INVARIANT DEFORMATIONS
OF ORBIT CLOSURES IN $\mathfrak{sl}(n)$

SÉBASTIEN JANSOU AND NICOLAS RESSAYRE

ABSTRACT. We study deformations of orbit closures for the action of a connected semisimple group G on its Lie algebra $\mathfrak{g}$, especially when G is the special linear group.

The tools we use are the invariant Hilbert scheme and the sheets of $\mathfrak{g}$. We show that when G is the special linear group, the connected components of the invariant Hilbert schemes we get are the geometric quotients of the sheets of $\mathfrak{g}$. These quotients were constructed by Katsylo for a general semisimple Lie algebra $\mathfrak{g}$; in our case, they happen to be affine spaces.

INTRODUCTION

Let G be a complex reductive group, and V a finite dimensional G-module. A fundamental problem is to endow some sets of orbits of G in V with a structure of variety. The geometric invariant theory is the classical answer in this context: the set of closed orbits of G in V has a natural structure of affine variety. We denote by $V /\!/ G$ this variety, equipped with a G-invariant quotient map $\pi : V \to V /\!/ G$.

Recently, Alexeev and Brion defined in [AB] a structure of quasiprojective scheme on some sets of G-stable closed affine subschemes of V. A natural question is to wonder what happens when one applies the Alexeev-Brion construction to the orbit closures of G in V. Here, we study this construction in the case of a well-known G-module, namely the adjoint representation of a semisimple group G, especially when G is the special linear group $\mathrm{SL}(n)$.

From now on, we assume that G is semisimple, and denote by $\mathfrak{g}$ its Lie algebra endowed with the adjoint action of G. Let us recall that a sheet of $\mathfrak{g}$ is an irreducible component of the set of points in $\mathfrak{g}$ whose G-orbit has a fixed dimension. Let us fix a sheet $\mathcal{S}$. We show that the G-module structure on the affine algebra $\mathbb{C}[\overline{G \cdot x}]$ of the orbit closure $\overline{G \cdot x}$ of x does not depend on x in $\mathcal{S}$. This allows us to define a set-theoretical application from $\mathcal{S}$ to some Alexeev-Brion invariant Hilbert scheme of $\mathfrak{g}$:

$$\pi_{\mathcal{S}} : \quad \mathcal{S} \quad \longrightarrow \quad \mathrm{Hilb}_{\mathcal{S}}^{G}(\mathfrak{g})$$
$$x \quad \longmapsto \quad \overline{G \cdot x}.$$

A unique sheet is open in $\mathfrak{g}$; we call it the *regular* one, and denote it by $\mathfrak{g}_{\mathrm{reg}}$.

In Section 2 we are interested in $\mathrm{Hilb}_{\mathfrak{g}_{\mathrm{reg}}}^{G}(\mathfrak{g})$. The graph of the quotient map $\pi : \mathfrak{g} \to \mathfrak{g} /\!/ G$ is a flat family of G-stable closed subschemes of $\mathfrak{g}$ over $\mathfrak{g} /\!/ G$. So, this family is the pullback of the universal one by a morphism. We prove that this

morphism is an isomorphism by showing that $\mathrm{Hilb}^G_{\mathfrak{g}_{\mathrm{reg}}}(\mathfrak{g})$ is smooth and applying Zariski's main theorem. So, we obtain that the application $\pi_{\mathfrak{g}_{\mathrm{reg}}}$ identifies with the restriction of the quotient map $\pi : \mathfrak{g} \to \mathfrak{g}//G$; in particular, it is a morphism.

In Section 3, we study any sheet $\mathcal{S}$ for $G = \mathrm{SL}(n)$. We explicitly construct a flat family over an affine space whose fibers are the closures in $\mathfrak{g}$ of the G-orbits in $\mathcal{S}$. Then, following the same method as in the case of $\mathfrak{g}_{\mathrm{reg}}$, we show that this family is universal. Let us denote by $\pi : \mathcal{S} \to \mathcal{S}/\mathrm{SL}(n)$ the geometric quotient of $\mathcal{S}$, constructed by Katsylo in [Ka]. We show that there is a canonical morphism

$$\begin{array}{rccc} \theta : & \mathcal{S}/\mathrm{SL}(n) & \longrightarrow & \mathrm{Hilb}^{\mathrm{SL}(n)}_{\mathcal{S}}(\mathfrak{g}) \\ & \mathrm{SL}(n) \cdot x & \longmapsto & \overline{\mathrm{SL}(n) \cdot x} \end{array}$$

which is actually an isomorphism onto a connected component of $\mathrm{Hilb}^{\mathrm{SL}(n)}_{\mathcal{S}}(\mathfrak{g})$.

Another motivation for this work is to understand examples of invariant Hilbert schemes. Indeed, the construction of Alexeev and Brion is indirect and only a few examples are known (see [J], [BC]). Here, the connected components of invariant Hilbert schemes we obtain happen to be affine spaces, as in [J] and [BC]. Note that this answers in the case of $\mathrm{SL}(n)$ a question of Katsylo who asked if the geometric quotient $\mathcal{S}/G$ is normal. Note that Bongartz already proved in [Bo] that the quotient $\mathcal{S}/G$ is homeomorphic to an affine space.

1. Hilbert's sheets

We consider schemes and affine algebraic groups over $\mathbb{C}$. Let G be a connected semisimple group. We choose a Borel subgroup B, and a maximal torus T contained in B. We denote by U the unipotent radical of B; we have $B = TU$.

We denote by Λ the character group of T. We denote by Λ^+ the set of elements of Λ that are dominant weights with respect to B. The set Λ^+ is in bijection with the set of isomorphism classes of simple rational G-modules. If λ is an element of Λ^+, we denote by $V(\lambda)$ an associated simple G-module that is of highest weight λ.

If V is a rational G-module, we denote by $V_{(\lambda)}$ its isotypical component of type λ, that is the sum of its submodules isomorphic to $V(\lambda)$. We have the decomposition $V = \bigoplus_{\lambda \in \Lambda^+} V_{(\lambda)}$.

In any decomposition of V as a direct sum of simple modules, the multiplicity of the simple module $V(\lambda)$ is the dimension of $V_{(\lambda)}^U$. We say that V *has finite multiplicities* if these multiplicities are finite (for any dominant weight λ).

Let us recall some definitions from [AB, §1]. A *family of affine G-schemes* over some scheme S is a scheme $\mathfrak{X}$ equipped with an action of G and with a morphism $\pi : \mathfrak{X} \to S$ that is affine, of finite type and G-invariant. We have a G-equivariant morphism of $\mathcal{O}_S$-modules

$$\pi_* \mathcal{O}_{\mathfrak{X}} \simeq \bigoplus_{\lambda \in \Lambda^+} \mathcal{F}_\lambda \otimes_{\mathbb{C}} V(\lambda),$$

where each $\mathcal{F}_\lambda := (\pi_* \mathcal{O}_{\mathfrak{X}})^U_{(\lambda)}$ is equipped with the trivial action of G. Let $h : \Lambda^+ \to \mathbb{N}$ be a function. The family $\mathfrak{X}$ is said to be *of Hilbert function h* if each $\mathcal{F}_\lambda$ is an $\mathcal{O}_S$-module locally free of rank $h(\lambda)$. (Then the morphism π is flat.)

Let X be an affine G-scheme, and $h : \Lambda^+ \longrightarrow \mathbb{N}$ a function. A *family of G-stable closed subschemes of X* over some scheme S is a G-stable closed subscheme $\mathfrak{X} \subseteq S \times X$. The projection $S \times X \to S$ induces a family of affine G-schemes $\mathfrak{X} \to S$. The contravariant functor: (Schemes) $\longrightarrow$ (Sets) that associates to every

scheme S the set of families $\mathfrak{X} \subseteq S \times X$ of Hilbert function h is represented by a quasiprojective scheme denoted by $\mathrm{Hilb}_h^G(X)$ ([AB, §1.2].

The dimension of an affine G-scheme whose affine algebra has finite multiplicities can be read off its Hilbert function:

Proposition 1.1. *Let* $h : \Lambda^+ \longrightarrow \mathbb{N}$ *be a function. Let* Y *and* Z *be two affine schemes of Hilbert function* h. *Then* $\dim Y = \dim Z$.

Proof. Let us denote by A the affine ring of Y.

If Y is *horospherical*, that is ([AB, Lemma 2.4]), if for any dominant weights λ, μ, we have $A_{(\lambda)} \cdot A_{(\mu)} \subseteq A_{(\lambda+\mu)}$, it is clear that the dimension of Y can be read off its Hilbert function. Indeed, let us denote by θ_0 the linear map from $\Lambda \otimes \mathbb{Q}$ to $\mathbb{Q}$ which associates to any fundamental weight the value 1. We denote by θ the group homomorphism from Λ to $\mathbb{Z}$, that is, the restriction of θ_0. We associate to θ a graduation of the algebra A by $\mathbb{N}$; its homogeneous component of degree d is

$$A_d := \bigoplus_{\lambda \in \Lambda^+, \ \theta(\lambda)=d} A_{(\lambda)}.$$

The dimension of A_d is finite, and depends only on h:

$$\dim A_d = \sum_{\lambda \in \Lambda^+, \ \theta(\lambda)=d} h(\lambda) \dim V(\lambda).$$

Therefore, the Hilbert polynomial of the graded algebra A depends only on h, and so does the dimension of Y.

We can deduce the proposition. Indeed, Y admits a flat degeneration over a connected scheme to a horospherical G-scheme Y' that admits the same Hilbert function (by [AB, Theorem 2.7]). So $\dim Y = \dim Y'$ depends only on h. $\square$

We will use the method of "asymptotic cones" of Borho and Kraft ([PV, §5.2]). Let V be a finite dimensional rational G-module and F the closure of an orbit in V (or, more generally, any G-stable closed subvariety contained in a fiber of the categorical quotient $V \to V /\!/ G$). We embed V into the projective space $\mathbb{P}(\mathbb{C} \oplus V)$ of vector lines of $\mathbb{C} \oplus V$ by the inclusion $v \mapsto [1 \oplus v]$. The closure of F in $\mathbb{P}(\mathbb{C} \oplus V)$ is denoted by $\overline{F}$. The affine cone in $\mathbb{C} \oplus V$ over $\overline{F}$ is the closed cone $\mathfrak{X}$ generated by F.

The vector space $\mathbb{C} \oplus V$, equipped with its natural scheme structure, is denoted by $\mathbb{A}^1 \times V$. The cone $\mathfrak{X} \subseteq \mathbb{A}^1 \times V$, viewed as a reduced closed subscheme, is a flat family of affine G-schemes. Its fibers over non-zero elements are homothetic to F. Its fiber over 0 is a reduced cone, denoted by $\hat{F}$. It is contained in the null-cone of V (that is the fiber of the categorical quotient $V \to V /\!/ G$ containing 0). Its dimension is the same as F.

We consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. If x is an element of $\mathfrak{g}$, the affine algebra of the closure of its orbit, viewed as a reduced scheme, has finite multiplicities. Let us denote by h_x its Hilbert function; we call it the Hilbert function associated to x. In this paper, we are interested in the connected component denoted Hilb_x^G of the scheme $\mathrm{Hilb}_{h_x}^G(\mathfrak{g})$ that contains $\overline{G \cdot x}$. It gives the G-invariant deformations of $\overline{G \cdot x}$ included in $\mathfrak{g}$. We determine it when x is in $\mathfrak{g}_{\mathrm{reg}}$ in §2, and for any x when G is the special linear group in §3.

Let us denote by G_x the stabilizer of x in G, and $\mathfrak{g}_x$ its Lie algebra. The coadjoint action of G_x is its natural action on the dual vector space $\mathfrak{g}_x^*$.

Proposition 1.2. *Let us assume the orbit closure $\overline{G \cdot x}$ to be normal. The tangent space $T_{\overline{G \cdot x}} \operatorname{Hilb}_x^G$ to Hilb_x^G at the point $\overline{G \cdot x}$ is canonically isomorphic to the space of invariants of the coadjoint action of G_x.*

Proof. The tangent space to $\overline{G \cdot x}$ at the point x is $\mathfrak{g}.x$; it is stable under the action of G_x. We denote by $[\mathfrak{g}/\mathfrak{g}.x]^{G_x}$ the space of invariants under the action of G_x on the quotient vector space $\mathfrak{g}/\mathfrak{g}.x$. According to [AB, Proposition 1.15 (iii)], we have a canonical isomorphism

$$(1) \qquad T_{\overline{G \cdot x}} \operatorname{Hilb}_x^G \cong [\mathfrak{g}/\mathfrak{g}.x]^{G_x}.$$

Indeed, the orbit closure $\overline{G \cdot x}$ is assumed to be normal. Moreover, every orbit in $\mathfrak{g}$ has even dimension, and has a finite number of orbits in its closure ([PV, Corollary 3, p. 198]), so the codimension of the boundary of $G \cdot x$ in $\overline{G \cdot x}$ is at least 2, and the proposition of [AB] can be applied.

To transform (1) into the isomorphism of the proposition, we will use the Killing form on $\mathfrak{g}$, denoted by κ. As $\mathfrak{g}$ is semisimple, its Killing form gives an isomorphism

$$\phi: \quad \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g}^* \\ y & \longmapsto & \kappa(y, \cdot). \end{array}$$

The isomorphism ϕ is G-equivariant, thus G_x-equivariant. It sends $\mathfrak{g}.x$ onto the space $\mathfrak{g}_x^\perp$ of linear forms on $\mathfrak{g}$ that vanish on $\mathfrak{g}_x$. Indeed, the common zeros of the elements of $\phi(\mathfrak{g}.x)$ are the elements y in $\mathfrak{g}$ such that

$$\forall z \in \mathfrak{g}, \ \kappa([z, x], y) = 0,$$

that is,

$$\forall z \in \mathfrak{g}, \ \kappa(z, [x, y]) = 0,$$

and this last condition means that y belongs to $\mathfrak{g}_x$ since κ is non-degenerate.

Thus the short exact sequence of G_x-modules

$$0 \longrightarrow \mathfrak{g}.x \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}.x \longrightarrow 0$$

identifies (thanks to ϕ) with

$$0 \longrightarrow \mathfrak{g}_x^\perp \longrightarrow \mathfrak{g}^* \longrightarrow (\mathfrak{g}_x)^* \longrightarrow 0,$$

and the proposition follows from (1). $\qquad\qquad\square$

A *sheet* of $\mathfrak{g}$ is a maximal irreducible subset of $\mathfrak{g}$ consisting of G-orbits of a fixed dimension. Every sheet of $\mathfrak{g}$ contains a unique nilpotent orbit. A *regular element* of $\mathfrak{g}$ is an element of $\mathfrak{g}$ whose orbit has maximal dimension. The open subset of $\mathfrak{g}$ whose elements are the regular elements is a sheet denoted by $\mathfrak{g}_{\mathrm{reg}}$.

Let us call *Hilbert's sheet* a maximal irreducible subset of $\mathfrak{g}$ consisting of elements admitting a fixed associated Hilbert function.

Proposition 1.3. *Hilbert's sheets of $\mathfrak{g}$ coincide with its sheets.*

Proof. According to Proposition 1.1, any Hilbert sheet is contained in some sheet. It just remains to check that two points of some sheet $\mathcal{S}$ have the same associated Hilbert function.

Let F be the closure of an orbit in $\mathcal{S}$. We recall that its asymptotic cone $\hat{F}$ is a degeneration of F. In particular, it is contained in the closure of $\mathcal{S}$. Moreover, $\hat{F}$ is contained in the null-cone of $\mathfrak{g}$, and its dimension is the same as F. So $\hat{F}$ is the closure of the nilpotent orbit of $\mathcal{S}$.

The affine algebra of $\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}^*$. Its graduation induces a G-invariant filtration on the affine algebra A of F. The affine algebra of the asymptotic cone $\hat{F}$ is isomorphic, as an algebra equipped with an action of G, to the graded algebra $\hat{A}$ associated to the filtered algebra A. In particular, A and $\hat{A}$ are isomorphic as G-modules, and their multiplicities are equal: the Hilbert function of F is equal to that of $\hat{F}$, and the proposition is proved. $\square$

Notice that in the case of the regular sheet, Proposition 1.3 is a direct consequence of [Ko, Theorem 0.9].

2. REGULAR CASE

Let us denote by h_{reg} the Hilbert function associated to the regular elements of $\mathfrak{g}$ (Proposition 1.3). In this section, we prove that the invariant Hilbert scheme $H_{\mathrm{reg}} := \mathrm{Hilb}^G_{h_{\mathrm{reg}}}(\mathfrak{g})$ is the categorical quotient $\mathfrak{g}/\!/G$, that is, an affine space whose dimension is the rank of G.

By [Ko, Theorem 0.1], all schematic fibers of the quotient morphism $\mathfrak{g} \to \mathfrak{g}/\!/G$ are reduced. This allows us to identify in the following the schematic fibers with the set-theoretical fibers.

2.1. **A morphism from $\mathfrak{g}/\!/G$ to H_{reg}.** Let $\mathfrak{X}_{\mathrm{reg}}$ be the graph of the canonical projection $\mathfrak{g} \to \mathfrak{g}/\!/G$. It is a family of G-stable closed subschemes of $\mathfrak{g}$ over $\mathfrak{g}/\!/G$.

Proposition 2.1. *The closed subscheme $\mathfrak{X}_{\mathrm{reg}}$ is a family of G-stable closed subschemes of $\mathfrak{g}$ with Hilbert function h_{reg}.*

Proof. Let us denote by $\pi : \mathfrak{X}_{\mathrm{reg}} \to \mathfrak{g}/\!/G$ the canonical projection, and by $\mathcal{R} := \pi_* \mathcal{O}_{\mathfrak{X}_{\mathrm{reg}}}$ the direct image by π of the structural sheaf of $\mathfrak{X}_{\mathrm{reg}}$. We have to prove that for any dominant weight λ, we have that $\mathcal{R}^U_{(\lambda)}$ is a locally free sheaf on $\mathfrak{g}/\!/G$ of rank $h(\lambda)$.

Let us first study the case where $\lambda = 0$. The morphism $\pi/\!/G : \mathfrak{X}_{\mathrm{reg}}/\!/G \to \mathfrak{g}/\!/G$ induced by π is clearly an isomorphism. So $\mathcal{R}^G = \mathcal{R}^U_{(0)}$ is a free module on $\mathfrak{g}/\!/G$ of rank $1 = h_{\mathrm{reg}}(0)$.

Let λ be a dominant weight. It is known (see [AB, Lemma 1.2]) that $\mathcal{R}^U_{(\lambda)}$ is a coherent $\mathcal{R}^G$-module. Thus it is a coherent module on $\mathfrak{g}/\!/G$. To see that it is locally free, we just have to check that its rank is constant. The fibers of π are those of the canonical projection $\mathfrak{g} \to \mathfrak{g}/\!/G$, so they are the orbit closures of the regular elements, and all of them admit h_{reg} as Hilbert function. So the rank of $\mathcal{R}^U_{(\lambda)}$ at any closed point of $\mathfrak{g}/\!/G$ is $h(\lambda)$, and the proposition is proved. $\square$

This gives us a canonical morphism

$$\phi_{\mathrm{reg}} : \mathfrak{g}/\!/G \longrightarrow H_{\mathrm{reg}}.$$

We will prove in the following that ϕ_{reg} is an isomorphism.

Lemma 2.2. *The morphism ϕ_{reg} realizes a bijection from the set of closed points of $\mathfrak{g}/\!/G$ to the set of closed points of H_{reg}.*

Proof. We remark that ϕ_{reg} is injective. Let us check if it is surjective; in other words, that any G-invariant closed subscheme of $\mathfrak{g}$ of Hilbert function h_{reg} is a fiber of $\mathfrak{g} \to \mathfrak{g}/\!/G$.

Let Y be such a subscheme. As $h_{\text{reg}}(0) = 1$, it has to be contained in some fiber F of $\mathfrak{g} \to \mathfrak{g}//G$ over a reduced closed point. But F already corresponds to a closed point of H_{reg} in the image of ϕ_{reg}. Moreover, F admits no proper closed subscheme admitting the same Hilbert function, so $F = Y$, and the lemma is proved. $\qquad \square$

Let us denote by r the rank of G. The quotient $\mathfrak{g}//G$ is an affine space of dimension r. A consequence of Lemma 2.2 is:

Corollary 2.3. *The dimension of* H_{reg} *is* r.

2.2. **Tangent space.** In this section, we prove:

Proposition 2.4. *The scheme* H_{reg} *is smooth.*

Proof. Let Z be a closed point of H_{reg}. We have to prove that the dimension of the tangent space $T_Z H_{\text{reg}}$ is r. We still denote by Z the closed subscheme of $\mathfrak{g}$ corresponding to Z. By Lemma 2.2, we know that Z is a fiber of the morphism $\mathfrak{g} \to \mathfrak{g}//G$, thus the closure of some regular element x. It is a normal variety. By Proposition 1.2, we have to prove that the dimension of

$$(\mathfrak{g}_x^*)^{G_x}$$

is r, or simply that it is lower or equal to r (by Corollary 2.3).

Let us prove that the dimension of the space

$$(\mathfrak{g}_x^*)^{\mathfrak{g}_x}$$

is r, and the proposition will be proved.

A linear form on $\mathfrak{g}_x$ is $\mathfrak{g}_x$-invariant if and only if it vanishes on the derived algebra $[\mathfrak{g}_x, \mathfrak{g}_x]$, so we have to prove that

$$(\mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x])^*$$

is r-dimensional. We will prove that $\mathfrak{g}_x$ is an r-dimensional abelian algebra, and the proposition will be proved. This is true if x is semisimple, because then $\mathfrak{g}_x$ is a Cartan subalgebra of $\mathfrak{g}$. If the regular element x is not assumed to be semisimple, the dimension of $\mathfrak{g}_x$ is still r, because this does not depend on the regular element x, by definition. Let us check that $\mathfrak{g}_x$ is abelian.

Let us denote by $\text{Grass}_r(\mathfrak{g})$ the grassmannian of r-dimensional subspaces of $\mathfrak{g}$, endowed with its projective variety structure. The subset of $\mathfrak{g}_{\text{reg}} \times \text{Grass}_r(\mathfrak{g})$:

$$\{(z, \mathfrak{h}) \in \mathfrak{g}_{\text{reg}} \times \text{Grass}_r(\mathfrak{g}) \mid \mathfrak{h} \cdot z = 0 \text{ and } [\mathfrak{h}, \mathfrak{h}] = 0\}$$

is closed, so its image by the natural projection into $\mathfrak{g}_{\text{reg}}$ is closed too. As its image contains the semisimple elements of $\mathfrak{g}_{\text{reg}}$, it is equal to $\mathfrak{g}_{\text{reg}}$. Thus $\mathfrak{g}_x$ is abelian for any regular x, and the proposition is proved. $\qquad \square$

2.3. **Conclusion.** We can now conclude that the family $\mathfrak{X}_{\text{reg}}$ of Proposition 2.1 is the universal family:

Theorem 2.5. *The morphism* ϕ_{reg} *from* $\mathfrak{g}//G$ *to* H_{reg} *is an isomorphism.*

Proof. The morphism ϕ_{reg} is bijective (Lemma 2.2) and H_{reg} is normal. According to Zariski's main theorem, ϕ_{reg} is an isomorphism. $\qquad \square$

Remark 2.6. One knows there is a canonical morphism

$$\psi_{\mathrm{reg}} : \mathrm{H}_{\mathrm{reg}} \longrightarrow \mathfrak{g} /\!/ G$$

that associates to any closed point F of $\mathrm{H}_{\mathrm{reg}}$ (viewed as a closed subscheme of $\mathfrak{g}$) its categorical quotient $F /\!/ G$ (viewed as a closed point of $\mathfrak{g} /\!/ G$). This morphism is a particular case of the morphism

$$\eta : \mathrm{Hilb}_h^G(V) \longrightarrow \mathrm{Hilb}_{h(0)}(V /\!/ G)$$

defined in [AB, §1.2], because $h_{\mathrm{reg}}(0) = 1$ and thus the punctual Hilbert scheme that parametrizes closed subschemes of length 1 in $\mathfrak{g} /\!/ G$ identifies with $\mathfrak{g} /\!/ G$ itself. The morphism ψ_{reg} is clearly the inverse morphism of ϕ_{reg}.

Remark 2.7. As pointed out to us by M. Brion, Theorem 2.5 admits the following generalization:

Let X be an irreducible affine G-variety such that $\pi : X \to X /\!/ G$ is flat. Let h be the Hilbert function of its fibers. Then the graph Γ of π is the universal family; in particular, $\mathrm{Hilb}_h^G(X)$ identifies with $X /\!/ G$.

The idea of his proof is to check that Γ represents the functor. Let $\mathfrak{X} \subseteq X \times S$ be a flat family of Hilbert function h, over some affine scheme S. Since $h(0) = 1$, the scheme S identifies with $\mathfrak{X} /\!/ G$ and maps on $X /\!/ G$ (by the morphism induced by the first projection $X \times S \to X$). We obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{\ p_2\ } & \Gamma \\
\downarrow & & \downarrow \\
\mathfrak{X} /\!/ G \simeq S & \longrightarrow & X /\!/ G.
\end{array}
$$

It remains to prove that $\mathfrak{X}$ is isomorphic (canonically) to the fiber product $\Gamma \times_{X /\!/ G} S$. This has only to be verified over the closed points of S. The assertion follows.

The Hilbert schemes we obtain by applying the above result to G-modules are always affine spaces. The representations V of a simple group G such that $V \to V /\!/ G$ is flat have been classified by G. Schwarz in [Sch].

Unfortunately, the sheets of $\mathfrak{sl}(n)$ are not affine in general and Katsylo's quotient cannot be extended to their closure. So, Brion's theorem cannot be applied, whereas the method we used to prove Theorem 2.5 can be used.

3. CASE OF $\mathfrak{sl}(n)$

Let us start this section by fixing notation for invariant factors of a square complex matrix (for more details see for example [La]).

We denote by t an indeterminate over $\mathbb{C}$, and I_n the identity matrix of size $n \times n$. If x is an element of $\mathfrak{sl}(n)$ and $i = 1, \ldots, n$, we denote by $Q_i^x(t)$ the monic greatest common divisor (in the ring $\mathbb{C}[t]$) of the $(n + 1 - i) \times (n + 1 - i)$-sized minors of $x - t I_n$, and $Q_{n+1}^x(t) := 1$.

Then we write

$$q_i^x(t) := Q_i^x(t) / Q_{i+1}^x(t).$$

The polynomials $q_1^x(t), \ldots, q_n^x(t)$ are the invariant factors of the matrix $x - t I_n$ with coefficients in the euclidean ring $\mathbb{C}[t]$, ordered in such a way that $q_{i+1}^x(t)$ divides $q_i^x(t)$.

If x, y are elements of $\mathfrak{sl}(n)$, then y is in the closure of the orbit $\mathrm{SL}(n) \cdot x$ of x if and only if for any $i = 1, \ldots, n$, the polynomial $Q_i^x(t)$ divides $Q_i^y(t)$. In other words, if and only if for any i, the polynomial $Q_i^x(t)$ divides the $(n+1-i) \times (n+1-i)$-sized minors of $y - tI_n$.

According to [W], when x is nilpotent, these conditions define the closure of $\mathrm{SL}(n) \cdot x$ as a reduced scheme; to be more precise, when one divides a $(n+1-i) \times (n+1-i)$-sized minor of $y - tI_n$ by $Q_i^x(t)$, then the remainder is a regular function of y. All such functions generate the ideal of the closure of $\mathrm{SL}(n) \cdot x$. We will deduce easily from this difficult result that the same remains true if x is no longer assumed to be nilpotent.

The set of sheets of $\mathfrak{sl}(n)$ is in bijection with the set of partitions n, that is, of sequences $\sigma = (b_1 \geq b_2 \geq b_3 \geq \cdots)$ of nonnegative integers such that $b_1 + b_2 + b_3 + \cdots = n$ (see [Bo, §2.3]). Namely, if σ is a partition of n, the elements of the corresponding sheet $\mathcal{S}_\sigma$ are those elements x such that for any i, the polynomial $q_i^x(t)$ is of degree b_i. We denote by $\widehat{\sigma} = (c_1 \geq c_2 \geq c_3 \geq \cdots)$ the conjugate partition, where c_j is the number of i such that $b_i \geq j$. We denote by h_σ the Hilbert function associated to the points of $\mathcal{S}_\sigma$ (Proposition 1.3). We denote by Z_σ the closure of the nilpotent orbit of $\mathcal{S}_\sigma$. The connected component of $\mathrm{Hilb}_{h_\sigma}^{\mathrm{SL}(n)}(\mathfrak{sl}(n))$ that contains Z_σ as a closed point is denoted H_σ. We will prove in this section that H_σ is an affine space of dimension $b_1 - 1$. The proof is similar to §2.

We recall that the sheets of $\mathfrak{sl}(n)$ are smooth ([Kr]).

3.1. A construction of the geometric quotient of $\mathcal{S}_\sigma$.

Katsylo showed in [Ka] that any sheet of a semisimple Lie algebra admits a geometric quotient. Although his proof contains an explicit construction, it does not make clear the geometric properties of the quotient. Here we present a simple description of the quotient in the case of the Lie algebra $\mathfrak{sl}(n)$. It is based on the invariant factors theory. We get that the quotient is an affine space (see also [Bo, Section 4]).

Lemma 3.1. *For all i, the application $\mathcal{S}_\sigma \longrightarrow \mathbb{A}^{b_i}$ that associates to any x the coefficients of $q_i^x(t) = t^{b_i} + \lambda_{b_i-1}^x t^{b_i-1} + \cdots + \lambda_0^x t^0$ is regular.*

Proof. Up to scalar multiplication, the polynomial $q_i^x(t)$ is the unique nonzero polynomial of degree less or equal to b_i such that

$$(2) \qquad \dim \ker q_i^x(x) \geq N := \sum_{j=1}^{b_i} c_j.$$

Thus the closed subset of $\mathcal{S}_\sigma \times \mathbb{P}^{b_i}$ consisting of elements $(x, [\mu_0 : \cdots : \mu_{b_i}])$ such that

$$\dim \ker(\sum_{j=0}^{b_i} \mu_j x^j) \geq N$$

is the graph of the application

$$\psi : \quad \begin{array}{ccc} \mathcal{S}_\sigma & \longrightarrow & \mathbb{P}^{b_i} \\ x & \longmapsto & [\lambda_0^x : \cdots : \lambda_{b_i-1}^x : 1]. \end{array}$$

According to [Hr, Exercise 7.8, p. 76], this graph is also the graph of a rational map ϕ from $\mathcal{S}_\sigma$ to $\mathbb{P}^{b_i}$. On the open subset Ω of $\mathcal{S}_\sigma$ where ϕ is regular, ϕ coincides with ψ, so the functions $x \mapsto \lambda_j^x$ are regular functions from Ω to $\mathbb{A}^1$. As $\mathcal{S}_\sigma$ is smooth, the complementary of Ω in $\mathcal{S}_\sigma$ has codimension at least 2 ([Sha, Thm. 3,

Chap. II.3.1]). We conclude that the functions extend to regular functions from $\mathcal{S}_\sigma$ to $\mathbb{A}^1$. By continuity, these extensions satisfy (2), so they coincide with the functions $x \mapsto \lambda_j^x$ on $\mathcal{S}_\sigma$. $\qquad\square$

Let us define, for any x in $\mathcal{S}_\sigma$, the monic polynomial of degree $b_i - b_{i+1}$:

$$p_i^x(t) := q_i^x(t)/q_{i+1}^x(t)$$

(where $q_{n+1}^x := 1$). It follows from the previous lemma that its coefficients, viewed as functions of x, are regular functions from $\mathcal{S}_\sigma$ to $\mathbb{A}^1$.

Given an x, the family $(p_1^x(t), \ldots, p_n^x(t))$ can be any family of monic polynomials of degrees $b_i - b_{i+1}$, provided the following relation is satisfied, where $S(p_i^x)$ denotes the sum of the roots of p_i^x, counted with multiplicities (given by the coefficient of the monomial of degree $\deg(p_i^x) - 1$):

$$\sum_{i=1}^n iS(p_i^x) = 0$$

(this relation simply means that the trace of x is zero).

Thus, associating to any x the coefficients of the family $(p_1^x(t), \ldots, p_n^x(t))$, we get a regular map π from $\mathcal{S}_\sigma$ to a linear hyperplane of $\mathbb{C}^{b_1}$, which we will denote by $\mathbb{A}^{b_1-1}$.

Proposition 3.2. *The map $\pi : \mathcal{S}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$ is the geometric quotient of $\mathcal{S}_\sigma$.*

Proof. This map is surjective, and its fibers are exactly the orbits of $\mathcal{S}_\sigma$ under the action of $\mathrm{SL}(n)$. Let us denote by $\mathcal{S}_\sigma / \mathrm{SL}(n)$ the geometric quotient of $\mathcal{S}_\sigma$ (whose existence is proved in [Ka]). The map π is the composite of the canonical projection from $\mathcal{S}_\sigma$ to $\mathcal{S}_\sigma / \mathrm{SL}(n)$ with a regular bijection

$$\mathcal{S}_\sigma / \mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}.$$

This last map is bijective (thus birational), and the space $\mathbb{A}^{b_1-1}$ is normal. According to Zariski's main theorem, it is an isomorphism. $\qquad\square$

3.2. **A morphism from $\mathcal{S}_\sigma \mathrm{SL}(n)$ to H_σ.** If $z = (p_1(t), \ldots, p_n(t))$ is a closed point of $\mathbb{A}^{b_1-1}$ corresponding to the orbit $\mathrm{SL}(n) \cdot x$ in $\mathcal{S}_\sigma$, the polynomial

$$Q_i^x(t) = p_i(t) \cdot (p_{i+1}(t))^2 \cdot \cdots \cdot (p_n(t))^{n-i+1}$$

depends on z only. Let us denote it by $Q_i^z(t)$. Its coefficients are regular functions from $\mathbb{A}^{b_1-1}$ to $\mathbb{A}^1$.

Let us consider the closed subscheme $\mathfrak{X}_\sigma$ of $\{(z,y) \in \mathbb{A}^{b_1-1} \times \mathfrak{sl}(n)\}$ defined by the vanishing, for $i = 1, \ldots, n$, of the remainders we get when we divide the $(n+1-i) \times (n+1-i)$-minors of $y - tI_n$ by $Q_i^z(t)$. We denote by I_σ the ideal generated by these remainders. The underlying set of $\mathfrak{X}_\sigma$ consists of all the couples (z,y) such that y is in the closure of the orbit corresponding to z.

Proposition 3.3. *The closed subscheme $\mathfrak{X}_\sigma$ is a family of $\mathrm{SL}(n)$-stable closed subschemes of $\mathfrak{sl}(n)$ with Hilbert function h_σ.*

Proof. The proof is similar to that of Proposition 2.1. The subscheme $\mathfrak{X}_\sigma$ is a family of $\mathrm{SL}(n)$-stable closed subschemes of $\mathfrak{sl}(n)$ over $\mathbb{A}^{b_1-1}$. Let us denote by π the morphism $\mathfrak{X}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$.

As previously, let us first recall that the morphism

$$\pi /\!/ \mathrm{SL}(n) : \mathfrak{X}_\sigma /\!/ \mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}$$

induced by π is an isomorphism. To do this, let us verify that the comorphism

$$(\pi /\!/ \operatorname{SL}(n))^* : \mathbb{C}[\mathbb{A}^{b_1-1}] \longrightarrow \mathbb{C}[\mathbb{A}^{b_1-1}] \otimes \mathbb{C}[\mathfrak{sl}(n)]^{\operatorname{SL}(n)} / I_\sigma^{\operatorname{SL}(n)}$$

is an isomorphism. It is injective, as π is surjective. Its surjectivity comes from the relations that define $\mathfrak{X}_\sigma$; they give, for $i = 1$, that $Q_1^z(t)$ divides the determinant of $t I_n - y$, that is, the characteristic polynomial of y. As their degrees are equal, $Q_1^z(t)$ and the characteristic polynomial of y are equal. This gives the surjectivity.

We go on as previously. Let λ be a dominant weight. The $R^{\operatorname{SL}(n)}$-module $R_{(\lambda)}^U$ is of finite type ([AB, Lemma 1.2]). Thus $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$ is a coherent $\mathcal{O}_{\mathbb{A}^{b_1-1}}$-module. To see that it is locally free, we just have to check that its rank is constant. Let us assume that the origin $0 \in \mathbb{A}^{b_1-1}$ corresponds to the nilpotent orbit in $\mathcal{S}_\sigma$. The fiber of π over 0 is the closure of this orbit, fitted with its structure of reduced scheme. Thus, the rank of $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$ at 0 is $h_\sigma(\lambda)$. If z is any point of $\mathbb{A}^{b_1-1}$, the fiber of π over z is as a set the closure in $\mathfrak{sl}(n)$ of the corresponding orbit. So, by Proposition 1.3 the rank of $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$ at z is at least $h_\sigma(\lambda)$. To conclude, we use the action of the multiplicative group on $\mathfrak{sl}(n)$ (by homotheties) and the induced action on $\mathbb{A}^{b_1-1}$, which makes π equivariant. The orbit of z goes arbitrarily close to 0, and the rank of a coherent sheaf is upper semicontinuous, so the rank of $(\pi_* \mathcal{O}_{\mathfrak{X}_\sigma})_{(\lambda)}^U$ is $h_\sigma(\lambda)$ at z. $\qquad\square$

3.3. Tangent space.

In this section, we compute the dimension of the tangent space to H_σ at the point Z_σ:

Proposition 3.4. *The dimension of $T_{\operatorname{Z}_\sigma} \operatorname{H}_\sigma$ is $b_1 - 1$.*

Proof. Let x be an element in the open orbit in Z_σ. It is known that Z_σ is normal ([KP]). So by Proposition 1.2, we just have to prove that the dimension of

$$(\mathfrak{sl}(n)_x^*)^{\operatorname{SL}(n)_x}$$

is $b_1 - 1$. Let us consider $\operatorname{SL}(n)$ as a closed subgroup of the general linear group $\operatorname{GL}(n)$, and $\mathfrak{sl}(n)$ as a subalgebra of $\mathfrak{gl}(n)$. The stabilizer $\operatorname{GL}(n)_x$ of x in $\operatorname{GL}(n)$ is generated by $\operatorname{SL}(n)_x$ and the center of $\operatorname{GL}(n)$. It is clearly equivalent to prove that the dimension of

$$(\mathfrak{gl}(n)_x^*)^{\operatorname{GL}(n)_x}$$

is b_1. The group $\operatorname{GL}(n)_x$ is connected, so the last space is isomorphic to

$$(\mathfrak{gl}(n)_x^*)^{\mathfrak{gl}(n)_x}.$$

A linear form on $\mathfrak{gl}(n)_x$ is $\mathfrak{gl}(n)_x$-invariant if and only if it vanishes on the derived algebra $[\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x]$, so we have to prove that

$$(\mathfrak{gl}(n)_x / [\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x])^*$$

is b_1-dimensional. This fact is the following elementary lemma. $\qquad\square$

Lemma 3.5. *Let $E = \bigoplus_{i=1}^{c_1} E_i$ be a graded vector space over $\mathbb{C}$, where each E_i is b_i-dimensional. We denote by $\mathfrak{h} := \mathfrak{gl}(E)$ the Lie algebra of endomorphisms of E. Let x be a nilpotent element of $\mathfrak{h}$ such that each subspace E_i is stabilized by x, and the restriction of x to each E_i is cyclic.*

Let us denote by $\mathfrak{h}_x$ the stabilizer of x in $\mathfrak{h}$. Then the codimension of the derived algebra $[\mathfrak{h}_x, \mathfrak{h}_x]$ in $\mathfrak{h}_x$ is b_1.

Proof. The graduation of E induces a graduation on the vector space $\mathfrak{h}$:

$$\mathfrak{h} = \bigoplus_{i,j} \operatorname{Hom}(E_i, E_j).$$

Let us denote by $p_i : E \longrightarrow E_i$ the natural projections. As they commute with x, the subspace $\mathfrak{h}_x$ of $\mathfrak{h}$ is homogeneous:

$$\mathfrak{h}_x = \bigoplus_{i,j} \operatorname{Hom}_x(E_i, E_j),$$

where $\operatorname{Hom}_x(E_i, E_j)$ denotes the space of homomorphisms that commute with x. Let us choose, for any i, an element e_i of E_i such that $x^{b_i-1}e_i \neq 0$. We put $n_{ij} := b_j - b_i$ if $j < i$ and 0 otherwise. We denote by $f_{ij} : E_i \to E_j$ the unique homomorphism that commutes with x and which sends e_i to $x^{n_{ij}}e_j$. Then any homomorphism from E_i to E_j that commutes with x is the composite of f_{ij} with a polynomial in x:

$$\operatorname{Hom}_x(E_i, E_j) = \mathbb{C}[x] \cdot f_{ij}.$$

We notice that if $i \neq j$, then $\operatorname{Hom}_x(E_i, E_j)$ is contained in $[\mathfrak{h}_x, \mathfrak{h}_x]$. Indeed, for any $u : E_i \to E_j$, we have $[u, p_i] = u$.

So we have to prove that the codimension in $\bigoplus_i \operatorname{Hom}_x(E_i, E_i)$ of

$$[\mathfrak{h}_x, \mathfrak{h}_x] \cap \bigoplus_i \operatorname{Hom}(E_i, E_i)$$

is b_1. The last vector space is generated by its elements of the form

$$P(x)[f_{ji}, f_{ij}] = P(x)x^{|b_i-b_j|}(\operatorname{id}_{E_i} - \operatorname{id}_{E_j}),$$

where $P(x)$ is a polynomial in x.

One checks easily that a basis of a supplementary in $\bigoplus_i \operatorname{Hom}_x(E_i, E_i)$ of this space is given by the family of elements

$$x^k \operatorname{id}_{E_i}$$

where $0 \leq k < b_i - b_{i+1}$, and the lemma is proved. $\qquad\square$

3.4. **Conclusion.** In this section, we prove that the family $\mathfrak{X}_\sigma$ of Proposition 3.3 is the universal family:

Theorem 3.6. *The morphism ϕ_σ from $\mathcal{S}_\sigma / \operatorname{SL}(n)$ to H_σ obtained in §3.2 is an isomorphism.*

We denote by $\overline{\mathcal{S}_\sigma}$ the closure of $\mathcal{S}_\sigma$ in $\mathfrak{sl}(n)$, equipped with its reduced scheme structure. The invariant Hilbert scheme $\operatorname{H}'_\sigma := \operatorname{Hilb}_{h_\sigma}^{\operatorname{SL}(n)}(\overline{\mathcal{S}_\sigma})$ which parametrizes the closed subschemes of $\overline{\mathcal{S}_\sigma}$ of Hilbert function h_σ is canonically identified with a closed subscheme of $\operatorname{Hilb}_{h_\sigma}^{\operatorname{SL}(n)}(\mathfrak{sl}(n))$. The morphism ϕ_σ factorizes through a morphism $\psi_\sigma : \mathcal{S}_\sigma / \operatorname{SL}(n) \to \operatorname{H}'_\sigma$.

To prove the theorem, we will get that the morphism ψ_σ is an isomorphism from $\mathcal{S}_\sigma / \operatorname{SL}(n)$ to H'_σ and that H'_σ is a connected component of H_σ (Corollary 3.10).

Lemma 3.7. *The morphism ψ_σ induces a bijection from the set of closed points of $\mathcal{S}_\sigma / \operatorname{SL}(n)$ to the set of closed points of H'_σ.*

Proof. We know that ψ_σ is injective. Let us see if it is surjective; in other words, that any $\mathrm{SL}(n)$-invariant closed subscheme of $\overline{\mathcal{S}_\sigma}$ with Hilbert function h_σ is the closure of some orbit in $\mathcal{S}_\sigma$.

Let X be such a subscheme. As $h_\sigma(0) = 1$, it has to be contained in some fiber F of the categorical quotient $\overline{\mathcal{S}_\sigma} \to \overline{\mathcal{S}_\sigma} /\!/ \mathrm{SL}(n)$ over a reduced closed point; but F already corresponds to a closed point of H'_σ in the image of ψ_σ. Moreover, F admits no proper closed subscheme admitting the same Hilbert function, so $F = X$, and the lemma is proved. $\square$

Corollary 3.8. *The dimension of H'_σ is $b_1 - 1$.*

The action of the multiplicative group $\mathbb{G}_m$ on $\mathfrak{sl}(n)$ by homotheties induces canonically an action of $\mathbb{G}_m$ on H_σ, and on H'_σ (because it stabilizes $\overline{\mathcal{S}_\sigma}$). The cone Z_σ is a $\mathbb{G}_m$-fixed point of H'_σ. In fact, it is in the closure of the $\mathbb{G}_m$-orbit of any point of H'_σ:

Proposition 3.9. *Let F be a closed point of H'_σ. The morphism $\eta : \mathbb{G}_m \longrightarrow \mathrm{H}'_\sigma$, $t \longmapsto t.F$ extends to a morphism $\mathbb{A}^1 \longrightarrow \mathrm{H}'_\sigma$, $0 \longmapsto \mathrm{Z}_\sigma$.*

Proof. The point F corresponds to a $\mathrm{SL}(n)$-invariant closed subscheme of $\overline{\mathcal{S}_\sigma}$ admitting Hilbert function h_σ. We still denote it by F. As $h_\sigma(0) = 1$, it is contained in the fiber of the categorical quotient $\mathfrak{sl}(n) \to \mathfrak{sl}(n) /\!/ \mathrm{SL}(n)$ over some closed point. Thus we can apply to it the method of asymptotic cones: we obtain a flat family over $\mathbb{A}^1$ whose fiber over 0 must be Z_σ (as in the proof of Proposition 1.3). It gives a morphism from $\mathbb{A}^1$ to H'_σ whose restriction outside 0 is η. $\square$

From the proposition, we deduce that the dimension of the tangent space to H_σ at any point of H'_σ is lower or equal to that at Z_σ, that is, $b_1 - 1$. As the dimension of H'_σ is $b_1 - 1$, we get:

Corollary 3.10.
- *The scheme H'_σ is reduced and smooth.*
- *It is a connected component of H_σ.*

The morphism ψ_σ is bijective (Lemma 3.7) and H'_σ is normal. According to Zariski's main theorem, ψ_σ is an isomorphism. So Theorem 3.6 is proved, thanks to the second point of Corollary 3.10.

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Le Mas des Landes, 87170 Isle, France

Department of Mathematics, University of Montpellier II, Place Eugène Bataillon, Montpellier, France