REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 13, Pages 141–164 (April 16, 2009)
S 1088-4165(09)00347-1

WHITTAKER MODULES FOR GENERALIZED WEYL ALGEBRAS

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Abstract. We investigate Whittaker modules for generalized Weyl algebras, a class of associative algebras which includes the quantum plane, Weyl algebras, the universal enveloping algebra of $\mathfrak{sl}_2$ and of Heisenberg Lie algebras, Smith's generalizations of $U(\mathfrak{sl}_2)$, various quantum analogues of these algebras, and many others. We show that the Whittaker modules $V = Aw$ of the generalized Weyl algebra $A = R(\phi, t)$ are in bijection with the ϕ-stable left ideals of R. We determine the annihilator $\mathrm{Ann}_A(w)$ of the cyclic generator w of V. We also describe the annihilator ideal $\mathrm{Ann}_A(V)$ under certain assumptions that hold for most of the examples mentioned above. As one special case, we recover Kostant's well-known results on Whittaker modules and their associated annihilators for $U(\mathfrak{sl}_2)$.

1. Introduction

In this work we study the notion of a Whittaker module in the setting of generalized Weyl algebras. Generalized Weyl algebras were introduced by Bavula [B1] and have been studied extensively since then (see for example, [B2], [R], [DGO]). We shall use the definition of a generalized Weyl algebra (GWA) given in [B2, 1.1].

Suppose that R is a unital associative algebra over a field $\mathbb{F}$ with $\phi = (\phi_i)_{i\in\mathcal{I}}$ a collection of pairwise commuting automorphisms of R indexed by the set $\mathcal{I}$ (which may be finite or infinite), and let $t = (t_i)_{i\in\mathcal{I}}$ be a collection of nonzero central elements of R also indexed by $\mathcal{I}$. The *generalized Weyl algebra* $A = R(\phi, t)$ with base ring R is the associative algebra generated over R by elements X_i and Y_i for $i \in \mathcal{I}$ with defining relations

$$Y_iX_i = t_i \qquad X_iY_i = \phi_i(t_i), \tag{1.1}$$

$$X_ir = \phi_i(r)X_i \qquad Y_i\phi_i(r) = rY_i \tag{1.2}$$

for $r \in R$, and

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Y_j] = 0 \tag{1.3}$$

for $i \neq j$, where $[\ ,\ ]$ denotes the commutator $[a, b] = ab - ba$.

We always assume that the algebra R is a domain which is left Noetherian. Thus, by [B2, Prop. 1.3], the algebra $A = R(\phi, t)$ is a domain, and A is left Noetherian if $\mathcal{I}$ is finite. This assumption forces the automorphisms ϕ_i to satisfy $\phi_i(t_j) = t_j$ for $j \neq i$, which can be seen from the calculation,

$$t_jX_i = Y_jX_jX_i = X_iY_jX_j = X_it_j = \phi_i(t_j)X_i.$$

Received by the editors March 25, 2008 and, in revised form, February 9, 2009.
2000 *Mathematics Subject Classification.* Primary 17B10; Secondary 16D60.

Weyl algebras provide the prototypical examples of generalized Weyl algebras. Let $R = \mathbb{F}[t_i \mid i \in \mathfrak{I}]$, the polynomial algebra over $\mathbb{F}$ in commuting variables t_i, and let ϕ_i for $i \in \mathfrak{I}$ be the automorphism of R defined by $\phi_i(t_j) = t_j - \delta_{i,j}1$. Assume $A = R(\phi, t)$, where the relations in (1.1), (1.2), and (1.3) hold for these choices. Then $[Y_i, X_j] = \delta_{i,j}1$ for all $i, j \in \mathfrak{I}$, and A is a Weyl algebra realized as a generalized Weyl algebra.

To construct a second family of examples, let $R = \mathbb{F}[c, t_i \mid i \in \mathfrak{I}]$, the polynomial algebra over $\mathbb{F}$ in commuting variables $c, t_i, i \in \mathfrak{I}$. Let ϕ_i be the automorphism given by $\phi_i(t_j) = t_j - \delta_{i,j}c$ and $\phi_i(c) = c$. In the generalized Weyl algebra $A = R(\phi, t)$ constructed from this data, $[Y_i, X_j] = \delta_{i,j}c$, and c is central in A. Thus, $A = R(\phi, t)$ is isomorphic to the universal enveloping algebra of a Heisenberg Lie algebra in this case. The Weyl and Heisenberg algebras are always generalized Weyl algebras, but $\mathfrak{I}$ needs to be finite for R to be Noetherian.

The notion of a generalized Weyl algebra encompasses many more examples such as the universal enveloping algebra $U(\mathfrak{sl}_2)$ and quantized enveloping algebras $U_q(\mathfrak{sl}_2)$, $U_{r,s}(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$, the Noetherian down-up algebras of [Be], [BR], generalized Heisenberg algebras, and quantum Weyl algebras. We will explain many of these examples later as we discuss results on their Whittaker modules.

Simple Whittaker modules for $\mathfrak{sl}_2$ were first studied by Arnal and Pinczon in [AP]. In [K], Kostant introduced a class of modules for finite-dimensional complex semisimple Lie algebras and named them Whittaker modules because of their connections with Whittaker equations in number theory. These modules have been studied subsequently in a variety of different settings. Miličić and Soergel [MS] investigated modules for semisimple Lie algebras induced from Whittaker modules for parabolic subalgebras. Whittaker modules for semisimple Lie algebras also appeared in the work of Brundan and Kleshchev [BK] on shifted Yangians and W-algebras. Christodoulopoulou [C] used Whittaker modules for Heisenberg Lie algebras to construct irreducible modules for affine Lie algebras.

In [Bl], Block showed that the simple modules for $\mathfrak{sl}_2$ over $\mathbb{C}$ are either highest (or lowest) weight modules, Whittaker modules, or modules obtained by localization. Whittaker modules for $U_q(\mathfrak{sl}_2)$ were investigated in [O1], [O2], where many analogues of Kostant's results on annihilators for Whittaker modules were shown to hold. Because of the prominent role that Whittaker modules play in the representation theory of $\mathfrak{sl}_2$ and of its quantum analogues, we were motivated to study them in the context of generalized Weyl algebras as a way of providing a unified approach to these modules.

Fix R, ϕ, t, and $A = R(\phi, t)$ as above, and let $\zeta = (\zeta_i)_{i \in \mathfrak{I}}$ be a set of nonzero elements of $\mathbb{F}$ indexed by $\mathfrak{I}$. We say that an A-module V is a *Whittaker module of type* ζ if there exists $w \in V$ such that

(1) $V = Aw$,
(2) $X_i w = \zeta_i w$ for all $i \in \mathfrak{I}$.

We refer to the pair (V, w) as a *Whittaker pair of type* ζ.

In what follows, any $v \in V$ such that $X_i v = \zeta_i v$ for all $i \in \mathfrak{I}$ will be called a *Whittaker vector of type* ζ. Such a vector is simply a common eigenvector for all the generators X_i with nonzero eigenvalues. For a Whittaker module V with cyclic Whittaker vector w of type ζ, let $Q = \mathrm{Ann}_R(w)$, the annihilator of w in R. Note that Q is a left ideal of R and $\mathrm{Ann}_A(w)$ is a left ideal of A, while $\mathrm{Ann}_A(V)$ is an ideal of A. We fix this notation for the remainder of the paper.

In Section 3 we construct a universal Whittaker module of type ζ for each generalized Weyl algebra $A = R(\phi, t)$. This module is used in the proof of Theorem 3.12 to show that the isomorphism classes of Whittaker modules of type ζ are in bijection with the ϕ-stable left ideals of R. In particular, simple Whittaker modules correspond to maximal ϕ-stable left ideals of R. For finite-dimensional complex semisimple Lie algebras $\mathfrak{g}$, the corresponding result in [K] states that the isomorphism classes of Whittaker modules of type ζ are in bijection with the ideals of the center of the universal enveloping algebra $U(\mathfrak{g})$. A similar result holds for the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$ (see [O1, O2]). For an arbitrary Whittaker module $V = Aw$ for a generalized Weyl algebra A, in Section 4 we obtain a description of the annihilator of w: $\mathrm{Ann}_A(w) = AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$, where $Q = \mathrm{Ann}_R(w)$. In Sections 5 and 6, we impose the assumption that R is commutative and determine the Whittaker vectors inside a Whittaker module. When R is commutative, Q is a prime ideal not containing any t_i, and the center of A is contained in R, then $\mathrm{Ann}_A(V) = AQ$ by Theorem 6.4. The final sections are devoted to illustrating what these results say for certain well-known algebras such as the (quantum) Weyl algebra, the quantum plane, and Smith's generalizations of $U(\mathfrak{sl}_2)$ and of $U_q(\mathfrak{sl}_2)$. We recover the results of [T1] and [JWZ] for the (quantum) Smith algebras of characteristic zero and determine the Whittaker modules for all of these algebras in the modular and root of unity cases.

2. Basic facts about generalized Weyl algebras and their Whittaker modules

Assume $A = R(\phi, t)$ is a generalized Weyl algebra as in Section 1. Let Γ denote the semigroup of tuples $\gamma = (\gamma_i)_{i\in\mathfrak{I}}$ of nonnegative integers with only finitely many nonzero entries under componentwise addition, $\gamma+\delta = \big((\gamma+\delta)_i\big)_{i\in\mathfrak{I}}$ where $(\gamma+\delta)_i = \gamma_i + \delta_i$. For $\gamma \in \Gamma$, set

$$X^\gamma = \prod_{i\in\mathfrak{I}} X_i^{\gamma_i}. \tag{2.1}$$

Because the various X_j commute, it follows that $X^\gamma X^\delta = X^{\gamma+\delta}$.

We adopt the notation

$$Z_i^\ell = \begin{cases} X_i^\ell & \text{if } \ell \geq 0, \\ Y_i^{-\ell} & \text{if } \ell < 0. \end{cases}$$

Observe that the defining relations give

$$Y_i^k X_i^\ell = \begin{cases} \phi_i^{-(k-1)}(t_i)\phi_i^{-(k-2)}(t_i)\cdots\phi_i^{-(k-\ell)}(t_i)Y_i^{k-\ell} & \text{if } k \geq \ell, \\ \phi_i^{-(k-1)}(t_i)\phi_i^{-(k-2)}(t_i)\cdots\phi_i^{-1}(t_i)\, t_i X_i^{\ell-k} & \text{if } k < \ell. \end{cases} \tag{2.2}$$

View $\mathbb{Z}^{\mathfrak{I}}$ under componentwise addition, and let Λ denote the subgroup of $\mathbb{Z}^{\mathfrak{I}}$ of all tuples $\alpha = (\alpha_i)_{i\in\mathfrak{I}}$ having only finitely many nonzero components. Set

$$Z^\alpha = \prod_i Z_i^{\alpha_i},$$

and note that this product is well-defined since the Z_i commute. Then it follows from (2.2) that every element $a \in A = R(\phi, t)$ can be written as a finite sum

$$a = \sum_{\alpha\in\Lambda} c_\alpha Z^\alpha$$

with coefficients c_α in R.

Lemma 2.3. *$A = R(\phi, t)$ is a free left (or right) R-module with basis $\{Z^\alpha \mid \alpha \in \Lambda\}$.*

Proof. We have observed already that these elements span A over R. Now suppose that $a = \sum_\alpha c_\alpha Z^\alpha = 0$, where $c_\alpha \in R$ for all $\alpha \in \Lambda$. Given such an expression, for each $i \in \mathcal{I}$ set

$$\gamma_i = \max\Big(\{-\alpha_i \mid \alpha_i < 0\} \cup \{0\}\Big),$$

where the maximum is taken over all α such that $c_\alpha \neq 0$. Then $\gamma = (\gamma_i) \in \Gamma$, and by (2.2) we have

$$aX^\gamma = \sum_{\alpha \in \Lambda} c_\alpha d_\alpha Z^{\alpha+\gamma} \tag{2.4}$$

for some nonzero $d_\alpha \in R$. The powers occurring in the monomials $Z^{\alpha+\gamma}$ are all nonnegative. Thus, the factors in $Z^{\alpha+\gamma}$ are just the X_i. Because the subalgebra of A generated by R and the X_i is a skew-polynomial ring, it is free over R with basis the monomials in the X_i. Therefore, $c_\alpha d_\alpha = 0$, and hence $c_\alpha = 0$ for all $\alpha \in \Lambda$. □

The next proposition is a generalization of a result of Kulkarni [Ku, Cor. 2.02] which treats the case that $|\mathcal{I}| = 1$.

Proposition 2.5. *Let $A = R(\phi, t)$ be a generalized Weyl algebra with R commutative. Then the center $\mathcal{Z}(A)$ of A is generated by the elements of R in $R^\phi := \{r \in R \mid \phi_i(r) = r \text{ for all } i \in \mathcal{I}\}$ and all the monomials Z^α for $\alpha \in \Lambda$ such that $\phi^\alpha := \prod_{i \in \mathcal{I}} \phi_i^{\alpha_i} = \mathrm{id}_R$.*

Proof. It is easy to see from (1.2) that R^ϕ is contained in $\mathcal{Z} = \mathcal{Z}(A)$. Moreover, if $\alpha \in \Lambda$, then $Z^\alpha r = \phi^\alpha(r) Z^\alpha$ for all $r \in R$, and thus Lemma 2.3 implies that $Z^\alpha \in \mathcal{Z}$ if and only if $\phi^\alpha = \mathrm{id}_R$. Now suppose $\sum_{\alpha \in \Lambda} r_\alpha Z^\alpha \in \mathcal{Z}$. Then for $s \in R$,

$$\sum_{\alpha \in \Lambda} s r_\alpha Z^\alpha = s\left(\sum_{\alpha \in \Lambda} r_\alpha Z^\alpha\right) = \left(\sum_{\alpha \in \Lambda} r_\alpha Z^\alpha\right) s = \sum_{\alpha \in \Lambda} r_\alpha \phi^\alpha(s) Z^\alpha. \tag{2.6}$$

Thus, if $r_\alpha \neq 0$, then $s = \phi^\alpha(s)$ for all $s \in R$ by Lemma 2.3, so that $\phi^\alpha = \mathrm{id}_R$ and $Z^\alpha \in \mathcal{Z}$. But then $\left(\sum_\alpha r_\alpha Z^\alpha\right) X_i = X_i \left(\sum_\alpha r_\alpha Z^\alpha\right)$ implies $\sum_\alpha r_\alpha X_i Z^\alpha = \sum_\alpha \phi_i(r_\alpha) X_i Z^\alpha$ since $Z^\alpha \in \mathcal{Z}$ for each nonzero r_α. This forces $r_\alpha = \phi_i(r_\alpha)$ for all i, so that $r_\alpha \in R^\phi$. □

Lemma 2.7. *Let V be a Whittaker module for A with cyclic Whittaker vector w of type ζ. Then $V = Rw$. If R is commutative, then $\mathrm{Ann}_R(V) = \mathrm{Ann}_R(w)$.*

Proof. Observe that $X_i w = \zeta_i w \in Rw$ and $Y_i w = \zeta_i^{-1} Y_i X_i w = \zeta_i^{-1} t_i w \in Rw$, and then apply the relations $X_i r = \phi_i(r) X_i$ and $Y_i r = \phi_i^{-1}(r) Y_i$ for $r \in R$. The assertion about annihilators is an immediate consequence of the fact that $V = Rw$ and the commutativity of R. □

Definition 2.8. If $J \subseteq R$ is an ideal or left ideal of R, we say that J is *ϕ-stable* if $\phi_i(J) \subseteq J$ for all $i \in \mathcal{I}$.

Examples 2.9. For a fixed Whittaker module $V = Aw$, if $r \in Q = \mathrm{Ann}_R(w)$, then $0 = X_i r w = \phi_i(r) X_i w = \zeta_i \phi_i(r) w$, so it follows that Q is a ϕ-stable left ideal of R. For another example, assume $z \in R$ is fixed by ϕ_i for all i. Then the left ideal $(z) = Rz$ of R generated by z is clearly ϕ-stable.

Remark 2.10. The sum of two ϕ-stable (left) ideals is again ϕ-stable. In addition, if R is commutative and J is a ϕ-stable ideal of R, it is easily seen that the radical $\sqrt{J}$ is also a ϕ-stable ideal of R.

If J is a ϕ-stable left ideal of R, applying ϕ_i^{-1} to the containment $\phi_i(J) \subseteq J$ gives $J \subseteq \phi_i^{-1}(J)$. Repeating this indefinitely yields $J \subseteq \phi_i^{-1}(J) \subseteq \phi_i^{-2}(J) \subseteq \phi_i^{-3}(J) \subseteq \cdots$. Since the $\phi_i^{-k}(J)$ $(k \geq 0)$ form an ascending chain of left ideals of the Noetherian ring R, it follows that $\phi_i^{-k}(J) = \phi_i^{-(k+1)}(J)$ for some $k \geq 0$. Applying an appropriate power of ϕ_i to each side gives the following lemma.

Lemma 2.11. *If J is a ϕ-stable left ideal of R, then $\phi_i(J) = J = \phi_i^{-1}(J)$ for all i.*

Let J be a ϕ-stable left ideal of R such that $t_i \notin J$ for all $i \in \mathcal{I}$. By $\overline{r}$, we mean the coset $\overline{r} = r + J$ for all $r \in R$. Thus, $\overline{t_i} \neq 0$ for $i \in \mathcal{I}$. Since J is ϕ-stable, we have an induced map $\overline{\phi_i}$ on the quotient R/J given by $\overline{\phi_i}(\overline{r}) = \phi_i(r) + J = \overline{\phi_i(r)}$. When R is commutative, then R/J is a ring, and $\overline{\phi_i}$ is an automorphism of R/J by Lemma 2.11. Thus, the following result holds in that case.

Proposition 2.12. *Let $A = R(\phi, t)$ be a generalized Weyl algebra. Assume R is commutative, and let J be a ϕ-stable prime ideal of R such that $t_i \notin J$ for all $i \in \mathcal{I}$. Then A/AJ is isomorphic to the generalized Weyl algebra $\overline{A} := (R/J)(\overline{\phi}, \overline{t})$.*

Proof. We will write bars on the X_i and Y_i in $\overline{A}$ to distinguish them from the generators in A, although the bar does not denote a coset reduction in this instance. Consider the $\mathbb{F}$-algebra homomorphism $\Phi : \mathcal{F} \to \overline{A}$ from the free algebra $\mathcal{F}$ generated by R, X_i, Y_i, $i \in \mathcal{I}$, to $\overline{A}$ given $r \mapsto \overline{r}$, $X_i \mapsto \overline{X}_i$, and $Y_i \mapsto \overline{Y}_i$. Then $\Phi(Y_i X_i) = \overline{Y}_i \overline{X}_i = \overline{t_i} = \Phi(t_i)$, so that $Y_i X_i - t_i \in \ker \Phi$. Similarly, $\Phi(X_i r) = \overline{X}_i \overline{r} = \overline{\phi_i}(\overline{r}) \overline{X}_i = \overline{\phi_i(r)}\, \overline{X}_i = \Phi(\phi_i(r) X_i)$, and $X_i r - \phi_i(r) X_i \in \ker \Phi$. Arguing in this way, we see that there is an induced algebra homomorphism $\overline{\Phi} : A \to \overline{A}$. Clearly, AJ is in the kernel. Now if some $\sum_{\alpha \in \Lambda} Z^\alpha r_\alpha$ maps to 0, then $\sum_{\alpha \in \Lambda} \overline{Z}^\alpha \overline{r_\alpha} = 0$ in $\overline{A}$, so by the freeness of $\overline{A}$ as a module for the domain R/J (see Lemma 2.3), we obtain $\overline{r_\alpha} = 0$ for each $\alpha \in \Lambda$. This implies that $r_\alpha \in J$ for all α, so that $\sum_{\alpha \in \Lambda} Z^\alpha r_\alpha \in AJ$. $\square$

3. Constructing a universal object

We continue to assume that $A = R(\phi, t)$ is a generalized Weyl algebra.

Definition 3.1. Let (V, w) be a Whittaker pair of type ζ. Suppose that V has the property that for any other Whittaker pair (V', w') of type ζ, there exists a unique surjective module homomorphism $\sigma' : V \to V'$ such that $\sigma'(w) = w'$. Then we say that (V, w) is a *universal Whittaker pair of type* ζ and V is a *universal Whittaker module of type* ζ.

Suppose that (V_1, w_1) and (V_2, w_2) are universal Whittaker pairs of type ζ. Then there are surjective A-module homomorphisms $\sigma_2 : V_1 \to V_2$ and $\sigma_1 : V_2 \to V_1$ such that $\sigma_2(w_1) = w_2$ and $\sigma_1(w_2) = w_1$. If $v \in V_1$, then we may write $v = rw_1$, and thus $\sigma_2(v) = \sigma_2(rw_1) = r\sigma_2(w_1) = rw_2$. Moreover, $\sigma_1(rw_2) = r\sigma_1(w_2) = rw_1 = v$, so we see that $\sigma_2 \circ \sigma_1 = \mathrm{id}_{V_2}$. Similarly, $\sigma_1 \circ \sigma_2 = \mathrm{id}_{V_1}$. Thus, the maps σ_1 and σ_2 are isomorphisms of A-modules, and it makes sense to refer to a universal Whittaker module of type ζ as *the* universal Whittaker module of type ζ.

To construct a universal Whittaker pair $(V_{\mathfrak{u}}, w_{\mathfrak{u}})$ of type ζ, we define an action of A on R via

$$r'.r = r'r, \tag{3.2}$$

$$X_i.r = \zeta_i \phi_i(r), \tag{3.3}$$

$$Y_i.r = \zeta_i^{-1}\phi_i^{-1}(r)t_i \tag{3.4}$$

for $r, r' \in R$ and $i \in \mathfrak{I}$. It is straightforward to verify that under this action, R is a Whittaker module of type ζ with cyclic Whittaker vector 1. When we regard R as a Whittaker module with the above action, we write $V_{\mathfrak{u}} = R$ and $w_{\mathfrak{u}} = 1$.

Lemma 3.5. *The module $V_{\mathfrak{u}}$ is the universal Whittaker module of type ζ and $\mathrm{Ann}_R(w_{\mathfrak{u}}) = 0$.*

Proof. It is clear that $\mathrm{Ann}_R(w_{\mathfrak{u}}) = 0$. Let (V, w) be an arbitrary Whittaker pair of type ζ, and define a map $\sigma : V_{\mathfrak{u}} \to V$ as follows. For $v \in V_{\mathfrak{u}}$, set $\sigma(v) = rw \in V$, where $r \in R$ is such that $v = rw_{\mathfrak{u}}$. If $s \in R$ satisfies $sw_{\mathfrak{u}} = v = rw_{\mathfrak{u}}$, then $s - r \in \mathrm{Ann}_R(w_{\mathfrak{u}}) = 0$, so $s = r$ and the map $\sigma : V_{\mathfrak{u}} \to V$ is well defined.

With $v = rw_{\mathfrak{u}} \in V_{\mathfrak{u}}$, we must verify that $\sigma(av) = a\sigma(v)$ for all $a \in A$. But since A is generated over R by X_i and Y_i for $i \in \mathfrak{I}$, it is sufficient to consider the cases $a \in R$, $a = X_i$, and $a = Y_i$, and these routine calculations are omitted.

Because $V = Aw = Rw$, we have that $\sigma(V_{\mathfrak{u}}) = \sigma(Rw_{\mathfrak{u}}) = R\sigma(w_{\mathfrak{u}}) = Rw = V$, and thus $\sigma : V_{\mathfrak{u}} \to V$ is surjective. The uniqueness of σ follows from the fact that $\sigma(w_{\mathfrak{u}}) = w$ and σ respects the action of R. $\square$

Remark 3.6. An $\mathbb{F}$-basis for $V_{\mathfrak{u}}$ is the set $\{b_\ell w_{\mathfrak{u}}\}_{\ell \in \mathfrak{L}}$, where $\{b_\ell\}_{\ell \in \mathfrak{L}}$ is any $\mathbb{F}$-basis of R.

Remark 3.7. Here, we describe an alternative construction of the universal Whittaker module of type ζ similar to that of [K, Thm. 3.3]. As a convenient shorthand in the construction, let $\mathbb{F}[X]$ denote the polynomial algebra over $\mathbb{F}$ generated by the X_i, $i \in \mathfrak{I}$, and regard $\mathbb{F}[X]$ as a subalgebra of A. Give the one-dimensional space $\mathbb{F}w_\zeta$ an $\mathbb{F}[X]$-module structure according to action $X_i w_\zeta = \zeta_i w_\zeta$. Set

$$V_\zeta = A \otimes_{\mathbb{F}[X]} \mathbb{F}w_\zeta, \tag{3.8}$$

and (using a slight abuse of notation) write w_ζ to denote $1 \otimes w_\zeta$. Then it is clear that $V_\zeta = Aw_\zeta$ and $X_i w_\zeta = \zeta_i w_\zeta$, so (V_ζ, w_ζ) is a Whittaker pair of type ζ. That this induced construction also gives the universal Whittaker module follows from the fact that the subalgebra of A generated by R and the X_i is free over $\mathbb{F}[X]$. We omit the details.

Lemma 3.9. *The A-submodules of $V_{\mathfrak{u}} = R$ are exactly the ϕ-stable left ideals of R.*

Proof. Suppose that $J \subseteq R$ is a submodule of $V_{\mathfrak{u}}$. Equation (3.2) shows that J is a left ideal of R. Since ζ_i is nonzero for all $i \in \mathfrak{I}$, (3.3) implies that $\phi_i(J) \subseteq J$ for all i, and thus J is ϕ-stable. It is routine to verify that any ϕ-stable left ideal of R is a submodule of V_u $\square$

Definition 3.10. If Q is a ϕ-stable ideal of R, let $V_Q = R/Q$, and regard V_Q as the quotient $V_{\mathfrak{u}}/QV_{\mathfrak{u}}$ with cyclic Whittaker vector $w_Q = 1 + Q$. Observe that $\mathrm{Ann}_R(w_Q) = \{r \in R \mid r(1 + Q) = 0 + Q\} = Q$.

Suppose now that (V, w) is an arbitrary Whittaker pair of type ζ, and let $Q = \mathrm{Ann}_R(w)$. Then there is a map $\sigma : V_{\mathfrak{u}} \to V$, $rw_{\mathfrak{u}} \mapsto rw$. If $0 = \sigma(rw_{\mathfrak{u}}) = rw$, then $r \in Q$, and thus $v = rw_{\mathfrak{u}} \in Qw_{\mathfrak{u}}$. Hence, $\ker(\sigma) = Qw_{\mathfrak{u}}$ and $V \cong V_{\mathfrak{u}}/Qw_{\mathfrak{u}} = V_Q$. Therefore, we have:

Lemma 3.11. *Assume (V, w) is an arbitrary Whittaker pair of type ζ, and let $Q = \mathrm{Ann}_R(w)$. Then $V \cong V_{\mathfrak{u}}/Qw_{\mathfrak{u}} = R/Q = V_Q$, where $(V_{\mathfrak{u}}, w_{\mathfrak{u}})$ is the universal Whittaker pair of type ζ.*

Theorem 3.12 below is the generalized Weyl algebra analogue of Kostant's result [K, Thm. 3.2] for finite-dimensional complex semisimple Lie algebras and Ondrus' result [O2, Cor. 4.1] for the quantum group $U_q(\mathfrak{sl}_2)$.

Theorem 3.12. *Let $A = R(\phi, t)$ be a generalized Weyl algebra. Then the map*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{Whittaker pairs of type } \zeta\end{array}\right\} \xrightarrow{\Psi} \{\phi\textit{-stable left ideals of } R\}$$

given by

$$(V, w) \mapsto \mathrm{Ann}_R(w)$$

is a bijection.

Proof. Suppose that (V_1, w_1) and (V_2, w_2) are Whittaker pairs of type ζ with $\mathrm{Ann}_R(w_1) = \mathrm{Ann}_R(w_2)$, and set $Q = \mathrm{Ann}_R(w_1) = \mathrm{Ann}_R(w_2)$. In Lemma 3.11, we have seen that $V_j \cong V_{\mathfrak{u}}/Qw_{\mathfrak{u}}$ for $j = 1, 2$, where $(V_{\mathfrak{u}}, w_{\mathfrak{u}})$ is the universal Whittaker pair of type ζ. Thus, $V_1 \cong V_2$. This implies that the map $\Psi : (V, w) \mapsto \mathrm{Ann}_R(w)$ is injective.

Now suppose $Q \subseteq R$ is a ϕ-stable left ideal of R, and let (V_Q, w_Q) be as in Definition 3.10. Since $\mathrm{Ann}_R(w_Q) = Q$, the map Ψ is surjective as well. □

Corollary 3.13. *Let (V, w) be a Whittaker pair of type ζ for a generalized Weyl algebra $A = R(\phi, t)$. Then V is simple if and only if $\mathrm{Ann}_R(w)$ is a maximal ϕ-stable left ideal of R.*

Example 3.14. Fix an element $q \in \mathbb{F}$ with $q \neq 0$ and $q^2 \neq 1$. Let $R = \mathbb{F}[c, K^{\pm 1}]$ and $t = c - \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} \in R$, and define $\phi : R \to R$ by $K \mapsto q^{-2}K$ and $c \mapsto c$. (Because $|\mathfrak{I}| = 1$ in this example, we are omitting the subscripts on ϕ and t.) Then $A = R(\phi, t) \cong U_q(\mathfrak{sl}_2)$. Since R is commutative, the simple Whittaker modules correspond to maximal ϕ-stable (two-sided) ideals of R. If $\xi \in \mathbb{F}$, then the ideal $R(c - \xi)$ generated by $c - \xi$ is clearly ϕ-stable. We shall see in Section 9 that this is a maximal ϕ-stable ideal when q^2 is not a root of unity.

Example 3.15. Assume $\mathbb{F}$ has characteristic 0, and let $A_n = R_n(\phi, t)$ where $R_n = \mathbb{F}[t_1, \ldots, t_n]$ and $\phi_i(t_j) = t_j - \delta_{i,j}1$. Assume $Y_iX_i = t_i$, $X_iY_i = \phi_i(t_i)$, $[X_i, X_j] = 0 = [Y_i, Y_j]$, and $[X_i, Y_j] = 0$ for $i \neq j$. Then A_n is the nth Weyl algebra realized as a generalized Weyl algebra. It is straightforward to show that R_n contains no proper ϕ-stable ideals, and thus every Whittaker module V for A_n is simple. In particular, the universal Whittaker module $V_{\mathfrak{u}}$ of type ζ is simple and is the unique Whittaker module of type ζ for A_n. The set $\{t^\gamma = t^\gamma w_{\mathfrak{u}} \mid t^\gamma = \prod_{i=1}^n t_i^{\gamma_i},\ \gamma_i \in \mathbb{Z}_{\geq 0}\}$

is a basis for $V_{\mathfrak{u}}$, and the A_n-action on $V_{\mathfrak{u}}$ is given by

$$\begin{aligned} t^\beta.t^\gamma &= t^{\beta+\gamma}, \\ X_i.t^\gamma &= \zeta_i(t_i-1)^{\gamma_i}\prod_{j\neq i} t_j^{\gamma_j}, \\ Y_i.t^\gamma &= \zeta_i^{-1} t_i(t_i+1)^{\gamma_i}\prod_{j\neq i} t_j^{\gamma_j}. \end{aligned} \tag{3.16}$$

If $\mathcal{K}$ is an ideal of the center $\mathcal{Z} = \mathcal{Z}(A)$ of a generalized Weyl algebra $A = R(\phi, t)$, then $\mathcal{K}V_{\mathfrak{u}}$ is a submodule of the universal Whittaker module $V_{\mathfrak{u}} = R$. Our next goal is to show that when R is commutative and $\mathcal{I}$ is finite, then under some assumptions,

$$V_{\mathfrak{u},\mathcal{K}} := V_{\mathfrak{u}} \Big/ \mathcal{K}V_{\mathfrak{u}} \tag{3.17}$$

is simple for every maximal ideal $\mathcal{K}$ of $\mathcal{Z}$.

Recall that a commutative ring is said to be a *Jacobson ring* if each prime ideal is the intersection of maximal ideals. We will use the following two results:

Theorem 3.18 ([E, Thm. 4.19])**.** *Let S be a Jacobson ring. If T is a finitely generated commutative S-algebra, then T is a Jacobson ring. Furthermore, if $\mathfrak{n} \subseteq T$ is a maximal ideal, then $\mathfrak{m} := \mathfrak{n} \cap S$ is a maximal ideal of S, and $T/\mathfrak{n}$ is a finite extension field of $S/\mathfrak{m}$.*

Theorem 3.19 ([Sw, Thm. 6.20])**.** *Let M be a finitely generated commutative monoid and let $\{f_i : M \to \mathbb{Z} \mid i = 1, ..., n\}$ be a finite collection of homomorphisms. Then $G = \{x \in M \mid f_i(x) \geq 0$ for all $i\}$ is a finitely generated monoid.*

Any field $\mathbb{F}$ is a Jacobson ring, and hence by Theorem 3.18, so is any finitely generated commutative $\mathbb{F}$-algebra. We intend to apply this theorem to the pair $S = R^\phi$ and $T = \mathcal{Z} = \mathcal{Z}(A)$, where our notation is that of Proposition 2.5. Thus, we need conditions under which $\mathcal{Z}$ is a finitely generated R^ϕ-algebra.

Let $\Delta = \{\alpha \in \Lambda \mid \phi^\alpha = \mathrm{id}_R\}$, and note that $\mathcal{Z} = \bigoplus_{\alpha\in\Delta} R^\phi Z^\alpha$ by Proposition 2.5. If $|\mathcal{I}|$ is finite, then the subgroup $\Delta \subseteq \Lambda$ is finitely generated. However, it may not be the case that $Z^\alpha Z^\beta = Z^{\alpha+\beta}$, so it is not immediately obvious that $\mathcal{Z}$ is a finitely generated R^ϕ-algebra.

Lemma 3.20. *If R is commutative and $|\mathcal{I}| < \infty$, then $\mathcal{Z}$ is a finitely generated R^ϕ-algebra.*

Proof. Assume that $|\mathcal{I}| = n < \infty$ so that $\Lambda = \mathbb{Z}^n$, and let $\Sigma = \{\pm 1\}^n$. For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \Sigma$, define homomorphisms $f_i^\varepsilon : \mathbb{Z}^n \to \mathbb{Z}$ (for $i = 1, \dots, n$) by $f_i^\varepsilon(\alpha) = \varepsilon_i\alpha_i$. With $\Delta = \{\alpha \in \Lambda \mid \phi^\alpha = \mathrm{id}_R\} \subseteq \mathbb{Z}^n$ as above, let $\Delta_\varepsilon \subseteq \Delta$ be the monoid defined by $\Delta_\varepsilon = \{\alpha \in \Delta \mid f_i^\varepsilon(\alpha) \geq 0 \text{ for } i = 1, \dots, n\}$. Note that $Z^\alpha Z^\beta = Z^{\alpha+\beta}$ for $\alpha, \beta \in \Delta_\varepsilon$, and by Theorem 3.19, there is a finite set $\mathcal{G}_\varepsilon$ of generators for the monoid Δ_ε. Observe that $|\Sigma| = 2^n$, and $\Delta = \bigcup_{\varepsilon\in\Sigma} \Delta_\varepsilon$. Thus, the set $\mathcal{G} = \bigcup_{\varepsilon\in\Sigma} \mathcal{G}_\varepsilon$ is finite, and the set $\{Z^\alpha \mid \alpha \in \mathcal{G}\}$ is a finite set of generators for $\mathcal{Z}$ over R^ϕ. □

Definition 3.21. We say that a ϕ-stable ideal Q of R is *centrally generated* if $Q = R(Q \cap \mathcal{Z})$.

Lemma 3.22. *Let A, $\mathcal{Z}$, $\mathcal{K}$, and $V_{\mathfrak{u},\mathcal{K}}$ be as above and set*

$$w_{\mathfrak{u},\mathcal{K}} = 1 + \mathcal{K}V_{\mathfrak{u}} \in V_{\mathfrak{u},\mathcal{K}} = V_{\mathfrak{u}}/\mathcal{K}V_{\mathfrak{u}}.$$

Assume $\mathcal{K}$ is a maximal ideal of $\mathcal{Z}$, and let $Q = \mathrm{Ann}_R(w_{\mathfrak{u},\mathcal{K}})$. Then $Q \cap \mathcal{Z} = R^\phi \cap \mathcal{K}$, where $R^\phi = R \cap \mathcal{Z} = \{r \in R \mid \phi_i(r) = r$ for all $i \in \mathcal{I}\}$.

Proof. It follows from the construction of $V_{\mathfrak{u},\mathcal{K}}$ that $\mathcal{K} \subseteq \mathrm{Ann}_{\mathcal{Z}}(w_{\mathfrak{u},\mathcal{K}})$. However, $\mathrm{Ann}_{\mathcal{Z}}(w_{\mathfrak{u},\mathcal{K}})$ is clearly a proper ideal of $\mathcal{Z}$, so since $\mathcal{K}$ is maximal, $\mathcal{K} = \mathrm{Ann}_{\mathcal{Z}}(w_{\mathfrak{u},\mathcal{K}})$ must hold. The proof of the remaining assertions is straightforward. □

Theorem 3.23. *Assume R is commutative and every maximal ϕ-stable ideal of R is centrally generated. Let $\mathcal{K}$ be a maximal ideal of the center $\mathcal{Z}$ of $A = R(\phi, t)$. If $|\mathcal{I}| < \infty$ and R^ϕ is a finitely generated $\mathbb{F}$-algebra, then the Whittaker module $V_{\mathfrak{u},\mathcal{K}}$ is simple. Moreover, if $Q = \mathrm{Ann}_R(w_{\mathfrak{u},\mathcal{K}})$, where $w_{\mathfrak{u},\mathcal{K}} = 1 + \mathcal{K}V_{\mathfrak{u}}$, then $Q = R(R^\phi \cap \mathcal{K})$.*

Proof. Since R^ϕ is a finitely generated $\mathbb{F}$-algebra, it is a Jacobson ring. Consequently, by Lemma 3.20 and Theorem 3.18, $\mathcal{Z}$ is a finitely generated R^ϕ-algebra, and $R^\phi \cap \mathcal{K}$ is a maximal ideal of R^ϕ. Let $Q = \mathrm{Ann}_R(w_{\mathfrak{u},\mathcal{K}})$, and recall from Lemma 3.22 that $Q \cap \mathcal{Z} = R^\phi \cap \mathcal{K}$. Since R is Noetherian, there exists a maximal ϕ-stable ideal Q' of R containing Q. By assumption $Q' = R(Q' \cap \mathcal{Z})$. But $Q' \cap \mathcal{Z}$ is a proper ideal of R^ϕ because $1 \notin Q' \cap \mathcal{Z}$, and $Q' \cap \mathcal{Z} \supseteq Q \cap \mathcal{Z}$. As $Q \cap \mathcal{Z} = R^\phi \cap \mathcal{K}$ is a maximal ideal of R^ϕ, it follows that $Q' \cap \mathcal{Z} = Q \cap \mathcal{Z}$, and so

$$Q' = R(Q' \cap \mathcal{Z}) = R(Q \cap \mathcal{Z}) \subseteq Q.$$

This implies that $Q = Q'$ is maximal among ϕ-stable ideals of R, hence $V_{\mathfrak{u},\mathcal{K}}$ is simple by Corollary 3.13. But then $Q = R(Q \cap \mathcal{Z}) = R(R^\phi \cap \mathcal{K})$, as claimed. □

Remark 3.24. All the examples in Sections 8–10 satisfy the hypothesis that R is commutative and R^ϕ is a finitely generated $\mathbb{F}$-algebra. Many of the examples satisfy the condition that every maximal ϕ-stable ideal of R is centrally generated, and thus the module $V_{\mathfrak{u},\mathcal{K}} = V_{\mathfrak{u}}/\mathcal{K}V_{\mathfrak{u}} \cong V_{\mathfrak{u}}/QV_{\mathfrak{u}} = R/Q$, where $Q = \mathrm{Ann}_R(w_{\mathfrak{u},\mathcal{K}})$, is a simple Whittaker module in those cases.

4. An expression for $\mathrm{Ann}_A(w)$

Let $A = R(\phi, t)$ be a generalized Weyl algebra as in Section 1, and suppose that $V = Aw$ is a Whittaker module of type ζ with $Q = \mathrm{Ann}_R(w)$. The map $A \to V$ given by $a \mapsto aw$ shows that $V \cong A/\mathrm{Ann}_A(w)$, and it is clear that $AQ + \sum_{i \in \mathcal{I}} A(X_i - \zeta_i) \subseteq \mathrm{Ann}_A(w)$. In this section, we prove that, in fact, these two left ideals of A always coincide.

As before, let Γ denote the semigroup of tuples $\gamma = (\gamma_i)_{i \in \mathcal{I}}$ of nonnegative integers with only finitely many nonzero entries under componentwise addition, and let $X^\gamma = \prod_{i \in \mathcal{I}} X_i^{\gamma_i}$. For $\gamma \in \Gamma$, set $|\gamma| = \sum_{i \in \mathcal{I}} \gamma_i$.

Lemma 4.1. *Let $I = AQ + \sum_{i \in \mathcal{I}} A(X_i - \zeta_i)$, where (V, w) is a Whittaker pair of type ζ and $Q = \mathrm{Ann}_R(w)$. Let $a \in A$, and suppose that there exists $\gamma \in \Gamma$ such that $aX^\gamma \in I$. Then $a \in I$.*

Proof. The proof is by induction on $|\gamma|$. We may assume that $|\gamma| > 0$ since there is nothing to prove if $\gamma_i = 0$ for all i. Assume that $\gamma_k > 0$ for some $k \in \mathcal{I}$, and thus we use the assumption that $aX^\gamma \in I$ to show that $aX^{\gamma'} \in I$, where $\gamma' = (\gamma'_i)$ is such that $\gamma'_i = \gamma_i$ for $i \neq k$ and $\gamma'_k = \gamma_k - 1$. The proof is essentially the same as in the case that $|\mathcal{I}| = 1$, so we give the proof in the degree 1 setting to avoid computation. Hence, we assume that $aX^m \in I$ for $m \geq 1$ and show that $aX^{m-1} \in I$. (We are omitting the subscripts on X and ζ, because of the reduction to the $|\mathcal{I}| = 1$ case.)

By the definition of I, it is clear that $a(X-\zeta)^m \in I$. Then it follows that $aX^m - a(X-\zeta)^m = a(X^m - (X-\zeta)^m) \in I$, and after simplification using the identity

$$x^m - y^m = (x-y)(x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}),$$

we have that

$$\zeta a(X^{m-1} + X^{m-2}(X-\zeta) + X^{m-3}(X-\zeta)^2 + \cdots + (X-\zeta)^{m-1}) \in I.$$

Since $\zeta a X^{m-i}(X-\zeta)^{i-1} \in I$ for all $i \geq 2$, it follows that $\zeta a X^{m-1} \in I$, and thus $aX^{m-1} \in I$. By induction on m, we may conclude that $a \in I$. $\square$

Lemma 4.2. *If $\delta \in \Gamma$, then $X^\delta \in R + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$.*

Proof. The proof is by induction on $|\delta|$. If $\delta_i = 0$ for all i, then $X^\delta = 1 \in R$. So we suppose that $\delta_k > 0$. Then

$$\begin{aligned} X^\delta &= \Big(\prod_{i\in\mathfrak{I}, i\neq k} X_i^{\delta_i}\Big) X_k^{\delta_k - 1}(X_k - \zeta_k + \zeta_k) \\ &= X^{\delta'}(X_k - \zeta_k) + \zeta_k X^{\delta'}, \end{aligned}$$

where $\delta_i' = \delta_i$ for $i \neq k$, and $\delta_k' = \delta_k - 1$. Since $X^{\delta'}(X_k - \zeta_k) \in \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$, and $|\delta'| < |\delta|$, we have by induction that $X^\delta \in R + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$. $\square$

Remark 4.3. It is evident from the proof of Lemma 4.2 that, in fact, $X^\delta \in \mathbb{F}1 + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$ for all $\delta \in \Gamma$. In applying the lemma, however, we only need that $X^\delta \in R + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$.

Theorem 4.4. *Suppose that $V = Aw$ is a Whittaker module of type ζ for $A = R(\phi, t)$, and let $Q = \mathrm{Ann}_R(w)$. Then $\mathrm{Ann}_A(w) = AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$.*

Proof. Let $a \in \mathrm{Ann}_A(w)$. As in the proof of Lemma 2.3 (in particular, as in (2.4)), there exists $\gamma \in \Gamma$ such that $aX^\gamma = \sum_{\delta\in\Gamma} r_\delta X^\delta$ for $r_\delta \in R$. Since $aX^\gamma w = \prod_i \zeta_i^{\gamma_i} aw = 0$, it follows that $aX^\gamma \in \mathrm{Ann}_A(w)$. Lemma 4.2 implies that $aX^\gamma = \sum_{\delta\in\Gamma} r_\delta X^\delta \in R + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$. Thus, we may write $aX^\gamma = r + b$, with $r \in R$ and $b \in \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i) \in \mathrm{Ann}_A(w)$. Since $aX^\gamma \in \mathrm{Ann}_A(w)$ and $b \in \mathrm{Ann}_A(w)$, it must be that $r \in \mathrm{Ann}_A(w) \cap R = Q$. Thus, $aX^\gamma \in AQ + \sum_{i=1}^n A(X_i - \zeta_i)$, and by Lemma 4.1, we have $a \in AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$. The other containment is clear. $\square$

Corollary 4.5. *If $V = Aw$ is a Whittaker module of type ζ for $A = R(\phi, t)$ and $Q = \mathrm{Ann}_R(w)$, then*

$$V \cong A \Big/ \left(AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i) \right).$$

Since $\mathrm{Ann}_A(V) \subseteq \mathrm{Ann}_A(w)$, we have the next corollary.

Corollary 4.6. *If $V = Aw$ is a Whittaker module of type ζ for $A = R(\phi, t)$, then $\mathrm{Ann}_A(V) \subseteq AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$.*

If $V = Aw$ is a one-dimensional Whittaker module, then $\mathrm{Ann}_A(V) = \mathrm{Ann}_A(w)$, which implies the following result.

Corollary 4.7. *Suppose that $V = Aw = \mathbb{F}w$ is a one-dimensional Whittaker module of type ζ for $A = R(\phi, t)$. Then $\mathrm{Ann}_A(V) = AQ + \sum_{i\in\mathfrak{I}} A(X_i - \zeta_i)$, and there exists an $\mathbb{F}$-algebra homomorphism $\theta : R \to \mathbb{F}$ such that $rw = \theta(r)w$ for all $r \in R$ and $\ker\theta = Q$.*

5. Whittaker vectors

Assume that (V, w) is a Whittaker pair of type ζ for the generalized Weyl algebra $A = R(\phi, t)$. Let $\mathrm{Wh}_\eta(V)$ denote the set of all Whittaker vectors of type $\eta = (\eta_i)_{i\in\mathfrak{I}}$ in V. In this section we describe how Whittaker vectors are related to eigenvalues of the automorphisms ϕ_i and how they can be used to deduce information about the module V. We note that Lemma 5.1 and Corollary 5.2 are true even if R is noncommutative. For the remaining results in this section, we must assume that R is commutative. The next result is apparent.

Lemma 5.1. *Let (V, w) be a Whittaker pair of type ζ with $Q = \mathrm{Ann}_R(w)$. Then for $v = rw \in V$ with $v \neq 0$ and $r \in R$, the following are equivalent:*

(a) *$X_i v = \eta_i v$, for all $i \in \mathfrak{I}$;*
(b) *$\eta_i rw = X_i rw = \phi_i(r)\zeta_i w$, for all $i \in \mathfrak{I}$;*
(c) *$\phi_i(r) \equiv \zeta_i^{-1}\eta_i r \mod Q$;*
(d) *$r + Q$ is an eigenvector for the induced linear transformation $\overline{\phi_i}$ on R/Q with eigenvalue $\zeta_i^{-1}\eta_i$ for all $i \in \mathfrak{I}$.*

If $\eta_i \neq 0$ for all $i \in \mathfrak{I}$, then $v \in \mathrm{Wh}_\eta(V)$.

Corollary 5.2. *For the universal Whittaker pair $(V_\mathfrak{u}, w_\mathfrak{u})$ of type ζ, $0 \neq rw_\mathfrak{u} \in \mathrm{Wh}_\eta(V_\mathfrak{u})$ if and only if r is an eigenvector of ϕ_i with eigenvalue $\zeta_i^{-1}\eta_i$ for all $i \in \mathfrak{I}$.*

Proof. This follows directly from Lemma 5.1 and the fact that $\mathrm{Ann}_R(w_\mathfrak{u}) = 0$ (see Lemma 3.5). □

Proposition 5.3. *Assume R is commutative, and let (V, w) be a Whittaker pair with $w \neq 0$. Set $Q = \mathrm{Ann}_R(w)$ and let*

$$P = \sqrt{Q} = \{r \in R \mid r^k \in Q \text{ for some } k\}.$$

Then Pw is a submodule of V and $Pw \neq V$.

Proof. It suffices to note that P is a ϕ-stable ideal of R containing Q, and $P \neq R$ since $1 \notin P$, and the rest follows from Lemmas 3.9 and 3.11. □

Corollary 5.4. *We adopt the notation of Lemma 5.1 and assume R is commutative. Assume $0 \neq v \in \mathrm{Wh}_\eta(V)$ and $\lambda_i := \zeta_i^{-1}\eta_i$ is not a root of unity for some $i \in \mathfrak{I}$. If V is simple, then V is infinite-dimensional.*

Proof. Since V is simple, we know by Proposition 5.3 that $\sqrt{Q}w = 0$, and hence that $\sqrt{Q} = Q$. Thus, if $rw \neq 0$, then $r^k w \neq 0$ for all $k \geq 1$. Now suppose that $v = rw$ is a nonzero Whittaker vector of type η and set $\lambda_i = \zeta_i^{-1}\eta_i$. By induction and Lemma 5.1 it follows that

$$X_i r^k w = \lambda_i^k \zeta_i r^k w \tag{5.5}$$

for all $k \geq 1$. Relation (5.5) implies that $r^k w$ is a nonzero eigenvector for X_i with eigenvalue $\lambda_i^k\zeta_i$. As these values are all distinct because λ_i is not a root of unity, the vectors $r^k w$ for $k \geq 1$ must be linearly independent. Thus, V is infinite-dimensional. □

Remark 5.6. It is evident from the proof of the previous result that if we replace the assumption that V is simple with the assumption that $Q = \sqrt{Q}$, the conclusion remains true.

For the remainder of the section we assume that (V, w) is a fixed Whittaker pair of type ζ with $Q = \mathrm{Ann}_R(w)$ for the generalized Weyl algebra $A = R(\phi, t)$, where R is commutative, and we set

$$\begin{aligned} S &= \{s \in R \,|\, X_i s w = \zeta_i s w \text{ for all } i \in \mathcal{I}\} \\ &= \{s \in R \,|\, s - \phi_i(s) \in Q \text{ for all } i \in \mathcal{I}\}. \end{aligned} \tag{5.7}$$

Note the second equality comes from Lemma 5.1, and $Sw = \mathrm{Wh}_\zeta(V)$ holds.

Lemma 5.8. *If R is commutative, and S is as in* (5.7)*, then S is a subring of R and $Q = \mathrm{Ann}_R(w)$ is an ideal of S.*

Proof. If $s_1, s_2 \in S$, then $X_i s_1 s_2 w = \phi_i(s_1) X_i s_2 w = \phi(s_1)\zeta_i s_2 w = \zeta_i s_2 \phi(s_1) w = \zeta_i s_2 s_1 w = \zeta_i s_1 s_2 w$. □

Lemma 5.9. *Assume R is commutative and (V, w) is a Whittaker pair of type ζ, and let $\pi : A \to \mathrm{End}(V)$ be the corresponding representation of A. Then for S as in* (5.7)*, $\pi(S) = \mathrm{End}_A(V)$.*

Proof. It is clear that $srv = rsv$ for $s \in S, r \in R$, and $v \in V$. We must show that $sX_i v = X_i sv$ and $sY_i v = Y_i sv$ whenever $s \in S$ and $v \in V$. But $X_i sv = \phi_i(s) X_i v = sX_i v$, as $s - \phi_i(s) \in Q \subseteq \mathrm{Ann}_A(V)$. Similarly, $Y_i sv = \phi_i^{-1}(s) Y_i v = sY_i v$, since $s - \phi_i^{-1}(s) = \phi_i^{-1}\big(\phi_i(s) - s\big) \in Q \subseteq \mathrm{Ann}_A(V)$. Thus, $\pi(S) \subseteq \mathrm{End}_A(V)$.

For the other direction, let $\psi \in \mathrm{End}_A(V)$, and note that $X_i \psi w = \zeta_i \psi w$, so $\psi w \in \mathrm{Wh}_\zeta(V) = Sw$. Write $\psi w = sw$ for $s \in S$. It is easy to see that the action of ψ on V is determined by its action on w, and thus $\psi = \pi(s)$. □

The map $S \to \mathrm{End}_A(V)$ defined by $s \mapsto \pi(s)$ gives the following.

Corollary 5.10. $\mathrm{End}_A(V) \cong S/Q$.

The following rendition of Schur's lemma enables us to say more in the simple case.

Lemma 5.11. *Suppose $\mathbb{F}$ is an uncountable algebraically closed field and A is a $\mathbb{F}$-algebra. If V is a simple A-module of countable dimension over $\mathbb{F}$, then $\mathrm{End}_A(V) = \mathbb{F}\mathrm{id}_V$.*

Corollary 5.12. *Assume R is commutative and is of countable dimension over an uncountable algebraically closed field $\mathbb{F}$, and let (V, w) be a Whittaker pair of type ζ for $A = R(\phi, t)$. If V is simple, then $\mathrm{Wh}_\zeta(V) = Sw = \mathcal{Z}w = \mathbb{F}w$, where $\mathcal{Z}$ is the center of A.*

Proof. If V is simple, $\mathrm{Wh}_\zeta(V) = Sw = \mathbb{F}w$ since $\pi(S) = \mathrm{End}_A(V) = \mathbb{F}\mathrm{id}_V$ by Schur's lemma. But then $\mathbb{F}w \subseteq \mathcal{Z}w \subseteq Sw = \mathbb{F}w$, forcing equality. □

6. An expression for $\mathrm{Ann}_A(V)$

Proposition 6.1. *Assume that*

$$\textit{if} \quad \lambda \in \Lambda \subseteq \mathbb{Z}^{\mathcal{I}} \quad \textit{and} \quad \phi^\lambda := \prod_{i \in \mathcal{I}} \phi_i^{\lambda_i} = \mathrm{id}_R, \quad \textit{then} \quad \lambda_i = 0 \textit{ for all } i \in \mathcal{I}. \tag{6.2}$$

(or equivalently by Proposition 2.5 *that the center of the generalized Weyl algebra* $A = R(\phi, t)$ *is contained in* R*). If* R *is commutative and* B *is a nonzero ideal of* A*, then* $B \cap R \neq 0$.

Proof. Let $0 \neq a \in B$. As in (2.4), there exists some X^γ for $\gamma \in \Gamma$ so that

$$aX^\gamma = \sum_{\delta \in \Gamma} r_\delta X^\delta \in B.$$

Thus, B contains some nonzero polynomial in the X_i with coefficients in R, and we may assume $b = \sum_{\varrho \in \Gamma} b_\varrho X^\varrho \in B$ is such a polynomial having the least number of nonzero terms. Then for $r \in R$, $br = \sum_{\varrho \in \Gamma} b_\varrho \phi^\varrho(r) X^\varrho \in B$. Suppose $b_\sigma \neq 0$. Then the element

$$br - \phi^\sigma(r)b = \sum_{\varrho \in \Gamma} b_\varrho \big(\phi^\varrho(r) - \phi^\sigma(r)\big) X^\varrho \in B$$

would have fewer nonzero terms unless $\phi^\varrho(r) = \phi^\sigma(r)$ for all ϱ with $b_\varrho \neq 0$. However, (6.2) implies that there must exist an r such that $\phi^\varrho(r) \neq \phi^\sigma(r)$ for $\varrho \neq \sigma$. Thus, a nonzero polynomial in the X_i belonging to B and having a minimal number of terms has the form sX^σ for some $s \in R$. But then $sX^\sigma Y^\sigma$ is a nonzero element of $B \cap R$. □

Corollary 6.3. *Let* $V = Rw$ *be a Whittaker module for a generalized Weyl algebra* $A = R(\phi, t)$ *with* R *commutative such that* (6.2) *holds. Then* $\mathrm{Ann}_A(V) \cap R = \mathrm{Ann}_R(w)$*, so that if* $\mathrm{Ann}_A(V) \neq 0$*, then* $\mathrm{Ann}_R(w) \neq 0$.

Let $A = R(\phi, t)$ be a generalized Weyl algebra with R commutative, and assume J is a ϕ-stable ideal of R such that $t_i \notin J$ for all $i \in \mathfrak{I}$. As before, let $\overline{r}$ mean the coset $\overline{r} = r + J$ for all $r \in R$. Thus, $\overline{t_i} \neq 0$. Since J is ϕ-stable, we have the induced automorphisms $\overline{\phi_i}$ on the quotient R/J. By Proposition 2.12, A/AJ is isomorphic to the GWA $\overline{A} := (R/J)(\overline{\phi}, \overline{t})$ whenever J is a prime ideal of R. Now if $V = Rw$ is a Whittaker module of type $\zeta = (\zeta_i)_{i \in \mathfrak{I}}$ for A, and if $Q = \mathrm{Ann}_R(w)$, then AQ is always a 2-sided ideal contained in $\mathrm{Ann}_A(V)$. So if Q is a prime ideal of R, we may always pass to the GWA $\overline{A} = (R/Q)(\overline{\phi}, \overline{t})$ (provided $\overline{t_i} \neq 0$ for all i) and regard V as a module for the (possibly) different GWA $\overline{A}$. The annihilator $\mathrm{Ann}_{\overline{A}}(V)$ may be nontrivial (if $AQ \neq \mathrm{Ann}_A(V)$), but here is a situation where that does not happen.

Theorem 6.4. *Let* $A = R(\phi, t)$ *be a generalized Weyl algebra with* R *commutative. Assume* $V = Rw$ *is a Whittaker module of type* ζ*, and suppose that* $Q = \mathrm{Ann}_R(w) = \mathrm{Ann}_R(V)$ *is a prime ideal such that* $t_i \notin Q$ *and the induced automorphisms* $\overline{\phi_i}$ *on* R/Q *satisfy* (6.2)*. Then* $\mathrm{Ann}_A(V) = AQ$.

Proof. By the above considerations, we may suppose that V is a Whittaker module of type ζ for the GWA $\overline{A} = A/AQ = (R/Q)(\overline{\phi}, \overline{t})$. Note that $\mathrm{Ann}_{R/Q}(w) = \mathrm{Ann}_{R/Q}(V) = 0$.

Consider the ideal $B = \mathrm{Ann}_A(V) + AQ$ in $A/AQ = (R/Q)(\overline{\phi}, \overline{t})$. Then R/Q is a commutative domain since Q is a prime ideal of R. Now we have seen from Proposition 6.1 that when a GWA has a commutative Noetherian domain as its coefficient ring and when the automorphisms satisfy (6.2), then any ideal intersects the coefficient ring nontrivially. Since $B \cap (R/Q) = 0$, it must be that $B = 0$ in A/AQ. That is, $\mathrm{Ann}_A(V) = AQ$. □

7. The case $R = \mathbb{F}[t]$

In the remainder of the paper, we apply results of the previous sections to determine the Whittaker modules for the quantum plane and the (quantum) Weyl algebra, and for certain generalizations of the universal enveloping algebra $U(\mathfrak{sl}_2)$ introduced by S.P. Smith and their quantum analogues. The algebras considered here have a realization as a generalized Weyl algebra $A = R(\phi, t)$, where R is commutative and ϕ is a single automorphism. The special case of $R = \mathbb{F}[t]$ is essential in considering the first set of examples.

An automorphism ϕ of the polynomial algebra $R = \mathbb{F}[t]$ is necessarily given by $\phi(t) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq 0$. Let

$$\tilde{t} = (\alpha - 1)t + \beta,$$

and note that $R = \mathbb{F}[\tilde{t}\,]$ as long as $\alpha \neq 1$. Since $\phi(\tilde{t}\,) = \alpha\tilde{t}$, it is evident that $\phi^\ell = \mathrm{id}_R$ if $\alpha \neq 1$ is a primitive ℓth root of unity, and ϕ has infinite order if α is not a root of unity. The next result is straightforward to show.

Lemma 7.1. *Assume J is a ϕ-stable ideal of $R = \mathbb{F}[\tilde{t}\,]$, where $\phi(\tilde{t}\,) = \alpha\tilde{t}$ and $\alpha \neq 1$. Let $f(\tilde{t}\,)$ be the unique monic generator of J.*

(a) *If α is not a root of unity, then there exists $n > 0$ such that $f(\tilde{t}\,) = \tilde{t}^n$. If α is a primitive ℓth root of unity, then there exist $n \geq 0$ and scalars $c_k \in \mathbb{F}$ such that $f(\tilde{t}\,) = \tilde{t}^n \sum_{k \geq 0} c_k \tilde{t}^{k\ell}$.*
(b) *Suppose $\mathbb{F}$ is algebraically closed. Then J is a proper maximal ϕ-stable ideal if and only if $f(\tilde{t}\,) = \tilde{t}$ when α is not a root of unity, and $f(\tilde{t}\,) = \tilde{t}$ or $f(\tilde{t}\,) = \tilde{t}^\ell - \xi$ for some nonzero $\xi \in \mathbb{F}$ when α is a primitive ℓth root of unity.*

Next we examine in detail the Whittaker modules for the generalized Weyl algebra $A = R(\phi, t)$ constructed from $R = \mathbb{F}[t]$ and the automorphism ϕ. Thus, $YX = t$, $XY = \phi(t) = \alpha t + \beta$, and as before, we assume $\alpha \neq 1$. Let (V, w) be a Whittaker pair of type ζ for A, and let $Q = \mathrm{Ann}_R(w)$. By Theorem 3.12, if $Q = 0$, then V is isomorphic to the universal Whittaker module $V_{\mathfrak{u}}$ of type ζ, so we will assume $Q \neq 0$.

7.2. $\boldsymbol{\alpha}$ is not a root of unity. When α is not a root of unity, then $Q = R\tilde{t}^n$ for some $n \geq 1$. Since $V = Rw \cong V_Q = R/Q$, it is clear that $\{v_k := \tilde{t}^k w \mid 0 \leq k \leq n-1\}$ is a basis of V and $\dim_{\mathbb{F}} V = n$. The action of A on V is given as follows:

$$\begin{aligned} tv_k &= (\alpha - 1)^{-1}\big(v_{k+1} - \beta v_k\big), \\ Xv_k &= \alpha^k \zeta v_k, \quad \text{and} \quad Yv_k = \zeta^{-1}\alpha^{-k}(\alpha - 1)^{-1}\big(v_{k+1} - \beta v_k\big), \end{aligned} \tag{7.3}$$

where $v_n = 0$. Since submodules of V correspond to ϕ-stable ideals of R containing Q, the submodules of V are given by $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, where $V_k = Av_k = Rv_k$ is a Whittaker (sub)module with cyclic Whittaker vector v_k of type $\alpha^k\zeta$, and $\{v_k, v_{k+1}, \ldots, v_{n-1}\}$ is a basis for V_k.

Theorem 7.4. *Let $R = \mathbb{F}[t]$, and let $\phi : R \to R$ be the algebra automorphism given by $\phi(t) = \alpha t + \beta$, where α is not a root of unity. Let (V, w) be a Whittaker pair of type ζ for $A = R(\phi, t)$, and assume that $Q := \mathrm{Ann}_R(w) = R\tilde{t}^n$ for some $n \geq 1$, where $\tilde{t} = (\alpha - 1)t + \beta$. Then V has a basis $v_k = \tilde{t}^k w$, $k = 0, 1, \ldots, n-1$, and the* *action of A on V is given by* (7.3)*. Moreover,*

$$\mathrm{Ann}_A(V) = \sum_{j=0}^{n} A\tilde{t}^{n-j}\left(\prod_{k=0}^{j-1}\left(X - \alpha^k\zeta\right)\right).$$

If V is simple, then $n = 1$, $\dim_{\mathbb{F}} V = 1$, and $\mathrm{Ann}_A(V) = A\tilde{t} + A(X - \zeta)$. Any n-dimensional space V with a basis v_k, $k = 0, 1, \ldots, n-1$, and A-action given by (7.3) *is a Whittaker module of type ζ with cyclic Whittaker vector $w = v_0$ and with $Q = R\tilde{t}^n$.*

Proof. Let $K = \sum_{j=0}^{n} A\tilde{t}^{n-j}\left(\prod_{k=0}^{j-1}\left(X - \alpha^k\zeta\right)\right)$. All that remains to be shown is that $\mathrm{Ann}_A(V) = K$. It is easy to verify that any $a \in K$ annihilates every basis vector $\tilde{t}^k w$, $k = 0, 1, \ldots, n-1$, and therefore annihilates V.

For the other inclusion, the proof is by induction on $\dim V$. If $\dim V > 1$, let $a \in \mathrm{Ann}_A(V) \subseteq \mathrm{Ann}_A(w) = AQ + A(X - \zeta)$, and write $a = a_1 q + a_2(X - \zeta)$ with $q \in Q \subseteq K$ and $a_1, a_2 \in A$. It is straightforward to show that a_2 annihilates the proper Whittaker submodule $V_1 := A\tilde{t}w \subset V$ of type $\alpha\zeta$. Hence, by induction, $a_2 \in \sum_{j=0}^{n-1} A\tilde{t}^{n-1-j}\left(\prod_{k=0}^{j-1}\left(X - \alpha^k\alpha\zeta\right)\right) = \sum_{j=1}^{n} A\tilde{t}^{n-j}\left(\prod_{k=1}^{j}\left(X - \alpha^k\zeta\right)\right)$ which implies that $a = a_1 q + a_2(X - \zeta) \in K$, as desired. □

7.5. α is a root of unity, $\alpha \neq 1$. Now suppose that α is a primitive ℓth root of unity. Since $\phi^\ell = \mathrm{id}_R$ in this case, it follows from Proposition 2.5 that the center $\mathcal{Z}$ of A is generated by X^ℓ, Y^ℓ and the set R^ϕ of elements of R fixed by ϕ, which are the polynomials in $\mathbb{F}[\tilde{t}^\ell]$. We will assume that $\mathbb{F}$ is algebraically closed, and the Whittaker module V is simple. Then since $Q = \mathrm{Ann}_R(w)$ is a maximal ϕ-stable ideal, it follows from Lemma 7.1(b), that $Q = R(\tilde{t}^\ell - \vartheta^\ell)$ for some nonzero $\vartheta \in \mathbb{F}$ or $Q = R\tilde{t}$. In the former case, $V \cong V_{\mathrm{u}}/QV_{\mathrm{u}} = R/Q$ is ℓ-dimensional, and the vectors $u_k := \vartheta^{-k}\tilde{t}^k w, k = 0, 1, \ldots, \ell-1$, determine a basis for V. Moreover, from (3.3) and (3.4) we see that $X^\ell = \zeta^\ell \mathrm{id}_V$, $Y^\ell = \zeta^{-\ell}(\alpha-1)^{-\ell}\alpha^{-(\ell-1)\ell/2}(\vartheta^\ell - \beta^\ell)\mathrm{id}_V$ and the following hold:

$$\begin{aligned} tu_k &= (\alpha-1)^{-1}(\vartheta u_{k+1} - \beta u_k), \\ Xu_k &= \zeta\alpha^k u_k, \quad \text{and} \quad Yu_k = \zeta^{-1}\alpha^{-k}(\alpha-1)^{-1}(\vartheta u_{k+1} - \beta u_k), \end{aligned} \tag{7.6}$$

where subscripts should be read mod ℓ. Now when $Q = R\tilde{t}$, then $V \cong R/Q = \mathbb{F}1$, and $V = \mathbb{F}w$. In this case, $tw = -(\alpha-1)^{-1}\beta w$, $Xw = \zeta w$, and $Yw = -\zeta^{-1}(\alpha - 1)^{-1}\beta w$. In summary we have:

Theorem 7.7. *Let $R = \mathbb{F}[t]$ where $\mathbb{F}$ is an algebraically closed field, and let $\phi : R \to R$ be the algebra automorphism given by $\phi(t) = \alpha t + \beta$, where α is a primitive ℓth root of unity. Assume V is a simple Whittaker module of type ζ for $A = R(\phi, t)$. Then either*

- (i) *$\dim V = \ell$, $Q = \mathrm{Ann}_R(w) = R(\tilde{t}^\ell - \vartheta^\ell)$ for some $\vartheta \neq 0$, and V has a basis $u_k, k = 0, 1, \ldots, \ell-1$, with A-action given by* (7.6)*; or*
- (ii) *$\dim V = 1$, $Q = \mathrm{Ann}_R(w) = R\tilde{t}$, and $V = \mathbb{F}w$, where*

$$tw = -(\alpha-1)^{-1}\beta w, \quad Xw = \zeta w, \quad Yw = -\zeta^{-1}(\alpha-1)^{-1}\beta w. \tag{7.8}$$

Conversely, any $\mathbb{F}$-vector space V of dimension ℓ (or 1) having an A-action given by (7.6) *(or by* (7.8)*) determines a simple Whittaker module of type ζ for $\zeta \neq 0$.*

7.9. $\boldsymbol{\alpha = 1}$. If $\alpha = 1$ and $\beta = 0$, the ϕ-stable ideals are just ordinary ideals J and every $V = R/J$ with A-action inherited from (3.2)–(3.4) is a Whittaker module of type ζ. If $\alpha = 1$ and $\beta \neq 0$, then R contains no nontrivial proper ϕ-stable ideals when $\mathsf{char}(\mathbb{F}) = 0$. Thus, there is, up to isomorphism, only one Whittaker module, namely the universal one $V_\mathfrak{u} = R$, and it is necessarily simple. In particular, when $\beta = -1$, the algebra $A = R(\phi, t)$ is the Weyl algebra A_1, and (3.16) gives the A-action on $V_\mathfrak{u}$ in this special case.

When $\mathsf{char}(\mathbb{F}) = p > 2$ and $\phi(t) = t + \beta$, where $\beta \neq 0$, then $\phi^p(t) = t + p\beta = t$, and ϕ has order p. In this case, the center $\mathcal{Z}$ of $A = R(\phi, t)$ is generated by X^p, Y^p, and the set R^ϕ of elements of R fixed by ϕ. It is straightforward to verify that $t^p - \beta^{p-1}t$ is fixed by ϕ. Define

$$z_n = \begin{cases} t^n & \text{if } n \not\equiv 0 \mod p, \\ (t^p - \beta^{p-1}t)^{n/p} & \text{if } n \equiv 0 \mod p, \end{cases}$$

so that the set $\{z_n \mid n \in \mathbb{Z}_{\geq 0}\}$ is a basis for R. Now let $g = \sum_{k=0}^m g_k z_k$, where $g_0, \ldots, g_m \in \mathbb{F}$, and assume that g is fixed by ϕ. It can be shown that $g - \phi(g) \neq 0$ unless $g = \sum_{k \equiv 0 \mod p} g_k z_k$, which is a polynomial in $t^p - \beta^{p-1}t$. Thus, when $\mathsf{char}(\mathbb{F}) = p > 2$, the center of A is generated by $X^p, Y^p, t^p - \beta^{p-1}t$. Observe that $z_p^j = z_{jp}$, so that $R^\phi = R \cap \mathcal{Z}$ is the polynomial algebra $\mathbb{F}[z_p]$, and R is a free R^ϕ-module with basis $1, t, \ldots, t^{p-1}$.

We claim that if J is a ϕ-stable ideal of R, then $J = R(J \cap \mathcal{Z})$. If this assertion is false, then there is a polynomial $f = \sum_{j=0}^n s_j t^j \in J \setminus R(J \cap \mathcal{Z})$ with coefficients in R^ϕ of least degree in t. Thus, $0 < n \leq p - 1$, $s_n \neq 0$, and $s_j \in R^\phi$ for all $0 \leq j \leq n$. Then $(\mathrm{id}_R - \phi)^n (f) = n! s_n \in J$. But this implies $s_n \in J \cap \mathcal{Z}$ and hence that $s_n t^n \in R(J \cap \mathcal{Z})$. By the minimality of n, we have $f - s_n t^n \in R(J \cap \mathcal{Z})$, and so $f = s_n t^n + (f - s_n t^n) \in R(J \cap \mathcal{Z})$, a contradiction. Thus, $J = R(J \cap \mathcal{Z})$, so that every ϕ-stable ideal of R is centrally generated.

Now let $\mathbb{F}$ be algebraically closed and let V be a simple Whittaker module of type ζ for A with $Q := \mathrm{Ann}_R(w)$. Since Q is ϕ-stable, $Q = R(Q \cap \mathcal{Z})$, where $Q \cap \mathcal{Z}$ is a maximal ideal of $R^\phi = R \cap \mathcal{Z} = \mathbb{F}[z_p]$. We can find $\lambda \in \mathbb{F}$ so that $Q = R\big(z_p - (\lambda^p - \beta^{p-1}\lambda)\big) = R\big(t^p - \beta^{p-1}t - (\lambda^p - \beta^{p-1}\lambda)\big)$, and $V \cong V_\mathfrak{u}/QV_\mathfrak{u} = R/Q$ is p-dimensional. Since $(t - \lambda)^p - \beta^{p-1}(t - \lambda) = \prod_{k=0}^{p-1} \big(t - (\lambda - k\beta)\big)$, the eigenvalues of t on V are of the form $\lambda - k\beta$ for $k = 0, 1, \ldots, p-1$. Let $v_0 = \prod_{k=1}^{p-1} \big(t - (\lambda - k\beta)\big) + Q \in R/Q$, and observe that $(t - \lambda)v_0 = 0$. Since X^p is central, it acts as a scalar on V, and from $X^p w = \zeta^p w$ we see that scalar is ζ^p. The vectors $v_k := \zeta^{-k} X^k v_0$ for $k = 0, 1, \ldots, p - 1$ determine a basis for V and relative to this basis, the A-action is given by

$$\text{(7.10)} \quad \begin{aligned} &t v_k = (\lambda - k\beta) v_k, \\ &X v_k = \zeta v_{k+1}, \quad \text{and} \quad Y v_k = \zeta^{-1}(\lambda - (k-1)\beta) v_{k-1} \quad (\text{subscripts mod } p). \end{aligned}$$

In particular, $Y^p = \zeta^{-p} \prod_{k=0}^{p-1} (\lambda - k\beta) = \zeta^{-p}(\lambda^p - \beta^{p-1}\lambda)\mathrm{id}_V$. To summarize, we have:

Theorem 7.11. *Assume $\mathbb{F}$ is algebraically closed, and let $A = R(\phi, t)$ be constructed from the automorphism $\phi : R \to R$ of $R = \mathbb{F}[t]$ given by $\phi(t) = t + \beta$ for $\beta \neq 0$. If $\mathsf{char}(\mathbb{F}) = 0$, then any Whittaker module of type ζ is isomorphic to the universal Whittaker module $V_\mathfrak{u}$ of type ζ, which is simple. If $\mathsf{char}(\mathbb{F}) = p > 2$ and V is a simple Whittaker module of type ζ, then V has a basis $v_k, k = 0, 1, \ldots, p-1$,* *so that the action of A on V is given by* (7.10) *for some scalar $\lambda \in \mathbb{F}$. The vector $w = v_0 + v_1 + \cdots + v_{p-1}$ is a cyclic Whittaker vector of type ζ for V. Moreover, $Q = \mathrm{Ann}_R(w) = R\big(t^p - \beta^{p-1}t - (\lambda^p - \beta^{p-1}\lambda)\big)$, and $\mathrm{Ann}_A(w) = AQ + A(X - \zeta)$.*

Remark 7.12. In the characteristic p case of Theorem 7.11, the ϕ-stable ideals of R are centrally generated. Thus, by Theorem 3.23, $V_{\mathfrak{u},\mathcal{K}} = V_{\mathfrak{u}}/\mathcal{K}V_{\mathfrak{u}}$ is a simple Whittaker module for every maximal ideal $\mathcal{K}$ of $\mathcal{Z}$, and $V_{\mathfrak{u},\mathcal{K}} \cong V_{\mathfrak{u}}/QV_{\mathfrak{u}} = R/Q$, where $Q = R(Q \cap \mathcal{Z}) = R(R^\phi \cap \mathcal{K}) = R\big(t^p - \beta^{p-1}t - (\lambda^p - \beta^{p-1}\lambda)\big)$ for some $\lambda \in \mathbb{F}$ as in Theorem 7.11.

Next we consider some well-known generalized Weyl algebras that fit into the pattern of arising from the polynomial algebra $R = \mathbb{F}[t]$.

7.13. The quantum plane: $R = \mathbb{F}[t]$ and $\phi(t) = \alpha t$ for $\alpha \neq 0, 1$. In the generalized Weyl algebra $A = R(\phi, t)$ constructed from the data $R = \mathbb{F}[t]$ and $\phi(t) = \alpha t$, we have $YX = t$, $XY = \alpha t$ so that A is a quantum plane. When α is not a root of unity, the simple Whittaker modules are one-dimensional, $V = \mathbb{F}w$, with the action of A given by

$$Xw = \zeta w, \quad Yw = 0, \quad tw = 0,$$

and $\mathrm{Ann}_A(V) = A\tilde{t} + A(X - \zeta)$, where $\tilde{t} = (\alpha - 1)t$.

When α is a primitive ℓth root of unity and $\mathbb{F}$ is algebraically closed, Theorem 7.7 implies that the simple Whittaker modules of type ζ are ℓ-dimensional with basis u_k, $k = 0, 1, \ldots, \ell - 1$, and A-action given by

$$\begin{aligned} &tu_k = (\alpha - 1)^{-1}\vartheta u_{k+1}, \\ &Xu_k = \zeta\alpha^k u_k, \text{ and } Yu_k = \zeta^{-1}\alpha^{-k}(\alpha - 1)^{-1}\vartheta u_{k+1} \quad \text{(subscripts mod } \ell) \end{aligned}$$

for some scalar $\vartheta \neq 0$, or they are one-dimensional $V = \mathbb{F}w$ with $Xw = \zeta w$, $Yw = 0$, and $tw = 0$. In the first case $\mathrm{Ann}_A(w) = A(\tilde{t}^\ell - \vartheta^\ell) + A(X - \zeta)$, while in the second, $\mathrm{Ann}_A(V) = A\tilde{t} + A(X - \zeta)$.

7.14. The quantum Weyl algebra: $R = \mathbb{F}[t]$ and $\phi(t) = q^{-1}(t - 1)$. Fix $q \in \mathbb{F}^\times$. Let $R = \mathbb{F}[t]$, and define $\phi : R \to R$ by $\phi(t) = q^{-1}(t - 1)$. The algebra $A = R(\phi, t)$ is commonly referred to as the quantum Weyl algebra and is often denoted $A_{q,1}$. We may view A as the unital algebra generated by elements X and Y over the field $\mathbb{F}$ with the relation $YX - qXY = 1$. In the notation $\phi(t) = \alpha t + \beta$ of the previous section, we have $\alpha = q^{-1}$, $\beta = -q^{-1}$.

When q is not a root of unity, the simple Whittaker modules of type ζ are one-dimensional, $V = \mathbb{F}w$, by Theorem 7.4 with the A-action given by

$$Xw = \zeta w, \quad Yw = \zeta^{-1}(1 - q)^{-1}w, \quad tw = (1 - q)^{-1}w, \tag{7.15}$$

and $\mathrm{Ann}_A(V) = A\tilde{t} + A(X - \zeta)$, where $\tilde{t} = (q^{-1} - 1)t - q^{-1} = q^{-1}\big((1 - q)t - 1\big)$.

When q is a primitive ℓth root of unity for $\ell \geq 2$, and $\mathbb{F}$ is algebraically closed, Theorem 7.7 implies that the simple Whittaker modules of type ζ are ℓ-dimensional with a basis $u_0, u_1, \ldots, u_{\ell-1}$ and A-action given by

$$\begin{aligned} &Xu_k = \zeta q^{-k}u_k, \quad Yu_k = \zeta^{-1}q^{k+1}(1 - q)^{-1}(\vartheta u_{k+1} + q^{-1}u_k), \\ &tu_k = q(1 - q)^{-1}\vartheta u_{k+1} + (1 - q)^{-1}u_k, \end{aligned} \tag{7.16}$$

where subscripts should be read mod ℓ and $\vartheta \neq 0$, or they are one-dimensional, $V = \mathbb{F}w$, with the A-action

$$tw = (1 - q)^{-1}w, \quad Xw = \zeta w, \quad Yw = \zeta^{-1}(1 - q)^{-1}w. \tag{7.17}$$

In the first case $\mathrm{Ann}_A(w) = A(\tilde{t}^\ell - \vartheta^\ell) + A(X - \zeta)$, and in the second $\mathrm{Ann}_A(V) = A\tilde{t} + A(X - \zeta)$, where $\tilde{t} = q^{-1}\big((1-q)t - 1\big)$.

When $q = 1$, then $A = R(\phi, t)$ is the (first) Weyl algebra A_1, and $\phi(t) = t - 1$. As we have discussed earlier, when $\mathsf{char}(\mathbb{F}) = 0$, the universal Whittaker module $V_{\mathfrak{u}} = R$ of type ζ is simple and the A-action is given by (3.16). When $q = 1$ and $\mathbb{F}$ is algebraically closed of characteristic $p > 2$, we may apply Theorem 7.11 with $\beta = -1$. Thus, in this setting, the simple Whittaker module V has a basis $\{v_0, \dots, v_{p-1}\}$ with A-action

$$tv_k = (\lambda + k)v_k, \quad Xv_k = \zeta v_{k+1}, \quad Yv_k = \zeta^{-1}(\lambda + (k-1))v_{k-1}. \tag{7.18}$$

8. Smith algebras

In [S], S.P. Smith introduced a family of associative algebras $\mathcal{A}$ which generalize the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$. These algebras are Noetherian domains with Gelfand-Kirillov dimension 3. Smith defined a notion of weight module for the algebra $\mathcal{A}$ and showed there is a category of $\mathcal{A}$-modules analogous to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Under special assumptions, the finite-dimensional modules for $\mathcal{A}$ are completely reducible. In [T1], Tang studied Whittaker modules for the algebra $\mathcal{A}$ over $\mathbb{C}$ and obtained exact analogues of the results by Kostant in [K] for $U(\mathfrak{sl}_2)$ and by Ondrus in [O2] for $U_q(\mathfrak{sl}_2)$.

Smith's algebras $\mathcal{A}$ have a realization as generalized Weyl algebras, and here we show how the results we have obtained can be specialized to recover Tang's results on Whittaker modules for these algebras. As a very special case, we obtain Kostant's results for Whittaker modules for $\mathfrak{sl}_2$. We also apply our results to determine the Whittaker modules in the modular case, which was not treated in the papers of Kostant and Tang.

Fix a nonzero polynomial $s(x)$ in the algebra $\mathbb{F}[x]$ of polynomials in x over a field $\mathbb{F}$ of characteristic not 2, and consider a unital associative algebra $\mathcal{A}$ over $\mathbb{F}$ with generators e, f, h, which satisfy the defining relations

$$he - eh = e, \qquad hf - fh = -f, \qquad ef - fe = s(h). \tag{8.1}$$

In particular, when $s(h) = 2h$, the algebra $\mathcal{A}$ is isomorphic to $U(\mathfrak{sl}_2)$. Smith showed that there is a polynomial $r(x)$ such that

$$s(x) = \frac{1}{2}(r(x+1) - r(x)), \tag{8.2}$$

and the "*Casimir element*", $c = 2fe + r(h+1)$, is central in $\mathcal{A}$. When $\mathsf{char}(\mathbb{F}) = 0$, the center $\mathcal{Z}$ of $\mathcal{A}$ consists just of polynomials in c.

To realize $\mathcal{A}$ as a GWA, let $R = \mathbb{F}[c, h]$, the polynomial algebra over $\mathbb{F}$ in commuting variables c, h, and let ϕ be the automorphism of R specified by $\phi(c) = c$, $\phi(h) = h - 1$. Set $t = \frac{1}{2}(c - r(h+1))$. Then $\phi(t) = \frac{1}{2}(c - r(h))$, and we obtain an isomorphism between $A = R(\phi, t)$ and Smith's algebra $\mathcal{A}$ by identifying X with e and Y with f. In what follows we will use the GWA realization to describe the Whittaker modules for Smith's algebra.

First suppose that $\mathsf{char}(\mathbb{F}) = 0$. Then $R^\phi = R \cap \mathcal{Z} = \mathbb{F}[c]$, and $R = \mathbb{F}[h, c]$ is a free R^ϕ-module with basis $\{h^j \mid j = 0, 1, \dots\}$. It can be shown, by an argument similar to that used in 7.9, that if J is a ϕ-stable ideal of R, then $J = R(J \cap \mathcal{Z})$. In this case, the elements h^j replace the elements t^j from 7.9.

Now assume (V, w) is a Whittaker pair for A of type ζ and let $Q = \mathrm{Ann}_R(w)$. As Q is ϕ-stable, $Q = R(Q \cap \mathcal{Z})$. Observe that $\mathcal{Z}_V := \mathrm{Ann}_A(V) \cap \mathcal{Z} = \mathrm{Ann}_R(V) \cap \mathcal{Z} = \mathrm{Ann}_R(w) \cap \mathcal{Z} = Q \cap \mathcal{Z}$, and by Theorem 3.12, the map

$$V \mapsto Q = \mathrm{Ann}_R(w) \mapsto Q \cap \mathcal{Z} = \mathcal{Z}_V \tag{8.3}$$

is a bijection. By Theorem 4.4,

$$\mathrm{Ann}_A(w) = AQ + A(X - \zeta) = A\mathcal{Z}_V + A(X - \zeta), \tag{8.4}$$

which is Theorem 2.2 of [T1]. Theorem 2.3 of [T1] establishes a one-to-one correspondence between isomorphism classes of Whitaker modules for A and ideals of the center $\mathcal{Z} = \mathbb{F}[c]$ given by $V \to \mathcal{Z}_V = \mathrm{Ann}_R(w)$, as above. (Tang assumes $\mathbb{F} = \mathbb{C}$, but the same results hold for any field of characteristic 0.)

Suppose now that $\mathbb{F}$ is algebraically closed and $\mathsf{char}(\mathbb{F}) = 0$, and let V be a simple Whittaker module for A of type ζ. Then $Q = \mathrm{Ann}_R(w) = R(Q \cap \mathcal{Z})$, where $Q \cap \mathcal{Z}$ is a maximal ideal of $R^\phi = R \cap \mathcal{Z} = \mathbb{F}[c]$. Thus, there is $\vartheta \in \mathbb{F}$ so that $Q = R(c - \vartheta)$. Since $V \cong R/Q = \mathbb{F}[h]$, the elements $h^k w$, $k \in \mathbb{Z}_{\geq 0}$, give a basis for V and the following hold:

$$\begin{aligned} &c.h^k w = \vartheta h^k w, \qquad h.h^k w = h^{k+1} w, \\ &X.h^k w = \phi(h)^k \zeta w = \zeta (h-1)^k w, \\ &Y.h^k w = \phi(h)^{-k} Y w = \zeta^{-1}(h+1)^k Y X w = \frac{1}{2}\zeta^{-1}(h+1)^k(\vartheta - r(h+1))w. \end{aligned} \tag{8.5}$$

Now $R/Q \cong \mathbb{F}[h]$ is a domain, and the induced automorphism $\overline{\phi} : R/Q \to R/Q$ clearly has infinite order since $\overline{\phi}(h) = h - 1$. Thus, if $t \notin Q$, Theorem 6.4 implies that $\mathrm{Ann}_A(V) = A(c - \vartheta)$. Recall that $t = \frac{1}{2}(c - r(h+1))$, where r is the polynomial defined by $s(x) = \frac{1}{2}(r(x+1) - r(x))$. If $t \in Q$, then $t - \phi(t)$ belongs to Q and is a nonzero polynomial in h since $t - \phi(t) = \frac{1}{2}(r(h) - r(h+1)) = \frac{1}{2}s(h) \neq 0$. Consequently, there exists $k > 0$ such that $(\mathrm{id}_R - \phi)^k(t)$ is a nonzero scalar contained in Q, contradicting the fact that Q is a proper ideal. Therefore, $t \notin Q$, and $\mathrm{Ann}_A(V) = A(c - \vartheta)$.

Now assume $\mathsf{char}(\mathbb{F}) = p > 2$. Observe that $\phi^p = \mathrm{id}_R$, so the center $\mathcal{Z}$ of the algebra $A = R(\phi, t)$ is generated by X^p, Y^p and all the elements R^ϕ of R fixed by ϕ. It is clear that c and $h^p - h$ are fixed by ϕ. Let us define

$$z_n = \begin{cases} h^n & \text{if } n \not\equiv 0 \mod p, \\ (h^p - h)^{n/p} & \text{if } n \equiv 0 \mod p. \end{cases}$$

Then R has a basis consisting of the monomials $c^j z_k$ for $j, k \in \mathbb{Z}_{\geq 0}$. Copying the argument of the characteristic p Weyl algebra case (with t replaced by h and the coefficients g_k in that argument assumed to lie in $\mathbb{F}[c]$ here), we see that $\mathcal{Z}$ is generated by $X^p, Y^p, c, h^p - h$. Observe that $z_p = h^p - h$ and $z_p^j = z_{jp}$. Thus, $R^\phi = R \cap \mathcal{Z}$ is the polynomial algebra $\mathbb{F}[c, z_p]$, and R is a free R^ϕ-module with basis $1, h, \ldots, h^{p-1}$. Exactly the same proof as in the Weyl algebra case proves that $J = R(J \cap \mathcal{Z})$ for any ϕ-stable ideal of R.

Now if $\mathbb{F}$ is an algebraically closed field and $\mathsf{char}(\mathbb{F}) = p > 2$, and V is a simple Whittaker module, then $Q = \mathrm{Ann}_R(w)$ is a maximal ϕ-stable ideal. Since $Q \cap \mathcal{Z}$ is a maximal ideal of $R^\phi = R \cap \mathcal{Z} = \mathbb{F}[c, z_p]$, there exist $\vartheta, \lambda \in \mathbb{F}$ so that $Q \cap \mathcal{Z} = R^\phi\big(c - \vartheta, z_p - (\lambda^p - \lambda)\big)$. Thus, $Q = R\big(c - \vartheta, h^p - h - (\lambda^p - \lambda)\big)$.

The vectors $h^k w$ for $k = 0, 1, \ldots, p-1$ form a basis for the simple module $V \cong R/Q$. Since the center of A must act as scalars on V, there exists $\mu \in \mathbb{F}$ with $X^p = \mu^p \mathrm{id}_V$. But then $(X - \mu \mathrm{id}_V)^p = 0$, and the only eigenvalue of X on V is μ, which must equal ζ. Since $(h-\lambda)^p - (h-\lambda) = \prod_{k=0}^{p-1}\big(h - (\lambda + k)\big)$, we see that the vector $v_0 := \prod_{k=1}^{p-1}\big(h - (\lambda+k)\big)w$ satisfies $(h-\lambda)v_0 = 0$. Set $v_k = \zeta^{-k}X^k v_0$ and note that $v_k \neq 0$ since $\zeta \neq 0$. Then the defining relations for A imply that

$$\begin{aligned} (8.6)\quad & cv_k = \vartheta v_k, \qquad hv_k = (\lambda + k)v_k, \\ & Xv_k = \zeta v_{k+1} \ \text{ and } \ Yv_k = \frac{1}{2}\zeta^{-1}\Big(\vartheta - r(\lambda+k)\Big)v_{k-1} \quad \text{(subscripts mod } p). \end{aligned}$$

Note that $Y^p = \frac{1}{2}\zeta^{-p}\prod_{k=0}^{p-1}\Big(\vartheta - r(\lambda+k)\Big)\mathrm{id}_V$ must hold. Therefore, we have:

Theorem 8.7. *Assume $\mathbb{F}$ is algebraically closed, and let $A = R(\phi, t)$ be a generalized Weyl algebra over $\mathbb{F}$ coming from a Smith algebra with defining relations* (8.1)*, where $r(x)$ is as in* (8.2)*. Let $V = Aw$ be a simple Whittaker module for A of type ζ with $Q = \mathrm{Ann}_R(w)$.*

(a) *If $\mathsf{char}(\mathbb{F}) = 0$, then $V = R/Q \cong \mathbb{F}[h]$, where $Q = R(c - \vartheta)$ for some $\vartheta \in \mathbb{F}$, and V has a basis $h^k w$, $k \in \mathbb{Z}_{\geq 0}$, with A-action given by* (8.5)*.*

(b) *If $\mathsf{char}(\mathbb{F}) = p > 2$, then V has a basis $v_k, k = 0, 1, \ldots, p-1$, so that the action of A is given by* (8.6) *for scalars λ, ϑ. The vector $w = v_0 + v_1 + \cdots + v_{p-1}$ is a cyclic Whittaker vector of type ζ for V; $Q = R\big(c - \vartheta, h^p - h - (\lambda^p - \lambda)\big)$; and $\mathrm{Ann}_A(w) = AQ + A(X - \zeta)$.*

Remark 8.8. We have shown for the Smith algebras that the ϕ-stable ideals of $R = \mathbb{F}[c, h]$ are centrally generated. By Theorem 3.23, $V_{\mathfrak{u},\mathcal{K}} = V_{\mathfrak{u}}/\mathcal{K}V_{\mathfrak{u}} = Aw_{\mathfrak{u},\mathcal{K}}$ is a simple Whittaker module for every maximal ideal $\mathcal{K}$ of the center $\mathcal{Z}$ of $A = R(\phi, t)$. Thus, when $\mathbb{F}$ is algebraically closed, $V_{\mathfrak{u},\mathcal{K}}$ is as in Theorem 8.7, and $Q := \mathrm{Ann}_R(w_{\mathfrak{u},\mathcal{K}}) = R(R^\phi \cap \mathcal{K})$ and $\mathrm{Ann}_A(w_{\mathfrak{u},\mathcal{K}}) = AQ + A(X - \zeta)$.

Remark 8.9. Suppose $A = R(\phi, t)$ is the Smith algebra defined using $s(x) = -1$ and $r(x) = -2x$. The quotient $\mathfrak{A} := A/Ac$ is isomorphic to the Weyl algebra A_1. Let $t' = h + 1 = YX$ in $\mathfrak{A}$. If in (8.6) we set $\vartheta = 0$, then there is an induced action of $\mathfrak{A}$ on V. Letting $\lambda' = \lambda - 1$, we have $t'v_k = (\lambda' + k)v_k$, $\quad Xv_k = \zeta v_{k+1}$, $\quad Yv_k = \zeta^{-1}(\lambda' + k - 1)v_k$, which are precisely the relations we obtained in (7.18) for the Whittaker modules of a Weyl algebra in characteristic p.

9. Quantum Smith algebras

In [JWZ] Ji, Wang, and Zhou introduced a family of associative algebras $\mathcal{A}$ which generalize the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ and which are quantum versions of the Smith algebras with defining polynomial $s(h) = h^{m+1} - h^m$ for some m. They showed that when the underlying field is $\mathbb{C}$, the finite-dimensional $\mathcal{A}$-modules are completely reducible. In [T2], Tang constructed all simple weight modules for $\mathcal{A}$ and obtained analogues for the Whittaker $\mathcal{A}$-modules of the results by Ondrus in [O2].

The algebras $\mathcal{A}$ have a realization as generalized Weyl algebras, and here we illustrate how results we have obtained can be specialized to recover results in [T2] and [O2]. We also determine all the simple Whittaker modules in the root of unity case, which is not considered either in [T2] or in [O2].

Assume $\mathsf{char}(\mathbb{F}) \neq 2$ and fix an integer $m \geq 1$. Assume $q \in \mathbb{F}, q \neq 0, \pm 1$, and q^2 is not an mth root of unity. Consider a unital associative algebra $\mathcal{A}$ over $\mathbb{F}$ with generators $E, F, K^{\pm 1}$, which satisfy the defining relations

$$\begin{aligned} &KE = q^2 EK, \qquad KF = q^{-2}FK, \qquad KK^{-1} = 1 = K^{-1}K, \\ &EF - FE = \frac{K^m - K^{-m}}{q - q^{-1}}, \end{aligned} \tag{9.1}$$

where $m \in \mathbb{Z}_{\geq 1}$. In particular, when $m = 1$, the algebra $\mathcal{A}$ is isomorphic to $U_q(\mathfrak{sl}_2)$. Tang gave a realization of this algebra as a hyperbolic algebra (as defined in [R]). Here we realize it as a generalized Weyl algebra (the two realizations are equivalent). The element

$$c = FE + \frac{q^m K^m + q^{-m}K^{-m}}{(q^m - q^{-m})(q - q^{-1})}$$

is central, and it generates the center of $\mathcal{A}$ when q^2 is not a root of unity (see [T2, Prop. 3.2]).

Let $R = \mathbb{F}[K^{\pm 1}, c]$ and define an automorphism ϕ on R by setting $\phi(K^{\pm 1}) = q^{\mp 2}K^{\pm 1}$ and $\phi(c) = c$. Let

$$t = c - \frac{q^m K^m + q^{-m}K^{-m}}{(q^m - q^{-m})(q - q^{-1})}, \tag{9.2}$$

and assume $A = R(\phi, t)$ is the generalized Weyl algebra constructed from this data. Thus, in A we have $YX = t$, and

$$XY = \phi(t) = c - \frac{q^{-m}K^m + q^m K^{-m}}{(q^m - q^{-m})(q - q^{-1})},$$

and A can be seen to be isomorphic to $\mathcal{A}$ by identifying X with E and Y with F.

We begin by applying Proposition 2.5 to describe the center $\mathcal{Z}$ of $A = R(\phi, t)$. If q^2 is not a root of unity, then $\mathcal{Z} = \mathbb{F}[c]$. If q^2 is a primitive ℓth root of unity for $\ell \neq m$, then $\phi^\ell = \mathrm{id}_R$, and $\mathcal{Z}$ is generated by X^ℓ, Y^ℓ and all the elements of R^ϕ. It is clear that $\phi(K^{\pm\ell}) = K^{\pm\ell}$. Suppose $h = \sum_{j=-r}^{s} h_j(c)K^j$ is fixed by ϕ. Then for each j with $h_j(c) \neq 0$, we must have $q^{-2j} = 1$, or $j \equiv 0 \mod \ell$. Thus, in this case, the center is generated by $X^\ell, Y^\ell, c, K^{\pm\ell}$.

Let $\Xi = \{q^{2k} \mid k \in \mathbb{Z}\}$, and note that the algebra R decomposes into eigenspaces R_ξ, $\xi \in \Xi$, relative to ϕ, where $R_\xi = \{r \in R \mid \phi(r) = \xi r\}$. Thus, $R_\xi = \mathrm{span}_{\mathbb{F}}\{c^i K^j \mid i \in \mathbb{Z}_{\geq 0} \text{ and } q^{-2j} = \xi\}$, and $R = \bigoplus_\xi R_\xi$ gives a grading of R. If J is a ϕ-stable ideal of R, then $J = \bigoplus_\xi J_\xi$ where $J_\xi = J \cap R_\xi$. For $\xi = q^{-2n} \in \Xi$, note that if $x \in J_\xi$, then $\phi(K^{-n}x) = q^{2n}q^{-2n}K^{-n}x = K^{-n}x$. Thus, $K^{-n}J_\xi \subseteq J \cap R^\phi \subseteq J \cap \mathcal{Z}$. As this holds for all ξ, we have the following.

Lemma 9.3. *Let J be a ϕ-stable ideal of $R = \mathbb{F}[c, K^{\pm 1}]$, where $\phi(c) = c$ and $\phi(K) = q^{-2}K$. Then $J = R(J \cap \mathcal{Z})$, where $\mathcal{Z}$ is the center of the corresponding generalized Weyl algebra $A = R(\phi, t)$ with t as in (9.2).*

Now let (V, w) be a Whittaker pair for A of type ζ with $Q = \mathrm{Ann}_R(w)$. As Q is ϕ-stable, Q is centrally generated and

$$Q = \mathrm{Ann}_R(V) = \begin{cases} R\mathcal{Z}_V & \text{if } q^2 \text{ is not a root of unity,} \\ R(\mathcal{Z}_V \cap R) & \text{if } q^2 \text{ is a root of unity,} \end{cases} \tag{9.4}$$

where $\mathcal{Z}_V = \mathrm{Ann}_A(V) \cap \mathcal{Z}$. Note that $A\mathcal{Z}_V \subseteq \mathrm{Ann}_A(w)$ regardless of whether or not q^2 is a root of unity. Thus, by Theorem 4.4,

$$\mathrm{Ann}_A(w) = AQ + A(X - \zeta) = A\mathcal{Z}_V + A(X - \zeta). \tag{9.5}$$

When q^2 is not a root of unity, this is Theorem 3.2 of [T2]. Theorem 3.3 of [T2] establishes a one-to-one correspondence between isomorphism classes of Whitaker modules for A of type ζ and ideals of the center $\mathcal{Z} = \mathbb{F}[c]$ given by $V \to \mathcal{Z}_V$. In the present setting (with no assumption on q^2), Theorem 3.12 gives a bijection $V \mapsto \mathrm{Ann}_R(w) = Q$ between isomorphism classes of Whitaker modules for A of type ζ and ϕ-stable ideals of R. But there is a bijection between ϕ-stable ideals J and ideals of $\mathcal{Z}$ given by $J \mapsto J \cap \mathcal{Z}$ since $J = R(J \cap \mathcal{Z})$.

Assume that q^2 is not a root of unity, and let V be a simple Whittaker module, with $\mathbb{F}$ algebraically closed. Since $Q := \mathrm{Ann}_R(w)$ is generated by its intersection with $\mathbb{F}[c]$, there must exist $\vartheta \in \mathbb{F}$ such that $Q = R(c - \vartheta)$. Notice that $R/Q \cong \mathbb{F}[K^{\pm 1}]$ is a domain, and it is clear that the induced automorphism $\overline{\phi} : R/Q \to R/Q$ has infinite order since $\overline{\phi}(K) = q^{-2}K$. Thus, as long as $t \notin Q$, Theorem 6.4 implies that $\mathrm{Ann}_A(V) = A(c - \vartheta)$. Recall that $t = c - \frac{q^m K^m + q^{-m} K^{-m}}{(q^m - q^{-m})(q - q^{-1})}$. If $t \in Q$, then $(\phi - \mathrm{id}_R)(\phi - q^{2m}\mathrm{id}_R)(t)$ is a nonzero multiple of K^m belonging to Q, contradicting the fact that Q is a proper ideal. Therefore, $t \notin Q$, and $\mathrm{Ann}_A(V) = A(c - \vartheta)$.

Suppose now that q^2 is a primitive ℓth root of unity for $\ell \neq m$. Assume $\mathbb{F}$ is algebraically closed and let V be a simple Whittaker module for A with Whittaker vector w of type ζ and with $Q = \mathrm{Ann}_R(w)$. Then $Q \cap \mathcal{Z}$ is a maximal ideal of $\mathbb{F}[c, K^{\pm \ell}]$, and there exist scalars ϑ, λ, with $\lambda \neq 0$, so that $Q \cap \mathcal{Z} = R^{\phi}(c - \vartheta, K^{\pm \ell} - \lambda^{\pm \ell})$, and $Q = R(c - \vartheta, K^{\pm \ell} - \lambda^{\pm \ell})$. Thus, $V \cong R/Q$ has a basis consisting of the vectors $K^j w$ for $j = 0, 1, \ldots, \ell - 1$. Since $XK^j w = q^{-2j}\zeta K^j w$, we see that $X^\ell = \zeta^\ell \mathrm{id}_V$. If $v_0 := \sum_{j=0}^{\ell-1} \lambda^{-j} K^j w$, then $Kv_0 = \lambda v_0$. The vectors $v_j = \zeta^{-j} X^j v_0$ for $j = 0, 1, \ldots, \ell - 1$ are eigenvectors for K ($Kv_j = \lambda q^{2j} v_j$) corresponding to different eigenvalues. Hence, they are linearly independent and comprise a basis for V. The action of A relative to this basis is given by

$$\begin{aligned} &cv_j = \vartheta v_j, \qquad Kv_j = \lambda q^{2j} v_j, \qquad Xv_j = \zeta v_{j+1}, \\ &Yv_j = \zeta^{-1}\Big(\vartheta - \frac{\lambda^m q^{(2j+1)m} + \lambda^{-m} q^{-(2j+1)m}}{(q^m - q^{-m})(q - q^{-1})}\Big) v_{j-1} \quad \text{(subscripts mod } \ell\text{).} \end{aligned} \tag{9.6}$$

Note that $Y^\ell = \zeta^{-\ell} \prod_{j=0}^{\ell-1} \Big(\vartheta - \frac{\lambda^m q^{(2j+1)m} + \lambda^{-m} q^{-(2j+1)m}}{(q^m - q^{-m})(q - q^{-1})}\Big) \mathrm{id}_V$ must hold.

Theorem 9.7. *Assume $\mathbb{F}$ is algebraically closed with $\mathsf{char}(\mathbb{F}) \neq 2$, and let $A = R(\phi, t)$ be a generalized Weyl algebra over $\mathbb{F}$ coming from a quantum Smith algebra with defining relations (9.1), where q^2 is a primitive ℓth root of unity, and $\ell \neq m$. If V is a simple Whittaker module for A of type ζ, then V has a basis $v_j, j = 0, 1, \ldots, \ell - 1$, with the action of A on V given by (9.6) for scalars λ, ϑ. The vector $w = v_0 + v_1 + \cdots + v_{\ell-1}$ is a cyclic Whittaker vector of type ζ for V. Then $Q = \mathrm{Ann}_R(w) = R\big(c - \vartheta, K^{\pm \ell} - \lambda^{\pm \ell}\big)$, and $\mathrm{Ann}_A(w) = AQ + A(X - \zeta)$.*

Remark 9.8. Since R^ϕ is always finitely generated over $\mathbb{F}$ for quantum Smith algebras, and every ϕ-stable ideal of R is centrally generated by Lemma 9.3, $V_{\mathfrak{u},\mathcal{K}}$ is a simple Whittaker module for any maximal ideal $\mathcal{K}$ of the center of A; hence, it is one of the modules in Theorem 9.7.

Acknowledgments

This paper was completed while the first author was a Simons Visiting Professor in the Combinatorial Representation Theory Program at the Mathematical Sciences Research Institute. She acknowledges with gratitude the warm hospitality of MSRI and the support from the Simons Foundation, as well as from National Science Foundation grant #DMS–0245082. The authors also thank Louis Solomon and Samuel Lopes for their insights and helpful comments.

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