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REMARKS ON SPRINGER'S REPRESENTATIONS

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Abstract. We give an explicit description of a set of irreducible representations of a Weyl group which parametrizes the nilpotent orbits in the Lie algebra of a connected reductive group in arbitrary characteristic. We also answer a question of Serre concerning the conjugacy class of a power of a unipotent element in a connected reductive group.

Introduction

0.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic exponent $p \geq 1$. Let G be a connected reductive algebraic group over $\mathbf{k}$ and let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathcal{U}_G$ be the variety of unipotent elements of G and let $\mathcal{N}_\mathfrak{g}$ be the variety of nilpotent elements of $\mathfrak{g}$ (we say that $x \in \mathfrak{g}$ is nilpotent if for some/any closed imbedding $G \subset GL(\mathbf{k}^n)$, the image of x under the induced map of Lie algebras $\mathfrak{g} \to \mathrm{End}(\mathbf{k}^n)$ is nilpotent as an endomorphism). Note that G acts on G and $\mathfrak{g}$ by the adjoint action. Let $\mathcal{X}_G$ (resp. $\mathcal{X}_\mathfrak{g}$) be the set of G-orbits on $\mathcal{U}_G$ (resp. on $\mathcal{N}_\mathfrak{g}$). We fix a prime number l, $l \neq p$. Let $\hat{\mathcal{X}}_G$ (resp. $\hat{\mathcal{X}}_\mathfrak{g}$) be the set of pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O} \in \mathcal{X}_G$ (resp. $\mathcal{O} \in \mathcal{X}_\mathfrak{g}$) and $\mathcal{L}$ is an irreducible G-equivariant $\bar{\mathbf{Q}}_l$-local system on $\mathcal{O}$ up to isomorphism. Let $\mathbf{W}$ be the Weyl group of G. For any Weyl group W let $\mathrm{Irr}(W)$ be the set of isomorphism classes of irreducible representations of W over $\mathbf{Q}$. In [Sp], Springer defined (assuming that $p = 1$ or $p \gg 0$) natural injective maps $S_G : \mathrm{Irr}(\mathbf{W}) \to \hat{\mathcal{X}}_G$, $S_\mathfrak{g} : \mathrm{Irr}(\mathbf{W}) \to \hat{\mathcal{X}}_\mathfrak{g}$ (each of these two maps determines the other since in this case we have canonically $\hat{\mathcal{X}}_G = \hat{\mathcal{X}}_\mathfrak{g}$). In [L2] a new definition of the map S_G (based on intersection homology) was given which applies without restriction on p. A similar method can be used to define $S_\mathfrak{g}$ without restriction on p (see [X1], [X2] and 2.2 below); note that in general $\hat{\mathcal{X}}_G, \hat{\mathcal{X}}_\mathfrak{g}$ cannot be identified. Now for any $\mathcal{O} \in \mathcal{X}_G$ (resp. $\mathcal{O} \in \mathcal{X}_\mathfrak{g}$), $(\mathcal{O}, \bar{\mathbf{Q}}_l)$ is in the image of S_G (resp. $S_\mathfrak{g}$); hence there is a well-defined injective map $S'_G : \mathcal{X}_G \to \mathrm{Irr}(\mathbf{W})$ (resp. $S'_\mathfrak{g} : \mathcal{X}_\mathfrak{g} \to \mathrm{Irr}(\mathbf{W})$) such that for any $\mathcal{O} \in \mathcal{X}_G$ (resp. $\mathcal{O} \in \mathcal{X}_\mathfrak{g}$) we have $S'_G(\mathcal{O}) = E$ (resp. $S'_\mathfrak{g}(\mathcal{O}) = E$) where $E \in \mathrm{Irr}(\mathbf{W})$ is given by $S_G(E) = (\mathcal{O}, \bar{\mathbf{Q}}_l)$ (resp. $S_\mathfrak{g}(E) = (\mathcal{O}, \bar{\mathbf{Q}}_l)$). Let $\mathfrak{S}_G$ be the image of $S'_G : \mathcal{X}_G \to \mathrm{Irr}(\mathbf{W})$. Let $\mathfrak{S}_\mathfrak{g}$ be the image of $S'_\mathfrak{g} : \mathcal{X}_\mathfrak{g} \to \mathrm{Irr}(\mathbf{W})$.

In [L5], we gave an a priori definition (in the framework of Weyl groups) of the subset $\mathfrak{S}_G$ of $\mathrm{Irr}(\mathbf{W})$ which parametrizes the unipotent G-orbits in G. In this paper we give an a priori definition (in a similar spirit) of the subset $\mathfrak{S}_\mathfrak{g}$ of $\mathrm{Irr}(\mathbf{W})$ which parametrizes the nilpotent G-orbits in $\mathfrak{g}$. (See Proposition 3.2.) This relies heavily on work of Spaltenstein [S2], [S3] and on [HS]. As an application we define a

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natural injective map from the set of unipotent G-orbits in G to the set of nilpotent G-orbits in $\mathfrak{g}$ (see 3.3); this map preserves the dimension of an orbit.

In [Se], Serre asked whether a power u^n (where n is an integer not divisible by p, $p \geq 2$) of a unipotent element $u \in G$ is conjugate to u under G. This is well known to be true when $p \gg 0$. In §2 we answer positively this question in general using the theory of Springer's representations; we also discuss an analogous property of nilpotent elements.

1. Combinatorics

1.1. For $k \in \mathbf{N}$ let $\mathcal{E}_k = \{a_* = (a_0, a_1, \dots, a_k) \in \mathbf{N}^{k+1}; a_0 \leq a_1 \leq \dots \leq a_k\}$. For $a_* \in \mathcal{E}_k$ let $|a_*| = \sum_i a_i$. For $a_*, a'_* \in \mathcal{E}_k$ we set $a_* + a'_* = (a_0 + a'_0, a_1 + a'_1, \dots, a_k + a'_k)$. For any $n \in \mathbf{N}$ let $\mathcal{E}_k^n = \{a_* \in \mathcal{E}_k; |a_*| = n\}$. We have an imbedding $\mathcal{E}_k^n \to \mathcal{E}_{k+1}^n$, $(a_0, a_1, \dots, a_k) \mapsto (0, a_0, a_1, \dots, a_k)$. This is a bijection if k is sufficiently large with respect to n. For $n \in \mathbf{N}$ let

$$\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{E}_k \times \mathcal{E}_k; |a_*| + |a'_*| = n\},$$
$$\mathcal{D}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; \text{ either } |a_*| > |a'_*| \text{ or } a_* = a'_*\}.$$

Here k is large (relative to n), fixed. Let

$${}^b\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i + 2 \quad \forall i \in [0, k]\},$$
$${}^{b1}\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i + 2 \quad \forall i \in [0, k], a_i \leq a'_{i+1} \quad \forall i \in [0, k-1]\},$$
$${}^{b2}\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; a'_i \leq a_i + 2 \quad \forall i \in [0, k], a_i \leq a'_{i+1} + 2 \quad \forall i \in [0, k-1]\},$$
$${}^{c1}\mathcal{C}_k^n = \{(a_*, a'_*) \in \mathcal{C}_k^n; a_i \leq a'_{i+1} + 1 \quad \forall i \in [0, k-1], a'_i \leq a_i + 1 \quad \forall i \in [0, k]\},$$
$${}^{d}\mathcal{D}_k^n = \{(a_*, a'_*) \in \mathcal{D}_k^n; a'_i \leq a_i \quad \forall i \in [0, k]\},$$
$${}^{d1}\mathcal{D}_k^n = \{(a_*, a'_*) \in \mathcal{D}_k^n; a'_i \leq a_i \quad \forall i \in [0, k], a_i \leq a'_{i+1} + 2 \quad \forall i \in [0, k-1]\},$$
$${}^{d2}\mathcal{D}_k^n = \{(a_*, a'_*) \in \mathcal{D}_k^n; a'_i \leq a_i \quad \forall i \in [0, k], a_i \leq a'_{i+1} + 4 \quad \forall i \in [0, k-1]\}.$$

Note that

$${}^{b1}\mathcal{C}_k^n \subset {}^{b2}\mathcal{C}_k^n \subset {}^{b}\mathcal{C}_k^n,$$
$${}^{c1}\mathcal{C}_k^n \subset {}^{b2}\mathcal{C}_k^n \subset \mathcal{C}_k^n,$$
$${}^{d1}\mathcal{D}_k^n \subset {}^{d2}\mathcal{D}_k^n \subset {}^{d}\mathcal{D}_k^n.$$

The following statements are obvious. If $(a_*, a'_*) \in \mathcal{C}_k^m$, $(b_*, b'_*) \in \mathcal{C}_k^{m'}$, then $(a_* + b_*, a'_* + b'_*) \in \mathcal{C}_k^{m+m'}$. If $(a_*, a'_*) \in {}^b\mathcal{C}_k^m$, $(b_*, b'_*) \in {}^d\mathcal{D}_k^{m'}$, then $(a_* + b_*, a'_* + b'_*) \in {}^b\mathcal{C}_k^{m+m'}$. If $(a_*, a'_*) \in {}^d\mathcal{D}_k^m$, $(b_*, b'_*) \in {}^d\mathcal{D}_k^{m'}$, then $(a_* + b_*, a'_* + b'_*) \in {}^d\mathcal{C}_k^{m+m'}$.

In the following result we assume that k is large relative to n.

Proposition 1.2. *(a) Let $(c_*, c'_*) \in \mathcal{C}_k^n$. Then either $(c_*, c'_*) \in {}^{c1}\mathcal{C}_k^n$ or there exist $m \geq 1, m' \geq 1$ such that $m + m' = n$ and $(a_*, a'_*) \in \mathcal{C}_k^m$, $(b_*, b'_*) \in \mathcal{C}_k^{m'}$ such that $(c_*, c'_*) = (a_* + b_*, a'_* + b'_*)$.*

(b) Let $(c_*, c'_*) \in {}^b\mathcal{C}_k^n$. Then either $(c_*, c'_*) \in {}^{b1}\mathcal{C}_k^n$ or there exist $m \geq 0, m' \geq 2$ such that $m + m' = n$ and $(a_*, a'_*) \in {}^b\mathcal{C}_k^m$, $(b_*, b'_*) \in {}^d\mathcal{D}_k^{m'}$, such that $(c_*, c'_*) = (a_* + b_*, a'_* + b'_*)$.

(c) Let $(c_*, c'_*) \in {}^d\mathcal{C}_k^n$. Then either $(c_*, c'_*) \in {}^{d1}\mathcal{C}_k^n$ or there exist $m \geq 2, m' \geq 2$ such that $m + m' = n$ and $(a_*, a'_*) \in {}^d\mathcal{D}_k^m$, $(b_*, b'_*) \in {}^d\mathcal{D}_k^{m'}$ such that $(c_*, c'_*) = (a_* + b_*, a'_* + b'_*)$.

We prove (a). Assume first that $c_s < c_{s+1}$ for some $s \in [0, k-1]$. Define $(b_*, b'_*) \in \mathcal{C}^k_r$, $r = k - s > 0$, by $b_i = 1$ for $i \in [s+1, k]$, $b_i = 0$ for $i \in [0, s]$, $b'_i = 0$ for $i \in [0, k]$. Define $(a_*, a'_*) \in \mathcal{C}^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1, k]$, $a_i = c_i$ in $[0, s]$, $a'_* = c'_*$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. If $r < n$ we see that (a) holds. If $r = n$, then $(c_*, c'_*) = (b_*, b'_*) \in {}^{c1}\mathcal{C}^n_k$ and (a) holds again.

Next we assume that $c'_s < c'_{s+1}$ for some $s \in [0, k-1]$. Define $(b_*, b'_*) \in \mathcal{C}^k_r$, $r = k - s > 0$, by $b_i = 0$ for $i \in [0, k]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_*, a'_*) \in \mathcal{C}^k_{n-r}$ by $a_* = c_*$, $a'_i = c'_i - 1$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. If $r < n$ we see that (a) holds. If $r = n$, then $(c_*, c'_*) = (b_*, b'_*) \in {}^{c1}\mathcal{C}^n_k$ and (a) holds again.

Finally, we assume that $c_0 = c_1 = \cdots = c_r$, $c'_0 = c'_1 = \cdots = c'_r$. Since k is large we can assume that $c_0 = 0$, $c'_0 = 0$. Then $n = 0$ and $(c_*, c'_*) \in {}^{c1}\mathcal{C}^n_k$.

We prove (b). If $n = 0$ we have clearly $(c_*, c'_*) \in {}^{b1}\mathcal{C}^n_k$. Hence we can assume that $n > 0$ and that the result is true when n is replaced by $n' \in [0, n-1]$.

Assume first that we can find $0 < t \le s \le k$ such that $c'_j = c_j + 2$ for $j \in [s+1, k]$, $c'_j < c_j + 2$ for $j \in [t, s]$, $c_{t-1} < c_t$. Note that if $s < k$, then $c'_s < c'_{s+1}$; indeed, $c'_s < c_s - 2 \le c_{s+1} - 2 = c'_{s+1}$. Define $(b_*, b'_*) \in {}^d\mathcal{D}^k_r$, $r = 2k - t - s + 1 > 0$ by $b_i = 1$ for $i \in [t, k]$, $b_i = 0$ for $i \in [0, t-1]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_*, a'_*) \in {}^b\mathcal{C}^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [t, k]$, $a_i = c_i$ for $i \in [0, t-1]$, $a'_i = c'_i - 1$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. If $r \ge 2$, we see that (b) holds. If $r = 1$, then $t = s = k$ and $a_k = c_k - 1$, $a_i = c_i$ for $i \in [0, k-1]$, $a'_i = c'_i$ for $i \in [0, k]$. The induction hypothesis is applicable to $(a_*, a'_*) \in {}^b\mathcal{C}^k_{n-1}$. If $(a_*, a'_*) \in {}^{b1}\mathcal{C}^k_{n-1}$, then clearly $(c_*, c'_*) \in {}^{b1}\mathcal{C}^k_{n-1}$ and (b) holds. If $(a_*, a'_*) \notin {}^{b1}\mathcal{C}^k_{n-1}$, then we can find $m \ge 0, m' \ge 2$ such that $m + m' = n - 1$ and $(\tilde{a}_*, \tilde{a}'_*) \in {}^b\mathcal{C}^m_k$, $(\tilde{b}_*, \tilde{b}'_*) \in {}^d\mathcal{D}^{m'}_k$ such that $(a_*, a'_*) = (\tilde{a}_* + \tilde{b}_*, \tilde{a}'_* + \tilde{b}'_*)$. Then $(c_*, c'_*) = (\tilde{a}_* + \tilde{b}_* + b_*, \tilde{a}'_* + \tilde{b}'_* + b'_*)$ where $(\tilde{a}_*, \tilde{a}'_*) \in {}^b\mathcal{C}^m_k$, $(\tilde{b}_* + b_*, \tilde{b}'_* + b'_*) \in {}^d\mathcal{D}^{m'+1}_k$ so that (b) holds.

Next we assume that $c_i > 0$ for some i. Then we have $0 = c_0 = c_1 = \cdots = c_{l-1} < c_l$ for some $l \in [0, k]$. If $c'_s < c_s + 2$ for some $s \in [l, k]$, then we can assume that s is maximum possible with this property and there are two possibilities. Either $c'_i < c_i + 2$ for all $i \in [l, s]$ and then by the previous paragraph (with $t = l$) we see that (b) holds; or $c'_i = c_i + 2$ for some $i \in [l, s]$ and letting $t - 1$ be the largest such i we have $0 < t \le s$, $c'_j < c_j + 2$ for $j \in [t, s]$, $c'_j = c_j + 2$ for $j = t - 1$ and $c_{t-1} = c'_{t-1} - 2 \le c'_t - 2 < c_t$; using again the previous paragraph we see that (b) holds. Thus we may assume that $c'_i = c_i + 2$ for all $i \in [l, k]$. Assume, in addition, that $c'_s < c'_{s+1}$ for some $s \in [l, k-1]$. We can assume that s is maximum possible so that $c'_s < c'_{s+1} = \cdots = c'_k$. We have $c_{s+1} = c'_{s+1} - 2 > c'_s - 2 = c_s$; hence $c_s < c_{s+1}$. Define $(b_*, b'_*) \in {}^d\mathcal{D}^k_r$, $r = 2k - 2s \ge 2$, by $b_i = 1$ for $i \in [s+1, k]$, $b_i = 0$ for $i \in [0, s]$, $b'_i = 1$ for $i \in [s+1, k]$, $b'_i = 0$ for $i \in [0, s]$. Define $(a_*, a'_*) \in {}^b\mathcal{C}^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1, k]$, $a_i = c_i$ for $i \in [0, s]$, $a'_i = c'_i - 1$ for $i \in [s+1, k]$, $a'_i = c'_i$ for $i \in [0, s]$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. We see that (b) holds. Thus we can assume that $c'_l = c'_{l+1} = \cdots = c'_k = N + 2$ so that $c_l = c_{l+1} = \cdots = c_k = N$. Note that $c'_i \le 2$ for $i \in [0, l-1]$. We have $(c_*, c'_*) \in {}^{b1}\mathcal{C}^n_k$ so that (b) holds.

Finally, we assume that $c_0 = c_1 = \cdots = c_k = 0$. Then $c'_i \le 2$ for $i \in [0, k]$ and $(c_*, c'_*) \in {}^{b1}\mathcal{C}^n_k$ so that (b) holds. This completes the proof of (b).

We prove (c). If $n = 0$ we have clearly $(c_*, c'_*) \in {}^{d1}\mathcal{D}^n_k$. Hence we can assume that $n > 0$ and that the result is true when n is replaced by $n' \in [0, n-1]$.

Assume first that we can find $0 < t \le s \le k$ such that $c'_j = c_j$ for $j \in [s+1,k]$, $c'_j < c_j$ for $j \in [t,s]$, $c_{t-1} < c_t$. Note that if $s < k$, then $c'_s < c'_{s+1}$; indeed, $c'_s < c_s \le c_{s+1} = c'_{s+1}$. Define $(b_*, b'_*) \in {}^d\mathcal{D}^k_r$, $r = 2k - t - s + 1 > 0$ by $b_i = 1$ for $i \in [t,k]$, $b_i = 0$ for $i \in [0,t-1]$, $b'_i = 1$ for $i \in [s+1,k]$, $b'_i = 0$ for $i \in [0,s]$. Define $(a_*, a'_*) \in {}^d\mathcal{D}^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [t,k]$, $a_i = c_i$ for $i \in [0,t-1]$, $a'_i = c'_i - 1$ for $i \in [s+1,k]$, $a'_i = c'_i$ for $i \in [0,s]$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. If $n-2 \ge r \ge 2$ we see that (c) holds. If $r = 1$, then $t = s = k$ and $a_k = c_k - 1$, $a_i = c_i$ for $i \in [0,k-1]$, $a'_i = c'_i$ for $i \in [0,k]$. The induction hypothesis is applicable to $(a_*, a'_*) \in {}^d\mathcal{D}^k_{n-1}$. If $(a_*, a'_*) \in {}^{d1}\mathcal{D}^k_{n-1}$, then clearly $(c_*, c'_*) \in {}^{d1}\mathcal{D}^k_{n-1}$ and (c) holds. If $(a_*, a'_*) \notin {}^{d1}\mathcal{D}^k_{n-1}$, then we can find $m \ge 2, m' \ge 2$ such that $m + m' = n-1$ and $(\tilde a_*, \tilde a'_*) \in {}^d\mathcal{D}^m_k$, $(\tilde b_*, \tilde b'_*) \in {}^d\mathcal{D}^{m'}_k$ such that $(a_*, a'_*) = (\tilde a_* + \tilde b_*, \tilde a'_* + \tilde b'_*)$. Then $(c_*, c'_*) = (\tilde a_* + \tilde b_* + b_*, \tilde a'_* + \tilde b'_* + b'_*)$ where $(\tilde a_*, \tilde a'_*) \in {}^d\mathcal{D}^m_k$, $(\tilde b_* + b_*, \tilde b'_* + b'_*) \in {}^d\mathcal{D}^{m'+1}_k$ so that (c) holds. If $r = n-1$, then $a_i = 0$ for $i \in [0,k-1]$, $a_k = 0$, $a'_i = 0$ for $i \in [0,k]$; hence $c_i = 1$ for $i \in [t,k-1]$, $c_k = 2$, $c_i = 0$ for $i \in [0,t-1]$, $c'_i = 1$ for $i \in [s+1,k]$, $c'_i = 0$ for $i \in [0,s]$. Hence $(c_*, c'_*) \in {}^d\mathcal{D}^n_k$ so that (c) holds. If $r = n$, then $(c_*, c'_*) = (b_*, b'_*) \in {}^d\mathcal{D}^n_k$ so that (c) holds.

Next we assume that $c_i > 0$ for some i. Then we have $0 = c_0 = c_1 = \cdots = c_{l-1} < c_l$ for some $l \in [0,k]$. If $c'_s < c_s$ for some $s \in [l,k]$, then we can assume that s is maximum possible with this property and there are two possibilities. Either $c'_i < c_i$ for all $i \in [l,s]$ and then by the previous paragraph (with $t = l$) we see that (c) holds; or $c'_i = c_i$ for some $i \in [l,s]$ and letting $t-1$ be the largest such i we have $0 < t \le s$, $c'_j < c_j$ for $j \in [t,s]$, $c'_j = c_j$ for $j = t-1$ and $c_{t-1} = c'_{t-1} \le c'_t < c_t$; using again the previous paragraph we see that (c) holds. Thus we may assume that $c'_i = c_i$ for all $i \in [l,k]$. Assume, in addition, that $c'_s < c'_{s+1}$ for some $s \in [l,k-1]$. We can assume that s is maximum possible so that $c'_s < c'_{s+1} = \cdots = c'_k$. We have $c_{s+1} = c'_{s+1} > c'_s = c_s$ hence $c_s < c_{s+1}$. Define $(b_*, b'_*) \in {}^d\mathcal{D}^k_r$,$r = 2k - 2s \ge 2$, by $b_i = 1$ for $i \in [s+1,k]$, $b_i = 0$ for $i \in [0,s]$, $b'_i = 1$ for $i \in [s+1,k]$, $b'_i = 0$ for $i \in [0,s]$. Define $(a_*, a'_*) \in {}^d\mathcal{D}^k_{n-r}$ by $a_i = c_i - 1$ for $i \in [s+1,k]$, $a_i = c_i$ for $i \in [0,s]$, $a'_i = c'_i - 1$ for $i \in [s+1,k]$, $a'_i = c'_i$ for $i \in [0,s]$. We have $a_* + b_* = c_*$, $a'_* + b'_* = c'_*$. If $r \le n-2$, we see that (c) holds. If $r = n-1$, then $a_i = 0$ for $i \in [0,k-1]$, $a_k = 0$, $a'_i = 0$ for $i \in [0,k]$; hence $c_i = 1$ for $i \in [s+1,k-1]$, $c_k = 2$, $c_i = 0$ for $i \in [0,s]$, $c'_i = 1$ for $i \in [s+1,k]$, $c'_i = 0$ for $i \in [0,s]$. Hence $(c_*, c'_*) \in {}^d\mathcal{D}^n_k$ so that (c) holds. If $r = n$, then $(c_*, c'_*) = (b_*, b'_*) \in {}^d\mathcal{D}^n_k$ so that (c) holds. Thus we can assume that $c'_l = c'_{l+1} = \cdots = c'_k = N$ so that $c_l = c_{l+1} = \cdots = c_k = N$. Note that $c'_i = 0$ for $i \in [0,l-1]$. We have $(c_*, c'_*) \in {}^{d1}\mathcal{D}^n_k$ so that (c) holds.

Finally, we assume that $c_0 = c_1 = \cdots = c_k = 0$. Then $c'_i = 0$ for $i \in [0,k]$. In this case we have $n = 0$ and $(c_*, c'_*) \in {}^{d1}\mathcal{D}^n_k$ so that (c) holds. This completes the proof of (c).

2. On Serre's questions

2.1. For any affine algebraic group H over $\mathbf{k}$ we denote by Lie$H$ the Lie algebra of H. For any $\mathcal{O} \in \mathcal{X}_G$ (or $\mathcal{O} \in \mathcal{X}_{\mathfrak{g}}$) we set $d_{\mathcal{O}} = 2\dim\mathcal{B} - \dim\mathcal{O}$.

2.2. We recall the definition of Springer's representations following [L2]. Let $\mathcal{B}$ be the variety of Borel subgroups of G. Let $\tilde{\mathcal{B}} = \{(g,B) \in G \times \mathcal{B}; g \in B\}$ and let $f : \tilde{\mathcal{B}} \to G$ be the first projection. Let $K = f_!\bar{\mathbf{Q}}_l$. In [L2] it was observed that K is an intersection cohomology complex on G coming from a local system on the open dense subset of G consisting on regular semisimple elements. Moreover, $\mathbf{W}$

acts naturally on this local system and hence, by "analytic continuation", on K. In particular, if $\mathcal{O} \in \mathcal{X}_G$ and $i \in \mathbf{Z}$, then $\mathbf{W}$ acts naturally on the i-th cohomology sheaf $\mathcal{H}^i K|_{\mathcal{O}}$ of $K|_{\mathcal{O}}$, an irreducible G-equivariant local system on $\mathcal{O}$; hence if $\mathcal{L}$ is an irreducible G-equivariant local system on $\mathcal{O}$, then $\mathbf{W}$ acts naturally on the $\bar{\mathbf{Q}}_l$-vector space $\mathrm{Hom}(\mathcal{L}, \mathcal{H}^i K|_{\mathcal{O}})$. We denote this $\mathbf{W}$-module (with $i = d_{\mathcal{O}}$) by $V_{\mathcal{O},\mathcal{L}}$. As shown in [L4], $V_{\mathcal{O},\mathcal{L}}$ is either 0 or of the form $\bar{\mathbf{Q}}_l \otimes E$ where $E \in \mathrm{Irr}(\mathbf{W})$; moreover, any $E \in \mathrm{Irr}(\mathbf{W})$ arises in this way from a unique $(\mathcal{O}, \mathcal{L})$ and $E \mapsto (\mathcal{O}, \mathcal{L})$ is an injective map

$$S_G : \mathrm{Irr}(\mathbf{W}) \to \hat{\mathcal{X}}_G.$$

We would like to define a similar map from $\mathrm{Irr}(\mathbf{W})$ to $\hat{\mathcal{X}}_{\mathfrak{g}}$. Let $\tilde{\mathcal{B}}' = \{(x, B) \in \mathfrak{g} \times \mathcal{B}; x \in \mathrm{Lie}B\}$ and let $f' : \tilde{\mathcal{B}}' \to \mathfrak{g}$ be the first projection. Let $K' = f'_! \bar{\mathbf{Q}}_l$. Now if p is small the set of regular semisimple elements in $\mathfrak{g}$ may be empty (this is the case for example if $G = SL_2(\mathbf{k})$, $p = 2$) so the method of [L4] cannot be used directly. However, T. Xue [X1], [X2] has observed that the method of [L4], [L2] can be applied if G is a classical group of adjoint type and $p = 2$ (in that case the set of regular semisimple elements in $\mathfrak{g}$ is open dense in $\mathfrak{g}$). More generally, for any G which is adjoint, the set of regular semisimple elements in $\mathfrak{g}$ is open dense in $\mathfrak{g}$. (Here is a proof. We must only check that if T is a maximal torus of G and $\mathfrak{t} = \mathrm{Lie}T$, then the set $\mathfrak{t}_{reg}$ of regular semisimple elements in $\mathfrak{t}$ is open dense in $\mathfrak{t}$. Let $Y = \mathrm{Hom}(\mathbf{k}^*, T)$. We have $\mathfrak{t} = \mathbf{k} \otimes Y$. Now $\mathfrak{t}_{reg}$ is the set of all $x \in \mathfrak{t}$ such that for any root $\alpha : \mathfrak{t} \to \mathbf{k}$ we have $\alpha(x) \neq 0$. It is enough to show that any root $\alpha : \mathfrak{t} \to \mathbf{k}$ is $\neq 0$. We have $\alpha = 1 \otimes \alpha_0$ where $\alpha_0 : Y \to \mathbf{Z}$ is a well-defined homomorphism. It is enough to show that α_0 is surjective. This follows from the adjointness of G.) As in the group case it now follows that K' is an intersection cohomology complex on $\mathfrak{g}$ coming from a local system on $\mathfrak{g}_{reg}$. Moreover, $\mathbf{W}$ acts naturally on this local system and hence, by "analytic continuation", on K'. In particular, if $\mathcal{O} \in \mathcal{X}_{\mathfrak{g}}$ and $i \in \mathbf{Z}$, then $\mathbf{W}$ acts naturally on the i-th cohomology sheaf $\mathcal{H}^i K'|_{\mathcal{O}}$ of $K'|_{\mathcal{O}}$, an irreducible G-equivariant local system on $\mathcal{O}$; hence if $\mathcal{L}$ is an irreducible G-equivariant local system on $\mathcal{O}$, then $\mathbf{W}$ acts naturally on the $\bar{\mathbf{Q}}_l$-vector space $\mathrm{Hom}(\mathcal{L}, \mathcal{H}^i K'|_{\mathcal{O}})$. We denote this $\mathbf{W}$-module (with $i = d_{\mathcal{O}}$) by $V_{\mathcal{O},\mathcal{L}}$. As in [L4], [X1], $V_{\mathcal{O},\mathcal{L}}$ is either 0 or of the form $\bar{\mathbf{Q}}_l \otimes E$ where $E \in \mathrm{Irr}(\mathbf{W})$; moreover, any $E \in \mathrm{Irr}(\mathbf{W})$ arises in this way from a unique $(\mathcal{O}, \mathcal{L})$ and $E \mapsto (\mathcal{O}, \mathcal{L})$ is an injective map

$$S_{\mathfrak{g}} : \mathrm{Irr}(\mathbf{W}) \to \hat{\mathcal{X}}_{\mathfrak{g}}.$$

If G is not assumed to be adjoint, let G_{ad} be the adjoint group of G and let $\mathfrak{g}_{ad} = \mathrm{Lie}G_{ad}$. The obvious map $\pi : \mathfrak{g} \to \mathfrak{g}_{ad}$ induces a bijective morphism $\mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\mathfrak{g}_{ad}}$ and a bijection $\mathcal{X}_{\mathfrak{g}} \to \mathcal{X}_{\mathfrak{g}_{ad}}$. Now any G_{ad}-equivariant irreducible $\bar{\mathbf{Q}}_l$-local system on a G_{ad}-orbit in $\mathcal{N}_{\mathfrak{g}_{ad}}$ can be viewed as an irreducible G-equivariant $\bar{\mathbf{Q}}_l$-local system on the corresponding G-orbit in $\mathcal{N}_{\mathfrak{g}}$. This yields an injective map $\hat{\mathcal{X}}_{\mathfrak{g}_{ad}} \to \hat{\mathcal{X}}_{\mathfrak{g}}$. We define an injective map $S_{\mathfrak{g}} : \mathrm{Irr}(\mathbf{W}) \to \hat{\mathcal{X}}_{\mathfrak{g}}$ as the composition of the last map with $S_{\mathfrak{g}_{ad}}$.

2.3. For any $u \in \mathcal{U}_G$, let $\mathcal{B}_u = \{B \in \mathcal{B}; u \in B\}$ and let $\mathcal{O}$ be the G-orbit of u in $\mathcal{U}_G$. Note that $\mathcal{B}_u$ is a non-empty subvariety of $\mathcal{B}$ of dimension $d_{\mathcal{O}}/2$; see [S1]. Using this and the definition of S_G we see that $(\mathcal{O}, \bar{\mathbf{Q}}_l)$ is in the image of S_G. Hence there is a well-defined injective map $S'_G : \mathcal{X}_G \to \mathrm{Irr}(\mathbf{W})$ such that for any $\mathcal{O} \in \mathcal{X}_G$ we have $S'_G(\mathcal{O}) = E$ where $E \in \mathrm{Irr}(\mathbf{W})$ is given by $S_G(E) = (\mathcal{O}, \bar{\mathbf{Q}}_l)$.

Similarly, for any $x \in \mathcal{N}_{\mathfrak{g}}$ let $\mathcal{B}_x = \{B \in \mathcal{B}; x \in \mathrm{Lie}B\}$ and let $\mathcal{O}$ be the G-orbit of x in $\mathcal{N}_{\mathfrak{g}}$. Note that $\mathcal{B}_x$ is a non-empty subvariety of $\mathcal{B}$ of dimension $d_{\mathcal{O}}/2$; see [HS]. Using this and the definition of $S_{\mathfrak{g}}$ we see that $(\mathcal{O}, \bar{\mathbf{Q}}_l)$ is in the image of $S_{\mathfrak{g}}$. Hence there is a well-defined injective map $S'_{\mathfrak{g}} : \mathcal{X}_{\mathfrak{g}} \to \mathrm{Irr}(\mathbf{W})$ such that for any $\mathcal{O} \in \mathcal{X}_{\mathfrak{g}}$ we have $S'_{\mathfrak{g}}(\mathcal{O}) = E$ where $E \in \mathrm{Irr}(\mathbf{W})$ is given by $S_{\mathfrak{g}}(E) = (\mathcal{O}, \bar{\mathbf{Q}}_l)$.

The maps $S'_G, S'_{\mathfrak{g}}$ can be described directly as follows. For $i \in \mathbf{Z}$, we may identify $H^i(\mathcal{B})$ (l-adic cohomology) with the stalk of $\mathcal{H}^i K$ at $1 \in G$; hence the $\mathbf{W}$-action on K induces a $\mathbf{W}$-action on the vector space $H^i(\mathcal{B})$. If $\mathcal{O} \in \mathcal{X}_G$ and $u \in \mathcal{O}$, then the inclusion $\mathcal{B}_u \to \mathcal{B}$ induces a linear map $f_u : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_u)$ whose kernel is $\mathbf{W}$-stable; hence there is an induced action of $\mathbf{W}$ on the image I_u of f_u. The $\mathbf{W}$-module I_u is of the form $\bar{\mathbf{Q}}_l \otimes E$ for a well-defined $E \in \mathrm{Irr}(\mathbf{W})$. We have $S'_G(\mathcal{O}) = E$. Similarly, if $\mathcal{O} \in \mathcal{X}_{\mathfrak{g}}$ and $x \in \mathcal{O}$, then the inclusion $\mathcal{B}_x \to \mathcal{B}$ induces a linear map $\phi_x : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_x)$ whose kernel is $\mathbf{W}$-stable; hence there is an induced action of $\mathbf{W}$ on the image I_x of ϕ_x. The $\mathbf{W}$-module I_x is of the form $\bar{\mathbf{Q}}_l \otimes E$ for a well-defined $E \in \mathrm{Irr}(\mathbf{W})$. We have $S'_{\mathfrak{g}}(\mathcal{O}) = E$.

Let $\mathfrak{S}_G$ be the image of $S'_G : \mathcal{X}_G \to \mathrm{Irr}(\mathbf{W})$. Let $\mathfrak{S}_{\mathfrak{g}}$ be the image of $S'_{\mathfrak{g}} : \mathcal{X}_{\mathfrak{g}} \to \mathrm{Irr}(\mathbf{W})$.

2.4. Any automorphism $a : G \to G$ induces a Lie algebra automorphism $a' : \mathfrak{g} \to \mathfrak{g}$ and an automorphism $\underline{a}$ of $\mathbf{W}$ as a Coxeter group. Now a (resp. a') induces a permutation $\mathcal{O} \mapsto a(\mathcal{O})$ (resp. $\mathcal{O} \mapsto a'(\mathcal{O})$) of $\mathcal{X}_G$ (resp. $\mathcal{X}_{\mathfrak{g}}$) denoted again by a (resp. a'). Also $\underline{a}$ induces in an obvious way a permutation of $\mathrm{Irr}(W)$ denoted again by $\underline{a}$. From the definitions we see that $\underline{a}S'_G = S'_G a$, $\underline{a}S'_{\mathfrak{g}} = S'_{\mathfrak{g}} a'$.

Let $x \mapsto x^p$ be the p-th power map $\mathfrak{g} \to \mathfrak{g}$ (if $p > 1$) and the 0 map $\mathfrak{g} \to \mathfrak{g}$ (if $p = 1$). The r-th iteration of this map is denoted by $x \mapsto x^{p^r}$; this restricts to a map $\mathcal{N}_{\mathfrak{g}} \to \mathcal{N}_{\mathfrak{g}}$ which is 0 for $r \gg 0$. The following result answers questions of Serre [Se].

Proposition 2.5. *(a) Let $u \in \mathcal{U}_G$ and let $n \in \mathbf{Z}$ be such that $nn' = 1$ in $\mathbf{k}$ for some $n' \in \mathbf{Z}$. Then u^n and u are G-conjugate.*

(b) Let $x \in \mathcal{N}_{\mathfrak{g}}$ and let $x' = a_0 x + a_1 x^p + a_2 x^{p^2} + \dots$ where $a_0, a_1, a_2, \dots \in \mathbf{k}$, $a_0 \neq 0$ (so that $x' \in \mathcal{N}_{\mathfrak{g}}$). Then x', x are G-conjugate.

We prove (a). Let $\mathcal{O}$ be the G-orbit of u and let $\mathcal{O}'$ be the G-orbit of $u' := u^n$. Clearly, $\mathcal{B}_u \subset \mathcal{B}_{u'}$. Since u' is a power of u we have also $\mathcal{B}_{u'} \subset \mathcal{U}$; hence $\mathcal{B}_{u'} = \mathcal{B}_u$. From $\dim \mathcal{B}_u = \dim \mathcal{B}_{u'}$ we see that $d_{\mathcal{O}} = d_{\mathcal{O}'}$. The map $f_u : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_u)$ in 2.3 remains the same if u is replaced by u'. From the description of S'_G given in 2.3 we deduce that $S'_G(\mathcal{O}) = S'_G(\mathcal{O}')$. Since S'_G is injective we deduce that $\mathcal{O} = \mathcal{O}'$. This proves (a).

We prove (b). Let $\mathcal{O}$ be the G-orbit of x and let $\mathcal{O}'$ be the G-orbit of x'. Clearly, $\mathcal{B}_x \subset \mathcal{B}_{x'}$. Since $x = a'_0 x' + a'_1 x'^p + a'_2 x'^{p^2} + \dots$ with $a'_0, a'_1, a'_2, \dots \in \mathbf{k}$, $a'_0 = a_0^{-1}$, we have $\mathcal{B}_{x'} \subset \mathcal{B}_x$; hence $\mathcal{B}_{x'} = \mathcal{B}_x$. From $\dim \mathcal{B}_x = \dim \mathcal{B}_{x'}$ we see that $d_{\mathcal{O}} = d_{\mathcal{O}'}$. The map $\phi_x : H^{d_{\mathcal{O}}}(\mathcal{B}) \to H^{d_{\mathcal{O}}}(\mathcal{B}_x)$ in 2.3 remains the same if x is replaced by x'. From the description of S'_G given in 2.3 we deduce that $S'_{\mathfrak{g}}(\mathcal{O}) = S'_{\mathfrak{g}}(\mathcal{O}')$. Since $S'_{\mathfrak{g}}$ is injective we deduce that $\mathcal{O} = \mathcal{O}'$. This proves (b).

Parts (a), (b) of the following result answer questions of Serre [Se]; the proof of (b) below (assuming (a)) is due to Serre [Se].

Proposition 2.6. *Let $c: G \to G$ be an automorphism such that for some maximal torus T of G we have $c(t) = t^{-1}$ for all $t \in T$. Let $\tilde{c}: \mathfrak{g} \to \mathfrak{g}$ be the automorphism of $\mathfrak{g}$ induced by c.*

(a) For any $u \in \mathcal{U}_G$, $c(u), u$ are G-conjugate.

(b) For any $g \in G$, $c(g), g^{-1}$ are G-conjugate.

(c) For any $x \in \mathcal{N}_{\mathfrak{g}}$, $\tilde{c}(x), x$ are G-conjugate.

(d) For any $x \in \mathfrak{g}$, $\tilde{c}(x), -x$ are G-conjugate.

We prove (a). Let $\underline{c}: \mathbf{W} \to \mathbf{W}$ be the automorphism induced by c. If $B \in \mathcal{B}$ contains T, then $T \subset c(B)$ and $B, c(B)$ are in relative position w_0, the longest element of $\mathbf{W}$. Hence if B, B' in $\mathcal{B}$ contain T and are in relative position $w \in \mathbf{W}$, then $c(B), c(B')$ contain T and are in relative position $w_0 w w_0^{-1}$. They are also in relative position $\underline{c}(w)$. It follows that $\underline{c}(w) = w_0 w w_0^{-1}$ for all $w \in \mathbf{W}$. Hence the induced permutation $\underline{c}: \mathrm{Irr}(\mathbf{W}) \to \mathrm{Irr}(\mathbf{W})$ is the identity map. Let $\mathcal{O}$ be the G-orbit of $u \in \mathcal{U}_G$. Then $c(\mathcal{O})$ is the G-orbit of $c(u)$. By 2.4 we have $S'_G(c(\mathcal{O})) = \underline{c}(S'_G(\mathcal{O})) = S'_G(\mathcal{O})$. Since S'_G is injective it follows that $\mathcal{O} = c(\mathcal{O})$. This proves (a).

Following [Se], we prove (b) by induction on $\dim(G)$. If $\dim G = 0$ the result is trivial. Now assume that $\dim G > 0$. Write $g = su = us$ with s semisimple, u unipotent. If the result holds for $g_1 \in G$, then it holds for any G-conjugate of g_1. Hence by replacing g by a conjugate we can assume that $s \in T$ so that $c(s) = s^{-1}$. Let $Z(s)^0$ be the connected centralizer of s, a connected reductive subgroup of G containing T. Note that c restricts to an automorphism of $Z(s)^0$ of the same type as $c: G \to G$. Moreover, we have $g \in Z(s)^0$. If $Z(s)^0 \neq G$, then by the induction hypothesis we see that $c(g), g^{-1}$ are conjugate under $Z(s)^0$; hence they are conjugate under G. If $Z(s)^0 = G$, then by (a), $c(u), u$ are conjugate in G. By 2.5(a), u, u^{-1} are conjugate in G. Hence $c(u), u^{-1}$ are conjugate in G. In other words, for some $h \in G$ we have $c(u) = hu^{-1}h$. Since s is central in G and $c(s) = s^{-1}$ we have $c(s) = hs^{-1}h^{-1}$. It follows that $c(g) = c(s)c(u) = hs^{-1}h^{-1}hu^{-1}h = hs^{-1}u^{-1}h^{-1} = hg^{-1}h^{-1}$. This proves (b).

The proof of (c) is completely similar to that of (a); it uses $S'_{\mathfrak{g}}$ instead of S_G. The proof of (d) is completely similar to that of (b); it uses (c) and 2.5(b) instead of (b) and 2.5(a).

3. A parametrization of the set of nilpotent G-orbits in $\mathfrak{g}$

3.1. Let V be a finite dimensional $\mathbf{Q}$-vector space. Let $R \subset V^* = \mathrm{Hom}(V, \mathbf{Q})$ be a (reduced) root system and let $W \subset GL(V)$ be the Weyl group of R. Let Π be a set of simple roots for R. Let $\Theta = \{\beta \in R; \beta - \alpha \notin R \text{ for all } \alpha \in \Pi\}$. For any integer $r \geq 1$ let $\mathcal{A}_r$ (resp. $\mathcal{A}'_r$) be the set of all $J \subset \Theta$ such that J is linearly independent in V^* and $\sum_{\alpha \in \Pi} \mathbf{Z}\alpha / \sum_{\beta \in J} \mathbf{Z}\beta$ is finite of order r^k for some $k \in \mathbf{N}$ (resp. $k \in \mathbf{Z}_{>0}$). For $J \in \mathcal{A}_r$ let W_J be the subgroup of W generated by the reflections with respect to the roots in J. For any $E \in \mathrm{Irr}(W)$ let b_E be the smallest integer ≥ 0 such that E appears with multiplicity $m_E > 0$ in the b_E-th symmetric power of V regarded as a W-module. Let $\mathrm{Irr}(W)^\dagger = \{E \in \mathrm{Irr}(W); m_E = 1\}$. Replacing here (V, W) by (V, W_J) with $J \in \mathcal{A}_r$ we see that b_E is defined for any $E \in \mathrm{Irr}(W_J)$ and that $\mathrm{Irr}(W_J)^\dagger$ is defined. For $J \in \mathcal{A}_r$ and $E \in \mathrm{Irr}(W_J)^\dagger$ there is a unique $\tilde{E} \in \mathrm{Irr}(W)$ such that $\tilde{E}$ appears with multiplicity 1 in $\mathrm{Ind}_{W_J}^W E$ and $b_E = b_{\tilde{E}}$; moreover, we have $\tilde{E} \in \mathrm{Irr}(W)^\dagger$. We set $\tilde{E} = j_{W_J}^W E$. Define $\mathcal{S}_W^1 \subset \mathrm{Irr}(W)^\dagger$ as in [L5, 1.3].

Replacing (V, W) by (V, W_J) with $J \in \mathcal{A}_r$ we obtain a subset $\mathcal{S}^1_{W_J} \subset \mathrm{Irr}(W_J)^\dagger$. For any integer $r \geq 1$ let $\mathcal{S}^r_W$ be the set of all $E \in \mathrm{Irr}(W)$ such that $E = j^W_{W_J} E_1$ for some $J \in \mathcal{A}_r$ and some $E_1 \in \mathcal{S}^1(W_J)$ (see [L5, 1.3]). If $r = 1$ this agrees with the earlier definition of $\mathcal{S}^1_W$ since in this case $W_J = W$ for any $J \in \mathcal{A}'_r$. For any integer $r \geq 1$ we define a subset $\mathcal{T}^r_W$ of $\mathrm{Irr}(W)^\dagger$ by induction on $|W|$ as follows. If $W = \{1\}$, we set $\mathcal{T}^r_W = \mathrm{Irr}(W)$. If $W \neq \{1\}$, then $\mathcal{T}^r_W$ is the set of all $E \in \mathrm{Irr}(W)$ such that either $E \in \mathcal{S}^1_W$ or $E = j^W_{W_J} E_1$ for some $J \in \mathcal{A}'_r$ and some $E_1 \in \mathcal{T}^r(W_J)$. From the definition it is clear that

$$\mathcal{S}^1_W \subset \mathcal{S}^r_W \subset \mathcal{T}^r_W.$$

When $r = 1$ we have $\mathcal{S}^1_W = \mathcal{T}^r_W$.

We apply these definitions in the case where $r = p$, $V = \mathbf{Q} \otimes \mathbf{Y}_G$ (with $\mathbf{T}$ being "the maximal torus" of G and $\mathbf{Y}_G = \mathrm{Hom}(\mathbf{k}^*, \mathbf{T})$), R is "the root system" of G (a subset of V^*) with its canonical set of simple roots and $W = \mathbf{W}$ viewed as a subgroup of $GL(V)$. Then the subsets $\mathcal{S}^1_{\mathbf{W}} \subset \mathcal{S}^p_{\mathbf{W}} \subset \mathcal{T}^p_{\mathbf{W}}$ of $\mathrm{Irr}(\mathbf{W})$ are defined. We can now state the following result.

Proposition 3.2. *(a) We have* $\mathfrak{S}_G = \mathcal{S}^p_{\mathbf{W}}$.
(b) We have $\mathfrak{S}_{\mathfrak{g}} = \mathcal{T}^p_{\mathbf{W}}$.

For (a) see [L5, 1.4]. The proof of (b) is given in 3.5.

Corollary 3.3. *There is a unique (injective) map* $\tau : \mathcal{X}_G \to \mathcal{X}_{\mathfrak{g}}$ *such that* $S'_G(\xi) = S'_{\mathfrak{g}}(\tau(\xi))$ *for all* $\xi \in \mathcal{X}_G$.

The existence and uniqueness of τ follows from $\mathfrak{S}_G \subset \mathfrak{S}_{\mathfrak{g}}$ which in turn follows from 3.2 and the inclusion $\mathcal{S}^p_{\mathbf{W}} \subset \mathcal{T}^p_{\mathbf{W}}$.

It is known that when $p \neq 2$ we have $\mathrm{card}\mathfrak{S}_G = \mathrm{card}\mathfrak{S}_{\mathfrak{g}}$; hence in this case τ is a bijection.

3.4. For $n \in \mathbf{N}$ let W_n be the group of all permutations of the set

$$\{1, 2, \dots, n, n', \dots, 2', 1'\}$$

which commute with the involution $i \mapsto i'$, $i' \mapsto i$; let W'_n be the subgroup of W_n consisting of the even permutations. Assume that $k \in \mathbf{N}$ is large relative to n. When G is adjoint simple of type B_n or C_n ($n \geq 2$) we identify $\mathbf{W} = W_n$ in the standard way; we have a bijection $[a_*, a'_*] \leftrightarrow (a_*, a'_*)$, $\mathrm{Irr}(\mathbf{W}) = \mathrm{Irr}(W_n) \leftrightarrow \mathcal{C}^n_k$ as in [L1, 2.3]; moreover, $\mathrm{Irr}(\mathbf{W}) = \mathrm{Irr}(\mathbf{W})^\dagger$; see [L1, 2.4]. When G is adjoint simple of type D_n ($n \geq 4$) we identify $\mathbf{W} = W'_n$ in the standard way; we have a surjective map $\zeta : \mathrm{Irr}(\mathbf{W})^\dagger = \mathrm{Irr}(W'_n)^\dagger \to \mathcal{D}^k_n$ such that for any $\rho \in \mathrm{Irr}(W'_n)$ we have $\zeta(\rho) = (a_*, a'_*)$ where $(a_*, a'_*) \in \mathcal{D}^k_n$ is such that ρ appears in the restriction of $[a_*, a'_*]$ from W_n to W'_n (the set $\mathrm{Irr}(W'_n)^\dagger$ is determined by [L1, 2.5]); note that $|\zeta^{-1}(a_*, a'_*)|$ is 2 if $a_* = a'_*$ and is 1 otherwise.

3.5. In this subsection we prove 3.2(b). We can assume that G is adjoint, simple. If $p = 1$ or p is a good prime for G, then $\mathfrak{S}_{\mathfrak{g}} = \mathfrak{S}_G$ hence using 3.2(a) we have $\mathfrak{S}_{\mathfrak{g}} = \mathcal{S}^p_{\mathbf{W}}$; in our case we have $\mathbf{W}_J = \mathbf{W}$ for any $J \in \mathcal{A}_p$ hence from the definitions we have $\mathcal{S}^p_{\mathbf{W}} = \mathcal{S}^1_{\mathbf{W}} = \mathcal{T}^p_{\mathbf{W}}$ and the result follows. In the rest of this subsection we assume that p is a bad prime for G. In this case $\mathfrak{S}_{\mathfrak{g}}$ has been described explicitly by Spaltenstein [S2], [S3], [HS] as follows (assuming that the theory of Springer correspondence holds; this assumption can be removed in view of [X1], [X2] and the remarks in 2.2.)

If G is of type C_n, $n \geq 2$ ($p = 2$), then we have $\mathfrak{S}_\mathfrak{g} = \mathrm{Irr}(\mathbf{W})$. If G is of type B_n, $n \geq 2$ ($p = 2$), then, according to [S1], $\mathfrak{S}_\mathfrak{g} = \{[a_*, a'_*] \in \mathrm{Irr}(\mathbf{W}); (a_*, a'_*) \in {}^b\mathcal{C}^n_k\}$. (Here k is large and fixed.) If G is of type D_n, $n \geq 4$ ($p = 2$), then $\mathfrak{S}_\mathfrak{g} = \zeta^{-1}({}^d\mathcal{D}^n_k)$. If G is of type G_2 ($p = 2$ or 3), of type F_4 ($p = 3$), of type E_6 ($p = 2$ or 3), of type E_7 ($p = 3$), or of type E_8 ($p = 3$ or 5), then $\mathfrak{S}_\mathfrak{g} = \mathfrak{S}_G$. If G is of type F_4 ($p = 2$), then $\mathfrak{S}_\mathfrak{g} = \mathfrak{S}_G \sqcup \{1_3, 2_3\}$ (notation as in [L3, 4.10]); note that $b_{1_3} = 12$, $b_{2_3} = 4$). If G is of type E_7 ($p = 2$), then $\mathfrak{S}_\mathfrak{g} = \mathfrak{S}_G \sqcup \{84'_a\}$ (notation as in [L3, 4.12]; we have $b_{84'_a} = 15$). If G is of type E_8 ($p = 2$), then $\mathfrak{S}_\mathfrak{g} = \mathfrak{S}_G \sqcup \{50_x, 700_{xx}\}$ (notation as in [L3, 4.13]; we have $b_{50_x} = 8$, $b_{700_{xx}} = 16$).

On the other hand, for types B, C, D, $\mathcal{T}^2_\mathbf{W}$ is computed by induction using 1.2, the formulas for the maps $j^W_{W_J}()$ given in [L6, 4.5, 5.3, 6.3] and the known description of $\mathcal{S}^1_\mathbf{W}$; for exceptional types, $\mathcal{T}^p_\mathbf{W}$ is computed by induction using the tables in [A] and the known description of $\mathcal{S}^1_\mathbf{W}$.

In each case, the explicitly described subset $\mathfrak{S}_\mathfrak{g}$ of $\mathrm{Irr}(\mathbf{W})$ coincides with the explicitly described subset $\mathcal{T}^p_\mathbf{W}$. This completes the proof of 3.2(b).

To illustrate the inclusion $\mathfrak{S}_\mathfrak{g} \subset \mathcal{T}^p_\mathbf{W}$ we note that:

if G is of type E_8 ($p = 2$) then 50_x, 700_{xx} in $\mathfrak{S}_\mathfrak{g} - \mathfrak{S}_G$ are obtained by applying $j^\mathbf{W}_{\mathbf{W}_J}$ (where $\mathbf{W}_J$ is of type $E_7 \times A_1$) to $15'_a \boxtimes \mathrm{sgn}$, $84'_a \boxtimes \mathrm{sgn}$ (which belong to $\mathcal{T}^2_{\mathbf{W}_J} - \mathcal{S}^2_{\mathbf{W}_J}$, $\mathcal{S}^2_{\mathbf{W}_J} - \mathcal{S}^1_{\mathbf{W}_J}$ respectively);

if G is of type F_4 ($p = 2$) then $1_3, 2_3$ in $\mathfrak{S}_\mathfrak{g} - \mathfrak{S}_G$ are obtained by applying $j^\mathbf{W}_{\mathbf{W}_J}$ (where $\mathbf{W}_J$ is of type $B_4, C_3 \times A_1$) to an object in $\mathcal{S}^2_{\mathbf{W}_J} - \mathcal{S}^1_{\mathbf{W}_J}$.

3.6. If G is of type B_n or C_n, $n \geq 2$ ($p = 2$), then, according to [LS], $\mathfrak{S}_G = \{[a_*, a'_*] \in \mathrm{Irr}(\mathbf{W}); (a_*, a'_*) \in {}^{b2}\mathcal{C}^n_k\}$. (Here k is large and fixed.) If G is of type D_n, $n \geq 4$ ($p = 2$), then according to [LS], $\mathfrak{S}_G = \zeta^{-1}({}^{d2}\mathcal{D}^n_k)$.

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