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ASYMPTOTIC K-SUPPORT AND RESTRICTIONS OF REPRESENTATIONS

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Abstract. The restriction, from a compact Lie group K to a closed subgroup, of a polynomially bounded representation remains polynomially bounded provided a geometric assumption on the asymptotic K-support of the representation is satisfied. This is a theorem of T. Kobayashi. We give a proof of this theorem using microlocal analysis in the setting of distribution rather than hyperfunction theory. The proof is based on a characterization, up to the natural $K \times K$ action, of the wavefront set of a distribution on K in terms of the asymptotic behavior of its Fourier coefficients.

1. Introduction

In the late 1990s T. Kobayashi wrote a series of papers in which he established a criterion for the discrete decomposablity of restrictions of unitary representations of reductive Lie groups to reductive subgroups. A key tool in the proof of sufficiency of his criterion was the use of the theory of hyperfunctions to study the microlocal behavior of characters of restrictions to compact subgroups; see [6]. In this paper we show how to replace this tool by microlocal analysis in the C^∞ category.

In the following, K denotes a connected, compact Lie group with Lie algebra $\mathfrak{k}$. We fix a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$ and an associated positive system. By $\bar{C} \subset i\mathfrak{t}^*$, $i = \sqrt{-1}$, we denote the closure of the (dual) Weyl chamber. We identify equivalence classes of irreducible representations with their highest weights. Thus we write $\hat{K} = \Lambda \cap \bar{C}$, where Λ denotes the weight lattice in $i\mathfrak{t}^*$. We also assume an Ad-invariant inner product on $\mathfrak{k}$, extended to an Ad-invariant hermitian inner product on the complexification $\mathfrak{k}_{\mathbb{C}}$. We denote the norm of $\lambda \in \mathfrak{k}_{\mathbb{C}}^*$ by $|\lambda|$. Using the inner products we identify $\mathfrak{t}^*$ and $\mathfrak{t}_{\mathbb{C}}^*$ with subsets of $\mathfrak{k}^*$ and of $\mathfrak{k}_{\mathbb{C}}^*$, respectively.

The Fourier series $u = \sum_{\lambda \in \hat{K}} u_\lambda$ of any square integrable function u converges in $L^2(K)$. A (formal) Fourier series $\sum_{\lambda \in \hat{K}} u_\lambda$ converges to a distribution $u \in C^{-\infty}(K)$ iff the L^2 norms $\|u_\lambda\|$ of the Fourier coefficients are polynomially bounded as functions of $\lambda \in \hat{K}$. Smooth functions, $u \in C^\infty(K)$, are characterized by the rapid decrease of their Fourier coefficients, $\|u_\lambda\| = \mathrm{O}(|\lambda|^{-\infty})$ as $\lambda \to \infty$. We shall define, for every distribution u, a closed cone $\operatorname{afsupp}(u) \subset \bar{C} \setminus 0$, the asymptotic Fourier support of u. Essentially this is the smallest cone outside of which the Fourier coefficients decrease rapidly. The asymptotic Fourier support is empty for C^∞ functions.

The wavefront set is a fundamental notion in the microlocal analysis of distributions. Given a closed cone $\Gamma \subset T^*K \setminus 0$ one defines the space $C_\Gamma^{-\infty}(K)$ which

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consists of all $u \in C^{-\infty}(K)$ having their wavefront sets contained in Γ, $\mathrm{WF}\, u \subset \Gamma$. Under appropriate geometric conditions on Γ some operations can be extended by continuity to $C_\Gamma^{-\infty}(K)$. The wavefront set was used by Howe [4] in a related setting.

The group $K \times K$ acts on the cotangent bundle T^*K via left and right translations.

Theorem 1. *Let $u \in C^{-\infty}(K)$. Then*

$$(K \times K) \cdot \mathrm{WF}(u) = (K \times K) \cdot i^{-1} \operatorname{afsupp}(u). \tag{1}$$

The Fourier series of u converges in $C^{-\infty}_{(K\times K)\cdot i^{-1} \operatorname{afsupp}(u)}(K)$.

Kashiwara and Vergne [5, 4.5] proved the first assertion in the hyperfunction setting and noticed the C^∞ analogue in a remark. The importance of the second assertion is that it implies for subgroups satisfying geometric assumptions that restriction commutes with Fourier series.

A representation π of K in a Hilbert space is said to be polynomially bounded if the K-multiplicity $m_K(\lambda : \pi) = \dim \mathrm{Hom}_K(\lambda, \pi)$ of λ in π is polynomially bounded as a function of $\lambda \in \hat{K}$. In particular, the multiplicities are then finite. The asymptotic K-support of π is a closed cone $\mathrm{AS}_K(\pi) \subset \bar{C} \setminus 0$ which approximates the support of $m_K(\cdot : \pi)$ as $\lambda \to \infty$. (See [6, (2.7.1)].)

Theorem 2 ([6]). *Let M be a closed subgroup of K. Denote its Lie algebra by $\mathfrak{m}$, and by $\mathfrak{m}^\perp \subset \mathfrak{k}^*$ the space of conormals. Let π be a unitary representation of K which is polynomially bounded and which satisfies*

$$\mathrm{AS}_K(\pi) \cap i \operatorname{Ad}^*(K)\mathfrak{m}^\perp = \emptyset. \tag{2}$$

Then the restriction $\pi|_M$ of π to M is a polynomially bounded representation of M. The asymptotic M-support $\mathrm{AS}_M(\pi|_M)$ is contained in the image of $\operatorname{Ad}^*(K)\, \mathrm{AS}_K(\pi)$ under the canonical projection $i\mathfrak{k}^* \to i\mathfrak{m}^*$.

It is known that the restriction of an irreducible unitary representation π of a real reductive Lie group G to a maximal compact subgroup is polynomially bounded. For closed subgroups $G' \subset G$, which are stable under the Cartan involution, a criterion on G'-admissability of $\pi|_{G'}$ is given in [6, Theorem 2.9]. Theorem 2 contains all the microlocal information needed to rewrite the proof of [6, Theorem 2.9] without having to invoke the theory of hyperfunctions. Thus we offer an alternative approach to Kobayashi's theorem for readers without a strong background in hyperfunction theory.

The proof of Theorem 2 is centered around the notion of the K-character of π, $\sum_{\lambda \in \hat{K}} m_K(\lambda : \pi) \operatorname{tr} \lambda$. The asumptions imply that this series converges in $C^{-\infty}(K)$, and that the K-character possesses a restriction to M which turns out to be the M-character of $\pi|_M$. Theorem 1 is used to prove this. The continuity statement given in Theorem 1 simplifies the proof Theorem 2 when compared with the original argument.

The paper is organized as follows. In Section 2 we recall the expansion in eigenfunctions of a positive elliptic operator and its application to Fourier series on K. The asymptotic Fourier support is defined in this section. In Section 3 we study, for central distributions, wavefront sets and the convergence of Fourier series. The theorems are proved in Sections 4 and 5.

2. Asymptotic Fourier support

The space $C^{-\infty}(K)$ of distributions on K is, by definition, the dual space of $C^{\infty}(K)$. Functions are identified with distributions, $L^2(K) \subset C^{-\infty}(K)$, using the normalized Haar measure dk on K. The L^2 scalar product $(\cdot|\cdot)$ extends to an anti-duality between $C^{-\infty}(K)$ and $C^{\infty}(K)$. We recall how the theory of Fourier series of distributions and of smooth functions follows from results on eigenfunction expansions of elliptic selfadjoint differential operators.

The Sobolev space $H^m(K)$ consists of all distributions which are mapped into $L^2(K)$ by differential operators with order $\leq m$. We assume differential operators to be linear with C^{∞} coefficients. $H^m(K)$ is equipped with a norm, making it a Banach space. Let A be a second order, elliptic differential operator. Regard A as an unbounded operator on $L^2(K)$ with domain $D(A) = H^2(K)$. Its Hilbert space adjoint A^* has, by elliptic regularity theory, the domain $D(A^*) = H^2(K)$. Assume, in addition, that $(Au|u) > 0$ if $0 \neq u \in D(A)$. Then A is positive selfadjoint. The eigenfunctions of A are in $C^{\infty}(K)$.

Proposition 3 ([7, §10])**.** *Let A be a positive selfadjoint second order elliptic differential operator on K. Let $Au = \sum_j \mu_j^2 (u|e_j)e_j$ denote its spectral resolution where $(e_j) \subset L^2(K)$ is an orthonormal basis of eigenfunctions and $0 < \mu_j \uparrow \infty$ the corresponding sequence of eigenvalues of $\sqrt{A}$. A series $\sum_j \alpha_j e_j$ converges in $C^{\infty}(K)$ iff $\sum_j \mu_j^{2N} |\alpha_j|^2 < \infty$ for all $N \in \mathbb{N}$. It converges in $C^{-\infty}(K)$ iff $\sum_j \mu_j^{-2N}|\alpha_j|^2 < \infty$ for some $N \in \mathbb{N}$. The coefficients are $\alpha_j = (u|e_j)$ if $u \in C^{-\infty}(K)$ denotes the sum of the series.*

Proof. The domain $D(A^k)$ of the k-th power of A consists of all u such that $\sum_j \mu_j^{2k} |(u|e_j)|^2 < \infty$. We equip $D(A^k)$ with the corresponding norm. The norm is equivalent to the graph norm. Hence $D(A^k)$ is a Banach space. Obviously, $H^{2k}(K) \subset D(A^k)$. By elliptic regularity we have equality $D(A^k) = H^{2k}(K)$. This holds also topologically because of Banach's theorem. By the Sobolev lemma, $C^{\infty}(K) = \bigcap_k H^{2k}(K)$ as a projective limit. Hence the norms on $D(A^k)$ define the Fréchet space topology of $C^{\infty}(K)$. The asserted convergence criterion for $C^{\infty}(K)$ follows from this. Using duality between weighted ℓ^2 sequence spaces we obtain the convergence criterion for $C^{-\infty}(K)$. Finally, the formula for the coefficients follows from the (separate) continuity of the anti-duality bracket. □

The ℓ^2 estimates in Proposition 3 can be replaced by supremum estimates because $\sum_j \mu_j^{-k} < \infty$ for some $k \in \mathbb{N}$. The latter property holds because $A^{-k/2}$ is of trace class if $k > \dim K$.

Denote by d_λ, $\chi_\lambda = \operatorname{tr} \lambda$, and $M_\lambda \subset L^2(K)$ the dimension, the character and the space of matrix coefficients of $\lambda \in \hat{K}$. The convolution with $d_\lambda \chi_\lambda$ is the orthoprojector from $L^2(K)$ onto M_λ. If $u \in L^2(K)$, then its Fourier series $\sum_{\lambda \in \hat{K}} u_\lambda$, $u_\lambda = d_\lambda u * \chi_\lambda$, converges to u in $L^2(K)$ by the Peter-Weyl theorem. The (formal) Fourier series of a distribution $u \in C^{-\infty}(K)$ is defined by the same formula using the convolution of a distribution with a C^{∞} function, i.e., $(u * \psi)(x) = \int_K u(y)\psi(y^{-1}x)\, dy$ for $\psi \in C^{\infty}(K)$ with the integral representing the duality bracket. Observe that $\chi_\lambda, u * \chi_\lambda \in M_\lambda \subset C^{\infty}(K)$. In general, we call a series $\sum_{\lambda \in \hat{K}} u_\lambda$ with $u_\lambda \in M_\lambda$ a Fourier series with coefficients u_λ.

We use left translation, $L_x(k) = xk$, to trivialize the tangent bundle $TK = K \times \mathfrak{k}$ and the cotangent bundle $T^*K = K \times \mathfrak{k}^*$. Under this identication left translation is the identity on the second components. Right translation $R_x(y) = yx$ acts, on the second components, as the adjoint action, $dR_{x^{-1}} : X \mapsto \operatorname{Ad}(x)X$, and as the co-adjoint action, ${}^t dR_{x^{-1}} : \xi \mapsto \operatorname{Ad}^*(x)\xi$. Bi-invariant subsets of T^*K are of the form $K \times \operatorname{Ad}^*(K)S$ for some $S \subset \mathfrak{k}^*$. Then formula (1) reads

$$(K \times K) \cdot \operatorname{WF}(u) = K \times i^{-1} \operatorname{Ad}^*(K) \operatorname{afsupp}(u).$$

Elements $X \in U(\mathfrak{k}_{\mathbb{C}})$ of the universal enveloping algebra act as left invariant differential operators $\widetilde{X}$ on $C^{-\infty}(K)$. The principal symbol of the first order differential operator $\widetilde{X}$ associated with $X \in \mathfrak{k}_{\mathbb{C}}$ is $\sigma_1(\widetilde{X})(x, \xi) = \langle \xi, X \rangle$. Denote the Ad-invariant hermitian inner product on $\mathfrak{k}_{\mathbb{C}}$ by Q. We assume that Q equals the negative Killing form on $[\mathfrak{k}, \mathfrak{k}]$ and that the center of the Lie algebra is orthogonal to $[\mathfrak{k}, \mathfrak{k}]$. Choose, consistent with this orthogonal decomposition, an orthonormal basis $\{X_j\}$ of $\mathfrak{k}$. Define the second order differential operator $A = 1 - \sum_j \widetilde{X}_j^2$. The principal symbol of A is $\sigma_2(A)(x, \xi) = Q^*(\xi)$, where Q^* is the dual form of Q. Hence A is elliptic. It follows from the left invariance of $\widetilde{X}$ and the invariance of Haar measure that $\int_K \widetilde{X} v(y)\, dy = 0$ for all $v \in C^\infty(K)$, $X \in \mathfrak{k}_{\mathbb{C}}$. Therefore, A is positive selfadjoint with domain $H^2(K)$. Furthermore, A is bi-invariant. Therefore, each M_λ, $\lambda \in \hat{K}$, is contained in an eigenspace of A with an eigenvalue $\mu = \mu(\lambda)$. There exists a constant $C > 0$ such that

$$1 + |\lambda + \rho|^2 - |\rho|^2 \leq \mu \leq C(1 + |\lambda|)^2 \quad \text{for all } \lambda \in \hat{K}.$$

Here ρ is the half sum of positive roots. The left inequality holds because $A - 1$ is the sum of a non-negative operator B and the Casimir operator. It is well known that the Casimir operator contains M_λ in its eigenspace with eigenvalue $|\lambda + \rho|^2 - |\rho|^2$. Since B is a sum of $-\widetilde{X}^2$, $X \in \mathfrak{t}$, the right inequality follows from $(-\widetilde{X}^2 u|u) = \|\widetilde{X} u\|^2 = \|\langle \lambda, X \rangle u\|^2$, which holds for any highest weight vector $u \in M_\lambda$.

Summarizing we have the following.

Corollary 4. *A Fourier series $\sum_{\lambda \in \hat{K}} u_\lambda$ converges in $C^\infty(K)$, resp. in $C^{-\infty}(K)$, iff*

$$\sum_{\lambda \in \hat{K}} (1 + |\lambda|)^{2N} \|u_\lambda\|^2 < \infty$$

for all, resp. for some, $N \in \mathbb{Z}$. If $u \in C^{-\infty}(K)$, then its Fourier series $\sum_{\lambda \in \hat{K}} u_\lambda$, $u_\lambda = d_\lambda u * \chi_\lambda$, converges to u in $C^{-\infty}(K)$.

Smoothness properties of a distribution correspond to decaying properties of its Fourier coefficients. We define an approximating cone to the directions of those $\lambda \in \hat{K} \subset \bar{C}$ such that the Fourier coefficients u_λ do not decay rapidly as $\lambda \to \infty$. A subset of a (finite dimensional) real vector space V (or of a vector bundle) is called conic or a *cone* iff it is invariant under multiplication with positive reals.

Let $u \in C^{-\infty}(K)$ and let $\sum_{\lambda \in \hat{K}} u_\lambda$ be its Fourier series. The *asymptotic Fourier support* of u is the closed cone $\operatorname{afsupp}(u) \subset \bar{C} \setminus 0$ which is defined as follows. A point $\mu \in \bar{C} \setminus 0$ is in the complement of $\operatorname{afsupp}(u)$ iff there is a conic neighbourhood $S \subset \bar{C} \setminus 0$ of μ such that

$$\sum_{\lambda \in S \cap \hat{K}} |\lambda|^{2N} \|u_\lambda\|^2 < \infty, \quad \forall N \in \mathbb{N}.$$

By Corollary 4, $u \in C^\infty(K)$ iff $\operatorname{afsupp}(u) = \emptyset$. More generally, if $S \subset \bar{C} \setminus 0$ is a closed cone which is disjoint from $\operatorname{afsupp}(u)$, then the Fourier series $\sum_{\lambda \in S \cap \hat{K}} u_\lambda$ converges in $C^\infty(K)$.

Remark 5. Instead of working with ℓ^2-estimates we can work with supremum estimates such as $\sup_{\lambda \in S \cap \hat{K}} |\lambda|^N \|u_\lambda\| < \infty$. This follows from the observation made after the proof of Proposition 3.

With a subset $S \subset V$ one associates the closed cone $S_\infty \subset V \setminus 0$ as follows. A point is in the complement of S_∞ if it has a conic neighbourhood which intersects S in a relatively compact set. Equivalently, $v \in S_\infty$ iff there exist sequences $(v_j) \subset S$ and $\varepsilon_j \downarrow 0$ such that $\lim_j \varepsilon_j v_j = v$. The cone S_∞ approximates S at infinity.

The K-support $\operatorname{supp}_K(\pi)$ of a representation π of K in a Hilbert space is the set of all $\lambda \in \hat{K} \subset \bar{C}$ such that λ occurs in π, i.e., $m_K(\lambda : \pi) > 0$. The set $\operatorname{AS}_K(\pi) = \operatorname{supp}_K(\pi)_\infty \subset \bar{C} \setminus 0$ is the *asymptotic K-support* of π.

3. Wavefront convergence of central Fourier series

The definition of the wavefront set of a distribution is based on the calculus of pseudodifferential operators. We collect, in our context, some definitions and results, referring to [2, Section 2.5], [1], and [3, Section 18.1] for details.

With every pseudodifferential operator $A \in \Psi^m(K)$ one associates its set of characteristic points, $\operatorname{Char} A \subset T^*K \setminus 0$. A point is non-characteristic if there is a symbol $b \in S^{-m}$ such that $ab - 1 \in S^{-1}$ in a conic neighbourhood of that point. Here $a \in S^m(T^*K)$ is, modulo $S^{m-1}(T^*K)$, a principal symbol of A. The operator is said to be elliptic at a non-characteristic point. An operator $A : C^\infty(K) \to C^{-\infty}(K)$ is a pseudodifferential operator iff its Schwartz kernel $K_A \in C^{-\infty}(K \times K)$ is a conormal distribution with respect to the diagonal. More explicitly, $A \in \Psi^m(K)$ iff the singular support of K_A is contained in the diagonal and K can be covered with open sets $U \subset K$ such that the kernel is given by an oscillatory integral

$$K_A(y', y) = \int e^{i\varphi(y',\eta) - i\varphi(y,\eta)} a(y', y, \eta)\, d\eta, \tag{3}$$

$y', y \in U$. The phase function $\varphi \in C^\infty(U \times \mathfrak{k})$ is real-valued, linear in the second variable, and non-degenerate, i.e., $\det \varphi''_{y\eta} \neq 0$. The amplitude a belongs to the symbol space $S^m(U \times U \times \mathfrak{k}^*)$. A is elliptic at $\xi = \varphi'_x(x, \zeta) \in T^*_x K \setminus 0$, $x \in U$, iff there is a neighbourhood $U_0 \subset U$ of x, a conic neighbourhood V of ζ, and $C > 0$ such that $|a(y, y, \eta)| \geq |\eta|^m / C$ for $y \in U_0$, $\eta \in V$, $|\eta| > C$.

Let $u \in C^{-\infty}(K)$. The wavefront set $\operatorname{WF} u \subset T^*K \setminus 0$ equals $\bigcap \operatorname{Char} A$, where the intersection is taken over all pseudodifferential operators A which satisfy $Au \in C^\infty(K)$. Let $\Gamma \subset T^*K \setminus 0$ be a closed cone. The space $C^{-\infty}_\Gamma(K)$ of distributions on K which have their wavefront sets contained in Γ is equipped with a locally convex topology. It contains $C^\infty(K)$ as a sequentially dense subspace. Convergence of a sequence, $u_j \to u$ in $C^{-\infty}_\Gamma(K)$, is equivalent to $u_j \to u$ (weakly) in $C^{-\infty}(K)$ and the existence, for every $(x, \xi) \in (T^*K \setminus 0) \setminus \Gamma$, of a pseudodifferential operator $A \in \Psi^m(K)$ such that $(x, \xi) \notin \operatorname{Char} A$, and $Au_j \to Au$ in $C^\infty(K)$. If $u_j \to u$ in $C^{-\infty}_\Gamma(K)$, then $Au_j \to Au$ in $C^\infty(K)$ for every $A \in \Psi^m(K)$ which satisfies $\operatorname{WF}(A) \cap \Gamma = \emptyset$. Here $\operatorname{WF}(A)$ is the smallest conic subset of $T^*K \setminus 0$ such that A is of order $-\infty$ in the complement. (See the remark following Theorem 18.1.28 of [3].)

Let K act on $C^\infty(K)$ via the right regular representation, $R_x f(y) = f(yx)$. The corresponding action of the Lie algebra $\mathfrak{k}_\mathbb{C}$ is by left invariant vector fields, $dR_e(X)f = \widetilde{X}f$.

The following lemma should be compared with [5, 3.1].

Lemma 6. *Let $\sum_{\lambda\in\hat{K}} u_\lambda$ be a Fourier series which converges in $C^{-\infty}(K)$. Assume that each u_λ is a highest weight vector for the right regular representation acting irreducibly on a subspace of M_λ. Let S be a closed cone $\subset \bar{C}\setminus 0$. Then $\sum_{\lambda\in S\cap\hat{K}} u_\lambda$ converges in $C^{-\infty}_{K\times i^{-1}S}(K)$.*

Proof. The differential equations $\widetilde{X}u_\lambda = 0$ and $\widetilde{X}u_\lambda = \langle\lambda, X\rangle u_\lambda$ hold for $X\in\mathfrak{n}$ and $X\in\mathfrak{t}$, respectively. Here $\mathfrak{n}\subset\mathfrak{k}_\mathbb{C}$ denotes the sum of positive root spaces.

Let $(x,\xi)\in K\times\mathfrak{k}^*\setminus 0$, $\xi\notin i^{-1}S$. It suffices to find a pseudodifferential operator A, elliptic at (x,ξ), such that the series $\sum_{\lambda\in S\cap\hat{K}} Au_\lambda$ converges in $C^\infty(K)$. If $\xi\notin\mathfrak{t}^*$, then there exists $X\in\mathfrak{n}$ with $\langle\xi, X\rangle\neq 0$; the first order differential operator $A=\widetilde{X}$ has the desired properties. Now assume $\xi\in\mathfrak{t}^*$. Then the cone $S-\mathbb{R}_+ i\xi$ is a closed subset of $i\mathfrak{t}^*\setminus 0$. It follows by a simple compactness argument that $|\lambda|+|\xi|\le C|\lambda - i\xi|$ with a constant $C>0$ independent of $\lambda\in S$. Assume that S is convex. Choose $X\in\mathfrak{t}$, which strictly separates the disjoint convex cones $-\mathbb{R}_+\xi$ and iS. We infer that there exists $c>0$ such that

$$\big|\langle\lambda - i\eta, X\rangle\big| > c(|\lambda|+|\eta|)\quad\text{for all }\lambda\in S,\ \eta\in\Gamma, \tag{4}$$

where $\Gamma=\mathbb{R}_+\xi$. By continuity, (4) also holds in a conic neighbourhood $\Gamma\subset\mathfrak{k}^*\setminus 0$ of ξ.

Let $U\subset K$ be an open neighbourhood of x and $H\subset U$ a hypersurface containing x such that the following holds. The real vector field $\widetilde{X}$ is transversal to H and every maximally extended integral curve of $\widetilde{X}$ in U hits H at a unique point. Furthermore, $y\mapsto \exp^{-1}(x^{-1}y)$ maps U diffeomorphically onto an open neighbourhood of the origin in $\mathfrak{k}$. Using the method of characteristics we solve, for every $\eta\in\mathfrak{k}^*$, the initial value problem

$$\widetilde{X}\varphi(\cdot,\eta) = \langle\eta, X\rangle \text{ in } U,\quad \varphi(y,\eta) = \langle\eta, \exp^{-1}(x^{-1}y)\rangle \text{ at } y\in H.$$

The solution $\varphi\in C^\infty(U\times\mathfrak{k}^*)$ is linear in the second variable and $\varphi'_x(x,\eta)=\eta$ holds in $T^*_xK=\mathfrak{k}^*$ for all η. In particular, $\varphi''_{y\eta}$ is non-degenerate at $y=x$. We have $\widetilde{X}e^{-i\varphi(\cdot,\eta)}u_\lambda = \langle\lambda - i\eta, X\rangle e^{-i\varphi(\cdot,\eta)}u_\lambda$. Since $\widetilde{X}$ is left invariant, $\int_K \widetilde{X}v(y)\,dy = 0$ holds for all $v\in C^\infty(K)$. Therefore, we can perform partial integration as follows:

$$\langle\lambda - i\eta, X\rangle\int_K e^{-i\varphi(y,\eta)}u_\lambda(y)\chi(y)\,dy = -\int_K e^{-i\varphi(y,\eta)}u_\lambda(y)\widetilde{X}\chi(y)\,dy$$

if $\chi\in C_c^\infty(U)$. Iterating N times and estimating the integral on the right using the Cauchy-Schwarz inequality we obtain

$$\Big|\langle\lambda - i\eta, X\rangle^N\int_K e^{-i\varphi(y,\eta)}u_\lambda(y)\chi(y)\,dy\Big| \le C_N\|u_\lambda\|$$

with a constant $C_N>0$ independent of $\lambda\in S\cap\hat{K}$ and $\eta\in\Gamma$. In view of (4) we get

$$\sup_{\eta\in\Gamma}|\eta|^N\Big|\int_K e^{-i\varphi(y,\eta)}\chi(y)u_\lambda(y)\,dy\Big| \le C_{2N}c^{-2N}|\lambda|^{-N}\|u_\lambda\|,$$

for all $\lambda \in S \cap \hat{K}$ and $N \in \mathbb{N}$. Since the L^2 norms of the Fourier coefficients are polynomially bounded we obtain, for every $\chi \in C_c^\infty(U)$,

$$\sum_{\lambda \in S \cap \hat{K}} \sup_{\eta \in \Gamma} |\eta|^N \Big| \int_K e^{-i\varphi(y,\eta)} \chi(y) u_\lambda(y)\, dy \Big| < \infty, \quad N \in \mathbb{N}. \tag{5}$$

We can assume that, making U and Γ smaller if necessary, $\det \varphi''_{y\eta} \neq 0$ in $U \times \mathfrak{k}^*$, and $\varphi'_y(U \times \Gamma) \cap i^{-1}S = \emptyset$. Fix $\chi \in C_c^\infty(U)$ with $\chi(x) = 1$. Choose a symbol $b \in S^0(\mathfrak{k}^*)$ with $\operatorname{supp} b \subset \Gamma$ and $b = 1$ in a conic neighbourhood of ξ minus a compact set. Define the pseudodifferential operator $A \in \Psi^0(K)$ with kernel K_A supported in $U \times U$ and given by (3) with amplitude $a(y', y, \eta) = \chi(y')b(\eta)\chi(y)$. It follows from (5) that $\sum_{\lambda \in S \cap \hat{K}} A u_\lambda$ converges in $C^\infty(K)$. Furthermore, A is elliptic at (x, ξ). Hence we have proved the assertion under the additional assumption that S is convex. To remove this assumption observe that S can be covered by finitely many closed convex cones each not containing $i\xi$. Decompose the Fourier series correspondingly. □

Pullback and pushforward of distributions is well defined and continuous under assumptions on the wavefront sets. One Associates with any C^∞ map $f : X \to Y$ map between smooth manifolds its canonical relation

$$C_f = \{(y, \eta; x, \xi) \,;\, y = f(x), \xi = {}^t f'(x)\eta\} \subset T^*Y \times T^*X.$$

For a closed cone $\Gamma \subset T^*Y \setminus 0$ define its pullback cone $f^*\Gamma = C_f^{-1} \circ \Gamma \subset T^*X$. If $f^*\Gamma$ does not intersect the zero section, then the pullback $f^*u = u \circ f$ extends from $C^\infty(X)$ to a (sequentially) continuous pullback operator $f^* : C_\Gamma^{-\infty}(Y) \to C_{f^*\Gamma}^{-\infty}(X)$. If f is a proper map, then the pushforward operator $f_* : C^{-\infty}(X) \to C^{-\infty}(Y)$ is defined by duality. If, in addition, f is a submersion and $\Gamma \subset T^*X \setminus 0$ is a closed cone, then $f_*\Gamma := C_f \circ \Gamma \subset T^*Y \setminus 0$ and the pushforward restricts to a (sequentially) continuous map $f_* : C_\Gamma^{-\infty}(X) \to C_{f_*\Gamma}^{-\infty}(Y)$.

An important example of a pullback operator is the restriction to a submanifold $M \subset K$. It is defined on distributions having wavefront sets disjoint from the conormal bundle of M. The pushforward by a projection $(x, y) \mapsto x$ is integration along fibers.

Lemma 7. *Let $\Gamma \subset T^*K \setminus 0$ be a $K \times K$-invariant closed cone. Taking the average $\operatorname{Av} f(x) = \int_K f(yxy^{-1})\, dy$ of a function f extends uniquely from $C^\infty(K)$ to an operator $\operatorname{Av} : C_\Gamma^{-\infty}(K) \to C_\Gamma^{-\infty}(K)$.*

Proof. Define $g : K \times K \to K$, $g(x, y) = yxy^{-1}$, and $p : K \times K \to K$, $p(x, y) = x$. Then $\operatorname{Av} = p_* g^*$ on $C^\infty(K)$. By assumption $\Gamma = K \times S$ where $S \subset \mathfrak{k}^* \setminus 0$ is an Ad^*-invariant closed cone. A computation shows that $((yxy^{-1}, \zeta), (x, y, \xi, \eta)) \in C_g$ iff $\xi = \operatorname{Ad}^*(y^{-1})\zeta$, and $\eta = \operatorname{Ad}^*(xy^{-1})\zeta - \operatorname{Ad}^*(y^{-1})\zeta$. Clearly, $g^*\Gamma$ does not intersect the zero section. Hence the pullback operator g^* is defined. Composing C_g^{-1} with the relation C_p leads to $\eta = 0$ and $p_* g^*\Gamma \subset K \times \operatorname{Ad}^*(K)T$. The assertion follows from this. □

A distribution on K is called central if it is invariant under conjugation. The Fourier coefficients of central distributions are multiples of characters.

Proposition 8 ([5, 4.5])**.** *Let S be a closed cone $\subset \bar{C} \setminus 0$. Let $u \in C^{-\infty}(K)$ be central. Assume $(u|\chi_\lambda) = 0$ if $\lambda \notin S$. Then the Fourier series of u converges in $C^{-\infty}_{K \times i^{-1} \operatorname{Ad}^*(K) S}(K)$.*

Proof. For each $\lambda \in \hat{K}$ we choose a highest weight vector $w_\lambda \in M_\lambda$ of an irreducible subrepresentation $\subset M_\lambda$ of the right regular representation. We may view w_λ as a matrix coefficient of the form $w_\lambda(x) = (R_x v|v)$ with $v \in M_\lambda$, $\|v\| = 1$. Then $w_\lambda(e) = 1$, and $\|w_\lambda\| \leq \sup_K |w_\lambda| \leq 1$ by the Cauchy-Schwarz inequality. The central function $\operatorname{Av} w_\lambda$ is a multiple of χ_λ. Comparing values at e we get $\operatorname{Av} w_\lambda = d_\lambda^{-1}\chi_\lambda$.

The dimension d_λ and, in view of Corollary 4, the Fourier coefficients $(u|\chi_\lambda)$ of u grow at most polynomially in λ. Hence $w = \sum_{\lambda\in S\cap\hat{K}} d_\lambda (u|\chi_\lambda) w_\lambda$ converges in $C^{-\infty}(K)$. By Lemma 6 the series converges in $C^{-\infty}_{K\times i^{-1}S}(K)$. The assertion follows from Lemma 7 since $u = \operatorname{Av} w$. □

4. Proof of Theorem 1

Every $v \in C^{-\infty}(K)$ defines a convolution operator $C^\infty(K) \to C^\infty(K)$, $w \mapsto v*w$. This is a continuous linear map which commutes with right translations. Conversely, every such map is given by convolution with a unique element $v \in C^{-\infty}(K)$. Composition of maps defines the convolution $u * v \in C^{-\infty}(K)$ of distributions $u, v \in C^{-\infty}(K)$ by $(u * v) * w = u * (v * w)$, $w \in C^\infty(K)$. We have the formula $u * v = p_* f^*(u \otimes v)$, where f is the diffeomorphism $f : K \times K \to K \times K$, $f(x, y) = (y, y^{-1}x)$, and p the projection $p : K \times K \to K$, $p(x, y) = x$. The formula is evident for smooth functions and extends to distributions by separate sequential continuity. The composition $C_p \circ C_f^{-1}$ of the canonical relations consists of all

$$((x,\xi),(y,y^{-1}x,\operatorname{Ad}^*(x^{-1}y)\xi,\xi)) \in T^*K \times T^*(K\times K).$$

The wavefront of a tensor product satisfies

$$\operatorname{WF}(u\otimes v) \subset (\operatorname{WF} u \times \operatorname{WF} v) \cup (0 \times \operatorname{WF} v) \cup (\operatorname{WF} u \times 0).$$

Moreover, as a bilinear map, the tensor product satisfies corresponding separate continuity properties. It follows that, for any two cones S_1 and S_2 in $\mathfrak{k}^* \setminus 0$, the convolution $(u_1, u_2) \mapsto u_1 * u_2$ defines a separately sequentially continuous bilinear map

$$* : C^{-\infty}_{K\times S_1}(K) \times C^{-\infty}_{K\times S_2}(K) \to C^{-\infty}_{K\times(S_1\cap \operatorname{Ad}^*(K)S_2)}(K). \qquad (6)$$

Convolution with the Dirac distribution $\delta = \sum_{\lambda\in\hat{K}} d_\lambda\chi_\lambda \in C^{-\infty}(K)$ is the identity, $\delta * u = u$. In the proof of the theorem we need $\delta_S = \sum_{\lambda\in S\cap\hat{K}} d_\lambda\chi_\lambda \in C^{-\infty}(K)$ where $S \subset \bar{C}\setminus 0$. If S is a closed cone, then it follows from Proposition 8 that the series also converges to δ_S in $C^{-\infty}_{K\times i^{-1}\operatorname{Ad}^*(K)S}(K)$.

Now, turning to the proof of the theorem, let $u \in C^{-\infty}(K)$. Assume that $S \subset \bar{C}\setminus 0$ is a closed cone which contains $\operatorname{afsupp}(u)$ in its interior. Then the series of $\delta_{\bar{C}\setminus S} * u$ converges in $C^\infty(K)$. Using (6) with $S_1 = i^{-1}\operatorname{Ad}^*(K)S$ we deduce from the above that the Fourier series of $\delta_S * u$ converges in $C^{-\infty}_{K\times i^{-1}\operatorname{Ad}^*(K)S}(K)$. It follows that the Fourier series of $u = \delta_S * u + \delta_{\bar{C}\setminus S} * u$ converges in this space, too. In particular, we have $(K\times K)\cdot\operatorname{WF}(u) \subset K \times i^{-1}\operatorname{Ad}^*(K)S$. This implies that the left-hand side in (1) is contained in the right-hand side.

To prove the opposite inclusion let S be a closed cone $\subset \bar{C}$ such that $\operatorname{WF}(u) \cap (K \times i^{-1}\operatorname{Ad}^*(K)S) = \emptyset$. We apply (6) to $\delta_S * u$ and deduce that the Fourier series $\sum_{\lambda\in S\cap\hat{K}} u_\lambda$ converges in $C^\infty(K)$. This implies that S is disjoint from the asymptotic Fourier support of u. Since the closure of a Weyl chamber is a fundamental domain for the coadjoint action on $\mathfrak{k}^*$, this implies $K \times i^{-1}\operatorname{afsupp}(u) \subset (K\times K)\cdot\operatorname{WF} u$.

5. Proof of Theorem 2

The polynomial boundedness of π implies the finiteness of the multiplicities $m_K(\lambda : \pi)$ and the convergence of its K-character

$$\Theta_\pi^K := \sum_{\lambda \in \hat{K}} m_K(\lambda : \pi) \operatorname{tr} \lambda \quad \text{in } C^{-\infty}(K). \tag{7}$$

The support of Θ_π^K equals $\operatorname{supp}_K(\pi)$. We have

$$\operatorname{AS}_K(\pi) = \operatorname{supp}_K(\pi)_\infty = \operatorname{afsupp}(\Theta_\pi^K).$$

The second equality holds because the L^2-norm of each non-zero summand in (7) is ≥ 1. From Theorem 1 it follows that (7) converges in $C_\Gamma^{-\infty}(K)$ where $\Gamma = K \times i^{-1} \operatorname{Ad}^*(K) \operatorname{AS}_K(\pi)$. Assumption (2) implies that the conormal bundle of M, which is a subset of $K \times \mathfrak{m}^\perp$, is disjoint from Γ. Hence the restriction

$$\Theta_\pi^K|_M = \sum_{\lambda \in \hat{K}} m_K(\lambda : \pi) \operatorname{tr} \lambda|_M \quad \text{converges in } C^{-\infty}(M), \tag{8}$$

and $\operatorname{WF}(\Theta_\pi^K|_M)$ is contained in $M \times i^{-1}A \subset T^*M \setminus 0$, where $A \subset i\mathfrak{m}$ denotes the image of $\operatorname{Ad}^*(K) \operatorname{AS}_K(\pi)$ under the projection $i\mathfrak{k}^* \to i\mathfrak{m}^*$. From Theorem 1 it follows that

$$\operatorname{afsupp}_M(\Theta_\pi^K|_M) \subset A = \operatorname{Ad}^*(M)A.$$

Consider, in $C^{-\infty}(M)$, the Fourier series $\Theta_\pi^K|_M = \sum_{\mu \in \hat{M}} c_\mu \operatorname{tr} \mu$. By Corollary 4 the map $\mu \mapsto c_\mu$ is polynomially bounded. We prove that

$$c_\mu = m_M(\mu : \pi|_M) \quad \text{for all } \mu \in \hat{M}.$$

The assertions of the theorem will follow from this. Moreover, it says that $\Theta_{\pi|_M}^M = \Theta_\pi^K|_M$.

Let $\mu \in \hat{M}$. Fix a representation space H_μ. Let $\rho = \operatorname{ind}_M^K(\mu)$ denote the unitary representation of K induced by μ. We view the representation space H_ρ of ρ as the subspace of $L^2(K, H_\mu)$ defined by $f(xm) = \mu(m^{-1})f(x)$, $m \in M$, almost every $x \in K$. Then $f \in L^2(K, H_\mu)$ belongs to H_ρ only if it satisfies, in the sense of distributions, the first order system of differential equations $\widetilde{Y} f + \mu_*(Y)f = 0$, $Y \in \mathfrak{m}$. Here μ_* is the Lie algebra representation induced by μ. The characteristic variety of $\widetilde{Y} + \mu_*(Y)$ is contained in $K \times Y^\perp$. Hence $\operatorname{WF}(f) \subset K \times \operatorname{Ad}^*(K)\mathfrak{m}^\perp$ if $f \in H_\rho$. Theorem 1, generalized to vector-valued distributions, implies that $\operatorname{afsupp}_K(f) \subset i \operatorname{Ad}^*(K)\mathfrak{m}^\perp$ for every $f \in H_\rho$. This implies $\operatorname{AS}_K(\rho) \subset i \operatorname{Ad}^*(K)\mathfrak{m}^\perp$. Indeed, if this were not true, we could find a closed cone $S \subset \bar{C} \setminus 0$, $S \cap i \operatorname{Ad}^*(K)\mathfrak{m}^\perp = \emptyset$, and $f = \sum_{\lambda \in \hat{K} \cap S} f_\lambda \in H_\rho$, $f_\lambda \in W_\lambda$, such that $\sum_{\lambda \in \hat{K} \cap S} |\lambda|^{2N} \|f_\lambda\|^2 = \infty$ for some $N \in \mathbb{N}$. Here W_λ denotes the λ-isotypical subspace of H_ρ. Using assumption (2) we deduce $\operatorname{AS}_K(\rho) \cap \operatorname{AS}_K(\pi) = \emptyset$. Therefore, $\operatorname{supp}_K(\rho) \cap \operatorname{supp}_K(\pi)$ is relatively compact, hence finite. By Frobenius reciprocity we get, with sums having only finitely many non-zero summands,

$$\begin{aligned} m_M(\mu : \pi|_M) &= \dim \operatorname{Hom}_K(\rho, \pi) \\ &= \sum_\lambda m_K(\lambda : \pi) m_K(\lambda : \rho) \\ &= \sum_\lambda m_K(\lambda : \pi) m_M(\mu : \lambda|_M) \\ &= \sum_\lambda m_K(\lambda : \pi) \int_M \overline{\operatorname{tr} \mu(m)} \operatorname{tr} \lambda|_M(m)\, dm \\ &= \big(\Theta_\pi^K|_M \big| \operatorname{tr} \mu\big)_{L^2(M)}. \end{aligned}$$

The last equation follows from (8).

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