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A FORMULA FOR THE R-MATRIX USING A SYSTEM OF WEIGHT PRESERVING ENDOMORPHISMS

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ABSTRACT. We give a formula for the universal R-matrix of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. This is similar to a previous formula due to Kirillov-Reshetikhin and Levendorskii-Soibelman, except that where they use the action of the braid group element T_{w_0} on each representation V, we show that one can instead use a system of weight preserving endomorphisms. One advantage of our construction is that it is well defined for all symmetrizable Kac-Moody algebras. However, we have only established that the result is equal to the universal R-matrix in finite type.

1. Introduction

Let $\mathfrak{g}$ be a finite type complex simple Lie algebra and $U_q(\mathfrak{g})$ the corresponding quantized universal enveloping algebra. In [KR] and [LS], Kirillov-Reshetikhin and Levendorskii-Soibelman developed a formula for the universal R-matrix

$$R = (X^{-1} \otimes X^{-1})\Delta(X), \tag{1}$$

where X belongs to a completion of $U_q(\mathfrak{g})$. The element X is constructed using the braid group element T_{w_0} corresponding to the longest word of the braid group, and as such only makes sense when $\mathfrak{g}$ is of finite type.

The element X in (1) defines a vector space endomorphism X_V on each representation V of $U_q(\mathfrak{g})$, and in fact X is defined by the system of endomorphisms $\{X_V\}$. Furthermore, any natural system of vector space endomorphisms $\{E_V\}$ can be represented as an element E in a certain completion of $U_q(\mathfrak{g})$ (see [KT]). The action of the coproduct $\Delta(E)$ on a tensor product $V \otimes W$ is then simply $E_{V\otimes W}$. Thus the right side of (1) is well defined if X is replaced by $E = \{E_V\}$.

In this note we consider the case where $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra. We define a system of weight preserving endomorphisms $\Theta = \{\Theta_V\}$ of all integrable highest weight representations V of $U_q(\mathfrak{g})$. When $\mathfrak{g}$ is of finite type, we show that

$$R = (\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta), \tag{2}$$

where the equality means that, for any type **1** finite dimensional modules V and W, the actions of the two sides of (2) on $V \otimes W$ agree. We expect this remains true in other cases, although this has not been proven.

Our endomorphisms Θ_V are not linear over the field $\mathbb{C}(q)$, but are instead compatible with the automorphism which inverts q. For this reason, Θ cannot be realized using an element in a completion of $U_q(\mathfrak{g})$, and it is crucial to work with

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systems of endomorphisms. There is a further technicality in that Θ_V actually depends on a choice of global basis for V. Nonetheless, we give a precise meaning to (2).

This note is organized as follows. In Section 2 we fix notation and conventions. In Section 3 we review the universal R-matrix. In Section 4 we review a method developed by Henriques and Kamnitzer [HK] to construct isomorphisms $V \otimes W \to W \otimes V$. In Section 5 we state some background results on crystal bases and global bases. In Section 6 we construct our endomorphism Θ. In Section 7 we prove our main theorem (Theorem 7.11), which establishes (2) when $\mathfrak{g}$ is of finite type. In Section 8 we briefly discuss future directions for this work.

2. Conventions

We must first fix some notation. For the most part we follow [CP].

• $\mathfrak{g}$ is a symmetrizable Kac-Moody algebra with Cartan matrix $A = (a_{ij})_{i,j \in I}$ and Cartan subalgebra $\mathfrak{h}$.

• $\langle \cdot, \cdot \rangle$ denotes the paring between $\mathfrak{h}$ and $\mathfrak{h}^\star$ and $(\cdot, \cdot)$ denotes the usual symmetric bilinear form on either $\mathfrak{h}$ or $\mathfrak{h}^\star$. Fix the usual elements $\alpha_i \in \mathfrak{h}^\star$ and $H_i \in \mathfrak{h}$, and recall that $\langle H_i, \alpha_j \rangle = a_{ij}$.

• $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1} a_{ij}$ and, for all $\lambda \in \mathfrak{h}^*$, $(\alpha_i, \lambda) = d_i \langle H_i, \lambda \rangle$.

• B is the symmetric matrix $(d_j^{-1} a_{ij})$.

• $\rho \in \mathfrak{h}^*$ satisfies $\langle H_i, \rho \rangle = 1$ for all i. Note that this implies $(\alpha_i, \rho) = d_i$. If A is not invertible this condition does not uniquely determine ρ, and we simply choose any one solution.

• H_ρ is the element of $\mathfrak{h}$ such that, for any $\lambda \in \mathfrak{h}^*$, $\langle H_\rho, \lambda \rangle = (\rho, \lambda)$. In particular, $\langle H_\rho, \alpha_i \rangle = d_i$ for all i.

• $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra associated to $\mathfrak{g}$, generated over $\mathbb{C}(q)$ by E_i, F_i for all $i \in I$, and K_H for H in the coweight lattice of $\mathfrak{g}$. As usual, let $K_i = K_{d_i H_i}$. For convenience, we recall the exact formula for the coproduct

$$\begin{cases} \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \\ \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ \Delta K_H = K_H \otimes K_H, \end{cases} \tag{3}$$

and the commutation relations

$$K_H E_i K_H^{-1} = q^{\langle H, \alpha_i \rangle} E_i \quad \text{and} \quad K_H F_i K_H^{-1} = q^{-\langle H, \alpha_i \rangle} F_i. \tag{4}$$

At times it will be necessary to adjoin a fixed kth root of q to the base field $\mathbb{C}(q)$, where k is twice the dual Coxeter number of $\mathfrak{g}$.

• $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, and $X^{(n)} = \frac{X^n}{[n][n-1]\cdots[2]}$.

• Fix a representation V of $U_q(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$. We say $v \in V$ is a weight vector of weight λ if, for all $H \in \mathfrak{h}$, $K_H(v) = q^{\langle H, \lambda \rangle} v$.

• $\lambda \in \mathfrak{h}^*$ is called a dominant integral weight if $\langle H_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ for all i.

• For each dominant integral weight λ, V_λ is the type **1** irreducible integrable representation of $U_q(\mathfrak{g})$ with highest weight λ.

• B_λ is a fixed global basis for V_λ, in the sense of Kashiwara (see [K]). b_λ and b_λ^{low} are the highest weight and lowest weight elements of B_λ, respectively.

3. The R-matrix

We briefly recall the definition of a universal R-matrix, and the related notion of a braiding.

Definition 3.1. A braided monoidal category is a monoidal category $\mathcal{C}$, along with a natural system of isomorphisms $\sigma^{br}_{V,W} : V \otimes W \to W \otimes V$ for each pair $V, W \in \mathcal{C}$, such that, for any $U, V, W \in \mathcal{C}$, the following two equalities hold:

$$\begin{aligned} (\sigma^{br}_{U,W} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \sigma^{br}_{V,W}) &= \sigma^{br}_{U\otimes V,W}, \\ (\mathrm{Id} \otimes \sigma^{br}_{U,W}) \circ (\sigma^{br}_{U,V} \otimes \mathrm{Id}) &= \sigma^{br}_{U,V\otimes W}. \end{aligned} \tag{5}$$

The system $\sigma^{br} := \{\sigma^{br}_{V,W}\}$ is called a braiding on $\mathcal{C}$.

Let $U_q(\widetilde{\mathfrak{g}) \otimes U_q}(\mathfrak{g})$ be the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of representations $V_\lambda \otimes V_\mu$, for all ordered pairs of dominant integral weights (λ, μ).

Definition 3.2. A universal R-matrix is an element R of $U_q(\widetilde{\mathfrak{g}) \otimes U_q}(\mathfrak{g})$ such that $\sigma^{br}_{V,W} := \mathrm{Flip} \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations.

Note in particular that, since the braiding is an isomorphism, R must be invertible. It is central to the theory of quantized universal enveloping algebras that, for any symmetrizable Kac-Moody algebra $\mathfrak{g}$, $U_q(\mathfrak{g})$ has a universal R-matrix. The universal R-matrix is not truly unique, but there is a well-studied standard choice. See [CP] for a thorough discussion when $\mathfrak{g}$ is of finite type, and [L] for the general case.

When $\mathfrak{g}$ is of finite type, the R-matrix can be described explicitly as follows. Note that the expression below is presented in the h-adic completion of $U_h(\mathfrak{g})$, whereas here we are working in $U_q(\mathfrak{g})$. However, it is straightforward to check that this gives a well-defined endomorphism of $V \otimes W$ for any integrable highest weight $U_q(\mathfrak{g})$-representations V and W, with the only difficulty being that certain fractional powers of q can appear.

Theorem 3.3 (see [CP, Theorem 8.3.9])**.** *Assume $\mathfrak{g}$ is of finite type. Then the standard universal R-matrix for $U_q(\mathfrak{g})$ is given by the expression*

$$R_h = \exp\left(h \sum_{i,j} (B^{-1})_{ij} H_i \otimes H_j \right) \prod_\beta \exp_{q_\beta} \left[(1 - q_\beta^{-2}) E_\beta \otimes F_\beta \right], \tag{6}$$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that β_r appears to the left of β_s if $r > s$. □

We will not explain all of the notation in (6), since the only thing we use is the fact that E_β acts as 0 on any highest weight vector, and so the product in the expression acts as the identity on $b_\lambda \otimes c \in V_\lambda \otimes V_\mu$.

4. Constructing isomorphisms using systems of endomorphisms

Here and throughout this note a representation of $U_q(\mathfrak{g})$ will mean a direct sum of possibly infinitely many of the irreducible integrable type **1** representations V_λ. We note that the category of such representations is closed under the tensor product. When $\mathfrak{g}$ is of finite type, we can restrict to finite direct sums, or equivalently finite dimensional type **1** modules, since this category is already closed under the tensor product.

In this section we review a method for constructing natural systems of isomorphisms $\sigma_{V,W} : V \otimes W \to W \otimes V$. This idea was used by Henriques and Kamnitzer in [HK], and was further developed in [KT]. The data needed to construct such a system is:

(i) an algebra automorphism C_ξ of $U_q(\mathfrak{g})$ which is also a coalgebra anti-automorphism;
(ii) a natural system of invertible vector space endomorphisms ξ_V of each representation V of $U_q(\mathfrak{g})$ which is compatible with C_ξ in the sense that the following diagram commutes for all V:

$$\begin{array}{ccc} V & \xrightarrow{\quad \xi_V \quad} & V \\ \circlearrowright & & \circlearrowright \\ U_q(\mathfrak{g}) & \xrightarrow{\quad C_\xi \quad} & U_q(\mathfrak{g}). \end{array}$$

It follows immediately from the definition of coalgebra anti-automorphism that

$$\sigma^{\xi}_{V,W} := \mathrm{Flip} \circ (\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W} \tag{7}$$

is an isomorphism of $U_q(\mathfrak{g})$ representations from $V \otimes W$ to $W \otimes V$, where Flip is the map from $V \otimes W$ to $W \otimes V$ defined by $\mathrm{Flip}(v \otimes w) = w \otimes v$.

We will normally denote the system $\{\xi_V\}$ simply by ξ, and we will denote the action of ξ on the tensor product of two representations by $\Delta(\xi)$. This is justified since, as explained in [KT], ξ in fact belongs to a completion of $U_q(\mathfrak{g})$, and the action of ξ on $V \otimes W$ is calculated using the coproduct. With this notation $\sigma^\xi := \{\sigma^\xi_{V,W}\}$ can be expressed as

$$\sigma^\xi = \mathrm{Flip} \circ (\xi^{-1} \otimes \xi^{-1}) \circ \Delta(\xi). \tag{8}$$

In the current work we require a little more freedom: we will sometimes use automorphisms C_ξ of $U_q(\mathfrak{g})$ which are not linear over $\mathbb{C}(q)$, but instead are bar-linear (i.e. invert q). This causes some technical difficulties, which we deal with in Section 6. Once we make this precise, we will use all the same notation for a bar-linear C_ξ and compatible system of $\mathbb{C}$ vector space automorphisms ξ as we do in the linear case, including using $\Delta(\xi)$ to denote ξ acting on a tensor product.

Comment 4.1. Since the representations we are considering are all completely reducible, to describe the data $(\mathcal{C}_\xi, \xi)$ it is sufficient to describe C_ξ and to give the action of ξ_{V_λ} on any one vector v in each irreducible representation V_λ. This is usually more convenient than describing ξ_{V_λ} explicitly. Of course, the choice of C_ξ imposes a restriction on $\xi_{V_\lambda}(v)$, so when we give such a description of ξ, we must check that the action on our chosen vector in each V_λ is compatible with C_ξ.

Comment 4.2. If C_ξ is an coalgebra automorphism as opposed to a coalgebra anti-automorphism, the same arguments show that $(\xi_V^{-1} \otimes \xi_W^{-1}) \circ \xi_{V \otimes W} : V \otimes W \to V \otimes W$ is an isomorphism.

5. Crystal bases and global bases

In order to extend the construction described in Section 4 to include bar-linear ξ, we will need to use some results concerning crystal bases and global bases. We state only what is relevant to us, and refer the reader to [K] for a more complete exposition. Unfortunately, the conventions in [K] and [CP] do not quite agree. In particular, the theorems from [K] that we will need are stated in terms of a different coproduct, so we have modified them to match our conventions.

Definition 5.1. Fix an integrable highest weight representation V of $U_q(\mathfrak{g})$. Define the Kashiwara operators $\tilde{F}_i, \tilde{E}_i : V \to V$ by linearly extending

$$\begin{cases} \tilde{F}_i(F_i^{(n)}(v)) = F_i^{(n+1)}(v), \\ \tilde{E}_i(F_i^{(n)}(v)) = F_i^{(n-1)}(v). \end{cases} \tag{9}$$

for all $v \in V$ such that $E_i(v) = 0$.

Definition 5.2. Let $\mathcal{A}_\infty = \mathbb{C}[q^{-1}]_0$ be the algebra of rational functions in q^{-1} over $\mathbb{C}$ whose denominators are not divisible by q^{-1}.

Definition 5.3. A crystal basis of a representation V (at $q = \infty$) is a pair $(\mathcal{L}, \widetilde{B})$, where $\mathcal{L}$ is an $\mathcal{A}_\infty$-lattice of V and $\widetilde{B}$ is a basis for $\mathcal{L}/q^{-1}\mathcal{L}$, such that:

(i) $\mathcal{L}$ and $\widetilde{B}$ are compatible with the weight decomposition of V.

(ii) $\mathcal{L}$ is invariant under the Kashiwara operators and $\widetilde{B} \cup 0$ is invariant under their residues $e_i := \tilde{E}_i^{(\mathrm{mod}\, q^{-1}\mathcal{L})}$, $f_i := \tilde{F}_i^{(\mathrm{mod}\, q^{-1}\mathcal{L})} : \mathcal{L}/q^{-1}\mathcal{L} \to \mathcal{L}/q^{-1}\mathcal{L}$.

(iii) For any $b, b' \in \widetilde{B}$, we have $e_i b = b'$ if and only if $f_i b' = b$.

Definition 5.4. Let $(\mathcal{L}, \widetilde{B})$ be a crystal basis for V. The highest weight elements of $\widetilde{B}$ are those $b \in \widetilde{B}$ such that, for all i, $e_i(b) = 0$.

Proposition 5.5 (see [K])**.** *Each V_λ has a crystal basis $(\mathcal{L}_\lambda, \widetilde{B}_\lambda)$. Furthermore, $(\mathcal{L}_\lambda, \widetilde{B}_\lambda)$ has a unique highest weight element, and this occurs in the λ weight space.* □

Theorem 5.6 ([K, Theorem 1])**.** *Let V, W be representations with crystal bases $(\mathcal{L}, \widetilde{A})$ and $(\mathcal{M}, \widetilde{B})$, respectively. Then $(\mathcal{L} \otimes \mathcal{M}, \widetilde{A} \otimes \widetilde{B})$ is a crystal basis of $V \otimes W$. Furthermore, the highest weight elements of $\widetilde{A} \otimes \widetilde{B}$ are all of the form $a^{\mathrm{high}} \otimes b$, where a^{high} is a highest weight element of $\widetilde{A}$.* □

Definition 5.7. Let $(\mathcal{L}_\lambda, \widetilde{B}_\lambda)$ and $(\mathcal{L}_\mu, \widetilde{B}_\mu)$ be crystal bases for V_λ and V_μ. Set

$$S^\nu_{\lambda,\mu} := \{b \in \widetilde{B}_\mu : b_\lambda \otimes b \text{ is a highest weight element of } \widetilde{B}_\lambda \otimes \widetilde{B}_\mu \text{ of weight } \nu\}.$$

For any V_λ, and any choice of highest weight vector $b_\lambda \in V_\lambda$, there is a canonical choice of basis B_λ for V_λ, which contains b_λ, and such that $(B_\lambda + q\mathcal{L}, \mathcal{L})$ is a crystal basis for V, where $\mathcal{L}$ is the $\mathcal{A}_\infty$-span of B_λ. That is not to say there is a unique basis for V_λ satisfying these two conditions, only that one can find a canonical "good" choice. This is known as the global basis for V_λ. A complete construction can be found in [K], although here we more closely follow the presentation from [CP, Chapter 14.1C]. In the present work we simply use the fact that the global basis exists, and state the properties of B_λ that we need.

Definition 5.8. $C_{\mathrm{bar}} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra involution defined by

$$\begin{cases} C_{\mathrm{bar}}(E_i) = E_i, \\ C_{\mathrm{bar}}(F_i) = F_i, \\ C_{\mathrm{bar}}(K_i) = K_i^{-1}, \\ C_{\mathrm{bar}}(q) = q^{-1}. \end{cases} \tag{10}$$

Theorem 5.9 (Kashiwara [K]). *Fix a highest weight vector $b_\lambda \in V_\lambda$. There is a canonical choice of a "global" basis B_λ of V_λ. This has the properties (although it is not defined by these alone) that:*

(i) $b_\lambda \in B_\lambda$.
(ii) *B_λ is a weight basis for V_λ.*
(iii) *Let $\mathcal{L}$ be the $\mathcal{A}_\infty$ span of B_λ. Then $(B_\lambda + q^{-1}\mathcal{L}, \mathcal{L})$ is a crystal basis for V_λ.*
(iv) *Define the involution $\mathrm{bar}_{(V_\lambda, B_\lambda)}$ of V_λ by $\mathrm{bar}_{(V_\lambda, B_\lambda)}(f(q)b) = f(q^{-1})b$ for all $f(q) \in \mathbb{C}(q)$ and $b \in B_\lambda$. Then $\mathrm{bar}_{(V_\lambda, B_\lambda)}$ is compatible with C_{bar}, in the sense discussed in Section 4.*

Furthermore, if a different highest weight vector is chosen, B_λ is multiplied by an overall scalar. □

Definition 5.10. If V is any (possibly reducible) representation of $U_q(\mathfrak{g})$, we say a basis B of V is a global basis if there is a decomposition of V into irreducible components such that B is a union of global bases for the irreducible pieces.

6. The system of endomorphisms Θ

We now introduce a $\mathbb{C}$-algebra automorphism C_Θ of $U_q(\mathfrak{g})$. Notice that this inverts q, so it is not a $\mathbb{C}(q)$-algebra automorphism, but is instead bar-linear:

$$\begin{cases} C_\Theta(E_i) = E_i K_i^{-1}, \\ C_\Theta(F_i) = K_i F_i, \\ C_\Theta(K_i) = K_i^{-1}, \\ C_\Theta(q) = q^{-1}. \end{cases} \tag{11}$$

One can check that C_Θ is a well-defined algebra involution and a coalgebra anti-involution. In order to use the methods of Section 4, we must define a $\mathbb{C}$-vector space automorphism Θ_{V_λ} of each V_λ which is compatible with C_Θ. This is complicated by the fact that C_Θ does not preserve the $\mathbb{C}(q)$-algebra structure, but instead inverts q. We must actually work in the category of representations with chosen global bases. An element of this category will be denoted (V, B), where B is the chosen global basis of V.

Definition 6.1. Fix a global basis B_λ for V_λ. The action of $\Theta_{(V_\lambda, B_\lambda)}$ on V_λ is defined by requiring that it be compatible with C_Θ, and that $\Theta_{(V_\lambda, B_\lambda)}(b_\lambda) = q^{-(\lambda,\lambda)/2+(\lambda,\rho)} b_\lambda$. This is extended by naturality to define $\Theta_{(V,B)}$ for any (possibly reducible) V.

Comment 6.2. To ensure that Definition 6.1 makes sense, one must check that there is a map which sends b_λ to $q^{-(\lambda,\lambda)/2+(\lambda,\rho)} b_\lambda$ and is compatible with C_Θ. This amounts to checking that b_λ is still a highest weight vector if the action of $U_q(\mathfrak{g})$ is twisted by the automorphism C_Θ, and is not difficult.

Comment 6.3. In some cases Θ acts on a weight vector as multiplication by a fractional power of q. To be completely precise we should adjoin a fixed kth root of unity to the base field $\mathbb{C}(q)$, where k is twice the dual Coxeter number of $\mathfrak{g}$. This causes no significant difficulties.

The construction described in Section 4 uses the action of $\xi_{V \otimes W}$ on $V \otimes W$. Thus we will need to define how Θ acts on a tensor product. In particular, we need a well-defined notion of tensor product in the category of representations with chosen global bases.

Definition 6.4. Let $V^{\nu}_{\lambda,\mu}$ denote the isotypic component of $V_\lambda \otimes V_\mu$ with highest weight ν. Let $V^{>\nu}_{\lambda,\mu} := \bigcup_{\gamma>\nu} V^{\gamma}_{\lambda,\mu}$, $V^{\geq\nu}_{\lambda,\mu} := \bigcup_{\gamma\geq\nu} V^{\gamma}_{\lambda,\mu}$, and $Q^{\nu}_{\lambda,\mu} := V^{\geq\nu}_{\lambda,\mu}/V^{>\nu}_{\lambda,\mu}$. Here we use the partial order of the weight lattice where $\gamma \geq \nu$ iff $\gamma - \nu$ is a nonnegative linear combination of the α_i.

Comment 6.5. It is clear that the inclusion $V^{\nu}_{\lambda,\mu} \hookrightarrow V^{\geq\nu}_{\lambda,\mu}$ descends to an isomorphism from $V^{\nu}_{\lambda,\mu}$ to $Q^{\nu}_{\lambda,\mu}$.

Definition 6.6. The tensor product $(V_\lambda, B_\lambda) \otimes (V_\mu, B_\mu)$ is defined to be $(V_\lambda \otimes V_\mu, A)$, where A is the unique global basis of $V \otimes W$ such that the projections of the highest weight elements of A of weight ν in $Q^{\nu}_{\lambda,\mu}$ are equal to the projections of $b_\lambda \otimes b$ for those $b \in S^{\nu}_{\lambda,\mu}$. This is well defined by Comment 6.5. Extend by naturality a tensor product $(V, B) \otimes (W, C)$ for possibly reducible V and W.

7. Proof that we obtain the R-matrix when $\mathfrak{g}$ is of finite type

The proof of our main theorem uses a relationship between the R-matrix and the braid group element T_{w_0} first observed in [KR] and [LS]. Thus for this section we must restrict to a finite type. We hope the result will prove to be true in greater generality, but establishing this would certainly require a different approach. We start by introducing a few more automorphisms of $U_q(\mathfrak{g})$ and of its representations.

Definition 7.1. Let θ to be the diagram automorphism such that $w_0(\alpha_i) = -\alpha_{\theta(i)}$, where w_0 is the longest element in the Weyl group.

Definition 7.2. C_Γ is the $\mathbb{C}$-Hopf algebra automorphism of $U_q(\mathfrak{g})$ defined by

$$\begin{cases} C_\Gamma(E_i) = -K_{\theta(i)}F_{\theta(i)}, \\ C_\Gamma(F_i) = -E_{\theta(i)}K^{-1}_{\theta(i)}, \\ C_\Gamma(K_i) = K_{\theta(i)}, \\ C_\Gamma(q) = q^{-1}. \end{cases} \tag{12}$$

Define the action of $\Gamma_{(V_\lambda,B_\lambda)}$ on V_λ to be the unique $\mathbb{C}$-linear endomorphism of each V_λ which is compatible with C_Γ, and which is normalized so that $\Gamma(b_\lambda) = b^{\text{low}}_\lambda$. Extend this by naturality to get the action of $\Gamma_{(V,B)}$ on any (possibly reducible) representation V with chosen global basis B.

Comment 7.3. It is a simple exercise to check that C_Γ is in fact a Hopf algebra automorphism, and is compatible with a $\mathbb{C}$-vector space automorphism of V_λ which takes b_λ to b^{low}_λ.

Definition 7.4. $C_{T_{w_0}}$ and C_J are the $\mathbb{C}(q)$-algebra automorphisms of $U_q(\mathfrak{g})$ defined by

$$\begin{cases} C_{T_{w_0}}(E_i) = -F_{\theta(i)}K_{\theta(i)}, \\ C_{T_{w_0}}(F_i) = -K^{-1}_{\theta(i)}E_{\theta(i)}, \\ C_{T_{w_0}}(K_H) = K_{w_0(H)}, \text{ so that } C_{T_{w_0}}(K_i) = K^{-1}_{\theta(i)}, \end{cases} \tag{13}$$

$$\begin{cases} C_J(E_i) = K_iE_i, \\ C_J(F_i) = F_iK_i^{-1}, \\ C_J(K_H) = K_H. \end{cases} \tag{14}$$

The systems of $\mathbb{C}(q)$-vector space automorphisms T_{w_0} and J of each V_λ are the unique automorphisms which are compatible with $C_{T_{w_0}}$ and C_J, respectively, and such that $T_{w_0}(b_\lambda^{\text{low}}) = b_\lambda$ and $J(b_\lambda) = q^{(\lambda,\lambda)/2+(\lambda,\rho)} b_\lambda$, where b_λ and b_λ^{low} are the highest and lowest weight elements in some global basis B_λ.

Comment 7.5. It is straightforward exercise to show that the formulas in Definition 7.4 do define algebra automorphisms of $U_q(\mathfrak{g})$ and compatible vector space automorphisms of each V_λ. There is an action of the braid group on each V_λ, and T_{w_0} is in fact the action of the longest element (for an appropriate choice of conventions). Note also that J and T_{w_0} do not depend on the choice of global basis as they are stable under simultaneously rescaling b_λ and b_λ^{low}. All of this is discussed in [KT].

Lemma 7.6. *The following identities hold:*

(i) $\Gamma_{(V,B)} = \text{bar}_{(V,B)} \circ T_{w_0}^{-1}$,

(ii) $\Theta_{(V,B)} = K_{2H_\rho} \circ \text{bar}_{(V,B)} \circ J$,

(iii) *for any weight vector* $v \in V$ *with* $\text{wt}(v) = \mu$, $J(v) = q^{(\mu,\mu)/2+(\mu,\rho)} v$,

(iv) *for any* $b \in B$ *with* $\text{wt}(b) = \mu$, $\Theta_{(V,B)}(b) = q^{-(\mu,\mu)/2+(\mu,\rho)} b$,

(v) $\Gamma_{(V,B)}^{-1} \circ \Theta_{(V,B)} = JT_{w_0}$.

Here $\text{bar}_{(V,B)}$ *is the involution defined in Theorem* 5.9, *part* (iv).

Proof. Let $C_{K_{2H_\rho}}$ be the algebra automorphism of $U_q(\mathfrak{g})$ defined by $C_{K_{2H_\rho}}(X) = K_{2H_\rho} X K_{2H_\rho}^{-1}$. It follows directly from (4) that

$$C_{K_{2H_\rho}}(K_i^{-1}E_i) = E_i K_i^{-1} \quad \text{and } C_{K_{2H_\rho}}(F_i K_i) = K_i F_i. \tag{15}$$

Using (15) and the relevant definitions, a simple check on generators shows that

$$C_\Gamma = C_{\text{bar}} \circ C_{T_{w_0}}^{-1}, \;\; C_\Theta = C_{K_{2H_\rho}} \circ C_{\text{bar}} \circ C_J, \quad \text{and} \quad C_\Gamma^{-1} \circ C_\Theta = C_J \circ C_{T_{w_0}}. \tag{16}$$

Thus, to prove (i), (ii) and (v), it suffices to check each identity when each side acts on any one chosen vector b in each V_λ. For parts (i) and (ii), choose $b = b_\lambda$ and the identity is immediate from the definitions.

For part (iii), it is sufficient to consider $V = V_\lambda$. By Definition 7.4, (iii) holds for $b = b_\lambda$. Furthermore, vectors of the form $F_{i_k} \cdots F_{i_1} b_\lambda$ generate V_λ as a $\mathbb{C}(q)$ module. Assume that v is a weight vector of weight μ, and $J(v) = q^{(\mu,\mu)/2+(\mu,\rho)} v$. Fix $i \in I$. Then

$$\begin{aligned} J(F_i v) &= C_J(F_i) J(v) = F_i K_i^{-1} q^{(\mu,\mu)/2+(\mu,\rho)} v = F_i q^{-\langle d_i H_i, \mu \rangle} q^{(\mu,\mu)/2+(\mu,\rho)} v \\ &= q^{-(\alpha_i,\mu)} q^{(\mu,\mu)/2+(\mu,\rho)} v = q^{(\mu-\alpha_i,\mu-\alpha_i)/2+(\mu-\alpha_i,\rho)} v. \end{aligned} \tag{17}$$

The claim now follows by induction on k.

Part (iv) follows by directly calculating the action of the right side of (ii) on b and using part (iii) to evaluate the action of J.

The definitions of $\Theta_{(V,B)}$ and $\Gamma_{(V,B)}$, along with parts (iii) and (iv), now immediately imply that $\Gamma_{(V_\lambda,B_\lambda)}^{-1} \circ \Theta_{(V_\lambda,B_\lambda)}(b_\lambda^{\text{low}}) = JT_{w_0}(b_\lambda^{\text{low}}) = q^{(\lambda,\lambda)/2+(\lambda,\rho)} b_\lambda$, completing the proof of (v). □

We also need the following construction of the R-matrix due to Kirillov-Reshetikhin and Levendorskii-Soibelman. Due to a different choice of conventions, our T_{w_0} is $K_{H_\rho}^{-1} T_{w_0}^{-1}$ in those papers, so we have modified the statement accordingly. As with Theorem 7.7, this expression is written using the h-adic completion of

$U_h(\mathfrak{g})$, but gives a well-defined action on $V \otimes W$ for any finite dimensional type **1** $U_q(\mathfrak{g})$-module.

Theorem 7.7 ([KR, Theorem 3], [LS, Theorem 1])**.** *The standard universal R-matrix can be realized as*

$$R = \exp\left(h \sum_{i,j\in I} (B^{-1})_{ij} H_i \otimes H_j\right) (T_{w_0}^{-1} \otimes T_{w_0}^{-1})\Delta(T_{w_0}). \tag{18}$$

□

Corollary 7.8. $(T_{w_0}^{-1} \otimes T_{w_0}^{-1})\Delta(T_{w_0}) = \prod_\beta \exp_{q_\beta}\left[(1-q_\beta^{-2})E_\beta \otimes F_\beta\right],$

where the product is over all the positive roots of $\mathfrak{g}$, and the order of the terms is such that β_r appears to the left of β_s if $r > s$.

Proof. This follows immediately from Theorems 3.3 and 7.7, since the action of R on $V_\lambda \otimes V_\mu$ is invertible. □

As discussed in [KT], the following is equivalent to Theorem 7.7:

Corollary 7.9 (see [KT, Comment 7.3])**.** *Let $X = JT_{w_0}$. Then*

$$R = (X^{-1} \otimes X^{-1})\Delta(X).$$

□

Lemma 7.10. *Fix type* **1** *finite dimensional $U_q(\mathfrak{g})$ representations with chosen global bases (V, B) and (W, C). The operator $(\Gamma_{(V,B)} \otimes \Gamma_{(W,C)})\Gamma_{(V\otimes W,A)})^{-1}$ acts on $V \otimes W$ as the identity, where A is the global basis of $V \otimes W$ constructed from B and C in Definition* 6.6.

Proof. It suffices to consider the case when $V = V_\lambda$ and $W = V_\mu$ are irreducible. Set

$$m^\Gamma := (\Gamma_{(V_\lambda,B_\lambda)} \otimes \Gamma_{(V_\mu,B_\mu)})(\Gamma_{(V_\lambda\otimes V_\mu,A)})^{-1} : V_\lambda \otimes V_\mu \to V_\lambda \otimes V_\mu. \tag{19}$$

We must show that m^Γ is the identity. C_Γ is a Hopf algebra automorphism of $U_q(\mathfrak{g})$, so, as in Section 4, it follows that m^Γ is an automorphism of $U_q(\mathfrak{g})$ representations. In particular, m^Γ preserves isotypic components of $V_\lambda \otimes V_\mu$ and acts on each subquotient $Q^\nu_{\lambda,\mu}$ (see Definition 6.4). It is sufficient to show that the action on $Q^\nu_{\lambda,\mu}$ is the identity for all ν. In fact, it is sufficient to consider the action on the highest weight space of $Q^\nu_{\lambda,\mu}$, since this generates $Q^\nu_{\lambda,\mu}$. This highest weight space has a basis consisting of $\{\overline{\overline{b_\lambda \otimes b}} : b \in S^\nu_{\lambda,\mu}\}$, where $S^\nu_{\lambda,\mu}$ is as in Definition 5.7 and we use the notation $\overline{\overline{a \otimes b}}$ to denote the image of $a \otimes b$ in $Q^\nu_{\lambda,\mu}$.

By Lemma 7.6 part (i) and Corollary 7.8, we get

$$\begin{aligned} m^\Gamma &= (\mathrm{bar}_{(V_\lambda,B_\lambda)} \otimes \mathrm{bar}_{(V_\mu,B_\mu)})(T_{w_0}^{-1} \otimes T_{w_0}^{-1})\Delta(T_{w_0})\mathrm{bar}_{(V_\lambda\otimes V_\mu,A)} \\ &= (\mathrm{bar}_{(V_\lambda,B_\lambda)} \otimes \mathrm{bar}_{(V_\mu,B_\mu)}) \prod_\beta \exp_{q_\beta}\left[(1-q_\beta^{-2})E_\beta \otimes F_\beta\right] \mathrm{bar}_{(V_\lambda\otimes V_\mu,A)}. \end{aligned} \tag{20}$$

For convenience, set

$$\Psi := (\mathrm{bar}_{(V_\lambda,B_\lambda)} \otimes \mathrm{bar}_{(V_\mu,B_\mu)}) \prod_\beta \exp_{q_\beta}\left[(1-q_\beta^{-2})E_\beta \otimes F_\beta\right]. \tag{21}$$

Both m^Γ and $\mathrm{bar}_{(V_\lambda\otimes V_\mu,A)}$ act in a well-defined way on each $Q^\nu_{\lambda,\mu}$, which implies that Ψ does as well.

The global basis A was chosen so that $\mathrm{bar}_{(V_\lambda \otimes V_\mu, A)}(\overline{\overline{b_\lambda \otimes b}}) = \overline{\overline{b_\lambda \otimes b}}$ (see Definition 6.6). Since all E_β kill b_λ and $(\mathrm{bar}_{(V_\lambda, B_\lambda)} \otimes \mathrm{bar}_{(V_\mu, B_\mu)})$ preserves $b_\lambda \otimes b$ by definition, we see that $\Psi(b_\lambda \otimes b) = b_\lambda \otimes b$, and, taking the image in $Q^\nu_{\lambda,\mu}$, $\Psi(\overline{\overline{b_\lambda \otimes b}}) = \overline{\overline{b_\lambda \otimes b}}$. Thus, using (20), we see that m^Γ acts on $\overline{\overline{b_\lambda \otimes b}}$ as the identity. The lemma follows. □

Theorem 7.11. *Fix type* **1** *finite dimensional* $U_q(\mathfrak{g})$ *representations with chosen global bases* (V, B) *and* (W, C)*. Then* $\big(\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}\big)\Theta_{(V\otimes W,A)}$ *acts on* $V \otimes W$ *as the standard R-matrix, where* A *is the global basis of* $V \otimes W$ *constructed from* B *and* C *in Definition* 6.6*. This holds independently of the choices of global bases* B *and* C*.*

Proof. By Corollary 7.9 and Lemma 7.6(v), we get

$$\begin{aligned} R &= ((JT_{w_0})^{-1} \otimes (JT_{w_0})^{-1})\Delta(JT_{w_0}) \\ &= (\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)})(\Gamma_{(V,B)} \otimes \Gamma_{(W,C)})(\Gamma_{(V\otimes W,A)})^{-1}\Theta_{(V\otimes W,A)}. \end{aligned} \tag{22}$$

By Lemma 7.10, the expression $(\Gamma_{(V,B)} \otimes \Gamma_{(W,C)})(\Gamma_{(V\otimes W,A)})^{-1}$ acts as the identity. □

Comment 7.12. By Theorem 7.11, the composition

$$\big(\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}\big)\Theta_{(V\otimes W,A)} \tag{23}$$

does not depend on the choices on global bases B and C. Introducing the notation $\Delta(\Theta)$ to mean $\Theta_{(V\otimes W,A)}$ and dropping the subscripts, we can interpret $(\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta)$ as (23) calculated using any global bases B and C. Then Theorem 7.11 becomes (2) from the introduction. We also note that $\Theta_{(V,B)}$ is easily seen to be an involution, so the inverses in (23) are perhaps unnecessary.

8. Future directions

Although we have only proven Theorem 7.11 when $\mathfrak{g}$ is of finite type, much of the construction works in greater generality. We did not assume $\mathfrak{g}$ was a finite type in Section 6, so the expression $\big(\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}\big)\Theta_{(V\otimes W,A)}$ makes sense for any symmetrizable Kac-Moody algebra. Since C_Θ is a coalgebra-antiautomorphism, the methods from Section 4 imply that

$$\mathrm{Flip} \circ \big(\Theta^{-1}_{(V,B)} \otimes \Theta^{-1}_{(W,C)}\big)\Theta_{(V\otimes W,A)} \tag{24}$$

is an isomorphism of representations. Furthermore, it is true in general that (24) does not depend on the choice of B and C. To see why, it is sufficient to consider the case when $V = V_\lambda$ and $W = V_\mu$ are irreducible. Then the global bases B_λ and B_μ are unique up to multiplication by an overall scalar. It is straightforward to see that if B_λ (or B_μ) is scaled by a constant z, then A is scaled by z as well, and from there we see that both $\Theta_{(V_\lambda, B_\lambda)}$ and $\Theta_{(V_\lambda\otimes V_\mu, A)}$ are scaled by $z/\bar{z}$, where $\bar{z}$ is obtained from z by inverting q. Thus the composition is unchanged.

As in Comment 7.12, we can now make sense of the expression $(\Theta^{-1}\otimes\Theta^{-1})\Delta(\Theta)$ for all symmetrizable Kac-Moody algebras $\mathfrak{g}$. The fact that (24) defines an isomorphism is one of the properties required of a universal R-matrix. However, we have not proven the crucial equalities (5). Thus we ask:

Question 1. Is $(\Theta^{-1} \otimes \Theta^{-1})\Delta(\Theta)$ a universal R-matrix for $U_q(\mathfrak{g})$ if $\mathfrak{g}$ is a general symmetrizable Kac-Moody algebra? If yes, is it the standard R-matrix?

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