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COMPUTATION OF WEYL GROUPS OF G-VARIETIES

IVAN V. LOSEV

Abstract. Let G be a connected reductive group. To any irreducible G-variety one assigns a certain linear group generated by reflections called the *Weyl group*. Weyl groups play an important role in the study of embeddings of homogeneous spaces. We establish algorithms for computing Weyl groups for homogeneous spaces and affine homogeneous vector bundles. For some special classes of G-varieties (affine homogeneous vector bundles of maximal rank, affine homogeneous spaces, homogeneous spaces of maximal rank with a discrete group of central automorphisms) we compute Weyl groups more or less explicitly.

Contents

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1. Introduction

Throughout the whole paper, the base field is the field $\mathbb{C}$ of complex numbers. In this section, G denotes a connected reductive group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. X denotes an irreducible G-variety.

1.1. Definition of the Weyl group of a G-variety.

The main object considered in the present paper is the Weyl group of an irreducible G-variety. Before giving the definition we would like to make some historical remarks.

The Weyl group of G was essentially defined in Herman Weyl's paper [Wey]. A little while later, E. Cartan, [Ca], generalized the notion of the Weyl group to symmetric spaces (the so-called *little Weyl groups*). From the algebraic viewpoint, a symmetric space is a homogeneous space G/H, where $(G^\sigma)^\circ \subset H \subset G^\sigma$ for some involutory automorphism σ of G.

To move forward we need the notion of complexity.

Definition 1.1.1. The *complexity* of X is the codimension of a general B-orbit in X, or, equivalently, $\operatorname{tr.deg} \mathbb{C}(X)^B$. We denote the complexity of X by $c_G(X)$. A normal irreducible G-variety of complexity 0 is said to be *spherical*.

In particular, every symmetric space is a spherical G-variety, [Vu1]. In [Bri], Brion constructed the Weyl group for a spherical homogeneous space G/H with $\# N_G(H)/H < \infty$. Brion's Weyl group generalizes that of symmetric spaces. In view of results of [BP], the restriction $\# N_G(H)/H < \infty$ is not essential. Knop, [Kn1], found another way to define the Weyl group for an arbitrary irreducible G-variety, also generalizing the Weyl group of a symmetric space. In [Kn3], he extended Brion's definition to arbitrary G-varieties. Finally, in [Kn4], Knop gave a third definition of the Weyl group and proved the equivalence of all three definitions.

Now we are going to introduce the definition of the Weyl group following [Kn3].

Consider the sublattice $\mathfrak{X}_{G,X} \subset \mathfrak{X}(T)$ consisting of the weights of B-semi-invariant elements of $\mathbb{C}(X)$. It is called the *weight lattice* of X.

Definition 1.1.2. Put $\mathfrak{a}_{G,X} = \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{C}$. We call the subspace $\mathfrak{a}_{G,X} \subset \mathfrak{t}^*$ the *Cartan space* of X. The dimension of $\mathfrak{a}_{G,X}$ is called the *rank* of X and is denoted by $\operatorname{rk}_G(X)$.

Let $\mathfrak{t}$ denote the Lie algebra of T, $\mathfrak{t}(\mathbb{R})$ the real form of $\mathfrak{t}$, and $W(\mathfrak{g})$ the Weyl group of $\mathfrak{g}$. Fix a $W(\mathfrak{g})$-invariant scalar product on $\mathfrak{t}(\mathbb{R})$. This induces the scalar product on $\mathfrak{a}_{G,X}(\mathbb{R}) := \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{R}$ and on $\mathfrak{a}_{G,X}(\mathbb{R})^*$. The Weyl group of X will act on $\mathfrak{a}_{G,X}$ preserving the weight lattice and the scalar product. To define the action we will describe its Weyl chamber. To this end we need the notion of a central G-valuation.

Definition 1.1.3. By a G-valuation of X we mean a discrete $\mathbb{R}$-valued G-invariant valuation of $\mathbb{C}(X)$. A G-valuation is called *central* if it vanishes on $\mathbb{C}(X)^B$.

In particular, if X is spherical, then any G-invariant valuation is central. A central valuation v determines an element $\varphi_v \in \mathfrak{a}_{G,X}(\mathbb{R})^*$ by $\langle \varphi_v, \lambda\rangle = v(f_\lambda)$, where $\lambda \in \mathfrak{X}_{G,X}$, $f_\lambda \in \mathbb{C}(X)^{(B)}_\lambda \setminus \{0\}$. The element φ_v is well defined because v is central.

Theorem 1.1.4.
1. *The map $v \mapsto \varphi_v$ is injective. Its image is a finitely generated rational convex cone in $\mathfrak{a}_{G,X}(\mathbb{R})^*$ called the central valuation cone of X and denoted by $\mathcal{V}_{G,X}$.*
2. *The central valuation cone is simplicial (that is, there are linearly independent vectors $\alpha_1, \ldots, \alpha_s \in \mathfrak{a}_{G,X}(\mathbb{R})$ such that the cone coincides with $\{x | \langle \alpha_i, x\rangle \geqslant 0, i = 1, \ldots, s\}$). Moreover, the reflections corresponding to its facets generate a finite group. This group is called the Weyl group of X and is denoted by $W_{G,X}$.*
3. *The lattice $\mathfrak{X}_{G,X} \subset \mathfrak{a}_{G,X}(\mathbb{R})$ is $W_{G,X}$-stable.*

The proof of the first part of the theorem is relatively easy. It is obtained (in a greater generality) in [Kn3], Korollare 3.6, 4.2, 5.2, 6.5. The second assertion is much more complicated. It was proved by Brion in [Bri] in the spherical case. Knop, [Kn3], used Brion's result to prove the assertion in the general case. Later, he gave an alternative proof in [Kn4]. The third assertion of Theorem 1.1.4 follows easily from the construction of the Weyl group in [Kn4].

Note that the Weyl group does not depend on the scalar product used in its definition. Indeed, the set of W-invariant scalar products on $\mathfrak{t}(\mathbb{R})^*$ is convex. The Weyl group fixes $\mathfrak{X}_{G,X}$, so does not change under small variations of the scalar product.

1.2. **Main problem.** Our general problem is to find an algorithm to compute $\mathfrak{a}_{G,X}, W_{G,X}$ for an irreducible G-variety X. However, for such an algorithm to exist, the variety X should have some good form. It is reasonable to restrict ourselves to the following two classes of G-varieties:

1. Homogeneous spaces G/H, where H is an algebraic subgroup of G.
2. Homogeneous vector bundles over affine homogeneous spaces (= affine homogeneous vector bundles) $G *_H V$. Here H is a reductive subgroup of G and V is an H-module.

There are several reasons to make these restrictions. First of all, these G-varieties have "group-theoretic" and "representation-theoretic" structure, so one may hope to find algorithms with "group-" or "representation-theoretic" steps. Secondly, the computation of the Cartan space and the Weyl group of an arbitrary G-variety can be reduced to the computation for homogeneous spaces. Namely, for an irreducible G-variety X and a point $x \in X$ in general position the equalities $\mathfrak{a}_{G,X} = \mathfrak{a}_{G,Gx}$,

$W_{G,X} = W_{G,Gx}$ hold (see Proposition 3.2.1). Moreover, if X is affine and smooth and $x \in X$ is a point with closed G-orbit, then $\mathfrak{a}_{G,X} = \mathfrak{a}_{G,G*_H V}, W_{G,X} = W_{G,G*_H V}$, where $H = G_x$ and V is the slice module at x, that is, $V = T_xX/\mathfrak{g}_*x$ (Corollary 3.2.3). The reason why the case of homogeneous spaces is not sufficient is two-fold. First, even if X is smooth and affine, the stabilizer G_x for x in general position is often difficult to compute, while it is relatively easy to find a point x with closed orbit and compute its stabilizer and the slice module $T_xX/\mathfrak{g}_*x$. Second, the algorithm for homogeneous vector bundles is a part of that for homogeneous spaces.

So the main results of the paper are algorithms computing the Cartan spaces and the Weyl groups for G-varieties of types (1) and (2). Moreover, we compute the Weyl groups of affine homogeneous spaces more or less explicitly.

Our algorithms are quite complicated, so we do not give them here. They will be presented (in a brief form) in Section 7. Roughly speaking, all our steps consist of computing some "structure characteristics" for pairs (an algebraic Lie algebra, a subalgebra), (a reductive Lie algebra, a module over this algebra). The computation of the normalizer or the unipotent radical is an example of an operation from the first group. An operation from the second group is, for instance, the decomposition of the restriction of an irreducible representation to a Levi subalgebra together with the determination of all highest vectors of the restriction.

1.3. **Motivations and known results.** Our main motivation comes from the theory of embeddings of homogeneous spaces.

One may say that the theory of algebraic transformation groups studies the category of varieties acted on by some algebraic group. Because of technical reasons, one usually considers actions of a connected reductive group G on normal irreducible varieties. The first problem in the study of a category is the classification of its objects up to an isomorphism. In our case, the problem may be divided into two parts, birational and regular. The birational part is the classification of G-varieties up to a birational equivalence, or, in the algebraic setting, the classification of all finitely generated fields equipped with an action of G by automorphisms. An important special case here is the birational classification of quasi-homogeneous G-varieties, i.e. those possessing an open G-orbit. Of course, an equivalent problem is the classification of algebraic subgroups of G up to conjugacy.

The regular part is the classification of G-varieties in a given class of birational equivalence. In the quasi-homogeneous case, this is equivalent to the classification of all open embeddings of a given homogeneous space into normal varieties. The program to perform this classification was proposed by Luna and Vust, [LV]. Note that in that paper only the quasi-homogeneous case was considered. However, the Luna-Vust theory can be generalized to the general case, where one considers all normal G-varieties with a given G-field of rational functions; see [T1]. Using the Luna-Vust theory one obtains a combinatorial (in a certain sense) description of G-varieties of complexity not exceeding 1. The spherical case was considered already in [LV]. A more self-contained and plain exposition is given, for example, in [Kn2]. The case of complexity 1 is due to Timashev, [T1]. The classification for complexity greater than 1 seems to be a wild problem.

Now we sketch the classification theory of spherical varieties. Clearly, a spherical G-variety has an open orbit. So the birational part of the classification is just the classification of all spherical homogeneous spaces. To describe all embeddings of a given spherical homogeneous space one needs to know the following data:

(1) The rational vector space $\mathfrak{a}_{G,X}(\mathbb{Q}) := \mathfrak{X}_{G,X} \otimes_{\mathbb{Z}} \mathbb{Q}$.
(2) The (central) valuation cone $\mathcal{V}_{G,X} \subset \mathfrak{a}_{G,X}(\mathbb{R})^*$.
(3) Certain *colored* vectors in $\mathfrak{a}_{G,X}(\mathbb{Q})^*$ that are in one-to-one correspondence with B-divisors of the spherical homogeneous space. Namely, given a prime B-divisor D we define the colored vector φ_D by $\langle \varphi_D, \lambda \rangle = \mathrm{ord}_D(f_\lambda)$, where $\lambda \in \mathfrak{X}_{G,X}$, $f_\lambda \in \mathbb{C}(X)^{(B)}_\lambda \setminus \{0\}$.

It turns out that normal embeddings of a spherical homogeneous space X are in one-to-one correspondence with certain *admissible* sets of cones in $\mathfrak{a}_{G,X}(\mathbb{R})^*$. Every cone from an admissible set is generated by colored vectors and elements of $\mathcal{V}_{G,X}$ and satisfies certain combinatorial requirements. An admissible set is one that satisfies some additional requirements of a combinatorial nature. So the solution of the regular part of the classification problem consists in the determination of $\mathfrak{a}_{G,X}(\mathbb{Q})$, $\mathcal{V}_{G,X}$ and colored vectors.

Note that $\mathfrak{a}_{G,X}(\mathbb{Q}) = \mathfrak{a}_{G,X} \cap \mathfrak{t}(\mathbb{Q})^*$. The computation of $\mathfrak{a}_{G,X}$ is not very difficult. After $\mathfrak{a}_{G,X}$ is computed one can proceed to the computation of the valuation cone $\mathcal{V}_{G,X}$. Despite the fact that a group generated by reflections has several Weyl chambers, the Weyl group determines the cone $\mathcal{V}_{G,X}$ uniquely. Namely, $\mathcal{V}_{G,X}$ is a unique Weyl chamber of $W_{G,X}$ containing the image of the negative Weyl chamber of $\mathfrak{t}$ under the projection $\mathfrak{t}(\mathbb{R}) \to \mathfrak{a}_{G,X}(\mathbb{R})^*$.

Now we discuss results concerning the computation of Weyl groups and Cartan spaces of G-varieties.

D.I. Panyushev in [Pa1], see also [Pa4], reduced the computation of weight lattices for G-varieties of two types mentioned above to that for affine homogeneous spaces (in fact, together with some auxiliary datum). In [Lo1], Cartan spaces for affine homogeneous spaces were computed.

We now proceed to results on the computation of Weyl groups. They are formulated in three possible ways:

(1) In terms of the Weyl group itself.
(2) In terms of the central valuation cone.
(3) In terms of primitive linearly independent elements $\beta_1, \ldots, \beta_r \in \mathfrak{X}_{G,X}$ such that the central valuation cone is given by the inequalities $\beta_i \leqslant 0$. We denote the set $\{\beta_1, \ldots, \beta_r\}$ by $\Pi_{G,X}$. For spherical X such elements β_i are called spherical roots of X.

The Weyl group is trivial if and only if X is *horospherical*, i.e., the stabilizer of any point contains a maximal unipotent subgroup. In full generality, this was proved by Knop, [Kn1].

The other results (at least known to the author) on the computation of Weyl groups relate to the spherical case.

In [Bri], Brion proposed a technique allowing one to extract the Weyl group $W_{G,G/H}$, where G/H is a spherical homogeneous space, from the algebra $g\mathfrak{h}g^{-1}$ for some special $g \in G$. It is an open problem to describe the set of all suitable g.

As we have already mentioned above, the Weyl group of a symmetric space coincides with its little Weyl group; see [Kn1], [Bri], [Vu2]. The Weyl groups of spherical G-modules were computed by Knop in [Kn6]. Note that in that paper the notation W_V is used for the Weyl group of V^*.

There are also computations of Weyl groups for some other special classes of spherical homogeneous spaces. Spherical roots for *wonderful varieties* of rank 2 were computed in [Wa]. The computation is based on some structure theorems on wonderful varieties. The computation for other interesting classes of homogeneous spaces can be found in [Sm]. It uses the method of formal curves, established in [LV].

Of all the results mentioned above we use only Wasserman's (see, however, Remark 6.1.1).

Finally, let us make a remark on the classification of spherical varieties. The first step of the classification is describing all spherical homogeneous spaces. Up to now there is only one approach to this problem due to Luna, who applied it to classify spherical subgroups in groups of type A (a connected reductive group is said to be of type A if all simple ideals of its Lie algebra are of type A). Using Luna's approach, the full classification for groups of types A-D ([Bra]) and a partial one for types A-C ([Pe]) were obtained. The basic idea of the Luna classification is to establish a one-to-one correspondence between spherical homogeneous spaces and certain combinatorial data that are almost equivalent to those listed above. The computation of combinatorial data (in particular, the Weyl group) for certain homogeneous spaces plays an important role in this approach.[1]

1.4. **The structure of the paper.** Every section is divided into subsections. Theorems, lemmas, definitions, etc. are numbered within each subsection, while formulae and tables are numbered within each section. The first subsection of Sections 3–6 describes their content in detail.

2. Notation and conventions

For an algebraic group denoted by a capital Latin letter we denote its Lie algebra by the corresponding small German letter. For example, the Lie algebra of $\widetilde{L}_0$ is denoted by $\widetilde{\mathfrak{l}}_0$.

By a unipotent Lie algebra we mean the Lie algebra of a unipotent algebraic group.

H-morphisms, H-subvarieties, etc. Let H be an algebraic group. We say that a variety X is an H-variety if an action of H on X is given. By an H-subset (resp., subvariety) in a given H-variety we mean an H-stable subset (resp., subvariety). A morphism of H-varieties is said to be an H-morphism if it is H-equivariant. The term "H-bundle" means a principal bundle with the structure group H.

Borel subgroups and maximal tori. While considering a reductive group G, we always fix its Borel subgroup B and a maximal torus $T \subset B$. In accordance with this choice, we fix the root system $\Delta(\mathfrak{g})$ and the system of simple roots $\Pi(\mathfrak{g})$ of $\mathfrak{g}$. The Borel subgroup of G containing T and opposite to B is denoted by B^-.

If G_1, G_2 are reductive groups with the fixed Borel subgroups $B_i \subset G_i$ and maximal tori $T_i \subset B_i$, then we take $B_1 \times B_2, T_1 \times T_2$ for the fixed Borel subgroup and maximal torus in $G = G_1 \times G_2$.

[1]Since the present paper had been submitted, two other papers completing the classification of spherical homogeneous spaces appeared. The uniqueness of a spherical variety with given combinatorial data was proved in [Lo4]. The existence part of Luna's conjecture was very recently obtained by S. Cupit-Foutou in [Cu].

Suppose G_1 is a reductive algebraic group. Fix an embedding $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{g}_1)$. Then $\mathfrak{t} \cap \mathfrak{g}_1$ is a Cartan subalgebra and $\mathfrak{b} \cap \mathfrak{g}_1$ is a Borel subalgebra of $\mathfrak{g}_1$. For fixed Borel subgroup and maximal torus in G_1 we take those with the Lie algebras $\mathfrak{b}_1, \mathfrak{t}_1$.

Homomorphisms and representations. All homomorphisms of reductive algebraic Lie algebras (for instance, representations) are assumed to be differentials of homomorphisms of the corresponding algebraic groups.

Identification $\mathfrak{g} \cong \mathfrak{g}^*$. Let G be a reductive algebraic group. There is a G-invariant symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that its restriction to $\mathfrak{t}(\mathbb{R})$ is positive definite. For instance, if V is a locally effective G-module, then $(\xi, \eta) = \mathrm{tr}_V(\xi\eta)$ has the required properties. Note that if H is a reductive subgroup of G, then the restriction of $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate, so one may identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Parabolic subgroups and Levi subgroups. A parabolic subgroup of G is called *standard* (resp., *antistandard*) if it contains B (resp., B^-). It is known that any parabolic subgroup is G-conjugate to a unique standard (and antistandard) parabolic subgroup. Standard (as well as antistandard) parabolic subgroups are in one-to-one correspondence with subsets of $\Pi(\mathfrak{g})$. Namely, one assigns to $\Sigma \subset \Pi(\mathfrak{g})$ the standard (resp., antistandard) parabolic subgroup, whose Lie algebra is generated by $\mathfrak{b}$ and $\mathfrak{g}^{-\alpha}$ for $\alpha \in \Sigma$ (resp., by $\mathfrak{b}^-$ and $\mathfrak{g}^{\alpha}, \alpha \in \Sigma$).

By a standard Levi subgroup in G we mean the Levi subgroup containing T of a standard (or an antistandard) parabolic subgroup.

Simple Lie algebras, their roots and weights. Simple roots of a simple Lie algebra $\mathfrak{g}$ are denoted by α_i. The numeration is described below. By π_i we denote the fundamental weight corresponding to α_i.

Classical algebras. In all cases for $\mathfrak{b}$ (resp. $\mathfrak{t}$) we take the algebra of all upper triangular (resp., diagonal) matrices in $\mathfrak{g}$.

$\mathfrak{g} = \mathfrak{sl}_n$. Let $e_1, \ldots, e_n$ denote the standard basis in $\mathbb{C}^n$ and $e^1, \ldots, e^n$ the dual basis in $\mathbb{C}^{n*}$. Choose the generators $\varepsilon_i, i = 1, \ldots, n$, given by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n)\rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1$.

$\mathfrak{g} = \mathfrak{so}_{2n+1}$. Let $e_1, \ldots, e_{2n+1}$ be the standard basis in $\mathbb{C}^{2n+1}$. We suppose $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{2n+1} x_i y_{2n+2-i}$. Define $\varepsilon_i \in \mathfrak{t}^*, i = 1, \ldots, n$, by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n, 0, -x_n, \ldots, -x_1)\rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n = \varepsilon_n$.

$\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $e_1, \ldots, e_{2n}$ be the standard basis in $\mathbb{C}^{2n}$. We suppose that $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{n}(x_i y_{2n+1-i} - y_i x_{2n+1-i})$. Let us define $\varepsilon_i \in \mathfrak{t}^*, i = 1, \ldots, n$, by $\langle \varepsilon_i, \mathrm{diag}(x_1, \ldots, x_n, -x_n, \ldots, -x_1)\rangle = x_i$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n = 2\varepsilon_n$.

$\mathfrak{g} = \mathfrak{so}_{2n}$. Let $e_1, \ldots, e_{2n}$ be the standard basis in $\mathbb{C}^{2n}$. We suppose that $\mathfrak{g}$ annihilates the form $(x, y) = \sum_{i=1}^{2n} x_i y_{2n+1-i}$. Define $\varepsilon_i \in \mathfrak{t}^*, i = 1, \ldots, n$, in the same way as for $\mathfrak{g} = \mathfrak{sp}_{2n}$. Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$.

Exceptional algebras. For roots and weights of exceptional Lie algebras we use the notation from [OV]. The numeration of simple roots is also taken from [OV].

Subalgebras in semisimple Lie algebras. For semisimple subalgebras of exceptional Lie algebras we use the notation from [D]. Below we explain the notation for classical algebras.

Suppose $\mathfrak{g}=\mathfrak{sl}_n$. By $\mathfrak{sl}_k, \mathfrak{so}_k, \mathfrak{sp}_k$ we denote the subalgebras of $\mathfrak{sl}_n$ annihilating a subspace $U\subset \mathbb{C}^n$ of dimension $n-k$, leaving its complement V invariant, and (for $\mathfrak{so}_k, \mathfrak{sp}_k$) annihilating a nondegenerate orthogonal or symplectic form on V.

The subalgebras $\mathfrak{so}_k\subset\mathfrak{so}_n, \mathfrak{sp}_k\subset\mathfrak{sp}_n$ are defined analogously. The subalgebra $\mathfrak{gl}_k^{diag}$ is embedded into $\mathfrak{so}_n,\mathfrak{sp}_n$ via the direct sum of τ,τ^* and a trivial representation (here τ denotes the tautological representation of $\mathfrak{gl}_k$). The subalgebras $\mathfrak{sl}_k^{diag},\mathfrak{so}_k^{diag},\mathfrak{sp}_k^{diag}\subset\mathfrak{so}_n,\mathfrak{sp}_n$ are defined analogously. The subalgebra G_2 (resp., $\mathfrak{spin}_7$) in $\mathfrak{so}_n$ is the image of G_2 (resp., $\mathfrak{so}_7$) under the direct sum of the 7-dimensional irreducible (resp., spinor) and the trivial representations.

Finally, let $\mathfrak{h}_1,\mathfrak{h}_2$ be subalgebras of $\mathfrak{g}=\mathfrak{sl}_n,\mathfrak{so}_n,\mathfrak{sp}_n$ described above. While writing $\mathfrak{h}_1\oplus\mathfrak{h}_2$, we always mean that $(\mathbb{C}^n)^{\mathfrak{h}_1}+(\mathbb{C}^n)^{\mathfrak{h}_2}=\mathbb{C}^n$.

The description above determines a subalgebra uniquely up to conjugacy in $\mathrm{Aut}(\mathfrak{g})$.

Now we list some notation used in the text.

$\sim_G$	the equivalence relation induced by an action of a group G.
$A^{(B)}$	the subset of all B-semi-invariant functions in a G-algebra A.
$A^\times$	the group of all invertible elements of an algebra A.
$\mathrm{Aut}(\mathfrak{g})$	the group of automorphisms of a Lie algebra $\mathfrak{g}$.
e_α	a nonzero element of the root subspace $\mathfrak{g}^\alpha$.
(G,G)	the derived subgroup of a group G.
$[\mathfrak{g},\mathfrak{g}]$	the derived subalgebra of a Lie algebra $\mathfrak{g}$.
G°	the connected component of unit of an algebraic group G.
$G*_H V$	the homogeneous bundle over G/H with the fiber V.
$[g,v]$	the equivalence class of (g,v) in $G*_H V$.
G_x	the stabilizer of $x\in X$ under an action $G:X$.
$G_1\rightthreetimes G_2$	a semidirect product of groups G_1,G_2 (G_1 is normal).
$G_2\leftthreetimes G_1$	a semidirect product of groups G_1,G_2 (G_1 is normal).
$\mathfrak{g}^\alpha$	the root subspace of $\mathfrak{g}$ corresponding to a root α.
$\mathfrak{g}^{(A)}$	the subalgebra $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha$ with $\alpha\in A\cup -A$.
$G^{(A)}$	the connected subgroup of G with Lie algebra $\mathfrak{g}^{(A)}$.
$\mathrm{Gr}(V,d)$	the Grassmanian of d-dimensional subspaces of a vector space V.
$\mathrm{Int}(\mathfrak{h})$	the group of inner automorphisms of a Lie algebra $\mathfrak{g}$.
$N_G(H)$	the normalizer of a subgroup H in a group G.
$N_G(\mathfrak{h})$	the normalizer of $\mathfrak{h}\subset\mathfrak{g}$ in G.
$\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$	the normalizer of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$.
$\mathrm{Quot}(A)$	the fraction field of A.
$\mathrm{rk}(G)$	the rank of an algebraic group G.
$R(\lambda)$	the irreducible representation of a reductive algebraic group (or a reductive Lie algebra) with highest weight λ.
$\mathrm{R}_u(H)$	the unipotent radical of an algebraic group H (of an algebraic Lie algebra $\mathfrak{h}$).
$\mathrm{R}_u(\mathfrak{h})$	the unipotent radical of an algebraic Lie algebra $\mathfrak{h}$.
s_α	the reflection in a Euclidean space corresponding to a vector α.
$\mathrm{tr.deg}\, A$	the transcendence degree of an algebra A.
$U^\angle$	the skew-orthogonal complement to a subspace $U\subset V$ of a symplectic vector space V.
$V^\mathfrak{g}$	$=\{v\in V\vert \mathfrak{g}v=0\}$, where $\mathfrak{g}$ is a Lie algebra and V is a $\mathfrak{g}$-module.

$V(\lambda)$	the irreducible module of the highest weight λ over a reductive algebraic group or a reductive Lie algebra.
$W(\mathfrak{g})$	the Weyl group of a reductive Lie algebra $\mathfrak{g}$.
$\mathfrak{X}(G)$	the character lattice of an algebraic group G.
$\mathfrak{X}_G$	the weight lattice of a reductive algebraic group G.
X^G	the fixed point set for an action $G : X$.
$X//G$	the categorical quotient for an action $G : X$, where G is a reductive group and X is an affine G-variety.
$\#X$	the number of elements in a set X.
$Z_G(H)$	the centralizer of a subgroup H (of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$) in an algebraic group G.
$Z_G(\mathfrak{h})$	the centralizer of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in G.
$Z(G)$	$:= Z_G(G)$.
$\mathfrak{z}_\mathfrak{g}(\mathfrak{h})$	the centralizer of a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.
$\mathfrak{z}(\mathfrak{g})$	$:= \mathfrak{z}_\mathfrak{g}(\mathfrak{g})$.
$\alpha^\vee$	the dual root to α.
$\Delta(\mathfrak{g})$	the root system of a reductive Lie algebra $\mathfrak{g}$.
λ^*	the dual highest weight to λ.
$\Lambda(\mathfrak{g})$	the root lattice of a reductive Lie algebra $\mathfrak{g}$.
$\Pi(\mathfrak{g})$	the system of simple roots for a reductive Lie algebra $\mathfrak{g}$.
$\pi_{G,X}$	the (categorical) quotient morphism $X \to X//G$.

3. Known results and constructions

In this section G is a connected reductive group and X is an irreducible G-variety. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Put $U = \mathrm{R}_u(B)$.

3.1. **Introduction.** In this section we quote known results and constructions related to Cartan spaces, Weyl groups and weight lattices. We also prove some more or less easy results for which it is difficult to give a reference.

In Subsection 3.2 we present results on the computation of Cartan spaces. In the beginning of the subsection we establish an important notion of a tame inclusion of a subgroup of G into a parabolic subgroup. Then we present a reduction of the computation from the general case to the case of affine homogeneous spaces. This reduction is due to Panyushev. Then we partially present results of [Lo1] on the computation of the Cartan spaces for affine homogeneous spaces. Finally, at the end of the section we study the behavior of Cartan spaces, Weyl groups, etc. under the twisting of the action $G : X$ by an automorphism.

The most important part of this section is Subsection 3.3, where we review some definitions and results related to Weyl groups of G-varieties. We start with results of F. Knop, [Kn1], [Kn3], [Kn4], [Kn7]. Then we quote results of [Lo2] that provide certain reductions for computing Weyl groups. These results allow us to reduce the computation of the groups $W_{G,X}$ to the case when G is simple and $\mathrm{rk}_G(X) = \mathrm{rk}\, G$.

Until the end of the subsection we deal with that special case. Here we have two types of restrictions on the Weyl group. Restrictions of the first type are valid for smooth affine varieties. They are derived from results of [Lo3]. The computation in Section 5 is based on these restrictions. Their main feature is that they describe the class of conjugacy of $W_{G,X}$ and do not answer the question whether a given reflection lies in $W_{G,X}$.

On the other hand, we have some restrictions on the form of a reflection lying in $W_{G,X}$. They are used in Section 6. These restrictions are derived from the observation that the Weyl group of an arbitrary X coincides with a Weyl group of a certain *wonderful variety.* This observation follows mainly from results of Knop, [Kn3].

3.2. **Computation of Cartan spaces.** In this subsection, G is a connected reductive group. The definitions of the Cartan space and the Weyl group of X given in Subsection 1.1 are compatible with those given in [Kn1], [Kn4]; see [Kn4], Theorem 7.4 and Corollary 7.5. In particular, $W_{G,X}$ is a subquotient of $W(\mathfrak{g})$ (i.e., there exist subgroups $\Gamma_1, \Gamma_2 \subset W(\mathfrak{g})$ such that $\mathfrak{a}_{G,X}$ is Γ_1-stable, Γ_2 is the inefficiency kernel of the action $\Gamma_1 : \mathfrak{a}_{G,X}$, and $W_{G,X} = \Gamma_1/\Gamma_2$). The following proposition describes functorial properties of Cartan spaces and Weyl groups.

Proposition 3.2.1 ([Kn1], Satz 6.5)**.** *Let X_1, X_2 be irreducible G-varieties and $\varphi : X_1 \to X_2$ a G-morphism.*

(1) *Suppose φ is dominant. Then $\mathfrak{a}_{G,X_2} \subset \mathfrak{a}_{G,X_1}$ and W_{G,X_2} is a subquotient of W_{G,X_1}.*
(2) *Suppose φ is generically finite (i.e., there is a dense open subset in the image of φ such that the fiber of any point from this subset is finite). Then $\mathfrak{a}_{G,X_1} \subset \mathfrak{a}_{G,X_2}$ and W_{G,X_1} is a subquotient of W_{G,X_2}.*
(3) *If φ is dominant and generically finite (e.g. étale), then $\mathfrak{a}_{G,X_1} = \mathfrak{a}_{G,X_2}$ and $W_{G,X_1} = W_{G,X_2}$.*
(4) *Let X be an irreducible G-variety. There is an open G-subvariety $X^0 \subset X$ such that $\mathfrak{a}_{G,Gx} = \mathfrak{a}_{G,X}, W_{G,Gx} = W_{G,X}$ for any $x \in X^0$.*

Corollary 3.2.2. *Let $H_1 \subset H_2$ be algebraic subgroups in G. Then $\mathfrak{a}_{G,G/H_2} \subset \mathfrak{a}_{G,G/H_1}$ and $W_{G,G/H_2}$ is a subquotient of $W_{G,G/H_1}$. If $H_1^\circ = H_2^\circ$, then $\mathfrak{a}_{G,G/H_2} = \mathfrak{a}_{G,G/H_1}$ and $W_{G,G/H_2} = W_{G,G/H_1}$.*

In the sequel we write $\mathfrak{a}(\mathfrak{g}, \mathfrak{h})$ instead of $\mathfrak{a}_{G,G/H}$.

Corollary 3.2.3. *Let X be smooth and affine, x a point of X with closed G-orbit. Put $H = G_x, V = T_xX/\mathfrak{g}_*x, X' = G *_H V$. Then $\mathfrak{a}_{G,X} = \mathfrak{a}_{G,X'}$, $W_{G,X} = W_{G,X'}$.*

Proof. This is a direct consequence of the Luna slice theorem for smooth points ([PV], Subsection 6.5) and assertion (3) of Proposition 3.2.1. □

Corollary 3.2.4. *Let H be a reductive subgroup of G and V an H-module. Then*

(1) *$\mathfrak{a}_{G,G*_HV} \subset \mathfrak{a}_{G,G*_{H_0}V}$ for any normal subgroup $H_0 \subset H$. If $\mathfrak{a}_{G,G*_HV} = \mathfrak{a}_{G,G*_{H_0}V}$, then $W_{G,G*_HV} \subset W_{G,G*_{H_0}V}$. If $H^\circ \subset H_0$, then $W_{G,G*_HV} = W_{G,G*_{H_0}V}$.*
(2) *$\mathfrak{a}_{G,G*_H(V/V^H)} = \mathfrak{a}_{G,G*_HV}$, $W_{G,G*_H(V/V^H)} = W_{G,G*_HV}$.*

Proof. Assertion (1) follows from assertions (1), (3) of Proposition 3.2.1. To prove assertion (2) note that there is an isomorphism of G-varieties $G *_H V \cong G *_H (V/V^H) \times V^H$ (G is assumed to act trivially on V^H). It remains to apply assertion (4) of Proposition 3.2.1. □

In the sequel we write $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}, V), W(\mathfrak{g}, \mathfrak{h}, V)$ instead of $\mathfrak{a}_{G,G*_HV}, W_{G,G*_HV}$.

Now we reduce the computation of Cartan spaces for homogeneous spaces to that for affine homogeneous vector bundles. To this end we need one fact about subgroups in G due to Weisfeller, [Wei].

Proposition 3.2.5. *Let H be an algebraic subgroup of G. Then there exists a parabolic subgroup $Q \subset G$ such that $\mathrm{R}_u(H) \subset \mathrm{R}_u(Q)$.*

Definition 3.2.6. Under the assumptions of the previous proposition, we say that the inclusion $H \subset Q$ is *tame*.

Algorithm 7.1.1 allows one to construct a tame inclusion.

Fix Levi decompositions $H = \mathrm{R}_u(H) \leftthreetimes S, Q = \mathrm{R}_u(Q) \leftthreetimes M$. Conjugating H by an element of Q, one may assume that $S \subset M$. Besides, conjugating Q, M, H by an element of G, one may assume that Q is an antistandard parabolic subgroup and M is its standard Levi subgroup.

The following lemma and remark seem to be standard.

Lemma 3.2.7. *Let Q, M, H, S be such as in the previous discussion and V an H-module, $Q^- := BM$. Put $X = G *_H V, \underline{X} := Q *_H V$. Then the fields $\mathbb{C}(\underline{X}), \mathbb{C}(X)^{\mathrm{R}_u(Q^-)}$ are M-equivariantly isomorphic.*

Proof. Consider the map $\iota : X^0 := \mathrm{R}_u(Q^-) \times \underline{X} \to X, (q, [m, x]) \mapsto [qm, x]$. Define the action of $Q^- = M \rightthreetimes \mathrm{R}_u(Q^-)$ on X^0 as follows: $q_1.(q, [g, v]) = (qq_1, [g, v])$, $m_1.(q, [g, v]) = (m_1 q m_1^{-1}, [m_1 g, v]), q_1, q \in \mathrm{R}_u(Q^-), m_1 \in M, g \in Q, v \in V$. The morphism ι becomes Q-equivariant. One easily checks that ι is injective. Since $\dim X^0 = \dim X$, the morphism ι is dominant. Taking into account that X is smooth, we see that ι is an open embedding. So $\mathbb{C}(X^0), \mathbb{C}(X)$ are Q^--equivariantly isomorphic, whence the claim of the lemma. □

Remark 3.2.8. Let Q, M, H, S, V be such as in Lemma 3.2.7. Let us construct an M-isomorphism of $Q *_H V$ and $M *_S ((\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})) \oplus V)$. Consider the decreasing chain of ideals $\mathrm{R}_u(\mathfrak{q}) = \mathfrak{q}^{(1)} \supset \mathfrak{q}^{(2)} \supset \ldots \supset \mathfrak{q}^{(m)} = \{0\}$, where $\mathfrak{q}^{(i+1)} = [\mathrm{R}_u(\mathfrak{q}), \mathfrak{q}^{(i)}]$. Choose an S-submodule $V_i \subset \mathfrak{q}^{(i)}, i = 1, \ldots, m-1$, complementary to $\mathfrak{q}^{(i+1)} + (\mathfrak{q}^{(i)} \cap \mathfrak{h})$. The map $M *_S (V_1 \oplus \cdots \oplus V_{m-1} \oplus V) \to Q *_H V, [m, (v_1, \ldots, v_{m-1}, v)] \mapsto [m \exp(v_1) \ldots \exp(v_{m-1}), v]$ is a well-defined M-isomorphism.

The following proposition stems from Lemma 3.2.7 and Remark 3.2.8. It is also a direct generalization of a part of Theorem 1.2 from [Pa1] (see also [Pa4], Theorem 2.5.20).

Proposition 3.2.9. *Let Q be an antistandard parabolic subgroup of G, M its standard Levi subgroup and H an algebraic subgroup of Q such that the inclusion $H \subset Q$ is tame and $S := M \cap H$ is a Levi subgroup in H. Then $\mathfrak{X}_{G, G *_H V} = \mathfrak{X}_{M, M *_S ((\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})) \oplus V)}$.*

Next, we reduce the case of affine homogeneous vector bundles to that of affine homogeneous spaces. To state the main result (Proposition 3.2.12) we need the notion of the distinguished component.

First of all, set

$$L_{G,X} := Z_G(\mathfrak{a}_{G,X}), \tag{3.1}$$

$$L_{0\,G,X} := \{g \in L | \chi(g) = 1, \forall \chi \in \mathfrak{X}_{G,X}\}. \tag{3.2}$$

Proposition 3.2.10 ([Lo2, Proposition 8.4]). *Suppose X is smooth and quasi-affine.*

(1) *Let L_1 be a normal subgroup of $L_{0\,G,X}$. Then there exists a unique irreducible component $\underline{X} \subset X^{L_1}$ such that $\overline{U\underline{X}} = X$.*

(2) *Set* $P := L_{G,X}B$. *Let* S *be a locally closed* $L_{G,X}$*-stable subvariety of* X *such that* $(L_{G,X}, L_{G,X})$ *acts trivially on* S *and the map* $P *_{L_{G,X}} S \to X, [p,s] \mapsto ps$, *is an embedding (such an* S *always exists; see* [Kn4], *Section 2, Lemma 3.1). Then* $\underline{X} := \overline{FS}$, *where* $F := \mathrm{R}_u(P)^{L_1}$.

Definition 3.2.11. The component $\underline{X} \subset X^{L_1}$ satisfying the assumptions of Proposition 3.2.10 is said to be *distinguished*.

The distinguished component for $L_1 = L_{0\,G,X}$ was considered by Panyushev in [Pa2].

Proposition 3.2.12. *Let* H *be a reductive subgroup in* G, V *an* H*-module and* π *the natural projection* $G *_H V \to G/H$. *Put* $L_1 := L^{\circ}_{0\,G,G/H}$. *Let* x *be a point from the distinguished component of* $(G/H)^{L_1}$. *Then* $\mathfrak{l}_{0\,G,G*_HV} = \mathfrak{l}_{0\,L_1,\pi^{-1}(x)}$.

Proof. Let x_1 be a canonical point in general position in the sense of [Pa4], Definition 5 of Subsection 2.1. This means that $B_{x_1} = B \cap L_1$. It follows from Theorem 2.5.20 from [Pa4] that $L^{\circ}_{0\,G,G*_HV} \cap B = L^{\circ}_{0\,L_1,\pi^{-1}(x)} \cap B$. Thus $L^{\circ}_{0\,G,G*_HV} = L^{\circ}_{0\,L_1,\pi^{-1}(x)}$. Note that the L_1-module $\pi^{-1}(x)$ does not depend on the choice of a point x from the distinguished component. Now it remains to note that the distinguished component of $(G/H)^{L_1}$ contains a canonical point in general position. Indeed, let S be an $L_{G,G/H}$-subvariety in G/H mentioned in assertion 2 of Proposition 3.2.10 (for $X = G/H$). Such an S consists of canonical points in general position. □

Algorithm 7.1.1 computes the subgroups $L_{0\,G,V}$ for a G-module V; see [Pa4].

Thus the computation of $\mathfrak{a}_{G,X}$ is reduced to the computation of the following data:

(1) The spaces $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$, where H is a reductive subgroup of G.
(2) A point from the distinguished component of $(G/H)^{L_1}$, where $L_1 = L^{\circ}_{0\,G,G/H}$, for a reductive subgroup $H \subset G$.

Now we are going to present results of the paper [Lo1] concerning the computation of $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$. Until further notice, H denotes a reductive subgroup in G.

To state our main results we need some definitions. We begin with a standard one.

Definition 3.2.13. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be *indecomposable* if $(\mathfrak{h}\cap\mathfrak{g}_1)\oplus(\mathfrak{h}\cap\mathfrak{g}_2) \subsetneq \mathfrak{h}$ for any pair of ideals $\mathfrak{g}_1,\mathfrak{g}_2 \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Since $\mathfrak{a}(\mathfrak{g}_1\oplus\mathfrak{g}_2, \mathfrak{h}_1\oplus\mathfrak{h}_2) = \mathfrak{a}(\mathfrak{g}_1,\mathfrak{h}_1)\oplus\mathfrak{a}(\mathfrak{g}_2,\mathfrak{h}_2)$, the computation of $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ can be easily reduced to the case when $\mathfrak{h}\subset\mathfrak{g}$ is indecomposable.

By virtue of Corollary 3.2.2, $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$ for any ideal $\mathfrak{h}_1 \subset \mathfrak{h}$. This observation motivates the following definition.

Definition 3.2.14. A reductive subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is called $\mathfrak{a}$-*essential* if $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) \subsetneq \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$ for any ideal $\mathfrak{h}_1 \subsetneq \mathfrak{h}$.

To make the presentation of our results more convenient, we introduce one more class of subalgebras.

Definition 3.2.15. An $\mathfrak{a}$-essential subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is said to be *saturated* if $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}})$, where $\widetilde{\mathfrak{h}} := \mathfrak{h} + \mathfrak{z}(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}))$.

Finally, we note that Lemmas 3.2.18, 3.2.19 below allow us to perform the computation of $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ just for one subalgebra $\mathfrak{h}$ in a given class of $\mathrm{Aut}(\mathfrak{g})$-conjugacy.

The next proposition is a part of Theorem 1.3 from [Lo1].

Proposition 3.2.16. (1) *There is a unique ideal $\mathfrak{h}^{ess}\subset\mathfrak{h}$ such that $\mathfrak{h}^{ess}$ is an $\mathfrak{a}$-essential subalgebra of $\mathfrak{g}$ and $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{a}(\mathfrak{g},\mathfrak{h}^{ess})$. The ideal $\mathfrak{h}^{ess}$ is maximal (with respect to inclusion) among all ideals of $\mathfrak{h}$ that are $\mathfrak{a}$-essential subalgebras of $\mathfrak{g}$.*

(2) *All semisimple indecomposable $\mathfrak{a}$-essential subalgebras in $\mathfrak{g}$ up to* $\mathrm{Aut}(\mathfrak{g})$-*conjugacy are listed in Table* 3.1.

(3) *All nonsemisimple saturated indecomposable $\mathfrak{a}$-essential subalgebras in $\mathfrak{g}$ up to* $\mathrm{Aut}(\mathfrak{g})$-*conjugacy are listed in Table* 3.2. *In all cases, $\mathfrak{h}=[\mathfrak{h},\mathfrak{h}]\oplus\mathfrak{z}(\mathfrak{z}_\mathfrak{g}([\mathfrak{h},\mathfrak{h}]))$ and $\dim\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]=1$.*

In Theorem 1.3 from [Lo1] all essential subalgebras are classified and a way to compute the Cartan spaces for them is given. Note that, as soon as this is done, assertion (1) of Proposition 3.2.16 provides an effective method for the determination of $\mathfrak{h}^{ess}$.

TABLE 3.1. Semisimple indecomposable $\mathfrak{a}$-essential subalgebras $\mathfrak{h}\subset\mathfrak{g}$

N	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{a}(\mathfrak{g},\mathfrak{h})$
1	$\mathfrak{sl}_n, n\geqslant 2$	$\mathfrak{sl}_k, \frac{n+2}{2}\leqslant k\leqslant n$	$\langle\pi_i,\pi_{n-i}; i\leqslant n-k\rangle$
2	$\mathfrak{sl}_n, n\geqslant 4$	$\mathfrak{sl}_k\times\mathfrak{sl}_{n-k}, \frac{n}{2}\leqslant k\leqslant n-2$	$\langle\pi_i+\pi_{n-i},\pi_k,\pi_{n-k}; i<n-k\rangle$
3	$\mathfrak{sl}_{2n}, n\geqslant 2$	$\mathfrak{sp}_{2n}$	$\langle\pi_{2i}; i\leqslant n-1\rangle$
4	$\mathfrak{sp}_{2n}, n\geqslant 2$	$\mathfrak{sp}_{2k}, \frac{n+1}{2}\leqslant k\leqslant n$	$\langle\pi_i; i\leqslant 2(n-k)\rangle$
5	$\mathfrak{sp}_{2n}, n\geqslant 2$	$\mathfrak{sp}_{2k}\times\mathfrak{sp}_{2(n-k)}, \frac{n}{2}\leqslant k<n$	$\langle\pi_{2i}; i\leqslant n-k\rangle$
6	$\mathfrak{sp}_{2n}, n\geqslant 4$	$\mathfrak{sp}_{2n-4}\times\mathfrak{sl}_2\times\mathfrak{sl}_2$	$\langle\pi_2,\pi_4,\pi_1+\pi_3\rangle$
7	$\mathfrak{sp}_6$	$\mathfrak{sl}_2\times\mathfrak{sl}_2\times\mathfrak{sl}_2$	$\langle\pi_2,\pi_1+\pi_3\rangle$
8	$\mathfrak{so}_n, n\geqslant 7$	$\mathfrak{so}_k, \frac{n+2}{2}\leqslant k\leqslant n$	$\langle\pi_i; i\leqslant n-k\rangle$
9	$\mathfrak{so}_{4n}, n\geqslant 2$	$\mathfrak{sl}_{2n}$	$\langle\pi_{2i}; i\leqslant n\rangle$
10	$\mathfrak{so}_{4n+2}, n\geqslant 2$	$\mathfrak{sl}_{2n+1}$	$\langle\pi_{2i},\pi_{2n+1}; i\leqslant n\rangle$
11	$\mathfrak{so}_9$	$\mathfrak{spin}_7$	$\langle\pi_1,\pi_4\rangle$
12	$\mathfrak{so}_{10}$	$\mathfrak{spin}_7$	$\langle\pi_1,\pi_2,\pi_4,\pi_5\rangle$
13	$\mathfrak{so}_7$	G_2	$\langle\pi_3\rangle$
14	$\mathfrak{so}_8$	G_2	$\langle\pi_1,\pi_3,\pi_4\rangle$
15	G_2	A_2	$\langle\pi_1\rangle$
16	F_4	B_4	$\langle\pi_1\rangle$
17	F_4	D_4	$\langle\pi_1,\pi_2\rangle$
18	E_6	F_4	$\langle\pi_1,\pi_5\rangle$
19	E_6	D_5	$\langle\pi_1,\pi_5,\pi_6\rangle$
20	E_6	B_4	$\langle\pi_1,\pi_2,\pi_4,\pi_5,\pi_6\rangle$
21	E_6	A_5	$\langle\pi_1+\pi_5,\pi_2+\pi_4,\pi_3,\pi_6\rangle$
22	E_7	E_6	$\langle\pi_1,\pi_2,\pi_6\rangle$
23	E_7	D_6	$\langle\pi_2,\pi_4,\pi_5,\pi_6\rangle$

N	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{a}(\mathfrak{g},\mathfrak{h})$
24	E_8	E_7	$\langle\pi_1,\pi_2,\pi_3,\pi_7\rangle$
25	$\mathfrak{h}\times\mathfrak{h}$	$\mathfrak{h}$	$\langle\pi_i^*+\pi_i';i\leqslant \operatorname{rk}\mathfrak{h}\rangle$
26	$\mathfrak{sp}_{2n}\times\mathfrak{sp}_{2m}, m>n>1$	$\mathfrak{sp}_{2n-2}\times\mathfrak{sl}_2\times\mathfrak{sp}_{2m-2}$	$\langle\pi_2,\pi_2',\pi_1+\pi_1'\rangle$
27	$\mathfrak{sp}_{2n}\times\mathfrak{sl}_2, n>1$	$\mathfrak{sp}_{2n-2}\times\mathfrak{sl}_2$	$\langle\pi_2,\pi_1+\pi_1'\rangle$

If $\mathfrak{g}$ has two simple ideals (rows 25–27), then by π_i (resp., π_i') we denote fundamental weights of the first (resp., the second) one.

TABLE 3.2. Nonsemisimple saturated $\mathfrak{a}$-essential indecomposable subalgebras $\mathfrak{h}\subset\mathfrak{g}$

$(\mathfrak{g},[\mathfrak{h},\mathfrak{h}])$	$\mathfrak{a}(\mathfrak{g},\mathfrak{h})$
$(\mathfrak{sl}_n,\mathfrak{sl}_k), k>\frac{n}{2}$	$\{\sum_{i=1}^{n-k}(x_i\pi_i+x_{n-i}\pi_{n-i});\sum_{i=1}^{n-k}i(x_i-x_{n-i})=0\}$
$(\mathfrak{sl}_n,\mathfrak{sl}_k\times\mathfrak{sl}_{n-k}), k>\frac{n}{2}$	$\langle\pi_i+\pi_{n-i};i\leqslant n-k\rangle$
$(\mathfrak{sl}_{2n+1},\mathfrak{sp}_{2n})$	$\{\sum_{i=1}^{2n}x_i\pi_i;\sum_{i=0}^{n-1}(n-i)x_{2i+1}-\sum_{i=1}^{n}ix_{2i}=0\}$
$(\mathfrak{so}_{4n+2},\mathfrak{sl}_{2n+1})$	$\langle\pi_{2i},\pi_{2n}+\pi_{2n+1};i\leqslant n-1\rangle$
(E_6,D_5)	$\langle\pi_1+\pi_5,\pi_6\rangle$

Remark 3.2.17. Note that for all subalgebras $\mathfrak{h}$ from Tables 3.1, 3.2 except N8 ($n=8$), N9, N25, the class of $\mathrm{Aut}(\mathfrak{g})$-conjugacy of $\mathfrak{h}$ coincides with the class of $\mathrm{Int}(\mathfrak{g})$-conjugacy. In case $8, n=8$ (resp., 9,25), the class of $\mathrm{Aut}(\mathfrak{g})$-conjugacy is the union of 3 (resp., 2,$\#\mathrm{Aut}(\mathfrak{h})/\mathrm{Int}(\mathfrak{h})$) classes of $\mathrm{Int}(\mathfrak{g})$-conjugacy.

At the end of this subsection we consider the behavior of Cartan spaces, Weyl groups, etc. under the twisting of the action $G:X$ by an automorphism.

Let $\tau\in\mathrm{Aut}(G)$. By ${}^\tau X$ we denote the G-variety coinciding with X as a variety, the action of G being defined by $(g,x)\mapsto\tau^{-1}(g)x$. The identity map is an isomorphism of ${}^\sigma({}^\tau X)$ and ${}^{\sigma\tau}X$ for $\sigma,\tau\in\mathrm{Aut}(G)$. If τ is an inner automorphism, $\tau(g)=hgh^{-1}$, then $x\mapsto hx:{}^\tau X\mapsto X$ is a G-isomorphism. Hence the G-variety ${}^\tau X$ is determined up to isomorphism by the image of τ in $\mathrm{Aut}(G)/\mathrm{Int}(G)$. In particular, considering G-varieties of the form ${}^\tau X$, one may assume that $\tau(B)=B,\tau(T)=T$.

Lemma 3.2.18. *Let $\tau\in\mathrm{Aut}(G),\tau(B)=B,\tau(T)=T$. Then $\mathfrak{X}_{G,{}^\tau X}=\tau(\mathfrak{X}_{G,X})$, $\mathfrak{a}_{G,{}^\tau X}=\tau(\mathfrak{a}_{G,X})$, $L_{G,{}^\tau X}=\tau(L_{G,X})$, $L_{0\,G,{}^\tau X}=\tau(L_{0\,G,X})$, $W_{G,{}^\tau X}=\tau W_{G,X}\tau^{-1}$. Further, if L_1 is a normal subgroup in $L_{0\,G,X}$, then the distinguished components in X^{L_1} and ${}^\tau X^{\tau(L_1)}$ coincide.*

Proof. Let $f\in\mathbb{C}(X)$ be a B-semi-invariant function of weight χ. Then f considered as an element of $\mathbb{C}({}^\tau X)$ is B-semi-invariant of weight $\tau(\chi)$. Assertions on $\mathfrak{a}_{G,\bullet},\mathfrak{X}_{G,\bullet},L_{G,\bullet},L_{0\,G,\bullet}$ follow immediately from this observation. U-orbits of the actions $G:X,G:{}^\tau X$ coincide, whence the equalities for the distinguished components. Now let v be a central valuation of X. Since $\mathbb{C}(X)^{(B)}=\mathbb{C}({}^\tau X)^{(B)}$, we see that v is a central valuation of ${}^\tau X$. Let $\varphi_v,{}^\tau\varphi_v$ be the corresponding elements in $\mathfrak{a}_{G,X}(\mathbb{R})^*,\mathfrak{a}_{G,{}^\tau X}(\mathbb{R})^*$. Then $\langle\varphi_v,\lambda\rangle=v(f)=\langle{}^\tau\varphi_v,\tau(\lambda)\rangle$, where $f\in\mathbb{C}(X)^{(B)}_\lambda$, whence the equality for the Weyl groups. $\square$

Lemma 3.2.19. *Let $\tau \in \mathrm{Aut}(G)$.*

1. *If H is an algebraic subgroup of G, then ${}^{\tau}(G/H) \cong G/\tau(H)$.*
2. *Let H be a reductive subgroup of G and let V be an H-module. Then ${}^{\tau}(G *_H V) = G *_{\tau(H)} V$, where $\tau(H)$ acts on V via the isomorphism $\tau^{-1} : \tau(H) \to H$.*

Proof. Required isomorphisms are given by $gH \mapsto \tau(g)\tau(H)$ and $[g, v] \mapsto [\tau(g), v]$. □

3.3. **Results about Weyl groups.**

Proposition 3.3.1. *Let $\mathfrak{h}$ be an algebraic subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}_0$ be a subalgebra of $\mathfrak{g}$ lying in the closure of $G\mathfrak{h}$ in $\mathrm{Gr}(\mathfrak{g}, \dim \mathfrak{h})$. Then $\mathfrak{h}_0$ is an algebraic subalgebra of $\mathfrak{g}$, $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{a}(\mathfrak{g}, \mathfrak{h}_0)$ and $W(\mathfrak{g}, \mathfrak{h}_0) \subset W(\mathfrak{g}, \mathfrak{h})$.*

Proof. This stems from [Kn7], Lemmas 3.1, 4.2. □

Let T_0 be a torus and $\pi : \widetilde{X} \to X$ be a principal locally trivial T_0-bundle, where T_0 is a torus. In particular, T_0 acts freely on $\widetilde{X}$ and X is a quotient for this action. Now let H be an algebraic group acting on X. The bundle $\pi : \widetilde{X} \to X$ is said to be *H-equivariant* if $\widetilde{X}$ is equipped with an action $H : \widetilde{X}$ commuting with that of T_0 and such that π is an H-morphism. We can consider $\widetilde{X}, X$ as $H \times T_0$-varieties (T_0 acts trivially on X) and π as an $H \times T_0$-morphism.

Proposition 3.3.2 ([Kn4, Theorem 5.1])**.** *Let X be an irreducible G-variety, T_0 a torus and $\pi : \widetilde{X} \to X$ a G-equivariant principal locally trivial T_0-bundle. Set $\widetilde{G} := G \times T_0$. Then $\mathfrak{a}_{\widetilde{G},\widetilde{X}} = \mathfrak{a}_{G,X} \oplus \mathfrak{t}_0$, $\mathfrak{t}_0, \mathfrak{a}_{G,X}$ are $W_{\widetilde{G},\widetilde{X}}$-subspaces in $\mathfrak{a}_{\widetilde{G},\widetilde{X}}$, and $W_{\widetilde{G},\widetilde{X}}$ acts trivially on $\mathfrak{t}_0$ and as $W_{G,X}$ on $\mathfrak{a}_{G,X}$.*

Now we want to establish a relation between the linear part of the cone $\mathcal{V}_{G,X}$, which coincides with $\mathfrak{a}^*_{G,X}(\mathbb{R})^{W_{G,X}}$, and a certain subgroup of $\mathrm{Aut}^G(X)$.

Definition 3.3.3. A G-automorphism φ of X is said to be *central* if φ acts on $\mathbb{C}(X)^{(B)}_{\lambda}$ by multiplication by a constant for any $\lambda \in \mathfrak{X}_{G,X}$. Central automorphisms of X form the subgroup of $\mathrm{Aut}^G(X)$ denoted by $\mathfrak{A}_G(X)$.

Lemma 3.3.4 ([Kn5, Corollary 5.6])**.** *A central automorphism commutes with any G-automorphism of X.*

It turns out that $\mathfrak{A}_G(X)$ is not a birational invariant of X. However, by Corollary 5.4 from [Kn5], there is an open G-subvariety X_0 such that $\mathfrak{A}_G(X_0) = \mathfrak{A}_G(X_1)$ for any open G-subvariety $X_1 \subset X_0$. We denote $\mathfrak{A}_G(X_0)$ by $\mathfrak{A}_{G,X}$.

Put $A_{G,X} := \mathrm{Hom}(\mathfrak{X}_{G,X}, \mathbb{C}^\times)$. The group $\mathfrak{A}_{G,X}$ is embedded into $A_{G,X}$ as follows. We assign $a_{\varphi,\lambda}$ to $\varphi \in \mathfrak{A}_{G,X}, \lambda \in \mathfrak{X}_{G,X}$ by the formula $\varphi f_\lambda = a_{\varphi,\lambda} f_\lambda$, $f \in \mathbb{C}(X)^{(B)}_{\lambda}$. The map $\iota_{G,X} : \mathfrak{A}_{G,X} \to A_{G,X}$ is given by $\lambda(\iota_{G,X}(\varphi)) = a_{\varphi,\lambda}$. Clearly, $\iota_{G,X}$ is a well-defined homomorphism.

Lemma 3.3.5 ([Kn5, Theorem 5.5])**.** *$\iota_{G,X}$ is injective and its image is closed. In particular, $\mathfrak{A}^{\circ}_{G,X}$ is a torus.*

In the sequel we identify $\mathfrak{A}_{G,X}$ with $\mathrm{im}\, \iota_{G,X}$.

The following proposition is a corollary of Satz 8.1 and (in fact, the proof of) Satz 8.2 from [Kn3].

Proposition 3.3.6. *The Lie algebra of $\mathfrak{A}_{G,X}$ coincides with $\mathfrak{a}_{G,X}^{W_{G,X}}$.*

Until further notice, we assume that X is normal. Let us study a relation between the central valuation cones of X and of certain G-divisors of X. Our goal is to prove Corollary 3.3.9, which will be used at the end of the subsection to obtain restrictions on possible Weyl groups of homogeneous spaces (Proposition 3.3.23).

Let v_0 be a nonzero $\mathbb{R}$-valued discrete geometric valuation of $\mathbb{C}(X)^B$ (by definition, v_0 is *geometric* if it is a nonnegative multiple of the valuation induced by a divisor on some model of $\mathbb{C}(X)^B$). We denote by $\mathcal{V}_{v_0}$ the set of all geometric G-valuations v of $\mathbb{C}(X)$ such that $v|_{\mathbb{C}(X)^B} = kv_0$ for some $k \geqslant 0$.

Now we construct a map from $\mathcal{V}_{v_0}$ to a finite-dimensional vector space similar to that from Subsection 1.1. Namely, we have an exact sequence of abelian groups

$$\{1\} \to \mathbb{C}(X)^{B\times} \to \mathbb{C}(X)^{(B)} \setminus \{0\} \to \mathfrak{X}_{G,X} \to \{0\}.$$

This sequence splits because $\mathfrak{X}_{G,X}$ is free. Fix a splitting $\lambda \mapsto f_\lambda$. We assign an element $(\varphi_v, k_v) \in \mathfrak{a}_{G,X} \times \mathbb{R}_{\geqslant 0}$ to $v \in \mathcal{V}_{v_0}$ by the following formula:

$$\langle \varphi_v, \lambda \rangle = v(f_\lambda), v|_{\mathbb{C}(X)^B} = k_v v_0.$$

There is a statement similar to Theorem 1.1.4.

Proposition 3.3.7. (1) *The map $v \mapsto (\varphi_v, k_v)$ is injective, so we may identify $\mathcal{V}_{v_0}$ with its image.*
(2) *$\mathcal{V}_{v_0} \subset \mathfrak{a}_{G,X} \times \mathbb{R}_{\geqslant 0}$ is a simplicial cone. One of its faces is the central valuation cone considered as a subspace in $\mathfrak{a}_{G,X} \times \{0\} \subset \mathfrak{a}_{G,X} \times \mathbb{R}_{\geqslant 0}$.*

In particular, there is $v \in \mathcal{V}_{v_0}$ such that $\mathcal{V}_{v_0}$ coincides with the cone spanned by v and $\mathcal{V}_{G,X}$. Such a v is defined uniquely up to rescaling and the shift by an element of $\mathcal{V}_{G,X} \cap -\mathcal{V}_{G,X}$.

The first part of this proposition stems from [Kn3], Korollare 3.6, 4.2. The second one is a reformulation of the second part of Satz 9.2 from [Kn3].

The following result is a special case of Satz 7.5 from [Kn3].

Proposition 3.3.8. *Let v be as above and D be a prime G-divisor on X such that its valuation is a positive multiple of v. Then $\mathfrak{a}_{G,D} = \mathfrak{a}_{G,X}, W_{G,D} = W_{G,X}$.*

Corollary 3.3.9. *Let X be an irreducible G-variety. Then there exists a spherical G-variety X' such that $\mathfrak{a}_{G,X} = \mathfrak{a}_{G,X'}, W_{G,X} = W_{G,X'}$.*

Proof. Replacing X with some birationally equivalent G-variety, we may assume that there is a divisor $D \subset X$ satisfying the assumptions of Proposition 3.3.8. Since the valuation corresponding to D is noncentral, we have $c_G(D) = c_G(X) - 1$ ([Kn3], Satz 7.3). Now it remains to use the induction on complexity. $\square$

Now we quote some results from [Lo2] providing some reduction procedures for computing Weyl groups.

Until further notice, X is a smooth quasi-affine G-variety. Set $L_0 := L^\circ_{0\,G,X}$. Let $\underline{X}$ denote the distinguished component of $X^{L_0^\circ}$; see Definition 3.2.11. Note that $\underline{X}$ is a smooth quasi-affine variety. Its smoothness is a standard fact; see, for example, [Lo2], Lemma 8.6. Put $\underline{G} := N_G(L_0^\circ, \underline{X})/L_0^\circ$. It is a reductive group acting on $\underline{X}$. Note that its tangent algebra $\underline{\mathfrak{g}}$ can be naturally identified with $\mathfrak{l}_0^\perp \cap \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = \mathfrak{g}^{L_0^\circ} \cap \mathfrak{z}(\mathfrak{l}_0)^\perp \subset \mathfrak{g}$. Further, note that $\mathfrak{t} \subset \mathfrak{n}_\mathfrak{g}(\mathfrak{g}_1)$. Thus there are the distinguished maximal torus $\underline{T}$ and the Borel subgroup $\underline{B}$ in $\underline{G}$; see Section 2. Let Γ denote the image of $N_{\underline{G}}(\underline{T}) \cap N_{\underline{G}}(\underline{B})$ in $\mathrm{GL}(\underline{\mathfrak{t}})$.

Theorem 3.3.10 ([Lo2, Theorem 8.7, Proposition 8.1])**.** *Let X be a smooth quasi-affine G-variety, and let $\underline{X}, \underline{G}, \Gamma$ be such as in the previous paragraph. Then* $\mathfrak{a}_{G,X} = \mathfrak{a}_{\underline{G}^\circ,\underline{X}} = \underline{\mathfrak{t}}$, $W_{G,X} = W_{\underline{G}^\circ,\underline{X}} \leftthreetimes \Gamma$.

It is easy to prove, see Section 4, that if X is a homogeneous space (resp., affine homogeneous vector bundle), then $\underline{X}$ is a homogeneous space (resp., affine homogeneous vector bundle) with respect to the action of $\underline{G}^\circ$.

Proposition 3.3.11. *Let H be an algebraic subgroup of G, $H = S \rightthreetimes \mathrm{R}_u(H)$ its Levi decomposition, Q an antistandard parabolic subgroup in G and M its standard Levi subgroup. Suppose that $\mathrm{R}_u(H) \subset \mathrm{R}_u(Q), S \subset M$. Then*

(1) *If G/H is quasi-affine and $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}$, then $\mathfrak{a}(\mathfrak{m},\mathfrak{s},\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})) = \mathfrak{t}$ and $W(\mathfrak{g},\mathfrak{h}) \cap M/T = W(\mathfrak{m},\mathfrak{s},\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}))$.*
(2) *Suppose $H = S$. Let V be an H-module such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h},V) = \mathfrak{t}$. Then $\mathfrak{a}(\mathfrak{m},\mathfrak{h},\mathrm{R}_u(\mathfrak{q}) \oplus V) = \mathfrak{t}$ and $W(\mathfrak{g},\mathfrak{h},V) \cap M/T = W(\mathfrak{m},\mathfrak{h},\mathrm{R}_u(\mathfrak{q}) \oplus V)$.*

Proof. This follows from [Lo2], Proposition 8.13, and Lemma 3.2.7. □

The following proposition may be considered as a weakened version of Proposition 3.3.6.

Proposition 3.3.12 ([Lo2, Proposition 8.3])**.** *Let T_0 be a torus acting on X by G-equivariant automorphisms. Put $\widetilde{G} = G \times T_0$. Then $\mathfrak{a}_{\widetilde{G},X} \cap \mathfrak{g} \subset \mathfrak{a}_{G,X}$ and $\mathfrak{a}_{G,X} \cap (\mathfrak{a}_{\widetilde{G},X} \cap \mathfrak{g})^\perp \subset \mathfrak{a}_{G,X}^{W_{G,X}}$.*

Corollary 3.3.13. *Let H_1, H_2 be subgroups of G such that $H_2 \subset N_G(H_1)$ and H_2/H_1 is a torus. Suppose that G/H_1 is quasi-affine. Then $W(\mathfrak{g},\mathfrak{h}_2) \cong W(\mathfrak{g},\mathfrak{h}_1)$, the inclusion $\mathfrak{a}(\mathfrak{g},\mathfrak{h}_2) \hookrightarrow \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$ (induced by the natural epimorphism $G/H_1 \to G/H_2$) is $W(\mathfrak{g},\mathfrak{h}_2)$-equivariant and the action of $W(\mathfrak{g},\mathfrak{h}_2)$ on $\mathfrak{a}(\mathfrak{g},\mathfrak{h}_2)^\perp \cap \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$ is trivial.*

Proof. The morphism $G/H_1 \to G/H_2$ is a G-equivariant principal H_2/H_1-bundle. Put $\widetilde{G} = G \times H_2/H_1$. According to Proposition 3.3.2, $W(\mathfrak{g},\mathfrak{h}_2)$ is identified with $W_{\widetilde{G},G/H_1}$ and the embedding $\mathfrak{a}(\mathfrak{g},\mathfrak{h}_2) \hookrightarrow \mathfrak{a}(\mathfrak{g},\mathfrak{h}_2) \oplus \mathfrak{t}_0 = \mathfrak{a}_{\widetilde{G},G/H_1}$ is $W(\mathfrak{g},\mathfrak{h}_2)$-equivariant, where the action $W(\mathfrak{g},\mathfrak{h}_2) : \mathfrak{t}_0$ is trivial. The embedding $\mathfrak{a}(\mathfrak{g},\mathfrak{h}_2) \hookrightarrow \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$ is the composition of the embedding $\mathfrak{a}(\mathfrak{g},\mathfrak{h}_2) \hookrightarrow \mathfrak{a}_{\widetilde{G},G/H_1}$ and the orthogonal projection $\mathfrak{a}_{\widetilde{G},G/H_1} \to \mathfrak{a}(\mathfrak{g},\mathfrak{h}_1)$. The required claims follow now from Proposition 3.3.12 applied to the action $\widetilde{G} : G/H_1$. □

The next well-known statement describes the behavior of the Weyl group under a so-called parabolic induction. The only case we need is that of homogeneous spaces of rank equal to $\mathrm{rk}\, G$. We give the proof only to illustrate our techniques.

Corollary 3.3.14. *Let Q be an antistandard parabolic subgroup of G and M the standard Levi subgroup of Q. Further, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ such that $\mathrm{R}_u(\mathfrak{q}) \subset \mathfrak{h}, \mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}$. Finally, assume that G/H is quasi-affine. Then $\mathfrak{a}(\mathfrak{m},\mathfrak{m}/\mathfrak{h}) = \mathfrak{t}$, $W(\mathfrak{g},\mathfrak{h}) = W(\mathfrak{m},\mathfrak{h}/\mathrm{R}_u(\mathfrak{q}))$.*

Proof. Using Proposition 3.3.11, we obtain $\mathfrak{a}(\mathfrak{m},\mathfrak{m} \cap \mathfrak{h}) = \mathfrak{t}, W(\mathfrak{m},\mathfrak{h}/\mathrm{R}_u(\mathfrak{q})) \subset W(\mathfrak{g},\mathfrak{h})$. On the other hand, $Z(M)^\circ$ acts on G/H by G-automorphisms. By Proposition 3.2.9, $\mathfrak{a}_{G\times Z(M)^\circ, G/\mathrm{R}_u(Q)} \cap \mathfrak{g} = \mathfrak{t} \cap [\mathfrak{m},\mathfrak{m}]$. Therefore $\mathfrak{a}_{G\times Z(M)^\circ, G/H} \cap \mathfrak{g} \subset \mathfrak{t} \cap [\mathfrak{m},\mathfrak{m}]$. The inclusion $W(\mathfrak{m},\mathfrak{h}/\mathrm{R}_u(\mathfrak{q})) \supset W(\mathfrak{g},\mathfrak{h})$ follows from Proposition 3.3.12. □

Now let $G_1,\ldots,G_k$ be simple normal subgroups in G so that $G=Z(G)^\circ G_1\ldots G_k$. Put $T_i=T\cap G_i$. This is a maximal torus in G_i.

Proposition 3.3.15. (1) *Let H be an algebraic subgroup of G such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}$ and G/H is quasi-affine. Then $G_i/(G_i\cap H)$ is quasi-affine too and $\mathfrak{a}(\mathfrak{g}_i,\mathfrak{g}_i\cap\mathfrak{h})=\mathfrak{t}_i, W(\mathfrak{g},\mathfrak{h})=\prod_i W(\mathfrak{g}_i,\mathfrak{g}_i\cap\mathfrak{h})$.*

(2) *Let H be a reductive subgroup of G and let V be an H-module. Suppose that $\mathfrak{a}(\mathfrak{g},\mathfrak{h},V)=\mathfrak{t}$. Then $\mathfrak{a}(\mathfrak{g}_i,\mathfrak{g}_i\cap\mathfrak{h},V)=\mathfrak{t}_i$ and $W(\mathfrak{g},\mathfrak{h},V)=\prod_i W(\mathfrak{g}_i,\mathfrak{g}_i\cap\mathfrak{h},V)$.*

Proof. To prove the first assertion we note that the stabilizer of any point of G/H in G_i is conjugate to $G_i\cap H$ and use assertion 4 of Proposition 3.2.1 and [Lo2], Proposition 8.8.

Let us proceed to assertion 2. All G_i-orbits in G/H are of the same dimension, whence closed. Choose $x\in G/H$ with $(G_i)_x=G_i\cap H$. The G_x-module $T_xX/\mathfrak{g}_*^{(i)}x$ is isomorphic to $V\oplus V_0$, where V_0 denotes a trivial G_x-module. By Lemma 3.2.3, $W_{G_i,G*_HV}=W(\mathfrak{g}_i,\mathfrak{g}_i\cap\mathfrak{h},V)$. It remains to use Corollary 3.2.4. □

Until the end of the section, G is simple, and X is a quasi-affine irreducible smooth G-variety such that $\mathrm{rk}_G(X)=\mathrm{rk}\,G$. In this case $W_{G,X}\subset W(\mathfrak{g})$.

At first, we consider the case when X is affine. Here we present some results on $W_{G,X}$ obtained in [Lo3]. Those results can be applied here because $W_{G,X}$ coincides up to conjugacy with the Weyl group of the Hamiltonian G-variety T^*X (see [Kn4]).

Definition 3.3.16. A subgroup $\Gamma\subset W(\mathfrak{g})$ is said to be *large* if for any roots $\alpha,\beta\in\Delta(\mathfrak{g})$ such that $\beta\neq\pm\alpha,(\alpha,\beta)\neq 0$ there exists $\gamma\in\mathbb{R}\alpha+\mathbb{R}\beta$ with $s_\gamma\in\Gamma$.

For a subgroup $\Gamma\subset W(\mathfrak{g})$ we denote by Δ_Γ the subset of $\Delta(\mathfrak{g})$ consisting of all α with $s_\alpha\in\Gamma$.

Proposition 3.3.17 ([Lo3, Corollaries 4.16,4.19])**.**

(1) *The subgroup $W_{G,X}\subset W(\mathfrak{g})$ is large.*
(2) *Suppose $\mathfrak{g}$ is a simple classical Lie algebra. Then $\Gamma\subset W(\mathfrak{g})$ is large if and only if Δ_Γ is listed in Table* 3.3.

Note that some subsets Δ_Γ appear in Table 3.3 more than once.

Now we obtain a certain restriction on $W_{G,X}$ in terms of the action $G:T^*X$. To do this we introduce the notions of a $\mathfrak{g}$-*stratum* and a *completely perpendicular* subset of $\Delta(\mathfrak{g})$.

Definition 3.3.18. A pair $(\mathfrak{h},V)$, where $\mathfrak{h}$ is a reductive subalgebra of $\mathfrak{g}$ and V is an $\mathfrak{h}$-module, is said to be a $\mathfrak{g}$-stratum. Two $\mathfrak{g}$-strata $(\mathfrak{h}_1,V_1)$, $(\mathfrak{h}_2,V_2)$ are called *equivalent* if there exist $g\in G$ and a linear isomorphism $\varphi:V_1/V_1^{\mathfrak{h}_1}\to V_2/V_2^{\mathfrak{h}_2}$ such that $\mathrm{Ad}(g)\mathfrak{h}_1=\mathfrak{h}_2$ and $(\mathrm{Ad}(g)\xi)\varphi(v_1)=\varphi(\xi v_1)$ for all $\xi\in\mathfrak{h}_1,v_1\in V_1/V_1^{\mathfrak{h}_1}$.

Definition 3.3.19. Let Y be a smooth affine variety and $y\in Y$ a point with closed G-orbit. The pair $(\mathfrak{g}_y,T_yY/\mathfrak{g}_*y)$ is called the $\mathfrak{g}$-stratum of y. We say that $(\mathfrak{h},V)$ is a $\mathfrak{g}$-stratum of Y if $(\mathfrak{h},V)$ is equivalent to a $\mathfrak{g}$-stratum of a point of Y. In this case we write $(\mathfrak{h},V)\rightsquigarrow_{\mathfrak{g}} Y$.

Remark 3.3.20. Let us justify the terminology. Pairs $(\mathfrak{h},V)$ do define some stratification of $Y/\!/G$ by varieties with quotient singularities. Besides, analogous objects were called "strata" in [Sch2], where the term is borrowed from.

TABLE 3.3. Subsets Δ_Γ for large subgroups $\Gamma \subset W(\mathfrak{g})$ when $\mathfrak{g}$ is classical

$\mathfrak{g}$	Δ_Γ
$A_l, l \geqslant 2$	$\{\varepsilon_i - \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\}, I \subsetneq \{1, \ldots, n+1\}, I \neq \varnothing$
$B_l, l \geqslant 3$	(a) $\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm\varepsilon_i \mid i \in I\}, I \subsetneq \{1, \ldots, n\}$ (b) $\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm\varepsilon_i \mid i \in \{1, 2, \ldots, n\}\}$, $I \subsetneq \{1, \ldots, n\}, I \neq \varnothing$ (c) $\{\varepsilon_i - \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm(\varepsilon_i + \varepsilon_j), i \in I, j \notin I)\}$, $I \subset \{1, \ldots, n\}$
$C_l, l \geqslant 2$	(a) $\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm 2\varepsilon_i \mid i \in I\}, I \subsetneq \{1, \ldots, n\}$ (b) $\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm 2\varepsilon_i \mid i \in \{1, 2, \ldots, n\}\}$, $I \subsetneq \{1, \ldots, n\}, I \neq \varnothing$ (c) $\{\varepsilon_i - \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm(\varepsilon_i + \varepsilon_j), i \in I, j \notin I)\}$, $I \subset \{1, \ldots, n\}$
$D_l, l \geqslant 3$	(a) $\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\}, I \neq \{1, \ldots, n\}, \varnothing$ (b) $\{\varepsilon_i - \varepsilon_j \mid i \neq j, i, j \in I$ or $i, j \notin I\} \cup \{\pm(\varepsilon_i + \varepsilon_j), i \in I, j \notin I)\}$, $I \subset \{1, \ldots, n\}$

Definition 3.3.21. A subset $A \subset \Delta(\mathfrak{g})$ is called *completely perpendicular* if the following two conditions take place:

(1) $(\alpha, \beta) = 0$ for any different $\alpha, \beta \in A$.
(2) $\mathrm{Span}_{\mathbb{R}}(A) \cap \Delta(\mathfrak{g}) = A \cup -A$.

For example, any one-element subset of $\Delta(\mathfrak{g})$ is completely perpendicular.

Let A be a nonempty completely perpendicular subset of $\Delta(\mathfrak{g})$. By $S^{(A)}$ we denote the $\mathfrak{g}$-stratum $(\mathfrak{g}^{(A)}, \sum_{\alpha \in A} V^\alpha)$, where V^α is, by definition, the direct sum of two copies of the two-dimensional irreducible $\mathfrak{g}^{(A)}/\mathfrak{g}^{(A \setminus \{\alpha\})}$-module.

Proposition 3.3.22 ([Lo3, Corollary 4.14])**.** *If* $W_{G,X} \cap W(\mathfrak{g}^{(A)}) = \{1\}$*, then* $S^{(A)} \rightsquigarrow_{\mathfrak{g}} X$*.*

In particular, if $W(\mathfrak{g})$ contains all reflections $W(\mathfrak{g})$-conjugate to $s_\alpha, \alpha \in \Delta(\mathfrak{g})$, then $S^{(\alpha)} \not\rightsquigarrow T^*X$.

Until the end of the subsection, G is simple and X is a homogeneous G-space of rank $\mathrm{rk}(G)$ (not necessarily quasi-affine). In this case, any element of $\Pi_{G,X}$ is a positive multiple of a unique positive root from $\Delta(\mathfrak{g})$. The set of all positive roots arising in this way is denoted by $\widehat{\Pi}_{G,X}$. Note that $\widehat{\Pi}_{G,G/H}$ depends only on $(\mathfrak{g}, \mathfrak{h})$. Therefore in the sequel we write $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$ instead of $\widehat{\Pi}_{G,G/H}$.

Proposition 3.3.23. *Let* G, X *be such as above and* $G \neq G_2$*. Then the pair* $(\mathrm{Supp}(\alpha), \alpha)$*, where* $\alpha \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$ *is considered as an element of the root system associated with* $\mathrm{Supp}(\alpha)$*, is contained in the following list:*

(1) (A_1, α_1);
(2) $(A_2, \alpha_1 + \alpha_2)$;
(3) $(B_2, \alpha_1 + \alpha_2)$.

Proof. For an arbitrary G-variety X let $P_{G,X}$ denote the intersection of the stabilizers of all B-stable prime divisors of X.

Thanks to Corollary 3.3.9, we may assume that X is spherical. There is a subgroup $\widetilde{H} \subset G$ containing H such that $\Pi_{G,G/H} = \Pi_{G,G/\widetilde{H}}$ and $\mathfrak{X}_{G,G/\widetilde{H}} = \mathrm{Span}_{\mathbb{Z}}(\Pi_{G,G/H})$. Furthermore, the homogeneous space $G/\widetilde{H}$ possesses a so-called wonderful embedding $G/\widetilde{H} \hookrightarrow \overline{X}$ (these two facts follow from results of [Kn5], Sections 6,7; for definitions and results concerning wonderful varieties, see [Lu2] or [T2], Section 30). Since $\mathfrak{a}_{G,G/H} = \mathfrak{t}$, we see that $P_{G,X} = B$. So a unique closed G-orbit on X is isomorphic to G/B. For $\alpha \in \Pi_{G,X}$, let X_α denote the wonderful subvariety of X of rank 1 corresponding to α. Since the closed G-orbit in X_α is isomorphic to G/B, we see that $P_{G,X_\alpha} = B$. Now the proposition stems from the classification of wonderful varieties of rank 1 and the computation of their spherical roots. These results are gathered in [Wa], Table 1. □

4. Determination of distinguished components

4.1. **Introduction.** In this section we find an algorithm to determine the distinguished component of $X^{L_0^\circ{}_{G,X}}$ in the case when X is a homogeneous space or an affine homogeneous vector bundle. This will complete the algorithm computing $\mathfrak{a}_{G,X}$ for the indicated classes of varieties; see Subsection 3.2.

Put $L_0 = L^\circ_{0\,G,X}$. By $\underline{X}$ we denote the distinguished component of X^{L_0}.

Our first task is to describe the structure of distinguished components.

Proposition 4.1.1. *Here L_0 denotes one of the groups $L_{0,G,X}, L^\circ_{0,G,X}$ and $\underline{X}$ is the distinguished component of X^{L_0}.*

1. *Let $X = G/H$ be a quasi-affine homogeneous space. Then the action $N_G(L_0)^\circ : \underline{X}$ is transitive. If $eH \in \underline{X}$, then $N_G(L_0, \underline{X}) = N_G(L_0)^\circ N_H(L_0)$.*
2. *Let H be a reductive subgroup of G, V an H-module, $X = G *_H V$. The distinguished component Y of $(G/H)^{L^\circ_0{}_{G,G/H}}$ is contained in $\underline{X}$. If $eH \in Y$, then $\underline{X} = N_G(L_0)^\circ *_{H\cap N_G(L)^\circ} V^{L_0}$, $N_G(L_0, \underline{X}) = N_G(L_0)^\circ N_H(L_0)$.*

So in both cases of interest the distinguished component is recovered from an arbitrary point of the appropriate distinguished component in a homogeneous space. The next problem is to reduce the determination of distinguished components for homogeneous spaces to the case of affine homogeneous vector bundles. Let $H \subset G$ be an algebraic subgroup. Recall that we may assume that H is tamely contained in an antistandard parabolic subgroup $Q \subset G$ and for the standard Levi subgroup $M \subset Q$ the intersection $M \cap H$ is a Levi subgroup in H. By Remark 3.2.8, Q/H is an affine homogeneous vector bundle over $M/(M \cap H)$.

Proposition 4.1.2. *Let H, Q, M be as above, $X = G/H$ be quasi-affine. The distinguished component of the M-variety $(Q/H)^{L^\circ_0{}_{M,Q/H}}$ is contained in $\underline{X}$.*

Proposition 4.1.2 together with assertion 2 of Proposition 4.1.1 allows us to reduce the determination of distinguished components to the case of affine homogeneous spaces.

To state the result concerning affine homogeneous spaces, we need some notation. Let $\mathfrak{h}$ be an $\mathfrak{a}$-essential subalgebra of $\mathfrak{g}$ (see Definition 3.2.14), $\widetilde{\mathfrak{h}} := \mathfrak{h} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}))$. Put $\mathfrak{h}^{sat} = \widetilde{\mathfrak{h}}^{ess}$. By properties of the mapping $\mathfrak{h} \mapsto \mathfrak{h}^{ess}$, see [Lo1], Corollary 2.8, $\mathfrak{h} \subset \mathfrak{h}^{sat}$. It follows directly from the construction that $\mathfrak{h}^{sat}$ is a saturated subalgebra of $\mathfrak{g}$ (Definition 3.2.15).

The next proposition allows one to find a point from a distinguished component of an affine homogeneous space.

Proposition 4.1.3. *Let H be a reductive subgroup of G and $X = G/H$.*

(1) *Let H^{ess} denote the connected subgroup of G corresponding to $\mathfrak{h}^{ess}$, π the projection $G/H^{ess} \to G/H, gH^{ess} \mapsto gH$, and $\underline{X}'$ the distinguished component of $(G/H^{ess})^{L^\circ_{0\,G,G/H^{ess}}}$. Then $\underline{X} \supset \pi(\underline{X}')$.*
(2) *Suppose $\mathfrak{h}$ is $\mathfrak{a}$-essential and H is connected. Let H^{sat} denote the connected subgroup of G with Lie algebra $\mathfrak{h}^{sat} \subset \mathfrak{g}$, π be the projection $G/H \to G/H^{sat}, gH \mapsto gH^{sat}$, and let $\underline{X}'$ be the distinguished component of $(G/H^{sat})^{L^\circ_{0\,G,G/H^{sat}}}$. Then $\pi^{-1}(\underline{X}') \subset \underline{X}$.*
(3) *Let $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{g}_1,\mathfrak{h}_1) \oplus (\mathfrak{g}_2,\mathfrak{h}_2)$, G_1, G_2, H_1, H_2 denote the connected subgroups of G corresponding to $\mathfrak{g}_1,\mathfrak{g}_2,\mathfrak{h}_1,\mathfrak{h}_2$, respectively, π denote the projection $G_1/H_1 \times G_2/H_2 \to G/H, (g_1H_1, g_2H_2) \mapsto g_1g_2H$, and $\underline{X}_i, i = 1,2$, denote the distinguished component of $(G_i/H_i)^{L^\circ_{0\,G_i,G_i/H_i}}$. Then $\underline{X} = \pi(\underline{X}_1 \times \underline{X}_2)$.*
(4) *Suppose $\mathfrak{h}$ is one of the subalgebras listed in Tables* 3.1*,* 3.2*, and suppose H is connected. Then $\underline{X} \subset X$ is stable under the action of $N_G(L_0)$.*
(5) *Let $\mathfrak{h}, H$ be such as indicated in Subsection* 4.4*. Then $eH \in \underline{X}$.*

The first three assertions reduce the problem of finding a point from $(G/H)^{L_0}$ to the case when H is connected and $\mathfrak{h}$ is $\mathfrak{a}$-essential, saturated and indecomposable. All such subalgebras $\mathfrak{h}$ are listed in Tables 3.1, 3.2; see Proposition 3.2.16. Assertion (5) solves the problem in this case. Assertion (4) is auxiliary.

Let us give a brief description of the section. In Subsection 4.2 we prove Propositions 4.1.1, 4.1.2. In Subsection 4.3 we find some necessary condition (Proposition 4.3.2) for a component of X^{L_0}, where X is smooth and affine, to be distinguished. Subsection 4.4 describes some embeddings $\mathfrak{h} \hookrightarrow \mathfrak{g}$, where $\mathfrak{h}$ is presented in Tables 3.1, 3.2. For these embeddings the point eH lies in the component of $(G/H)^{L^\circ_{0\,G,G/H}}$, satisfying the necessary condition of Proposition 4.3.2. In Subsection 4.5 we complete the proof of Proposition 4.1.3.

4.2. **Proofs of Propositions 4.1.1, 4.1.2.**

Proof of Proposition 4.1.1. Suppose F is a reductive subgroup of G. Then for any subgroup $H \subset G$ the variety $(G/H)^F$ is a finite union of $N_G(F)^\circ$-orbits. Indeed, let $x \in (G/H)^F$. Then for any $x \in (G/H)^F$,

$$T_x\left((G/H)^F\right) \subset (T_x(G/H))^F = (\mathfrak{g}/\mathfrak{g}_x)^F = \mathfrak{g}^F/(\mathfrak{g}_x)^F = T_x(N_G(F)^\circ x).$$

Therefore $T_x\left((G/H)^F\right) = T_x(N_G(F)^\circ x)$. To prove the claim we remark that $N_G(F)^\circ x$ is a smooth irreducible subvariety of $(G/H)^F$. Thence in assertion (1), $\underline{X}$ is a homogeneous $N_G(L_0)^\circ$-space. The claim describing $N_G(L_0,\underline{X})$ is obvious.

We now proceed to the proof of assertion (2). Clearly, the projection $X^{L_0} \to (G/H)^{L_0}$ is surjective, and its fiber over $x \in (G/H)^{L_0}$ coincides with $\pi^{-1}(x)^{L_0}$. Here π denotes the natural projection $X \to G/H$. It follows that, as an $N_G(L_0,\underline{X})$-variety, $\underline{X}$ is a homogeneous vector bundle over some component of $(G/H)^{L_0}$. Note also that $N_G(L_0,\underline{X}) = N_G(L_0,\pi(\underline{X}))$. The last equality yields the claim on the structure of $N_G(L_0,\underline{X})$.

It remains to verify that the distinguished component Y of $(G/H)^{L_1}$, where $L_1 = L^\circ_{0\,G,G/H}$, is contained in $\underline{X}$. This claim will follow if we check that $U(\pi^{-1}(Y)^{L_0})$

is dense in X. Thanks to Proposition 3.2.12, $(U\cap L_1)\pi^{-1}(x)^{L_0}$ is dense in $\pi^{-1}(x)$ for all $x\in Y$. It remains to recall that $\overline{UY}=G/H$. □

Proof of Proposition 4.1.2. By Proposition 3.2.9, $L_0=L^\circ_{0\,M,Q/H}$. Let Y denote the distinguished component in $(Q/H)^{L_0}$. By definition, $(U\cap M)Y$ is dense in Q/H. To complete the proof, note that $U(Q/H)$ is dense in X. □

4.3. **An auxiliary result.** Here we study components of the variety X^{L_0} in the case when X is smooth and affine.

Definition 4.3.1. Let H be an algebraic group acting on an irreducible variety Y. A subgroup $H_0\subset H$ is said to be the *stabilizer in general position* (in short, s.g.p.) for the action $H:Y$ if there is an open subset $Y^0\subset Y$ such that $H_y\sim_H H_0$ for all $y\in Y^0$. By the *stable subalgebra in general position* (in short, s.s.g.p.) we mean the Lie algebra of the s.g.p.

Proposition 4.3.2. *Let X be a smooth affine G-variety, L_0 one of the groups $L_{0\,G,X}, L^\circ_{0\,G,X}$, and $\underline{X}$ the distinguished component of X^{L_0}. Then* $\dim X^{L_0}=\dim X-\frac{\dim G-\dim N_G(L_0)}{2}$. *A component of X^{L_0} is $N_G(L_0)$-conjugate to $\underline{X}$ if and only if its dimension is equal to* $\dim X^{L_0}$.

Proof. Proposition 3.2.10 implies (in the notation of that proposition) that $\dim\underline{X}=\dim X-\dim \mathrm{R}_u(P)+\dim F$. Since F is a maximal unipotent subgroup in $N_G(L_0)/L_0$, the equality $\dim\underline{X}=\dim X-(\dim G-\dim N_G(L_0))/2$ holds.

By [Kn1], Korollar 8.2, $L_{0\,G,X}$ is the s.g.p. for the action $G:T^*X$. Let us show that the action $G:T^*X$ is stable, that is, an orbit in general position is closed. Choose a point $x\in X$ with closed G-orbit. Put $H=G_x$, $V=T_xX/\mathfrak{g}_*x$. Then $T_x(T^*X)/\mathfrak{g}_*x\cong V\oplus V^*\oplus\mathfrak{h}^\perp$. Using the Luna slice theorem, we see that the action $G:T^*X$ is stable if and only if so is the representation $H:\mathfrak{h}^\perp\oplus V\oplus V^*$. Since $\mathfrak{h}^\perp\oplus V\oplus V^*$ is an orthogonal H-module, we are done by results of [Lu1].

Thanks to results of [LR], $N_G(L_0)$ permutes transitively irreducible (= connected) components of $(T^*X)^{L_0}$ whose image in $(T^*X)/\!/G$ is dense. A component of $(T^*X)^{L_0}$ has dense image in $(T^*X)/\!/G$ if it contains a point x with closed G-orbit and $G_x=L_0$ (when $L_0=L_{0\,G,X}$) or $(G_x)^\circ=L_0$ (when $L_0=L^\circ_{0\,G,X}$). Let Z be such a component and $G_0=N_G(L_0,Z)$. Then the morphism $G*_{G_0}Z\to T^*X$ is birational. Indeed, this morphism is dominant because a point in general position in T^*X is G-conjugate to a point of Z. On the other hand, for $z\in Z$ in general position the equality $G_z=L_0$ (for $L_0=L_{0\,G,X}$) or $(G_z)^\circ=L_0$ (for $L_0=L^\circ_{0\,G,X}$) holds. Therefore the inclusion $gz\in Z$ implies $g\in G_0$. We conclude that $\dim Z=2\dim\underline{X}$. Note that the unit component of the s.g.p. for the action $N_G(L_0)^\circ:Z$ coincides with L_0°.

Denote by π the projection $T^*X\to X$. Clearly, $\pi((T^*X)^{L_0})=X^{L_0}$. By a corollary in [PV], Subsection 6.5, X^{L_0} is smooth. Let Z_0 be a component of X^{L_0}. Then $Z:=\pi^{-1}(Z_0)^{L_0}$ is an irreducible component of $(T^*X)^{L_0}$.

It remains to prove that $\dim Z_0\leqslant\dim X-(\dim G-\dim N_G(L_0))/2$ and that if the equality holds, then Z contains a point z with closed G-orbit and $G_z=L_0$ (or $G_z^\circ=L_0$).

Note that the restriction of π to Z is the cotangent bundle $T^*Z_0\to Z_0$. Therefore the action $N_G(L_0)^\circ:Z$ is stable. Let L_Z denote the s.g.p. for this action. Obviously, $L_0\subset L_Z$. One also has the equality $\dim Z/\!/N_G(L_0)^\circ=\dim Z-\dim N_G(L_0)+\dim L_Z$.

The morphism $Z//N_G(L_0)^\circ \to T^*X//G$ induced by the embedding $Z \hookrightarrow T^*X$ is finite (see [PV], Theorem 6.16). It follows that $\dim Z - \dim N_G(L_0) + \dim L_Z \leqslant \dim T^*X - \dim G + \dim L_0$, or, equivalently,

$$\dim Z_0 \leqslant \dim X - \frac{\dim G - \dim N_G(L_0)}{2} - \frac{\dim L_Z - \dim L_0}{2}.$$

This completes the proof. □

4.4. **Embeddings of subalgebras.** In this subsection we construct embeddings $\mathfrak{h} \hookrightarrow \mathfrak{g}$ for the pairs $(\mathfrak{g}, \mathfrak{h})$ from Tables 3.1, 3.2. In the next subsection we will see that the corresponding points $eH \in G/H$ lie in the distinguished components of $(G/H)^{L_0}, L_0 = L^\circ_{0\,G,G/H}$. If eH lies in this distinguished component, then, obviously, $\mathfrak{l}_0 \subset \mathfrak{h}$ and, by Proposition 4.3.2,

$$2(\dim \mathfrak{h} - \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0)) = \dim \mathfrak{g} - \dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0). \tag{4.1}$$

Proposition 4.4.1. *Suppose that one of the following two assumptions holds:*

(1) $(\mathfrak{g}, \mathfrak{h})$ *is one of the pairs from Table* 3.1.
(2) $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}([\mathfrak{h}, \mathfrak{h}]))$, *and* $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}])$ *is listed in Table* 3.2.

If $[\mathfrak{h}, \mathfrak{h}] \hookrightarrow \mathfrak{g}$ *is embedded into* $\mathfrak{g}$ *as described below in this subsection, then* $\mathfrak{l}_0 \subset \mathfrak{h}$ *and* (4.1) *holds.*

We check this assertion case by case.

The case $\mathfrak{g} = \mathfrak{sl}_n$.

1) The subalgebra $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{sl}_k, \frac{n}{2} \leqslant k < n$ (the first rows of Tables 3.1, 3.2) is embedded into $\mathfrak{g}$ as the annihilator of $e_1, \ldots, e_{n-k}, e^1, \ldots, e^{n-k}$. In both cases in interest ($\dim \mathfrak{z}(\mathfrak{h}) = 0$ or 1) we get $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 4(n-k)^2 + (2k-n)^2 - 1, \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = (n-k)^2 + (2k-n)^2 - 1 + \dim \mathfrak{z}(\mathfrak{h})$.

2) The subalgebra $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{sl}_k \times \mathfrak{sl}_{n-k}, \frac{n}{2} \leqslant k < n-1$ (the second rows of both tables) is embedded into $\mathfrak{g}$ as $[\mathfrak{n}_{\mathfrak{g}}(\mathfrak{f}), \mathfrak{n}_{\mathfrak{g}}(\mathfrak{f})]$, where $\mathfrak{f} \cong \mathfrak{sl}_k$ is embedded as described in 1). We have $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 4(n-k) + (2k-n)^2 - 1, \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = 2(n-k) + (2k-n)^2 - 1 + \dim \mathfrak{z}(\mathfrak{h})$.

3) Let $n = 2m$. The subalgebra $\mathfrak{h} = \mathfrak{sp}_{2m}$ (row N3 of Table 3.1) is embedded into $\mathfrak{sl}_n$ as the annihilator of $e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2m-1} \wedge e^{2m}$. We have $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 4m - 1, \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = 3m$.

4) Let $n = 2m+1$. The subalgebra $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{sp}_{2m}$ (the third row of Table 3.2) is embedded into $\mathfrak{g} = \mathfrak{sl}_{2m+1}$ as the annihilator of e_1, e^1 and the 2-form $e^2 \wedge e^3 + \cdots + e^{2m} \wedge e^{2m+1}$. In this case $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 2m^2 + 2m, \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = m^2$.

The case $\mathfrak{g} = \mathfrak{sp}_{2n}$.

1) The subalgebra $\mathfrak{sp}_{2k}, k \geqslant \frac{n}{2}$ (row N4 of Table 3.1) is embedded into $\mathfrak{sp}_{2n}$ as the annihilator of $2(n-k)$ vectors $e_1 + e_{2n-1}, e_2 - e_{2n}, e_3 + e_{2n-3}, e_4 - e_{2n-2}, \ldots, e_{2(n-k)} - e_{2(n+k+1)}$. Here $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 8(n-k)^2 + 2(2k-n)^2 + n, \dim \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = 2(n-k)^2 + 2(2k-n)^2 + k$.

2) The subalgebra $\mathfrak{sp}_{2n-2k} \times \mathfrak{sp}_{2k}, k \geqslant \frac{n}{2}$, is embedded into $\mathfrak{sp}_{2n}$ as $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{f})$, where $\mathfrak{f} \cong \mathfrak{sp}_{2k}$ is embedded as described in 1). We have $\dim \mathfrak{n}_{\mathfrak{g}}(\mathfrak{l}_0) = 4(n-k) + 2(2k-n)^2 + (2k-n), \mathfrak{n}_{\mathfrak{h}}(\mathfrak{l}_0) = 3(n-k) + 2(2k-n)^2 + (2k-n)$.

Remark 4.4.2. Let us explain why we choose this strange (at the first glance) embedding. The pair $(\mathfrak{g},\mathfrak{h})=(\mathfrak{sp}_{2n},\mathfrak{sp}_{2k}\times\mathfrak{sp}_{2(n-k)})$ is symmetric. The corresponding involution σ acts on the annihilator of $\mathfrak{sp}_{2(n-k)}\subset\mathfrak{h}$ (resp. $\mathfrak{sp}_{2k}\subset\mathfrak{h}$) in $\mathbb{C}^{2n}$ identically (resp., by -1). Under the chosen embedding, σ acts on $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ by -1; in other words, $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ is the Cartan space of $(\mathfrak{g},\mathfrak{h})$ in the sense of the theory of symmetric spaces.

Further, we note that there is an embedding $\mathfrak{h}\hookrightarrow\mathfrak{g}$ such that $\mathfrak{l}_0\subset\mathfrak{h}$ but $2(\dim\mathfrak{h}-\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0))<\dim\mathfrak{g}-\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)$. Indeed, take the (most obvious) embedding $\mathfrak{sp}_{2k}\times\mathfrak{sp}_{2(n-k)}\hookrightarrow\mathfrak{sp}_{2n}$ such that $\mathfrak{sp}_{2k}$ is the annihilator of $\langle e_1,\ldots,e_{n-k},e_{2n},\ldots,e_{n-k+1}\rangle$.

3) The subalgebra $\mathfrak{sl}_2\times\mathfrak{sl}_2\times\mathfrak{sp}_{2n-4}, n\geqslant 3$ (rows N6, N7 of Table 3.1) is embedded into $\mathfrak{sp}_{2n}$ as follows: the ideal $\mathfrak{sp}_{2n-4}\subset\mathfrak{h}$ (resp., one of the ideals $\mathfrak{sl}_2\subset\mathfrak{h}$) is the annihilator of e_1,e_2,e_{2n},e_{2n-1} (resp., of $e_2,e_3,\ldots,e_{2n-1}$). If $n\geqslant 8$, then $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=16+2(n-4)^2+(n-4),\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=6+2(n-4)^2+(n-4)$. For $n=3$ we have $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=9,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=3$.

The case $\mathfrak{g}=\mathfrak{so}_n, n\geqslant 7$.

1) The subalgebra $\mathfrak{so}_k, k\geqslant\frac{n+2}{2}$ (row N8 of Table 3.1) is embedded into $\mathfrak{so}_n$ as the annihilator of the vectors $e_1,e_2,\ldots,e_l,e_n,\ldots,e_{n-l+1}$ (for $n-k=2l$) or of the vectors $e_1,e_2,\ldots,e_l,e_n,\ldots,e_{n-l+1},e_{l+1}+e_{n-l}$ (for $n-k=2l+1$). We have $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=\frac{(2k-n)(2k-n-1)}{2}+(n-k)(2n-2k-1),\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=\frac{(2k-n)(2k-n-1)}{2}+\frac{(n-k)(n-k-1)}{2}$.

2) Let $n=2m$. The subalgebra $[\mathfrak{h},\mathfrak{h}]=\mathfrak{sl}_m$ (rows N9, N10 of Table 3.1 and the fourth row of Table 3.2) is embedded into $\mathfrak{g}=\mathfrak{so}_{2m}$ as $\mathfrak{g}^{(\alpha_1,\ldots,\alpha_{m-1})}$. If m is even, then $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=3m,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=2m-1$. For odd m we have $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=3m-2,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=2m-2+\dim\mathfrak{z}(\mathfrak{h})$.

3) The subalgebra $\mathfrak{spin}_7$ is embedded into $\mathfrak{g}=\mathfrak{so}_9$ (row N11 of Table 3.1) as the annihilator of the sum of a highest vector and a lowest vector in the $\mathfrak{so}_9$-module $V(\pi_4)$. Under this choice of the embedding, $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=12,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=9$.

4) For the embedding $\mathfrak{spin}_7\hookrightarrow\mathfrak{so}_{10}$ (row N12 of Table 3.1) we take the composition of the embeddings $\mathfrak{spin}_7\hookrightarrow\mathfrak{so}_9,\mathfrak{so}_9\hookrightarrow\mathfrak{so}_{10}$ defined above. In this case $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=21,\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=9$.

5) G_2 is embedded into $\mathfrak{so}_7$ (row N13 of Table 3.1) as the annihilator of the sum of a highest vector and a lowest vector of the $\mathfrak{so}_7$-module $V(\pi_3)$. The equalities $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=9,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=8$ hold.

6) For the embedding $G_2\hookrightarrow\mathfrak{so}_8$ (row N14 of Table 3.1) we take the composition of the embeddings $G_2\hookrightarrow\mathfrak{so}_7,\mathfrak{so}_7\hookrightarrow\mathfrak{so}_8$ described above. Here $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=12,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=6$.

The case $\mathfrak{g}=G_2$.

The subalgebra A_2 (row N15 of Table 3.1) is embedded into G_2 as $\mathfrak{g}^{(\Delta_{max})}$, where Δ_{max} denotes the subsystem of all long roots. In this case $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=6,\dim\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)=4$.

The case $\mathfrak{g}=F_4$.

1) The subalgebra B_4 is embedded into F_4 (row N16 of Table 3.1) as follows. The equality $\dim\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)=22$ holds. (4.1) takes place if and only if $\mathfrak{l}_0\cong\mathfrak{so}_7$ is embedded into $\mathfrak{h}\cong\mathfrak{so}_9$ as $\mathfrak{spin}_7$. Put $\alpha=\frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2}$, $x=e_\alpha+e_{-\alpha},\mathfrak{h}_0:=\mathfrak{g}^{(\{\varepsilon_i,i=1,\ldots,4\})}\cong\mathfrak{sp}_9$. Let $x\in\mathfrak{h}_0^\perp$ be the sum of a highest vector and a lowest vector of the $\mathfrak{h}_0$-module $\mathfrak{h}_0^\perp$, so that $\mathfrak{z}_{\mathfrak{g}_0}(x)\cap\mathfrak{h}_0$ is the subalgebra $\mathfrak{spin}_7$ in $\mathfrak{h}_0\cong\mathfrak{so}_9$. Now put $\mathfrak{h}=\mathrm{Ad}(g)\mathfrak{h}_0$,

where $g \in G$ is such that $\mathrm{Ad}(g)x = \varepsilon_1^\vee$. One may take for such an element g the product $g = g_2 g_1$, where $g_1 \in N_G(\mathfrak{t})$ is such that $g_1 \frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4}{2} = \varepsilon_1$ and g_2 is an appropriate element of the connected subgroup of G corresponding to the $\mathfrak{sl}_2$-triple $(e_{\varepsilon_1}, \varepsilon_1^\vee, e_{-\varepsilon_1})$.

2) The subalgebra D_4 (row N17 of Table 3.1) is embedded into $\mathfrak{g}$ as $\mathfrak{g}^{(\Delta_{max})}$, where Δ_{max} denotes the subsystem of all long roots. The equalities $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 16, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 10$ hold.

The case $\mathfrak{g} = E_6$.

1) The subalgebra F_4 (row N18 of Table 3.1) is embedded into $\mathfrak{g}$ as the annihilator of the vector $v_{\pi_1} + v_{-\pi_5} + v_{\pi_5-\pi_1} \in V(\pi_1)$ (where v_λ denotes a nonzero vector of weight λ). We have $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 30, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 28$.

2) The subalgebra D_5 (row N19 of Table 3.1 and row N5 of Table 3.2) is embedded into E_6 as $\mathfrak{g}^{(\{\alpha_i\}_{i=2,\ldots,6})}$. Here $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 22, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 17 + \dim \mathfrak{z}(\mathfrak{h})$.

3) For the embedding $B_4 \hookrightarrow E_6$ (row N20 of Table 3.1) we take the composition of the embeddings $B_4 \hookrightarrow D_5$, $D_5 \hookrightarrow E_6$ described above. The equalities $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 38, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 16$ take place.

4) The subalgebra A_5 (row N21 of Table 3.1) is embedded into E_6 as $\mathfrak{g}^{(\{\alpha_i\}_{i=1,\ldots,5})}$. We have $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 30, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 11$.

The case $\mathfrak{g} = E_7$.

1) The subalgebra $\mathfrak{h} = E_6$ (row N22 of Table 3.1) is embedded into $\mathfrak{g}$ as $\mathfrak{g}^{(\{\alpha_i\}_{i=2,\ldots,7})}$. Under this choice of an embedding, we get $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 37, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 30$.

2) The subalgebra $\mathfrak{h} = D_6$ (row N23 of Table 3.1) is embedded into $\mathfrak{g}$ as $\mathfrak{g}^{(\{\alpha_i\}_{i=1,\ldots,6})}$. We have $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 37, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 18$.

The case $\mathfrak{g} = E_8$.

$\mathfrak{h} = E_7$ (row N24 in Table 3.1) is embedded into $\mathfrak{g}$ as $\mathfrak{g}^{(\{\alpha_i\}_{i=2,\ldots,8})}$. The equalities $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 56, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 37$ hold.

The case $\mathfrak{g} = \mathfrak{h} \times \mathfrak{h}$.

Fix a Borel subgroup $B_0 \subset H$ and a maximal torus $T_0 \subset B_0$ and construct from them the Borel subgroup and the maximal torus of G (Section 2). The embedding $\mathfrak{h} \hookrightarrow \mathfrak{h} \times \mathfrak{h}$ is given by $\xi \mapsto (\xi, w_0\xi)$. Here w_0 is an element $N_H(T_0)$ mapping into the element of maximal length of the Weyl group. We have $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 2\,\mathrm{rk}\,\mathfrak{h}, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = \mathrm{rk}\,\mathfrak{h}$.

The case $\mathfrak{g} = \mathfrak{sp}_{2n} \times \mathfrak{sp}_{2m}, m \geqslant 1, n > 1$.

Let $e_1, \ldots, e_{2n}, e_1', \ldots, e_{2m}'$ be the standard frames in $\mathbb{C}^{2n}, \mathbb{C}^{2m}$, resp., and let the symplectic forms on $\mathbb{C}^{2n}, \mathbb{C}^{2m}$ be as indicated in Section 2. We embed $\mathfrak{h} = \mathfrak{sp}_{2n-2} \times \mathfrak{sl}_2 \times \mathfrak{sp}_{2m-2}$ into $\mathfrak{g}$ as the stabilizer of the subspace $\mathrm{Span}_\mathbb{C}(e_1 + e_1', e_{2n} + e_{2m}')$. We get $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 2(n-2)^2 + n - 2 + 2(m-2)^2 + (m-2) + 8, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 2(n-2)^2 + n - 2 + 2(m-2)^2 + (m-2) + 3$ for $m > 1$ and $\dim \mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0) = 2(n-2)^2 + n - 2 + 5, \dim \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0) = 2(n-2)^2 + n - 2 + 2(m-2)^2 + (m-2) + 2$ for $m = 1$.

In Table 4.1 we list the pairs $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0, \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0$ for the embeddings $\mathfrak{h} \hookrightarrow \mathfrak{g}$ of subalgebras from Table 3.1 constructed above. In the first column the number of a pair in Table 3.1 is given; this number is denoted by "N". The fourth column contains elements of $\mathfrak{a}(\mathfrak{g}, \mathfrak{h})$ constituting a system of simple roots in $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$. The table will be used in Subsection 5.5 to compute the Weyl groups of affine homogeneous spaces.

TABLE 4.1. The pairs $(\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0, \mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0)$

N	$\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$	$\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0$	simple roots of $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$
1	$\mathfrak{sl}_{2(n-k)}$	$\mathfrak{sl}_{n-k}$	$\varepsilon_i-\varepsilon_{i+1}, i<n-k$ or $i\geqslant k$, $\varepsilon_{n-k}-\varepsilon_k$
2	$\mathfrak{sl}_2^{n-k}\oplus\mathbb{C}$	$\mathbb{C}^{n-k}:(\mathbb{C},\ldots,\mathbb{C},0)$	$\varepsilon_i-\varepsilon_{n+1-i}, i\leqslant n-k.$
3	$\mathbb{C}^n$	0	
4	$\mathfrak{sp}_{4(n-k)}$	$\mathfrak{sp}_{2(n-k)}$	$\varepsilon_i-\varepsilon_{i+1}, i<2(n-k), \varepsilon_{2(n-k)}$
5	$\mathbb{C}^{n-k}$	0	
6	$\mathfrak{sl}_4$	$\mathfrak{sl}_2\times\mathbb{C}^2$	$\varepsilon_1-\varepsilon_4, \varepsilon_3+\varepsilon_4, \varepsilon_2-\varepsilon_3$
7	$\mathfrak{sl}_3$	$\mathbb{C}^2$	$\varepsilon_1+\varepsilon_3, \varepsilon_2-\varepsilon_3$
8	$\mathfrak{so}_{2(n-k)}$	$\mathfrak{so}_{n-k}$	$\varepsilon_i-\varepsilon_{i+1}, i<n-k, \varepsilon_{n-k+1}+\varepsilon_{n-k}$
9	$\mathfrak{sl}_2^n$	$\mathbb{C}^n$	$\varepsilon_{2i-1}+\varepsilon_{2i}, i=1,\ldots,n$
10	$\mathfrak{sl}_2^n\oplus\mathbb{C}$	$\mathbb{C}^{n+1}$	$\varepsilon_{2i-1}+\varepsilon_{2i}, i=1,\ldots,n$
11	$\mathfrak{sl}_2\times\mathbb{C}$	$\mathbb{C}$:$(\mathbb{C},\mathbb{C})$	ε_1
12	$\mathfrak{so}_6\times\mathfrak{sl}_2$	$\mathfrak{sl}_2\times\mathfrak{sl}_2$:$(\mathfrak{so}_4,\mathfrak{sl}_2)$	$\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_5, -\varepsilon_1-\varepsilon_2, \varepsilon_3+\varepsilon_4$
13	$\mathbb{C}$	0	
14	$\mathfrak{sl}_2^3$	$\mathfrak{sl}_2$: $(\mathfrak{sl}_2,\mathfrak{sl}_2,\mathfrak{sl}_2)$	$\varepsilon_1-\varepsilon_4, \varepsilon_1+\varepsilon_4, \varepsilon_2+\varepsilon_3$
15	$\mathfrak{sl}_2$	$\mathbb{C}$	ε_1
16	$\mathfrak{sl}_2$	$\mathbb{C}$	ε_1
17	$\mathfrak{sl}_3$	$\mathbb{C}^2$	$(\varepsilon_1\pm(\varepsilon_2+\varepsilon_3\pm\varepsilon_4))/2$
18	$\mathbb{C}^2$	0	
19	$\mathfrak{sl}_2^2\times\mathbb{C}$	$\mathbb{C}^2:(\mathbb{C},\mathbb{C},\mathbb{C})$	$\varepsilon_1-\varepsilon_6, 2\varepsilon$
20	$\mathfrak{sl}_6$	$\mathfrak{sp}_4\times\mathfrak{sl}_2$	$\varepsilon-\varepsilon_1-\varepsilon_3-\varepsilon_4, \varepsilon_1-\varepsilon_2,$ $\varepsilon_2-\varepsilon_5, \varepsilon_5-\varepsilon_6, \varepsilon+\varepsilon_3+\varepsilon_4+\varepsilon_6$
21	$\mathfrak{so}_8$	$\mathfrak{sl}_2^3$	$\varepsilon_1-\varepsilon_6, \varepsilon+\varepsilon_4+\varepsilon_5+\varepsilon_6, \varepsilon_2-\varepsilon_5, \varepsilon_3-\varepsilon_4$
22	$\mathfrak{sl}_2^3$	$\mathbb{C}^2:(\mathbb{C},\mathbb{C},\mathbb{C})$	$\varepsilon_1-\varepsilon_2, \varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6, \varepsilon_8-\varepsilon_7$
23	$\mathfrak{so}_8$	$\mathfrak{sl}_2^3$	$\varepsilon_3+\varepsilon_4+\varepsilon_7+\varepsilon_8, \varepsilon_6-\varepsilon_7,$ $\varepsilon_5-\varepsilon_6, \varepsilon_1+\varepsilon_2+\varepsilon_7+\varepsilon_8$
24	$\mathfrak{so}_8$	$\mathfrak{sl}_2^3$	$\varepsilon_2-\varepsilon_3, \varepsilon_1-\varepsilon_2, -\varepsilon_1-\varepsilon_2-\varepsilon_9, \varepsilon_2+\varepsilon_3+\varepsilon_8$
25	$\mathbb{C}^{\mathrm{rk}\,\mathfrak{h}}$	0	
26	$\mathfrak{sl}_2^2\times\mathbb{C}$	$\mathbb{C}^2:(\mathbb{C},\mathbb{C},\mathbb{C})$	$\varepsilon_1+\varepsilon_2, \varepsilon_1'+\varepsilon_2'$
27	$\mathfrak{sl}_2\times\mathbb{C}$	$\mathbb{C}:(\mathbb{C},\mathbb{C})$	$\varepsilon_1+\varepsilon_2$

In the brackets after $\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0$ in the third column we indicate the projections of $\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0$ to simple ideals of $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$ provided the last algebra is not simple and $\mathfrak{n}_\mathfrak{h}(\mathfrak{l}_0)/\mathfrak{l}_0\neq 0$.

4.5. **Proof of Proposition 4.1.3.**

Proof. Assertion 1. Note that $L^\circ_{0\,G,G/H^{ess}}=L^\circ_{0\,G,G/H}$. It is clear that $\pi(\underline{X}')\subset X^{L_0}$. Moreover, $\overline{U\pi(\underline{X}')}=\pi(\overline{U\underline{X}'})=\pi(G/H^{ess})=X$. Therefore $\pi(\underline{X}')\subset\underline{X}$.

Assertion 2. Put $L=L_{G,G/H}, P=LB, \widetilde{L}_0=L^\circ_{0\,G,G/H^{sat}}$. Since $L_{G,G/H^{sat}}=L$, there is an L-stable subvariety $S\subset G/H^{sat}$ such that (L,L) acts trivially on S and the natural morphism $P*_L S\to G/H^{sat}$ is an open embedding; see Proposition 3.2.10. Thence there is an open P-embedding $P*_L\pi^{-1}(S)\hookrightarrow X$. It follows that $\mathfrak{X}_{G,X}=\mathfrak{X}_{L,\pi^{-1}(S)}$. Hence the action $L_0:\pi^{-1}(S)$ is trivial. By Proposition 3.2.10,

$\underline{X}' = \overline{\mathrm{R}_u(P)^{\widetilde{L}_0} S}, \underline{X} = \overline{\mathrm{R}_u(P)^{L_0}\pi^{-1}(S)}$. Therefore $\mathrm{R}_u(P)^{\widetilde{L}_0}\pi^{-1}(S)$ is a dense subset of $\pi^{-1}(\underline{X}')$. This proves assertion 2.

Assertion 3. This is obvious.

Assertion 4. Put $L = L_{G,G/H}$. By Proposition 4.1.1, the assertion will follow if we prove that

$$N_G(L_0) = N_G(L_0)^\circ N_H(L_0). \tag{4.2}$$

Note that $Z(G) \subset L \subset N_G(L_0)^\circ N_H(L_0)$. Therefore in the proof we may always consider any group G with Lie algebra $\mathfrak{g}$.

The proof of (4.2) will be carried out in two steps. Put $N_G(L_0)_0 = Z_G(L_0)L_0$, $N_H(L_0)_0 = Z_H(L_0)L_0$. These are the subgroups in the corresponding normalizers consisting of all elements acting on $\mathfrak{l}_0$ by inner automorphisms. In the first step we show that $N_G(L_0)_0 = N_H(L_0)_0 N_G(L_0)^\circ$ and in the second one that $N_G(L_0) = N_G(L_0)_0 N_H(L_0)$. The claim of the first step will follow if we check that $Z_G(\mathfrak{l}_0)/Z_H(\mathfrak{l}_0)$ is connected.

Step 1.

Lemma 4.5.1. *Let $\mathfrak{m}_0$ be a subalgebra of $\mathfrak{g}$ contained in some Levi subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and containing $[\mathfrak{m},\mathfrak{m}]$. Suppose that any element of $\Delta(\mathfrak{m})$ is a long root in $\Delta(\mathfrak{g})$ (more precisely, a long root in $\Delta(\mathfrak{g}_i)$, where $\mathfrak{g}_i$ is some simple ideal of $\mathfrak{g}$). Then $Z_G(\mathfrak{m}_0)M_0$ is connected, where M_0 denotes the connected subgroup of G corresponding to $\mathfrak{m}_0$.*

Proof of Lemma 4.5.1. The claim is well known in the case when $\mathfrak{m}_0$ is commutative (see [OV], ch.3, §3). Using induction on $\dim \mathfrak{m}_0$, we may assume that $\mathfrak{m}_0$ is simple. Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{m}$. Any connected component of $Z_G(M_0)M_0$ contains an element from $N_G(\mathfrak{t}) \cap Z_G(\mathfrak{t}\cap\mathfrak{m}_0)$. An element of the last group acts on $\mathfrak{t}$ as a composition of reflections corresponding to elements of $\Delta(\mathfrak{m})^\perp \cap \Delta(\mathfrak{g})$. Since β is a long root, we see that $\alpha+\beta \notin \Delta(\mathfrak{g})$ for any $\alpha \in \Delta(\mathfrak{m})^\perp, \beta \in \Delta(\mathfrak{m})$. Therefore $\Delta(\mathfrak{m})^\perp \cap \Delta(\mathfrak{g}) \subset \Delta(\mathfrak{z}_\mathfrak{g}(\mathfrak{m}_0))$. In particular, $Z_G(\mathfrak{m}_0 \cap \mathfrak{t}) \cap N_G(\mathfrak{t}) \subset Z_G(\mathfrak{m}_0)^\circ M_0$. □

It remains to consider only pairs $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{g}$ contains a simple ideal isomorphic to B_l, C_l, F_4, G_2 and $\Delta(\mathfrak{l})$ contains a short root. Such a pair $(\mathfrak{g},\mathfrak{h})$ is one of the following pairs from Table 3.1: N4 ($2k-n>1$), N5, N6 ($n>5$), N8 (n is odd), N16, N26 ($n \geqslant 3$), N27 ($n \geqslant 3$). In cases N4, N6, N27 the subalgebra $[\mathfrak{l}_0,\mathfrak{l}_0]$ coincides with $\mathfrak{sp}_{2l}$, while in case N26 it coincides with the direct sum of the subalgebras $\mathfrak{sp}_{2l_1}, \mathfrak{sp}_{2l_2}$ embedded into different simple ideals of $\mathfrak{g}$. The centralizer of $\mathfrak{sp}_{2l} \subset \mathfrak{sp}_{2n}$ in $\mathrm{Sp}(2n)$ is connected for all l,n. It remains to consider cases N5, N8, N16.

Put $\mathfrak{g} = \mathfrak{sp}_{2m}, \mathfrak{h} = \mathfrak{sp}_{2k} \times \mathfrak{sp}_{2(m-k)}, k \geqslant n/2$. As in the previous paragraph, it is enough to consider the case $n = 2k$. Let us describe the embedding $L_0 \hookrightarrow \mathrm{Sp}(2n)$. The space $\mathbb{C}^{4k}$ is decomposed into the direct sum of two-dimensional spaces $V_1,\ldots,V_{2k}$. We may assume that $H = \mathrm{Sp}(V_1 \oplus V_3 \oplus \ldots \oplus V_{2k-1}) \times \mathrm{Sp}(V_2 \oplus V_4 \oplus \ldots \oplus V_{2k})$. The subgroup $L_0 \subset \mathrm{Sp}_{2n}$ is isomorphic to the direct product of k copies of SL_2. The i-th copy of SL_2 (we denote it by L_0^i) acts diagonally on $V_{2i-1} \oplus V_{2i}$ and trivially on $V_j, j \neq 2i-1, 2i$. It can be seen directly that $Z_G(L_0) = Z_{\mathrm{Sp}(V_1\oplus V_2)}(L_0^1) \times \ldots \times Z_{\mathrm{Sp}(V_{2k-1}\oplus V_{2k})}(L_0^k)$. Therefore $Z_G(L_0)/Z_H(L_0)$ is the k-dimensional torus.

In case N8, both groups $Z_G(L_0), Z_H(L_0)$ contain two connected components. The components of the unit in both cases consist of all elements acting trivially on $\mathbb{C}^{2n+1}/(\mathbb{C}^{2n+1})^{\mathfrak{l}_0}$. This proves the required claim.

In case N16 we have $Z_H(\mathfrak{l}_0)/Z_H(\mathfrak{l}_0)^\circ \cong \mathbb{Z}_2$. An element $\sigma \in (Z_H(\mathfrak{l}_0) \setminus Z_H(\mathfrak{l}_0)^\circ) \cap N_G(\mathfrak{t})$ acts on $\mathfrak{t} \cap \mathfrak{l}_0^\perp$ by -1. Any element of $Z_G(\mathfrak{l}_0) \cap N_G(\mathfrak{t})$ acts on $\mathfrak{t} \cap \mathfrak{l}_0^\perp$ by ± 1. Since $Z_G(\mathfrak{l}_0) = Z_G(\mathfrak{l}_0)^\circ (Z_G(\mathfrak{l}_0) \cap N_G(\mathfrak{t}))$, we see that the group $Z_G(L_0)/Z_H(L_0)$ is connected.

Step 2. It remains to check that $N_G(L_0) = N_G(L_0)_0 N_H(L_0)$; equivalently, the images of $N_G(\mathfrak{l}_0), N_H(\mathfrak{l}_0)$ in $\mathrm{GL}(\mathfrak{l}_0)$ coincide. For the pair N1 in Table 3.1 the group $N_G(\mathfrak{l}_0)$ is connected. If $(\mathfrak{g}, \mathfrak{h})$ is one of the pairs N4, N8 (n is odd), N12, N14, N15, N16, N20, then the group of outer automorphisms of $\mathfrak{l}_0$ is trivial. For the pairs N11, N13, N17, N18, N19, N22, N24 from Table 3.1, the algebra $\mathfrak{l}_0$ is simple and $N_H(\mathfrak{l}_0)$ contains all automorphisms of $\mathfrak{l}_0$ (in the cases N22, N24 one proves the last claim using the embeddings $D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$). In the cases N3, N9, N22, N5 ($n = 2k$) the algebra $\mathfrak{l}_0$ is isomorphic to the direct product of some copies of $\mathfrak{sl}_2$. The group of outer automorphisms of $\mathfrak{l}_0$ is the symmetric group on the set of simple ideals of $\mathfrak{l}_0$. The image of $N_H(\mathfrak{l}_0)$ in $\mathrm{Aut}(\mathfrak{l}_0)/\mathrm{Int}(\mathfrak{l}_0)$ coincides with the whole symmetric group. One considers case N5, $n > 2k$, analogously. Here $\mathrm{Aut}(\mathfrak{l}_0)/\mathrm{Int}(\mathfrak{l}_0)$ is isomorphic to the symmetric group on the set of simple ideals of $\mathfrak{l}_0$ that are isomorphic to $\mathfrak{sl}_2$ and have Dynkin index 2. In case N8 (with even n), the images of both $N_H(\mathfrak{l}_0), N_G(\mathfrak{l}_0)$ in $\mathrm{Aut}(\mathfrak{l}_0)$ contain an outer automorphism of $\mathfrak{l}_0$ induced by an element from $\mathrm{O}_{2k-n} \setminus \mathrm{SO}_{2k-n}$ and do not contain elements from the other nontrivial connected components of $\mathrm{Aut}(\mathfrak{l}_0)$ (which exist only when $2k - n = 8$).

It remains to consider the pairs N2, N6, N7, N21, N25, N26, N27 from Table 3.1 and the pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{z}_\mathfrak{g}([\mathfrak{h}, \mathfrak{h}]))$ and $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}])$ is listed in Table 3.2. In all these cases $\mathfrak{z}(\mathfrak{l}_0) \neq \{0\}$. If $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) \neq (\mathfrak{so}_{4n+2}, \mathfrak{sl}_{2n+1}), (E_6, D_5)$, then $N_G([\mathfrak{l}_0, \mathfrak{l}_0])$ is connected. As we have seen above, in these two cases the equality $N_G([\mathfrak{l}_0, \mathfrak{l}_0]) = N_G([\mathfrak{l}_0, \mathfrak{l}_0])_0 N_H([\mathfrak{l}_0, \mathfrak{l}_0])$ holds. Therefore it is enough to show that

$$N_{Z_G([\mathfrak{l}_0, \mathfrak{l}_0])}(\mathfrak{z}(\mathfrak{l}_0)) = Z_G(\mathfrak{l}_0) N_{Z_H([\mathfrak{l}_0, \mathfrak{l}_0])}(\mathfrak{z}(\mathfrak{l}_0)). \tag{4.3}$$

If a pair $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}])$ is contained in Table 3.2, then, using Lemma 4.5.1, one can see that $Z_G([\mathfrak{l}_0, \mathfrak{l}_0])$ is connected. If, additionally, $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) \neq (\mathfrak{sl}_n, \mathfrak{sl}_{n-k} \times \mathfrak{sl}_k)$, then the group $N_{Z_G([\mathfrak{l}_0, \mathfrak{l}_0])}(\mathfrak{z}(\mathfrak{l}_0))$ is connected too. In the cases N6, N7, N26, N27, the algebra $\mathfrak{z}(\mathfrak{l}_0)$ is one-dimensional and the groups on both sides of (4.3) act on $\mathfrak{z}(\mathfrak{l}_0)$ as $\mathbb{Z}_2$. This observation yields (4.3) in these cases. It remains to check (4.3) for $(\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]) = (\mathfrak{h} \times \mathfrak{h}, \mathfrak{h}), (E_6, A_5), (\mathfrak{sl}_n, \mathfrak{sl}_{n-k} \times \mathfrak{sl}_k)$. The first case is obvious.

The case $(\mathfrak{g}, \mathfrak{h}) = (E_6, A_5)$. We suppose that the invariant symmetric form on $\mathfrak{g}$ is chosen in such a way that the length of a root equals 2.

We have $\mathfrak{l}_0 = \langle \alpha_1 - \alpha_5, \alpha_2 - \alpha_4 \rangle$. The action of $N_H(\mathfrak{l}_0)$ on $\mathfrak{l}_0$ coincides with the action of the symmetric group S_3 on its unique 2-dimensional module. To see this, note that $\mathfrak{l}_0$ is embedded into $\mathfrak{sl}_6$ as $\{\mathrm{diag}(x, y, -x-y, -x-y, y, x)\}$.

Any element of length 4 lying in the intersection of $\mathfrak{l}_0$ and the root lattice of $\mathfrak{g}$ is one of the following elements: $\pm(\alpha_1 - \alpha_5), \pm(\alpha_2 - \alpha_4), \pm(\alpha_1 + \alpha_2 - \alpha_4 - \alpha_5)$. Assume that the images of $N_G(\mathfrak{l}_0)$ and $N_H(\mathfrak{l}_0)$ in $\mathrm{GL}(\mathfrak{l}_0)$ differ. The image N of $N_G(\mathfrak{l}_0) \cap N_G(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{l}_0)$ permutes the six elements of length 4 listed above and preserves the scalar product on $\mathfrak{l}_0 \cap \mathfrak{t}(\mathbb{R})$. It follows that $-id \in N$. Let us check that $g \in N_G(\mathfrak{t}) \cap N_G(\mathfrak{l}_0)$ cannot act on $\mathfrak{l}_0$ by -1.

Assume the converse; let g be such an element. Clearly, $g \in N_G(\mathfrak{z}_\mathfrak{g}(\alpha_1 - \alpha_5))$. Note that $\mathfrak{z}_\mathfrak{g}(\alpha_1 - \alpha_5) \cong \mathbb{C} \times \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_4$. Simple roots of the ideals isomorphic to $\mathfrak{sl}_2$ are $\varepsilon_1 - \varepsilon_6, \varepsilon_2 - \varepsilon_5$. Thus g preserves the pair of lines $\{\langle \varepsilon_1 - \varepsilon_6 \rangle, \langle \varepsilon_2 - \varepsilon_5 \rangle\}$.

Replacing $\alpha_1-\alpha_5$ with $\alpha_2-\alpha_4$ and $\alpha_1+\alpha_2-\alpha_4-\alpha_5$ in the previous argument, we see that g preserves the pairs $\{\langle\varepsilon_2-\varepsilon_5\rangle,\langle\alpha_3\rangle\}$ and $\{\langle\varepsilon_1-\varepsilon_6\rangle,\langle\alpha_3\rangle\}$. Thence $\langle\varepsilon_1-\varepsilon_6\rangle,\langle\varepsilon_2-\varepsilon_5\rangle,\langle\alpha_3\rangle$ are g-stable. Since g is an orthogonal transformation of $\mathfrak{t}$ leaving $\langle\alpha_1-\alpha_5,\alpha_2-\alpha_4\rangle$ invariant, we see that the line $\langle\varepsilon\rangle$ is g-stable. Replacing g with $gs_{2\varepsilon}$, we may assume that $g\in Z_G(\langle\varepsilon\rangle)$. The last group is the product of $A_5\subset E_6$ and a one-dimensional torus and does not contain an element acting on $\mathfrak{l}_0$ by -1.

The case $(\mathfrak{g},[\mathfrak{h},\mathfrak{h}])=(\mathfrak{sl}_n,\mathfrak{sl}_k\times\mathfrak{sl}_{n-k})$. The subalgebra $\mathfrak{l}_0\subset\mathfrak{h}$ consists of all matrices of the form $\mathrm{diag}(x_1,\ldots,x_{n-k},A,x_{n-k},\ldots x_1)$, where $x_1,\ldots,x_{n-k}\in\mathbb{C}, A\in\mathfrak{gl}_{2k-n}$, $\mathrm{tr}(A)=-2\sum_{i=1}^{n-k}x_i$ for $\mathfrak{z}(\mathfrak{h})\neq 0$ and $\mathrm{tr}(A)=\sum_{i=1}^{n-k}x_i=0$ for $\mathfrak{z}(\mathfrak{h})=0$. The groups $N_G(\mathfrak{l}_0),N_H(\mathfrak{l}_0)$ act on $\mathfrak{z}(\mathfrak{l}_0)$ as S_{n-k} (permuting x_i).

Assertion 5. This follows directly from assertion 4 and Propositions 4.1.1, 4.3.2. □

In Subsection 5.5 we will need to know the image of $N_G(\mathfrak{l}_0)\cap N_G(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{a}(\mathfrak{g},\mathfrak{h}))$ for pairs $(\mathfrak{g},\mathfrak{h})$ from Table 3.1. This information is extracted mostly from the previous proof and Table 4.1. It is presented in Table 4.2.

TABLE 4.2. The image of $N_G(\mathfrak{l}_0)\cap N_G(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{a}_{\mathfrak{g},\mathfrak{h}})$

N	The group
1	W
2	$s_{\varepsilon_i-\varepsilon_j-\varepsilon_{n+1-i}+\varepsilon_{n+1-j}}, 1\leqslant i<j\leqslant n-k$
3	$s_{\varepsilon_{2i-1}+\varepsilon_{2i}\pm(\varepsilon_{2j-1}+\varepsilon_{2j})}, 1\leqslant i,j\leqslant n$
4	W
5	$s_{\varepsilon_{2i-1}+\varepsilon_{2i}\pm(\varepsilon_{2j-1}+\varepsilon_{2j})}, 1\leqslant i,j\leqslant n-k$
6	A
7	A
8	A
9	A
10	A
11	$A\oplus\mathbb{Z}_2$
12	W
13	$\mathbb{Z}_2$
14	W
15	$\mathbb{Z}_2$
16	$\mathbb{Z}_2$
17	A
18	s_{π_1}, s_{π_5}
19	A
20	W
21	A
22	A
23	A
24	A
25	$W(\mathfrak{h})$
26	$W\oplus\mathbb{Z}_2$
27	$W\oplus\mathbb{Z}_2$

Let us explain the notation used in Table 4.2. In rows 2, 3, 5, 18 a set of generators of the group is given. The symbols A, W mean the automorphism group (resp., the Weyl group) of the root system of $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$ (we assume that this group acts trivially on $\mathfrak{z}(\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0)$). The symbol $\mathbb{Z}_2$ denotes the group acting by ± 1 on the center of $\mathfrak{n}_\mathfrak{g}(\mathfrak{l}_0)/\mathfrak{l}_0$ and trivially on the semisimple part.

5. Computation of Weyl groups for affine homogeneous vector bundles

5.1. **Introduction.** The main goal of this section is to compute the group $W_{G,X}$, where $X = G*_H V$ is a homogeneous vector bundle over an affine homogeneous space G/H. We recall that $W_{G,X}$ depends only on the triple $(\mathfrak{g}, \mathfrak{h}, V)$ (see Corollary 3.2.4), so we write $W(\mathfrak{g}, \mathfrak{h}, V)$ instead of $W_{G,G*_H V}$.

Let $\underline{X}$ be the distinguished component of X^{L_0}, where $L_0 = L^\circ_{0\,G,X}$, and $\underline{G} = N_G(L_0, \underline{X})/L_0$. Theorem 3.3.10 allows us to recover $W_{G,X}$ from $W_{\underline{G}^\circ,\underline{X}}$. Besides, the results of Section 4 show that the $\underline{G}^\circ$-variety $\underline{X}$ is an affine homogeneous vector bundle and allow us to determine it. So to compute Weyl groups one may restrict to the case $\mathrm{rk}_G(X) = \mathrm{rk}\, G$. Further, Proposition 3.3.15 reduces the computation of $W_{G,X}$ to the case of simple G. Finally, we may assume that G is simply connected. Below in this subsection we always assume that these conditions are satisfied.

By an *admissible triple* we mean a triple $(\mathfrak{g}, \mathfrak{h}, V)$, where $\mathfrak{g}$ is a simple Lie algebra, $\mathfrak{h}$ its reductive subalgebra, and V is a module over the connected subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$ such that $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}, V) = \mathfrak{t}$.

Let us state our main result. There is a minimal ideal $\mathfrak{h}_0 \subset \mathfrak{h}$ such that $W(\mathfrak{g}, \mathfrak{h}, V) = W(\mathfrak{g}, \mathfrak{h}_0, V/V^{\mathfrak{h}_0})$. We say that the corresponding triple $(\mathfrak{g}, \mathfrak{h}_0, V/V^{\mathfrak{h}_0})$ is a *W-essential part* of the triple $(\mathfrak{g}, \mathfrak{h}, V)$. The problem of computing $W(\mathfrak{g}, \mathfrak{h}, V)$ may be divided into three parts:

a) To find all admissible triples $(\mathfrak{g}, \mathfrak{h}, V)$ such that $V^\mathfrak{h} = \{0\}$ and $W(\mathfrak{g}, \mathfrak{h}, V) \neq W(\mathfrak{g}, \mathfrak{h}_0, V)$ for any ideal $\mathfrak{h}_0 \subsetneq \mathfrak{h}$. Such a triple $(\mathfrak{g}, \mathfrak{h}, V)$ is called *W-essential*. Clearly, a triple is W-essential if and only if it coincides with its W-essential part.

b) To compute the groups $W(\mathfrak{g}, \mathfrak{h}, V)$ for all triples $(\mathfrak{g}, \mathfrak{h}, V)$ found in the previous step.

c) To show how one can determine a W-essential part of a given admissible triple.

Definition 5.1.1. Two triples $(\mathfrak{g}, \mathfrak{h}_1, V_1), (\mathfrak{g}, \mathfrak{h}_2, V_2)$ are said to be *isomorphic* (resp., *equivalent*), if there exist $\sigma \in \mathrm{Aut}(\mathfrak{g})$ (resp., $\sigma \in \mathrm{Int}(\mathfrak{g})$) and a linear isomorphism $\varphi : V_1/V_1^{\mathfrak{h}_1} \to V_2/V_2^{\mathfrak{h}_2}$ such that $\sigma(\mathfrak{h}_1) = \mathfrak{h}_2$ and $\varphi(\xi v) = \sigma(\xi)\varphi(v)$ for all $\xi \in \mathfrak{h}, v \in V_1/V_1^{\mathfrak{h}_1}$.

Let H_1, H_2 be the connected subgroups of G corresponding to $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$. Put $X_{01} = G *_{H_1} V_1, X_{02} = G *_{H_2} V_2$. The triples $(\mathfrak{g}, \mathfrak{h}_1, V_1), (\mathfrak{g}, \mathfrak{h}_2, V_2)$ are isomorphic (resp., equivalent) if and only if $X_{01} \cong^\tau X_{02}$ for some $\tau \in \mathrm{Aut}(G)$ (resp., $X_{01} \cong X_{02}$). Lemma 3.2.18 allows us to compute the Weyl group only for one triple in a given class of isomorphisms.

There is a trivial W-essential triple $(\mathfrak{g}, 0, 0)$ and $W(\mathfrak{g}, 0, 0) = W(\mathfrak{g})$.

Theorem 5.1.2. *Let $(\mathfrak{g}, \mathfrak{h}, V)$ be an admissible triple. If there exists an ideal $\mathfrak{h}_1 \subset \mathfrak{h}$ such that the triple $(\mathfrak{g}, \mathfrak{h}_1, V/V^{\mathfrak{h}_1})$ is isomorphic to a triple from Table* 5.1*, then $(\mathfrak{g}, \mathfrak{h}_1, V/V^{\mathfrak{h}_1})$ is a W-essential part of $(\mathfrak{g}, \mathfrak{h}, V)$ and the Weyl group is presented in the fifth column of Table* 5.1*. Otherwise, $(\mathfrak{g}, 0, 0)$ is a W-essential part of $(\mathfrak{g}, \mathfrak{h}, V)$.*

Table 5.1. W-essential triples $(\mathfrak{g}, \mathfrak{h}, V)$ and the corresponding Weyl groups

N	$\mathfrak{g}$	$\mathfrak{h}$	V	$W(\mathfrak{g}, \mathfrak{h}, V)$
1	$\mathfrak{sl}_n$ $n > 1$	$\mathfrak{sl}_{n-k}$ $k < \frac{n}{2}$	$l\tau + m\tau^*$ $l + m = n - 2k - 1, l \geqslant m$	$\varepsilon_i - \varepsilon_{i-1}, \varepsilon_{k+m} - \varepsilon_{k+m+2}$ $i \neq k+m, k+m+1$
2	$\mathfrak{sl}_n$ $n > 3$	$\mathfrak{sl}_n$	$\bigwedge^2 \tau + \tau$	$\varepsilon_i - \varepsilon_{i+2}$ $i = 1, \dots, n-2$
3	$\mathfrak{sl}_n$ even $n \geqslant 4$	$\mathfrak{sl}_n$	$\bigwedge^2 \tau + \tau^*$	N2
4	$\mathfrak{sl}_n$ odd $n \geqslant 5$	$\mathfrak{sl}_n$	$\bigwedge^2 \tau + \tau^*$	$\varepsilon_i - \varepsilon_{i+2}, \varepsilon_1 - \varepsilon_2$ $i = 2, \dots, n-1$
5	$\mathfrak{sl}_n$ even $n \geqslant 4$	$\mathfrak{sl}_{n-1}$	$\bigwedge^2 \tau$	$\varepsilon_i - \varepsilon_{i+2}, \varepsilon_1 - \varepsilon_2$ $i = 2, \dots, n-1$
6	$\mathfrak{sl}_n$ odd $n \geqslant 3$	$\mathfrak{sl}_{n-1}$	$\bigwedge^2 \tau$	N2
7	$\mathfrak{sl}_n$ even $n \geqslant 4$	$\mathfrak{sp}_n$	τ	N2
8	$\mathfrak{sl}_n$ odd $n \geqslant 5$	$\mathfrak{sp}_{n-1}$	0	N2
9	$\mathfrak{so}_{2n+1}$ $n \geqslant 3$	$\mathfrak{sl}_n^{diag}$	0	$\varepsilon_i - \varepsilon_{i+2}, \varepsilon_{n-1}, \varepsilon_n$ $i = 1, \dots, n-2$
10	$\mathfrak{so}_7$	$\mathfrak{so}_7$	$l\tau + (2-l)R(\pi_3), l = 0, 1$	N9
11	$\mathfrak{so}_7$	$\mathfrak{so}_6$	$kR(\pi_3) + (2-k)R(\pi_1), k > 0$	$k = 1$:N9 $k = 2 : \varepsilon_1 - \varepsilon_2, \varepsilon_2, \varepsilon_3$
12	$\mathfrak{so}_7$	G_2	$R(\pi_1)$	N9
13	$\mathfrak{so}_9$	$\mathfrak{so}_9$	$R(\pi_4) + \tau$	$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_4, \varepsilon_3, \varepsilon_4$
14	$\mathfrak{so}_9$	$\mathfrak{so}_8$	$(2-k-l)\tau + lR(\pi_3) + kR(\pi_4)$ $(k,l) = (1,0), (2,0), (1,1)$	N13, $(k,l) \neq (2,0)$ N9, $(k,l) = (2,0)$
15	$\mathfrak{so}_9$	$\mathfrak{so}_7$	$R(\pi_3)$	N13
16	$\mathfrak{so}_9$	$\mathfrak{spin}_7$	$R(\pi_1)$	N9
17	$\mathfrak{so}_9$	$\mathfrak{spin}_7$	$R(\pi_3)$	N13
18	$\mathfrak{so}_9$	G_2	0	N13
19	$\mathfrak{so}_{11}$	$\mathfrak{so}_{11}$	$R(\pi_5)$	N9
20	$\mathfrak{so}_{11}$	$\mathfrak{so}_{10}$	$R(\pi_1) + R(\pi_4)$	$\varepsilon_i - \varepsilon_{i+1}, i = 1, 2, 3$ $\varepsilon_4, \varepsilon_5$
21	$\mathfrak{so}_{11}$	$\mathfrak{so}_9$	$R(\pi_4)$	N20

N	$\mathfrak{g}$	$\mathfrak{h}$	V	$W(\mathfrak{g},\mathfrak{h},V)$
22	$\mathfrak{so}_{11}$	$\mathfrak{so}_8$	$R(\pi_3)$	N20
23	$\mathfrak{so}_{11}$	$\mathfrak{spin}_7$	0	N20
24	$\mathfrak{so}_{13}$	$\mathfrak{so}_{10}$	$R(\pi_4)$	$\varepsilon_i-\varepsilon_{i+1}, i=1,2,3$ $\varepsilon_4,\varepsilon_5$
25	$\mathfrak{sp}_{2n}$ $n\geqslant 2$	$\mathfrak{sp}_{2k}$ $k\geqslant \frac{n}{2}$	$(2k-n)\tau$	$\varepsilon_i-\varepsilon_{i+1},\varepsilon_{n-1}+\varepsilon_n$ $i=1,\ldots,n-1$
26	$\mathfrak{so}_{4n+2}$ $n\geqslant 2$	$\mathfrak{sl}_{2n+1}^{diag}$	τ^*	$\varepsilon_i+\varepsilon_{i+1}$ $i=1,\ldots,2n$
27	$\mathfrak{so}_{4n}$ $n\geqslant 2$	$\mathfrak{sl}_{2n}^{diag}$	τ^*	$\varepsilon_i+\varepsilon_{i+1}$ $i=1,\ldots,2n-1$
28	G_2	A_2	τ	$\varepsilon_1,\varepsilon_2$
29	F_4	B_4	2τ	$\varepsilon_3,\frac{\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4}{2},$ $\varepsilon_4,\varepsilon_2-\varepsilon_4$
30	F_4	D_4	τ	N29
31	F_4	B_3	0	N29
32	$\mathfrak{sp}_{4n}$ $n\geq 1$	$\mathfrak{sp}_{2n}\times\mathfrak{sp}_{2n}$	$R(\pi_1)$	$\varepsilon_i+\varepsilon_{i+1}$ $i=1,\ldots,2n-1$
33	$\mathfrak{sp}_{4n+2}$ $n\geqslant 1$	$\mathfrak{sp}_{2n+2}\times\mathfrak{sp}_{2n}$	$R(\pi_1)$	$\varepsilon_i+\varepsilon_{i+1}$ $i=1,\ldots,2n$

Let us explain the notation used in the table. The first column is used for numbering. In the fourth column the representation of $\mathfrak{h}$ in V is given. τ denotes the tautological representation of $\mathfrak{h}$. In column 5 we list roots such that the corresponding reflections generate $W(\mathfrak{g},\mathfrak{h},V)$. "N$k$" in column 5 means that the corresponding Weyl group coincides with that from row Nk. If in row 1 the lower index j of ε_j is less than 1 or greater than n, then the corresponding root should be omitted. In row N27 we suppose that $\mathfrak{h}=\mathfrak{g}^{(\alpha_1,\ldots,\alpha_{2n-1})}$.

Remark 5.1.3. Inspecting Table 5.1, we deduce from Theorem 5.1.2 that a W-essential part of $(\mathfrak{g},\mathfrak{h},V)$ is uniquely determined.

Remark 5.1.4. Let $(\mathfrak{g},\mathfrak{h},V)$ be a triple listed in Table 5.1. The isomorphism class of $(\mathfrak{g},\mathfrak{h},V)$ consists of more than one equivalence class precisely for the following triples: N1 ($l\neq m$), N2–N6, N26, N27. In these cases an isomorphism class consists of two different equivalence classes.

Now we describe the content of this section. In Subsection 5.2 we classify *W-quasi-essential triples*.

Definition 5.1.5. An admissible triple $(\mathfrak{g},\mathfrak{h},V)$ is called *W-quasi-essential* if for any proper ideal $\mathfrak{h}_1\subset\mathfrak{h}$ there exists a root $\alpha\in\Delta(\mathfrak{g})$ such that $S^{(\alpha)}\rightsquigarrow_{\mathfrak{g}} T^*(G*_H V)$ but $S^{(\alpha)}\not\rightsquigarrow_{\mathfrak{g}} T^*(G*_{H_1}V)$. Here H,H_1 are connected subgroups of G corresponding to $\mathfrak{h},\mathfrak{h}_1$.

Below we will see that W-quasi-essential triples are precisely those listed in Table 5.1.

In Subsection 5.3 we will compute the Weyl groups for all triples listed in Table 5.1. Subsection 5.4 completes the proof of Theorem 5.1.2. Finally, in Subsection 5.5 we compute the Weyl groups of affine homogeneous spaces (without restrictions on the rank) more or less explicitly.

5.2. **Classification of W-quasi-essential triples.** In this subsection $\mathfrak{g}$ is a simple Lie algebra.

Proposition 5.2.1. (1) *An admissible triple $(\mathfrak{g},\mathfrak{h},V)$ is W-quasi-essential if and only if it is listed in Table* 5.1.
(2) *Let $(\mathfrak{g},\mathfrak{h},V)$ be a W-quasi-essential triple. If $\mathfrak{h}$ is simple and $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G *_H V)$, then α is a long root. If $\mathfrak{h}$ is not simple, then $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G *_H V)$ for all roots α.*

To prove Proposition 5.2.1 we need some technical results.

Let us introduce some notation. Let H be a reductive algebraic group, $\mathfrak{s}$ a subalgebra of $\mathfrak{h}$ isomorphic to $\mathfrak{sl}_2$. We denote by $S^{\mathfrak{s}}$ the $\mathfrak{h}$-stratum consisting of $\mathfrak{s}$ and the direct sum of two copies of the two-dimensional irreducible $\mathfrak{s}$-module.

Remark 5.2.2. Let H be a reductive subgroup of G, U an H-module, $(\mathfrak{s},V)$ a $\mathfrak{g}$-stratum. Then $(\mathfrak{s},V) \rightsquigarrow_{\mathfrak{g}} G *_H U$ if and only if there exists $g\in G$ such that $\mathrm{Ad}(g)s\subset\mathfrak{h}$ and $(\mathrm{Ad}(g)\mathfrak{s},V)\rightsquigarrow_{\mathfrak{h}} U$ (the algebra $\mathrm{Ad}(g)\mathfrak{s}$ is represented in V via the isomorphism $\mathrm{Ad}(g^{-1}):\mathrm{Ad}(g)\mathfrak{s}\to\mathfrak{s}$). Conversely, if $(\mathfrak{s},V)\rightsquigarrow_{\mathfrak{h}} U$, then $(\mathfrak{s},V)\rightsquigarrow_{\mathfrak{g}} G *_H U$.

Now we recall the definition of the Dynkin index ([D]). Let $\mathfrak{h}$ be a simple subalgebra of $\mathfrak{g}$. We fix an invariant nondegenerate symmetric bilinear form $K_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $K_{\mathfrak{g}}(\alpha^\vee,\alpha^\vee)=2$ for a root $\alpha\in\Delta(\mathfrak{g})$ of maximal length. Analogously define a form $K_{\mathfrak{h}}$ on $\mathfrak{h}$. The *Dynkin index* of the embedding $\iota:\mathfrak{h}\hookrightarrow\mathfrak{g}$ is, by definition, $K_{\mathfrak{g}}(\iota(x),\iota(x))/K_{\mathfrak{h}}(x,x)$ (the last fraction does not depend on the choice of $x\in\mathfrak{h}$ such that $K_{\mathfrak{h}}(x,x)\neq 0$). For brevity, we denote the Dynkin index of ι by $i(\mathfrak{h},\mathfrak{g})$. It turns out that $i(\mathfrak{h},\mathfrak{g})$ is a positive integer (see [D]).

The following lemma seems to be standard.

Lemma 5.2.3. *Let $\mathfrak{h}$ be a simple Lie algebra and $\mathfrak{s}$ a subalgebra of $\mathfrak{h}$ isomorphic to $\mathfrak{sl}_2$. Then the following conditions are equivalent:*
(1) $\iota(\mathfrak{s},\mathfrak{h})=1$;
(2) $\mathfrak{s}\sim_{\mathrm{Int}(\mathfrak{h})}\mathfrak{h}^{(\alpha)}$ *for a long root $\alpha\in\Delta(\mathfrak{h})$.*

Proof. Clearly, (2) $\Rightarrow$ (1). Let us check (1) $\Rightarrow$ (2). Choose the standard basis e,h,f in $\mathfrak{s}$. We may assume that h lies in the fixed Cartan subalgebra of $\mathfrak{h}$. It follows from the representation theory of $\mathfrak{sl}_2$ that $\langle\pi,h\rangle$ is an integer for any weight π of $\mathfrak{h}$. Thus $h\in Q^\vee$, where $Q^\vee$ denotes the dual root lattice. Since $i(\mathfrak{s},\mathfrak{h})=i(\mathfrak{h}^{(\alpha)},\mathfrak{h})$ for a long root $\alpha\in\Delta(\mathfrak{h})$, the lengths of $h,\alpha^\vee\in Q^\vee$ coincide. It can be seen directly from the constructions of the root systems, that all elements $h\in Q^\vee$ with $(h,h)=(\alpha^\vee,\alpha^\vee)$ are short dual roots; see [Bou]. Thus we may assume that $h=\alpha^\vee$. It follows from the standard theorems on the conjugacy of $\mathfrak{sl}_2$-triples, see, for example, [McG], that $\mathfrak{s}$ and $\mathfrak{h}^{(\alpha)}$ are conjugate. □

In Table 5.2 we list all simple subalgebras of index 1 in simple classical Lie algebras.

TABLE 5.2. Simple subalgebras $\mathfrak{h}$ in classical Lie algebras $\mathfrak{g}$ with $\iota(\mathfrak{h},\mathfrak{g})=1$

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}_n, n\geqslant 2$	$\mathfrak{sl}_k, k\leqslant n$
$\mathfrak{sl}_n, n\geqslant 4$	$\mathfrak{sp}_{2k}, 2\leqslant k\leqslant n/2$
$\mathfrak{so}_n, n\geqslant 7$	$\mathfrak{so}_k, k\leqslant n, k\neq 4$
$\mathfrak{so}_n, n\geqslant 7$	$\mathfrak{sl}_k^{diag}, k\leqslant n/2$
$\mathfrak{so}_n, n\geqslant 8$	$\mathfrak{sp}_{2k}^{diag}, 2\leqslant k\leqslant n/4$
$\mathfrak{so}_n, n\geqslant 7$	G_2
$\mathfrak{so}_n, n\geqslant 9$	$\mathfrak{spin}_7$
$\mathfrak{sp}_{2n}, n\geqslant 2$	$\mathfrak{sp}_{2k}, k\leqslant n$

Lemma 5.2.4. *Let $\alpha\in\Delta(\mathfrak{g})$ and $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^{(\alpha)}$. Suppose that there is no proper ideal of $\mathfrak{h}$ containing $\mathfrak{g}^{(\alpha)}$. Then*

(1) *If α is a long root, then $\mathfrak{h}$ is simple and $\iota(\mathfrak{g}^{(\alpha)},\mathfrak{h})=\iota(\mathfrak{h},\mathfrak{g})=1$.*
(2) *Suppose α is a short root and $\mathfrak{h}$ is not simple. Then*
   (a) *$\mathfrak{g}\cong\mathfrak{so}_{2l+1},\mathfrak{sp}_{2l}, l\geqslant 2, F_4$, $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_1,\mathfrak{h}_2$ are simple ideals of $\mathfrak{g}$ with $i(\mathfrak{h}_1,\mathfrak{g})=i(\mathfrak{h}_2,\mathfrak{g})=1$. If $\mathfrak{s}_i$ denotes the projection of $\mathfrak{g}^{(\alpha)}$ to $\mathfrak{h}_i, i=1,2$, then $i(\mathfrak{s}_i,\mathfrak{h}_i)=1$.*
   (b) *If $\mathfrak{g}\cong\mathfrak{so}_{2l+1}, l\geqslant 2$, then $\mathfrak{h}=\mathfrak{so}_4$.*
   (c) *If $\mathfrak{g}\cong\mathfrak{sp}_{2l}$, then $\mathfrak{h}=\mathfrak{sp}_{2k}\oplus\mathfrak{sp}_{2m}, k+m\leqslant l$.*

Proof. Since $\mathfrak{g}^{(\alpha)}\subset[\mathfrak{h},\mathfrak{h}]$, we see that $\mathfrak{h}$ is semisimple. Let $\mathfrak{h}_1,\mathfrak{h}_3$ be simple Lie algebras and $\mathfrak{h}_2$ a semisimple Lie algebra, $\mathfrak{h}_1\subset\mathfrak{h}_2\subset\mathfrak{h}_3$. Let $\mathfrak{h}_2=\mathfrak{h}_2^1\oplus\cdots\oplus\mathfrak{h}_2^k$ be the decomposition of $\mathfrak{h}_2$ into the direct sum of simple ideals and $\iota^i, i=1,\ldots,k$, the composition of the embedding $\mathfrak{h}_1\hookrightarrow\mathfrak{h}_2$ and the projection $\mathfrak{h}_2\to\mathfrak{h}_2^i$. It is shown in [D] that

$$i(\mathfrak{h}_1,\mathfrak{h}_3)=\sum_{i=1}^k i(\iota_i(\mathfrak{h}_1),\mathfrak{h}_2^i)i(\mathfrak{h}_2^i,\mathfrak{h}_3). \tag{5.1}$$

This implies assertion (1).

In assertion (2)(a) it remains to check that $\mathfrak{g}\neq G_2$. Indeed, there is a unique up to conjugacy semisimple but not simple subalgebra of G_2, namely $\mathfrak{g}^{(\alpha)}\times\mathfrak{g}^{(\beta)}$, where β is a long root and $(\alpha,\beta)=0$. Since $i(\mathfrak{g}^{(\alpha)},\mathfrak{g})=1, i(\mathfrak{g}^{(\beta)},\mathfrak{g})=3$, the claim follows from (5.1).

Let us proceed to assertion (2)(b). Note that the representation of $\mathfrak{g}^{(\alpha)}$ in the tautological $\mathfrak{g}$-module V is the sum of the trivial $(2l-2)$-dimensional and the 3-dimensional irreducible representations. Since $\mathfrak{g}^{(\alpha)}$ is not contained in a proper ideal of $\mathfrak{h}$, we see that the representation of $\mathfrak{h}$ in $V/V^{\mathfrak{h}}$ is irreducible. Thus $V/V^{\mathfrak{h}}=V_1\otimes V_2$, where V_i is an irreducible $\mathfrak{h}_i$-module, $i=1,2$. Note that the representation of $\mathfrak{g}^{(\alpha)}$ in V_i is nontrivial because the projection of $\mathfrak{g}^{(\alpha)}$ to $\mathfrak{h}_i$ is nontrivial. From the equality $\dim(V_1\otimes V_2)/(V_1\otimes V_2)^{\mathfrak{g}^{(\alpha)}}=3$ and the Clebsch-Gordan formula it follows that $\dim V_1=\dim V_2=2$. Thence $\mathfrak{h}_1\cong\mathfrak{h}_2\cong\mathfrak{sl}_2$, $\mathfrak{h}=\mathfrak{so}_4$.

We now proceed to assertion (2)(c). It follows from assertion (2)(a) that $i(\mathfrak{h}_1,\mathfrak{g})=i(\mathfrak{h}_2,\mathfrak{g})=1$. Then $\mathfrak{h}_1=\mathfrak{sp}_{2m},\mathfrak{h}_2=\mathfrak{sp}_{2k}$; see Table 5.2. $\square$

Let us recall the notion of the *index* of a module over a simple Lie algebra; see [AEV]. Let $\mathfrak{h}$ be a simple Lie algebra, U an $\mathfrak{h}$-module. We define a symmetric invariant bilinear form $(\cdot,\cdot)_U$ on $\mathfrak{h}$ by $(x,y)_U = \operatorname{tr}_U(xy)$. The form $(\cdot,\cdot)_U$ is nondegenerate whenever U is nontrivial. By the *index* of U we mean the fraction $\frac{(x,y)_U}{(x,y)_{\mathfrak{h}}}$. Since $\mathfrak{h}$ is simple, the last fraction does not depend on the choice of $x,y\in\mathfrak{h}$ with $(x,y)_{\mathfrak{h}}\neq 0$. We denote the index of U by $l_{\mathfrak{h}}(U)$.

Lemma 5.2.5. *Let H be a semisimple algebraic group and $\mathfrak{s}$ a subalgebra in $\mathfrak{h}$ isomorphic to $\mathfrak{sl}_2$. Let V be an H-module such that $S^{\mathfrak{s}} \rightsquigarrow_{\mathfrak{h}} V$.*

(1) *Let e,h,f be the standard basis of $\mathfrak{s}$. Then $(h,h)_V = (h,h)_{\mathfrak{h}} - 4$.*
(2) *If $\mathfrak{h}$ is simple, then $l_{\mathfrak{h}}(V) = 1 - \frac{4}{i(\mathfrak{s},\mathfrak{h})k_{\mathfrak{h}}}$, where $k_{\mathfrak{h}} = (\alpha^\vee,\alpha^\vee)_{\mathfrak{h}}$ for a long root $\alpha\in\Delta(\mathfrak{h})$.*

Proof. There is an isomorphism $V\cong V^{\mathfrak{s}}\oplus\mathfrak{h}/\mathfrak{s} + (\mathbb{C}^2)^{\oplus 2}$ of $\mathfrak{s}$-modules, where $\mathbb{C}^2$ denotes the tautological $\mathfrak{s}$-module. Thus $\operatorname{tr}_V h^2 = \operatorname{tr}_{\mathfrak{h}} h^2 - \operatorname{tr}_{\mathfrak{s}} h^2 + 2\operatorname{tr}_{\mathbb{C}^2} h^2 = \operatorname{tr}_{\mathfrak{h}} h^2 + 8 - 4$. Assertion (2) stems from

$$\frac{\operatorname{tr}_{\mathfrak{h}} h^2}{\operatorname{tr}_{\mathfrak{h}} \alpha^{\vee 2}} = \frac{i(\mathfrak{s},\mathfrak{h})}{i(\mathfrak{h}^{(\alpha)},\mathfrak{h})} = i(\mathfrak{s},\mathfrak{h}). \qquad \square$$

The numbers $k_{\mathfrak{h}}$ for all simple Lie algebras are given in Table 5.3.

Table 5.3. $k_{\mathfrak{h}}$.

$\mathfrak{h}$	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$k_{\mathfrak{h}}$	$4l+4$	$8l-4$	$4l+4$	$8l-8$	48	72	120	36	16

Let H be a reductive group, V an H-module. There exists the s.g.p. for the action $H:V$; see [PV], Theorem 7.2. Recall that the action $H:V$ is called *stable* if an orbit in general position is closed. In this case the s.g.p. is reductive.

Lemma 5.2.6. *Let $\mathfrak{h}$ be a semisimple Lie algebra, V an $\mathfrak{h}$-module, $V_1\subset V$ an H-submodule such that the action $H:V_1$ is stable. Let H_1 denote the s.g.p. for the action $H:V_1$ and let $\mathfrak{s}$ be a subalgebra of $\mathfrak{h}_1$ isomorphic to $\mathfrak{sl}_2$. If $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}_1} V/V_1$, then $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}} V$.*

Proof. Let $v_1\in V_1$ be such that Hv_1 is closed and $H_{v_1}=H_1$. The slice module at v_1 is the direct sum of V/V_1 and a trivial H_{v_1}-module. The claim of the lemma follows from the Luna slice theorem and Remark 5.2.2. $\square$

Proof of Proposition 5.2.1. The proof is in three steps.

Step 1. Here we suppose that $\mathfrak{h}$ is simple. An admissible triple $(\mathfrak{g},\mathfrak{h},V)$ is W-quasi-essential if and only if $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}} V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{s}$ is a subalgebra in $\mathfrak{h}$ such that $\mathfrak{s}\sim_G\mathfrak{g}^{(\alpha)}$ for some $\alpha\in\Delta(\mathfrak{g})$.

Suppose $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}} V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h}$ for some subalgebra $\mathfrak{s}\subset\mathfrak{h}$ isomorphic to $\mathfrak{sl}_2$. From assertion 2 of Lemma 5.2.5 it follows that

$$l_{\mathfrak{h}}(V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h}) = 1 - \frac{4}{i(\mathfrak{s},\mathfrak{h})k_{\mathfrak{h}}}. \tag{5.2}$$

Thence $l_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h})<1$. In Section 3 of [Lo1] it was shown that $i(\mathfrak{h},\mathfrak{g})=1$. Equivalently, $i(\mathfrak{s},\mathfrak{h}) = i(\mathfrak{s},\mathfrak{g})$. All simple subalgebras $\mathfrak{h}\subset\mathfrak{g}$ with $l_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h})<1$ are listed in [Lo1], Table 5. The list (up to $\operatorname{Aut}(\mathfrak{g})$-conjugacy) is presented in Table 5.4. In column 4 the nontrivial part of the representation of $\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$ is given. By τ we denote the tautological representation of a classical Lie algebra.

TABLE 5.4. Simple subalgebras $\mathfrak{h} \subsetneq \mathfrak{g}$ with $l_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h}) < 1$

N	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}/\mathfrak{h}_+$
1	$\mathfrak{sl}_n, n > 1$	$\mathfrak{sl}_k, n/2 < k < n$	$(n-k)(\tau + \tau^*)$
2	$\mathfrak{sl}_{2n}, n \geqslant 2$	$\mathfrak{sp}_{2n}$	$\bigwedge^2 \tau$
3	$\mathfrak{sl}_{2n+1}, n \geqslant 2$	$\mathfrak{sp}_{2n}$	$2\tau + \bigwedge^2 \tau$
4	$\mathfrak{sp}_{2n}, n \geqslant 2$	$\mathfrak{sp}_{2k}, n/2 \leqslant k < n$	$2(n-k)\tau$
5	$\mathfrak{so}_n, n \geqslant 7$	$\mathfrak{so}_k, \frac{n+2}{2} < k < n, k \neq 4$	$(n-k)\tau$
6	$\mathfrak{so}_{2n}, n \geqslant 5$	$\mathfrak{sl}_n$	$\bigwedge^2 \tau + \bigwedge^2 \tau^*$
7	$\mathfrak{so}_{2n+1}, n \geqslant 3$	$\mathfrak{sl}_n$	$\tau + \tau^* + \bigwedge^2 \tau + \bigwedge^2 \tau^*$
8	$\mathfrak{so}_n, 9 \leqslant n \leqslant 11$	$\mathfrak{spin}_7$	$\tau + (n-8)R(\pi_3)$
9	$\mathfrak{so}_n, 7 \leqslant n \leqslant 9$	G_2	$(n-3)R(\pi_1)$
10	G_2	A_2	$\tau + \tau^*$
11	F_4	B_4	$R(\pi_4)$
12	F_4	D_4	$\tau + R(\pi_3) + R(\pi_4)$
13	F_4	B_3	$2\tau + 2R(\pi_3)$
14	E_6	F_4	$R(\pi_1)$
15	E_6	D_5	$R(\pi_4) + R(\pi_5)$
16	E_6	B_4	$\tau + 2R(\pi_4)$
17	E_7	E_6	$R(\pi_1) + R(\pi_5)$
18	E_7	D_6	$2R(\pi_6)$
19	E_8	E_7	$2R(\pi_1)$

All orthogonal H-modules U such that $m_H(U) = \dim H$ and $S^{\mathfrak{s}} \rightsquigarrow_{\mathfrak{h}} U$ were found by G. Schwarz in [Sch2], Tables I-V ("orthogonal representations with an $\mathbb{S}^3$-stratum", in his terminology). These modules are given in Table 5.5. Note that one of them is omitted in [Sch2], namely N11, $k = 1$. To see that $S^{\mathfrak{s}} \rightsquigarrow U$, where $\mathfrak{s} = \mathfrak{h}^{(\alpha)}$ and α is a long root in $\Delta(\mathfrak{h})$, one applies Lemma 5.2.6 to $\mathbb{C}^{12} \subset U$. The corresponding pair $(\mathfrak{h}_1, V/(V_1 \oplus V^{\mathfrak{h}_1}))$ is N6 of Table 5.5.

TABLE 5.5. Orthogonal H-modules U with $m_H(U) = \dim H$ and $S^{\mathfrak{s}} \rightsquigarrow_{\mathfrak{h}} U$

N	$\mathfrak{h}$	$U/U^{\mathfrak{h}}$
1	$\mathfrak{sl}_n, n > 1,$	$(n-1)(\tau + \tau^*)$
2	$\mathfrak{sl}_n, n > 3,$	$\tau + \tau^* + \bigwedge^2 \tau + \bigwedge^2 \tau^*$
3	$\mathfrak{sl}_4$	$2(\tau + \tau^*) + \bigwedge^2 \tau$
4	$\mathfrak{so}_7$	$(4-k)\tau + kR(\pi_3), k > 0$

N	$\mathfrak{h}$	U
5	$\mathfrak{so}_9$	$kR(\pi_4)+(6-2k)\tau, k>0$
6	$\mathfrak{so}_{11}$	$2R(\pi_5)$
7	$\mathfrak{sp}_{2m}, m>1$	$2m\tau$
8	$\mathfrak{sp}_{2m}, m>1$	$2\tau+R(\pi_2)$
9	$\mathfrak{so}_8$	$k\tau+lR(\pi_3)+mR(\pi_4), k+l+m=5, k,l,m<5$
10	$\mathfrak{so}_{10}$	$3\tau+R(\pi_4)+R(\pi_5)$
11	$\mathfrak{so}_{12}$	$\tau+kR(\pi_5)+(2-k)R(\pi_6)$
12	G_2	$3R(\pi_1)$

It follows from (5.2) that $\iota(\mathfrak{s},\mathfrak{h})=1$ for all modules from Table 5.5. This proves assertion 2 of the proposition for simple $\mathfrak{h}$. Inspecting Tables 5.4, 5.5, we find all admissible triples $(\mathfrak{g},\mathfrak{h},V)$ with $S^{\mathfrak{s}} \rightsquigarrow_{\mathfrak{h}} V\oplus V^*\oplus \mathfrak{g}/\mathfrak{h}$. This completes the proof of assertion 1 when $\mathfrak{h}$ is simple.

Step 2. It remains to consider the situation when $\mathfrak{h}$ is not simple. In this step we assume that $\mathfrak{h}$ possesses the following property:

(*) $S^{\mathfrak{s}} \rightsquigarrow_{\mathfrak{h}} \mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$, where $\mathfrak{s}\subset\mathfrak{h}$ is such that

(1) $\mathfrak{s}\subset \mathfrak{h}_1\oplus\mathfrak{h}_2$, where $\mathfrak{h}_1,\mathfrak{h}_2$ are simple ideals of $\mathfrak{h}$;
(2) $\mathfrak{s}\sim_G \mathfrak{g}^{(\alpha)}$, where α is a short root of $\mathfrak{g}$;
(3) $i(\mathfrak{s}_i,\mathfrak{h}_i)=1$, where $\mathfrak{s}_i$ denotes the projection of $\mathfrak{s}$ to $\mathfrak{h}_i, i=1,2$.

It follows from Lemma 5.2.4 that a subalgebra $\mathfrak{s}\subset\mathfrak{h}$ satisfying (1)–(3) is defined uniquely up to $\mathrm{Int}(\mathfrak{h})$-conjugacy.

Let us check that

$$k_{\mathfrak{g}}=k_{\mathfrak{h}_1}+k_{\mathfrak{h}_2}-2-k_{\mathfrak{h}_1}l_{\mathfrak{h}_1}(V)-k_{\mathfrak{h}_2}l_{\mathfrak{h}_2}(V). \tag{5.3}$$

Let $h\in\mathfrak{s}$ be a dual root, $h=h_1+h_2, h_i\in\mathfrak{h}_i$. By Lemma 5.2.5, $(h,h)_V+(h,h)_{V^*}+(h,h)_{\mathfrak{g}/\mathfrak{h}}=(h,h)_{\mathfrak{h}}-4$. Equivalently,

$$2(h,h)_V+(h,h)_{\mathfrak{g}}=2(h,h)_{\mathfrak{h}}-4. \tag{5.4}$$

Since $(\cdot,\cdot)_V,(\cdot,\cdot)_{\mathfrak{h}}$ are $\mathfrak{h}$-invariant forms on $\mathfrak{h}$, we see that $\mathfrak{h}_1,\mathfrak{h}_2$ are orthogonal with respect to $(\cdot,\cdot)_V,(\cdot,\cdot)_{\mathfrak{h}}$. Therefore (5.4) is equivalent to

$$(h,h)_{\mathfrak{g}}+2((h_1,h_1)_V+(h_2,h_2)_V)=2((h_1,h_1)_{\mathfrak{h}_1}+(h_2,h_2)_{\mathfrak{h}_2})-4. \tag{5.5}$$

By assertion (2)(a) of Lemma 5.2.4 and Lemma 5.2.3, $h_i\sim_{\mathrm{Int}(\mathfrak{h}_i)}\alpha^\vee$, where α is a long root in $\Delta(\mathfrak{h}_i)$. Therefore $(h_i,h_i)_{\mathfrak{h}_1}=k_{\mathfrak{h}_i}, i=1,2$. It follows from the choice of $\mathfrak{s}$ that $i(\mathfrak{s},\mathfrak{h})=2$, whence $(h,h)_{\mathfrak{g}}=2k_{\mathfrak{g}}$. So (5.5) and (5.3) are equivalent.

Let us show that $\mathfrak{g}\not\cong F_4,\mathfrak{so}_{2l+1}, l\geqslant 3$. Assume that $\mathfrak{g}\cong F_4$. By (5.3), $k_{\mathfrak{h}_1}+k_{\mathfrak{h}_2}\geqslant 38$. Since $\mathrm{rk}\,\mathfrak{h}_1+\mathrm{rk}\,\mathfrak{h}_2\leqslant 4$, this is impossible (see Table 5.3). Assume that $\mathfrak{g}\cong\mathfrak{so}_{2l+1}, l\geqslant 3$. By assertion (2)(b) of Lemma 5.2.4, $\mathfrak{h}_1\cong\mathfrak{h}_2\cong\mathfrak{sl}_2$. This again contradicts (5.3).

It follows from assertions (2)(a), (2)(c) of Lemma 5.2.4 that $\mathfrak{g}\cong\mathfrak{sp}_{2n}$, $\mathfrak{h}_1\oplus\mathfrak{h}_2=\mathfrak{sp}_{2m_1}\oplus\mathfrak{sp}_{2m_2}, m_1+m_2\leqslant n, m_1\leqslant m_2$. One can rewrite (5.3) as

$$2(m_1+m_2-n)+1=(2m_1+2)l_{\mathfrak{h}_1}(V)+(2m_2+2)l_{\mathfrak{h}_2}(V). \tag{5.6}$$

From (5.6) it follows that $m_1+m_2=n$, $l_{\mathfrak{h}_i}(V)\leqslant\frac{1}{2m_i+2}$. Therefore the $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$ is the tensor product of the tautological $\mathfrak{sp}_{2m_1}$- and $\mathfrak{sp}_{2m_2}$-modules and, see the table from [AEV], V is the tautological $\mathfrak{sp}_{2m_j}$-module for some $j\in\{1,2\}$. Recall that $m_H(V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h})=\dim H$. In particular, $m_{H_i}(V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h})=\dim H_i, i=1,2$. By the above, the H_i-module $V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h}$ is the direct sum of tautological $\mathrm{Sp}(2m_i)$-modules. Thus if $m_1<m_2$, then H_1 acts trivially on V and $m_2=m_1+1$. This shows that a W-quasi-essential triple $(\mathfrak{g},\mathfrak{h},V)$ satisfying (*) is one of N32, N33 of Table 5.1.

Let us show that the triples N32, N33 satisfy (*). Put $V_0=\mathfrak{g}/\mathfrak{h}$. This is an orthogonal, whence stable, see, for instance, [Lu1], H-module. The s.s.g.p. $\mathfrak{h}_0$ for the H-module V_0 is the direct sum of m_1 copies of $\mathfrak{sl}_2$ embedded diagonally into $\mathfrak{h}_1\oplus\mathfrak{h}_2$ and m_2-m_1 copies of $\mathfrak{sl}_2$ embedded into $\mathfrak{h}_2$ (see [E2]). We are done by Lemma 5.2.6 applied to $\mathfrak{g}/\mathfrak{h}\subset\mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$.

Finally, we see that $S^{(\mathfrak{h}_2^{(\alpha)})}\rightsquigarrow_{\mathfrak{h}_2}V\oplus V^*\oplus\mathfrak{g}/\mathfrak{h}$ (if $\mathfrak{h}_1\cong\mathfrak{h}_2$, then for $\mathfrak{h}_2$ we take the ideal of $\mathfrak{h}$ acting on V trivially), where α is a long root of $\mathfrak{h}_2$.

Step 3. It remains to show that any W-quasi-essential triple $(\mathfrak{g},\mathfrak{h},V)$ satisfies (*) provided $\mathfrak{h}$ is not simple. Assume the converse. Let us show that

(**) $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}}\mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$ for some subalgebra $\mathfrak{s}\subset\mathfrak{h},\mathfrak{s}\cong\mathfrak{sl}_2$, not lying in any simple ideal of $\mathfrak{h}$.

Indeed, let $\mathfrak{h}_1$ be a simple ideal of $\mathfrak{h}$ and $\mathfrak{s}_1$ a subalgebra of $\mathfrak{h}_1$ isomorphic to $\mathfrak{sl}_2$ such that $S^{\mathfrak{s}_1}\rightsquigarrow_{\mathfrak{h}}\mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$, or equivalently, $S^{\mathfrak{s}_1}\rightsquigarrow_{\mathfrak{h}_1}\mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$. Then $(\mathfrak{g},\mathfrak{h}_i,V/V^{\mathfrak{h}_i})$ is one of the triples N1–N31 from Table 5.1 and $\mathfrak{s}_1\sim_G\mathfrak{g}^{(\alpha)}$, where α is a long root in $\Delta(\mathfrak{g})$. If (**) does not hold, then, by step 1, $S^{(\alpha)}\rightsquigarrow_{\mathfrak{g}}T^*(G*_H V)$ implies that α is a long root. Since $\mathfrak{h}$ is not simple, we see that $(\mathfrak{g},\mathfrak{h},V)$ is not W-quasi-essential. So (**) is checked.

It follows from assertion (2)(a) of Lemma 5.2.5 that there is a subalgebra $\mathfrak{s}\subset\mathfrak{h}$ and simple ideals $\mathfrak{h}_1,\mathfrak{h}_2$ of $\mathfrak{h}$ such that $(\mathfrak{g},\mathfrak{h}_1\oplus\mathfrak{h}_2,V/V^{\mathfrak{h}_1\oplus\mathfrak{h}_2})$ satisfies condition (*) of step 2. But in this case $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}_1\oplus\mathfrak{h}_2)=\mathfrak{h}_1\oplus\mathfrak{h}_2$, whence $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$, a contradiction. □

5.3. **Computation of Weyl groups.** In this subsection we compute the groups $W(\mathfrak{g},\mathfrak{h},V)$ for triples $(\mathfrak{g},\mathfrak{h},V)$ listed in Table 5.1 proving that the groups $W(\mathfrak{g},\mathfrak{h},V)$ are as indicated in column 5 of that table.

At first, we reduce the problem to the case when $\mathbb{C}[V]^H=\mathbb{C}$. Let $(\mathfrak{g},\mathfrak{h},V)$ be one of the triples from Table 5.1. Let H_0 denote the unit component of the principal isotropy subgroup for the action $H:V$. Put $V_0=V/(V^{H_0}+\mathfrak{h}v)$, where $v\in V$ is such that Hv is closed and $H_v^\circ=H_0$. Thanks to Corollary 3.2.3, $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g},\mathfrak{h}_0,V_0)$. The triple $(\mathfrak{g},\mathfrak{h}_0,V_0)$ is said to be *reduced* from $(\mathfrak{g},\mathfrak{h},V)$. Clearly, $(\mathfrak{g},\mathfrak{h},V)=(\mathfrak{g},\mathfrak{h}_0,V_0)$ if and only if $\mathbb{C}[V]^H=\mathbb{C}$. Since H is semisimple, we get $\mathrm{Quot}(\mathbb{C}[V]^H)=\mathbb{C}(V)^H$. So $\mathbb{C}[V]^H=\mathbb{C}$ if and only if H has a dense orbit in V. In Table 5.6 we present all triples that are reduced from nonreduced triples from Table 5.1. We use the same notation as in Table 5.1. In column 2 we give the number of $(\mathfrak{g},\mathfrak{h},V)$ in Table 5.1 and, in some cases, restrictions on the algebra $\mathfrak{g}$ or the $\mathfrak{h}$-module V. It turns out that $(\mathfrak{g},\mathfrak{h}_0,V_0)$ is again contained in Table 5.1.

Table 5.6. Reduced triples

N	$(\mathfrak{g},\mathfrak{h},V)$	$\mathfrak{h}_0$	V_0
1	N1, $m>0$	$\mathfrak{sl}_{n-k-m}$	$(l-m)\tau$
2	N2, even n	$\mathfrak{sp}_n$	τ
3	N2, odd n	$\mathfrak{sp}_{n-1}$	0
4	N3	$\mathfrak{sp}_n$	τ
5	N6, odd n	$\mathfrak{sp}_{n-1}$	0
6	N10	$\mathfrak{sl}_3$	0
7	N11, $k=1$	$\mathfrak{sl}_3^{diag}$	0
8	N12	$\mathfrak{sl}_3$	0
9	N13	G_2	0
10	N14, $(k,l)=(2,0)$	$\mathfrak{sl}_4$	0
11	N14, $(k,l)\neq(2,0)$	G_2	0
12	N15	G_2	0
13	N16	$\mathfrak{sl}_4^{diag}$	0
14	N17	G_2	0
15	N19	$\mathfrak{sl}_5^{diag}$	0
16	N20	$\mathfrak{spin}_7$	0
17	N21	$\mathfrak{spin}_7$	0
18	N22	$\mathfrak{spin}_7$	0
19	N25, even n	$\mathfrak{sp}_n$	0
20	N25, odd n	$\mathfrak{sp}_{n+1}$	τ
21	N30	B_3	0
22	N31	B_3	0

Lemma 5.3.1. *All triples $(\mathfrak{g},\mathfrak{h},V)$ from Table 5.1 such that $\mathbb{C}[V]^H\neq\mathbb{C}$ are listed in the second column of Table 5.6. The reduced triple for $(\mathfrak{g},\mathfrak{h},V)$ coincides with $(\mathfrak{g},\mathfrak{h}_0,V_0)$.*

Proof. The triples N32, N33 are reduced. So we may assume that $\mathfrak{h}$ is simple. The list of all simple linear groups with a dense orbit is well known; see, for example, [Vi]. It follows from the classification of the paper [E1] that the s.s.g.p.'s for the H-module $\mathfrak{g}/\mathfrak{h}\oplus V\oplus V^*$ are simple for all triples $(\mathfrak{g},\mathfrak{h},V)$, except N1, N2, N4, N20. By Popov's criterion, see [Po], the actions $H:V$ are stable whenever the s.s.g.p. is reductive. For the remaining four triples, the reduced triples are easily found case by case. □

Below in this subsection we consider only reduced triples $(\mathfrak{g},\mathfrak{h},V)$ from Table 5.1.

The group $T_1\times T_2$, where $T_1=Z(Z_G(H))^\circ, T_2=Z(\mathrm{GL}(V)^H)$, is a torus naturally acting on $X=G*_H V$ by G-automorphisms. Namely, we define the action morphism by $(t_1,t_2,[g,v])\mapsto[gt_1^{-1},t_2v], t_1\in T_1, t_2\in T_2, g\in G, v\in V$. Put $\widetilde{G}=G\times T_1\times T_2$, $\widetilde{H}=H\times T_1\times T_2$.

For some triples $(\mathfrak{g},\mathfrak{h},V)$ the inequality $\mathrm{rk}_{\widetilde{G}}(X)<\mathrm{rk}\,\widetilde{G}$ holds. These triples are presented in Table 5.7. The matrix in column 3 of the first row is the diagonal matrix, whose $(k+1)$-th entry is $(n-1)x$ and the other entries are $-x$.

TABLE 5.7. The projections of $\mathfrak{l}_{0\,\widetilde{G},X}$ to $\mathfrak{g}$

N	$(\mathfrak{g},\mathfrak{h},V)$	the projection of $\mathfrak{l}_{0\,\widetilde{G},X}$ to $\mathfrak{g}$
1	N1, $m=0$	$diag(-x^k,(n-1)x,-x^{n-k-1})$
2	N4	$diag(-\frac{x}{n+1},-\frac{x}{n+1},\frac{x}{n-1},-\frac{x}{n+1}\ldots,\frac{x}{n-1})$
3	N5	$diag(\frac{x}{n+2},\frac{x}{n+2},-\frac{x}{(n-2)},\frac{x}{n+2},\ldots,-\frac{x}{(n-2)},\frac{x}{n+2})$
4	N7	$diag(-x,x,-x,\ldots,x)$
5	N8	$diag(\frac{x}{n+1},-\frac{x}{n-1},\frac{x}{n+1},\ldots,-\frac{x}{n-1},\frac{x}{n+1})$
6	N26	$x\sum_{i=1}^{2n+1}(-1)^i\varepsilon_i$
7	N27	$x\sum_{i=1}^{2n}(-1)^i\varepsilon_i$
8	N32	$x\sum_{i=1}^{2n}(-1)^i\varepsilon_i$
9	N33	$x\sum_{i=1}^{2n+1}(-1)^i\varepsilon_i$

Lemma 5.3.2. *If $(\mathfrak{g},\mathfrak{h},V)$ is a triple from the first column of Table 5.7, then $W(\mathfrak{g},\mathfrak{h},V)$ coincides with the group indicated in Table 5.1.*

Proof. At first, we will check that the projections of $\mathfrak{l}_{0\,\widetilde{G},\widetilde{X}_0}$ to $\mathfrak{g}$ for the triples in consideration are given in column 3 of Table 5.7.

Let us note that $\dim T_1=1$ for rows N1 (with $k\neq 0$), N3, N5–N8 of Table 5.7; otherwise $\dim T_1=0$. The dimension of T_2 equals 2 for the second row, 1 for rows N1, N3, N4, N6–N9, and 0 otherwise.

We embed $\widetilde{H}$ into $\widetilde{G}$ via $(h,t_1,t_2)\mapsto(ht_1,t_1,t_2)$. Equip V with a natural structure of an $\widetilde{H}$-module: H acts on V as before, T_1 trivially, and T_2 via the identification $T_2\cong Z(\mathrm{GL}(V)^H)$. The $\widetilde{G}$-varieties $X,\widetilde{G}*_{\widetilde{H}}V$ are isomorphic.

Since $\mathfrak{a}_{G,X}=\mathfrak{t}$, the projection $\mathfrak{a}_{\widetilde{G},X}\to\mathfrak{t}$ is surjective. Thus $\mathfrak{a}_{\widetilde{G},X},\mathfrak{l}_{0\,\widetilde{G},X}$ are mutually orthogonal and $\mathfrak{a}_{\widetilde{G},X}\oplus\mathfrak{l}_{0\,\widetilde{G},X}=\mathfrak{t}\oplus\mathfrak{t}_1\oplus\mathfrak{t}_2$. To prove the claim, it is enough to compute $\mathfrak{a}_{\widetilde{G},X}\subset\mathfrak{t}\oplus\mathfrak{t}_1\oplus\mathfrak{t}_2$. By virtue of the isomorphism $X\cong\widetilde{G}*_{\widetilde{H}}V$, the computation can be done by using the algorithm from Subsection 7.1. However, in some cases one can simplify the computation. Namely, for rows N1–N3, N6, N7 there is an antistandard parabolic subgroup $Q\subset\widetilde{G}$ with the standard Levi subgroup $M\subset Q$ such that $H=(M,M)$. Applying Proposition 3.2.9, we reduce the computation of $\mathfrak{a}_{\widetilde{G},X}$ to computing the spaces $\mathfrak{a}_{\bullet,\bullet}$ for certain linear actions. For rows N2, N3, N6, N7 these linear actions are spherical, so the Cartan spaces can be extracted, for example, from the second table in [Le], Section 2.

Now we proceed to the claim on the Weyl groups. The projection $\mathfrak{a}_0$ of $\mathfrak{l}_{0\,\widetilde{G},X}$ to $\mathfrak{g}$ lies in $\mathfrak{t}\cap(\mathfrak{a}_{\widetilde{G},X}\cap\mathfrak{g})^\perp$. It follows from Proposition 3.3.12 that $\mathfrak{a}_0\subset\mathfrak{t}^{W_{G,X}}$. Therefore $W_{G,X}\subset W(\mathfrak{g})_\xi$ for $\xi\in\mathfrak{a}_0\setminus\{0\}$. By Proposition 3.3.17, $W_{G,X}$ is one of the groups listed in Table 3.3. Now the equalities $W_{G,X}=W(\mathfrak{g})_\xi$ are easily checked case by case. $\square$

It remains to compute the Weyl groups for triples N9, N11 (with $k=2$), N18, N23–N25, N28, N31 from Table 5.1.

Note that $s_\alpha \in W(\mathfrak{g},\mathfrak{h},V)$ for a short root $\alpha\in\Delta(\mathfrak{g})$. Indeed, Proposition 5.2.1 implies that $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G*_H V)$ if and only if $\alpha\in\Delta(\mathfrak{g})$ is a long root. Our claim follows from Corollary 3.3.22.

N9. Here $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{h}=\mathfrak{sl}_n, V=\{0\}$. Put $\mathfrak{m}:=\mathfrak{t}+\mathfrak{g}^{(\alpha_1,\ldots,\alpha_{n-1})}$ (so that $\mathfrak{h}=[\mathfrak{m},\mathfrak{m}]$), $\mathfrak{q}=\mathfrak{m}+\mathfrak{b}^-$ and let M,Q be the corresponding Levi and parabolic subgroups. The representation of $H\cong \mathrm{SL}_n$ in $\mathrm{R}_u(\mathfrak{q})$ coincides with $\bigwedge^2\tau^*+\tau^*$. From Corollary 3.3.11 it follows that $W(\mathfrak{g},\mathfrak{h},V)\cap M/T=W(\mathfrak{sl}_n,\mathfrak{sl}_n,\bigwedge^2\mathbb{C}^{n*}\oplus\mathbb{C}^{n*})$. By the above, the last Weyl group is generated by $s_{\varepsilon_i-\varepsilon_{i+2}}, i=1,\ldots,n-2$. Thence $W(\mathfrak{g},\mathfrak{h},V)$ coincides with the group indicated in Table 5.1.

N11. Here $\mathfrak{g}=\mathfrak{so}_7,\mathfrak{h}=\mathfrak{so}_6$ and V is the direct sum of two copies of the semispinor $\mathfrak{h}$-module $V(\pi_3)$. We assume that $\mathfrak{h}$ is embedded into $\mathfrak{so}_7$ as the annihilator of e_4. It follows from assertion (3) of Proposition 3.2.1 that $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g},\mathfrak{h}_0)$, where $\mathfrak{h}_0$ denotes the s.s.g.p. of the $\mathfrak{h}$-module V. For $\mathfrak{h}_0$ one may take the following subalgebra:

$$\mathfrak{h}_0=\{\begin{pmatrix}0&0&0&0&0&0&0\\ x&a&b&0&0&0&0\\ y&c&-a&0&0&0&0\\ 0&0&0&0&0&0&0\\ -t&0&0&0&a&-b&0\\ -z&0&0&0&-c&-a&0\\ 0&z&t&0&-y&-x&0\end{pmatrix}, x,y,z,t,a,b,c\in\mathbb{C}\}.$$

Put $\mathfrak{m}:=\mathfrak{t}+\mathfrak{g}^{(\varepsilon_2-\varepsilon_3)},\mathfrak{q}=\mathfrak{b}^-+\mathfrak{m}$. It is seen directly that $\mathfrak{s}:=\mathfrak{h}_0\cap\mathfrak{m}\cong\mathfrak{sl}_2$ is a Levi subalgebra in $\mathfrak{h}_0$ and $\mathrm{R}_u(\mathfrak{h}_0)\subset \mathrm{R}_u(\mathfrak{q})$. The nontrivial part of the $\mathfrak{s}$-module $\mathfrak{q}/\mathfrak{h}_0$ is two-dimensional. Applying Corollary 3.3.11 to Q and M, we see that $s_{\varepsilon_2-\varepsilon_3}\notin W(\mathfrak{g},\mathfrak{h}_0)$.

It remains to show that $s_{\varepsilon_1-\varepsilon_2}\in W(\mathfrak{g},\mathfrak{h}_0)$. Put

$$\widetilde{\mathfrak{h}}_0=\{\begin{pmatrix}u&0&0&0&0&0&0\\ x&a&b&0&0&0&0\\ y&c&d&0&0&0&0\\ 0&0&0&0&0&0&0\\ -t&v&0&0&-d&-b&0\\ -z&0&-v&0&-c&-a&0\\ 0&z&t&0&-y&-x&-u\end{pmatrix}, a,b,c,d,u,v,x,y,z,t\in\mathbb{C}\}.$$

By [Wa], Table B, row N7, $\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}}_0)=\langle\varepsilon_1,\varepsilon_2\rangle$ and $W(\mathfrak{g},\widetilde{\mathfrak{h}}_0)$ is generated by $s_{\varepsilon_1-\varepsilon_2},s_{\varepsilon_2}$. From assertion (1) of Proposition 3.2.1 it follows that $W(\mathfrak{g},\widetilde{\mathfrak{h}}_0)$ is a subquotient in $W(\mathfrak{g},\mathfrak{h}_0)$. Thus $s_{\varepsilon_1-\varepsilon_2}\in W(\mathfrak{g},\mathfrak{h}_0)$.

N18. Here $\mathfrak{g}=\mathfrak{so}_9,\mathfrak{h}=G_2,V=\{0\}$.

We may assume that $H\subset M:=TG^{(\alpha_2,\alpha_3,\alpha_4)}$. Put $\mathfrak{q}:=\mathfrak{b}^-+\mathfrak{m}$. Applying Corollary 3.3.11 to Q,M, we see that $W(\mathfrak{g},\mathfrak{h})\cap M/T=W(\mathfrak{so}_7,\mathfrak{h},V(\pi_3))$. The last group has already been computed. It is generated by $s_{\varepsilon_2-\varepsilon_4},s_{\varepsilon_3},s_{\varepsilon_4}$. Therefore $W(\mathfrak{g},\mathfrak{h})$ is generated either by $s_{\varepsilon_1-\varepsilon_2},s_{\varepsilon_2-\varepsilon_4},s_{\varepsilon_3},s_{\varepsilon_4}$, or by $s_{\varepsilon_1-\varepsilon_3},s_{\varepsilon_3}, s_{\varepsilon_2-\varepsilon_4},s_{\varepsilon_4}$. According to Corollary 3.3.22, it remains to check $S^{(A)}\not\rightsquigarrow_{\mathfrak{g}} T^*(G/H)$, where $A=\{\varepsilon_1-\varepsilon_2,\varepsilon_3-\varepsilon_4\}$. Indeed, there is no $G^{(A)}$-fixed point in $T^*(G/H)$ because $G^{(A)}\subset G$ is not conjugate to a subgroup in $H=G_2$.

N23. Here $\mathfrak{g}=\mathfrak{so}_{11}, \mathfrak{h}=\mathfrak{spin}_7, V=\{0\}$.

As above, we may assume that $H\subset M:=TG^{(\alpha_2,\ldots,\alpha_5)}$. Analogously to the previous case, we obtain $W(\mathfrak{g},\mathfrak{h})\cap M/T=W(\mathfrak{so}_9,\mathfrak{h},V(\pi_3))$. The last Weyl group is generated by $s_{\varepsilon_2-\varepsilon_3}, s_{\varepsilon_3-\varepsilon_5}, s_{\varepsilon_4}, s_{\varepsilon_5}$. As above, one should check that $S^{(A)}\not\twoheadrightarrow_{\mathfrak{g}} T^*(G/H)$, where $A=\{\varepsilon_1-\varepsilon_2,\varepsilon_3-\varepsilon_4\}$. Assume the converse. If $\mathfrak{g}^{(A)}\sim\mathfrak{s}$ for some subalgebra $\mathfrak{s}\subset\mathfrak{h}$, then both simple ideals of $\mathfrak{s}$ are of index 1 in $\mathfrak{h}$. Such a subalgebra $\mathfrak{s}\subset\mathfrak{h}\cong\mathfrak{so}_7$ is unique; it coincides with $\mathfrak{so}_4\subset\mathfrak{so}_7$. By Remark 5.2.2, $(\mathfrak{s},V_0)\rightsquigarrow_{\mathfrak{h}} U:=\mathfrak{g}/\mathfrak{h}$, where $\dim V_0=8$. Therefore $\operatorname{codim}_U(\mathfrak{h}v+U^{\mathfrak{s}})=8$ for some $v\in U^{\mathfrak{s}}$. From this observation we deduce that $\dim U^{\mathfrak{s}}+\dim\mathfrak{h}/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{s})\geqslant \dim U-8$. There is an isomorphism of $\mathfrak{h}$-modules $U\cong V(\pi_1)\oplus V(\pi_3)^{\oplus 3}$. It can be easily seen that $\dim V(\pi_1)^{\mathfrak{s}}=3, \dim V(\pi_3)^{\mathfrak{s}}=0, \dim\mathfrak{h}/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{s})=21-9=12$, a contradiction.

N24. Here $\mathfrak{g}=\mathfrak{so}_{13},\mathfrak{h}=\mathfrak{so}_{10}, V=V(\pi_4)$.

Again, we may assume that $H\subset(M,M)$, where $M:=TG^{(\alpha_2,\ldots,\alpha_6)}$. Using Corollary 3.3.11, we have $W(\mathfrak{g},\mathfrak{h},V)\cap M/T=W([\mathfrak{m},\mathfrak{m}],\mathfrak{h},V(\pi_1)\oplus V(\pi_4))$. As above, it remains to show that $S^{(A)}\not\twoheadrightarrow_{\mathfrak{g}} G*_H(\mathfrak{h}^\perp\oplus V\oplus V^*)$, where $A=\{\varepsilon_1-\varepsilon_2,\varepsilon_4-\varepsilon_5\}$. There is a unique (up to H-conjugacy) subalgebra $\mathfrak{s}\subset\mathfrak{h}$ that is G-conjugate with $\mathfrak{g}^{(A)}$, namely, $\mathfrak{s}=\mathfrak{h}^{(A')}$, where $(A')=\{\varepsilon_1-\varepsilon_2,\varepsilon_3-\varepsilon_4\}$ (the roots are taken in the root system $\Delta(\mathfrak{so}_{10})$).

Analogously to the previous case, one should prove that $\dim\widetilde{V}^{\mathfrak{s}}+\dim\mathfrak{h}/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{s})<\dim\widetilde{V}-8$, where $\widetilde{V}:=\mathfrak{h}^\perp\oplus V\oplus V^*\cong V(\pi_1)^{\oplus 3}\oplus V(\pi_4)\oplus V(\pi_5)$. It can be easily seen that $\dim V(\pi_1)^{\mathfrak{s}}=2$ and $\dim\mathfrak{h}/\mathfrak{n}_{\mathfrak{h}}(\mathfrak{s})=32$. To compute $\dim(V(\pi_4)\oplus V(\pi_5))^{\mathfrak{s}}$ we note that the weight system of the $\mathfrak{h}$-module $V(\pi_4)\oplus V(\pi_5)$ consists of elements $\frac{1}{2}\sum_{i=1}^5\pm\varepsilon_i$ without multiplicities. The subspace $(V(\pi_4)\oplus V(\pi_5))^{\mathfrak{s}}$ is the direct sum of weight subspaces. A weight vector v_λ of weight λ is annihilated by $\mathfrak{s}$ if and only if $(\lambda,\varepsilon_1-\varepsilon_2)=(\lambda,\varepsilon_3-\varepsilon_4)=0$. Therefore $\dim(V(\pi_4)\oplus V(\pi_5))^{\mathfrak{s}}=8$, whence the required inequality.

N25. Here $\mathfrak{g}=\mathfrak{sp}_{2n},\mathfrak{h}=\mathfrak{sp}_{2m}, m=\left[\frac{n+1}{2}\right], V=V(\pi_1)$ for odd n and 0 otherwise.

It is enough to check $W(\mathfrak{g},\mathfrak{h},V)\neq W(\mathfrak{g})$. We do this by induction on n. For $n=1$ we get $\mathfrak{g}=\mathfrak{h}=\mathfrak{sl}_2, V=V(\pi_1)$ and $W(\mathfrak{g},\mathfrak{h},V)=\{1\}$. Now let $n>1$ and set $M:=TG^{(\alpha_2,\ldots,\alpha_n)}$. One may assume that $H\subset(M,M)$. Using Proposition 3.3.11, we see that $W(\mathfrak{g},\mathfrak{h},V)\cap M/T=W([\mathfrak{m},\mathfrak{m}],\mathfrak{h},V(\pi_1)\oplus V)$. The reduced triple for $([\mathfrak{m},\mathfrak{m}],\mathfrak{h},V(\pi_1)\oplus V)$ coincides with $(\mathfrak{sp}_{2(n-1)},\mathfrak{sp}_{2[n/2]},V(\pi_1)^{\oplus(2\{n/2\})})$. We are done by the inductive assumption.

N28. Here $\mathfrak{g}=G_2,\mathfrak{h}=A_2,V=V(\pi_1)$.

Again, it is enough to show that $W(\mathfrak{g},\mathfrak{h},V)\neq W(\mathfrak{g})$. Note that $\alpha_2,\alpha_2+3\alpha_1$ is a simple root system in $\mathfrak{h}$. The subalgebra $\mathfrak{h}_0=\mathfrak{g}^{(\alpha_2)}\oplus\mathfrak{g}^{-\alpha_2-3\alpha_1}\oplus\mathfrak{g}^{-2\alpha_2-3\alpha_1}$ is the s.s.g.p. for the $\mathfrak{h}$-module V. Note that $\mathfrak{h}_0$ is tamely embedded into $\mathfrak{q}:=\mathfrak{g}^{\alpha_2}\oplus\mathfrak{b}^-$. Set $M:=TG^{(\alpha_2)}$. Note that the restriction of the $\mathfrak{h}_0$-module $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}_0)$ to $\mathfrak{g}^{(\alpha_2)}$ is the irreducible two-dimensional module. Applying Corollary 3.3.11 to the homogeneous space G/H_0 and the pair (Q,M), we see that $W(\mathfrak{g},\mathfrak{h}_0)\cap M/T=\{1\}$. But $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g},\mathfrak{g}/\mathfrak{h}_0)$ by assertion 3 of Proposition 3.2.1.

N31. Here $\mathfrak{g}=F_4,\mathfrak{h}=B_3,V=\{0\}$.

Set $M:=TG^{(\alpha_2,\alpha_3,\alpha_4)}$. We may assume that $\mathfrak{h}=[\mathfrak{m},\mathfrak{m}]$. Using Corollary 3.3.11, we see that $W(\mathfrak{g},\mathfrak{h},V)\cap M/T=W([\mathfrak{m},\mathfrak{m}],\mathfrak{h},V(\pi_3)\oplus V(\pi_1))$. As we have shown above, the latter is generated by $s_{\varepsilon_2-\varepsilon_4},s_{\varepsilon_3},s_{\varepsilon_4}$. In particular, $W(\mathfrak{g},\mathfrak{h},V)\neq W(\mathfrak{g})$. Recall that $s_\alpha\in W(\mathfrak{g},\mathfrak{h},V)$ for all simple roots α. The reflections s_α, where α is a short root, and $s_{\varepsilon_2-\varepsilon_4}$ generate the subgroup of $W(\mathfrak{g})$ indicated in Table 5.1. This subgroup is maximal.

5.4. **Completing the proof of Theorem 5.1.2.** To prove Theorem 5.1.2 it remains

(1) To check that any W-quasi-essential triple is W-essential and vice versa.
(2) To show how to determine a W-essential part of a given admissible triple.

The following lemma solves both problems.

Lemma 5.4.1. (1) *Any W-quasi-essential triple is W-essential.*
(2) *Let $(\mathfrak{g},\mathfrak{h},V)$ be an admissible triple and $\mathfrak{h}_0$ a minimal ideal of $\mathfrak{h}$ such that the conditions $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G*_H V)$ and $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G*_{H_0} V)$ are equivalent for any root α. Then $(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$ is a W-quasi-essential triple and $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$.*

Assertion 2 of Lemma 5.4.1 implies that any W-essential triple is W-quasi-essential. Moreover, a triple $(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$ from assertion 2 is either the trivial triple $(\mathfrak{g},0,0)$ or one of the triples from Table 5.1. Inspecting Table 5.1, we note that an ideal $\mathfrak{h}_0\subset\mathfrak{h}$ is determined uniquely. Therefore the triple $(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$ is a W-essential part of $(\mathfrak{g},\mathfrak{h},V)$.

Proof. Suppose an admissible triple $(\mathfrak{g},\mathfrak{h},V)$ is W-quasi-essential but not W-essential. Then, by the definition of a W-essential triple, there exists a proper ideal $\mathfrak{h}_0\subset\mathfrak{h}$ such that $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g},\mathfrak{h}_0,V)$. On the other hand, there is $\alpha\in\Delta(\mathfrak{g})$ such that $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G*_H V)$, $S^{(\alpha)} \not\rightsquigarrow_{\mathfrak{g}} T^*(G*_{H_0} V)$. Corollary 3.3.22 implies that $s_{w\alpha}\in W(\mathfrak{g},\mathfrak{h}_0,V)$ for all $w\in W$. By the computation of the previous subsection, there is $w\in W$ such that $s_{w\alpha}\notin W(\mathfrak{g},\mathfrak{h},V)$, a contradiction.

Let us proceed to assertion 2. The triple $(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$ is W-quasiessential, thanks to the minimality condition for $\mathfrak{h}_0$. If $\mathfrak{h}_0=\{0\}$, then, by Corollary 3.3.22, $W(\mathfrak{g},\mathfrak{h},V)=W(\mathfrak{g})$. If $(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$ is one of the triples N28, N32, N33 from Table 5.1, then $\mathfrak{h}=\mathfrak{h}_0$, whence the equality of the Weyl groups. So we may assume that $\mathfrak{g}\neq G_2$ and that $S^{(\alpha)} \rightsquigarrow_{\mathfrak{g}} T^*(G*_H V)$ if and only if α is a long root. By Corollary 3.3.22, $W(\mathfrak{g},\mathfrak{h},V)$ contains all reflections corresponding to short roots. Assume, at first, that $\mathfrak{g}$ is a classical Lie algebra. By Proposition 3.3.17, $W(\mathfrak{g},\mathfrak{h},V)$ is one of the subgroups listed in Table 3.3. But any of those groups containing all reflections corresponding to short roots is maximal among all proper subgroups of $W(\mathfrak{g})$ generated by reflections. Since $W(\mathfrak{g},\mathfrak{h},V)\subset W(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})$, the previous subsection yields $W(\mathfrak{g},\mathfrak{h}_0,V/V^{\mathfrak{h}_0})=W(\mathfrak{g},\mathfrak{h},V)$. It remains to consider the case $\mathfrak{g}=F_4$. Analogously to the classical case, $W(\mathfrak{g},\mathfrak{h},V)$ contains a reflection corresponding to a long root. Otherwise, $W(\mathfrak{g},\mathfrak{h},V)$ is not large in $W(\mathfrak{g})$ (in Definition 3.3.16 take $\alpha=\varepsilon_1-\varepsilon_2,\beta=\varepsilon_2-\varepsilon_3$). Any subgroup in $W(\mathfrak{g})$ containing all reflections corresponding to short roots and some reflection corresponding to a long root is maximal. □

5.5. **Weyl groups of affine homogeneous spaces.** In this subsection, $\mathfrak{g}$ is a reductive Lie algebra, $\mathfrak{h}$ its reductive subalgebra.

Before stating the main result we make the following remark. The Weyl group $W(\mathfrak{g},\mathfrak{h})$ depends only on the subalgebra $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{g}$; see Corollary 3.3.13. Therefore we may assume that $\mathfrak{h}$ is semisimple. The strategy of the computation is similar to that above: we compute the Weyl groups only for so-called *W-essential* subalgebras $\mathfrak{h}\subset\mathfrak{g}$ and then show how to reduce the general case to this one.

Definition 5.5.1. A semisimple subalgebra $\mathfrak{h}\subset\mathfrak{g}$ is called W-essential if any ideal $\mathfrak{h}_0\subset\mathfrak{h}$ such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{a}(\mathfrak{g},\mathfrak{h}_0), W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g},\mathfrak{h}_0)$ coincides with $\mathfrak{h}$.

Proposition 5.5.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ its semisimple subalgebra. Then the following claims hold.*

(1) *Let $\mathfrak{h}$ be a W-essential subalgebra of $\mathfrak{g}$. Then the pair $(\mathfrak{g},\mathfrak{h})$ is contained either in Table 5.1 or in Table 3.1. In the latter case, $W(\mathfrak{g},\mathfrak{h})$ coincides with the group indicated in Table 4.2.*
(2) *There is a unique minimal ideal $\mathfrak{h}^{W-ess}\subset\mathfrak{h}$ such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h}^{W-ess})=\mathfrak{a}(\mathfrak{g},\mathfrak{h}), W(\mathfrak{g},\mathfrak{h}^{W-ess})=W(\mathfrak{g},\mathfrak{h})$. It coincides with a maximal ideal of $\mathfrak{h}$ that is a W-essential subalgebra of $\mathfrak{g}$.*

Proof. The case $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$ is clear. By definition, any $\mathfrak{a}$-essential semisimple subalgebra of $\mathfrak{g}$ is W-essential. Now let $\mathfrak{h}$ be a W-essential subalgebra in $\mathfrak{g}$ with $\mathfrak{a}(\mathfrak{g},\mathfrak{h})\subsetneqq\mathfrak{t}$. Let $L_0=L^\circ_{0\,G,G/H}$, $\underline{X}$ be the distinguished component in $(G/H)^{L_0}$, $\underline{G}=N_G(\mathfrak{l}_0,\underline{X})/L_0, \underline{H}=N_H(\mathfrak{l}_0)/L_0$, $\underline{\mathfrak{t}}=\underline{\mathfrak{g}}\cap\mathfrak{t}$. Denote by F the subgroup of $\underline{G}$ consisting of all elements leaving invariant the distinguished Borel and Cartan subalgebras in $\underline{\mathfrak{g}}$. It follows from assertion (1) of Proposition 4.1.1 that $\underline{X}=\underline{G}/\underline{H}$. By Theorem 3.3.10, $W(\mathfrak{g},\mathfrak{h})=W(\underline{\mathfrak{g}},\underline{\mathfrak{h}})\leftthreetimes F/\underline{T}$. From assertion 4 of Proposition 4.1.3 it follows that $N_G(L_0)=N_G(L_0)^\circ N_{H^{ess}}(L_0)$. Therefore $N_G(L_0)=N_G(L_0)^\circ N_H(L_0)$ or, equivalently, $N_G(L_0,\underline{X})=N_G(L_0)$. Hence $N_G(L_0)^\circ F=N_G(L_0)$. Inspecting Table 4.1 and using Theorem 5.1.2, we see that for any subalgebra $\underline{\mathfrak{h}}_1\subset\underline{\mathfrak{g}}$ such that $\mathfrak{h}^{ess}$ is an ideal in $\underline{\mathfrak{h}}_1$ and $\mathfrak{a}(\underline{\mathfrak{g}},\underline{\mathfrak{h}}_1)=\underline{\mathfrak{t}}$, the equality $W(\underline{\mathfrak{g}},\underline{\mathfrak{h}}_1)=W(\underline{\mathfrak{g}})$ holds. Thence $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g},\mathfrak{h}^{ess})=W(\underline{\mathfrak{g}})\leftthreetimes F/\underline{T}$. In particular, if $\mathfrak{h}^{ess}\neq\{0\}$, then $\mathfrak{h}^{W-ess}=\mathfrak{h}^{ess}$. □

6. Computation of the Weyl groups for homogeneous spaces

6.1. **Introduction.** In this section we complete the computation of the groups $W(\mathfrak{g},\mathfrak{h})$.

At first, note that Corollary 3.3.13 allows one to reduce the computation to the case when a maximal reductive subalgebra of $\mathfrak{h}$ is semisimple. In this case G/H is quasiaffine, thanks to Sukhanov's criterion; see [Su].

Now recall that we have reduced the computation of $W(\mathfrak{g},\mathfrak{h})$ to the case when $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$ (Theorem 3.3.10 and results of Section 4) and $\mathfrak{g}$ is simple (Proposition 3.3.15).

The computation of $W(\mathfrak{g},\mathfrak{h})$ for $\mathfrak{g}=\mathfrak{so}_5,G_2$ is carried out in Subsection 6.2. Basically, it is built upon Proposition 3.3.1. In the beginning of Subsection 6.3 we show how to compute the space $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$. It turns out that for $\mathfrak{g}$ of type A,D,E the group $W(\mathfrak{g},\mathfrak{h})$ is determined uniquely by $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$. This is the main result of Subsection 6.3; its proof is based on the restrictions on $W(\mathfrak{g},\mathfrak{h})$ obtained in Proposition 3.3.23. The computation for algebras of type C,B,F and rank greater than 2 is carried out in Subsections 6.4–6.6. Remark 6.3.2 allows us to make an additional restriction on $\mathfrak{h}$: we require $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}=0$.

Remark 6.1.1. The computation of $W(\mathfrak{g},\mathfrak{h})$ for all groups of rank 2 essentially does not use Proposition 3.3.23 and results of [Wa] on the classification of all wonderful varieties of rank 2. Let us note that the classification of all spherical varieties of rank 2 is not very difficult. The reductions described above allow us to reduce the computation of the Weyl groups of varieties of rank 2 to the case when G itself has rank 2.

6.2. **Types B_2, G_2.** In this subsection $G=\mathrm{SO}_5, G_2$, and $\mathfrak{h}$ denotes a subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$ and a maximal reductive subalgebra $\mathfrak{h}_0$ of $\mathfrak{h}$ is semisimple.

Suppose that $\mathrm{R}_u(\mathfrak{q})\subset\mathfrak{h}$ for some parabolic subalgebra $\mathfrak{q}\subset\mathfrak{g}$. We may assume that $\mathfrak{q}$ is antistandard; let $\mathfrak{m}$ be its standard Levi subalgebra. Then, by Corollary 3.3.14, $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{m},\mathfrak{m}\cap\mathfrak{h})$. Below in this subsection we suppose that $\mathfrak{h}$ does not contain the unipotent radical of a parabolic subalgebra.

Proposition 6.2.1. *Let $\mathfrak{g}=\mathfrak{so}_5$ and a subalgebra $\mathfrak{h}\subset\mathfrak{g}$ satisfy the above assumptions. Then the following conditions are equivalent:*

(1) $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$;
(2) $\mathfrak{h}$ *is conjugate to* $\mathfrak{sl}_2^{diag}$.

Under conditions (1),(2), $W(\mathfrak{g},\mathfrak{h})$ is generated by $s_{\varepsilon_i}, i=1,2$.

Proof. We have already computed the groups $W(\mathfrak{g},\mathfrak{h})$ for reductive subalgebras $\mathfrak{h}$. Suppose now that $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$. Put $\mathfrak{q}_1=\mathfrak{b}^-+\mathfrak{g}^{\varepsilon_2}, \mathfrak{q}_2=\mathfrak{b}^-+\mathfrak{g}^{\varepsilon_1-\varepsilon_2}$.

Firstly, we consider the case when H is unipotent. Then $\dim\mathfrak{h}\leqslant 4$. Since $\mathfrak{h}$ does not contain the unipotent radical of a parabolic subalgebra, we see that $\dim\mathfrak{h}\leqslant 3$ and if $\dim\mathfrak{h}=3$, then $\mathfrak{h}\sim_G\widetilde{\mathfrak{h}}:=\mathrm{Span}_{\mathbb{C}}(e^{-\varepsilon_1-\varepsilon_2}, e^{-\varepsilon_1}, e^{\varepsilon_2-\varepsilon_1}+e^{-\varepsilon_2})$. The closure of $G\widetilde{\mathfrak{h}}$ in $\mathrm{Gr}(\mathfrak{g},3)$ contains $G\,\mathrm{R}_u(\mathfrak{q}_i), i=1,2$. Applying Corollary 3.3.14 and Proposition 3.3.1, we get $s_{\alpha_1}, s_{\alpha_2}\in W(\mathfrak{g},\mathfrak{h})$.

If $\dim\mathfrak{h}=1$, then there exists a unipotent subalgebra $\widehat{\mathfrak{h}}$ of dimension 2 containing $\mathfrak{h}$. Indeed $\mathfrak{n}_{\mathfrak{u}_1}(\mathfrak{h})\neq\mathfrak{h}$, where $\mathfrak{u}_1$ denotes the maximal unipotent subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Let us check that $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})$ whenever $\mathfrak{h}$ is a unipotent subalgebra of dimension 2. By Proposition 3.3.1, we may assume that $G\mathfrak{h}\subset\mathrm{Gr}(\mathfrak{g},2)$ is closed or, equivalently, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ is a parabolic subalgebra of $\mathfrak{g}$. From this one easily deduces that $\mathfrak{h}\sim_G\mathfrak{h}_0:=\mathrm{Span}_{\mathbb{C}}(e^{-\varepsilon_1-\varepsilon_2}, e^{-\varepsilon_1})$. But $\mathfrak{h}_0$ is conjugate to a subalgebra in $\widetilde{\mathfrak{h}}$, whence, thanks to Corollary 3.2.2, $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})$.

It remains to consider the case $\mathfrak{h}_0\cong\mathfrak{sl}_2$. It is easily deduced from $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$ that $\mathfrak{h}$ is semisimple. □

Proposition 6.2.2. *Suppose $\mathfrak{g}=G_2$ and $\mathfrak{h}$ satisfies the above assumptions. Then the following conditions are equivalent:*

(1) $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$;
(2) $\mathfrak{h}$ *is conjugate to one of the following subalgebras:* $\mathfrak{h}_1:=\mathfrak{g}^{(\alpha_2)}\oplus\mathfrak{g}^{-3\alpha_1-\alpha_2}\oplus\mathfrak{g}^{-3\alpha_1-2\alpha_2}$, $\mathfrak{h}_2:=\mathfrak{h}_1+\mathfrak{g}^{-2\alpha_1-\alpha_2}$.

$W(\mathfrak{g},\mathfrak{h}_1)$ is generated by $s_{\alpha_1}, s_{\alpha_1+\alpha_2}$, and $W(\mathfrak{g},\mathfrak{h}_2)$ is generated by $s_{\alpha_1+\alpha_2}$.

Proof. Again, put $\mathfrak{q}_1=\mathfrak{b}^-+\mathfrak{g}^{\alpha_1}, \mathfrak{q}_2=\mathfrak{b}^-+\mathfrak{g}^{\alpha_2}$ (α_2 is a long root).

The subalgebra $\mathfrak{h}_1$ coincides with the s.s.g.p. for the action $G:G*_{A_2}\mathbb{C}^3$, and the equality for $W(\mathfrak{g},\mathfrak{h}_1)$ is deduced from Theorem 5.1.2. Now put $\widetilde{\mathfrak{h}}_2:=\mathfrak{h}_2+\mathbb{C}\alpha_1^\vee$. Clearly, $\widetilde{\mathfrak{h}}_2\subset\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}_2)$. The subalgebra $\widetilde{\mathfrak{h}}_2$ is tamely contained in $\mathfrak{q}_2$. Applying the results of Subsection 3.2, we obtain $\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}}_2)=\mathbb{C}(\alpha_1+\alpha_2)$. Since $\widetilde{\mathfrak{h}}_2$ does not contain a maximal unipotent subalgebra, we have $W(\mathfrak{g},\widetilde{\mathfrak{h}}_2)\neq\{1\}$. The required equality for $W(\mathfrak{g},\mathfrak{h}_2)$ stems from Corollary 3.3.13.

Now suppose (1) holds. Firstly, we show that $\mathfrak{h}_0\not\sim_G\mathfrak{g}^{(\alpha_2)}$ implies $\mathfrak{h}_0=\{0\}$. Assume the converse. Since $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$, we have $\mathfrak{h}_0\neq A_2$. So if $\mathfrak{h}_0\neq 0$, then $\mathfrak{h}_0$ contains an ideal conjugate to $\mathfrak{g}^{(\alpha_1)}$. We easily check that $\mathfrak{h}$ is conjugate to a subalgebra in $\mathfrak{g}^{(\alpha_1)}\oplus\mathfrak{g}^{(3\alpha_1+2\alpha_2)}$. The Weyl group of the latter coincides with $W(\mathfrak{g})$. So $\mathfrak{h}_0=\{0\}$; in other words, $\mathfrak{h}$ is unipotent.

We have $\dim\mathfrak{h}\leqslant 5$. If $\dim\mathfrak{h}=5$, then, thanks to $\mathfrak{h}\not\sim_G \mathrm{R}_u(\mathfrak{q}_1),\mathrm{R}_u(\mathfrak{q}_2)$, we get $\mathfrak{h}\sim_G\widetilde{\mathfrak{h}}:=[\mathfrak{b}^-,\mathfrak{b}^-]+\mathbb{C}(e^{-\alpha_1}+e^{-\alpha_2})$. As in the proof of Proposition 6.2.2, we get $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})$.

Now let $\dim\mathfrak{h}\leqslant 4$. Let us check that $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})$. By Proposition 3.3.1, it is enough to check the claim only when $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$ is a parabolic subalgebra of $\mathfrak{g}$. But in this case $\mathfrak{h}\sim_G[\mathfrak{b}^-,\mathfrak{b}^-]\subset\widetilde{\mathfrak{h}}$. From Proposition 3.2.1 it follows that $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})$.

So it remains to consider the case $\mathfrak{h}_0=\mathfrak{g}^{(\alpha_2)}$. By Theorem 5.1.2, $\mathfrak{h}$ is not reductive. We may assume that $\mathfrak{h}$ is tamely contained in $\mathfrak{q}_2$. Since $\mathfrak{h}\not\subset\mathfrak{g}^{(\alpha_2)}\oplus\mathfrak{g}^{(\alpha_2+2\alpha_1)}$ and $\mathrm{R}_u(\mathfrak{q}_2)\not\subset\mathfrak{h}$, we get the required list of subalgebras $\mathfrak{h}$. □

6.3. **Types A,D,E.** In this subsection, $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ its subalgebra such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$ and a maximal reductive subalgebra of $\mathfrak{h}$ is simple.

Firstly, we show how to compute the space $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$.

Proposition 6.3.1. *Let $\mathfrak{g},\mathfrak{h}$ be such as above and $\mathfrak{z}$ a commutative reductive subalgebra of $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})/\mathfrak{h}$ containing all semisimple elements of $\mathfrak{z}(\mathfrak{n}_\mathfrak{g}(\mathfrak{h})/\mathfrak{h})$. Let $\widetilde{\mathfrak{h}}$ denote the inverse image of $\mathfrak{z}$ in $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$ under the natural epimorphism $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})\twoheadrightarrow\mathfrak{n}_\mathfrak{g}(\mathfrak{h})/\mathfrak{h}$. Then $\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}})$ is the orthogonal complement to $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$ in $\mathfrak{t}$.*

Proof. By Corollary 3.3.13, $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})\perp}\subset\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}})$. Let $\widehat{H}$ denote the inverse image of the subgroup $\mathfrak{A}^\circ_{G,G/H}\subset N_G(H)/H$ (see Definition 3.3.3) in $N_G(H)/H$. By Proposition 3.3.6, $\mathfrak{a}(\mathfrak{g},\widehat{\mathfrak{h}})=\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})\perp}$. By Lemma 3.3.4, $\widehat{H}/H\subset Z(N_G(H)/H)$. By Lemma 3.3.5, $\widehat{H}/H$ is a torus. These two observations yield $\widehat{H}\subset\widetilde{H}$, whence $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})\perp}\supset\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}})$. □

Remark 6.3.2. We use the notation of Proposition 6.3.1 and for $\mathfrak{z}$ take the ideal of $\mathfrak{z}(\mathfrak{n}_\mathfrak{g}(\mathfrak{h})/\mathfrak{h})$ consisting of all semisimple elements. Suppose that $\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}})\neq\mathfrak{t}$. Then we can reduce the computation of $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g},\widetilde{\mathfrak{h}})$ to the computation of $W(\underline{\mathfrak{g}},\underline{\mathfrak{h}})$, where $\mathrm{rk}[\underline{\mathfrak{g}},\underline{\mathfrak{g}}]<\mathrm{rk}\,\mathfrak{g}$, $\mathrm{rk}\,\underline{G}/\underline{H}=\mathrm{rk}\,\underline{G}$, as follows.

Put $\widetilde{G}=G\times\mathbb{C}^\times$. There is a quasi-affine homogeneous space $\widetilde{X}$ of $\widetilde{G}$ such that there is a G-equivariant principal $\mathbb{C}^\times$-bundle $\widetilde{X}\to G/\widetilde{H}$; see Subsection 7.2 for details. The stable subalgebra $\widetilde{\mathfrak{h}}_1$ of $\widetilde{X}$ is naturally identified with $\widetilde{\mathfrak{h}}$. By Proposition 3.3.2, $W(\mathfrak{g},\widetilde{\mathfrak{h}})\cong W(\widetilde{\mathfrak{g}},\widetilde{\mathfrak{h}}_1)$. Then we apply Theorem 3.3.10 to $\widetilde{X}$.

However, for some algebras $\mathfrak{g}$, the group $W(\mathfrak{g},\mathfrak{h})$ is uniquely determined by $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$, see below, so the reduction described above is unnecessary.

Proposition 6.3.3. *Let $\mathfrak{g}=\mathfrak{sl}_n, n\geqslant 3,\mathfrak{so}_{2n}, n\geqslant 4$ or $E_l, l=6,7,8$, and $\mathfrak{h}$ be as above. Then $W(\mathfrak{g},\mathfrak{h})=W(\mathfrak{g})\cap Z_G(\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})})/T$.*

Proof. Set $\Delta_0:=\Delta(\mathfrak{g})\cap\mathrm{Span}_\mathbb{C}(\widehat{\Pi}(\mathfrak{g},\mathfrak{h}))$. Our goal is to prove that $\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ is a system of simple roots in Δ_0. Let Π_0 be the system of simple roots in Δ_0 positive on the dominant Weyl chamber of $\mathfrak{g}$. Then for any $\alpha\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ there are positive integers $n_\gamma,\gamma\in\Pi_0$, such that $\alpha=\sum_{\gamma\in\Pi_0}n_\gamma\gamma$ and for any $\gamma\in\Pi_0$ there are rationals m_α and $\alpha\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ such that $\gamma=\sum_{\alpha\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})}m_\alpha\alpha$. So for any $\gamma\in\Pi_0$ there is $\alpha\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ such that $\mathrm{Supp}(\gamma)\subset\mathrm{Supp}(\alpha)$.

Set $\Pi_1:=\widehat{\Pi}(\mathfrak{g},\mathfrak{h})\cap\Pi(\mathfrak{g}),\widehat{\Pi}_2:=\widehat{\Pi}(\mathfrak{g},\mathfrak{h})\setminus\Pi_1,\Pi_2:=\bigcup_{\beta\in\widehat{\Pi}_2}\mathrm{Supp}(\beta)$. Recall (Proposition 3.3.23) that for any $\beta\in\widehat{\Pi}_2$ there are adjacent simple roots β_1,β_2 such that $\beta=\beta_1+\beta_2$. It follows that $\Pi_0\subset\widehat{\Pi}(\mathfrak{g},\mathfrak{h})\cup\Pi_2$. It is enough to prove that $\Pi_0\cap\Pi_2=\varnothing$.

Since $(\alpha,\beta)\leqslant 0$ for any $\alpha,\beta\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$, we see that $\alpha\not\in\operatorname{Supp}(\beta)$ for any $\alpha\in\Pi_1,\beta\in\widehat{\Pi}_2$. Our claim will follow if we check that $\operatorname{Span}_{\mathbb{C}}(\widehat{\Pi}_2)\cap\Pi(\mathfrak{g})=\varnothing$. Assume the converse, there is a subset $\Sigma\subset\widehat{\Pi}_2$ such that $\alpha:=\sum_{\beta\in\Sigma}n_\beta\beta\in\Pi(\mathfrak{g})$ with all n_β nonzero. There are at least two simple roots in Π_2 such that no more than one of their neighbors lies in $\bigcup_{\beta\in\Sigma}\operatorname{Supp}(\beta)$. Let β_1,β_1' be such roots. Then there are $\beta,\beta'\in\Sigma$ such that $\beta_1\not\in\operatorname{Supp}(\gamma)$ (resp. $\beta_1'\not\in\operatorname{Supp}(\gamma)$) for any $\gamma\in\Pi_2,\gamma\neq\beta$ (resp., $\gamma\neq\beta'$). It follows that $\beta_1,\beta_1'\in\operatorname{Supp}(\sum_{\beta\in\Sigma}n_\beta\beta)$, a contradiction. □

6.4. **Type $C_l, l>2$.** In this subsection $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ its subalgebra such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t}$, and a maximal reductive subalgebra of $\mathfrak{h}$ is semisimple.

Proposition 6.4.1. *Suppose $\mathfrak{g}\cong\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n},n>2,F_4$. Choose roots $\alpha,\beta\in\Delta(\mathfrak{g})$ as follows: $(\alpha,\beta)=(\varepsilon_{n-1},\varepsilon_n)$ for $\mathfrak{g}=\mathfrak{so}_{2n+1}$, $(\alpha,\beta)=(\varepsilon_{n-1}+\varepsilon_n,\varepsilon_{n-1}-\varepsilon_n)$ for $\mathfrak{g}=\mathfrak{sp}_{2n}$, $(\alpha,\beta)=(\varepsilon_3,\varepsilon_4)$ for $\mathfrak{g}=F_4$. Suppose $s_\alpha,s_\beta\in W(\mathfrak{g},\mathfrak{h}), s_{\alpha+\beta}\not\in W(\mathfrak{g},\mathfrak{h})$. Then there is a subalgebra $\mathfrak{s}\subset\mathfrak{g}$ satisfying the following two conditions:*

(1) *$\mathfrak{s}\sim_G\mathfrak{g}^{(\alpha+\beta)}$;*
(2) *$\mathfrak{g}/(\mathfrak{g}^{\mathfrak{s}}\oplus\mathfrak{s})\cong(\mathfrak{h}/\mathfrak{h}^{\mathfrak{s}})^{\oplus 2}\oplus(\mathbb{C}^2)^{\oplus 2}$ (an isomorphism of $\mathfrak{s}$-modules).*

Proof. For two $\mathfrak{s}$-modules V_1,V_2 we write $V_1\sim V_2$ if $V_1/V_1^{\mathfrak{s}}\cong V_2/V_2^{\mathfrak{s}}$. Set $M:=TG^{(\beta,\alpha-\beta)},Q:=B^-M,X:=G/H$. Let Z_0 denote a rational quotient for the action $\mathrm{R}_u(Q):X$. Modifying Z_0 if necessary, one may assume that M acts regularly on Z_0. By Proposition 8.2 from [Lo2], W_{M,Z_0} is generated by s_α,s_β. Thanks to assertion 4 of Proposition 3.2.1, $W_{M,Z_0}=W(\mathfrak{m},\mathfrak{m}_z)$ for a general point $z\in Z_0$. From Proposition 6.2.1 it follows that $\mathfrak{m}_z$ is reductive and $\mathfrak{s}:=[\mathfrak{m}_z,\mathfrak{m}_z]\sim_M\mathfrak{m}^{(\alpha+\beta)}$. By [Lo2], Lemma 7.12, the action $\mathrm{R}_u(Q):G/H$ is locally free. Since Z_0 is a rational quotient for the action $\mathrm{R}_u(Q):G/H$, we see that $\mathfrak{q}_x\sim_Q\mathfrak{g}^{(\alpha+\beta)}$ for $x\in X$ in general position. Thence there is $x\in X$ such that $\mathfrak{s}\subset\mathfrak{g}_x$ and $T_xX\sim\mathfrak{q}/\mathfrak{s}$. We may assume that $x=eH$. It follows that $\mathfrak{g}/\mathfrak{h}\sim\mathfrak{q}/\mathfrak{s}\sim\mathrm{R}_u(\mathfrak{q})+(\mathbb{C}^2)^{\oplus 2}$. Note that $\mathfrak{g}\sim\mathrm{R}_u(\mathfrak{q})^{\oplus 2}\oplus(\mathbb{C}^2)^{\oplus 2}\oplus\mathfrak{s}$, whence (2). □

Lemma 6.4.2. *Let $\mathfrak{g},\mathfrak{h}$ satisfy the assumptions of the previous proposition and $\mathfrak{s}\subset\mathfrak{h}$ possess properties (1), (2). Further, let Q be a parabolic subgroup of G such that $\mathfrak{h}$ is tamely contained in $\mathfrak{q}$ and M is a Levi subgroup of Q such that $\mathfrak{h}_0:=\mathfrak{m}\cap\mathfrak{h}$ is a maximal reductive subalgebra of $\mathfrak{h}$ containing $\mathfrak{s}$. Then there are simple ideals $\mathfrak{h}_1\subset\mathfrak{h}_0,\mathfrak{m}_1\subset\mathfrak{m}_0$ such that $\mathfrak{s}\subset\mathfrak{h}_1\subset\mathfrak{m}_1$ and $S^{\mathfrak{s}}\rightsquigarrow_{\mathfrak{h}_1}\mathfrak{m}_1/\mathfrak{h}_1\oplus\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})\oplus(\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}))^*$.*

Proof. We preserve the notation of Proposition 6.4.1. Note that $\alpha+\beta$ is a long root. Therefore $\mathfrak{s}$ is contained in a simple ideal $\mathfrak{m}_1\subset\mathfrak{m}$ and $i(\mathfrak{s},\mathfrak{m}_1)=1$, by (5.1). Moreover, $\mathfrak{s}\sim_M\mathfrak{m}^{(\gamma)}$ for some root $\gamma\in\Delta(\mathfrak{m})$ (see Lemma 5.2.3). Analogously, $\mathfrak{s}$ is contained in some simple ideal $\mathfrak{h}_1\subset\mathfrak{h}_0$. Since $\mathfrak{s}\subset\mathfrak{m}_1$, we get $\mathfrak{h}_1\subset\mathfrak{m}_1$. Property 2 of Proposition 6.4.1 implies that the nontrivial parts of the $\mathfrak{s}$-modules $(\mathfrak{m}/\mathfrak{h}_0)\oplus V\oplus V^*$ and $(\mathfrak{h}_1/\mathfrak{s})\oplus(\mathbb{C}^2)^{\oplus 2}$ coincide, where $V:=\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})$. Thence, see the proof of Lemma 5.2.5, $l_{\mathfrak{h}_1}(\mathfrak{m}/\mathfrak{h}_0\oplus V\oplus V^*)=1-\frac{4}{k_{\mathfrak{h}_1}}$. Besides, we remark that the $\mathfrak{h}_1$-module $(\mathfrak{m}/\mathfrak{h}_0)\oplus V\oplus V^*$ is orthogonal and its s.s.g.p. is trivial. The latter follows easily from the observation that the s.s.g.p. for the action $M:M*_H(\mathfrak{m}/\mathfrak{h}\oplus V\oplus V^*)$ is trivial. One can list all such modules by using a table from [AEV] and results from [E1]. It turns out that the nontrivial part of such a module is presented in Table 5.5. □

Our algorithm for computing $W(\mathfrak{g},\mathfrak{h})$ for $\mathfrak{g}\cong\mathfrak{sp}_{2n}$ is based on the following proposition.

Proposition 6.4.3. *Suppose $\mathfrak{g}\cong\mathfrak{sp}_{2n}, n>2$. Assume, in addition, that $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}=0$. Further, suppose $\mathfrak{h}$ is tamely contained in an antistandard parabolic subalgebra $\mathfrak{q}\subset\mathfrak{g}$ and $\mathfrak{m}\cap\mathfrak{h}$ is a maximal reductive subalgebra of $\mathfrak{h}$, where $\mathfrak{m}$ is the standard Levi subalgebra of $\mathfrak{q}$. Then the following conditions are equivalent:*

1. *$W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$;*
2. *$\mathfrak{g}^{(\alpha_{n-1},\alpha_n)}\subset\mathfrak{q}$, there are simple ideals $\mathfrak{h}_1\subset\mathfrak{h}_0,\mathfrak{m}_1\subset\mathfrak{m}$ such that $\mathfrak{m}_1=\mathfrak{g}^{(\alpha_k,\ldots,\alpha_n)}, k\leqslant n$, $\mathfrak{h}_1=\mathfrak{sp}_{2l}\subset\mathfrak{m}_1\cong\mathfrak{sp}_{2(n-k)}$, and the $\mathfrak{h}_1$-modules $\mathrm{R}_u(\mathfrak{q})/(\mathrm{R}_u(\mathfrak{h})+\mathrm{R}_u(\mathfrak{q})^{\mathfrak{h}_1})$ and $(\mathbb{C}^{2l})^{\oplus 4l-2(n-k)}$ are isomorphic.*

Under conditions (1),(2), $W(\mathfrak{g},\mathfrak{h})$ is generated by all reflections corresponding to short roots.

Proof. Let us introduce some notation. Let I be a finite set. By V we denote the vector space with the basis $\varepsilon_i, i\in I$. Let A_I (resp., B_I,C_I,D_I) be a linear group acting on V as the Weyl group of type $A_{\#I}$ (resp. $B_{\#I},C_{\#I},D_{\#I}$).

Since $W(\mathfrak{g},\mathfrak{h})$ is generated by reflections and $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}=0$, we see that there exists a partition $I_1,\ldots,I_k$ of $\{1,2,\ldots,n\}$ such that $W(\mathfrak{g},\mathfrak{h})=\prod_{i=1}^k\Gamma_i$, where $\Gamma_j=D_{I_j}$ or C_{I_j}. From Proposition 3.3.23 it follows that $\varepsilon_i+\varepsilon_j\not\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ unless $(i,j)=(n,n),(n,n-1)$ or $(n-1,n)$. Thus $k=1$. (2)⇒(1) stems from Corollary 3.3.11 applied to $\mathfrak{q}$ and results of the previous section. Now assume that $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$, whence $W(\mathfrak{g},\mathfrak{h})=D_n$. The assumptions of Proposition 6.4.1 are satisfied. By that proposition, $\mathfrak{h}_0$ contains a subalgebra conjugate to $\mathfrak{g}^{(\alpha_n)}$. Since α_n is a long root, any subalgebra of $\mathfrak{m}$ that is G-conjugate to $\mathfrak{g}^{(\alpha_n)}$ is contained in an ideal $\mathfrak{m}_1$ of the form $\mathfrak{g}^{(\alpha_k,\ldots,\alpha_n)}$. Now (1)⇒(2) follows from Lemma 6.4.2 and assertion (1) of Proposition 5.2.1. □

6.5. **Type** $B_l, l>2$**.** In this subsection, if not indicated otherwise, $\mathfrak{g}=\mathfrak{so}_{2n+1}, n\geqslant 3$, and $\mathfrak{h}\subset\mathfrak{g}$ is such that $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t},\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}=\{0\}$ and the maximal reductive subalgebra of $\mathfrak{h}$ is semisimple.

Lemma 6.5.1. *If $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$, then $W(\mathfrak{g},\mathfrak{h})=B_I\times B_J$, where I is one of the following subsets of $\{1,2,\ldots,n\}$ and $J=\{1,\ldots,n\}\setminus I$:*

1. *$I=\{n,n-2,\ldots,n-2k\}, 0\leqslant k<\frac{n-1}{2}$;*
2. *$I=\{n-1,\ldots,n-2k+1\}, 0<k<\frac{n}{2}$.*

Proof. As in the proof of Proposition 6.4.3, there is a partition $I_1,\ldots,I_l$ of $\{1,2\ldots,n\}$ such that $W(\mathfrak{g},\mathfrak{h})=\prod_{j=1}^l\Gamma_j$, where $\Gamma_j=B_{I_j}$ or D_{I_j}. Since $\varepsilon_i+\varepsilon_j,\varepsilon_k\not\in\widehat{\Pi}(\mathfrak{g},\mathfrak{h})$ for arbitrary i,j and $k<n-1$ (Proposition 3.3.23), we get $l=2,\Gamma_j=B_{I_j}$. It follows that $n\in I_1, n-1\in I_2$. By Proposition 3.3.23, if $k,k+1\in I_j, j=1,2$ for $1<k<n$, then $k-1$ or $k+2$ lies in I_j. This implies the claim of the present proposition. □

We remark that, by definition, $1\not\in I$.

Lemma 6.5.2. *Let $\mathfrak{g}\cong\mathfrak{so}_{2n+1}, n\geqslant 3$ or F_4. Suppose $W(\mathfrak{g},\mathfrak{h})$ is a maximal proper subgroup generated by reflections in $W(\mathfrak{g})$ (for $\mathfrak{g}=\mathfrak{so}_{2n+1}$ we have checked this in*

Lemma 6.5.1*). Let* $\mathfrak{q},\mathfrak{m},\mathfrak{h}_1,\mathfrak{m}_1$ *be such as in Lemma* 6.4.2*. Then:*

(1) *Subalgebras* $\mathfrak{h}_1\subset\mathfrak{h}$*,* $\mathfrak{m}_1\subset\mathfrak{m}$ *are determined uniquely.*
(2) *Let* $\underline{\mathfrak{h}}$ *denote the subalgebra in* $\mathfrak{h}$ *generated by* $\mathfrak{h}_1$ *and* $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{h})]$*. Then* $W(\mathfrak{g},\underline{\mathfrak{h}})=W(\mathfrak{g},\mathfrak{h})$*.*

Proof. It follows from assertion (1) of Proposition 5.2.1 that $\mathfrak{m}_1\neq\mathfrak{m}_2$ for two different pairs $(\mathfrak{m}_1,\mathfrak{h}_1),(\mathfrak{m}_2,\mathfrak{h}_2)$ satisfying the assumptions of Lemma 6.4.2. Thanks to Corollary 3.2.2, $W(\mathfrak{g},\mathfrak{h})\subset W(\mathfrak{g},\underline{\mathfrak{h}})$. The subalgebra $\underline{\mathfrak{h}}$ is tamely contained in $\mathfrak{q}$. Applying Corollary 3.3.11 to $\mathfrak{q}$, we see that $W(\mathfrak{m}_2)\not\subset W(\mathfrak{g},\mathfrak{h})$, $W(\mathfrak{m}_2)\subset W(\mathfrak{g},\underline{\mathfrak{h}}),W(\mathfrak{m}_1)\not\subset W(\mathfrak{g},\underline{\mathfrak{h}})$. The latter holds because the nontrivial parts of the $\mathfrak{h}_1$-modules $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}),\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\underline{\mathfrak{h}})$ are the same. Since $W(\mathfrak{g},\mathfrak{h})$ is maximal, this proves both assertions. □

For indices i_1,i_2 we write $i_1\sim i_2$ whenever $i_1,i_2\in I$ or $i_1,i_2\in J$.

Lemma 6.5.3. *Suppose that* $\mathfrak{h}$ *is tamely contained in an antistandard parabolic subalgebra* $\mathfrak{q}\subset\mathfrak{g}$ *and that some maximal reductive subalgebra* $\mathfrak{h}_0\subset\mathfrak{h}$ *is contained in the standard Levi subalgebra* $\mathfrak{m}\subset\mathfrak{g}$*. Then*

(1) $\mathfrak{m}_1=\mathfrak{g}^{(\alpha_{k+1},\ldots,\alpha_l)}$*, where* $n-2\leqslant l\leqslant n$*,* $i\sim j$ *for* $i,j\leqslant k$*. Further, for* $i,j\in\{k+1,k+2,\ldots,\min(l+1,n)\}$ *we have* $s_{\varepsilon_i-\varepsilon_j}\in W(\mathfrak{m}_1,\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}))$ *if and only if* $i\sim j$*.*
(2) *Suppose* $k+1\sim k+2$*. Let* $m\in\{k+1,k+2,\ldots,\min(l,n-1)\}$ *be the maximal integer such that* $m\sim m+1$*. Then* $I=\{m+2k,k=1,\ldots,[(n-m)/2]\}$*.*

Proof. Thanks to Lemma 6.5.1, $W(\mathfrak{g},\mathfrak{h})$ is maximal among all proper subgroups of $W(\mathfrak{g})$ generated by reflections. There are k,l such that $\mathfrak{m}_1=\mathfrak{g}^{(\alpha_{k+1},\ldots,\alpha_l)}$. Let $\underline{\mathfrak{q}}\subset\mathfrak{g}$ denote the antistandard parabolic subalgebra of $\mathfrak{g}$ corresponding to simple roots $\alpha_1,\ldots,\alpha_{k-1},\ \alpha_{k+1},\ldots,\alpha_l,\alpha_{l+2},\ \ldots,\alpha_n$. Since $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{q})]\subset\mathrm{R}_u(\underline{\mathfrak{q}})$, we see that $\underline{\mathfrak{h}}$ is tamely contained in $\underline{\mathfrak{q}}$. Now both assertions of this lemma stem from Corollary 3.3.11 applied to $\underline{\mathfrak{q}}$ and from the explicit form of $W(\mathfrak{g},\mathfrak{h})$ indicated in Lemma 6.5.1. □

Summarizing results obtained above in this subsection, we see that it remains to compute $W(\mathfrak{g},\mathfrak{h})$ under the following assumptions on $(\mathfrak{g},\mathfrak{h})$:

(a) $\mathfrak{a}(\mathfrak{g},\mathfrak{h})=\mathfrak{t},\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}=0$.
(b) A maximal reductive subalgebra $\mathfrak{h}_1\subset\mathfrak{h}$ is simple and the Lie algebra $\mathrm{R}_u(\mathfrak{h})$ is generated by $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{h})]$.
(c) There are integers $l\in\{n-2,n-1,n\},k<l$ such that $\mathfrak{h}$ is tamely contained in the antistandard parabolic subalgebra $\mathfrak{q}\subset\mathfrak{g}$ corresponding to $\Pi(\mathfrak{g})\setminus\{\alpha_k,\alpha_{l+1}\},\mathfrak{h}_1\subset\mathfrak{m}_1:=\mathfrak{g}^{(\alpha_{k+1},\ldots,\alpha_l)}$.
(d) $\mathfrak{h}_1$ is not contained in a proper Levi subalgebra of $\mathfrak{m}_1$.
(e) The group $W(\mathfrak{m}_1,\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}))$ does not contain a reflection of the form $s_{\varepsilon_i-\varepsilon_{i+1}}$, $k<i<l$.

Proposition 6.5.4. *Let* $\mathfrak{h},\mathfrak{h}_1,\mathfrak{q},\mathfrak{m}_1$ *be such as in (a)-(e). The inequality* $W(\mathfrak{g},\mathfrak{h})\neq W(\mathfrak{g})$ *holds if and only if* $\mathfrak{h}$ *is* G*-conjugate to a subalgebra from the following list.*

1) $\mathfrak{h}$ consists of all block matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{31} & 0 & x_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{51} & 0 & x'_{31} & 0 & 0 \end{pmatrix}.$$

The sizes of the blocks (from left to right and from top to bottom) are $k-1,1,7,1,k-1$; x'_{ij} denotes the matrix $-I_p x_{ij}^T I_q$, where $I_p = (\delta_{i+j,p+1})_{i,j=1}^p, I_q = (\delta_{i+j,q+1})_{i,j=1}^q$, and x_{33}, x_{51} are arbitrary elements of $G_2 \hookrightarrow \mathfrak{so}_7, \mathfrak{so}_{k-1}$. In this case $I = \{n-1\}$.

2) $\mathfrak{h}$ consists of all block matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{41} & 0 & 0 & x'_{22} & 0 \\ x_{51} & x'_{41} & 0 & x'_{21} & 0 \end{pmatrix}.$$

Here the sizes of the blocks are $k, n-k, 1, n-k, k$; x_{22}, x_{51} are arbitrary elements of $\mathfrak{sl}_{n-k}, \mathfrak{so}_k$. We have $I = \{k+2, k+4, \ldots, k+2i, \ldots\}$.

3) $\mathfrak{h}$ consists of all block matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{51} & x_{52} & \iota(x_{32}) & 0 & x'_{33} & 0 & 0 \\ x_{61} & 0 & x'_{52} & 0 & x'_{32} & 0 & 0 \\ x_{71} & x'_{61} & x'_{51} & 0 & x'_{31} & 0 & 0 \end{pmatrix}. \tag{6.1}$$

The sizes of the blocks (from left to right and from top to bottom) are $k-1,1,3,1,3,1,k-1$; ι denotes an isomorphism of $\mathfrak{sl}_3$-modules $\mathbb{C}^3$ and $\bigwedge^2 \mathbb{C}^{3*}$; x_{33}, x_{71} are arbitrary elements of $\mathfrak{sl}_3$ and $\mathfrak{so}_{k-1}$. Here $I = \{n-1\}$.

4) $\mathfrak{h}$ consists of all block matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & 0 & x'_{22} & 0 \\ x_{51} & x'_{41} & 0 & x'_{21} & 0 \end{pmatrix}.$$

Here the sizes of the blocks are $k, n-k, 1, n-k, k$, and $n-k$ is even; x_{22}, x_{51} are arbitrary elements of $\mathfrak{sp}_{n-k}, \mathfrak{so}_k$, while x_{42} lies in the nontrivial part of the $\mathfrak{sp}_{n-k}$-module $\bigwedge^2 \mathbb{C}^{n-k}$. Here $I = \{k+2, k+4, \ldots, n\}$.

5) $\mathfrak{h}$ consists of all block matrix of the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & 0 & 0 \\ x_{41} & x_{42} & \iota(x_{32}) & 0 & 0 & 0 & 0 \\ x_{51} & x_{52} & x_{53} + x_{42}\omega & \iota(x_{32})' & x'_{33} & 0 & 0 \\ x_{61} & 0 & x'_{52} & x'_{42} & x'_{32} & 0 & 0 \\ x_{71} & x'_{61} & x'_{51} & x'_{41} & x'_{31} & 0 & 0 \end{pmatrix}. \tag{6.2}$$

The sizes of the blocks are $k-1, 1, r, 1, r, 1, k-1$; x'_{ij} *have the same meaning as in* (6.1); ι *denotes an isomorphism of* $\mathfrak{sp}_r$*-modules* $\mathbb{C}^r, \mathbb{C}^{*r}$; $x_{71} \in \mathfrak{so}_{k-1}, x_{33} \in \mathfrak{sp}_r$, x_{53} *lies in the nontrivial part of the* $\mathfrak{sp}_r$*-module* $\bigwedge^2 \mathbb{C}^{r*}$, *and* ω *is an appropriate nonzero element in* $(\bigwedge^2 \mathbb{C}^{r*})^{\mathfrak{h}_1}$. *We have* $I = \{k+1, k+3, \ldots, n-1\}$.

Note that in all five cases the equality $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = 0$ can be checked by using Proposition 6.3.1 (compare with the proof of Proposition 6.6.2 below).

Proof. Note that $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})$ is an $\mathfrak{h}_1$-submodule in $\mathrm{R}_u(\mathfrak{q})$. Using Theorem 5.1.2, we obtain the following list of possible triples $(\mathfrak{m}_1, \mathfrak{h}_1, \mathrm{R}_u(\mathfrak{q})/(\mathrm{R}_u(\mathfrak{h}) + \mathrm{R}_u(\mathfrak{q})^{\mathfrak{h}_1}))$:

- $(\mathfrak{sl}_r, \mathfrak{sl}_r, \bigwedge^2 \mathbb{C}^{r*} \oplus \mathbb{C}^{r*}), r \geqslant 3$.
- $(\mathfrak{sl}_r, \mathfrak{sl}_r, \bigwedge^2 \mathbb{C}^{r*} \oplus \mathbb{C}^{r})$, $r \geqslant 4$ is even.
- $(\mathfrak{sl}_r, \mathfrak{sp}_r, \mathbb{C}^r)$, $r \geqslant 2$ is even.
- $(\mathfrak{so}_7, G_2, V(\pi_1))$.

The case $(\mathfrak{sl}_r, \mathfrak{sl}_r, \bigwedge^2 \mathbb{C}^{r*} \oplus \mathbb{C}^r)$. Let us check that this triple cannot occur. Since $r > 3$, we see that $\mathrm{R}_u(\mathfrak{q})$ contains a unique $\mathfrak{m}_1$-submodule isomorphic to $\bigwedge^2 \mathbb{C}^{r*}$. It is spanned by $e_{-\varepsilon_i-\varepsilon_j}$, $k+1 \leqslant i \leqslant k+r$. On the other hand, any $\mathfrak{m}_1$-submodule of $\mathrm{R}_u(\mathfrak{q})$ isomorphic to $\mathbb{C}^{*r}$ is contained in $\mathfrak{h}$. Vectors of the form $[\xi, \eta]$, where ξ, η lie in the $\mathbb{C}^{r*}$-isotypical component of the $\mathfrak{m}_1$-module $\mathrm{R}_u(\mathfrak{q})$, generate a submodule of $\mathrm{R}_u(\mathfrak{q})$ isomorphic to $\bigwedge^2 \mathbb{C}^{r*}$.

The case $(\mathfrak{so}_7, G_2, V(\pi_1))$. Here $\mathfrak{m}_1 = \mathfrak{g}^{(\alpha_{n-2}, \alpha_{n-1}, \alpha_n)}$. Conjugating $\mathfrak{h}$ by an appropriate element of $G^{(\alpha_1, \ldots, \alpha_{k-1})}$, we may assume that $\mathfrak{h}$ coincides with the subalgebra indicated in 1). In particular, $\mathfrak{h}$ is tamely contained in the antistandard parabolic subalgebra corresponding to simple roots $\alpha_{n-3}, \ldots, \alpha_n$. Applying Corollary 3.3.11 and then using Theorem 5.1.2, we obtain that $s_{\alpha_{n-3}} \in W(\mathfrak{g}, \mathfrak{h}) \cap W(\mathfrak{g}^{(\alpha_{n-3}, \ldots, \alpha_n)})$, whence $I = \{n-1\}$.

The case $(\mathfrak{sl}_r, \mathfrak{sl}_r, \bigwedge^2 \mathbb{C}^{r*} \oplus \mathbb{C}^{r*}), r \geqslant 3$. Let V_0 denote the $\mathfrak{m}_1$-submodule of $\mathrm{R}_u(\mathfrak{q})$ spanned by $e_{-\varepsilon_i-\varepsilon_j}$, $k+1 \leqslant i < j \leqslant k+r$. The isotypical component of type $\mathbb{C}^{r*}$ in the $\mathfrak{m}_1$-module $\mathrm{R}_u(\mathfrak{q})$ is the direct sum of $\mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_i})$, $\mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_i-\varepsilon_j}), j \leqslant k$, $\mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_i \pm \varepsilon_j}), j > k+r$, where i ranges from $k+1$ to $k+r$. If $k+r = n-1$, then, conjugating $\mathfrak{h}$ by an element from $G^{(\alpha_n)}$, one may assume that $\mathrm{Span}_{\mathbb{C}}(\varepsilon_{-i}, i = k+1, \ldots, k+r) \subset \mathfrak{h}$. Since $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, it follows that $V_0 \subset \mathfrak{h}$. But if $r > 3$, then V_0 is a unique submodule of $\mathrm{R}_u(\mathfrak{q})$ isomorphic to $\bigwedge^2 \mathbb{C}^{r*}$. So $k+r = n$ provided $r > 3$. Further, if $k+r = n$ and $V_0 \not\subset \mathfrak{h}$, then $\mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_i-\varepsilon_j}, i = k+1, \ldots, k+r)$ is the isotypical component of type $\mathbb{C}^{r*}$ in the $\mathfrak{h}_1$-module $\mathrm{R}_u(\mathfrak{h})$. Otherwise, the commutators of elements from the isotypical $\mathbb{C}^{r*}$-component in $\mathrm{R}_u(\mathfrak{h})$ generate V_0.

Suppose $\mathrm{R}_u(\mathfrak{h})$ is such as indicated in 2). By the previous paragraph, this is always the case when $r > 3$. Put $\widetilde{\mathfrak{h}} = \mathfrak{t} + \mathfrak{g}^{(\alpha_1, \ldots, \alpha_{k-1})} + \mathfrak{h} + V_0$. By [Wa], $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}}) = \mathrm{Span}_{\mathbb{C}}(\varepsilon_1, \varepsilon_{k+1})$ and $W(\mathfrak{g}, \widetilde{\mathfrak{h}}) = B_{\{1,k+1\}}$. Using Corollary 3.2.2, we see that $s_{\varepsilon_1-\varepsilon_{k+1}} \subset W(\mathfrak{g}, \mathfrak{h})$, whence $I = \{k+2, k+4, \ldots\}$.

Now suppose $r = 3$ and $\mathfrak{h}$ is not conjugate to the subalgebra from 2). Let us check that $h := \sum_{i=k+1}^{k+3} \varepsilon_i \notin \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. Assume the converse. Note that $V_0 \subset \mathrm{R}_u(\mathfrak{h})$. The subalgebra $\mathfrak{m}_1 + \mathbb{C}h$ acts trivially on $\mathrm{R}_u(\mathfrak{q})^{\mathfrak{m}_1}$ and as $\mathfrak{gl}_3$ on $\mathrm{R}_u(\mathfrak{q})/(\mathrm{R}_u(\mathfrak{h}) + \mathrm{R}_u(\mathfrak{q})^{\mathfrak{m}_1})$. Applying Proposition 3.2.9 to $\mathfrak{q}$, one easily gets that $\mathfrak{a}(\mathfrak{g}, \mathfrak{h} + \mathbb{C}h) \subset \varepsilon_{k+2}^{\perp}$, whence $\varepsilon_{k+2} \in \mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})\perp}$.

If $V_0\subset\mathfrak{h}$ (in particular, if $k+r=n-1$), then $h\in\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$. So we may assume that $k=n-3, r=3$ and $h\not\in\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$. Here $\mathfrak{h}$ is $G^{(\alpha_1,\ldots,\alpha_{k-1})}$-conjugate to the subalgebra indicated in 3), so we assume that $\mathfrak{h}$ coincides with that subalgebra. Put $\tau(t):=\mathrm{diag}(t,\ldots,t,1,\ldots,1,t^{-1},\ldots,t^{-1})$ (t occurs k times). The limit $\lim_{t\to 0}\tau(t)\,\mathrm{R}_u(\mathfrak{h})$ exists and is the subalgebra from 2). Using Proposition 3.3.1, we get $I=\{n-1\}$.

The case $(\mathfrak{sl}_r,\mathfrak{sp}_r,\mathbb{C}^r)$*, r is even.* Let us introduce some notation. Put

$$\begin{aligned}
&V_{21}:=\mathrm{Span}_\mathbb{C}(e_{\varepsilon_i-\varepsilon_j}, i=k+1,\ldots,k+r, j=1,\ldots,k),\\
&V_{31}:=\mathrm{Span}_\mathbb{C}(e_{-\varepsilon_j}, e_{-\varepsilon_j\pm\varepsilon_i}, j=1,\ldots,k, i>k+r),\\
&V_{41}:=\mathrm{Span}_\mathbb{C}(e_{-\varepsilon_i-\varepsilon_j}, i=k+1,\ldots,k+r, j=1,\ldots,k),\\
&V_{51}:=\mathrm{Span}_\mathbb{C}(e_{-\varepsilon_i-\varepsilon_j}, 1\leqslant i<j\leqslant k),\\
&V_{32}:=\mathrm{Span}_\mathbb{C}(e_{-\varepsilon_j}, e_{-\varepsilon_j\pm\varepsilon_i}, j=k+1,\ldots,k+r, i>k+r),\\
&V_{42}:=\mathrm{Span}_\mathbb{C}(e_{-\varepsilon_i-\varepsilon_j}, k+1\leqslant i<j\leqslant k+r).
\end{aligned}$$

Further, let V_{42}^+, V_{42}^0 denote the nontrivial and the trivial parts of the $\mathfrak{h}_1$-module V_{42}, respectively. Clearly,

$$\mathrm{R}_\mathfrak{u}(\mathfrak{q})=V_{21}\oplus V_{31}\oplus V_{41}\oplus V_{51}\oplus V_{32}\oplus V_{42}. \tag{6.3}$$

The isotypical component of type $\mathbb{C}^r$ (resp., of type $V(\pi_2)$, the trivial one) of the $\mathfrak{h}_1$-module $\mathrm{R}_u(\mathfrak{q})$ coincides with $V_{21}\oplus V_{41}\oplus V_{32}$ (resp., V_{42}^+, $V_{31}\oplus V_{51}\oplus V_{42}^0$). So $V_{42}^+\subset\mathrm{R}_u(\mathfrak{h})$.

Note that

$$\begin{aligned}
&[V_{32}',V_{32}']=V_{42},\\
&[V_{31}',V_{31}']=V_{51},\\
&[V_{21},V_{32}]=V_{31},\\
&[V_{21},V_{42}^0]=V_{41},\\
&[V_{21},V_{42}^+]=V_{41}, r>2,\\
&[V_{21},V_{41}]=V_{51},\\
&[V_{32},V_{31}]=V_{41},\\
&[V_{ij},V_{pq}]=0 \text{ for the other pairs } (i,j),(p,q).
\end{aligned}\tag{6.4}$$

In (6.4) we denote by V_{32}', resp. V_{31}', either the whole module V_{32}, resp. V_{31} (for $k+r=n$), or the direct sum of any two $\mathfrak{h}_1$-, resp., $\mathfrak{g}^{(\alpha_1,\ldots,\alpha_{k-1})}$-submodules in V_{32}, resp., in V_{31} (for $k+r=n-1$).

Suppose $h:=\sum_{i=k+1}^{k+r}\varepsilon_k\in\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$. In this case either $V_{21}\subset\mathrm{R}_u(\mathfrak{h})$ or $V_{32}\oplus V_{41}\subset\mathrm{R}_u(\mathfrak{h})$.

If $V_{42}^0\subset\mathfrak{h}$, then the nontrivial part of the $\mathfrak{h}_1+\mathbb{C}h$-module $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})$ does not contain a trivial $\mathfrak{h}_1$-submodule. In this case $\mathfrak{a}(\mathfrak{g},\mathfrak{h}+\mathbb{C}h)\neq\mathfrak{t}$. The latter is deduced from Proposition 3.2.9 applied to $\mathfrak{q}$. So $V_{42}^0\not\subset\mathfrak{h}$. Since $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$, we get

$$\mathrm{R}_u(\mathfrak{h})\subset V_{21}\oplus V_{31}\oplus V_{41}\oplus V_{51}\oplus V_{32}\oplus V_{42}^+. \tag{6.5}$$

Now let us suppose that (6.5) holds. By (6.4), the projection of $\mathrm{R}_u(\mathfrak{h})$ to V_{32} (with respect to decomposition (6.3)) is trivial, whence $\mathfrak{h}$ is the subalgebra described in 4). Let us check that $1\sim k+1$ for this $\mathfrak{h}$. By Proposition 3.3.1 and computations above it is enough to show that $\mathfrak{h}\in\overline{G\mathfrak{h}^1}$, where $\mathfrak{h}^1$ is the subalgebra 2). Let $\xi\in V_{42}^0$ be nonzero. Then $\lim_{t\to\infty}\exp(t\xi)\mathfrak{h}^1=\mathfrak{h}$.

So it remains to consider $\mathfrak{h}$ such that (6.5) does not hold and

$$\sum_{i=k+1}^{k+r} \varepsilon_i \notin \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}). \tag{6.6}$$

One can deduce from (6.4),(6.6) that

$$V_{51} \oplus V_{41} = [(V_{31} \oplus V_{42}^0) \cap \mathrm{R}_u(\mathfrak{h}), (V_{31} \oplus V_{42}^0 \oplus V_{21} \oplus V_{32}) \cap \mathrm{R}_u(\mathfrak{h})]. \tag{6.7}$$

From (6.7) it follows that

$$V_{51} \oplus V_{41} \oplus V_{42}^+ \subset \mathrm{R}_u(\mathfrak{h}). \tag{6.8}$$

Further, if $k + r = n - 1$, then (6.4) implies that

$$V_{42}^0 \subset \mathrm{R}_u(\mathfrak{h}). \tag{6.9}$$

It follows from (6.7)–(6.9) that

$$\mathrm{R}_u(\mathfrak{q})/\,\mathrm{R}_u(\mathfrak{h}) \cong \mathbb{C}^r. \tag{6.10}$$

Suppose (6.8)–(6.10) hold. In this case $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ contains $h_1 := 2\sum_{i=1}^k \varepsilon_i + \sum_{i=k+1}^r \varepsilon_i$. As above, one may check that $\mathfrak{a}(\mathfrak{g}, \mathfrak{h} + \mathbb{C}h_1) \neq \mathfrak{t}$, whence $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} \neq \{0\}$.

So $k + r = n$ and $\mathrm{R}_u(\mathfrak{h})$ projects surjectively onto $V_{32}, V_{21}, V_{42}^0, V_{31}$. From this observation and (6.8) it follows that $\mathfrak{h}$ is $G^{(\alpha_1,\ldots,\alpha_{k-1})}$ conjugate to the subalgebra 5).

Put $\tau(t) = \mathrm{diag}(1, \ldots, 1, t, \ldots, t, 1, t^{-1}, \ldots, t^{-1}, 1, \ldots, 1)$ (t occurs r times). Let $\mathfrak{h}^1$ denote $\lim_{t\to 0} \tau(t)\mathfrak{h}$. Note that $\mathfrak{h}^1$ is tamely contained in the antistandard parabolic subalgebra corresponding to $\Pi(\mathfrak{g}) \setminus \{\alpha_n, \alpha_{n-r-1}\}$. Applying Corollary 3.3.11 to this subalgebra and using Theorem 5.1.2, we see that $s_{\varepsilon_k - \varepsilon_{k+2}} \in W(\mathfrak{g}, \mathfrak{h}^1)$. By Proposition 3.3.1, $k \sim k + 2$, whence $I = k + 1, k + 3, \ldots, n - 1$. ☐

6.6. **Type F_4.** In this subsection $\mathfrak{g} = F_4$ and $\mathfrak{h}$ denotes a subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{t}, \mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$ and a maximal reductive subalgebra of $\mathfrak{h}$ is semisimple.

First, we describe all possible sets $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$.

Lemma 6.6.1. *Suppose* $W(\mathfrak{g}, \mathfrak{h}) \neq W(\mathfrak{g})$. *Then* $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_4\}$ *or* $\{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$.

Proof. It follows from Proposition 3.3.23 that $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) \subset \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_3 + \alpha_4\}$. Let us check that $\alpha_1 + \alpha_2 \notin \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$. Assume the converse. Then $\alpha_1, \alpha_2 \notin \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$ because $(\alpha_1, \alpha_1+\alpha_2) = (\alpha_2, \alpha_1+\alpha_2) > 0$. Since $\mathrm{Span}_{\mathbb{C}}(\widehat{\Pi}(\mathfrak{g}, \mathfrak{h})) = \mathfrak{t}$, we obtain $\alpha_2 + \alpha_3 \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$. As above, $\alpha_3 \notin \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$, whence $\alpha_3 + \alpha_4, \alpha_4 \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$. This is not possible, for $(\alpha_3 + \alpha_4, \alpha_4) > 0$. So $\alpha_1 + \alpha_2 \notin \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$ is proved. Thus $\alpha_1 \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$.

Let us check that $\alpha_2 \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$. Assume the converse. Then $\alpha_2 + \alpha_3 \in \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$, $\alpha_3+\alpha_4 \notin \widehat{\Pi}(\mathfrak{g}, \mathfrak{h})$ (the latter stems from $\#\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = 4, (\alpha_3+\alpha_4, \alpha_3) = (\alpha_3+\alpha_4, \alpha_4) > 0$). Therefore $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2 + \alpha_3, \alpha_3, \alpha_4\}$. But the Weyl group corresponding to this simple root system coincides with $W(\mathfrak{g})$, a contradiction.

Let us check that $\alpha_3 \notin \widehat{\Pi}(\mathfrak{g},\mathfrak{h})$. Indeed, otherwise $\alpha_4 \in \widehat{\Pi}(\mathfrak{g},\mathfrak{h})$, which contradicts $W(\mathfrak{g},\mathfrak{h}) \neq W(\mathfrak{g})$. Summarizing, we get $\{\alpha_1,\alpha_2\} \subset \widehat{\Pi}(\mathfrak{g},\mathfrak{h}) \subset \{\alpha_1,\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4,\alpha_4\}$. Now the claim of the proposition is easy. □

Both possible subgroups $W(\mathfrak{g},\mathfrak{h}) \subsetneq W(\mathfrak{g})$ are maximal among subgroups of $W(\mathfrak{g})$ generated by reflections. So the assertion of Lemma 6.5.2 holds. Let $\underline{\mathfrak{h}},\mathfrak{h}_1,\mathfrak{m}_1$ have the same meaning as in Lemma 6.5.2. Recall that $\alpha_3 \in \Delta(\mathfrak{m}_1)$. If $\alpha_4 \in \Delta(\mathfrak{m}_1)$, then $s_{\alpha_4} \in W(\mathfrak{g},\mathfrak{h})$ if and only if $s_{\alpha_4} \in W(\mathfrak{m}_1,\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h}))$, thanks to Corollary 3.3.11. Replacing $\mathfrak{h}$ with $\underline{\mathfrak{h}}$, we may assume that

(a) $\mathfrak{a}(\mathfrak{g},\mathfrak{h}) = \mathfrak{t}, \mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = 0$.
(b) A maximal reductive subalgebra $\mathfrak{h}_1 \subset \mathfrak{h}$ is simple, and the Lie algebra $\mathrm{R}_u(\mathfrak{h})$ is generated by $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{h})]$.
(c) $\mathfrak{m}_1 = \mathfrak{g}^{(\alpha_i,\alpha_{i+1},\ldots,\alpha_3)}, i=1,2,3$.
(d) $\mathfrak{h}_1$ is not contained in a proper Levi subalgebra of $\mathfrak{m}_1$.

Proposition 6.6.2. *Let $\mathfrak{h},\mathfrak{h}_1,\mathfrak{q},\mathfrak{m}_1$ be such as above. Then $W(\mathfrak{g},\mathfrak{h}) \neq W(\mathfrak{g})$ precisely in the following cases:*

1) $\mathfrak{h}_1 = \mathfrak{g}^{(\alpha_2,\alpha_3)}$, $\mathrm{R}_u(\mathfrak{h})$ is spanned by $e_\alpha, \alpha = -\varepsilon_1-\varepsilon_2, -\varepsilon_i, -\varepsilon_i \pm \varepsilon_j, i=1,2, j=3,4$. In this case, $\widehat{\Pi}(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4\}$.

2) $\mathfrak{h}_1 = \mathfrak{g}^{(\alpha_3)}$, $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{h})]$ is spanned by e_α with $\alpha = -\varepsilon_i \pm \varepsilon_j, i=1,2, j=3,4, (-\varepsilon_1-\varepsilon_2\pm\varepsilon_3\mp\varepsilon_4)/2$ and by $v_1 := x_1e_{-\varepsilon_3}+y_1e_{(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2}, v_2 := x_2e_{-\varepsilon_4}+y_2e_{(-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4)/2}$, where $(x_1,y_1),(x_2,y_2)$ are nonzero pairs of complex numbers such that $\mathrm{Span}_{\mathbb{C}}(v_1,v_2)$ is an $\mathfrak{h}_1$-submodule in $\mathfrak{g}$. Here we have three conjugacy classes of subalgebras corresponding to the collections x_i,y_i with $x_1=x_2=0$, $y_1=y_2=0$, $x_1x_2y_1y_2 \neq 0$. The equality $\widehat{\Pi}(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\alpha_2,\alpha_2+\alpha_3,\alpha_4\}$ holds.

3) $\mathfrak{h}_1 = \mathfrak{g}^{(\alpha_3)}$, $[\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{h})]$ is spanned by e_α with $\alpha = -\varepsilon_1\pm\varepsilon_i, -\varepsilon_2-\varepsilon_i, i=3,4, (-\varepsilon_1-\varepsilon_2\pm\varepsilon_3\mp\varepsilon_4)/2$ and by $v_1 := e_{\varepsilon_4-\varepsilon_2}+x_1e_{-\varepsilon_3}+y_1e_{(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2}$, $v_2 := e_{\varepsilon_3-\varepsilon_2}+x_2e_{-\varepsilon_4}+y_2e_{(-\varepsilon_1+\varepsilon_2-\varepsilon_4+\varepsilon_3)/2}$, $u_1 := x_1'e_{-\varepsilon_3}+y_1'e_{(-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4)/2}$, $u_2 := x_2'e_{-\varepsilon_4}+y_2'e_{(-\varepsilon_1+\varepsilon_2-\varepsilon_4+\varepsilon_3)/2}$. Here $x_i,y_i,x_i',y_i', i=1,2$ are complex numbers such that (x_1,y_1) and (x_1',y_1') are linearly independent and $\mathrm{Span}_{\mathbb{C}}(v_1,v_2), \mathrm{Span}_{\mathbb{C}}(u_1,u_2)$ are $\mathfrak{h}_1$-submodules. There are also three classes of conjugacy corresponding to the collections x_i,y_i,x_i',y_i' with $x_1'=x_2'=0$, $y_1'=y_2'=0$, $x_1'x_2'y_1'y_2' \neq 0$. Finally, $\widehat{\Pi}(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4\}$.

Proof. If $\operatorname{rk}\mathfrak{m}_1 > 1$, then $W(\mathfrak{m}_1,\mathfrak{h}_1,\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})) = W(\mathfrak{g},\mathfrak{h}) \cap M_1/T$ is generated by all reflections corresponding to short roots in $\Delta(\mathfrak{m}_1)$. By our conventions, $\mathfrak{h}_1$ is not contained in a proper Levi subalgebra of $\mathfrak{m}_1$. Using Theorem 5.1.2, we see that $\mathfrak{h}_1 = \mathfrak{m}_1$ and the nontrivial part of the $\mathfrak{m}_1$-module $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})$ is isomorphic to the direct sum of r copies of the tautological $\mathfrak{m}_1 \cong \mathfrak{sp}_{2r}$-module.

If $\mathfrak{m}_1 = \mathfrak{g}^{(\alpha_1,\alpha_2,\alpha_3)} \cong \mathfrak{sp}_6$, then the nontrivial part of the $\mathfrak{m}_1$-module $\mathrm{R}_u(\mathfrak{q})/\mathrm{R}_u(\mathfrak{h})$ coincides with $V(\pi_3)$. Thus in this case, $W(\mathfrak{g},\mathfrak{h}) = W(\mathfrak{g})$.

Now let $\mathfrak{m}_1 = \mathfrak{g}^{(\alpha_2,\alpha_3)} \cong \mathfrak{sp}_4$. The multiplicity of $\mathbb{C}^4$ in $\mathrm{R}_u(\mathfrak{q})$ equals 2. It follows that $\mathfrak{h}$ is the subalgebra indicated in 1). The ideal of $\mathfrak{z}(\mathfrak{n}_\mathfrak{g}(\mathfrak{h})/\mathfrak{h})$ consisting of all semisimple elements is one-dimensional, its preimage $\widetilde{\mathfrak{h}}$ in $\mathfrak{n}_\mathfrak{g}(\mathfrak{h})$ is spanned by $\mathfrak{h}$ and $\varepsilon_1-\varepsilon_2$. Applying Proposition 3.2.9, we see that $\mathfrak{a}(\mathfrak{g},\widetilde{\mathfrak{h}}) = \mathfrak{t}$. By Proposition 6.3.1, we get $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$.

Now we check $\widehat{\Pi}(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_4\}$. Note that $\mathfrak{h} \in \overline{G\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)}}$. Indeed, $\mathfrak{h} = \lim_{t\to\infty}\exp(te_{-\varepsilon_1-\varepsilon_2})\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)}$. Proposition 3.3.1 and Theorem 5.1.2 yield the claim.

Until the end of the proof, $\mathfrak{m}_1 = \mathfrak{g}^{(\alpha_3)}$. Put

$$\begin{aligned}
V_1 &:= \mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_3}, e_{-\varepsilon_4}),\\
V_{2i}^{\pm} &:= \mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_i\pm\varepsilon_3}, e_{-\varepsilon_i\pm\varepsilon_4}),\ i=1,2,\\
V_3^{\pm} &:= \mathrm{Span}_{\mathbb{C}}(e_{(-\varepsilon_1\pm\varepsilon_2+\varepsilon_3-\varepsilon_4)/2}, e_{(-\varepsilon_1\pm\varepsilon_2-\varepsilon_3+\varepsilon_4)/2}),\\
V_{21} &:= V_{21}^+ \oplus V_{21}^-,\\
V_2^+ &:= V_{21}^+ \oplus V_{22}^+,\\
V_{22} &:= V_{22}^+ \oplus V_{22}^-,\\
V_2^- &:= V_{21}^- \oplus V_{22}^-,\\
V_2 &:= V_{21} \oplus V_{22},\\
V_3 &:= V_3^+ \oplus V_3^-.
\end{aligned}$$

The isotypical component of type $\mathbb{C}^2$ in $\mathrm{R}_u(\mathfrak{q})$ coincides with $V_1 \oplus V_2 \oplus V_3$.

One easily verifies that

$$\begin{aligned}
&[V_1, V_1] = \mathbb{C}e_{-\varepsilon_3-\varepsilon_4},\\
&[V_1, V_{2i}^+] = \mathbb{C}e_{-\varepsilon_i},\\
&[V_1, V_3^{\pm}] = \mathbb{C}e_{(-\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2},\\
&[V_{21}^{\pm}, V_{22}^{\mp}] = [V_3^-, V_3^-] = \mathbb{C}e_{-\varepsilon_1-\varepsilon_2},\\
&[V_{22}^+, V_3^+] = \mathbb{C}e_{(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)/2},\\
&[V_{22}^-, V_3^+] = \mathbb{C}e_{(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2},\\
&[V_3^+, V_3^+] = \mathbb{C}e_{\varepsilon_2-\varepsilon_1},\\
&[V_3^+, V_3^-] = \mathbb{C}e_{-\varepsilon_1}.
\end{aligned} \tag{6.11}$$

The other commutators of $V_1, V_{2i}^{\pm}, V_3^{\pm}$ vanish.

Let us check that $e_{-\varepsilon_1-\varepsilon_2} \in \mathrm{R}_u(\mathfrak{h})$. Indeed, according to (6.11), one may consider the commutator on V_2 as a nondegenerate skew-symmetric form. The intersection $\mathrm{R}_u(\mathfrak{h}) \cap V_2$ is 6-dimensional, whence not isotropic.

Let us show that $e_{-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2} \in \mathrm{R}_u(\mathfrak{h})$. The dimension of $\mathrm{R}_u(\mathfrak{h}) \cap (V_1 \oplus V_{22}^- \oplus V_3)$ is not less than 6. By (6.11), it is enough to consider the situation when this dimension equals 6 and the projection of $\mathrm{R}_u(\mathfrak{h}) \cap (V_1 \oplus V_{22}^- \oplus V_3)$ to any of four submodules $V_1, V_{22}^-, V_3^{\pm}$ is nonzero. If $V_{22}^- \subset \mathrm{R}_u(\mathfrak{h})$, then $e_{-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2}$ lies in the commutator of V_{22}^- with an $\mathfrak{m}_1$-submodule of $\mathrm{R}_u(\mathfrak{h})$ not contained in $V_1 \oplus V_{22}^- \oplus V_3^-$. So we may assume that $\mathrm{R}_u(\mathfrak{h}) \cap (V_1 \oplus V_{22}^- \oplus V_3^-)$ contains a submodule projecting nontrivially to V_1 and $\mathrm{R}_u(\mathfrak{h}) \cap (V_{22}^- \oplus V_3^-)$ contains a submodule projecting nontrivially to V_3^-. The commutator of two such submodules contains an element of the form $e_{-(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2} + xe_{-\varepsilon_1-\varepsilon_2}$. It remains to recall that $e_{-\varepsilon_1-\varepsilon_2} \in \mathrm{R}_u(\mathfrak{h})$.

Analogously (by considering $V_1 \oplus V_{21}^+ \oplus V_3$ instead of $V_1 \oplus V_{22}^- \oplus V_3$), we get $e_{-\varepsilon_1} \in \mathrm{R}_u(\mathfrak{h})$.

Now we show that $V_2^- \oplus V_{21}^+ \oplus V_3^- \subset \mathrm{R}_u(\mathfrak{h})$. Assume the converse. Then the projection of $\mathrm{R}_u(\mathfrak{h})$ to $V_1 \oplus V_{22}^+ \oplus V_3^+$ is surjective. Note that the commutator of $V_1 \oplus V_2 \oplus V_3$ and $V_2^- \oplus V_{21}^+ \oplus V_3^-$ lies in $\mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_1-\varepsilon_2}, e_{(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)/2}, e_{-\varepsilon_1})$. It follows from (6.11) that $\mathrm{R}_u(\mathfrak{h})^{\mathfrak{m}_1}$ is spanned by the r.h.s.'s of (6.11). From this one can deduce that $V_2^- \oplus V_{21}^+ \oplus V_3^- \subset [\mathrm{R}_u(\mathfrak{h})^{\mathfrak{m}_1}, \mathrm{R}_u(\mathfrak{h})]$, a contradiction.

So $\mathrm{R}_u(\mathfrak{h}) = V_{21} \oplus V_{22}^- \oplus V_3^- \oplus U$, where $U \subset V_1 \oplus V_{22}^+ \oplus V_3^+$ is a four-dimensional $\mathfrak{m}_1$-submodule. (6.11) implies that

$$[U, U] \subset V' := \mathrm{Span}_{\mathbb{C}}(e_{-\varepsilon_3-\varepsilon_4}, e_{-\varepsilon_2}, e_{\varepsilon_2-\varepsilon_1}, e_{(-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4)/2}, e_{(-\varepsilon_1\pm\varepsilon_2-\varepsilon_3-\varepsilon_4)/2}).$$

Note that $\dim[U, U] = 2$ if $V_{22}^+ \subset U$ and 3 otherwise. The subalgebra $\mathfrak{h}$ such that $U \subset \mathrm{R}_u(\mathfrak{h})$ is denoted by $\mathfrak{h}_U$. The torus T_0 with the Lie algebra $\mathfrak{t}_0 := (\varepsilon_3 - \varepsilon_4)^\perp \subset \mathfrak{t}$ acts on $V_1 \oplus V_{22}^+ \oplus V_3^+$ and on V'. If U, U_0 are four-dimensional $\mathfrak{m}_1$-submodules in $V_1 \oplus V_{22}^+ \oplus V_3^+$ such that $\dim[U, U] = \dim[U_0, U_0]$ and $U_0 \in \overline{T_0 U}$, then $\mathfrak{h}_{U_0} \in \overline{T_0 \mathfrak{h}_U}$.

Let us suppose for a moment that $\mathfrak{t} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}_U)$. Let Σ_U denote the set of all roots β with $\mathfrak{g}^\beta \subset [V_1 \oplus V_2 \oplus V_3, V_1 \oplus V_2 \oplus V_3] \setminus \mathrm{R}_u(\mathfrak{h})$. For any $\Sigma \subset \Sigma_U$ the subspace $\widetilde{\mathfrak{h}} := \mathfrak{h}_U \oplus \bigoplus_{\beta\in\Sigma} \mathfrak{g}^\beta \subset \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ normalized by $\mathfrak{t}$. The equality $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$ holds if and only if $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}} + \mathfrak{t}_0) = \mathfrak{t}$ (Proposition 6.3.1). Applying Proposition 3.2.9 to $\mathfrak{q}$, one can show that $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}} + \mathfrak{t}_0) = \mathfrak{t}$ if and only if the subspace spanned by $\Sigma_U \setminus \Sigma$ and $(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$ is three-dimensional.

There are seven T_0-orbits of submodules $U \subset V_1 \oplus V_{22}^+ \oplus V_3^+$. To the T_0-orbit of U we associate the triple (x_1, x_2, x_3) consisting of 0 and 1 by the following rule: $V_1 \subset U$ (resp. $V_{22}^+ \subset U, V_3^+ \subset U$) if and only if $x_1 = 0$ (resp., $x_2 = 0, x_3 = 0$). Abusing the notation, we write $\mathfrak{h}_{(x_1,x_2,x_3)}$ instead of $\mathfrak{h}_U$. The T_0-orbit corresponding to (x_1, x_2, x_3) lies in the closure of that corresponding to (y_1, y_2, y_3) if and only if $x_i \leqslant y_i$. Consider the seven possible variants case by case.

The case $(1, 0, 0)$. Here $\mathfrak{t} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ and $\Sigma_U = \{-\varepsilon_3-\varepsilon_4, -\varepsilon_2, (-\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)/2\}$. By the above, $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$. Let us check that $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_4\}$. Let $\widetilde{\mathfrak{h}}$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and $e_{(-\varepsilon_1+\varepsilon_2\pm(\varepsilon_3+\varepsilon_4))/2}$; it is a subalgebra. Applying Proposition 3.2.9, we see that $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}}) = \mathfrak{t}$. Let $\widetilde{\mathfrak{q}}$ denote the antistandard parabolic subalgebra of $\mathfrak{g}$ corresponding to the simple roots $\alpha_2, \alpha_3, \alpha_4$. Note that $\mathrm{R}_u(\widetilde{\mathfrak{q}}) \subset \widetilde{\mathfrak{h}}$. Therefore $W(\mathfrak{g}, \widetilde{\mathfrak{h}}) = W(\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)}, \widetilde{\mathfrak{h}}/\mathrm{R}_u(\widetilde{\mathfrak{q}}))$ (Corollary 3.3.14). The last group was computed in Subsection 5.3; it is generated by $s_{\alpha_2}, s_{\alpha_2+\alpha_3}, s_{\alpha_4}$. It remains to recall that $W(\mathfrak{g}, \widetilde{\mathfrak{h}}) \subset W(\mathfrak{g}, \mathfrak{h})$.

The case $(1, 0, 1)$. As we have remarked above, $\mathfrak{h}_{(1,0,0)} \in \overline{T_0\mathfrak{h}}$, whence $W(\mathfrak{g}, \mathfrak{h}) = W(\mathfrak{g}, \mathfrak{h}_{(1,0,0)})$.

The case $(0, 0, 1)$. Here $\mathfrak{h} \in \overline{T_0\mathfrak{h}_{(1,0,1)}}$. By the above, $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$ ($\Sigma_U = \{\varepsilon_2 - \varepsilon_1, -\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2, (-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2\}$).

The case $(0, 1, 0)$. We have $\Sigma_U = \{-\varepsilon_2, (-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2\}\}$. Thus $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} \neq \{0\}$.

The case $(1, 1, 0)$. The subalgebra $\mathfrak{t}_1 = \mathrm{Span}_{\mathbb{C}}(\varepsilon_1, 2\varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ is contained in $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$. On the other hand, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}+\mathfrak{t}_1)/(\mathfrak{h}+\mathfrak{t}_1)$ does not contain a semisimple element, since $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{m}_1+\mathfrak{t}_1) = \mathfrak{m}_1+\mathfrak{t}$. To verify $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$ it is enough to check $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}+\mathfrak{t}_1) = \mathfrak{t}$. This is done by using Proposition 3.2.9. Let us show that $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$. Let $\widetilde{\mathfrak{q}}$ denote the antistandard parabolic subgroup of $\mathfrak{g}$ corresponding to $\alpha_2, \alpha_3, \alpha_4$. It is enough to check that $s_{\alpha_4} \notin W(\mathfrak{g}, \mathfrak{h}) \cap W(\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)})$. It follows from [Lo2], Proposition 8.2, that $W(\mathfrak{g}, \mathfrak{h}) \cap W(\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)}) = W_{G^{(\alpha_2,\alpha_3,\alpha_4)},\widetilde{Q}/H}$. By Proposition 6.5.4, $s_{\alpha_4} \notin W(\mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)}, \mathfrak{g}^{(\alpha_2,\alpha_3,\alpha_4)} \cap \mathfrak{h})$ and we are done.

The case $(1, 1, 1)$. Since $\mathfrak{h}_{(1,1,0)} \in \overline{T_0\mathfrak{h}}$, we have $\widehat{\Pi}(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$.

The case $(0,1,1)$. Analogously to the case $(1,1,0)$, one can show that $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$. Since $\mathfrak{h} \in \overline{T_0\mathfrak{h}_{(1,1,1)}}$, we see that $\widehat{\Pi}(\mathfrak{g},\mathfrak{h}) = \{\alpha_1, \alpha_2, \alpha_2+\alpha_3, \alpha_3+\alpha_4\}$. □

7. Algorithms

7.1. **Algorithms for computing Cartan spaces.** In this subsection, we present algorithms for computing Cartan spaces. Here G is a reductive group and X is a G-variety whose Cartan space $\mathfrak{a}_{G,X}$ we want to compute. We consider three different cases:

(1) $X = G/H$, where H is a reductive subgroup of G.
(2) $X = G *_H V$, where H is a reductive subgroup of G and V is an H-module.
(3) $X = G/H$, where H is a *nonreductive* subgroup of G.

We present algorithms for these three cases separately. We remark, however, that the algorithm in case (2) uses that for case (1), while the algorithm for case (3) uses case (2).

In each of these cases we also show how to determine the distinguished component of $X^{L^\circ_{0\,G,X}}$; see Definition 3.2.11. The reason why we need this computation is that it is necessary for the computations of Cartan spaces in subsequent cases as well as for the computations of Weyl groups; see the next subsection.

Case 1. We compute $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ using Theorem 1.3, [Lo1].

Computation of the distinguished component. Applying Proposition 4.1.3, we compute a point in the distinguished component of $(G/H)^{L^\circ_{0\,G,G/H}}$. Finally, using Proposition 4.1.1, we find the whole distinguished component.

Case 2. Let $\pi : X := G *_H V \to G/H$ denote the natural projection.

Step 1. Applying the algorithm of case 1 to G/H, we compute the space $\mathfrak{a}(\mathfrak{g},\mathfrak{h})$ and find a point x in the distinguished component of $(G/H)^{L_{0\,G,G/H}}$.

Step 2. Applying the following algorithm to the group $L_0 := L^\circ_{0\,G,G/H}$ and the module $V = \pi^{-1}(x)$, we compute $\mathfrak{a}_{L_0,V}$.

Algorithm 7.1.1. *Let G be a connected reductive algebraic group and V a G-module. Put $G_0 = G, V_0 = V$. Assume that we have already constructed a pair (G_i, V_i), where G_i is a connected subgroup in G_0, $B_i := B \cap G_i$. Choose a B_i-semi-invariant vector $\alpha \in V_i^*$. Put $V_{i+1} = (\mathfrak{u}_i^-\alpha)^0$, where $\mathfrak{u}_i^-$ is a maximal unipotent subalgebra of $\mathfrak{g}_i$ normalized by T and opposite to $\mathfrak{b}_i$ and the superscript ^{0} means the annihilator. Put $G_{i+1} = Z_{G_i}(\alpha)$. The group G_{i+1} is connected and $L_{0\,G_i,V_i} = L_{0\,G_{i+1},V_{i+1}}$. Note that* $\mathrm{rk}[\mathfrak{g}_{i+1},\mathfrak{g}_{i+1}] \leqslant \mathrm{rk}[\mathfrak{g}_i,\mathfrak{g}_i]$ *with the equality if and only if $\alpha \in V_i^{[\mathfrak{g}_i,\mathfrak{g}_i]}$. Thus if $[\mathfrak{g}_i,\mathfrak{g}_i]$ acts nontrivially on V_i, then we may assume that* $\mathrm{rk}[\mathfrak{g}_{i+1},\mathfrak{g}_{i+1}] < \mathrm{rk}[\mathfrak{g}_i,\mathfrak{g}_i]$. *So $V_k = V_k^{[\mathfrak{g}_k,\mathfrak{g}_k]}$ for some k. Here $L_{0\,G,V} = L_{0\,G_k,V_k}$ coincides with the unit component of the inefficiency kernel for the action $G_k : V_k$.*

Step 3. Thanks to Proposition 3.2.12, $\mathfrak{a}_{G,X} = \mathfrak{a}(\mathfrak{g},\mathfrak{h}) + \mathfrak{a}_{L_0,V}$.

Computation of the distinguished component. Using Proposition 4.1.1, we can recover the distinguished component of $X^{L^\circ_{0\,G,X}}$ from the point x.

Case 3. *Step 1.* We find a parabolic subgroup $Q \subset G$ tamely containing H by using the following algorithm.

Algorithm 7.1.2. *Put* $\mathfrak{n}_0 = \mathrm{R}_u(\mathfrak{h})$. *If an algebra* $\mathfrak{n}_i \subset \mathfrak{g}, i \geqslant 0$, *consisting of nilpotent elements is already constructed, we put* $\mathfrak{n}_{i+1} = \mathrm{R}_u(\mathfrak{n}_\mathfrak{g}(\mathfrak{n}_i))$. *Clearly,* $\mathfrak{n}_i \subset$

$\mathfrak{n}_{i+1}$. *It is known that if* $\mathfrak{n}_{i+1} = \mathfrak{n}_i$*, then* $\mathfrak{n}_i$ *is the unipotent radical of a parabolic subalgebra* $\mathfrak{q}$ *(see* [Bou]*, ch.8, § 10, Theorem 2). Clearly,* $\mathrm{R}_u(H) \subset \mathrm{R}_u(Q)$*. It is clear from construction that* $N_G(\mathfrak{n}_0) \subset Q$*. In particular,* $H \subset Q$*.*

Step 2. We choose a Levi subgroup $M \subset Q$ such that $M \cap H$ is a maximal reductive subgroup of H and an element $g \in G$ such that gQg^{-1} is an antistandard parabolic subgroup and gMg^{-1} is its standard Levi subgroup. Replace (Q, M, H) with $(gQg^{-1}, gMg^{-1}, gHg^{-1})$. Put $X = Q^-/H$. Using Remark 3.2.8, we construct an M-isomorphism of X and an affine homogeneous vector bundle. Using the algorithm for case (2), we compute $\mathfrak{a}_{M,X}$. By Proposition 3.2.9, $\mathfrak{a}_{G,G/H} = \mathfrak{a}_{M,X}$.

Computation of the distinguished component. Here we suppose that G/H is quasi-affine. We use Proposition 4.1.2 and obtain a point in the distinguished component of $(G/H)^{L^\circ_{0\,G,G/H}}$. Applying Proposition 4.1.1, we determine the whole distinguished component.

7.2. **Computation of Weyl groups.** Again we distinguish several cases.

(1) $X = G *_H V$, where H is a reductive subgroup of G, V is an H-module, and $\mathrm{rk}_G(X) = \mathrm{rk}(G)$.
(2) $X = G *_H V$, where H is a reductive subgroup of G, V is an H-module (the rank of X is arbitrary).
(3) $X = G/H$ is a nonaffine but quasi-affine homogeneous space, and $\mathrm{rk}_G(X) = \mathrm{rk}\, G$.
(4) $X = G/H$ is an arbitrary nonaffine homogeneous space.

Case 1. *Step 1.* Let $G = Z(G)^\circ G_1 \dots G_k$ be the decomposition into the locally direct product of the center and simple normal subgroups. Put $H_i = G_i \cap H$. By Proposition 3.3.15, the equality $W(\mathfrak{g}, \mathfrak{h}, V) = \prod_{i=1}^k W(\mathfrak{g}_i, \mathfrak{h}_i, V)$ holds.

Step 2. The computation of $W(\mathfrak{g}_i, \mathfrak{h}_i, V)$ is carried out by using Theorem 5.1.2.

Case 2. *Step 1.* We find $\mathfrak{a}_{G,X}$ and a point in the distinguished component of $\underline{X} \subset X^{L_0}$, where $L_0 = L^\circ_{0\,G,X}$, lying in G/H, by using the algorithm of case (2) of Subsection 7.1. We may assume that eH lies in the distinguished component.

Step 2. Put $\underline{G} = (N_G(L_0)^\circ N_H(L_0))/L_0, \underline{H} = N_H(L_0)/L_0, \underline{V} = V^{L_0}$. We have a $\underline{G}$-isomorphism $\underline{X} \cong \underline{G} *_{\underline{H}} \underline{V}$ (Proposition 4.1.1). Put $\Gamma = N_{\underline{G}}(\underline{B}, \underline{T})/\underline{T}$, where $\underline{B}, \underline{T}$ are the distinguished Borel subgroup and the maximal torus of $\underline{G}$. By Theorem 3.3.10, $\mathfrak{a}_{G,X} = \underline{\mathfrak{t}} = \mathfrak{a}_{\underline{G}^\circ, \underline{X}}, W_{G,X} = W_{\underline{G}^\circ, \underline{X}} \leftthreetimes \Gamma$. This reduces the computation of $W_{G,X}$ to case (1).

Case 3. Here, recall, $X = G/H$ is a quasi-affine homogeneous space with $\mathrm{rk}_G(G/H) = \mathrm{rk}(G)$.

Step 1. Let $G = Z(G)^\circ G_1 G_2 \dots G_k$ be the decomposition into the locally direct product of the center and simple normal subgroups. Put $H_i := G_i \cap H$. Then, thanks to Proposition 3.3.15, $W(\mathfrak{g}, \mathfrak{h}) = \prod W(\mathfrak{g}_i, \mathfrak{h}_i)$. So we reduce the computation to the case when G is simple.

Step 2. If $\mathfrak{g} \cong \mathfrak{sl}_2$, then $W(\mathfrak{g}, \mathfrak{h})$ is trivial if and only if $\mathfrak{h}$ is a one-dimensional unipotent subalgebra.

Suppose $\mathfrak{g} \cong \mathfrak{so}_5, G_2$. If $\mathfrak{h}$ does not contain the unipotent radical of a parabolic subalgebra, then $W(\mathfrak{g}, \mathfrak{h})$ can be computed by using Propositions 6.2.1, 6.2.2. Otherwise, applying Corollary 3.3.14, we reduce the computation of $W(\mathfrak{g}, \mathfrak{h})$ to the case $\mathrm{rk}[\mathfrak{g}, \mathfrak{g}] \leqslant 1$.

Next, we suppose $\mathfrak{g} \not\cong \mathfrak{sl}_2, \mathfrak{so}_5, G_2$.

Step 3. Let $\widetilde{\mathfrak{h}}$ be the inverse image of the reductive part of $\mathfrak{z}(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})/\mathfrak{h})$ under the canonical epimorphism $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \twoheadrightarrow \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})/\mathfrak{h}$. Compute $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}})$. By Proposition 6.3.1, $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}}) = \mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$.

When $\mathfrak{g}$ is of type A, D, E we recover $W(\mathfrak{g}, \mathfrak{h})$ from $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})}$ using Proposition 6.3.3.

Step 4. Now let $\mathfrak{g} \cong \mathfrak{so}_{2l+1}, \mathfrak{sp}_{2l}, l \geqslant 3, F_4$. Recall, see Corollary 3.3.13, that $W(\mathfrak{g}, \mathfrak{h})$ is identified with $W(\mathfrak{g}, \widetilde{\mathfrak{h}})$. If $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} \neq \{0\}$, then $\mathfrak{a}(\mathfrak{g}, \widetilde{\mathfrak{h}}) \neq \mathfrak{t}$ and we proceed to case (4). Note that under the reduction of case (4) to case (3), $\mathrm{rk}[\mathfrak{g}, \mathfrak{g}]$ decreases. So we may assume that $\mathfrak{t}^{W(\mathfrak{g},\mathfrak{h})} = \{0\}$.

Step 5. Let us find a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ tamely containing $\mathfrak{h}$ (see Algorithm 7.1.2). Conjugating $(\mathfrak{q}, \mathfrak{h})$ by an element of G, we may assume that $\mathfrak{q}$ is an antistandard parabolic subalgebra of $\mathfrak{g}$ and that $\mathfrak{h}_0 := \mathfrak{m} \cap \mathfrak{h}$ is a Levi subalgebra of $\mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{q}$ is the standard Levi subalgebra of $\mathfrak{q}$.

Step 6. If $\mathfrak{g} \cong \mathfrak{sp}_{2l}$, then the computation is carried out by using Proposition 6.4.3. In the remaining cases we inspect Table 5.5 and find simple ideals $\mathfrak{h}_1 \subset \mathfrak{h}_0, \mathfrak{m}_1 \subset \mathfrak{m}$ satisfying the assumptions of Lemma 6.4.2. Denote by $\underline{\mathfrak{h}}$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}_1$ and $[\mathfrak{h}_1, \mathrm{R}_u(\mathfrak{h})]$. According to Lemmas 6.5.1, 6.5.3, 6.6.1, $W(\mathfrak{g}, \underline{\mathfrak{h}}) = W(\mathfrak{g}, \mathfrak{h})$. The groups $W(\mathfrak{g}, \underline{\mathfrak{h}})$ for $\mathfrak{g} = \mathfrak{so}_{2l+1}$ are computed in Subsection 6.5; see especially Proposition 6.5.4 and the preceding discussion. The case $\mathfrak{g} = F_4$ is considered in Subsection 6.6 (Proposition 6.6.2).

Case 4. Here $X = G/H$ is an arbitrary (nonaffine) homogeneous space.

Step 1. We find a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ tamely containing $\mathfrak{h}$. Let us choose a subalgebra $\mathfrak{q}_0 \subset \mathfrak{q}$ annihilating a $\mathfrak{q}$-semi-invariant vector $v \in V$, where V is a G-module (if $\mathfrak{q}$ is standard, one can take a highest vector of an appropriate irreducible module for v). Let $\chi \in \mathfrak{X}(Q)$ be the character of $\mathfrak{q}$ such that $\xi v = \chi(\xi) v$ for all $\xi \in \mathfrak{q}$. If $\mathfrak{h} \subset \mathfrak{q}_0$, then $\mathfrak{h}$ is observable (i.e., the homogeneous space G/H is quasi-affine), thanks to Sukhanov's theorem, [Su]. If not, put $\widetilde{G} = G \times \mathbb{C}^\times$, and embed $\mathfrak{q}$ into $\widetilde{\mathfrak{g}}$ via $\iota : \xi \mapsto (\xi, -\chi(\xi))$. There is the natural representation of $\widetilde{G}$ in V such that $\widetilde{G}_v = \iota(Q)$. So $\iota(\mathfrak{h})$ is observable in $\widetilde{\mathfrak{g}}$. Further, we have a G-equivariant principal $\mathbb{C}^\times$-bundle $\widetilde{G}/\iota(H^\circ) \to G/H^\circ$. From Proposition 3.3.2, it follows that $W(\mathfrak{g}, \mathfrak{h}) = W(\widetilde{\mathfrak{g}}, \iota(\mathfrak{h}))$. So replacing $(\mathfrak{g}, \mathfrak{h})$ with $(\widetilde{\mathfrak{g}}, \iota(\mathfrak{h}))$, if necessary, we may (and will) assume that $\mathfrak{h}$ is observable.

Step 2. We compute $\mathfrak{a}(\mathfrak{g}, \mathfrak{h})$ and determine the distinguished component $\underline{X} \subset X^{L_0}$, where $L_0 := L^\circ_{0\,G,G/H}$. Put $\underline{G} := (N_G(L_0)^\circ N_H(L_0))/L_0, \underline{H} := N_H(L_0)/L_0$. Let $\underline{B}, \underline{T}, \Gamma$ have the same meaning as in case 2. According to Proposition 4.1.1, $\underline{X} \cong \underline{G}/\underline{H}$. Using Theorem 3.3.10, we get $\mathfrak{a}(\mathfrak{g}, \mathfrak{h}) = \underline{\mathfrak{t}} = \mathfrak{a}(\underline{\mathfrak{g}}, \underline{\mathfrak{h}}), W(\mathfrak{g}, \mathfrak{h}) = W(\underline{\mathfrak{g}}, \underline{\mathfrak{h}}) \leftthreetimes \Gamma$.

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Department of Mathematics, Massachusetts Institute of Technology. 77 Massachusetts Avenue, Cambridge, Massachusetts 02139

E-mail address: `ivanlosev@math.mit.edu`