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THE SATAKE ISOMORPHISM FOR SPECIAL MAXIMAL PARAHORIC HECKE ALGEBRAS

THOMAS J. HAINES AND SEAN ROSTAMI

ABSTRACT. Let G denote a connected reductive group over a nonarchimedean local field F. Let K denote a special maximal parahoric subgroup of $G(F)$. We establish a Satake isomorphism for the Hecke algebra $\mathcal{H}_K$ of K-bi-invariant compactly supported functions on $G(F)$. The key ingredient is a Cartan decomposition describing the double coset space $K\backslash G(F)/K$. As an application we define a transfer homomorphism $t : \mathcal{H}_{K^*}(G^*) \rightarrow \mathcal{H}_K(G)$ where G^* is the quasi-split inner form of G. We also describe how our results relate to the treatment of Cartier [Car], where K is replaced by a special maximal compact open subgroup $\widetilde{K} \subset G(F)$ and where a Satake isomorphism is established for the Hecke algebra $\mathcal{H}_{\widetilde{K}}$.

1. INTRODUCTION

The Satake isomorphism plays an important role in automorphic forms and in representation theory of p-adic groups. For global applications, one may often work with unramified groups. We begin by recalling the Satake isomorphism in this context. Let G denote an unramified group over a nonarchimedean local field F. Let v_F denote a special vertex in the Bruhat-Tits building $\mathcal{B}(G_{\rm ad}(F))$. Let $\widetilde{K} = \widetilde{K}_{v_F}$ denote a special maximal compact open subgroup of $G(F)$ which fixes v_F. Let

$$\mathcal{H}_{\widetilde{K}} = C_c^\infty(\widetilde{K}\backslash G(F)/\widetilde{K})$$

denote the Hecke algebra of $\widetilde{K}$-bi-invariant compactly-supported complex-valued functions on $G(F)$. Let A denote a maximal F-split torus in G whose corresponding apartment in $\mathcal{B}(G_{\rm ad}(F))$ contains v_F. Let $W = W(G, A)$ denote the relative Weyl group. Then the Satake isomorphism is a $\mathbb{C}$-algebra isomorphism

$$\mathcal{H}_{\widetilde{K}} \xrightarrow{\sim} \mathbb{C}[X_*(A)]^W.$$

(See [Car].) A key ingredient is the Cartan decomposition

$$\widetilde{K}\backslash G(F)/\widetilde{K} \;\cong\; W(G, A)\backslash X_*(A).$$

Now let G denote an arbitrary connected reductive group over F and let $\widetilde{K}, v_F$ and so on have the same meaning as above. A form of the Satake isomorphism for

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such G was described by Cartier [Car], but it is less explicit than that above. It identifies $\mathcal{H}_{\widetilde{K}}$ with the ring of functions

$$\mathbb{C}[M(F)/M(F)^1]^W,$$

where $M := \mathrm{Cent}_G(A)$ is a minimal F-Levi subgroup of G and $M(F)^1$ is the unique maximal compact open subgroup of $M(F)$. The quotient $M(F)/M(F)^1$ is a free abelian group $\widetilde{\Lambda}_M$ which contains $X_*(A)$ and has the same rank. (In [Car], our $\widetilde{\Lambda}_M$ is denoted $\Lambda(M)$ or simply Λ.) As Cartier explains, in this general context we have a Satake isomorphism

$$\mathcal{H}_{\widetilde{K}} \;\cong\; \mathbb{C}[\widetilde{\Lambda}_M]^W,$$

and a Cartan decomposition

$$\widetilde{K}\backslash G(F)/\widetilde{K} \;\cong\; W(G,A)\backslash\widetilde{\Lambda}_M.$$

However, Cartier does not identify $\widetilde{\Lambda}_M$ explicitly, except in special cases.

Now let $K = K_{v_F}$ denote the special maximal parahoric subgroup of $G(F)$ corresponding to v_F; it is a normal subgroup of $\widetilde{K}_{v_F}$ having finite index (see Section 8). This paper concerns the Hecke algebra $\mathcal{H}_K = C_c^\infty(K\backslash G(F)/K)$. In several situations, it is more appropriate to consider $\mathcal{H}_K$ instead of $\mathcal{H}_{\widetilde{K}}$, for example, in relation to Shimura varieties having parahoric level structure (see [Rap] and [H05]).

Let $M(F)_1 \subset M(F)$ denote the unique parahoric subgroup of $M(F)$; it is a finite-index normal subgroup of $M(F)^1$. Our main result is the following theorem.

Theorem 1.0.1. *Let $\Lambda_M := M(F)/M(F)_1$. There is a canonical isomorphism*

$$\mathcal{H}_K \xrightarrow{\sim} \mathbb{C}[\Lambda_M]^W.$$

The group Λ_M is a finitely generated abelian group which can be explicitly described and which has the property that $\widetilde{\Lambda}_M = \Lambda_M/torsion$. Moreover, $\widetilde{K}/K \cong \Lambda_{M,\mathrm{tor}}$, the torsion subgroup of Λ_M.

When G is unramified over F or when G is semi-simple and simply connected, it turns out that $\widetilde{K} = K$ and $\widetilde{\Lambda}_M \cong \Lambda_M$ (see Section 11) so that our theorem does not give any new information in those cases. However, our results are new in the case $\widetilde{K} \neq K$, and different methods from [Car] are needed to prove them. For ramified groups, in particular, our results are expected to play some role in the study of Shimura varieties with parahoric level structure at p. For more about ramified groups and Shimura varieties with parahoric level the reader should consult [Rap], [PR], and [Kr].

In order to describe Λ_M, we need to recall some notation and results of Kottwitz [Ko97]. Let F^s denote a separable closure of F, and let F^{un} denote the maximal unramified extension of F in F^s. Let $L = \widehat{F^{\mathrm{un}}}$ denote the completion of F^{un} with respect to the valuation on F^{un} which extends the normalized valuation on F. Let $I = \mathrm{Gal}(F^s/F^{\mathrm{un}}) \cong \mathrm{Gal}(L^s/L)$ denote the inertia subgroup of $\mathrm{Gal}(F^s/F)$, and let $\sigma \in \mathrm{Aut}(L/F)$ denote the Frobenius automorphism. In [Ko97] Kottwitz defined a surjective homomorphism

$$\kappa_G : G(L) \to X^*(Z(\widehat{G}))_I,$$

and in loc. cit. §7.7 he also proved that this induces a surjective homomorphism

$$\kappa_G : G(F) \to X^*(Z(\widehat{G}))_I^\sigma$$

of the groups of σ-invariants. Set $G(L)_1 := \ker(\kappa_G)$ and $G(F)_1 := G(F) \cap G(L)_1$. (When $G = M$, this is consistent with our definition of $M(F)_1$ above; see Lemmas 4.1.1, 4.2.1.)

The Iwahori-Weyl group $\widetilde{W}$ (defined in §2.3) for G carries a natural action under σ and contains a σ-invariant abelian subgroup Ω_G (the subgroup of *length-zero elements*). By choosing representatives in the normalizer of A we may embed $\widetilde{W}^\sigma$ set-theoretically into $G(F)$, and then Ω_G^σ is mapped by κ_G isomorphically onto $X^*(Z(\widehat{G}))_I^\sigma$ (see Section 2). The following is the sought-after explicit description of Λ_M:

Proposition 1.0.2. *The Kottwitz homomorphism induces an isomorphism*

$$\Lambda_M = M(F)/M(F)_1 \cong X^*(Z(\widehat{M}))_I^\sigma.$$

Via the Kottwitz isomorphism $\kappa_M : \Omega_M^\sigma \xrightarrow{\sim} X^*(Z(\widehat{M}))_I^\sigma$, we can also identify Λ_M with Ω_M^σ.

As before, the main step in the proof of Theorem 1.0.1 is an appropriate Cartan decomposition.

Theorem 1.0.3. *The embedding $\Omega_M^\sigma \subset \widetilde{W}^\sigma \hookrightarrow G(F)$ determines a bijection*

$$W(G,A)\backslash\Omega_M^\sigma \cong K\backslash G(F)/K.$$

Equivalently, via the isomorphism $\kappa_M : \Omega_M^\sigma \xrightarrow{\sim} X^*(Z(\widehat{M}))_I^\sigma$, we have a bijection

$$W(G,A)\backslash X^*(Z(\widehat{M}))_I^\sigma \xrightarrow{\sim} K\backslash G(F)/K.$$

We give additional information about the finitely generated abelian group Λ_M in Section 11. For example, we prove that if G is an inner form of a split group, then $\Lambda_M = X^*(Z(\widehat{M})) = X_*(T)_\sigma$ (see Corollary 11.3.2).

Finally, let G^* denote the quasi-split inner form of G, and consider special maximal parahoric subgroups $K^* \subset G^*(F)$ and $K \subset G(F)$. In Section 12, we define a canonical transfer homomorphism $t : \mathcal{H}_{K^*}(G^*) \to \mathcal{H}_K(G)$, and we establish some of its basic properties.

This article relies heavily on the ideas of Kottwitz, especially as they are manifested in the article [HR]. The main theorems of [HR] provide the starting points for the proof of Theorem 1.0.3.

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2. Notation

2.1. **Ring-theoretic notation.** Let $\mathcal{O} = \mathcal{O}_F$ (resp. $\mathcal{O}_L$) denote the ring of integers in the field F (resp. L). Let ϖ denote a uniformizer of F (resp. L), and let k_F denote the residue field of F. We may identify the residue field k_L with an algebraic closure of k_F. Let $\Gamma := \mathrm{Gal}(F^s/F)$.

Throughout this paper, if $J \subset G(F)$ denotes a compact open subgroup, we make

$$\mathcal{H}_J := C_c^\infty(J\backslash G(F)/J)$$

a convolution algebra by using the Haar measure on $G(F)$ which gives J volume 1.

2.2. **Buildings notation.** Let $\mathcal{B}(G(L))$ (resp. $\mathcal{B}(G(F))$) denote the Bruhat-Tits building of $G(L)$ (resp. $G(F)$). The building $\mathcal{B}(G(L))$ carries an action of σ. By [BT2], 5.1.25, we have an identification $\mathcal{B}(G(F)) = \mathcal{B}(G(L))^\sigma$. Moreover, there is a bijection $\mathbf{a}_J \mapsto \mathbf{a}_J^\sigma$ from the set of σ-stable facets in $\mathcal{B}(G(L))$ to facets in $\mathcal{B}(G(F))$ ([BT2], 5.1.28). This bijection sends alcoves to alcoves ([BT2], 5.1.14). It also follows from loc. cit. that every σ-stable facet $\mathbf{a}_J$ in $\mathcal{B}(G(L))$ is contained in the closure $\overline{\mathbf{a}}$ of a σ-stable alcove $\mathbf{a}$.

Let v_F denote a special vertex in $\mathcal{B}(G_{\mathrm{ad}}(F))$ ([Tits], 1.9). Let A denote a maximal F-split torus in G whose corresponding apartment in $\mathcal{B}(G_{\mathrm{ad}}(F))$ contains v_F. Let $\mathcal{A}$ (resp. $\mathcal{A}_{\mathrm{ad}}$) denote the apartment in $\mathcal{B}(G(F))$ (resp. $\mathcal{B}(G_{\mathrm{ad}}(F))$) corresponding to A. Let $V_{G(F)}$ denote the real vector space $X_*(Z(G))_\Gamma \otimes \mathbb{R}$. There exists a simplicial isomorphism ([Tits], 1.2)

$$\mathcal{A} \cong \mathcal{A}_{\mathrm{ad}} \times V_{G(F)}.$$

Therefore, there is a minimal dimensional facet $\mathbf{a}_0^\sigma$ in $\mathcal{A}$ associated to a σ-stable facet $\mathbf{a}_0 \subset \mathcal{B}(G(L))$, such that

$$\mathbf{a}_0^\sigma \cong \{v_F\} \times V_{G(F)}.$$

We consider parahoric (or Iwahori) subgroups in the sense of [BT2], 5.2. That is, to a facet $\mathbf{a}_J \subset \mathcal{B}(G(L))$ we associate an $\mathcal{O}_L$-group scheme $\mathcal{G}^\circ_{\mathbf{a}_J}$ with connected geometric fibers, whose group of $\mathcal{O}_L$-points fixes identically the points of $\mathbf{a}_J$. We often write $J(L) := \mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_L)$. By [BT2], 5.2, if $\mathbf{a}_J$ is σ-stable, we get a parahoric subgroup $J(F) := J(L)^\sigma$ in $G(F)$ and this is associated to the facet $\mathbf{a}_J^\sigma$ in $\mathcal{B}(G(F))$. Moreover, every parahoric subgroup of $G(F)$ is of this form for a unique σ-stable facet $\mathbf{a}_J$.

Now fix a σ-stable alcove $\mathbf{a}$ whose closure contains $\mathbf{a}_0$. Let $I(L)$ (resp. $K(L)$) denote the Iwahori (resp. parahoric) subgroup of $G(L)$ corresponding to the σ-stable alcove $\mathbf{a}$ (resp. facet $\mathbf{a}_0$). Then $I := I(F) = I(L)^\sigma$ is the Iwahori subgroup of $G(F)$ corresponding to $\mathbf{a}^\sigma$. Also, $K := K(F) = K(L)^\sigma$ is a special maximal parahoric subgroup of $G(F)$ corresponding to $\mathbf{a}_0^\sigma$ (or equivalently, to v_F).

2.3. **Weyl groups and Iwahori-Weyl groups.** For a torus S in G, let $N_G(S) = \mathrm{Norm}_G(S)$ denote its normalizer and $C_G(S) = \mathrm{Cent}_G(S)$ its centralizer. Let $W(G,S) := N_G(S)/C_G(S)$ denote its Weyl group.

Fix the torus A as before. From now on, let S be a maximal L-split torus that is defined over F and contains A ([BT2], 5.1.12). Let $T := C_G(S)$, a maximal torus of G (defined over F) since G_L is quasi-split by Steinberg's theorem.

We need to recall definitions and facts about Iwahori-Weyl groups; we refer the reader to [HR] for details. Let $T(L)_1 := \ker(\kappa_T)$, a normal subgroup of $N_G(S)(L)$. Let $\widetilde{W} := N_G(S)(L)/T(L)_1$ denote the *Iwahori-Weyl* group for G. It carries an obvious action of σ. Let $\mathcal{A}_L$ denote the apartment of $\mathcal{B}(G(L))$ corresponding to S, which we may assume contains the alcove $\mathbf{a}$ we fixed above. We let W_{aff} denote the *affine Weyl group*, which is a Coxeter group generated by the reflections through the walls of $\mathbf{a}$. The group $\widetilde{W}$ acts on the set of all alcoves in the apartment of $\mathcal{B}(G(L))$ corresponding to S; let $\Omega_G = \Omega_{G,\mathbf{a}}$ denote the stabilizer of $\mathbf{a}$. There exists a σ-equivariant decomposition

$$\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega_G.$$

We extend the Bruhat order $\leq$ and the length function ℓ from $W_{\rm aff}$ to $\widetilde{W}$ in the usual way: if $x, x' \in \widetilde{W}$ are written as $w \cdot \tau$ and $w' \cdot \tau'$ via the preceding decomposition, then we set $\ell(x) := \ell(w)$ and declare that $x \leq x'$ if and only if both $w \leq w'$ and $\tau = \tau'$. We can identify $W_{\rm aff}$ with the Iwahori-Weyl group associated to the pair $G_{\rm sc}, S_{\rm sc}$, where $S_{\rm sc}$ is the pull-back of $(S \cap G_{\rm der})^\circ$ via $G_{\rm sc} \to G_{\rm der}$.

We can embed $\widetilde{W}$ *set-theoretically* into $G(L)$ by choosing a set-theoretic section of the surjective homomorphism $N_G(S)(L) \to \widetilde{W}$. Since $T(L)_1 \subset \ker(\kappa_G)$, we easily see that the restriction of κ_G to $\widetilde{W} \hookrightarrow G(L)$ gives a *homomorphism*

$$\kappa_G : \widetilde{W} \to X^*(Z(\widehat{G}))_I,$$

which is surjective and σ-equivariant and whose kernel is $W_{\rm aff}$.

3. Cartan decomposition: reduction to the key lemma

Changing slightly the notation of [HR], we set

$$\widetilde{W}_K := (N_G(S)(L) \cap K(L))/T(L)_1.$$

We write $\widetilde{W}_K^\sigma := (\widetilde{W}_K)^\sigma$.

Our starting point is the following fact (see [HR], esp. Remark 9): the map $K(L)nK(L) \mapsto n \in \widetilde{W}$ induces a bijection

$$K(L)\backslash G(L)/K(L) \cong \widetilde{W}_K \backslash \widetilde{W} / \widetilde{W}_K,$$

and taking fixed-points under σ yields a bijection

$$K(F)\backslash G(F)/K(F) \cong \widetilde{W}_K^\sigma \backslash \widetilde{W}^\sigma / \widetilde{W}_K^\sigma. \tag{3.0.1}$$

The Cartan decomposition follows immediately from the key lemma below, which allows us to describe the right-hand side of (3.0.1) in the desired way. To state this we note that the σ-stable alcove $\mathbf{a}$ is contained in a unique σ-stable alcove $\mathbf{a}^M$ in the apartment $\mathcal{A}_L^M \subset \mathcal{B}(M(L))$ corresponding to S. As before, we define $\Omega_M \subset \widetilde{W}_M$ to be the stabilizer of $\mathbf{a}^M$ under the action of $\widetilde{W}_M$ on the alcoves in $\mathcal{A}_L^M$.

Lemma 3.0.1. (I) *There is a tautological isomorphism* $\widetilde{W}_K^\sigma \widetilde{\to} W(G,A)$ *which allows us to view* $W(G,A)$ *as a subgroup of* $\widetilde{W}^\sigma$.

(II) *There is a decomposition* $\widetilde{W}^\sigma = \widetilde{W}_M^\sigma \cdot W(G,A)$, *and* $W(G,A)$ *normalizes* $\widetilde{W}_M^\sigma$.

(III) *We have* $W_{M,\rm aff}^\sigma = 1$, *and hence because of the* σ*-equivariant decomposition*

$$\widetilde{W}_M = W_{M,\rm aff} \rtimes \Omega_M$$

we have $\widetilde{W}^\sigma = \Omega_M^\sigma \rtimes W(G,A)$.

The Kottwitz homomorphism gives an isomorphism

$$\kappa_M : \Omega_M^\sigma \widetilde{\to} X^*(Z(\widehat{M}))_I^\sigma$$

(cf. [Ko97], 7.7). Putting this together with the lemma we get Theorem 1.0.3.

The proof of Lemma 3.0.1 will occupy the next four sections.

4. Some ingredients about parahoric subgroups

4.1. **Parahoric subgroups of F-Levi subgroups.** As before, let A denote a maximal F-split torus in G, let $S \supseteq A$ be a maximal L-split torus which is defined over F, and let $T = C_G(S)$ be a maximal torus of G which is defined over F.

Let A_M denote any subtorus of A, and let $M = C_G(A_M)$. Thus M is a semi-standard F-Levi subgroup of G. The extended buildings $\mathcal{B}(M(L))$ and $\mathcal{B}(G(L))$ share an apartment (which corresponds to S), but the affine hyperplanes in the apartment $\mathcal{A}_L^M$ for $M(L)$ form a subset of those in the apartment $\mathcal{A}_L$ for $G(L)$. Hence any facet $\mathbf{a}_J$ in $\mathcal{A}_L$ is contained in a unique facet in $\mathcal{A}_L^M$, which we will denote by $\mathbf{a}_J^M$.

The following result was proved in [H09] in the special case where G splits over L.

Lemma 4.1.1. *Suppose $J(L) \subset G(L)$ is the parahoric subgroup corresponding to a facet $\mathbf{a}_J \subset \mathcal{A}_L$. Then $J(L) \cap M$ is a parahoric subgroup of $M(L)$, and corresponds to the facet $\mathbf{a}_J^M \subset \mathcal{A}_L^M$.*

Proof. The main result of [HR] is the following characterization of parahoric subgroups:

$$J(L) = \mathrm{Fix}(\mathbf{a}_J) \cap G(L)_1$$

where $\mathrm{Fix}(\mathbf{a}_J)$ is the set of elements of $G(L)$ that fixes every point of $\mathbf{a}_J$. Applying this for the groups M and G, we only need to show that

$$\mathrm{Fix}(\mathbf{a}_J) \cap G(L)_1 \cap M(L) = \mathrm{Fix}(\mathbf{a}_J^M) \cap M(L)_1.$$

The functoriality of the Kottwitz homomorphisms shows $M(L)_1 \subset G(L)_1$, and then the inclusion "$\supseteq$" is evident. Let $\mathbf{a}^M$ denote an alcove in $\mathcal{A}_L^M$ whose closure contains $\mathbf{a}_J^M$. Let I_M denote the Iwahori subgroup of $M(L)$ corresponding to $\mathbf{a}^M$.

Let S_{sc}^M, resp., T_{sc}^M denote the pull-back of the torus $(S \cap M_{\mathrm{der}})^\circ$, resp., $T \cap M_{\mathrm{der}}$ along the homomorphism $M_{\mathrm{sc}} \to M_{\mathrm{der}}$. To prove the inclusion "$\subseteq$" it is enough to prove the following claim, since $N_{M_{\mathrm{sc}}}(S_{\mathrm{sc}}^M)(L)$ and I_M belong to $M(L)_1$. Here and in what follows, we abuse notation slightly by writing $N_{M_{\mathrm{sc}}}(S_{\mathrm{sc}}^M)(L)$ where we really mean its image in $M(L)$.

Claim. Any element $m \in M(L) \cap G(L)_1$ which fixes a point in $\mathbf{a}_J^M$ belongs to

$$I_M \, N_{M_{\mathrm{sc}}}(S_{\mathrm{sc}}^M)(L) \, I_M$$

and fixes every point of $\mathbf{a}_J^M$.

Proof. Recall the decomposition

$$I_M \backslash M(L) / I_M \cong N_M(S)(L)/T(L)_1 \tag{4.1.1}$$

of [HR], Prop. 8. Using this we may assume $m \in N_M(S)(L)$.

We will show that for such an element m which fixes a point of $\mathbf{a}_J^M$ we have $m \in T(L)_1 \, N_{M_{\mathrm{sc}}}(S_{\mathrm{sc}}^M)(L)$, which will prove the first statement of the claim. It will also prove the second statement, since then m determines a type-preserving automorphism of the apartment $\mathcal{A}_L^M$, hence fixes every point of $\mathbf{a}_J^M$ if it fixes any of its points.

Choose a special vertex $\mathbf{a}_0^M$ contained in the closure of $\mathbf{a}^M$, and let K_0 denote the corresponding special maximal parahoric subgroup of $M(L)$. We may write $m = tn$, where $t \in T(L)$ and $n \in N_M(S)(L) \cap K_0$ (cf. [HR], Prop. 13). Define $\nu \in X_*(T)_I$ to be $\kappa_T(t)$ and $w \in W(M, S)$ to be the image of n under the projection

$N_M(S)(L) \to W(M,S)$. Thus m maps to the element $t_\nu\, w \in X_*(T)_I \rtimes W(M,S) \cong \widetilde{W}_M$, the Iwahori-Weyl group for M.

Let $\Sigma^\vee$ denote the coroots associated to the unique reduced root system Σ such that the set of affine roots $\Phi_{\text{af}}(G(L),S)$ on $\mathcal{A}_L$ are given by $\Phi_{\text{af}} = \{\alpha + k \mid \alpha \in \Sigma,\, k \in \mathbb{Z}\}$; cf. [HR]. Let $\Sigma_M^\vee$ denote the coroots for the corresponding root system Σ_M for $\Phi_{\text{af}}(M(L),S)$ on $\mathcal{A}_L^M$. Let $Q^\vee(\Sigma)$, resp., $Q^\vee(\Sigma_M)$ denote the lattice spanned by $\Sigma^\vee$, resp., $\Sigma_M^\vee$. Recall from [HR] that we have identifications $Q^\vee(\Sigma) \cong X_*(T_{\text{sc}})_I$ and $Q^\vee(\Sigma_M) \cong X_*(T_{\text{sc}}^M)_I$. Also, we have $\Phi_{\text{af}}(M(L),S) \subseteq \Phi_{\text{af}}(G(L),S)$, and therefore $Q^\vee(\Sigma_M) \subseteq Q^\vee(\Sigma)$.

Clearly, w is the image of an element from $N_{M_{\text{sc}}}(S_{\text{sc}}^M)(L) \cap K_0$, since the latter also surjects onto $W(M,S)$. Thus we need only show that $\nu \in Q^\vee(\Sigma_M)$, since $Q^\vee(\Sigma_M)$ is also in the image of $N_{M_{\text{sc}}}(S_{\text{sc}}^M)(L) \to \widetilde{W}_M$.

First, we will prove that $\nu \in Q^\vee(\Sigma)$. Indeed, by construction $t \in G(L)_1$, and using

$$X_*(T)_I / X_*(T_{\text{sc}})_I \cong X^*(Z(\widehat{G}))_I$$

(cf. [HR]) we see that $\nu \in X_*(T_{\text{sc}})_I = Q^\vee(\Sigma)$.

Next, let r denote the order of $w \in W(M,S)$. The element m^r maps to $(t_\nu w)^r \in \widetilde{W}_M$, which is the translation by the element $\mu := \sum_{i=0}^{r-1} w^i \nu \in Q^\vee(\Sigma)$. But as this translation fixes a point of $\mathbf{a}_J^M$, we must have $\mu = 0$. Since $w^i\nu \equiv \nu$ modulo $Q^\vee(\Sigma_M)$, it follows that

$$\nu \in Q^\vee(\Sigma_M)_{\mathbb{Q}} \cap Q^\vee(\Sigma) = Q^\vee(\Sigma_M).$$

This completes the proof of the claim, and thus the lemma. $\square$

4.2. **Parahoric subgroups of minimal F-Levi subgroups.** Now we return to the usual notation, where $M := C_G(A)$ is a minimal F-Levi subgroup of G. In this case M_{ad} is anisotropic over F, so the semi-simple building $\mathcal{B}(M_{\text{ad}}(F)) = \mathcal{B}(M_{\text{ad}}(L))^\sigma$ is a singleton and the apartment $(\mathcal{A}_L^M)^\sigma$ contains no affine hyperplanes. Therefore, $M(F)$ has only one parahoric subgroup.

Lemma 4.2.1. *Let J be any parahoric subgroup of $G(L)$ corresponding to a σ-invariant facet $\mathbf{a}_J$ in $\mathcal{A}_L$. Then $J(L) \cap M(F) = M(F)_1$.*

Proof. By Lemma 4.1.1, the inclusion "$\subseteq$" is clear. Let $m \in M(F)_1$. Since m acts trivially on the apartment $\mathcal{A}_L^\sigma$ in the building $\mathcal{B}(G(F)) = \mathcal{B}(G(L))^\sigma$, it fixes a point of the σ-invariant facet $\mathbf{a}_J$ (e.g. its barycenter). But then since $m \in G(F)_1$, by the Claim in the proof of Lemma 4.1.1 (taking $M = G$), m fixes every point in $\mathbf{a}_J$. Clearly, then $m \in \text{Fix}(\mathbf{a}_J) \cap G(L)_1 \cap M(F) = J(L) \cap M(F)$. $\square$

Lemma 4.2.2. *Let $K(L)$ denote the parahoric subgroup of $G(L)$ whose σ-fixed subgroup $K = K(L)^\sigma$ is the special maximal compact subgroup of $G(F)$ we fixed earlier. Then*

$$K \cap N_G(S)(L) \cap M(F) = T(F)_1.$$

Proof. Fix an Iwahori subgroup $I \subset G(L)$ corresponding to a σ-invariant alcove in $\mathcal{A}_L$. Note that by Lemma 4.2.1, we have $K \cap M(F) = I \cap M(F)$ and hence

$$K \cap N_G(S)(L) \cap M(F) = I \cap N_G(S)(L) \cap M(F).$$

By [HR], Lemma 6, the right-hand side is $T(L)_1 \cap M(F) = T(F)_1$. $\square$

5. The isomorphism $\widetilde{W}_K^\sigma \cong W(G,A)$

By [HR], Remark 9, any element of $\widetilde{W}_K^\sigma$ is represented by an element of $N_G(S)(F)$. Let $x \in N_G(S)(F)$. Then $xSx^{-1} = S$ contains xAx^{-1} and A, which being maximal F-split tori in S, must coincide. Thus, there is a tautological homomorphism

$$N_G(S)(F) \to N_G(A)(F).$$

By Lemma 4.2.2, this factors to give an injective homomorphism

$$\widetilde{W}_K^\sigma \hookrightarrow W(G,A).$$

The next statement furnishes the proof of Lemma 3.0.1, (I).

Lemma 5.0.1. *The homomorphism $\widetilde{W}_K^\sigma \to W(G,A)$ is an isomorphism. This allows us to regard $W(G,A)$ as a subgroup of $\widetilde{W}^\sigma$.*

Proof. It is enough to prove the domain and codomain have the same order. Let k_L denote the residue field of $\mathcal{O}_L$, which can be identified with an algebraic closure of k_F. Consider the special fiber $\overline{\mathcal{G}}_{\mathbf{a_0}}^\circ = \mathcal{G}_{\mathbf{a}_0}^\circ \times_{\mathcal{O}_L} k_L$ of the Bruhat-Tits group scheme $\mathcal{G}_{\mathbf{a}_0}^\circ$ over $\mathcal{O}_L$ which is associated to the facet $\mathbf{a}_0$ in the building $\mathcal{B}(G(L))$. Let $\overline{\mathcal{G}}_{\mathbf{a}_0}^{\circ,\mathrm{red}}$ denote the maximal reductive quotient of $\overline{\mathcal{G}}_{\mathbf{a}_0}^\circ$. By [HR], Prop. 12, $\widetilde{W}_K$ is the Weyl group of $\overline{\mathcal{G}}_{\mathbf{a_0}}^{\circ,\mathrm{red}}$. The group $\overline{\mathcal{G}}_{\mathbf{a}_0}^{\circ,\mathrm{red}}$ is defined over k_F, and in fact we have $\overline{\mathcal{G}}_{\mathbf{a}_0}^{\circ,\mathrm{red}} = \overline{\mathcal{G}}_{v_F}^{\circ,\mathrm{red}} \times_{k_F} k_L$, where $\overline{\mathcal{G}}_{v_F}^\circ$ is the special fiber of $\mathcal{G}_{v_F}^\circ$ (cf. [Land], Cor. 10.10). Since k_F is finite, $\overline{\mathcal{G}}_{v_F}^{\circ,\mathrm{red}}$ is automatically quasi-split over k_F, and it follows that $\widetilde{W}_K^\sigma$ is the Weyl group of $\overline{\mathcal{G}}_{v_F}^{\circ,\mathrm{red}}$ (this is well known, but one can also use the argument which yields Remark 6.1.3 below).

On the other hand, by [Tits], 3.5.1, the root system of $\overline{\mathcal{G}}_{v_F}^{\circ,\mathrm{red}}$ is Φ_{v_F}, the root system consisting of the vector parts of the affine roots for A which vanish on v_F (loc. cit. 1.9). Because v_F is special, $\Phi_{v_F} = \Phi(G,A)$, the relative root system. Thus the Weyl group of $\overline{\mathcal{G}}_{v_F}^{\circ,\mathrm{red}}$ is isomorphic to $W(G,A)$.

These remarks imply that $\widetilde{W}_K^\sigma$ and $W(G,A)$ are abstractly isomorphic groups and, in particular, they have the same order. □

6. A decomposition of the Iwahori Weyl group

The goal here is to prove Lemma 3.0.1, (II).

6.1. **A lemma on finite Weyl groups.** Let $w \in W(G,A)$ and choose a representative $g \in N_G(A)(F)$ for w; write $[g] = w$. The tori gSg^{-1} and S are both maximal L-split tori in M, hence there exists $m \in M(L)$ such that $mgSg^{-1}m^{-1} = S$. We claim that the map

$$W(G,A) \to W(G,S)/W(M,S)$$
$$w \mapsto [mg] \cdot W(M,S)$$

is well defined and injective. Indeed, suppose $g_0 \in N_G(A)(F)$ represents an element $w_0 \in W(G,A)$ and that $m_0 \in M(L)$ satisfies $m_0g_0Sg_0^{-1}m_0^{-1} = S$. To show the map is well defined, we suppose $w = w_0$ and we show that $(mg)^{-1}m_0g_0 \in N_M(S)$. It will suffice to show $(mg)^{-1}m_0g_0$ belongs to $M(L)$. Since g normalizes $M = C_G(A)$ and $g^{-1}g_0 \in M$, this is obvious. To show the map is injective we suppose $[mg]W(M,S) = [m_0g_0]W(M,S)$, that is, $(mg)^{-1}m_0g_0 \in N_M(S)$. Arguing as before, we deduce that $g^{-1}g_0 \in M$. This shows that $w = w_0$ and so we get the injectivity.

Remark 6.1.1. Here is another way to describe the map. For an element $w \in W(G, A)$, using Lemma 5.0.1 choose an element $x \in N_G(S)(F) \cap K$ whose image in $\widetilde{W}_K^\sigma$ maps to w under the isomorphism $\widetilde{W}_K^\sigma \widetilde{\to} W(G, A)$. Then the map sends w to the coset $[x]W(M, S)$.

Lemma 6.1.2. *The above map induces a bijection*

$$W(G, A) \widetilde{\to} [W(G, S)/W(M, S)]^\sigma.$$

Proof. First we prove the image $[mg]W(M, S)$ is σ-invariant. This follows because the element $(mg)^{-1}\sigma(m)g$ belongs to M, hence to $N_M(S)$.

Next, we prove the surjectivity. Suppose $x \in N_G(S)$ projects to an element in $W(G, S)$ which represents a σ-fixed coset C in $W(G, S)/W(M, S)$, that is, $x^{-1}\sigma(x) \in M$. Then the subtorus $xAx^{-1} \subset S$ is defined over F. The inner automorphism $\mathrm{Int}(x) : S \to S$, restricted to A gives an isomorphism $\mathrm{Int}(x) : A \widetilde{\to} xAx^{-1}$ which is defined over F. It follows that xAx^{-1} is F-split. Since A and xAx^{-1} are maximal F-split tori in S, they coincide. Thus $x \in N_G(A)$, and the image of x is the coset C. □

Remark 6.1.3. If G is quasi-split over F, then $M = T$ and we recover the well-known result that $W(G, A) = W(G, S)^\sigma$.

6.2. **Proof of the decomposition.** We keep the notation of the previous subsection. There is a commutative diagram of exact sequences with σ-equivariant morphisms and injective vertical maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_*(T)_I & \longrightarrow & \widetilde{W}_M & \longrightarrow & W(M,S) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_*(T)_I & \longrightarrow & \widetilde{W} & \longrightarrow & W(G,S) & \longrightarrow & 0
\end{array}$$

(see [HR], Prop. 13). The canonical map $\widetilde{W}_M \backslash \widetilde{W} \to W(M, S)\backslash W(G, S)$ is bijective and σ-equivariant, so we get

$$[\widetilde{W}_M \backslash \widetilde{W}]^\sigma \cong [W(M, S)\backslash W(G, S)]^\sigma.$$

Using the map $W(G, A) \hookrightarrow \widetilde{W}^\sigma$ constructed in Lemma 5.0.1 we get a commutative diagram

$$\begin{array}{ccc}
W(G,A) & \longrightarrow & \widetilde{W}_M^\sigma \backslash \widetilde{W}^\sigma \\
 & \searrow & \downarrow \\
 & & (\widetilde{W}_M \backslash \widetilde{W})^\sigma.
\end{array}$$

The commutativity of this diagram follows using Remark 6.1.1. Since the diagonal arrow is a bijection by the above discussion, and the vertical arrow is obviously an injection, it follows that all arrows in the diagram are bijections. The decomposition

$$\widetilde{W}^\sigma = \widetilde{W}_M^\sigma \cdot W(G, A)$$

follows. It is clear that $W(G, A)$ normalizes $\widetilde{W}_M^\sigma$. This completes the proof of Lemma 3.0.1,(II) .

7. End of proof of the Cartan decomposition

7.1. Invariants in the affine Weyl group of M.

Lemma 7.1.1. *Let M again denote a minimal F-Levi subgroup, and let $W_{M,\mathrm{aff}}$ denote the affine Weyl group associated to M. Then $W^\sigma_{M,\mathrm{aff}} = 1$.*

Proof. We identify $W_{M,\mathrm{aff}}$ with the Iwahori-Weyl group $N_{M_{\mathrm{sc}}}(S^M_{\mathrm{sc}})(L)/T^M_{\mathrm{sc}}(L)_1$. Let $I_{M_{\mathrm{sc}}}$ denote the Iwahori subgroup of $M_{\mathrm{sc}}(L)$ corresponding to a σ-invariant alcove $\mathbf{a}^{M_{\mathrm{sc}}}$ in the apartment $\mathcal{A}^{M_{\mathrm{sc}}}_L = X_*(S^M_{\mathrm{sc}})_{\mathbb{R}}$ of $\mathcal{B}(M_{\mathrm{sc}}(L))$ associated to the torus S^M_{sc}. By [HR], Remark 9, the set $W^\sigma_{M,\mathrm{aff}}$ is in bijective correspondence with

$$I_{M_{\mathrm{sc}}}(F)\backslash M_{\mathrm{sc}}(F)/I_{M_{\mathrm{sc}}}(F).$$

Therefore, it is enough to prove that $M_{\mathrm{sc}}(F) = I_{M_{\mathrm{sc}}}(F)$. But $M_{\mathrm{sc}}(F) = M_{\mathrm{sc}}(F)_1 = I_{M_{\mathrm{sc}}}$. (For the second equality, use Lemma 4.2.1 with $G = M_{\mathrm{sc}}$.) □

7.2. Conclusion of the proof of Theorem 1.0.3. We have fixed the σ-stable alcove $\mathbf{a}$ and this determines the σ-stable alcove $\mathbf{a}^M$ and the corresponding subgroup $\Omega_M \subset \widetilde{W}_M$. There is a canonical σ-equivariant decomposition $\widetilde{W}_M = W_{M,\mathrm{aff}} \rtimes \Omega_M$, so in view of the above lemma, we deduce that

$$\widetilde{W}^\sigma_M = \Omega^\sigma_M.$$

This completes the proof of the last part, namely (III), of Lemma 3.0.1. Since the Theorem 1.0.3 is a consequence of Lemma 3.0.1, we have proved Theorem 1.0.3. □

8. Characterization of special maximal compact subgroups

Let

$$v_G : G(L) \to X^*(Z(\widehat{G}))_I/torsion$$

denote the homomorphism derived from the Kottwitz homomorphism

$$\kappa_G : G(L) \to X^*(Z(\widehat{G}))_I$$

in the obvious way. Denote its kernel by $G(L)^1$ and let $G(F)^1 = G(L)^1 \cap G(F)$ (cf. [BT2], 5.1.29). Note that if M is a minimal F-Levi subgroup of G, then $M(F)^1$ is the unique maximal compact open subgroup of $M(F)$, consistent with the notation used in the introduction.

Let $K := \mathcal{G}^\circ_{v_F}(\mathcal{O}_F)$, the maximal parahoric subgroup of $G(F)$ corresponding to v_F. By [HR], Prop. 3 and Remark 9, we have the equality

$$K = G(F)_1 \cap \mathrm{Fix}(\mathbf{a}_0).$$

Using the Claim from the proof of Lemma 4.1.1 in the case $M = G$, we derive the equality

$$K = G(F)_1 \cap \mathrm{Fix}(v_F). \tag{8.0.1}$$

Our goal is to prove the analogous description of $\widetilde{K}$.

Lemma 8.0.1. *The special maximal compact subgroups of $G(F)$ are precisely the subgroups of the form*

$$\widetilde{K} = G(F)^1 \cap \mathrm{Fix}(v_F), \tag{8.0.2}$$

where v_F ranges over the special vertices in the building $\mathcal{B}(G_{\mathrm{ad}}(F))$.

Proof. A compact subgroup of $G(F)$ is automatically contained in $G(F)^1$. This follows from the alternative description of $G(L)^1$ as the intersection of the kernels of the homomorphisms $|\chi| : G(L) \to \mathbb{R}_{>0}$, where χ ranges over L-rational characters on G.

Thus, using [BT1], Cor. (4.4.1), every maximal compact subgroup $\widetilde{K}$ of $G(F)$ (equiv., of $G(F)^1$) is the stabilizer in $G(F)^1$ of a well-defined facet in the building $\mathcal{B}(G_{\rm der}(F))$. By definition, such a $\widetilde{K}$ is special if and only if the facet it stabilizes is a special vertex v_F. In that case, we have $\widetilde{K} = G(F)^1 \cap {\rm Fix}(v_F)$.

To show the converse, we must check that $G(F)^1 \cap {\rm Fix}(v_F)$ is compact (the argument above will then show it is (special) maximal compact). Recall $K = \mathcal{G}^\circ_{v_F}(\mathcal{O}_F)$ is compact and is given by (8.0.1). Since $G(F)_1 \cap {\rm Fix}(v_F)$ has finite index in $G(F)^1 \cap {\rm Fix}(v_F)$, and since the former is compact, so is the latter. This completes the proof. □

Remark 8.0.2. Equation (8.0.1) can be generalized. Let $\mathbf{a}_J$ denote any σ-stable alcove in $\mathcal{B}(G(L))$. Then

$$\mathcal{G}^\circ_{\mathbf{a}_J}(\mathcal{O}_F) = G(F)_1 \cap {\rm Fix}(\mathbf{a}_J^\sigma).$$

However, if $\mathcal{G}^\circ_{\mathbf{a}_J}$ is replaced with the "full-fixer" group scheme $\widehat{\mathcal{G}}_{\mathbf{a}_J}$ (cf. [BT2], 4.6.28, 5.1.29), the corresponding statement

$$\widehat{\mathcal{G}}_{\mathbf{a}_J}(\mathcal{O}_F) = G(F)^1 \cap {\rm Fix}(\mathbf{a}_J^\sigma)$$

is false. Indeed, the right-hand side, a general analogue of our $\widetilde{K}$ above, can be strictly larger than the left-hand side. For example, consider the anisotropic group $G = D^\times/F^\times$ of Remark 11.1.3, and let $\mathbf{a}_J^\sigma = v_F$, and $\mathbf{a}_J = \mathbf{a}$. Then the right-hand side is $G(F)$, but the left-hand side is a subgroup of index $n = \sqrt{\dim_F(D)}$, namely $\mathcal{O}_D^\times/\mathcal{O}_F^\times$.

9. Statement of the Satake isomorphism

In this section, let $P = MN$ denote any F-rational parabolic subgroup of G with unipotent radical N, which has M as a Levi factor.

9.1. **Iwasawa decomposition.**

In light of Lemma 8.0.1, the following version of the Iwasawa decomposition can be derived easily from similar statements in the literature (cf. [BT1], Rem. (4.4.5) or Prop. (7.3.1)):

Proposition 9.1.1. *There is an equality of sets*

$$G(F) = P(F) \cdot \widetilde{K}(F).$$

We need the variant of this where $\widetilde{K}(F)$ is replaced by $K(F)$. It will be enough to prove that

$$\widetilde{K}(F) = (\widetilde{K} \cap M(F)) \cdot K(F).$$

Using (3.0.1) together with Lemma 3.0.1, we see that any element $\tilde{k} \in \widetilde{K}(F)$ satisfies

$$\tilde{k} \in K(F) m K(F)$$

for some $m \in \Omega_M^\sigma \subset M(F)$. It follows that $m \in \widetilde{K}(F)$, and then since $\widetilde{K}(F)$ normalizes $K(F)$ (cf. e.g. Lemma 8.0.1), we see that $\tilde{k} \in mK(F)$ as desired.

We have thus proved the first part of the following corollary.

Corollary 9.1.2 (Iwasawa decomposition). *There is an equality of sets*

$$G(F) = P(F) \cdot K(F).$$

Moreover, $P(F) \cap K(F) = (M(F) \cap K) \cdot (N(F) \cap K)$.

Proof. We need only show the second equality, which can be rewritten as

$$P(F) \cap \mathcal{G}^{\circ}_{v_F}(\mathcal{O}_F) = (M(F) \cap \mathcal{G}^{\circ}_{v_F}(\mathcal{O}_F)) \cdot (N(F) \cap \mathcal{G}^{\circ}_{v_F}(\mathcal{O}_F)).$$

This follows from [BT2], 5.2.4 (taking the set denoted by Ω there to be $\{v_F\}$). □

9.2. **Construction of the Satake transform.** We will follow the approach taken in [HKP], which treated the case of F-split groups.

Recall that $\mathcal{H}_K := C_c(K(F)\backslash G(F)/K(F))$, the spherical Hecke algebra of $K(F)$-bi-invariant compactly-supported functions on $G(F)$. The convolution is defined using the Haar measure on $G(F)$ which gives $K(F)$ volume 1.

Set $R := \mathbb{C}[M(F)/M(F)_1]$. Since $M(F)_1$ is the unique parahoric subgroup of $M(F)$, this is just the Iwahori-Hecke algebra for $M(F)$. Let

$$\mathbf{M} := C_c(M(F)_1 N(F)\backslash G(F)/K(F)),$$

where the subscript "c" means we consider functions supported on finitely many double cosets. Then $\mathbf{M}$ carries an obvious right convolution action under $\mathcal{H}_K$. It also carries a left action by R given by normalized convolutions:

$$r \cdot \phi(m) := \int_{M(F)} \delta_P^{1/2}(m_1)\, r(m_1)\, \phi(m_1^{-1}m)\, dm_1.$$

Here dm_1 is the Haar measure on $M(F)$ giving $M(F)_1$ volume 1, and δ_P is the modular function on $P(F)$ given by the normalized absolute value of the determinant of the adjoint action on $\mathrm{Lie}(N(F))$. For $m \in M(F)$ we have

$$\delta_P(m) := |\det(\mathrm{Ad}(m)\ ;\ \mathrm{Lie}(N(F)))|_F.$$

The actions of R and $\mathcal{H}_K$ on $\mathbf{M}$ commute, so that $\mathbf{M}$ is an $(R, \mathcal{H}_K)$-bimodule.

Lemma 9.2.1. *The R-module $\mathbf{M}$ is free of rank 1, with canonical generator*

$$v_1 := \mathrm{char}(M(F)_1\, N(F)\, K(F)).$$

Proof. This follows directly from Corollary 9.1.2. □

Given $f \in \mathcal{H}_K$, let $f^\vee \in R$ denote the unique element satisfying the identity

$$v_1 f = f^\vee v_1. \tag{9.2.1}$$

It is obvious that

$$\begin{aligned} \mathcal{H}_K &\to R \\ f &\mapsto f^\vee \end{aligned}$$

is a $\mathbb{C}$-algebra homomorphism.

Evaluating both sides of (9.2.1) on $m \in M(F)$ and using the usual $G = MNK$ integration formula (see [Car]), we get the familiar expression

$$f^\vee(m) = \delta_P^{-1/2}(m) \int_{N(F)} f(nm)\, dn = \delta_P^{1/2}(m) \int_{N(F)} f(mn)\, dn, \tag{9.2.2}$$

where dn gives $N(F) \cap K(F)$ measure 1.

10. The Satake transform is an isomorphism

10.1. **Weyl group invariance.** The first step is to prove that $f^\vee$ belongs to the subring $R^{W(G,A)}$ of $W(G,A)$-invariants in R. Once this is proved, the functoriality of the Kottwitz homomorphism

$$\kappa_M : M(F)/M(F)_1 \widetilde{\to} X^*(Z(\widehat{M}))_I^\sigma$$

shows that $f^\vee \in \mathbb{C}[X^*(Z(\widehat{M}))_I^\sigma]^{W(G,A)}$, as well.

The argument is virtually the same as Cartier's [Car]. Define a function on $m \in M(F)$ by

$$D(m) = |\det(\mathrm{Ad}(m) - 1\,; \text{Lie } G(F)/\text{Lie } M(F))|^{1/2}.$$

Then exactly as in loc. cit. one can prove the formula

$$f^\vee(m) = D(m) \int_{G/A} f(gmg^{-1}) \, \frac{dg}{da} \tag{10.1.1}$$

on the Zariski-dense subset of elements $m \in M(F)$ which are regular semi-simple as elements in G. Here dg (resp. da) is the Haar measure on $G(F)$ (resp. $A(F)$) which gives K (resp. $K \cap A(F)$) volume 1. By Lemma 3.0.1 (I), every element $w \in W(G,A)$ can be represented by an $x \in N_G(A) \cap K$. Clearly, $D(m) = D(xmx^{-1})$. Since the measure on G/A is invariant under conjugation by x, we see as in loc. cit. that the integral in (10.1.1) is also invariant under $m \mapsto xmx^{-1}$. Thus (10.1.1) is similarly invariant, as desired.

Remark 10.1.1. As in the case of $\mathcal{H}_{\widetilde{K}}$, equation (10.1.1) also shows that $f^\vee$ is independent of the choice of F-rational parabolic subgroup P which contains M as a Levi factor.

10.2. **Upper triangularity.** The second step is to show that with respect to natural $\mathbb{C}$-bases of $\mathcal{H}_K$ and $R^{W(G,A)}$, the map $f \mapsto f^\vee$ is "invertible upper triangular", hence is an isomorphism of algebras.

The set $\widetilde{W}_K^\sigma \backslash \widetilde{W}^\sigma / \widetilde{W}_K^\sigma \cong W(G,A)\backslash\Omega_M^\sigma$ provides a natural $\mathbb{C}$-basis for $\mathcal{H}_K$ and for $R^{W(G,A)}$. Recall that $\widetilde{W}$ has a natural structure of a *quasi-Coxeter group*

$$\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$$

(cf. [HR], Lemma 14). We extend the Bruhat order $\leq$ and the length function ℓ from W_{aff} to $\widetilde{W}$ in the usual way (cf. loc. cit.). Given $x \in \widetilde{W}$, denote by $\tilde{x} \in \widetilde{W}$ the unique minimal element in $\widetilde{W}_K x \widetilde{W}_K$. (Note that $\widetilde{W}_K$ is finite and that the usual theory of such minimal elements for Coxeter groups goes over to handle quasi-Coxeter groups.)

By [HR], Remark 9, we may regard $\widetilde{W}_K^\sigma \backslash \widetilde{W}^\sigma / \widetilde{W}_K^\sigma$ as a subset (the σ-invariant elements) in $\widetilde{W}_K \backslash \widetilde{W} / \widetilde{W}_K$. For $y, y' \in W(G,A)\backslash\Omega_M^\sigma$, resp., $x, x' \in \widetilde{W}_K^\sigma \backslash \widetilde{W}^\sigma / \widetilde{W}_K^\sigma$, we define the partial order $\preceq$ by requiring

$$y \preceq y' \Leftrightarrow \tilde{y} \leq \tilde{y}', \quad \text{resp.}$$
$$x \preceq x' \Leftrightarrow \tilde{x} \leq \tilde{x}'.$$

The set $W(G,A)\backslash\Omega_M^\sigma$ is countable and every element y has only finitely many predecessors with respect to the partial order $\preceq$. Therefore, there is a total ordering $y_1, y_2, \ldots$ on this set which is compatible with $\preceq$, meaning that $y_i \preceq y_j$ only if

$i \leq j$. Similar remarks apply to the partially ordered set $\widetilde{W}_K^\sigma \backslash \widetilde{W}^\sigma / \widetilde{W}_K^\sigma$, and we get an analogous total ordering $x_1, x_2, \dots$ for it.

We claim that the matrix for $f \mapsto f^\vee$ in terms of the bases $\{y_i\}_1^\infty$ and $\{x_i\}_1^\infty$ is upper triangular and invertible. The upper triangularity is the content of the next lemma.

Lemma 10.2.1. *Suppose $x \in \widetilde{W}^\sigma$ and $y \in \Omega_M^\sigma$ and that*

$$N(F)yK(F) \cap K(F)xK(F) \neq \emptyset. \tag{10.2.1}$$

Then $\tilde{y} \leq \tilde{x}$.

Proof. Let I denote the Iwahori subgroup of $G(L)$ associated to the σ-stable alcove $\mathbf{a}$, as defined earlier. We shall need two BN-pair relations. The first is the relation

$$K(L) = I(L)\, \widetilde{W}_K\, I(L). \tag{10.2.2}$$

This follows easily using [HR], Prop. 8. The second is the relation

$$I(L)\, w\, I(L)\, w'\, I(L) \subseteq \coprod_{w'' \leq w'} I(L)\, w\, w''\, I(L). \tag{10.2.3}$$

This relation per se does not appear in the literature, but it follows easily from the BN-pair relations established in [BT2], 5.2.12 (cf. [HR], paragraph following Lemma 17).

Using (10.2.2) and (10.2.3) we see that (10.2.1) implies that

$$N(L)yI(L) \cap I(L)\, x'\, I(L) \neq \emptyset \tag{10.2.4}$$

for some $x' \in \widetilde{W}_K x \widetilde{W}_K$. Write

$$ny = i\, x'\, i' \tag{10.2.5}$$

for $n \in N(L)$, and $i, i' \in I(L)$. Choose a cocharacter $\lambda \in X_*(A)$ such that $\varpi^\lambda n \varpi^{-\lambda} \in I(L)$. Then multiplying (10.2.5) by ϖ^λ we see that

$$I(L)\, \varpi^\lambda y\, I(L) \subseteq I(L)\, \varpi^\lambda\, I(L)\, x'\, I(L).$$

Using (10.2.3) again we deduce that

$$I(L)\, \varpi^\lambda y\, I(L) = I(L)\, \varpi^\lambda x''\, I(L)$$

and hence $y = x''$ for some $x'' \in \widetilde{W}$ with $x'' \leq x'$. Thus $\tilde{y} \leq x'$. A standard argument then shows that $\tilde{y} \leq \tilde{x}$, which is what we wanted to prove. □

Finally, the invertibility follows from the obvious fact that

$$N(F)xK(F) \cap K(F)xK(F) \neq \emptyset.$$

This completes the proof that $f \mapsto f^\vee$ is an isomorphism. □

11. The structure of Λ_M

It is clear that $\Lambda_M = X^*(Z(\widehat{M}))_I^\sigma$ is a finitely-generated abelian group. In this section we make it more concrete in various situations.

11.1. **General results.** As before, in this subsection T denotes the centralizer in G of the torus S. Recall that we can assume S is defined over F, and so T is also defined over F. Recall also that T_{sc}^M denotes the pull-back of T via $M_{\mathrm{sc}} \to M$.

Lemma 11.1.1. *There is an embedding $X_*(T)_I^\sigma \hookrightarrow \Lambda_M$ whose cokernel is isomorphic to the finite abelian group* $\ker[X_*(T_{\mathrm{sc}}^M)_\Gamma \to X_*(T)_\Gamma]$.

Proof. Use the long exact sequence for $H^i(\langle\sigma\rangle, -)$ associated to the short exact sequence

$$0 \longrightarrow X_*(T_{\mathrm{sc}}^M)_I \longrightarrow X_*(T)_I \longrightarrow X^*(Z(\widehat{M}))_I \longrightarrow 0.$$

(For a discussion of this short exact sequence, see [HR], proof of Prop. 13.) Note that $X_*(T_{\mathrm{sc}}^M)_I^\sigma \subset W_{M,\mathrm{aff}}^\sigma = 1$ (cf. Lemma 7.1.1). Also, $X_*(T_{\mathrm{sc}}^M)_\Gamma$ is finite because M_{sc} is anisotropic over F. The lemma follows easily using these remarks. □

Corollary 11.1.2. (a) *If G is quasi-split over F, then $\Lambda_M = X_*(T)_I^\sigma$.*

(b) *If G is split over L, then Λ_M fits into the exact sequence*

$$1 \to X_*(A) \to \Lambda_M \to \ker[X_*(T_{\mathrm{sc}}^M)_\sigma \to X_*(T)_\sigma] \to 0.$$

(c) *If G is unramified over F, then $\Lambda_M = X_*(A)$.*

Proof. Part (a). Since G is quasi-split over F, we have $M = T$, and the desired formula follows directly from the definition of Λ_M.

Part (b) follows immediately from Lemma 11.1.1.

Part (c) follows as a special case of either (a) or (b). Part (c) was known previously (cf. [Bo], 9.5). □

Remark 11.1.3. If G is semi-simple and anisotropic, then Λ_M is finite. There are examples, namely $G = D^\times/F^\times$ for D a central simple division algebra over F with $\dim_F(D) > 1$, where $\Lambda_M \neq 0$.

At the opposite extreme, let E/F denote a finite totally ramified extension. Consider the "diagonal" embedding $\mathbb{G}_m \hookrightarrow \mathrm{R}_{E/F}\mathbb{G}_m$ and set $G = (\mathrm{R}_{E/F}\mathbb{G}_m)/\mathbb{G}_m$. Then Λ_G is torsion, and non-zero if $E \neq F$.

The next proposition tells us how to measure the difference between the subgroups K and $\widetilde{K}$ of $G(F)$ attached to a special vertex v_F. This will complete the proof of Theorem 1.0.1. For an abelian group H let H_{tor} denote its torsion subgroup.

Proposition 11.1.4. *There is a set-theoretic inclusion $\Omega_{M,\mathrm{tor}}^\sigma \subset \widetilde{K}$ which induces an isomorphism of groups*

$$\Lambda_{M,\mathrm{tor}} \xrightarrow{\sim} \widetilde{K}/K.$$

Proof. Clearly, $\Omega_{M,\mathrm{tor}}^\sigma$ lies in $M(F)^1$ hence in $G(F)^1$. Also, every element of $M(F)^1$ acts trivially on the apartment $\mathcal{A}_L^\sigma$, and in particular, fixes $\mathbf{a}_0^\sigma$. This shows that $\Omega_{M,\mathrm{tor}}^\sigma \subset \mathrm{Fix}^{G(F)}(v_F) \cap G(F)^1 = \widetilde{K}$ (cf. Lemma 8.0.1).

We claim the induced homomorphism $\Omega_{M,\mathrm{tor}}^\sigma \to \widetilde{K}/K$ is an isomorphism. It is injective because

$$\Omega_M \cap K = \Omega_M \cap M(F) \cap K = \Omega_M \cap M(F)_1 = \{1\}$$

(cf. Lemma 4.2.1).

Let us prove surjectivity. Any coset in $\widetilde{K}/K$ can be represented by an element $x \in \Omega_M^\sigma$. We need to show this element is torsion. Let r be such that $x^r \in K$. But then $x^r \in \Omega_M^\sigma \cap K = \{1\}$ (see above), and we are done. □

Corollary 11.1.5. *If M_L is L-split group and $M_{\mathrm{der}} = M_{\mathrm{sc}}$, then Λ_M is torsion-free, and for every special vertex v_F, we have $\widetilde{K}_{v_F} = K_{v_F}$.*

Proof. We have

$$X^*(Z(\widehat{M}))_I = X^*(Z(\widehat{M})) \tag{11.1.1}$$

and the latter is torsion free since $M_{\mathrm{der}} = M_{\mathrm{sc}}$ is equivalent to $Z(\widehat{M})$ being connected. □

Remark 11.1.6. The hypotheses on M hold if $G_{\mathrm{der}} = G_{\mathrm{sc}}$ and G_L is an L-split group.

Corollary 11.1.7. *If $G = G_{\mathrm{sc}}$, then $\widetilde{K} = K$ and Λ_M is torsion-free.*

Proof. Observe that since $Z(\widehat{G}) = 1$ we have $G(F)_1 = G(F)^1 = G(F)$. Then use (8.0.1) and (8.0.2). □

Of course, this corollary was already known (cf. [BT2], 4.6.32).

11.2. **Passing to inner forms.** It is of interest to describe Λ_M explicitly in terms of an appropriate maximal torus $\widehat{T}$ in $\widehat{G}$. For quasi-split groups this has been done in Corollary 11.1.2, (a), which proves that $\Lambda_M = X^*(\widehat{T})_I^\sigma = X^*(\widehat{T}^I)^\sigma$. Here we study the effect of passing to an inner form of a quasi-split group.

Thus, we fix a connected reductive group G^* which is quasi-split over F. Recall that an inner form of G^* is a pair (G, Ψ) consisting of a connected reductive F-group G and a Γ-stable $G^*_{\mathrm{ad}}(F^s)$-orbit Ψ of F^s-isomorphisms $\psi : G \to G^*$. The set of isomorphism classes of inner forms of G^* corresponds bijectively to the set $H^1(F, G^*_{\mathrm{ad}})$, by the rule which sends (G, Ψ) to the 1-cocycle $\tau \mapsto \psi \circ \tau(\psi)^{-1}$ for any $\psi \in \Psi$ (cf. [Ko97], 5.2).

Now assume (G, Ψ) is an inner form of G^*. Denote the action of $\tau \in \Gamma$ on $G(F^s)$ (resp. $G^*(F^s)$) by τ (resp. τ^*).

Let A be a maximal F-split torus in G, and let S denote a maximal F^{un}-split torus in G which is defined over F and contains A. Such a torus S exists by [BT2], 5.1.12, noting that any F-torus which is split over L is already split over F^{un}. Let $T = C_G(S)$ and $M = C_G(A)$. Then T is a maximal torus of G, since the group $G_{F^{\mathrm{un}}}$ is quasi-split. Let A^*, S^*, T^* have the corresponding meaning for the group G^*, and assume that T^* is contained in an F-rational Borel subgroup $B^* = T^*U^*$ of G^*. Of course $T^* = C_{G^*}(A^*)$ since G^* is quasi-split over F.

Let $P = MN$ be an F-rational parabolic subgroup of G having Levi factor M and unipotent radical N. Let P^* be the unique standard F-rational parabolic subgroup of G^* which is $G^*(F^s)$-conjugate to $\psi(P)$ for all $\psi \in \Psi$ (cf. [Bo], section 3). Let M^* denote the unique Levi factor of P^* which contains T^*. Let Ψ_M denote the set of $\psi \in \Psi$ such that $\psi(P) = P^*$ and $\psi(M) = M^*$. Then Ψ_M is a non-empty Γ-stable $M^*_{\mathrm{ad}}(F^s)$-orbit of F^s-isomorphisms $M \to M^*$; hence M is an inner form of the F-quasi-split group M^*.

It is clear that $G_{F^{\mathrm{un}}}$ and $G^*_{F^{\mathrm{un}}}$ are isomorphic, since they are inner forms of each other and are both quasi-split (cf. [Tits], 1.10.3). In fact, it is easy to see that any inner twisting $G_{F^{\mathrm{un}}} \xrightarrow{\sim} G^*_{F^{\mathrm{un}}}$ over F^{un} is $G^*(F^s)$-conjugate to an *isomorphism* of

$F^{\rm un}$-groups. For this a key fact is that the image $T^*_{\rm ad}$ of T^* in $G^*_{{\rm ad},F^{\rm un}}$ is an induced $F^{\rm un}$-torus. The same remarks obviously apply to $M_{F^{\rm un}}$ and $M^*_{F^{\rm un}}$. Hence we may choose $\psi_0 \in \Psi_M$ such that $\psi_0 : M \to M^*$ is an $F^{\rm un}$-isomorphism and $\psi_0(S) = S^*$ (and thus also $\psi_0(T) = T^*$). Since ψ_0 restricted to A is defined over F, we see that $\psi_0(A)$ is an F-split subtorus of T^* and hence $\psi_0(A) \subseteq A^*$.

Let $\tilde{\sigma}$ denote any lift in Γ of the Frobenius element $\sigma \in {\rm Gal}(F^{\rm un}/F)$. We may write

$$\psi_0 \circ \tilde{\sigma}(\psi_0)^{-1} = \psi_0 \circ \sigma(\psi_0)^{-1} = {\rm Int}(m^*_\sigma)$$

for an element $m^*_\sigma \in N_{M^*}(S^*)(F^s)$ whose image in $M^*_{\rm ad}(F^s)$ is well defined. As operators on $X_*(T^*) = X^*(\widehat{T^*})$, we may write

$$\psi_0 \circ \sigma(\psi_0)^{-1} = w^*_\sigma \tag{11.2.1}$$

for a well-defined element $w^*_\sigma \in W(M^*, S^*)(F^{\rm un})$. Denote by w_σ the preimage under the isomorphism $\psi_0 : W(M,S)(F^{\rm un}) \xrightarrow{\sim} W(M^*,S^*)(F^{\rm un})$ of w^*_σ. Then (11.2.1) translates into the equality

$$\sigma \circ \psi_0^{-1} \circ (\sigma^*)^{-1} \circ \psi_0 = w_\sigma \tag{11.2.2}$$

of operators on $X_*(T) = X^*(\widehat{T})$. In defining $w_\sigma \in W(M,S)$, we fixed the objects A and S (needed to specify the ambient group $W(M,S)$) and along the way we also chose several additional objects: P, A^*, S^*, B^*, and an element $\psi_0 \in \Psi_M$ such that $\psi_0(S) = S^*$ and $\psi_0 : M \to M^*$ is $F^{\rm un}$-rational. It is straightforward to check that the element $w_\sigma \in W(M,S)$ is independent of all of these additional choices.

11.3. **Inner forms of split groups.** In this subsection we assume G^* is F-split. Then $A^* = S^* = T^*$, and $G_{F^{\rm un}}$ and $M_{F^{\rm un}}$ are split groups. In particular, the relative Weyl group $W(M^*,S^*)$ coincides with the absolute Weyl group $W(M^*,T^*)$. Using ψ_0 as above, we may regard w_σ as an element of $W(M,S) = W(M,T)^I = W(\widehat{M},\widehat{T})^I$.

For the next lemma, we need to recall the notion of cuspidal elements of Weyl groups. Let (W,S) be any Coxeter group with a finite set S of simple reflections. We say $w \in W$ is *cuspidal* if every conjugate of w is elliptic, that is, every conjugate w' has the property that any reduced expression for w' contains every element of S. Note that the identity element of W is cuspidal if and only if $S = \emptyset$, in which case W itself is trivial.

Lemma 11.3.1. (a) *The element w_σ is a cuspidal element of the absolute Weyl group $W(M,T)$ of M.*

(b) *The group M is of type A and the element w_σ is a Coxeter element of $W(M,T)$.*

(c) *We have the equality $Z(\widehat{M}) = \widehat{T}^{w_\sigma}$.*

Proof. Part (a). We may assume $M \neq T$ and hence $W(M,T)$ is not trivial. Suppose the assertion is false. Then there is a notion of simple positive root for M, T and a corresponding Coxeter group structure on $W(M,T)$, for which w_σ is not an elliptic element. Let s_i denote a simple reflection in $W(M,T)$ which does not appear in a reduced expression for w_σ. Then the corresponding fundamental coweight $\lambda_i \in X_*(T/Z(M))$ for $M_{\rm ad}$ is fixed by w_σ. It is also fixed by $\psi_0^{-1} \circ (\sigma^*)^{-1} \circ \psi_0$. Thus by (11.2.2) λ_i is fixed by σ, and $\lambda_i(\mathbb{G}_m)$ is an F-split torus in $M_{\rm ad}$. This contradicts the fact that $M_{\rm ad}$ is anisotropic over F.

Part (b). Since every anisotropic F-group is type A (cf. Kneser [Kn] and Bruhat-Tits [BT3], 4.3), the group M is type A. For type A groups, every cuspidal element in the Weyl group is Coxeter, as may be seen using cycle decompositions of permutations. Thus, the cuspidal element w_σ is a Coxeter element of $W(M,T)$.

Part (c). It is enough to prove the following statement: if $\mathcal{G}$ is a type A connected reductive complex group with maximal torus $\mathcal{T}$, and if $w \in W(\mathcal{G},\mathcal{T})$ is a Coxeter element, then $Z(\mathcal{G}) = \mathcal{T}^w$. First, if $\mathcal{G} = \mathrm{PGL}_n$, a simple computation shows that $\mathcal{T}^w = 1 = Z(\mathcal{G})$. Since $\mathcal{G}_{\mathrm{ad}}$ is a product of projective linear groups and w corresponds to a product of Coxeter elements, this also handles the case of adjoint groups. In the general case, note that an element $t \in \mathcal{T}^w$ maps to $(\mathcal{T}_{\mathrm{ad}})^w = 1$ in $\mathcal{G}_{\mathrm{ad}}$, hence $t \in \ker(\mathcal{G} \to \mathcal{G}_{\mathrm{ad}}) = Z(\mathcal{G})$. □

Corollary 11.3.2. *If G is an inner form of an F-split group, then*

$$\Lambda_M = X^*(Z(\widehat{M})) = X^*(\widehat{T}^\sigma) = X_*(T)_\sigma.$$

Proof. The element σ^* acts trivially on $Z(\widehat{M}) \hookrightarrow \widehat{T^*}$, since T^* is F-split. Moreover, $w_\sigma \in W(M,T)$ acts trivially on $X^*(Z(\widehat{M}))$. Then using (11.2.2) it follows that σ acts trivially on $X^*(Z(\widehat{M}))_I = X^*(Z(\widehat{M}))$. This proves the first equality.

The second equality follows similarly using Lemma 11.3.1, (c), and the third equality is apparent. □

12. The transfer homomorphism

Now we return to the conventions and notation of subsection 11.2. Let $\mathcal{A}_L^S$ (resp. $\mathcal{A}_L^{S^*}$) denote the apartment of $\mathcal{B}(G(L))$ (resp. $\mathcal{B}(G^*(L))$) corresponding to S (resp. S^*). The twisting ψ_0 gives an isomorphism $X_*(S)_\mathbb{R} \to X_*(S^*)_\mathbb{R}$ of the real vector spaces underlying these apartments. Let K (resp. K^*) denote a special maximal parahoric subgroup of $G(F)$ (resp. $G^*(F)$) corresponding to a special vertex in $(\mathcal{A}_L^S)^\sigma$ (resp. $(\mathcal{A}_L^{S^*})^{\sigma^*}$). Then our goal is to define a canonical algebra homomorphism

$$t : \mathcal{H}_{K^*}(G^*) \to \mathcal{H}_K(G).$$

We expect t will play a role in the study of Shimura varieties with parahoric level structure and in some related problems in p-adic harmonic analysis. These issues will be addressed on another occasion.

12.1. Relating the relative Weyl groups for G^* and G.

Proposition 12.1.1. *Any twist $\psi_0 \in \Psi_M$ induces a map*

$$W(G,A) \to W(G^*,A^*)/W(M^*,A^*).$$

Proof. For $w \in W(G,A)$, choose a lift $n \in N_G(S)^\sigma$ (cf. Lemma 5.0.1). Write

$$\sigma \circ \psi_0^{-1} \circ (\sigma^*)^{-1} \circ \psi_0 = \mathrm{Int}(m_\sigma)$$

for an element $m_\sigma \in N_M(S)(F^s)$. Set $m_* = \psi_0(\sigma^{-1}(m_\sigma)) \in N_{M^*}(S^*)(F^s)$. Using $\sigma(n) = n$ and the fact that $\psi_0(n)$ normalizes M^*, we obtain

$$\begin{aligned}(\sigma^*)^{-1}(\psi_0(n)) &= m_*\, \psi_0(n)\, m_*^{-1} \\ &= \psi_0(n) \cdot (\psi_0(n)^{-1} m_* \psi_0(n) m_*^{-1}) \\ &\in \psi_0(n)\, N_{M^*}(S^*).\end{aligned}$$

Thus $n \mapsto \psi_0(n)$ induces a well-defined map

$$W(G, A) \to \Big(W(G^*, S^*)/W(M^*, S^*)\Big)^{\sigma^*}.$$

The natural map $W(G^*, S^*)^{\sigma^*} \to \Big(W(G^*, S^*)/W(M^*, S^*)\Big)^{\sigma^*}$ is surjective. Indeed, the choice of an F-rational Borel subgroup of G^* containing T^* gives us a notion of length on $W(G^*, S^*)$ which is preserved by σ^*, so that the minimal-length representatives of σ^*-fixed cosets in $W(G^*, S^*)/W(M^*, S^*)$ are fixed by σ^*. It follows that

$$W(G^*, S^*)^{\sigma^*}/W(M^*, S^*)^{\sigma^*} = \Big(W(G^*, S^*)/W(M^*, S^*)\Big)^{\sigma^*}.$$

Thus, we have a well-defined map

$$W(G, A) \to W(G^*, S^*)^{\sigma^*}/W(M^*, S^*)^{\sigma^*} = W(G^*, A^*)/W(M^*, A^*)$$

(cf. Remark 6.1.3). □

12.2. **Definition of $t : \mathcal{H}_{K^*}(G^*) \to \mathcal{H}_K(G)$.** The isomorphism

$$\widehat{\psi_0} : Z(\widehat{M^*}) \xrightarrow{\sim} Z(\widehat{M})$$

is Galois-equivariant. Combined with the canonical inclusion $Z(\widehat{M^*}) \hookrightarrow \widehat{T^*}$ we see that $\widehat{\psi_0}$ induces a homomorphism

$$\psi_0 : X^*(\widehat{T^*})_I^{\sigma^*} \to X^*(Z(\widehat{M}))_I^{\sigma}. \tag{12.2.1}$$

Since $W(M^*, A^*)$ induces the trivial action on $Z(\widehat{M^*})$, it follows using Proposition 12.1.1 that (12.2.1) is equivariant with respect to the map $W(G, A) \to W(G^*, A^*)/W(M^*, A^*)$, in an obvious sense. We thus get an algebra homomorphism

$$\psi_0 : \mathbb{C}[X^*(\widehat{T^*})_I^{\sigma^*}]^{W(G^*,A^*)} \to \mathbb{C}[X^*(Z(\widehat{M}))_I^{\sigma}]^{W(G,A)}. \tag{12.2.2}$$

Since Ψ_M is a torsor for M^*_{ad}, one can check that this homomorphism is independent of the choice of ψ_0 in Ψ_M. In fact, it depends only on the choice of A and A^*. Therefore, it makes sense to denote it by $t_{A^*,A}$ in what follows. It is easy to check that this homomorphism is surjective when G^* is split over F.

Definition 12.2.1. Fix A and A^* as above. Define $t : \mathcal{H}_{K^*}(G^*) \to \mathcal{H}_K(G)$ to be the unique homomorphism making the following diagram commute

$$\begin{array}{ccc}
\mathcal{H}_{K^*}(G^*) & \xrightarrow{\quad t \quad} & \mathcal{H}_K(G) \\
\wr\downarrow & & \wr\downarrow \\
\mathbb{C}[X^*(\widehat{T^*})_I^{\sigma^*}]^{W(G^*,A^*)} & \xrightarrow{t_{A^*,A}} & \mathbb{C}[X^*(Z(\widehat{M}))_I^{\sigma}]^{W(G,A)},
\end{array}$$

where the vertical arrows are the Satake isomorphisms.

Obviously, t depends on K and K^*. It is easy to see that t is independent of all other choices used in its construction. Also, if G^* is split over F, t is surjective.

12.3. **Compatibilities with constant term homomorphisms.** Let A, A^*, K, and K^* be fixed as above. Let H be a semi-standard F-Levi subgroup of G; this means that $H = C_G(A_H)$ for some subtorus $A_H \subseteq A$. Let H^* be a semi-standard F-Levi subgroup of G^*, so that $H^* = C_{G^*}(A^*_{H^*})$ for a subtorus $A^*_{H^*} \subset A^*$. We have $M \subseteq H$ and $T^* \subseteq H^*$. Let us suppose that some inner twist $G \to G^*$ restricts to give an inner twist $H \to H^*$.

For example, for any $\psi_0 \in \Psi_M$ as above, we could take A_H to be any subtorus of A and set $A^*_{H^*} = \psi_0(A_H)$ (recalling that $\psi_0(A) \subseteq A^*$).

Choose any F-rational parabolic subgroup $P_H = HN_H$ of G with unipotent radical N_H which contains H as a Levi factor. Recall the constant term map $c^G_H : \mathcal{H}_K(G) \to \mathcal{H}_{H\cap K}(H)$, which is defined by

$$c^G_H(f)(h) = \delta^{1/2}_{P_H}(h) \int_{N_H(F)} f(hn)\, dn, \tag{12.3.1}$$

for $h \in H(F)$, where the Haar measure dn on $N_H(F)$ gives $N_H(F) \cap K$ measure 1. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_K(G) & \xrightarrow{\sim} & \mathbb{C}[\Lambda_M]^{W(G,A)} \\ {\scriptstyle c^G_H}\downarrow & & \downarrow \\ \mathcal{H}_{H\cap K}(H) & \xrightarrow{\sim} & \mathbb{C}[\Lambda_M]^{W(H,A)}, \end{array} \tag{12.3.2}$$

where the horizontal arrows are the Satake isomorphisms, and the right vertical arrow is the inclusion homomorphism. It follows that c^G_H is an injective algebra homomorphism which is independent of the choice of F-rational parabolic subgroup $P_H \subseteq G$ which contains H as a Levi factor.

The following proposition is proved using (12.3.2) and the definitions.

Proposition 12.3.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{H}_{K^*}(G^*) & \xrightarrow{t} & \mathcal{H}_K(G) \\ {\scriptstyle c^{G^*}_{H^*}}\downarrow & & \downarrow{\scriptstyle c^G_M} \\ \mathcal{H}_{H^*\cap K^*}(H^*) & \xrightarrow{t} & \mathcal{H}_{H\cap K}(H). \end{array}$$

□

Taking $H = M$, the diagram shows that in order to compute t, it is enough to compute it in the case where G_{ad} is anisotropic. In that case, if $f \in \mathcal{H}_{K^*}(G^*)$, the function $t(f)$ is given by summing f over the fibers of the Kottwitz homomorphism $k_{G^*}(F)$.

References

[Bo] A. Borel, *Automorphic L-functions*, In: *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math., vol. **33**, part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 27-61. MR546608 (81m:10056)

[Bou] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 4, 5, et 6, Masson, 1981. MR647314 (83g:17001)

[BT1] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251. MR0327923 (48:6265)

[BT2] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. II*, Inst. Hautes Études Sci. Publ. Math. **60** (1984), 5-184.

[BT3] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. III*, Journal Fac. Science University Tokyo **34** (1987), 671-698. MR927605 (89b:20099)

[Car] P. Cartier, *Representations of* $\mathfrak{p}$*-adic groups: a survey* In: *Automorphic Forms, Representations and L-functions*, Proc. Sympos. Pure Math., vol. **33**, part 1, Amer. Math. Soc., Providence, RI, 1979, pp. 111–155. MR546593 (81e:22029)

[H05] T. Haines, *Introduction to Shimura varieties with bad reduction of parahoric type*, Clay Math. Proc. **4**, (2005), 583-642. MR2192017 (2006m:11085)

[H09] T. Haines, *The base change fundamental lemma for central elements in parahoric Hecke algebras*, Duke Math. J. **149**, no. 3 (2009), 569-643. MR2553880

[HKP] T. Haines, R. Kottwitz, A. Prasad, *Iwahori-Hecke algebras*, math.RT/0309168. Preprint.

[HR] T. Haines, M. Rapoport, *Appendix: On parahoric subgroups*, Advances in Math. **219** (1), (2008), 188-198; appendix to: G. Pappas, M. Rapoport, *Twisted loop groups and their affine flag varieties*, Advances in Math. **219** (1), (2008), 118-198. MR2435422 (2009g:22039)

[Kn] M. Kneser, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über* $\mathfrak{p}$*-adischen Körpern. II.* Math. Zeitschr. **89**, 250-272 (1965). MR0188219 (32:5658)

[Ko97] R. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109**, 255-339 (1997). MR1485921 (99e:20061)

[Kr] N. Krämer, *Local models for ramified unitary groups*, Abh. Math. Sem. Univ. Hamburg **73** (2003), 67-80. MR2028507 (2004k:14037)

[Land] E. Landvogt, *A compactification of the Bruhat-Tits building*, Lecture Notes in Mathematics **1619**, Springer, 1996, 152 pp. + vii. MR1441308 (98h:20081)

[PR] G. Pappas, M. Rapoport, *Local models in the ramified case. III. Unitary groups.*, J. Inst. Math. Jussieu **8** (2009), no. 3, 507–564. MR2516305

[Rap] M. Rapoport: A guide to the reduction modulo p of Shimura varieties. Astérisque **298** (2005), 271–318. MR2141705 (2006c:11071)

[Tits] J. Tits, *Reductive groups over local fields*, Proc. Symp. Pure. Math. **33** (1979), part. 1, pp. 29-69. MR546588 (80h:20064)

University of Maryland, Department of Mathematics, College Park, Maryland 20742-4015

E-mail address: `tjh@math.umd.edu`

University of Maryland, Department of Mathematics, College Park, Maryland 20742-4015

E-mail address: `srostami@math.umd.edu`