{E}_i$ annihilates every simple right module which is not isomorphic to $S$. The rest of the proof is analogous to the proof in part (1) if we replace $V$ by $S$. □

## 5. Decomposition maps and their duals

This section briefly recalls some notation and definitions needed in the following sections. All these concepts can be defined in a more general way, but we do not need them in full generality and thus can avoid some additional complications, especially due to the fact that we always assume our base fields to be splitting fields. We start by introducing some notation for decomposition maps.

For a finite-dimensional algebra $\mathcal{A}$ over a field we denote by $R_0(\mathcal{A})$ the Grothendieck group of the category of finite-dimensional right $\mathcal{A}$-modules and by $K_0(\mathcal{A})$ the Grothendieck group of the category of finite-dimensional projective right $\mathcal{A}$-modules. For a right $\mathcal{A}$-module $M$ we denote by $[M]$ its class in $R_0(\mathcal{A})$ or $K_0(\mathcal{A})$, depending on the context. Recall that there is a bilinear form $\langle -|-\rangle$ : $K_0(\mathcal{A}) \times R_0(\mathcal{A}) \to \mathbb{Z}$ given by $\langle [P]|[V]\rangle = \dim \mathrm{Hom}_{\mathcal{A}}(P, V)$. Then the set of classes of projective indecomposable $\mathcal{A}$-modules is the dual basis of the set of classes of simple $\mathcal{A}$-modules with respect to that form, because for every projective indecomposable $\mathcal{A}$-module $P$ the head $P/\mathrm{rad}(P)$ is isomorphic to a simple module and every isomorphism type of simple modules arises in this way (see [3, (6.9)]).

Let $K$ be a number field, that is, a finite extension of the field $\mathbb{Q}$ of rational numbers, and $R$ its ring of integers, thus $R$ is a Dedekind domain and $K$ is the field of fractions of $R$. For every prime ideal $\mathfrak{p} \lhd R$ the localisation $R_\mathfrak{p}$ of $R$ at $\mathfrak{p}$ is a discrete valuation ring (see [12, 11.2]) and the residue class field $k_\mathfrak{p} := R/\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ (see [3, (4.1)]) is a finite field of characteristic $\ell$ for $\ell$ being the rational prime contained in $\mathfrak{p}$.

Let $H$ be an associative $R$-free $R$-algebra of finite $R$-rank, and let $KH := K \otimes_R H$ be its extension of scalars. Note that we do not assume $KH$ to be semisimple and in fact the examples occurring in Section 7 will not be semisimple. However, we assume that $K$ is a splitting field for $KH$ and that $k_\mathfrak{p}$ is a splitting field for the modular reduction $k_\mathfrak{p}H := k_\mathfrak{p} \otimes_R H$.

In this situation, the natural map $R_\mathfrak{p} \to R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p} \cong k_\mathfrak{p}$ gives rise to a ($\mathbb{Z}$-linear) decomposition map $d_\mathfrak{p} : R_0(KH) \to R_0(k_\mathfrak{p}H)$ in the following way: A class $[V] \in R_0(KH)$ of a simple $\mathcal{A}$-module is mapped to $[k_\mathfrak{p} \otimes_{R_\mathfrak{p}} \tilde{V}] \in R_0(k_\mathfrak{p}H)$ where $\tilde{V}$ is an $R_\mathfrak{p}H$-module with $K \otimes_{R_\mathfrak{p}} \tilde{V} \cong V$ as $KH$-modules. Such a module $\tilde{V}$ exists because $R_\mathfrak{p}$ is a valuation ring and thus every finitely generated torsion-free $R_\mathfrak{p}$-module is free. See [8, 7.4] for details on why $d_\mathfrak{p}$ is well defined in this way.

We now define a linear map $e_\mathfrak{p} : K_0(k_\mathfrak{p}H) \to K_0(KH)$, which will turn out to be closely related to $d_\mathfrak{p}$ in the sequel:

**Definition 15** (The dual map of the decomposition map)**.** Let $P = fk_\mathfrak{p}H$ be a projective indecomposable $k_\mathfrak{p}H$-module where $f \in k_\mathfrak{p}H$ is a primitive idempotent. By [13, Satz 3.4.1] or [3, Exercise 6.16] there is an idempotent $e \in R_\mathfrak{p}H$ which is mapped to $f$ by the map $1_{k_\mathfrak{p}} \otimes_{R_\mathfrak{p}} - : R_\mathfrak{p}H \to k_\mathfrak{p}H$. We set $e_\mathfrak{p}([P]) := [eKH] \in K_0(KH)$. By standard arguments one shows that this is well defined.

The relation between $d_\mathfrak{p}$ and $e_\mathfrak{p}$ is described by the following proposition:

**Proposition 16** (Brauer reciprocity)**.** *The map*

$$e_\mathfrak{p} : K_0(k_\mathfrak{p}H) \to K_0(KH)$$

*is the dual map of the map*

$$d_\mathfrak{p} : R_0(KH) \to R_0(k_\mathfrak{p}H)$$

*with respect to the pairing* $\langle - | - \rangle$ *between* $K_0(KH)$ *and* $R_0(KH)$*, and* $K_0(k_\mathfrak{p}H)$ *and* $R_0(k_\mathfrak{p}H)$ *respectively. More precisely, we have*

$$\Big\langle e_{k_\mathfrak{p}}([P]), [V] \Big\rangle = \Big\langle [P], d_{k_\mathfrak{p}}([V]) \Big\rangle \tag{5.1}$$

*for all* $[P] \in K_0(k_\mathfrak{p}H)$ *and all* $[V] \in R_0(KH)$.

*Proof.* This is proved in exactly the same way as [3, Theorem 18.9]. Note that, in this reference, the algebra $KH$ is globally assumed to be semisimple (or even a group algebra), but this is completely irrelevant for the proof of (5.1). □

Taking the classes of simple modules as bases for $R_0(KH)$ and $R_0(k_\mathfrak{p}H)$ we can express the decomposition map $d_\mathfrak{p}$ as a matrix, the so-called "decomposition matrix of $d_\mathfrak{p}$", the rows of which are indexed by the basis of $R_0(KH)$ and the columns of which are indexed by the basis of $R_0(k_\mathfrak{p}H)$. A row of the decomposition matrix thus contains the multiplicities of the simple $k_\mathfrak{p}$-modules in a modular reduction of the corresponding simple $KH$-module.

Analogously, if we take the classes of projective indecomposable modules as bases for $K_0(k_\mathfrak{p}H)$ and $K_0(KH)$, we can express the map $e_\mathfrak{p}$ as a matrix, the rows of which are indexed by the basis of $K_0(k_\mathfrak{p}H)$ and the columns of which are indexed by the basis of $K_0(KH)$. A row of this matrix thus contains the multiplicities of the projective indecomposable $KH$-modules in a direct sum decomposition of a lift of a projective indecomposable $k_\mathfrak{p}H$-module.

Since our chosen bases of $K_0(KH)$ and $K_0(k_\mathfrak{p}H)$ are just the dual bases of our chosen bases of $R_0(KH)$ and $R_0(k_\mathfrak{p}H)$ with respect to the pairing $\langle -|-\rangle$, Proposition 16 states that the matrix of $e_\mathfrak{p}$ is just the transposed matrix of the decomposition matrix. Thus a column of the decomposition matrix contains the multiplicities of the projective indecomposable $KH$-modules in a direct sum decomposition of a lift of a projective indecomposable $k_\mathfrak{p}H$-module, which is nothing but the classical Brauer reciprocity. See [8, 7.5.2] for the corresponding result if $KH$ is semisimple, and [9, Section 2] for a more general and more complicated case not assuming splitting fields. Our exposition in the present paper is similar to [15, Kapitel V].

## 6. Application to decomposition maps

This section is motivated by James' conjecture on Iwahori-Hecke algebras. The setup presented here is a generalisation of the situation in Section 7.

Let $R$ be a Dedekind domain, $K$ its field of fractions, $H$ an associative $R$-free $R$-algebra with finite $R$-rank such that its extension of scalars $KH := K \otimes_R H$ is a Frobenius algebra with $K$-linear map $\tau : KH \to K$. Note that we do not assume $KH$ to be semisimple and in fact the examples occurring in Section 7 will not be semisimple.

We first formulate a criterion for a column in the decomposition matrix to be trivial:

**Lemma 17** (Trivial columns in the decomposition matrix)**.** *Let $\mathfrak{p}$ be a prime ideal of $R$ and $e \in KH$ a primitive idempotent that lies in $R_\mathfrak{p}H := R_\mathfrak{p} \otimes_R H$. Then the idempotent $f := 1_{k_\mathfrak{p}} \otimes_{R_\mathfrak{p}} e \in k_\mathfrak{p}H$ is primitive and the column of the decomposition matrix of $d_\mathfrak{p}$ (see the end of Section 5) corresponding to the simple $k_\mathfrak{p}H$-module $fk_\mathfrak{p}H/\mathrm{rad}(fk_\mathfrak{p}H)$ contains exactly one 1 in the row corresponding to the simple $KH$-module $eKH/\mathrm{rad}(eKH)$ and apart from that only zeros.*

*Proof.* We use Brauer reciprocity as described in Section 5: Since $R_\mathfrak{p}H$ is semiperfect by [13, Satz 4.3.1] or [3, Exercise 6.16] the idempotent $f := 1_{k_\mathfrak{p}} \otimes_{R_\mathfrak{p}} e \in k_\mathfrak{p}H$ is primitive and the class $[fk_\mathfrak{p}H] \in K_0(k_\mathfrak{p}H)$ is mapped to the class $[eKH] \in K_0(KH)$ of the projective indecomposable module $eKH$ by the map $e_\mathfrak{p}$. The interpretation of $e_\mathfrak{p}$ as the dual map of the decomposition map $d_\mathfrak{p}$ and the definition of the decomposition matrix of $d_\mathfrak{p}$ gives the statement in the lemma. □

Now we use a concrete representation of $KH$ on a projective indecomposable module $P$ to find an infinite set of prime ideals $\mathfrak{p} \lhd R$ for which Lemma 17 can be applied.

Let $(B_w)_{w\in W}$ be an $R$-basis of $H$. Then it is also a $K$-basis of $KH$ and we denote its dual basis with respect to $\tau$ by $(B_w^\vee)_{w\in W}$.

Choose a $K$-basis $(b_1, \dots, b_d, \dots, b_m, \dots, b_n)$ of $P$ as in Section 2. Scale the basis vectors $(b_{m+1}, \dots, b_n)$ corresponding to the head of $P$ by a common scalar in $K$, such that the constant $c(1, 1, 1, 1+m)$ (see Definition 8 and Remark 13) is equal to 1. For a fixed $1 \le i \le d$, write every number $a_w := b_i^*(b_{i+m} \cdot B_w)$ for $w \in W$ as a quotient $a_w =: s_w/t_w$ with $s_w \in R$ and $t_w \in R \setminus \{0\}$ and let $I_i \lhd R$ be the ideal generated by the product $\prod_{w\in W} t_w$ of all such denominators. Let $I$ be the ideal generated by all the $I_i$ for $1 \le i \le d$.

Note that the ideal $I$ depends on the choice of the numerators and denominators $s_w$ and $t_w$ and of course on the choice of basis $(b_j)_{1\le j\le n}$. However, for all such choices, we have:

**Theorem 18** (Criterion for trivial column I)**.** *Let the ideal $I$ be defined as above and $\mathfrak{p} \triangleleft R$ be a prime ideal such that $I$ is not contained in $\mathfrak{p}$. Then there is a primitive idempotent $e \in R_\mathfrak{p}H$ satisfying $eKH \cong P$ as $KH$-modules, such that the idempotent $f := 1_{k_\mathfrak{p}} \otimes_{R_\mathfrak{p}} e \in k_\mathfrak{p}H$ is also primitive. Furthermore, the column of the decomposition matrix of $d_\mathfrak{p}$ corresponding to the simple $k_\mathfrak{p}H$-module $fk_\mathfrak{p}H/\mathrm{rad}(fk_\mathfrak{p}H)$ contains only zeros, except in the row corresponding to the simple $KH$-module $eKH/\mathrm{rad}(eKH)$, where it contains a* 1.

*Proof.* At least one of the ideals $I_i$ is not contained in $\mathfrak{p}$ and thus all the denominators $t_w$ the product of which generates $I_i$ are not in $\mathfrak{p}$, because $\mathfrak{p}$ is a prime ideal. Therefore Theorem 14 gives an idempotent $e_i \in R_\mathfrak{p}H$ with $e_iKH \cong P$ as $KH$-modules and Lemma 17 shows the last statement in the theorem. □

*Remark.* As every ideal $I \triangleleft R$ is divided by only finitely many prime ideals $\mathfrak{p} \triangleleft R$, there is only a finite number of cases, in which the hypotheses of Theorem 18 is not fulfilled. Repeating this argument for every isomorphism type of projective indecomposable $KH$-modules shows that there are only finitely many prime ideals $\mathfrak{p} \triangleleft R$ for which the decomposition matrix is not equal to an identity matrix.

We now follow a different approach. Whereas for Theorem 18 we used only some entries of all representing matrices for a basis $(B_w)_{w\in W}$, we now consider all entries of representing matrices for a system of generators of $H$.

To this end, let $X \subseteq W$ be a set, such that $\{B_x \mid x \in W\}$ generates $H$ as an algebra. Write every entry $b_i^*(b_j \cdot B_x)$ of the representing matrices for all $1 \leq i, j \leq n$ and all $x \in X$ as a quotient $s_x/t_x$ and let $N$ be the product of all the denominators $t_x$. Let $J := NR \triangleleft R$ be the ideal of $R$ generated by $N$.

Note that the ideal $J$ depends on the choice of the numerators and denominators, of course on the choice of the basis $(b_j)_{1\leq j\leq n}$, and on the choice of the generating system $\{B_x \mid x \in X\}$. However, for all such choices we have:

**Theorem 19** (Criterion for trivial column II)**.** *Let the ideal $J$ be defined as above and $\mathfrak{p} \triangleleft R$ be a prime ideal such that $J$ is not contained in $\mathfrak{p}$. Then there is a primitive idempotent $e \in R_\mathfrak{p}H$ satisfying $eKH \cong P$ as $KH$-modules, such that the idempotent $f := 1_{k_\mathfrak{p}} \otimes_{R_\mathfrak{p}} e \in k_\mathfrak{p}H$ is also primitive. Furthermore, the column of the decomposition matrix of $d_\mathfrak{p}$ corresponding to the simple $k_\mathfrak{p}H$-module $fk_\mathfrak{p}H/\mathrm{rad}(fk_\mathfrak{p}H)$ contains only zeros, except in the row corresponding to the simple $KH$-module $eKH/\mathrm{rad}(eKH)$, where it contains a* 1.

*Proof.* All denominators of all entries of the representing matrices of all generators $B_x$ for $x \in X$ are not in $\mathfrak{p}$ and so these matrix entries lie in $R_\mathfrak{p}$. Since the $B_x$ generate $H$ as an algebra, the same holds for all representing matrices of all elements $B_w$ for $w \in W$. Thus Theorem 14 gives an idempotent $e_1 \in R_\mathfrak{p}H$ with $e_1KH \cong P$ as $KH$-modules and Lemma 17 shows the last statement in the theorem. □

*Remark.* The same argument as the one after Theorem 18 applies here, showing that the hypotheses of Theorem 19 are fulfilled for all but a finite number of prime ideals $\mathfrak{p}$.

## 7. A possible application to James' conjecture

In this section we present the situation in which the above results could be applied, provided one could construct certain representations, such that one could control the denominators of the entries in the representing matrices.

Let $(W,S)$ be a finite Coxeter system, that is, $W$ is a finite group with a subset $S$ such that we have a presentation of the form

$$W = \left\langle s \in S \mid s^2 = 1 \text{ and } (st)^{m_{s,t}} = 1 \text{ for } s,t \in S \right\rangle,$$

where $m_{s,t}$ is the order of $st$. Let $L : W \to \mathbb{N} \cup \{0\}$ be the length function on $W$, that is, $L(w)$ is the number of factors in the shortest expression of $w$ as a product of generators in $S$. An expression $w = s_1 \cdot \cdots \cdot s_{L(w)}$ with $s_i \in S$ is called *reduced*.

Let $A$ be any commutative ring with 1 and $v \in A$ invertible. We can now define the one-parameter Iwahori-Hecke algebra $\mathcal{H}_A(W,S,v)$ over $A$ to be the associative $A$-algebra with generators $\{T_w \mid w \in W\}$ subject to the relations

$$\begin{array}{rcll} T_s^2 &=& T_{\mathrm{id}} + (v - v^{-1})T_s & \text{for all } s \in S, \\ T_w &=& T_{s_1} \cdot \cdots \cdot T_{s_k} & \text{for every reduced expression} \\ && & w = s_1 \cdot \cdots \cdot s_k \text{ in } W \text{ with } s_i \in S \end{array}$$

where $\mathrm{id} \in W$ denotes the identity element.

The algebra $\mathcal{H}$ is free as an $A$-module with basis $(T_w)_{w \in W}$ (see [11, 3.3]) and has a symmetrising trace map $\tau : \mathcal{H} \to A, T_{\mathrm{id}} \mapsto 1, T_w \mapsto 0$ for $\mathrm{id} \neq w \in W$, which makes $\mathcal{H}$ into a symmetric algebra in the sense of [8, 7.1.1]. The dual basis of $(T_w)_{w \in W}$ with respect to $\tau$ is $(T_w^\vee)_{w \in W}$ with $T_w^\vee = T_{w^{-1}}$ (for all of this, see [11, 10.3, 10.4]). Note that in Section 2 we chose to define Frobenius algebras only over fields. If $A$ is a field, $(\mathcal{H}, \tau)$ is just a Frobenius algebra with the additional property that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{H}$.

This construction is functorial in the sense, that if $f : A \to B$ is a homomorphism into a commutative ring $B$, we can regard $B$ as an $A$-module via $f$ and then have a canonical isomorphism $B \otimes_A \mathcal{H}_A(W,S,v) \cong \mathcal{H}_B(W,S,f(v))$, where the latter is defined in exactly the same way as above as a finitely presented algebra, only over the ring $B$ with parameter $f(v)$ instead over $A$ with parameter $v$ (see [8, 8.1.2]).

We now consider three different base rings: First, let $\tilde{K} \supseteq \mathbb{Q}$ be a finite extension such that $\tilde{K}$ is a splitting field for the group algebra $\tilde{K}W$. Let $\tilde{R}$ be the ring of integers of $\tilde{K}$, it is a Dedekind domain and in particular integrally closed. Set $\hat{R} := \tilde{R}[v, v^{-1}]$, the ring of Laurent polynomials over $\tilde{R}$. Then $\hat{K} := \tilde{K}(v)$ is the field of fractions of $\hat{R}$. Note that $\hat{K}$ is a splitting field for the extension of scalars $\hat{K}\mathcal{H} := \mathcal{H}_{\hat{K}}(W,S,v) = \hat{K} \otimes_{\hat{R}} \mathcal{H}_{\hat{R}}(W,S,v)$. This follows from [8, 9.3.5] and the fact that our parameter $v$ is the square root of the parameter there. In addition, all irreducible characters of $\hat{K}\mathcal{H}$ can be realized over $\hat{K}$.

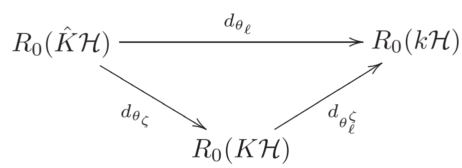
Second, we consider a finite field $k$ of characteristic $\ell > 0$ and a homomorphism $\theta_\ell : \hat{R} \to k$ such that $k$ is the field of fractions of the image $\theta_\ell(\hat{R})$. Since $v$ is invertible in $\hat{R}$, this also holds for the image $q := \theta_\ell(v)$, which thus has finite order. Let $e$ be the order of $q$ for $q \neq 1$, and otherwise set $e := \ell$.

Third, let $K$ be $\tilde{K}(\zeta)$ where $\zeta$ is a primitive $e$-th root of unity. Let $R$ be the ring of integers of $K$. Then $R$ is a Dedekind domain, which contains $\tilde{R}$ as a (possibly equal) subring. Note that the choice of $\zeta$ and thus of $R$ and $K$ is determined entirely by $k$ and $q$.

In this situation, we have a ring homomorphism $\theta_\zeta : \hat{R} \to R$ mapping $v$ to $\zeta$. Since both $R$ and $k$ are integral domains, the kernels of $\theta_\ell$ and $\theta_\zeta$ are both prime ideals. Furthermore, the kernel of $\theta_\zeta$ is generated by the minimal polynomial of $\zeta$ over $\tilde{R}$, which is an irreducible factor of the cyclotomic polynomial $\phi_e(v) \in R[v]$ having $\zeta$ as a root. Since $q = \theta_\ell(v)$ has multiplicative order $e$ (for $q \neq 1$), the field $k$ contains primitive $e$-th roots of unity and thus the $\ell$-modular reduction of the cyclotomic polynomial $\phi_e \in \mathbb{Z}[X]$ is equal to a product of linear factors over $k$. Therefore, the kernel of $\theta_\zeta$ is contained in the kernel of $\theta_\ell$ and there is a ring homomorphism $\theta_\ell^\zeta : R \to k$ with $\theta_\ell = \theta_\ell^\zeta \circ \theta_\zeta$ thus mapping $\zeta$ to $q = \theta_\ell(v) \in k$. The same holds for the case $e = \ell$ and $q = 1$.

These ring homomorphisms together with the functoriality above gives us three Iwahori-Hecke algebras $\mathcal{H}_{\hat{R}}(W, S, v)$, $\mathcal{H}_R(W, S, \zeta)$ and $k\mathcal{H} := \mathcal{H}_k(W, S, q)$ together with canonical maps between them. The first two are associative algebras over the rings $\hat{R}$ and $R$ respectively such that we can also consider the corresponding extensions of scalars $\hat{K}\mathcal{H} := \mathcal{H}_{\hat{K}}(W, S, v)$ and $K\mathcal{H} := \mathcal{H}_K(W, S, \zeta)$ to the respective fields of fractions $\hat{K}$ and $K$.

For this situation, Meinolf Geck and Raphaël Rouquier in [9, 2.5] have defined a commutative diagram of decomposition maps:




We do not want to go into the details of the definition of $d_{\theta_\ell}$ and $d_{\theta_\zeta}$ here, because we do not need those two maps in the sequel and our definition of decomposition maps in Section 6 would have to be generalised to do this.

However, we can now formulate Geck's version of James' conjecture (see [6, (3.4)]).

**Conjecture 20** (see [10, Section 4], [6, (3.4)])**.** *If in the situation above, $k = \mathbb{F}_\ell$ is the field of $\ell$ elements, and $\ell$ is coprime to the order of $W$, then $d_{\theta_\ell^\zeta}$ is an isomorphism that preserves the classes of simple modules.*

*Remark* 1. The authors do not see a reason for the restriction $k = \mathbb{F}_\ell$ in this conjecture. However, there does not seem to be much computational evidence available showing that the conjecture holds in cases $k \neq \mathbb{F}_\ell$. Thus we stick to this restriction.

*Remark* 2. The statement simply means that the decomposition matrix corresponding to $d_{\theta_\ell^\zeta}$ is an identity matrix. This in turn implies that the modular decomposition map $d_{\theta_\ell}$ is completely determined by the decomposition map $d_{\theta_\zeta}$ which is defined only involving rings of characteristic 0.

*Remark* 3. The definition of $d_{\theta_\ell^\zeta}$ in [6] coincides with our definition of a decomposition map $d_\mathfrak{p} : R_0(K\mathcal{H}) \to R_0(k\mathcal{H})$ using the kernel of the surjective ring homomorphism $\theta_\ell^\zeta$ as the prime ideal $\mathfrak{p} \triangleleft R$.

To the best knowledge of the authors, this conjecture is still open except in the defect 1 case and a few other special cases (see for example [4, 6.4], [5] and [7]).

Using any set $\{P_1, P_2, \ldots, P_t\}$ of representatives of the isomorphism classes of projective indecomposable $K\mathcal{H}$-modules together with the argument after the proof of Theorem 18 we can show that there are only finitely many prime ideals $\mathfrak{p} \triangleleft R$ and thus only finitely many pairs $(k, q)$ such that the decomposition map $d_\ell^\zeta$ does not fulfill the statement of Conjecture 20. This reasoning provides an alternative to the corresponding proof in [5, 5.5].

However, there is no explicit lower bound $B$ for $\ell$ known such that for all $\ell > B$ and all $(k, q)$ with char $k = \ell$ the decomposition map $d_\ell^\zeta$ is an isomorphism that preserves the classes of simple modules.

Our Theorems 18 and 19 could eventually be used to achieve such a bound in the following way: If one could explicitly construct realisations of projective indecomposable $K\mathcal{H}$-modules together with bases adapted to socle and radical, **and** one could control the denominators of the resulting representing matrix entries (either some specific entries for all representing matrices of a basis of $\mathcal{H}_R(W, S, \zeta)$ (as for Theorem 18) or for all entries for representing matrices of a generating system of $\mathcal{H}_R(W, S, \zeta)$ (as for Theorem 19)), then one would get a criterion for which prime ideals $\mathfrak{p} \lhd R$ **all** columns of the decomposition matrix are trivial and thus the decomposition map $d_\ell^\zeta$ is an isomorphism that preserves the classes of simple modules.

It is our hope that such projective modules can be constructed eventually thus yielding at least a part of a proof of James' conjecture. A slight hint in this direction is provided by some observations the first author has made in his Ph.D. thesis (see [15, Section VI.7]). There it is reported, that explicit matrix representations coming from projective modules for the Iwahori-Hecke algebra $\mathcal{H}_R(W, S, \zeta)$ over the ring of integers $R$ of the number field $\mathbb{Q}(\zeta)$ can be obtained using the Kazhdan-Lusztig basis and intervals in the poset of left cells and their corresponding non-simple cell modules (see [11, Chapter 8] or [16] for definitions of these concepts).

Although these observations are still only the result of a few computer calculations it seems not totally impossible that the Kazhdan-Lusztig theory and, in particular, the methods involving cells could eventually lead to the explicit construction of projective modules; in particular, since the Kazhdan-Lusztig basis has already been used extensively to study the representation theory of these algebras.

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## References

[1] Richard Brauer. *On hypercomplex arithmetic and a theorem of Speiser.* A. Speiser Festschrift, Zürich, 1945. MR0014082 (7:238b)

[2] Charles W. Curtis and Irving Reiner. *Representation Theory of Finite Groups and Associative Algebras.* John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1962. MR0144979 (26:2519)

[3] Charles W. Curtis and Irving Reiner. *Methods of Representation Theory*, volume I. John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1981. MR632548 (82i:20001)

[4] Joseph Chuang and Kai Meng Tan. Filtrations in Rouquier blocks of symmetric groups and Schur algebras. *Proc. London Math. Soc. (3)* **86** (2003), no. 3, 685–706. MR1974395 (2004g:20016)

[5] Meinolf Geck. Brauer trees of Hecke algebras. *Comm. Algebra* **20**, (1992), pp. 2937–2973. MR1179271 (94a:20019)

[6] Meinolf Geck. Representations of Hecke algebras at roots of unity. *Séminaire Bourbaki.* Vol. 1997/98, in *Astérisque No. 252*, (1998), Exp. No. 836, 3, 33–55. MR1685620 (2000g:20018)

[7] Matthew Fayers and Kay Meng Tan. *Adjustment matrices for weight three blocks of Iwahori-Hecke algebras. J. Algebra* **306**, (2006), pp. 76–103. MR2271573 (2007i:20010)

[8] Meinolf Geck and Götz Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, volume 21 in *London Mathematic Society, New Series*. Oxford University Press, Oxford, 2000. MR1778802 (2002k:20017)

[9] Meinolf Geck and Raphaël Rouquier. *Centers and Simple Modules for Iwahori-Hecke algebras*, in *Finite reductive groups (Luminy, 1994)*, Birkhäuser Boston, volume 141 in *Progr. Math.*, pp. 251–272, 1997. MR1429875 (98c:20013)

[10] Gordon James. The Decomposition Matrices of $\mathrm{GL}_n(q)$ for $n \leq 10$. *Proc. London Math. Soc. (3)*, 60:225–265, 1990. MR1031453 (91c:20024)

[11] George Lusztig. *Hecke algebras with unequal parameters*, Vol. 18 of *CRM Monograph Series*, American Mathematical Society, Providence, RI, 2003. MR1974442 (2004k:20011)

[12] Hideyuki Matsumura. *Commutative ring theory*. Cambridge University Press, Cambridge, London, New York, 1986. MR879273 (88h:13001)

[13] Jürgen Müller. *Zerlegungszahlen für generische Iwahori-Hecke-Algebren von exzeptionellem Typ*. Ph.D. thesis, RWTH Aachen, 1995. See `http://www.math. rwth-aachen.de/~Juergen.Mueller/preprints/jm3.pdf`

[14] Tadasi Nakayama. On Frobeniusean Algebras I. *The Annals of Mathematics*, 2nd Series, Vol. 40, No. 3, 1939, pp. 611–633. MR0000016 (1:3a)

[15] Max Neunhöffer. Untersuchungen zu James' Vermutung über Iwahori-Hecke-Algebren vom Typ A. Ph.D. thesis, RWTH Aachen, 2003. See `http://www.math.rwth-aachen.de/~Max.Neunhoeffer/Publications/phd.html`

[16] Max Neunhöffer. Kazhdan-Lusztig basis, Wedderburn decomposition, and Lusztig's homomorphism for Iwahori-Hecke algebras. *J. Algebra* **303** (2006), no. 1, pp. 430–446. MR2253671 (2008a:20012)

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