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TROPICAL R MAPS AND AFFINE GEOMETRIC CRYSTALS

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ABSTRACT. By modifying an earlier method of the authors (2008), certain affine geometric crystals are realized in affinization of the fundamental representation $W(\varpi_1)_l$, and the tropical R maps for the affine geometric crystals are described explicitly. We also define prehomogeneous geometric crystals and show that for a positive geometric crystal, the connectedness of the corresponding ultra-discretized crystal is the sufficient condition for prehomogeneity of the positive geometric crystal. Moreover, the uniqueness of tropical R maps is discussed.

1. INTRODUCTION

An R-matrix appears as a solution to the Yang-Baxter equation, which is a key to solve integrable lattice models in statistical mechanics. To understand R-matrices representation theoretically, Drinfeld [D1] and Jimbo [J] introduced the quantized universal enveloping algebra. Once its representation theory has been established, an R-matrix is interpreted as an intertwiner between the tensor product of finite-dimensional modules and the one with the order of the tensor product being reversed. In [KMN1], [KMN2] the notions of finite-dimensional modules and R-matrices acquired a combinatorial version by using the theory of crystal bases. In the papers, the objects, perfect crystals and combinatorial R-matrices, are introduced and play an important role to show that some physical quantities for particular vertex-type solvable models are equal to affine Lie algebra characters. Combinatorial R-matrices also play an essential role in some integrable cellular automata [HKOTY].

The notion of geometric crystals for semi-simple algebraic groups has been initiated by Berenstein and Kazhdan [BK], and it is extended for Kac-Moody groups in [N1]. Geometric crystals are constructed on some geometric objects, such as algebraic varieties, and has an analogous structure to crystals. The relation between crystals and geometric crystals is not only a simple analogy, but also a more direct functorial connection, called ultra-discretization/tropicalization. Indeed, in [N1], the geometric crystals on Schubert varieties of the corresponding Kac-Moody group are ultra-discretized to certain crystals (see 2.3). Applying this result, we constructed for an affine Lie algebra $\mathfrak{g}$ the affine geometric crystal $\mathcal{V} = \mathcal{V}(\mathfrak{g})$ in the fundamental $\mathfrak{g}$-representation $W(\varpi_1)$ in [KNO] . Ultra-discretizing the geometric

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crystal $\mathcal{V}(\mathfrak{g})$, we obtained the limit of the coherent family of perfect crystals $B_\infty(\mathfrak{g}^L)$ where $\mathfrak{g}^L$ is the Langlands dual of $\mathfrak{g}$ (see [KKM]).

A tropical R map is an analogous object to the set-theoretic R ([D2]) and defined as follows (see §9): For a family of geometric crystals $X := \{X_\lambda\}_{\lambda\in\Lambda}$ (parametrized by $\lambda \in \Lambda$) with a product structure, a birational map $R_{\lambda\mu} : X_\lambda \times X_\mu \to X_\mu \times X_\lambda$ ($\lambda, \mu \in \Lambda$) is said to be a tropical R map, if it satisfies the Yang-Baxter equation and preserves the geometric crystal structure. In [Y] and [KOTY], an explicit form of tropical R map is given for the geometric crystal $\mathcal{B}_l$ ($l \in \mathbb{C}^\times$) of type $A_n^{(1)}$ and $D_n^{(1)}$. They obtained the tropical R map as a unique solution (x', y') of the equation $M_l(x, z)M_m(y, z) = M_m(x', z)M_l(y', z)$ where $x \in \mathcal{B}_l, y \in \mathcal{B}_m$, z is an indeterminate and $M_l(x, z)$ is a square matrix called an *M-matrix* (see §8).

In this paper, we construct the affine geometric crystal $\mathcal{V}_l$ in $W(\varpi_1)_l$, the affinization of the fundamental representation $W(\varpi_1)$, for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}$, $D_{n+1}^{(2)}, A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$. The geometric crystals $\mathcal{B}_l$ constructed in [KOTY] are isomorphic to our geometric crystals $\mathcal{V}_l$. Thus, in these two cases we have the tropical R map for $\mathcal{V}_l$ through the isomorphism. By virtue of the "folding" method, we obtain the tropical R maps for the other cases; $B_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ from the tropical R map for $D_N^{(1)}$ with a suitable integer N. (The case $C_n^{(1)}$ cannot be obtained from folding $D_N^{(1)}$. This case will be discussed elsewhere.) Here note that the geometric crystal $\mathcal{V}(\mathfrak{g})$ is a special case of $\mathcal{V}(\mathfrak{g})_l$ for $l = 1$.

Let us explain the construction of the tropical R map more precisely. First, we take a Dynkin diagram automorphism σ for $D_N^{(1)}$, which induces the automorphism Σ of the geometric crystal $\mathcal{B}_l$ of type $D_N^{(1)}$, where $\mathcal{B}_l \cong (\mathbb{C}^\times)^{2N-2}$ as algebraic varieties. Let X_l be the fixed-point variety for Σ, that is, $X_l := \{x \in \mathcal{B}_l \mid \Sigma(x) = x\}$. There is an invertible matrix J such that $M_l(\Sigma(x), z) = JM_l(x, z)J^{-1}$ ($x \in \mathcal{B}_l$). By this formula and the uniqueness of the solution for the equation $M_l(x, z)M_m(y, z) = M_m(x', z)M_l(y', z)$ ($l, m \in \mathbb{C}^\times$), we deduce that the tropical R map sends $X_l \times X_m \to X_m \times X_l$. Furthermore, this fixed-point variety X_l is equipped with the $\mathfrak{g}^\sigma$-geometric crystal structure, where $\mathfrak{g}^\sigma$ is the affine Lie algebra obtained from $D_N^{(1)}$ by the folding associated with σ and it is isomorphic to $\mathcal{V}_l$ for $\mathfrak{g}^\sigma$. Hence, we get the tropical R map for $\{\mathcal{V}_l\}$ in the case of $\mathfrak{g}^\sigma$.

In the article, we also discuss the uniqueness of the tropical R maps. As for usual R-matrices, it is unique (up to constant) by Schur's lemma, if the corresponding modules are irreducible. A similar situation occurs in the geometric crystals. A geometric crystal is *prehomogeneous* if there exists an open dense orbit by the actions of the e_i^c's. For ϕ, ϕ' two isomorphisms of prehomogeneous geometric crystals, if there exists a point p in the open dense orbit which is sent to the same point by ϕ and ϕ', it is shown that $\phi = \phi'$ as rational maps. A crucial result in this article is that a positive geometric crystal $\mathbb{X}$ is prehomogeneous if the crystal ultra-discretized from $\mathbb{X}$ is connected (see Theorem 3.3). This uniqueness implies the following fact: For geometric crystals X, Y, Z, suppose that the product $X \times Y \times Z$ is prehomogeneous and that there exist tropical R maps for any two of X, Y, Z, then it follows from the uniqueness that we have the Yang-Baxter equation

$$R^{(12)}R^{(23)}R^{(12)} = R^{(23)}R^{(12)}R^{(23)}.$$

If $X \times Y$ is prehomogeneous, we also obtain the inversion formula

$$R_{YX}R_{XY} = \mathrm{id},$$

where $R^{(ij)}$ means R acting on the i-th and the j-th components of the product. Namely, important properties of the tropical R maps are deduced from the uniqueness. The tropical R map on the affine geometric crystal $\mathcal{V}_l$ introduced in this article is unique, since the ultra-discretized crystal $\mathcal{UD}(\mathcal{V}_l)$ and its tensor products are connected, which implies that $\mathcal{V}_l$ and its products are prehomogeneous.

2. Geometric crystals and crystals

In this section, we review Kac-Moody groups and geometric crystals following [BK], [Kac], [N1], [N2], [PK].

2.1. **Kac-Moody algebras and Kac-Moody groups.** Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations ([Kac]). There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*}\mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0,\ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ = \Delta \cap Q_+$. An element of Δ_+ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$ and $Q \subset P \subset \{\lambda \mid \lambda(Q^\vee) \subset \mathbb{Z}\}$, whose element is called a weight.

Define the simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. It induces the action of W on $\mathfrak{t}^*$ by $s_i(\lambda):=\lambda-\lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) \mid w \in W,\ i \in I\}$, whose element is called a real root.

Let G be the Kac-Moody group associated with $(\mathfrak{g}, P)$ ([PK]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+)$, i.e., $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\mathrm{re}}\rangle$.

For any $i \in I$, there exists a unique group homomorphism $\phi_i \colon SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(te_i),\ \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tf_i) \qquad (t \in \mathbb{C}).$$

Set $\alpha_i^\vee(c) := \phi_i\left(\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1}\end{smallmatrix}\right)\right)$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \alpha_i^\vee(\mathbb{C}^\times)$ and $N_i := N_{G_i}(T_i)$. Let T be the subgroup of G with P as its weight lattice which is called a *maximal torus* in G, and let $B^\pm = U^\pm T$ be the Borel subgroup of G. Let N be the subgroup of G generated by the N_i's. Then we have the isomorphism $\phi \colon W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_iT/T$. An element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\right)$ in $N_G(T)$ is a representative of $s_i \in W = N_G(T)/T$.

2.2. **Geometric crystals.** Let W be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \left\{(i_1, i_2, \ldots, i_l) \in I^l \mid w = s_{i_1}s_{i_2}\cdots s_{i_l}\right\},$$

where l is the length of w, i.e., $R(w)$ is the set of reduced expressions of w.

Let X be a variety, $\gamma_i \colon X \to \mathbb{C}$ and $\varepsilon_i \colon X \longrightarrow \mathbb{C}$ $(i \in I)$ rational functions on X, and $e_i \colon \mathbb{C}^\times \times X \longrightarrow X$ $((c, x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action. For $w \in W$ and

$\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$, set $\alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$\begin{aligned} e_{\mathbf{i}} \colon T \times X &\to X \\ (t, x) &\mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x). \end{aligned}$$

Definition 2.1. A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i,\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if:

(i) $\{1\} \times X \cap \mathrm{dom}(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$. Here $\mathrm{dom}(e_i)$ is the domain of definition of $e_i \colon \mathbb{C}^\times \times X \to X$.
(ii) $\gamma_j(e_i^c(x)) = c^{a_{ij}} \gamma_j(x)$.
(iii) $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$ and $\mathbf{i}$. $\mathbf{i}' \in R(w)$.
(iv) $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$.

Note that condition (iii) is equivalent to the following so-called *Verma relations*:

$$\begin{aligned}
&e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0,\\
&e_i^{c_1} e_j^{c_1c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1,\\
&e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^2c_2} e_i^{c_1} && \text{if } a_{ij} = -2,\ a_{ji} = -1,\\
&e_i^{c_1} e_j^{c_1^3c_2} e_i^{c_1^2c_2} e_j^{c_1^3c_2^2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^3c_2^2} e_i^{c_1^2c_2} e_j^{c_1^3c_2} e_i^{c_1} && \text{if } a_{ij} = -3,\ a_{ji} = -1,
\end{aligned}$$

Note that the last formula is different from the one in [BK], [N1], [N2] which seems to be incorrect. If $\mathbb{X} = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies conditions (i), (ii) and (iv), we call $\mathbb{X}$ a *pregeometric crystal*.

2.3. **Geometric crystal on a Schubert cell.** Let $X := G/B$ be the flag variety, which is the inductive limit of finite-dimensional projective varieties. For $w \in W$, let $X_w := BwB/B \subset X$ be the Schubert cell associated with w, which has a natural geometric crystal structure ([BK], [N1]). For $\mathbf{i} = (i_1, \ldots, i_k) \in R(w)$, set

$$B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \ldots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_l}(c_k) \mid c_1 \cdots, c_k \in \mathbb{C}^\times\} \subset B^- \tag{2.1}$$

where $Y_i(c) := y_i(c^{-1})\alpha_i^\vee(c) = \alpha_i^\vee(c) y_i(c)$. Then $B_{\mathbf{i}}^-$ is birationally isomorphic to X_w and endowed with the induced geometric crystal structure. Let ξ be an element in the torus T. Then we can define the geometric crystal structure on $B_{\mathbf{i}}^- \cdot \xi$ and we shall describe its explicit form: The action e_i^c on $B_{\mathbf{i}}^- \cdot \xi$ is given by

$$e_i^c(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)\xi) = Y_{i_1}(\mathcal{C}_1) \cdots Y_{i_l}(\mathcal{C}_k)\xi,$$

where

$$\mathcal{C}_j := c_j \cdot \frac{\displaystyle\sum_{1\le m\le j,\, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j<m\le k,\, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1\le m<j,\, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j\le m\le k,\, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}. \tag{2.2}$$

The explicit forms of rational functions ε_i and γ_i are:

$$\varepsilon_i(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)\xi) = \sum_{1\le m\le k,\, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m},$$

$$\gamma_i(Y_{i_1}(c_1) \cdots Y_{i_l}(c_k)\xi) = c_1^{a_{i_1,i}} \cdots c_k^{a_{i_k,i}} \alpha_i(\xi).$$

These will be needed in Section 5 to construct affine geometric crystals.

2.4. **Crystals.** We recall the notion of crystals, which is obtained by abstracting the combinatorial properties of crystal bases.

Definition 2.2. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned}
&\mathrm{wt}\colon B \longrightarrow P,\\
&\varepsilon_i\colon B \longrightarrow \mathbb{Z}\sqcup\{-\infty\}, \quad \varphi_i\colon B \longrightarrow \mathbb{Z}\sqcup\{-\infty\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i\colon B\sqcup\{0\} \longrightarrow B\sqcup\{0\}, \quad \tilde{f}_i\colon B\sqcup\{0\} \longrightarrow B\sqcup\{0\} \quad \text{for} \quad i\in I,\\
&\tilde{e}_i(0)=\tilde{f}_i(0)=0.
\end{aligned}$$

Those maps satisfy the following axioms: For all $b, b_1, b_2 \in B$, we have

$$\begin{aligned}
&\varphi_i(b)=\varepsilon_i(b)+\langle\alpha_i^\vee,\mathrm{wt}(b)\rangle,\\
&\mathrm{wt}(\tilde{e}_ib)=\mathrm{wt}(b)+\alpha_i \text{ if } \tilde{e}_ib\in B,\\
&\mathrm{wt}(\tilde{f}_ib)=\mathrm{wt}(b)-\alpha_i \text{ if } \tilde{f}_ib\in B,\\
&\tilde{e}_ib_2=b_1 \Longleftrightarrow \tilde{f}_ib_1=b_2,\\
&\varepsilon_i(b)=-\infty \Longrightarrow \tilde{e}_ib=\tilde{f}_ib=0.
\end{aligned}$$

Example 2.3. (i) *If (L, B) is a crystal base, then B is a crystal.*

(ii) *For the crystal base $(L(\infty), B(\infty))$ of the subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.*

(iii) *For $\lambda\in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on T_λ by*

$$\tilde{e}_i(t_\lambda)=\tilde{f}_i(t_\lambda)=0, \quad \varepsilon_i(t_\lambda)=\varphi_i(t_\lambda)=-\infty, \quad \mathrm{wt}(t_\lambda)=\lambda.$$

Definition 2.4. (i) To a crystal B, a colored oriented graph is associated by

$$b_1\xrightarrow{i}b_2 \Longleftrightarrow \tilde{f}_ib_1=b_2.$$

We call this graph the *crystal graph* of B.

(ii) A crystal B is said to be *connected*, if its crystal graph is connected as a graph.

(iii) A crystal B is *free* if $\tilde{e}_i^n(b)\neq 0$ and $\tilde{f}_i^n(b)\neq 0$ for any $b\in B$, $i\in I$ and $n>0$.

2.5. **Positive structure, ultra-discretization and tropicalization.** Let us recall the notions of positive structure and ultra-discretization/tropicalization.

Set $R := \mathbb{C}(c)$ and define

$$\begin{array}{cccc} v\colon & R\setminus\{0\} & \longrightarrow & \mathbb{Z}\\ & f(c) & \mapsto & \deg(f(c)).\end{array}$$

Here deg is the degree of poles at $c=\infty$. Note that for $f_1, f_2\in R\setminus\{0\}$, we have

$$v(f_1f_2)=v(f_1)+v(f_2), \quad v\left(\frac{f_1}{f_2}\right)=v(f_1)-v(f_2). \tag{2.3}$$

We say that a non-zero rational function $f(c)\in\mathbb{C}(c)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients. Note that $f\in\mathbb{C}(c)$ is positive if and only if any pole of f is not a positive number and $f(x)>0$ for any $x>0$.

If $f_1,\ f_2\in R$ are positive, then we have

$$v(f_1+f_2)=\max(v(f_1),v(f_2)). \tag{2.4}$$

Let $T\simeq(\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) := \mathrm{Hom}(T,\mathbb{C}^\times)$ (resp. $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T)$) the lattice of characters (resp. co-characters) of T. We denote by T_+ the set of points x in T such that $\chi(x) > 0$ for any character χ. Then $\big((\mathbb{C}^\times)^n\big)_+ = (\mathbb{R}_{>0})^n$.

A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are a positive linear combination of characters of T.

Definition 2.5. Let $f\colon T \to T'$ be a rational mapping between two algebraic tori T and T'. We say that f is *positive*, if $\mathbb{X}\circ f$ is positive for any character $\mathbb{X}\colon T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T,T')$ the set of positive rational mappings from T to T'.

Note that any $f \in \mathrm{Mor}^+(T,T')$ induces a real analytic map $f_+\colon T_+ \to T'_+$.

Lemma 2.6 ([BK])**.** *For any $f \in \mathrm{Mor}^+(T_1,T_2)$ and $g \in \mathrm{Mor}^+(T_2,T_3)$, the composition $g\circ f$ is well defined and belongs to* $\mathrm{Mor}^+(T_1,T_3)$*.*

By Lemma 2.6, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational mappings. The category $\mathcal{T}_+$ admits products. For two algebraic tori T and T', their product in $\mathcal{T}_+$ coincides with the usual product of T and T'.

Note that $T \mapsto T_+$ gives a functor from $\mathcal{T}_+$ to the category of real analytic manifolds.

Let $f\colon T \to T'$ be a positive rational mapping of algebraic tori T and T'. We define a map $\widehat{f}\colon X_*(T) \to X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi)\rangle = v(\chi\circ f\circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$. Note that $\chi\circ f\circ\xi$ is a rational map from $\mathbb{C}^\times$ to itself.

Lemma 2.7 ([BK])**.** *For any algebraic tori T_1, T_2, T_3, and positive rational mappings $f \in \mathrm{Mor}^+(T_1,T_2)$, $g \in \mathrm{Mor}^+(T_2,T_3)$, we have $\widehat{g\circ f} = \widehat{g}\circ\widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{array}{cccc}
\mathcal{U}D\colon & \mathcal{T}_+ & \longrightarrow & \mathrm{Set} \\[1em]
 & T & \mapsto & X_*(T) \\
 & (f\colon T \to T') & \mapsto & (\widehat{f}\colon X_*(T) \to X_*(T')).
\end{array}$$

Let us come back to the situation in §2.2. Hence G is a Kac-Moody group and T is its Cartan subgroup.

Definition 2.8 ([BK])**.** Let $\mathbb{X} = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ be a G (or $\mathfrak{g}$)-geometric crystal, T' an algebraic torus and $\theta\colon T' \to X$ a birational mapping. The mapping θ is called a *positive structure* on $\mathbb{X}$ if it satisfies:

(i) for any $i \in I$ the rational functions $\gamma_i\circ\theta\colon T' \to \mathbb{C}$ and $\varepsilon_i\circ\theta\colon T' \to \mathbb{C}$ are positive,

(ii) for any $i \in I$, the rational mapping $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c,t) := \theta^{-1}\circ e_i^c\circ\theta(t)$ is positive.

Let $\theta\colon T' \to X$ be a positive structure on a geometric crystal $\mathbb{X} = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{U}D$ to the positive rational mappings $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ and $\gamma_i\circ\theta,\ \varepsilon_i\circ\theta\colon T' \to \mathbb{C}^\times$, we obtain

$$\begin{aligned}
&\tilde{e}_i := \mathcal{U}D(e_{i,\theta})\colon \mathbb{Z}\times X_*(T') \to X_*(T'),\\
&\mathrm{wt}_i := \mathcal{U}D(\gamma_i\circ\theta),\ \varepsilon_i := \mathcal{U}D(\varepsilon_i\circ\theta)\colon X_*(T') \to \mathbb{Z}.
\end{aligned}$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a free pre-crystal structure (see [BK, 2.2]) and we denote it by $\mathcal{U}D_\theta(\mathbb{X}) = \mathcal{U}D_{\theta,T'}(\mathbb{X})$. As for the definition of crystal, see 4.11 or [KKM], [K1], [K2]. We thus have the following theorem:

Theorem 2.9 ([BK], [N1])**.** *For any geometric crystal $\mathbb{X} = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ and a positive structure $\theta\colon T' \to X$, the associated precrystal $\mathcal{U}D_{\theta,T'}(\mathbb{X})$ is a crystal* (see [BK, 2.2]).

Now, let $\mathcal{G}C^+(\mathfrak{g})$ be the category whose object is a triplet $(\mathbb{X}, T', \theta)$ where $\mathbb{X} = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a $\mathfrak{g}$-geometric crystal and $\theta\colon T' \to X$ is a positive structure on $\mathbb{X}$, and a morphism $f\colon (\mathbb{X}_1, T'_1, \theta_1) \to (\mathbb{X}_2, T'_2, \theta_2)$ is given by a morphism $\varphi\colon X_1 \longrightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1 \colon T'_1 \longrightarrow T'_2,$$

is a positive rational mapping. Let $\mathcal{C}R(\mathfrak{g})$ be the category of $\mathfrak{g}$-crystals. Then by the theorem above, we have

Corollary 2.10. *$\mathcal{U}D$ defines a functor*

$$\mathcal{U}D : \mathcal{G}C^+(\mathfrak{g}) \longrightarrow \mathcal{C}R(\mathfrak{g}^L),$$

$$(\mathbb{X}, T', \theta) \mapsto X_*(T'),$$
$$(f\colon (\mathbb{X}_1, T'_1, \theta_1) \to (\mathbb{X}_2, T'_2, \theta_2)) \mapsto (\widehat{f}\colon X_*(T'_1) \to X_*(T'_2)),$$

where $\mathfrak{g}^L$ is the Langlands dual for $\mathfrak{g}$.

We call the functor $\mathcal{U}D$ an *"ultra-discretization"* as in [N1], [N2]. While for a crystal $B \in \mathcal{C}R(\mathfrak{g})$, if there exists an object $(\mathbb{X}, T', \theta)$ in $\mathcal{G}C^+(\mathfrak{g}^L)$, we call $(\mathbb{X}, \theta)$ a *tropicalization* of B.

3. Prehomogeneous geometric crystals

Definition 3.1. Let $\mathbb{X} = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a geometric crystal. We say that $\mathbb{X}$ is *prehomogeneous* if there exists a Zariski open dense subset $\Omega \subset X$ which is an orbit by the actions of the e_i^c's.

The following lemma is obvious.

Lemma 3.2. *Let $\mathbb{X}_j = (X_j, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ $(j = 1, 2)$ be geometric crystals and $\mathbb{X}_1$ prehomogeneous. Let $\Omega_1 \subset X_1$ be an open dense orbit in X_1. For isomorphisms of geometric crystals ϕ, $\phi'\colon \mathbb{X}_1 \to \mathbb{X}_2$, suppose that there exists $p_1 \in \Omega_1$ such that $\phi(p_1) = \phi'(p_1) \in X_2$. Then, we have $\phi = \phi'$ as rational maps.*

The following is the criterion for the prehomogeneity of a geometric crystal.

Theorem 3.3. *Let $\mathbb{X} = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a finite-dimensional positive geometric crystal with the positive structure $\theta\colon T \to X$ and $B := \mathcal{U}D_\theta(\mathbb{X})$ the crystal obtained as the ultra-discretization of $\mathbb{X}$. If B is a connected crystal, then $\mathbb{X}$ is prehomogeneous.*

Proof. Set $m := \dim T = \dim X$. We identify T and X by θ, and take $\mathbf{a} \in T_+$ (see 2.5). Assume that B is connected and $\mathbb{X}(= T)$ is not prehomogeneous. Then there exists a nowhere dense closed subset Z of T such that Z contains $\mathbf{a}$ and is invariant by the actions of the e_i^c's.

Suppose that Z is contained in a variety $\{x \in T \mid \phi(x) = 0\}$ where

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} a_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \mathbb{C}[x_1, \ldots, x_m]$$

is a non-zero polynomial. Set $A := \{\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0} \mid a_\alpha \neq 0\} \neq \emptyset$ and let H be the convex hull of A in $\mathbb{Z}^m$. Let us take an end point $\tilde{\alpha}$ of H. Then $\tilde{\alpha} \in A$ and there exists $c = (c_1, \cdots, c_m) \in B = \mathbb{Z}^m$ such that $\sum_i \tilde{\alpha}_i c_i > \sum_i \alpha_i c_i$ for any $\alpha \in A \setminus \{\tilde{\alpha}\}$. Since the crystal B is connected, there exist $i_1, \ldots, i_l \in I$ and $k_1, \ldots, k_l \in \mathbb{Z}$ such that

$$c = (c_1, \ldots, c_m) = \tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_l}^{k_l}(0, 0, \ldots, 0).$$

We define rational functions $f_1(t), \ldots, f_m(t)$ by

$$(f_1(t), \ldots, f_m(t)) := e_{i_1}^{t^{k_1}} \cdots e_{i_l}^{t^{k_l}} \mathbf{a} \in Z.$$

Since $\tilde{e}_i = \mathcal{UD}(e_{i,\theta})$, we have $v(f_i(t)) = c_i$, where $v \colon \mathbb{C}(t) \setminus \{0\} \to \mathbb{Z}$ is as in 2.5. Thus we have

$$v(\phi(f_1(t), \cdots, f_m(t))) = \sum_i \tilde{\alpha}_i c_i,$$

which implies $\phi(f_1(t), \cdots, f_m(t)) \neq 0$. This is a contradiction. □

By the above proof, we obtain the following:

Corollary 3.4. *In the setting of Theorem* 3.3*, let Ω be the open dense orbit in $\mathbb{X}$ and identify T and X by θ. Then we have $T_+ \subset \Omega$.*

4. Fundamental representations and perfect crystals

4.1. **Affine weights.** Let $\mathfrak{g}$ be an affine Lie algebra and let the sets $\mathfrak{t}$, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ be as in 2.1. We take $\mathfrak{t}$ so that $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$

and let $\mathbf{c} \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i^\vee \subset \mathfrak{g}$ be a unique central element satisfying

$$\{h \in Q^\vee \mid \langle h, \alpha_i\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}.$$

We write ([Kac, 6.1])

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i. \tag{4.1}$$

Let $(\ ,\ )$ be the non-degenerate W-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda\rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\mathrm{cl}} := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl} \colon \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{\mathrm{cl}}$ be the canonical projection. Then we have $\mathfrak{t}^*_{\mathrm{cl}} \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}^*_0 := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda\rangle = 0\}$, $(\mathfrak{t}^*_{\mathrm{cl}})_0 := \mathrm{cl}(\mathfrak{t}^*_0)$. Then we have a positive-definite symmetric form on $(\mathfrak{t}^*_{\mathrm{cl}})_0$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}^*_{\mathrm{cl}}$ $(i \in I)$ be a weight such that $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{i,j}$, which is called a *fundamental weight*. We choose P so that $P_{\mathrm{cl}} := \mathrm{cl}(P)$ coincides with $\bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ and we call P_{cl} the *classical weight lattice*.

4.2. **Affinization.** Let $U_q(\mathfrak{g}) = \langle e_i, f_i, q^h | i \in I,\ h \in P\rangle$ be the quantum affine algebra associated with P and $U'_q(\mathfrak{g}) = \langle e_i, f_i, q^h | i \in I,\ h \in (P_{\rm cl})^*\rangle$ its subalgebra associated with $(P_{\rm cl})^*$. Set $\mathrm{Mod}^f(\mathfrak{g}, P_{\rm cl})$ the category of a finite-dimensional $U'_q(\mathfrak{g})$-module M with a weight decomposition $M = \bigoplus_{\lambda \in P_{\rm cl}} M_\lambda$.

Let M be an object in $\mathrm{Mod}^f(\mathfrak{g}, P_{\rm cl})$. For $l \in \mathbb{C}^\times$, define the $U'_q(\mathfrak{g})$-module M_l as follows: There exists a $\mathbb{C}$-linear bijective homomorphism $\Phi_l : M \longrightarrow M$, such that

$$\begin{aligned} &q^h \Phi_l(u) = \Phi_l(q^h u) \quad \text{for } h \in P^*_{\rm cl}, \\ &e_i \Phi_l(u) = l^{\delta_{i,0}} \Phi_l(e_i u), \\ &f_i \Phi_l(u) = l^{-\delta_{i,0}} \Phi_l(f_i u). \end{aligned}$$

The module $M_l := \Phi_l(M)$ is said to be the *affinization* of M ([KMN1], [K1]).

4.3. **Fundamental representation** $W(\varpi_1)_l$**.** Here we consider the following affine Lie algebras $\mathfrak{g} = A^{(1)}_n$, $B^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$, $A^{(2)}_{2n}$. Let $\{\Lambda_i \,|\, i \in I\}$ be the set of fundamental weights as in §4.1. Let $\varpi_1 := \Lambda_1 - a^\vee_1 \Lambda_0$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as below and $a^\vee_i$ is given in (4.1).

Let $W(\varpi_1)$ be the fundamental representation of $U'_q(\mathfrak{g})$ (see [K1, Section 5]). By [K1, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$-module, an object in $\mathrm{Mod}^f(\mathfrak{g}, P_{\rm lc})$ and has a global basis with a simple crystal. Thus, we can consider its affinization $W(\varpi_1)_l$ ($l \in \mathbb{C}^\times$) specialized at $q = 1$. Then we obtain a finite-dimensional $\mathfrak{g}$-module denoted also by $W(\varpi_1)_l$ in which we shall construct affine geometric crystals in the next section.

Note that $W(\varpi_1)_l$ is an irreducible $\mathfrak{g}$-module for any $\mathfrak{g}$ as above. But, for some i and $\mathfrak{g}$, $W(\varpi_i)_l$ is not necessarily an irreducible $\mathfrak{g}$-module.

Let us see the explicit description of $W(\varpi_1)_l$ for $\mathfrak{g} = A^{(1)}_n$, $B^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$, $A^{(2)}_{2n}$.

4.4. $A^{(1)}_n$ $(n \geq 2)$**.** The Cartan matrix $(a_{ij})_{i,j \in I}$ $(I := \{0, 1, \dots, n\})$ of type $A^{(1)}_n$ is

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv j \pm 1 \mod n+1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mathbf{c} = \sum_{i \in I} \alpha^\vee_i$ and $\delta = \sum_{i \in I} \alpha_i$.

The basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2, \cdots,\ v_{n+1}\}$, and we have

$$\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1} \quad (i = 1 \cdots, n+1),$$

where we understand $\Lambda_{n+1} = \Lambda_0$. The explicit actions of the f_i's are

$$\begin{aligned} &f_i v_i = v_{i+1} \quad (1 \leq i \leq n), \quad f_0 v_{n+1} = l^{-1} v_1, \\ &f_i v_j = 0 \qquad \text{otherwise,} \end{aligned}$$

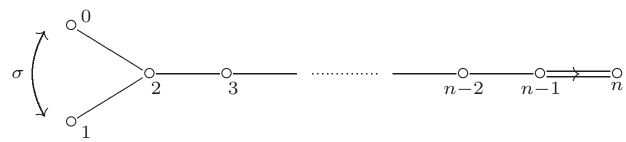
and the explicit actions of e_i's are

$$\begin{aligned} &e_i v_{i+1} = v_i \quad (1 \leq i \leq n), \quad e_0 v_1 = l v_{n+1}, \\ &e_i v_j = 0 \qquad \text{otherwise.} \end{aligned}$$

4.5. $B_n^{(1)}$ $(n \geq 2)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I := \{0, 1, \ldots, n\})$ of type $B_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i-j| = 1 \text{ and } (i,j) \neq (0,1),(1,0),(n,n-1) \text{ or } (i,j) = (0,2),(2,0), \\ -2 & (i,j) = (n,n-1), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma : \alpha_0 \leftrightarrow \alpha_1$ which will be needed later. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n-1} \alpha_i^\vee + \alpha_n^\vee, \qquad \delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n} \alpha_i.$$

The basis of $W(\varpi_1)_l$ is $\{v_1, v_2, \cdots, v_n, v_0, v_{\overline{n}}, \cdots, v_{\overline{2}}, v_{\overline{1}}\}$, and we have

$$\begin{aligned} &\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 0, 2, n), \\ &\mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}}) = \Lambda_0 + \Lambda_1 - \Lambda_2, \\ &\mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n, \qquad \mathrm{wt}(v_0) = 0. \end{aligned}$$

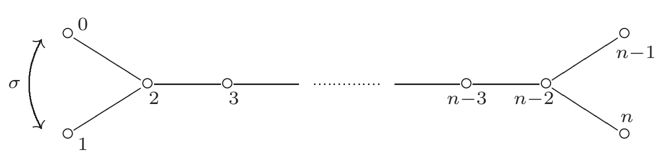
The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll} f_i v_i = v_{i+1}, & f_i v_{\overline{i+1}} = v_{\overline{i}} \ \ (1 \leq i < n), & e_i v_{i+1} = v_i, & e_i v_{\overline{i}} = v_{\overline{i+1}} \ \ (1 \leq i < n), \\ f_n v_n = v_0, & f_n v_0 = 2v_{\overline{n}}, & e_n v_0 = 2v_n, & e_n v_{\overline{n}} = v_0, \\ f_0 v_{\overline{2}} = l^{-1} v_1, & f_0 v_{\overline{1}} = l^{-} v_2, & e_0 v_1 = l v_{\overline{2}}, & e_0 v_2 = l v_{\overline{1}}, \\ f_i v_j = 0 & \text{otherwise,} & e_i v_j = 0 & \text{otherwise,} \end{array}$$

4.6. $D_n^{(1)}$ $(n \geq 4)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I := \{0, 1, \ldots, n\})$ of type $D_n^{(1)}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i-j| = 1 \text{ and } 1 \leq i,j \leq n-1 \\ & \qquad\qquad \text{or } (i,j) = (0,2),(2,0),(n-2,n),(n,n-2), \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon \alpha_0 \leftrightarrow \alpha_1$ and $\sigma\alpha_i = \alpha_i$ for $i \neq 0, 1$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n-2} \alpha_i^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee, \qquad \delta = \alpha_0 + \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n.$$

The basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2,\cdots,\ v_n,\ v_{\overline{n}},\cdots,\ v_{\overline{2}},\ v_{\overline{1}}\}$, and we have

$$\begin{aligned}
&\mathrm{wt}(v_i)=\Lambda_i-\Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}})=\Lambda_{i-1}-\Lambda_i \qquad (i\neq 2,n-1),\\
&\mathrm{wt}(v_2)=-\Lambda_0-\Lambda_1+\Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}})=\Lambda_0+\Lambda_1-\Lambda_2,\\
&\mathrm{wt}(v_{n-1})=\Lambda_{n-1}+\Lambda_n-\Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n-1}})=\Lambda_{n-2}-\Lambda_{n-1}-\Lambda_n.
\end{aligned}$$

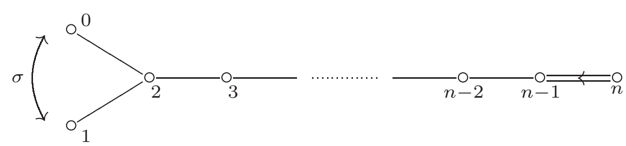
The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll}
f_iv_i=v_{i+1}, & f_iv_{\overline{i+1}}=v_{\overline{i}}\ \ (1\le i<n), & e_iv_{i+1}=v_i, & e_iv_{\overline{i}}=v_{\overline{i+1}}\ \ (1\le i<n),\\
f_nv_n=v_{\overline{n-1}}, & f_nv_{n-1}=v_{\overline{n}}, & e_nv_{\overline{n}}=v_{n-1}, & e_nv_{\overline{n-1}}=v_n,\\
f_0v_{\overline{2}}=l^{-1}v_1, & f_0v_{\overline{1}}=l^{-1}v_2, & e_0v_1=lv_{\overline{2}}, & e_0v_2=lv_{\overline{1}},\\
f_iv_j=0 & \text{otherwise}, & e_iv_j=0 & \text{otherwise}.
\end{array}$$

4.7. $A^{(2)}_{2n-1}$ $(n\ge 3)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I:=\{0,1,\ldots,n\})$ of type $A^{(2)}_{2n-1}$ is

$$a_{ij}=\begin{cases}2 & i=j,\\ -1 & |i-j|=1 \text{ and } 1\le i,j\le n-1 \text{ or } (i,j)=(0,2),(2,0),(n,n-1),\\ -2 & (i,j)=(n-1,n),\\ 0 & \text{otherwise.}\end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon\alpha_0\leftrightarrow\alpha_1$. We have

$$\mathbf{c}=\alpha_0^\vee+\alpha_1^\vee+2\sum_{i=2}^{n}\alpha_i^\vee, \qquad \delta=\alpha_0+\alpha_1+2\sum_{i=2}^{n-1}\alpha_i+\alpha_n.$$

The basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2,\cdots,\ v_n,\ v_{\overline{n}},\cdots,\ v_{\overline{2}},\ v_{\overline{1}}\}$, and we have

$$\begin{aligned}
&\mathrm{wt}(v_i)=\Lambda_i-\Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}})=\Lambda_{i-1}-\Lambda_i \qquad (i\neq 2),\\
&\mathrm{wt}(v_2)=-\Lambda_0-\Lambda_1+\Lambda_2, \qquad \mathrm{wt}(v_{\overline{2}})=\Lambda_0+\Lambda_1-\Lambda_2.
\end{aligned}$$

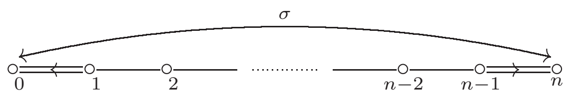
The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll}
f_iv_i=v_{i+1}, & f_iv_{\overline{i+1}}=v_{\overline{i}}\ \ (1\le i<n), & e_iv_{i+1}=v_i, & e_iv_{\overline{i}}=v_{\overline{i+1}}\ \ (1\le i<n),\\
f_nv_n=v_{\overline{n}}, & & e_nv_{\overline{n}}=v_n, & \\
f_0v_{\overline{2}}=l^{-1}v_1, & f_0v_{\overline{1}}=l^{-1}v_2, & e_0v_1=lv_{\overline{2}}, & e_0v_2=lv_{\overline{1}},\\
f_iv_j=0 & \text{otherwise}, & e_iv_j=0 & \text{otherwise}.
\end{array}$$

4.8. $D^{(2)}_{n+1}$ $(n\ge 2)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I:=\{0,1,\ldots,n\})$ of type $D^{(2)}_{n+1}$ is

$$a_{ij}=\begin{cases}2 & i=j,\\ -1 & |i-j|=1 \text{ and } (i,j)\neq(0,1),(n,n-1),\\ -2 & (i,j)=(0,1),(n,n-1),\\ 0 & \text{otherwise.}\end{cases}$$

The Dynkin diagram is



where σ is the Dynkin diagram automorphism $\sigma\colon \alpha_i \leftrightarrow \alpha_{n-i}$ $(i=0,1\cdots,n)$. We have

$$\mathbf{c} = \alpha_0^\vee + \alpha_1^\vee + 2\sum_{i=2}^{n-1}\alpha_i^\vee + \alpha_n^\vee, \qquad \delta = \sum_{i=0}^{n}\alpha_i.$$

The basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2,\cdots,\ v_n,\ v_0,\ v_{\overline{n}},\cdots,\ v_{\overline{2}},\ v_{\overline{1}},\ \phi\}$, and we have

$$\begin{aligned}
&\mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \qquad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \qquad (i \neq 0,1,n),\\
&\mathrm{wt}(v_1) = \Lambda_1 - 2\Lambda_0, \qquad \mathrm{wt}(v_{\overline{1}}) = 2\Lambda_0 - \Lambda_1,\\
&\mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, \qquad \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - 2\Lambda_n,\\
&\mathrm{wt}(v_0) = 0, \qquad \mathrm{wt}(\phi) = 0
\end{aligned}$$

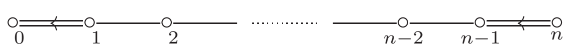
The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll}
f_iv_i = v_{i+1}, & f_iv_{\overline{i+1}} = v_{\overline{i}}\ \ (1\le i<n), & e_iv_{i+1} = v_i, & e_iv_{\overline{i}} = v_{\overline{i+1}}\ \ (1\le i<n),\\
f_nv_n = v_0, & f_nv_0 = 2v_{\overline{n}}, & e_nv_0 = 2v_n, & e_nv_{\overline{n}} = v_0,\\
f_0v_{\overline{1}} = l^{-1}\phi, & f_0\phi = 2l^{-1}v_1, & e_0v_1 = l\phi, & e_0\phi = 2lv_{\overline{1}},\\
f_iv_j = 0,\ \ f_i\phi = 0 & \text{otherwise}, & e_iv_j = 0, & e_i\phi = 0\ \ \text{otherwise}.
\end{array}$$

4.9. $A_{2n}^{(2)}$ $(n \geq 2)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I := \{0,1,\ldots,n\})$ of type $A_{2n}^{(2)}$ is

$$a_{ij} = \begin{cases} 2 & i=j,\\ -1 & |i-j|=1 \text{ and } (i,j)\neq(0,1),(n-1,n),\\ -2 & (i,j)=(0,1),(n-1,n),\\ 0 & \text{otherwise.}\end{cases}$$

Then the Dynkin diagram is



Note that there exists no Dynkin diagram automorphism in this case. We have

$$\mathbf{c} = \alpha_0^\vee + 2\sum_{i=1}^{n}\alpha_i^\vee, \qquad \delta = 2\sum_{i=0}^{n-1}\alpha_i + \alpha_n.$$

In this case, we denote this type by $(A_{2n}^{(2)},\varpi_1)$ in order to distinguish it from the type $(A_{2n}^{(2)}\dagger,\varpi_1)$. Then the basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2,\cdots,\ v_n,\ v_{\overline{n}},\cdots,\ v_{\overline{2}},\ v_{\overline{1}},\ \emptyset\}$, and we have

$$\begin{aligned}
&\mathrm{wt}(v_1) = \Lambda_1 - 2\Lambda_0, \quad \mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, \quad \mathrm{wt}(v_{\overline{i}}) = \Lambda_{i-1} - \Lambda_i \ \ (i=2,\ldots,n),\\
&\mathrm{wt}(v_{\overline{1}}) = 2\Lambda_0 - \Lambda_1, \quad \mathrm{wt}(\emptyset) = 0.
\end{aligned}$$

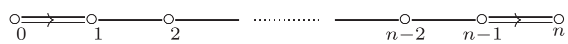
The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll} f_i v_i = v_{i+1}, & f_i v_{\overline{i+1}} = v_{\bar{i}} \ \ (1 \le i < n), & e_i v_{i+1} = v_i, & e_i v_{\bar{i}} = v_{\overline{i+1}} \ \ (1 \le i < n), \\ f_n v_n = v_{\bar{n}}, & & e_n v_n = v_n, & \\ f_0 v_{\bar{1}} = l^{-1}\emptyset, & f_0 \emptyset = 2l^{-1} v_1, & e_0 v_1 = l\emptyset, & e_0 \emptyset = 2l v_{\bar{1}}, \\ f_i v_j = 0 & \text{otherwise}, & e_i v_j = 0, & \text{otherwise}. \end{array}$$

4.10. $A_{2n}^{(2)\dagger}$ $(n \geq 2)$**.** The Cartan matrix $(a_{ij})_{i,j\in I}$ $(I := \{0, 1, \ldots, n\})$ of type $A_{2n}^{(2)\dagger}$ is

$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i-j| = 1 \text{ and } (i,j) \neq (1,0), (n, n-1), \\ -2 & (i,j) = (1,0), (n, n-1), \\ 0 & \text{otherwise.} \end{cases}$$

Then the Dynkin diagram is



Note that there exists no Dynkin diagram automorphism in this case. We have

$$\mathbf{c} = 2\sum_{i=0}^{n-1} \alpha_i^\vee + \alpha_n^\vee, \qquad \delta = \alpha_0 + 2\sum_{i=1}^{n} \alpha_i.$$

In this case, we denote this type by $(A_{2n}^{(2)\dagger}, \varpi_1)$ in order to distinguish it from the type $(A_{2n}^{(2)}, \varpi_1)$. Then the basis of $W(\varpi_1)_l$ is $\{v_1,\ v_2, \cdots,\ v_n,\ v_0,\ v_{\bar{n}}, \cdots,\ v_{\bar{2}},\ v_{\bar{1}}\}$, and we have

$$\begin{array}{lll} \mathrm{wt}(v_i) = \Lambda_i - \Lambda_{i-1}, & \mathrm{wt}(v_{\bar{i}}) = \Lambda_{i-1} - \Lambda_i & (i = 1, \ldots, n-1), \\ \mathrm{wt}(v_n) = 2\Lambda_n - \Lambda_{n-1}, & \mathrm{wt}(v_0) = 0, & \mathrm{wt}(v_{\bar{1}}) = \Lambda_{n-1} - 2\Lambda_n. \end{array}$$

The explicit forms of the actions by f_i's and e_i's are

$$\begin{array}{llll} f_i v_i = v_{i+1}, & f_i v_{\overline{i+1}} = v_{\bar{i}} \ \ (1 \le i < n), & e_i v_{i+1} = v_i, & e_i v_{\bar{i}} = v_{\overline{i+1}} \ \ (1 \le i < n), \\ f_n v_n = v_{\bar{0}}, & f_n v_0 = 2 v_{\bar{n}}, & e_n v_0 = 2 v_n, & e_n v_{\bar{n}} = v_0, \\ f_0 v_{\bar{1}} = l^{-1} v_1, & & e_0 v_1 = l v_{\bar{1}}, & \\ f_i v_j = 0, & \text{otherwise}, & e_i v_j = 0, & \text{otherwise}. \end{array}$$

4.11. **Limit of perfect crystals.** We review the limits of perfect crystals following [KKM]. (See also [KMN1], [KMN2].)

Let $\mathfrak{g}$ be an affine Lie algebra, P_{cl} the classical weight lattice as above and for $l \in \mathbb{Z}_{>0}$ set $(P_{\mathrm{cl}})_l^+ := \{\lambda \in P_{\mathrm{cl}} \mid \langle \mathbf{c}, \lambda \rangle = l,\ \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$.

Definition 4.1. We say that a crystal B is *perfect* of level l if:

(i) $B \otimes B$ is connected as a crystal graph.

(ii) There exists $\lambda_0 \in P_{\mathrm{cl}}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \mathrm{cl}(\alpha_i), \qquad \sharp B_{\lambda_0} = 1.$$

(iii) There exists a finite-dimensional $U'_q(\mathfrak{g})$-module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$.

(iv) Set $\varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b)\Lambda_i$. For any $b \in B$, $\langle \mathbf{c}, \varepsilon(b)\rangle \geq l$ and the maps $\varepsilon, \varphi\colon B^{\min} := \{b \in B \mid \langle c, \varepsilon(b)\rangle = l\} \longrightarrow (P_{\rm cl}^+)_l$ are bijective.

For an affine Lie algebra $\mathfrak{g}$, let $\{B_l(\mathfrak{g})\}_{l\geq 1}$ be a family of perfect crystals of level l and set $J := \{(l,b) \mid l > 0,\, b \in B_l^{\min}\}$.

Definition 4.2. A crystal $B_\infty = B_\infty(\mathfrak{g})$ with an element b_∞ is called the *limit of* $\{B_l\}_{l\geq 1}$ if:

(i) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.

(ii) For any $(l,b) \in J$, there exists an embedding of crystals:

$$f_{(l,b)}\colon \quad T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty, \qquad t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty.$$

(iii) $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$.

As for the crystal T_λ, see Example 2.3(iii). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

Remark. By the definition of perfect crystals, any perfect crystal is connected and then the limit of a coherent family of perfect crystals is also connected.

5. Affine geometric crystals

Following the method in [KNO], we shall construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l$ $(l \in \mathbb{C}^\times)$ in the $\mathfrak{g}$-module $W(\varpi_1)_l$, the affinization of the fundamental representation $W(\varpi_1)$.

5.1. **Translation $t(\widetilde{\varpi}_1)$.** For $\xi_0 \in (\mathfrak{t}^*_{\rm cl})_0$, let $t(\xi_0)$ be as in [K1, Section 4], that is,

$$t(\xi_0)(\lambda) := \lambda + (\delta,\lambda)\xi - (\xi,\lambda)\delta - \frac{(\xi,\xi)}{2}(\delta,\lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\mathrm{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of ξ, and it is well defined.

Let $c_i^\vee$ be as follows ([K2]):

$$c_i^\vee := \max(1, \frac{2}{(\alpha_i,\alpha_i)}). \tag{5.1}$$

Then $t(m\varpi_i)$ belongs to the extended Weyl group $\widetilde{W}$ if and only if $m \in c_i^\vee\mathbb{Z}$. Setting $\widetilde{\varpi}_i := c_i^\vee \varpi_i$ $(i \in I)$, $t(\widetilde{\varpi}_1)$ is expressed as follows (see e.g. [KOTUY]):

$$t(\widetilde{\varpi}_1) = \begin{cases} \iota(s_n s_{n-1}\cdots s_2 s_1) & A_n^{(1)} \text{ case},\\ \iota(s_1\cdots s_n)(s_{n-1}\cdots s_2 s_1) & B_n^{(1)},\ A_{2n-1}^{(2)} \text{ cases},\\ (s_0 s_1\cdots s_n)(s_{n-1}\cdots s_2 s_1) & C_n^{(1)},\ D_{n+1}^{(2)} \text{ cases},\\ \iota(s_1\cdots s_n)(s_{n-2}\cdots s_2 s_1) & D_n^{(1)} \text{ case},\\ (s_0 s_1\cdots s_n)(s_{n-1}\cdots s_2 s_1) & A_{2n}^{(2)},\ A_{2n}^{(2)\dagger} \text{ cases},\end{cases}$$

where ι is the Dynkin diagram automorphism

$$\iota = \begin{cases} \sigma & \mathfrak{g} = A_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)},\\ \alpha_0 \leftrightarrow \alpha_1 \text{ and } \alpha_{n-1} \leftrightarrow \alpha_n & \mathfrak{g} = D_n^{(1)}.\end{cases}$$

Now, we know that each $t(\widetilde{\varpi}_1)$ is in the form w_1 or $\iota\cdot w_1$ for some $w_1 \in W$, e.g., $w_1 = s_n\cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1\cdots s_n)(s_{n-1}\cdots s_1)$ for $B_n^{(1)}$, etc., ...

In the case $A^{(2)}_{2n}$ (resp. $\mathfrak{g}=A^{(2)\dagger}_{2n}$), $\eta:=\mathrm{wt}(v_{\overline{n}})=\Lambda_{n-1}-\Lambda_n$ (resp. $\Lambda_{n-1}-2\Lambda_n$) is a unique weight of $W(\varpi_1)_l$ which satisfies $\langle\alpha_i^\vee,\eta\rangle\geq 0$ for $i\neq n$. For this η we have

$$t(\eta)=(s_ns_{n-1}\cdots s_1)(s_0s_1\cdots s_{n-1})=:w_2, \tag{5.2}$$

which will be used later.

5.2. **Affine geometric crystals in $W(\varpi_1)_l$.** Let σ be the Dynkin diagram automorphism as in 4.4–4.8 and let $w_1=s_{i_1}\cdots s_{i_k}$ be as in the previous subsection. Let H be an element in $\mathfrak{t}$ such that

$$\alpha_i(H)=\begin{cases}1 & \text{if } i=1 \text{ and } \mathfrak{g}\neq D^{(2)}_{n+1},A^{(2)}_{2n},A^{(2)\dagger}_{2n},\\ 2 & \text{if } i=1 \text{ and } \mathfrak{g}= D^{(2)}_{n+1},A^{(2)}_{2n},A^{(2)\dagger}_{2n},\\ -1 & \text{if } i=0 \text{ and } \mathfrak{g}\neq D^{(2)}_{n+1},A^{(2)}_{2n},A^{(2)\dagger}_{2n},\\ -2 & \text{if } i=0 \text{ and } \mathfrak{g}= D^{(2)}_{n+1},A^{(2)}_{2n},A^{(2)\dagger}_{2n},\\ 0 & \text{otherwise.}\end{cases}$$

Set

$$\mathcal{V}(\mathfrak{g})_l:=\{v(x_1,\ldots,x_k):=Y_{i_1}(x_1)\cdots Y_{i_k}(x_k)l^Hv_1\,\big|\,x_1,\ldots,x_k\in\mathbb{C}^\times\}\subset W(\varpi_1)_l. \tag{5.3}$$

Let $\mathfrak{g}_0\subset\mathfrak{g}$ (resp. $G_0\subset G$) be a simple Lie algebra (resp. simple algebraic group) corresponding to the index set $I_0:=I\setminus\{0\}$. Since the vector v_1 satisfies $x_i(c)v_1=v_1$ for any $i\in I_0$, the actions of e_i^c $(i\in I_0)$ on $v(x)$ and $Y_{i_1}(x_1)\cdots Y_{i_k}(x_k)\cdot l^H$ coincide with each other. Therefore, $\mathcal{V}(\mathfrak{g})_l$ has a G_0-geometric crystal structure the same as that of $B^-_{i_1\cdots i_k}\cdot l^H$ (see 2.3), Moreover, $(\mathbb{C}^\times)^k\to\mathcal{V}(\mathfrak{g})_l$ given by $(x_1,\ldots,x_k)\mapsto v(x_1,\ldots,x_k)$ is a birational map. We shall define a G-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by using the Dynkin diagram automorphism σ except for $A^{(2)}_{2n}$. This σ induces an automorphism of $W(\varpi_1)_l$, which is denoted by $\sigma_l\colon W(\varpi_1)_l\longrightarrow W(\varpi_1)_l$. The following theorems are analogous results to Theorem 5.1 and 5.2 in [KNO].

Theorem 5.1. (i) *Case $\mathfrak{g}\neq A^{(2)}_{2n}$, $A^{(2)\dagger}_{2n}$. For $x=(x_1\cdots,x_k)\in(\mathbb{C}^\times)^k$, there exist a unique $y=(y_1,\ldots,y_k)\in(\mathbb{C}^\times)^k$ and a positive rational function $a(x)$ such that*

$$v(y)=a(x)\sigma_l(v(x)),\quad \varepsilon_{\sigma(i)}(v(y))=\varepsilon_i(v(x))\quad \textit{if } i,\sigma(i)\neq 0. \tag{5.4}$$

(ii) *Case $\mathfrak{g}=A^{(2)}_{2n}$ (resp. $A^{(2)\dagger}_{2n}$). Associated with w_1 and w_2 as in the previous section, we define*

$$\mathcal{V}(\mathfrak{g})_l:=\{v_1(x)=Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)l^Hv_1\,\big|\,x_i,\bar{x}_i\in\mathbb{C}^\times\},$$
$$\mathcal{V}_2(\mathfrak{g})_l:=\{v_2(y)=Y_n(y_n)\cdots Y_1(y_1)Y_0(y_0)Y_1(\overline{y}_1)\cdots Y_{n-1}(\overline{y}_{n-1})l^{H'}v_{\overline{n}}\,\big|\,y_i,\overline{y}_i\in\mathbb{C}^\times\},$$

where $\alpha_0(H')=2$ (resp. $\alpha_0(H')=2$), $\alpha_n(H')=-4$ (resp. $\alpha_n(H')=-1$) and $\alpha_i(H')=0$ otherwise. (Note that $\mathrm{wt}(v_1)(H)=\mathrm{wt}(v_{\overline{n}})(H')$.) For any $x\in(\mathbb{C}^\times)^{2n}$ there exist a unique $y\in(\mathbb{C}^\times)^{2n}$ and a rational function $a(x)$ such that $v_2(y)=a(x)v_1(x)$.

Now, using this theorem, we define the rational mapping

$$\overline{\sigma}\colon \mathcal{V}(\mathfrak{g})_l \longrightarrow \mathcal{V}(\mathfrak{g})_l, \quad v(x)\mapsto v(y) \quad (\mathfrak{g}\neq A^{(2)}_{2n}, {A^{(2)}_{2n}}^{\dagger}), \qquad \overline{\sigma}\colon \mathcal{V}(\mathfrak{g})_l \longrightarrow \mathcal{V}_2(\mathfrak{g})_l, \quad v_1(x)\mapsto v_2(y) \quad (\mathfrak{g}= A^{(2)}_{2n},\ {A^{(2)}_{2n}}^{\dagger}), \tag{5.5}$$

Theorem 5.2. *The rational mapping $\overline{\sigma}$ is birational. If we define a $\mathfrak{g}_0$-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by the one on $B^-_{i_1\cdots i_k}\cdot l^H$ $(w_1 = s_{i_1}\cdots s_{i_k})$ as in 2.3 and a rational $\mathbb{C}^\times$-action $e_0 : \mathbb{C}^\times\times\mathcal{V}(\mathfrak{g})_l \to \mathcal{V}(\mathfrak{g})_l$ and rational functions* wt_0 *and ε_0 on $\mathcal{V}(\mathfrak{g})_l$ by*

$$\begin{cases} e^c_0 := \overline{\sigma}^{-1}\circ e^c_{\sigma(0)}\circ\overline{\sigma}, \quad \varepsilon_0 := \varepsilon_{\sigma(0)}\circ\overline{\sigma}, \quad \gamma_0 := \gamma_{\sigma(0)}\circ\overline{\sigma}, \\ \qquad\qquad\qquad \text{for } \mathfrak{g}\neq A^{(2)}_{2n},\ {A^{(2)}_{2n}}^{\dagger}, \\ e^c_0 := \overline{\sigma}^{-1}\circ e^c_0\circ\overline{\sigma}, \quad \varepsilon_0 := \varepsilon_0\circ\overline{\sigma}, \quad \gamma_0 := \gamma_0\circ\overline{\sigma}, \\ \qquad\qquad\qquad \text{for } \mathfrak{g}= A^{(2)}_{2n},\ {A^{(2)}_{2n}}^{\dagger}. \end{cases} \tag{5.6}$$

then $(\mathcal{V}(\mathfrak{g})_l, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ turns out to be an affine $\mathfrak{g}$-geometric crystal.

Remark. In the case $\mathfrak{g} = A^{(2)}_{2n}$ and ${A^{(2)}_{2n}}^{\dagger}$, $\mathcal{V}_2(\mathfrak{g})_l$ has a $\mathfrak{g}_{I\setminus\{n\}}$-geometric crystal structure. Thus, $e_0, \gamma_0, \varepsilon_0$ are well defined on $\mathcal{V}_2(\mathfrak{g})_l$.

The following lemma shows Theorem 5.2 partially.

Lemma 5.3. *Suppose that $\mathfrak{g}\neq A^{(2)}_{2n}$, ${A^{(2)}_{2n}}^{\dagger}$. If there exists $\overline{\sigma}$ as above and*

$$e^c_{\sigma(i)} = \overline{\sigma}\circ e^c_i\circ\overline{\sigma}^{-1}, \quad \gamma_i = \gamma_{\sigma(i)}\circ\overline{\sigma}, \quad \varepsilon_i = \varepsilon_{\sigma(i)}\circ\overline{\sigma}, \tag{5.7}$$

for $i\neq\sigma^{-1}(0), 0$, then we obtain

(i) $e^{c_1}_0 e^{c_2}_i = e^{c_2}_i e^{c_1}_0$ *if $a_{0i} = a_{i0} = 0$,*
$e^{c_1}_0 e^{c_1c_2}_i e^{c_2}_0 = e^{c_2}_i e^{c_1c_2}_0 e^{c_2}_i$ *if $a_{0i}a_{i0} = 1$,*
$e^{c_1}_0 e^{c_1^2c_2}_i e^{c_1c_2}_0 e^{c_2}_i = e^{c_2}_i e^{c_1c_2}_0 e^{c_1^2c_2}_i e^{c_1}_0$ if $a_{0i} = -2$, $a_{i0} = -1$,
$e^{c_2}_0 e^{c_1c_2}_i e^{c_1^2c_2}_0 e^{c_1}_i = e^{c_1}_i e^{c_1^2c_2}_0 e^{c_1c_2}_i e^{c_2}_0$ if $a_{0i} = -1$, $a_{i0} = -2$.

(ii) $\gamma_0(e^c_i(v(x))) = c^{a_{i0}}\gamma_0(v(x))$ *and* $\gamma_i(e^c_0(v(x))) = c^{a_{0i}}\gamma_i(v(x))$.

(iii) $\varepsilon_0(e^c_0(v(x))) = c^{-1}\varepsilon_0(v(x))$.

Proof. For example, we have

$$\begin{aligned} \gamma_0(e^c_i(v(x))) &= \gamma_{\sigma(0)}(\overline{\sigma}e^c_i\overline{\sigma}^{-1}(\overline{\sigma}(v(x)))) \\ &= \gamma_{\sigma(0)}(e^c_{\sigma(i)}(\overline{\sigma}(v(x)))) = c^{a_{\sigma(i),\sigma(0)}}\gamma_{\sigma(0)}(\overline{\sigma}(v(x))) \\ &= c^{a_{i,0}}\gamma_0(v(x)), \end{aligned}$$

where we use $a_{\sigma(i),\sigma(0)} = a_{i0}$ in the last equality. The other assertions are obtained similarly. □

In the rest of this section, we shall prove Theorem 5.1 and Theorem 5.2 in case-by-case methods.

5.3. $A^{(1)}_n$**-case** $(n\geq 2)$**.** We have $w_1 := s_ns_{n-1}\cdots s_2s_1$, and

$$\mathcal{V}(A^{(1)}_n)_l := \{Y_n(x_n)\cdots Y_2(x_2)Y_1(x_1)l^Hv_1 \mid x_i\in\mathbb{C}^\times\}\subset W(\varpi_1)_l.$$

Since $y_i(\frac{1}{c}) = \exp(\frac{f_i}{c}) = 1 + c^{-1}f_i$ on $W(\varpi_1)$, $v(x) = Y_n(x_n)\cdots Y_2(x_2)Y_1(x_1)l^H v_1$ is explicitly written as

$$v(x) = v(x_1, \ldots, x_n) = l^m \left(\sum_{i=1}^{n} x_i v_i + v_{n+1}\right),$$

where $m = \varpi_1(H)$. Let $\sigma\colon \alpha_k \mapsto \alpha_{k+1}$ $(k \in I)$ be the Dynkin diagram automorphism for $A_n^{(1)}$, which gives rise to the automorphism $\sigma_l\colon W(\varpi_1)_l \to W(\varpi_1)_l$. We have

$$\sigma_l(v_i) = \begin{cases} v_{i+1} & i \neq n+1, \\ l^{-1}v_1 & i = n+1. \end{cases}$$

Then, we obtain

$$\sigma_l(v(x)) = l^m \left(l^{-1}v_1 + \sum_{i=1}^{n} x_i v_{i+1}\right).$$

Then the equation $v(y) = a(x)\sigma_l(v(x))$, i.e.,

$$\sum_{i=1}^{n} y_i v_i + v_{n+1} = a(x)\left(l^{-1}v_1 + \sum_{i=1}^{n} x_i v_{i+1}\right)$$

is solved by

$$a(x) = \frac{1}{x_n}, \quad y_1 = \frac{1}{lx_n}, \quad y_i = \frac{x_{i-1}}{x_n}\ (i = 2, \ldots, n), \tag{5.8}$$

that is,

$$\overline{\sigma}(v(x_1\cdots, x_n)) = v(\frac{1}{lx_n}, \frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}). \tag{5.9}$$

The A_n-geometric crystal structure on $\mathcal{V}(A_n^{(1)})_l$ induced from the one on $B^-_{w_1} \cdot l^H$ is given by:

$$e_i^c(v(x_1, \ldots, x_n)) = v(x_1, \ldots, cx_i, \ldots, x_n) \quad (i = 1, \ldots, n), \tag{5.10}$$

(5.11)

$$\gamma_1(v(x)) = \frac{lx_1^2}{x_2}, \quad \gamma_i(v(x)) = \frac{x_i^2}{x_{i-1}x_{i+1}} \quad (i = 2, \ldots, n-1), \quad \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}},$$

$$\varepsilon_i(v(x)) = \frac{x_{i+1}}{x_i} \quad (i = 1, \ldots, n-1), \quad \varepsilon_n(v(x)) = \frac{1}{x_n}. \tag{5.12}$$

Then we have

$$\varepsilon_{i+1}(\overline{\sigma}(v(x))) = \begin{cases} \frac{x_{i+1}}{x_i} & \text{if } i = 1\cdots, n-2, \\ \frac{x_n}{x_{n-1}} & \text{if } i = n-1, \end{cases}$$

which implies $\varepsilon_{\sigma(i)}(\overline{\sigma}(v(x))) = \varepsilon_i(v(x))$, and then we completed the proof of Theorem 5.1 for $A_n^{(1)}$.

Now, we define e_0^c, γ_0 and ε_0 by

$$e_0^c := \overline{\sigma}^{-1} \circ e_1^c \circ \overline{\sigma}, \quad \gamma_0 := \gamma_1 \circ \overline{\sigma}, \quad \varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}. \tag{5.13}$$

Their explicit forms are

$$e_0^c(v(x)) = v(\frac{x_1}{c}, \frac{x_2}{c}, \ldots, \frac{x_n}{c}), \tag{5.14}$$

$$\gamma_0(v(x)) = \frac{1}{lx_1x_n}, \quad \varepsilon_0(v(x)) = lx_1. \tag{5.15}$$

Thus, we can check (5.7) easily, and then Lemma 5.3 reduces the proof of Theorem 5.2 to the statements:

$$e_0^{c_1} e_n^{c_1 c_2} e_0^{c_2} = e_n^{c_2} e_0^{c_1 c_2} e_n^{c_1}, \tag{5.16}$$

$$\gamma_0(e_n^c(v(x))) = c^{-1}\gamma_0(v(x)), \quad \gamma_n(e_0^c(v(x))) = c^{-1}\gamma_n(v(x)). \tag{5.17}$$

These are immediate from (5.10)–(5.15). Thus, we obtain Theorem 5.2 for $A_n^{(1)}$.

Let us introduce the $A_n^{(1)}$-geometric crystal $\mathcal{B}_L(A_n^{(1)})$ ($L \in \mathbb{C}^\times$) ([KOTY]):

$$\mathcal{B}_L(A_n^{(1)}) := \{l = (l_1, \ldots, l_n, l_{n+1}) \in (\mathbb{C}^\times)^{n+1} | l_1 \cdots l_{n+1} = L\},$$
$$e_i^c(l) = (\cdots, cl_i, c^{-1}l_{i+1}, \ldots), \quad \gamma_i(l) = \frac{l_i}{l_{i+1}}, \quad \varepsilon_i(l) = l_{i+1} \quad (i = 0, 1, \ldots, n),$$

where we understand $l_0 = l_{n+1}$.

We have $\mathcal{B}_L(A_n^{(1)}) \cong \mathcal{V}(A_n^{(1)})_L$. Indeed, defining $\phi : \mathcal{B}_L(A_n^{(1)}) \to \mathcal{V}(A_n^{(1)})_L$ by $\phi(l_1, \cdots, l_{n+1}) = v(\frac{l_1}{L}, \frac{l_1 l_2}{L}, \cdots, \frac{l_1 \cdots l_n}{L})$, it is easy to see that ϕ is an isomorphism of geometric crystals.

5.4. $B_n^{(1)}$**-case** ($n \geq 2$)**.** We have $w_1 = s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_1$, and

$$\mathcal{V}(B_n^{(1)})_l := \{v(x) = Y_1(x_1) \cdots Y_n(x_n) Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)_l$ as in 4.5 that

$$v(x_1, \ldots, x_n, \overline{x}_{n-1}, \ldots, \overline{x}_1) = l^m \left\{ \left(\sum_{i=1}^n \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^n x_{i-1} v_{\bar{i}} \right) + v_{\bar{1}} \right\},$$

$$\text{where} \quad m := \varpi_1(H) \quad \text{and} \quad \xi_i(x) := \begin{cases} x_1 \overline{x}_1 & i = 1, \\ \dfrac{x_{i-1}\overline{x}_{i-1} + x_i \overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \dfrac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

The automorphism $\sigma_l : W(\varpi_1)_l \to W(\varpi_1)_l$ is given as

$$\sigma_l v_1 = l v_{\bar{1}}, \quad \sigma_l v_{\bar{1}} = l^{-1} v_1, \sigma_l v_k = v_k \text{ otherwise}. \tag{5.18}$$

Then we have

$$\sigma_l(v(x)) = l^m \left\{ l^{-1} v_1 + \left(\sum_{i=2}^n \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=2}^n x_{i-1} v_{\bar{i}} \right) + l x_1 \overline{x}_1 v_{\bar{1}} \right\},$$

The equation $v(y) = a(x)\sigma_l(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n-1})$ has a unique solution:

$$\begin{aligned} a(x) = \frac{1}{l x_1 \overline{x}_1}, \quad y_i = a(x) x_i = \frac{x_i}{l x_1 \overline{x}_1} \quad (1 \leq i \leq n), \\ \overline{y}_i = a(x)\overline{x}_i = \frac{\overline{x}_i}{l x_1 \overline{x}_1} \quad (1 \leq i < n). \end{aligned} \tag{5.19}$$

Hence we have the rational mapping

$$\overline{\sigma}(v(x)) := v(y) = v\left(\frac{x_1}{l x_1 \overline{x}_1}, \frac{x_2}{l x_1 \overline{x}_1}, \cdots, \frac{x_n}{l x_1 \overline{x}_1}, \frac{\overline{x}_{n-1}}{l x_1 \overline{x}_1}, \cdots, \frac{\overline{x}_1}{l x_1 \overline{x}_1} \right). \tag{5.20}$$

By the explicit form of $\overline{\sigma}$ in (5.20), we have $\overline{\sigma}^2 = \mathrm{id}$, which means that the morphism $\overline{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if i, $\sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 in this case is completed.

Now, we set $e_0^c := \overline{\sigma} \circ e_1^c \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$.

The explicit forms of e_i, ε_i and γ_i are

$$
\begin{aligned}
e_0^c\colon\quad & x_1 \mapsto \quad x_1\frac{cx_1\bar{x}_1+x_2\bar{x}_2}{c(x_1\bar{x}_1+x_2\bar{x}_2)} \qquad x_i\mapsto \frac{x_i}{c}\ (2\le i\le n),\\
& \bar{x}_1 \mapsto \quad \bar{x}_1\frac{x_1\bar{x}_1+x_2\bar{x}_2}{cx_1\bar{x}_1+x_2\bar{x}_2}, \qquad \bar{x}_i\mapsto \frac{\bar{x}_i}{c}\ (2\le i\le n-1),\\
e_i^c\colon\quad & x_i \mapsto \quad x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i\mapsto \overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto \quad x_j,\quad \overline{x}_j\mapsto\overline{x}_j\ (j\ne i) \qquad\qquad (1\le i< n-1),\\
e_{n-1}^c\colon\quad & x_{n-1} \mapsto \quad x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n^2}{x_{n-1}\overline{x}_{n-1}+x_n^2}, \quad \overline{x}_{n-1}\mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n^2)}{cx_{n-1}\overline{x}_{n-1}+x_n^2},\\
& x_j \mapsto \quad x_j,\quad \overline{x}_j\mapsto\overline{x}_j\ (j\ne n-1),\\
e_n^c\colon\quad & x_n \mapsto \quad cx_n, \qquad x_j\mapsto x_j \quad \overline{x}_j\mapsto\overline{x}_j\ (j\ne n),
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon_0(v(x))=\frac{l(x_1\overline{x}_1+x_2\overline{x}_2)}{x_1}, \quad \varepsilon_1(v(x))=\frac{1}{x_1}\left(1+\frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right),\\
&\varepsilon_i(v(x))=\frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right)\ (2\le i\le n-2),\\
&\varepsilon_{n-1}(v(x))=\frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right),\ \varepsilon_n(v(x))=\frac{x_{n-1}}{x_n},\\
&\gamma_0(v(x))=\frac{1}{lx_2\overline{x}_2},\ \gamma_1(v(x))=\frac{l(x_1\overline{x}_1)^2}{x_2\overline{x}_2},\\
&\gamma_i(v(x))=\frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2\le i\le n-2),\\
&\gamma_{n-1}(v(x))=\frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \quad \gamma_n(v(x))=\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}.
\end{aligned}
$$

Since $\sigma(i)=i$ for $i\ne 0,1$, the condition (5.7) in Lemma 5.3 can be easily seen by (5.20) and by the explicit form of e_i, γ_i and ε_i ($i\in I$). Thus, in order to prove Theorem 5.2, it suffices to show that

$$e_0^{c_1}e_1^{c_2}=e_1^{c_2}e_0^{c_1}, \tag{5.21}$$

$$\gamma_0(e_1^c(v(x)))=\gamma_0(v(x)), \quad \gamma_1(e_0^c(v(x)))=\gamma_1(v(x)). \tag{5.22}$$

It follows from the explicit formula above that

$$
\begin{aligned}
e_0^{c_1}e_1^{c_2}(v(x)) &= e_1^{c_2}e_0^{c_1}(v(x))\\
&= v\left(x_1\frac{c_1c_2x_1\overline{x}_1+x_2\overline{x}_2}{c_1(x_1\overline{x}_1+x_2\overline{x}_2)}, \frac{x_2}{c_1}, \cdots, \frac{\overline{x}_2}{c_1}, \overline{x}_1\frac{c_2(x_1\overline{x}_1+x_2\overline{x}_2)}{c_1c_2x_1\overline{x}_1+x_2\overline{x}_2}\right),
\end{aligned}
$$

which implies (5.21). We get (5.22) immediately from the formula above and we complete the proof of Theorem 5.2 for $B_n^{(1)}$.

5.5. $D_n^{(1)}$**-case** ($n \geq 4$)**.** We have $w_1 = s_1 s_2 \cdots s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_2 s_1$, and

$$\mathcal{V}(D_n^{(1)})_l := \{v(x) = Y_1(x_1) \cdots Y_{n-1}(x_{n-1}) Y_n(x_n) Y_{n-2}(\bar{x}_{n-2}) \cdots Y_1(\bar{x}_1) l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit form of $W(\varpi_1)_l$ in 4.6 that $y_i(c^{-1}) = \exp(c^{-1} f_i) = 1 + c^{-1} f_i$ on $W(\varpi_1)$. Thus, we have

$$v(x) = l^m \left\{ \left(\sum_{i=1}^{n-1} \xi_i(x) v_i \right) + x_n v_n + \left(\sum_{i=2}^{n} x_{i-1} v_{\bar{i}} \right) + v_{\bar{1}} \right\},$$

$$\text{where} \quad m := \varpi_1(H), \quad \xi_i(x) := \begin{cases} x_1 \overline{x}_1 & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i \overline{x}_i}{x_{i-1}} & i \neq 1, n-1, \\ \frac{x_{n-2}\overline{x}_{n-2} + x_{n-1} x_n}{x_{n-2}} & i = n-1. \end{cases}$$

The automorphism $\sigma_l : W(\varpi_1)_l \to W(\varpi_1)_l$ is given as

$$\sigma_l v_1 = l v_{\bar{1}}, \quad \sigma_l v_{\bar{1}} = l^{-1} v_1, \quad \sigma_l v_k = v_k \quad \text{otherwise.}$$

Then we have

$$\sigma_l(v(x)) = l^m \left\{ l^{-1} v_1 + \left(\sum_{i=2}^{n} \xi_i(x) v_i \right) + x_n v_n + \left(\sum_{i=1}^{n-1} x_{i-1} v_{\bar{i}} \right) + l \xi_1 v_{\bar{1}} \right\}.$$

Then the equation $v(y) = a(x)\sigma_l(v(x))$ ($x, y \in (\mathbb{C}^\times)^{2n-2}$) has the following unique solution:

$$a(x) = \frac{1}{l x_1 \overline{x}_1}, \quad y_i = a(x) x_i = \frac{x_i}{l x_1 \overline{x}_1} \quad (1 \leq i \leq n),$$
$$\overline{y}_i = a(x) \overline{x}_i = \frac{\overline{x}_i}{l x_1 \overline{x}_1} \quad (1 \leq i \leq n-2). \tag{5.23}$$

We define the rational mapping $\overline{\sigma} \colon \mathcal{V}(D_n^{(1)})_l \longrightarrow \mathcal{V}(D_n^{(1)})_l$ by

$$\overline{\sigma}(v(x)) = v\left(\frac{x_1}{l x_1 \overline{x}_1}, \frac{x_2}{l x_1 \overline{x}_1}, \cdots, \frac{x_n}{l x_1 \overline{x}_1}, \frac{\overline{x}_{n-2}}{l x_1 \overline{x}_1}, \cdots, \frac{\overline{x}_1}{l x_1 \overline{x}_1} \right). \tag{5.24}$$

It is immediate from (5.24) that $\overline{\sigma}^2 = \mathrm{id}$, which implies the birationality of the morphism $\overline{\sigma}$. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if $i, \sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $D_n^{(1)}$ is completed.

Now, we set $e_0^c := \overline{\sigma} \circ e_1^c \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$. The explicit forms of e_i, ε_i and γ_i are

$$
\begin{aligned}
e_0^c\colon & x_1 \mapsto x_1 \frac{c x_1\bar{x}_1 + x_2\bar{x}_2}{c(x_1\bar{x}_1 + x_2\bar{x}_2)}, \qquad x_i \mapsto \frac{x_i}{c}\ (2 \le i \le n),\\
& \bar{x}_1 \mapsto \bar{x}_1 \frac{x_1\bar{x}_1 + x_2\bar{x}_2}{c x_1\bar{x}_1 + x_2\bar{x}_2}, \qquad \bar{x}_i \mapsto \frac{\bar{x}_i}{c}\ (2 \le i \le n-2),\\
e_i^c\colon & x_i \mapsto x_i \frac{c x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i \mapsto \overline{x}_i \frac{c(x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1})}{c x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne i), \qquad (1 \le i \le n-3),\\
e_{n-2}^c\colon & x_{n-2} \mapsto x_{n-2} \frac{c x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n}{x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n}, \quad \overline{x}_{n-2} \mapsto \overline{x}_{n-2} \frac{c(x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n)}{c x_{n-2}\overline{x}_{n-2} + x_{n-1}x_n},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-2),\\
e_{n-1}^c\colon & x_{n-1} \mapsto c x_{n-1}, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-1),\\
e_n^c\colon & x_n \mapsto c x_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n),
\end{aligned}
$$

$$
\varepsilon_0(v(x)) = \frac{l(x_1\overline{x}_1 + x_2\overline{x}_2)}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1}\left(1 + \frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right),
$$

$$
\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \quad (2 \le i \le n-3),
$$

$$
\varepsilon_{n-2}(v(x)) = \frac{x_{n-3}}{x_{n-2}}\left(1 + \frac{x_{n-1}x_n}{x_{n-2}\overline{x}_{n-2}}\right), \quad \varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}, \quad \varepsilon_n(v(x)) = \frac{x_{n-2}}{x_n},
$$

$$
\gamma_0(v(x)) = \frac{1}{l x_2\overline{x}_2}, \ \gamma_1(v(x)) = \frac{l(x_1\overline{x}_1)^2}{x_2\overline{x}_2},
$$

$$
\gamma_i(v(x)) = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2 \le i \le n-3),
$$

$$
\gamma_{n-2}(v(x)) = \frac{(x_{n-2}\overline{x}_{n-2})^2}{x_{n-3}\overline{x}_{n-3}x_{n-1}x_n}, \quad \gamma_{n-1}(v(x)) = \frac{x_{n-1}^2}{x_{n-2}\overline{x}_{n-2}},
$$

$$
\gamma_n(v(x)) = \frac{x_n^2}{x_{n-2}\overline{x}_{n-2}}.
$$

By these formulas, we can show Theorem 5.2 for $D_n^{(1)}$ similarly to the one for $B_n^{(1)}$.

5.6. $A_{2n-1}^{(2)}$**-case** $(n \ge 3)$**.** We have $w_1 = s_1 s_2 \cdots s_n s_{n-1} \cdots s_2 s_1$, and

$$
\mathcal{V}(A_{2n-1}^{(2)})_l := \{v(x) := Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)l^H v_1 \bigm| x_i, \bar{x}_i \in \mathbb{C}^\times\}.
$$

In this case, $y_i(c^{-1}) = \exp(c^{-1}f_i) = 1 + c^{-1}f_i$ on $W(\varpi_1)$, and we have

$$
v(x_1, \ldots, x_n, \overline{x}_{n-1}, \ldots, \overline{x}_1) = l^m\left\{\left(\sum_{i=1}^n \xi_i v_i\right) + \left(\sum_{i=2}^n x_{i-1} v_{\bar{i}}\right) + v_{\bar{1}}\right\},
$$

$$
\text{where} \quad m := \varpi_1(H), \quad \xi_i := \begin{cases} x_1\overline{x}_1 & i = 1,\\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \ne 1, n,\\ \frac{x_{n-1}\overline{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases}
$$

The automorphism $\sigma_l : W(\varpi_1)_l \to W(\varpi_1)_l$ is given as

$$
\sigma_l v_1 = l v_{\bar{1}}, \quad \sigma_l v_{\bar{1}} = l^{-1} v_1, \sigma_l v_k = v_k \quad \text{otherwise}.
$$

Then we have

$$\sigma_l(v(x)) = l^m \left\{ l^{-1} v_1 + \left(\sum_{i=2}^{n} \xi_i v_i \right) + \left(\sum_{i=2}^{n} x_{i-1} v_{\bar{i}} \right) + l x_1 \overline{x}_1 v_{\bar{1}} \right\},$$

Solving $v(y) = a(x)\sigma_l(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n-1})$, we obtain a unique solution:

$$\begin{aligned} a(x) &= \frac{1}{l x_1 \overline{x}_1},\ y_i = a(x) x_i = \frac{x_i}{l x_1 \overline{x}_1}, \\ \overline{y}_i = a(x)\overline{x}_i &= \frac{\overline{x}_i}{l x_1 \overline{x}_1}\ (1 \le i \le n-1),\ y_n = a(x)^2 x_n = \frac{x_n}{(l x_1 \overline{x}_1)^2}. \end{aligned} \tag{5.25}$$

Here we have

$$\overline{\sigma}(v(x)) = v\left(\frac{x_1}{l x_1 \overline{x}_1}, \frac{x_2}{l x_1 \overline{x}_1}, \cdots, \frac{x_n}{(l x_1 \overline{x}_1)^2}, \frac{\overline{x}_{n-1}}{x_1 \overline{x}_1}, \cdots, \frac{\overline{x}_1}{x_1 \overline{x}_1} \right). \tag{5.26}$$

By the explicit form of $\overline{\sigma}$ in (5.26), we have $\overline{\sigma}^2 = \mathrm{id}$, which means that the morphism $\overline{\sigma}$ is birational. In this case, the second condition in Theorem 5.1 is trivial since $\sigma(i) = i$ if $i, \sigma(i) \neq 0$. Thus, the proof of Theorem 5.1 for $A^{(2)}_{2n-1}$ is completed.

Now, we set $e^c_0 := \overline{\sigma} \circ e^c_1 \circ \overline{\sigma}$, $\gamma_0 := \gamma_1 \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_1 \circ \overline{\sigma}$. The explicit forms of e_i, ε_i and γ_i are

$$\begin{aligned}
e^c_0\colon x_1 &\mapsto x_1 \frac{c x_1 \bar{x}_1 + x_2 \bar{x}_2}{c(x_1 \bar{x}_1 + x_2 \bar{x}_2)}, \qquad x_i \mapsto \frac{x_i}{c}\ (2 \le i \le n-1), \quad x_n \mapsto \frac{x_n}{c^2} \\
\bar{x}_1 &\mapsto \bar{x}_1 \frac{x_1 \bar{x}_1 + x_2 \bar{x}_2}{c x_1 \bar{x}_1 + x_2 \bar{x}_2}, \qquad \bar{x}_i \mapsto \frac{\bar{x}_i}{c}\ (2 \le i \le n-1), \\
e^c_i\colon x_i &\mapsto x_i \frac{c x_i \overline{x}_i + x_{i+1} \bar{x}_{i+1}}{x_i \overline{x}_i + x_{i+1} \bar{x}_{i+1}}, \qquad \overline{x}_i \mapsto \overline{x}_i \frac{c(x_i \overline{x}_i + x_{i+1} \bar{x}_{i+1})}{c x_i \overline{x}_i + x_{i+1} \bar{x}_{i+1}}, \\
x_j &\mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq i) \qquad (1 \le i < n-1), \\
e^c_{n-1}\colon x_{n-1} &\mapsto x_{n-1} \frac{c x_{n-1} \overline{x}_{n-1} + x_n}{x_{n-1} \overline{x}_{n-1} + x_n}, \qquad \overline{x}_{n-1} \mapsto \overline{x}_{n-1} \frac{c(x_{n-1} \overline{x}_{n-1} + x_n)}{c x_{n-1} \overline{x}_{n-1} + x_n}, \\
x_j &\mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq n-1), \\
e^c_n\colon x_n &\mapsto c x_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \neq n).
\end{aligned}$$

$$\varepsilon_0(v(x)) = \frac{l(x_1 \overline{x}_1 + x_2 \overline{x}_2)}{x_1}, \quad \varepsilon_1(v(x)) = \frac{1}{x_1}\left(1 + \frac{x_2 \overline{x}_2}{x_1 \overline{x}_1}\right),$$

$$\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1 + \frac{x_{i+1} \overline{x}_{i+1}}{x_i \overline{x}_i}\right) \quad (2 \le i \le n-2),$$

$$\varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1 + \frac{x_n}{x_{n-1} \overline{x}_{n-1}}\right), \quad \varepsilon_n(v(x)) = \frac{x^2_{n-1}}{x_n},$$

$$\gamma_0(v(x)) = \frac{1}{l x_2 \overline{x}_2}, \quad \gamma_1(v(x)) = \frac{l(x_1 \overline{x}_1)^2}{x_2 \overline{x}_2}, \quad \gamma_i = \frac{(x_i \overline{x}_i)^2}{x_{i-1} \overline{x}_{i-1} x_{i+1} \overline{x}_{i+1}} \quad (2 \le i \le n-2),$$

$$\gamma_{n-1}(v(x)) = \frac{(x_{n-1} \overline{x}_{n-1})^2}{x_{n-2} \overline{x}_{n-2} x_n}, \quad \gamma_n(v(x)) = \frac{x^2_n}{(x_{n-1} \overline{x}_{n-1})^2}.$$

We can show Theorem 5.2 for $A^{(2)}_{2n-1}$ similarly to the one for $B^{(1)}_n$.

5.7. $D^{(2)}_{n+1}$**-case** ($n \ge 2$)**.** We have $w_1 = s_0 s_1 \cdots s_n s_{n-1} \cdots s_2 s_1$, and

$$\mathcal{V}(D^{(2)}_{n+1})_l := \{v(x) := Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots Y_1(\bar{x}_1)l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

It follows from the explicit description of $W(\varpi_1)_l$ as in 4.8 that on $W(\varpi_1)_l$ we have

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \ne 0, n, \\ 1 + c^{-1}f_i + \frac{1}{2c^2}f_i^2 & i = 0, n. \end{cases}$$

Then we have

$$v(x) = l^m \left\{ \left(\sum_{i=1}^{n} \xi_i(x) v_i \right) + x_n v_0 + l^{-1} x_0 \phi + \left(\sum_{i=2}^{n} x_{i-1} v_{\bar{i}} \right) + x_0^2 v_{\bar{1}} \right\}$$

where

$$m := \varpi_1(H), \qquad \xi_i(x) := \begin{cases} \frac{l^{-2}x_0^2 + x_1\overline{x}_1}{x_0^2} & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \ne 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i = n. \end{cases}$$

The automorphism $\sigma_l : W(\varpi_1)_l \to W(\varpi_1)_l$ is given as

$$\sigma_l v_1 = l v_{\overline{n}}, \quad \sigma_l v_{\overline{1}} = l^{-1} v_n, \quad \sigma_l v_n = l v_{\overline{1}}, \quad \sigma_l v_{\overline{n}} = l^{-1} v_1,$$
$$\sigma_l v_0 = \phi, \quad \sigma_l \phi = v_0, \quad \sigma_l v_{i+1} = v_{\overline{n-i}}, \quad \sigma_l v_{\overline{i+1}} = v_{n-i} \quad (1 \le i < n-1).$$

Then we have

$$\sigma_l(v(x)) = l^m \left\{ l^{-1} x_{n-1} v_1 + \left(\sum_{i=2}^{n-1} x_{n-i} v_i \right) + l^{-1} x_0^2 v_n + l \xi_n v_{\overline{n}} + \left(\sum_{i=2}^{n-1} \xi_{n-i+1} v_{\bar{i}} \right) + l \xi_1 v_{\overline{n}} + x_n \phi + x_0 v_0 \right\},$$

Solving $v(y) = a(x)\sigma_l(v(x))$ $(x, y \in (\mathbb{C}^\times)^{2n})$, we get a unique solution:

$$a(x) = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{l x_{n-1} x_n^2},$$
$$y_0 = l a(x) x_n = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1} x_n},$$
$$y_i = \frac{(x_{n-i-1}\overline{x}_{n-i-1} + x_{n-i}\overline{x}_{n-i})(x_{n-1}\overline{x}_{n-1} + x_n^2)}{l x_{n-i-1} x_{n-1} x_n^2} \quad (1 \le i < n),$$
$$y_{n-1} = \frac{(l^{-2}x_0^2 + x_1\overline{x}_1)(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_0^2 x_{n-1} x_n^2},$$
$$y_n = \frac{x_0(x_{n-1}\overline{x}_{n-1} + x_n^2)}{l^2 x_{n-1} x_n^2},$$
$$\overline{y}_i = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2) x_{n-i-1} x_{n-i} \overline{x}_{n-i}}{l(x_{n-i-1}\overline{x}_{n-i-1} + x_{n-i}\overline{x}_{n-i}) x_{n-1} x_n^2} \quad (1 \le i \le n-2),$$
$$\overline{y}_{n-1} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2) x_0^2 x_1 \overline{x}_1}{(x_0^2 + l^2 x_1 \overline{x}_1) x_{n-1} x_n^2}.$$

Then we have the rational mapping $\overline{\sigma}\colon \mathcal{V}(D^{(2)}_{n+1})_l \longrightarrow \mathcal{V}(D^{(2)}_{n+1})_l$ defined by $v(x) \mapsto v(y)$. The explicit forms of ε_i $(1 \le i \le n)$ are as follows:

$$\varepsilon_1(v(x)) = \frac{x_0^2}{x_1}\left(1+\frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right), \quad \varepsilon_n(v(x)) = \frac{x_{n-1}}{x_n},$$
$$\varepsilon_i(v(x)) = \frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \ (2 \le i \le n-2),$$
$$\varepsilon_{n-1}(v(x)) = \frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right).$$

Then we get easily that $\varepsilon_{n-i}(v(y)) = \varepsilon_i(v(x))$ $(1 \le i \le n-1)$, which finishes the proof of Theorem 5.1 for $D^{(2)}_{n+1}$.

Let us define $e_0^c := \overline{\sigma} \circ e_n^c \circ \overline{\sigma}$ $(\overline{\sigma}^2 = \mathrm{id})$, $\gamma_0 := \gamma_n \circ \overline{\sigma}$ and $\varepsilon_0 := \varepsilon_n \circ \overline{\sigma}$. The explicit forms of e_i, γ_i and ε_0 are

$$\begin{aligned}
e_0^c\colon \quad & x_0 \mapsto x_0\frac{c^2x_0^2+l^2x_1\bar{x}_1}{c(x_0^2+l^2x_1\bar{x}_1)}, \qquad x_i \mapsto x_i\frac{c^2x_0^2+l^2x_1\bar{x}_1}{c^2(x_0^2+l^2x_1\bar{x}_1)} \ (1 \le i \le n),\\
& \bar{x}_i \mapsto \bar{x}_i\frac{c^2x_0^2+l^2x_1\bar{x}_1}{c^2(x_0^2+l^2x_1\bar{x}_1)} \ (1 \le i \le n-1),\\
e_i^c\colon \quad & x_i \mapsto x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \qquad \overline{x}_i \mapsto \overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne i), \quad (1 \le i < n-1),\\
e_{n-1}^c\colon \quad & x_{n-1} \mapsto x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n^2}{x_{n-1}\overline{x}_{n-1}+x_n^2}, \quad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n^2)}{cx_{n-1}\overline{x}_{n-1}+x_n^2},\\
& x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n-1),\\
e_n^c\colon \quad & x_n \mapsto cx_n, \quad x_j \mapsto x_j, \quad \overline{x}_j \mapsto \overline{x}_j \quad (j \ne n),
\end{aligned}$$

$$\gamma_0(v(x)) = \frac{x_0^2}{l^2x_1\overline{x}_1}, \quad \gamma_1(v(x)) = \frac{(lx_1\overline{x}_1)^2}{x_0^2x_2\overline{x}_2},$$
$$\gamma_i(v(x)) = \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}} \ (2 \le i \le n-2),$$
$$\gamma_{n-1}(v(x)) = \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \ \gamma_n(v(x)) = \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}},$$
$$\varepsilon_0(v(x)) = \frac{x_0^2+l^2x_1\overline{x}_1}{x_0^3}.$$

Let us check the condition (5.7) in Lemma 5.3. The following are useful for this purpose:

$$y_i\overline{y}_i = a(x)^2x_{n-i}\overline{x}_{n-i}, \quad y_0 = a(x)x_n, \tag{5.27}$$
$$a(v(y)) = a(\overline{\sigma}(v(x))) = \frac{1}{a(v(x))}. \tag{5.28}$$

Using these we can easily check the two conditions $\gamma_i = \gamma_{\sigma(i)} \circ \overline{\sigma}$ and $\varepsilon_i = \varepsilon_{\sigma(i)} \circ \overline{\sigma}$. The condition $e^c_{\sigma(i)} = \overline{\sigma} \circ e_i^c \circ \overline{\sigma}^{-1}$ for $i = 2, \cdots, n-2$ is also immediate from (5.27) and (5.28). Next let us see the case $i = 1, n-1$. We have

$$a(e_{n-1}^c(v(y))) = \frac{y_n^2+cy_{n-1}\overline{y}_{n-1}}{y_n^2y_{n-1}\frac{cy_{n-1}\overline{y}_{n-1}y+y_n^2}{y_{n-1}\overline{y}_{n-1}y+y_n^2}} = \frac{y_{n-1}\overline{y}_{n-1}y+y_n^2}{y_{n-1}y_n^2} = \frac{1}{a(v(x))}.$$

Using this, we can get $e^c_{n-i} = \overline{\sigma}\circ e^c_1 \circ \overline{\sigma}^{-1}$ and then $e^c_1 = \overline{\sigma}\circ e^c_{n-1}\circ\overline{\sigma}^{-1}$ since $\overline{\sigma}^2 = \mathrm{id}$. Now, it remains to show that

$$e_0^{c_1} e_n^{c_2} = e_n^{c_2} e_0^{c_1}, \quad \varepsilon_0(e_n^c(v(x))) = \varepsilon_0(v(x)), \quad \varepsilon_n(e_0^c(v(x))) = \varepsilon_n(v(x)).$$

They easily follow from the explicit form of e^c_0. Thus, the proof of Theorem 5.2 in this case is completed.

5.8. $A^{(2)}_{2n}$**-case** $(n \geq 2)$**.** As in the beginning of this section, we have

$$w_1 = s_0 s_1 \cdots s_n s_{n-1} \cdots s_2 s_1$$

and

$$\mathcal{V}(A^{(2)}_{2n})_l := \{v_1(x) = Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times\}.$$

By the explicit description of $W(\varpi_1)_l$ as in 4.9, on $W(\varpi_1)_l$ we have

$$y_i(c^{-1}) = \exp(c^{-1}f_i) = \begin{cases} 1 + c^{-1}f_i & i \neq 0, \\ 1 + c^{-1}f_0 + \frac{1}{2c^2}f_0^2 & i = 0. \end{cases}$$

Then we have

$$v_1(x) = l^m \left\{ \left(\sum_{i=1}^n \xi_i(x) v_i \right) + x_0^2 v_{\overline{1}} + \left(\sum_{i=2}^n x_{i-1} v_{\overline{i}} + \right) + \frac{x_0}{l}\emptyset \right\},$$

where

$$m := \varpi_1(H), \quad \xi_i(x) := \begin{cases} \frac{l^{-2}x_0^2 + x_1\overline{x}_1}{x_0^2} & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n}{x_{n-1}} & i = n. \end{cases}$$

Next, for $w_2 = s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$, we set

$$\mathcal{V}_2(A^{(2)}_{2n})_l := \{v_2(y) = Y_n(y_n)\cdots Y_1(y_1)Y_0(y_0)Y_1(\overline{y}_1)\cdots Y_{n-1}(\overline{y}_{n-1}) l^{H'} v_{\overline{n}} \mid y_i, \overline{y}_i \in \mathbb{C}^\times\}.$$

Then we have

$$v_2(y) = l^m \left\{ \left(\sum_{i=1}^n \frac{y_i}{l^2} v_i \right) + \left(\sum_{i=1}^n \eta_i(y) v_{\overline{i}} \right) + \frac{y_0}{l}\emptyset \right\}$$

where

$$\eta_i(y) := \begin{cases} \frac{y_0^2 + y_1\overline{y}_1}{y_1} & i = 1, \\ \frac{y_{i-1}\overline{y}_{i-1} + y_i\overline{y}_i}{y_i} & i \neq 1, n, \\ \frac{y_{n-1}\overline{y}_{n-1} + l^{-2}y_n}{y_n} & i = n. \end{cases}$$

Note that $\varpi_1(H) = m = \mathrm{wt}(v_{\overline{n}})(H') = (\varpi_n(H'))$. For $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y = (y_0, \dots, \overline{y}_1) \in (\mathbb{C}^\times)^{2n}$ and $a(x)$ such that $v_2(y) = a(x)v_1(x)$. They are given by

$$
\begin{aligned}
a(x) &= \frac{x_{n-1}\overline{x}_{n-1} + x_n}{l^2 x_{n-1} x_n},\\
y_0 &= a(x)x_0 = \frac{x_0(x_{n-1}\overline{x}_{n-1} + x_n)}{l^2 x_{n-1}x_n},\\
y_1 &= l^2 a(x)\xi_1(x) = \frac{(x_{n-1}\overline{x}_{n-1} + x_n)(x_0 + l^2 x_1\overline{x}_1)}{x_0^2 x_{n-1}x_n},\\
y_i &= l^2 a(x)\xi_i(x) = \frac{(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i)(x_{n-1}\overline{x}_{n-1} + x_n)}{x_{i-1}x_{n-1}x_n} \quad (1 < i < n),\\
y_n &= l^2 a(x)\xi_n = \frac{(x_{n-1}\overline{x}_{n-1} + x_n)^2}{x_{n-1}^2 x_n},\\
\overline{y}_1 &= a(x)\frac{l^2 x_0^2 x_1\overline{x}_1}{x_0 + l^2 x_1\overline{x}_1} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n)x_0^2 x_1\overline{x}_1}{(x_0 + l^2 x_1\overline{x}_1)x_{n-1}x_n},\\
\overline{y}_i &= a(x)\frac{x_{i-1}x_i\overline{x}_i}{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)x_{i-1}x_i\overline{x}_i}{l^2(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i)x_{n-1}x_n^2} \quad (1 < i \le n-1).
\end{aligned}
$$

It defines a rational mapping $\overline{\sigma}\colon \mathcal{V}(A_{2n}^{(2)})_l \longrightarrow \mathcal{V}_2(A_{2n}^{(2)})_l$ $(v_1(x) \mapsto v_2(y))$.
The inverse $\overline{\sigma}^{-1}\colon \mathcal{V}_2(A_{2n}^{(2)})_l \longrightarrow \mathcal{V}(A_{2n}^{(2)})_l$ $(v_2(y) \mapsto v_1(x))$ is given by

$$
\begin{aligned}
a(y) &:= \frac{y_0^2 y_1}{y_0^2 + y_1\overline{y}_1} (= a(x)),\\
x_0 &= \frac{y_0}{a(y)},\\
x_i &= a(y)^{-1}\frac{y_i\overline{y}_i + y_{i+1}\overline{y}_{i+1}}{y_{i+1}} \quad (1 \le i \le n-2),\\
x_{n-1} &= a(y)^{-1}\frac{y_{n-1}\overline{y}_{n-1} + l^{-2}y_n}{y_n},\\
x_n &= \frac{y_n}{l^4 a(y)^2},\\
\overline{x}_i &= (l^2 a(y))^{-1}\frac{y_i\overline{y}_i y_{i+1}}{y_i\overline{y}_i + y_{i+1}\overline{y}_{i+1}} \quad (1 \le i \le n-2),\\
\overline{x}_{n-1} &= \frac{y_{n-1}\overline{y}_{n-1}y_n}{a(y)(l^2 y_{n-1}\overline{y}_{n-1} + y_n)},
\end{aligned}
$$

which means that the morphism $\overline{\sigma}\colon \mathcal{V}(A_{2n}^{(2)})_l \longrightarrow \mathcal{V}_2(A_{2n}^{(2)})_l$ is birational. Thus, we obtain Theorem 5.1(ii).

The actions of e_i $(0 \le i < n)$ on $v_2(y)$ are induced from the ones on $Y_{\mathbf{i_2}}(y) \cdot l^{H'} := Y_n(y_n) \cdots Y_{n-1}(\overline{y}_{n-1}) \cdot l^{H'}$ $(\mathbf{i_2} = (n, \dots, 1, 0, 1, \cdots, n-1))$ since $e_i v_{\overline{n}} = 0$ for $i = 0, 1, \dots, n-1$. We also get $\gamma_i(v_2(y))$ and $\varepsilon_i(v_2(y))$ from the ones for $Y_{\mathbf{i_2}}(y) \cdot l^{H'}$

where $v_2(y)=\overline{\sigma}(v_1(x))$:

$$
\begin{aligned}
&e_0^c\colon y_0\mapsto cy_0,\quad y_i\mapsto y_i,\quad \overline{y}_i\mapsto\overline{y}_i\ (i\neq 0),\\
&e_1^c\colon y_1\mapsto y_1\frac{cy_1\overline{y}_1+y_0^2}{y_1\overline{y}_1+y_0^2},\quad \overline{y}_1\mapsto\overline{y}_1\frac{c(y_1\overline{y}_1+y_0^2)}{cy_1\overline{y}_1+y_0^2},\quad y_i\mapsto y_i,\quad \overline{y}_i\mapsto\overline{y}_i\ (i\neq 1),\\
&e_i^c\colon y_i\mapsto y_i\frac{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}},\quad \overline{y}_i\mapsto\overline{y}_i\frac{c(y_i\overline{y}_i+y_{i-1}\overline{y}_{i-1})}{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}},\\
&\quad y_j\mapsto y_j,\quad \overline{y}_j\mapsto\overline{y}_j\ (j\neq i),\quad (i=2,\ldots,n-1),\\
&\gamma_0(v_2(y))=\frac{y_0^2}{y_1\overline{y}_1},\quad \gamma_1(v_2(y))=\frac{(y_1\overline{y}_1)^2}{y_0^2y_2\overline{y}_2},\\
&\gamma_i(v_2(y))=\frac{(y_i\overline{y}_i)^2}{y_{i-1}\overline{y}_{i-1}y_{i+1}\overline{y}_{i+1}}\quad (i=2,\ldots,n-2),\\
&\gamma_{n-1}(v_2(y))=\frac{(y_{n-1}\overline{y}_{n-1})^2}{y_{n-2}\overline{y}_{n-2}y_n},\\
&\varepsilon_0(v_2(y))=\frac{y_1}{y_0},\ \varepsilon_1(v_2(y))=\frac{y_2}{y_1}\left(1+\frac{y_0^2}{y_1\overline{y}_1}\right),\\
&\varepsilon_{n-1}(v_2(y))=\frac{y_n}{y_{n-1}}\left(1+\frac{y_{n-2}\overline{y}_{n-2}}{y_{n-1}\overline{y}_{n-1}}\right),\\
&\varepsilon_i(v_2(y))=\frac{y_{i+1}}{y_i}\left(1+\frac{y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i}\right)\quad (i=2\cdots,n-1).
\end{aligned}
$$

The explicit forms of $\varepsilon_i(v_1(x))$ and $\gamma_i(v_1(x))$ $(1\le i\le n)$ are also induced from the ones for $Y_{\mathbf{i}_1}(x)\cdot l^H:=Y_0(x_0)\cdots Y_1(\overline{x}_1)\cdot l^H$ and, we define $\varepsilon_0(v_1(x)):=\varepsilon_0(v_2(y))$ and $\gamma_0(v_1(x)):=\gamma_0(v_2(y))$ $(v_2(y):=\overline{\sigma}(v_1(x)))$:

$$
\begin{aligned}
\varepsilon_0(v_1(x))&=\frac{1}{x_0}\left(1+\frac{l^2x_1\overline{x}_1}{x_0^2}\right)^2,\\
\varepsilon_i(v_1(x))&=\frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right)\quad (1<i\le n-2),\\
\varepsilon_1(v_1(x))&=\frac{x_0^2}{x_1}\left(1+\frac{x_2\overline{x}_2}{x_1\overline{x}_1}\right),\\
\varepsilon_{n-1}(v_1(x))&=\frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n}{x_{n-1}\overline{x}_{n-1}}\right),\quad \varepsilon_n(v_1(x))=\frac{x_{n-1}^2}{x_n}.\\
\gamma_0(v_1(x))&=\frac{x_0^2}{l^4x_1\overline{x}_1},\ \gamma_1(v_1(x))=\frac{(lx_1\overline{x}_1)^2}{x_0^2x_2\overline{x}_2},\\
\gamma_i(v_1(x))&=\frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2\le i\le n-2),\\
\gamma_{n-1}(v_1(x))&=\frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n},\quad \gamma_n(v_1(x))=\frac{x_n^2}{(x_{n-1}\overline{x}_{n-1})^2}.
\end{aligned}
$$

For $i=0$, we define $e_0^c(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c\circ\overline{\sigma}(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c(v_2(y))$. Then we get

$$\begin{aligned}
e_0^c\colon\quad & x_0\mapsto\quad x_0\frac{c^2x_0+l^2x_1\bar{x}_1}{c(x_0^2+l^2x_1\bar{x}_1)},\qquad x_i\mapsto x_i\frac{c^2x_0+l^2x_1\bar{x}_1}{c^2(x_0^2+l^2x_1\bar{x}_1)}\ (1\le i<n),\\
& \bar{x}_i\mapsto\quad \bar{x}_i\frac{c^2x_0+l^2x_1\bar{x}_1}{c^2(x_0^2+l^2x_1\bar{x}_1)}\ (1\le i\le n-1),\ x_n\mapsto x_n\frac{(c^2x_0+l^2x_1\bar{x}_1)^2}{c^4(x_0^2+l^2x_1\bar{x}_1)^2},\\
e_i^c\colon\quad & x_i\mapsto\quad x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\quad \overline{x}_i\mapsto\overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& \qquad\qquad x_j\mapsto x_j,\quad \overline{x}_j\mapsto\overline{x}_j\ (j\ne i)\quad (1\le i<n-1)\\
e_{n-1}^c\colon\quad & x_{n-1}\mapsto\quad x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n}{x_{n-1}\overline{x}_{n-1}+x_n},\qquad \overline{x}_{n-1}\mapsto\overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n)}{cx_{n-1}\overline{x}_{n-1}+x_n},\\
& \qquad\qquad x_i\mapsto x_i,\quad \overline{x}_i\mapsto\overline{x}_i\ (i\ne n-1),\\
e_n^c\colon\quad & x_n\mapsto\quad cx_n,\qquad x_i\mapsto x_i,\quad \overline{x}_i\mapsto\overline{x}_i\ (i\ne n).
\end{aligned}$$

In order to prove Theorem 5.2, it suffices to show the following:

$$e_i^c=\overline{\sigma}^{-1}\circ e_i^c\circ\overline{\sigma},\ \gamma_i=\gamma_i\circ\overline{\sigma},\ \varepsilon_i=\varepsilon_i\circ\overline{\sigma}\ (i\ne 0,n), \tag{5.29}$$

$$e_0^{c_1}e_n^{c_2}=e_n^{c_2}e_0^{c_1}, \tag{5.30}$$

$$\gamma_n(e_0^c(v_1(x)))=\gamma_n(v_1(x)),\quad \gamma_0(e_n^c(v_1(x)))=\gamma_0(v_1(x)), \tag{5.31}$$

$$\varepsilon_0(e_0^c(v_1(x)))=c^{-1}\varepsilon_0(v_1(x)), \tag{5.32}$$

which are immediate from the above formulae. Let us show (5.29). Set $v_2(y):=\overline{\sigma}(v_1(x))$ and $v_1(x'):=\overline{\sigma}^{-1}(e_i^c(v_2(y)))$ for $i=2,\dots,n-2$. Then $x'_j=x_j$ and $\overline{x}'_j=\overline{x}_j$ for $j\ne i-1,i$, and we have

$$\begin{aligned}
& a(v_2(y))=a(v_1(x)),\\
& x'_i=\frac{1}{a(v_2(y))}\left(\overline{y}_{i+1}+\frac{cy_i\overline{y}_i}{y_{i+1}}\right)=x_i\frac{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& \overline{x}'_i=\frac{1}{a(v_2(y))}\frac{cy_i\overline{y}_iy_{i+1}}{cy_i\overline{y}_i+y_{i+1}\overline{y}_{i+1}}=\overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}},\\
& x'_{i-1}=\frac{1}{a(v_2(y))}\left(\overline{y}_i+\frac{cy_{i-1}\overline{y}_{i-1}}{y_i}\right)=x_{i-1},\\
& \overline{x}'_{i-1}=\frac{1}{a(v_2(y))}\frac{cy_{i-1}\overline{y}_{i-1}y_i}{cy_i\overline{y}_i+y_{i-1}\overline{y}_{i-1}}=\overline{x}_{i-1},
\end{aligned}$$

where the formula $y_i\overline{y}_i=a(v_1(x))x_i\overline{x}_i$ is useful to obtain these results. Therefore we have $e_i^c=\overline{\sigma}^{-1}\circ e_i^c\circ\overline{\sigma}$ for $i=2,\dots,n-2$. Others are obtained similarly.

5.9. $A_{2n}^{(2)\dagger}$**-case** ($n\ge 2$)**.** As in the beginning of this section, we have

$$w_1=s_0s_1\cdots s_ns_{n-1}\cdots s_2s_1$$

and

$$\begin{aligned}
\mathcal{V}(A_{2n}^{(2)\dagger})_l:=\{v_1(x)=Y_0(x_0)Y_1(x_1)\cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1})\cdots\\
Y_1(\bar{x}_1)l^Hv_1\ \big|\ x_i,\bar{x}_i\in\mathbb{C}^\times\}.
\end{aligned}$$

By the explicit description of $W(\varpi_1)_l$ as in 4.9, on $W(\varpi_1)_l$ we have

$$y_i(c^{-1})=\exp(c^{-1}f_i)=\begin{cases}1+c^{-1}f_i & i\ne n,\\ 1+c^{-1}f_n+\frac{1}{2c^2}f_n^2 & i=n.\end{cases}$$

Then we have

$$v_1(x) = l^m \left\{ \left(\sum_{i=1}^{n} \xi_i(x) v_i \right) + x_n v_0 + \left(\sum_{i=1}^{n} x_{i-1} v_{\overline{i}} \right) \right\}$$

where

$$\xi_i(x) := \begin{cases} \frac{l^{-1}x_0 + x_1\overline{x}_1}{x_0} & i = 1, \\ \frac{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i}{x_{i-1}} & i \neq 1, n, \\ \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1}} & i = n, \end{cases}$$

and $m := \varpi_1(H)$. Next, for $w_2 = s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{n-1}$ and H' such that $m = \varpi_n(H')$, we set

$$\mathcal{V}_2(A_{2n}^{(2)\,\dagger})_l := \{ v_2(y) = Y_n(y_n) \cdots Y_1(y_1) Y_0(y_0) Y_1(\overline{y}_1) \cdots \\ Y_{n-1}(\overline{y}_{n-1}) l^{H'} v_{\overline{n}} \,\big|\, y_i, \overline{y}_i \in \mathbb{C}^\times \}.$$

Then we have

$$v_2(y) = l^m \left\{ \left(\sum_{i=1}^{n-1} \frac{y_i}{l} v_i \right) + \frac{y_n^2}{l} v_n + \frac{y_n}{l} v_0 + \left(\sum_{i=1}^{n} \eta_i(y) v_{\overline{i}} \right) \right\}$$

where

$$\eta_i(y) := \begin{cases} \frac{y_0 + y_1\overline{y}_1}{y_1} & i = 1, \\ \frac{y_{i-1}\overline{y}_{i-1} + y_i\overline{y}_i}{y_i} & i \neq 1, n, \\ \frac{y_{n-1}\overline{y}_{n-1} + l^{-1}y_n^2}{y_n^2} & i = n. \end{cases}$$

For $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y = (y_0, \ldots, \overline{y}_1) \in (\mathbb{C}^\times)^{2n}$ and $a(x)$ such that $v_2(y) = a(x)v_1(x)$. They are given by

$$\begin{aligned}
a(x) &= \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{l x_{n-1} x_n^2}, \\
y_0 &= a(x)^2 x_0 = \frac{x_0(x_{n-1}\overline{x}_{n-1} + x_n^2)^2}{(l x_{n-1} x_n^2)^2}, \\
y_1 &= l a(x) \xi_1(x) = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2)(l^{-1}x_0 + x_1\overline{x}_1)}{x_0 x_{n-1} x_n^2}, \\
y_i &= l a(x) \xi_i(x) = \frac{(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i)(x_{n-1}\overline{x}_{n-1} + x_n^2)}{x_{i-1} x_{n-1} x_n^2} \quad (1 < i < n), \\
y_n &= l a(x) x_n = \frac{x_{n-1}\overline{x}_{n-1} + x_n^2}{x_{n-1} x_n}, \\
\overline{y}_1 &= a(x) \frac{x_0 x_1 \overline{x}_1}{l^{-1}x_0 + x_1\overline{x}_1} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2) x_0 x_1 \overline{x}_1}{(x_0 + l x_1 \overline{x}_1) x_{n-1} x_n^2}, \\
\overline{y}_i &= a(x) \frac{x_{i-1} x_i \overline{x}_i}{x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i} = \frac{(x_{n-1}\overline{x}_{n-1} + x_n^2) x_{i-1} x_i \overline{x}_i}{l(x_{i-1}\overline{x}_{i-1} + x_i\overline{x}_i) x_{n-1} x_n^2} \quad (1 < i \leq n-1).
\end{aligned}$$

It defines a rational mapping $\overline{\sigma}\colon \mathcal{V}(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}_2(A_{2n}^{(2)\dagger})$ $(v_1(x) \mapsto v_2(y))$.
The inverse $\overline{\sigma}^{-1}\colon \mathcal{V}_2(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}(A_{2n}^{(2)\dagger})$ $(v_2(y) \mapsto v_1(x))$ is given by

$$
\begin{aligned}
&a(y) := \frac{y_0 y_1}{y_0 + y_1\overline{y}_1} (= a(x)),\\
&x_0 = a(y)^{-1}\frac{y_0 + y_1\overline{y}_1}{y_1},\\
&x_i = a(y)^{-1}\frac{y_i\overline{y}_i + y_{i+1}\overline{y}_{i+1}}{y_{i+1}} \quad (1 \le i \le n-2),\\
&x_{n-1} = a(y)^{-1}\frac{y_{n-1}\overline{y}_{n-1} + l^{-1}y_n^2}{y_n^2},\\
&x_n = \frac{y_n}{la(y)},\\
&\overline{x}_i = (la(y))^{-1}\frac{y_i\overline{y}_i y_{i+1}}{y_i\overline{y}_i + y_{i+1}\overline{y}_{i+1}} \quad (1 \le i \le n-2),\\
&\overline{x}_{n-1} = a(y)^{-1}\frac{y_{n-1}\overline{y}_{n-1}y_n^2}{ly_{n-1}\overline{y}_{n-1} + y_n^2},
\end{aligned}
$$

which means that the morphism $\overline{\sigma}\colon \mathcal{V}(A_{2n}^{(2)\dagger}) \longrightarrow \mathcal{V}_2(A_{2n}^{(2)\dagger})$ is birational. Thus, we obtain Theorem 5.1(ii).

The actions of e_i $(0 \le i < n)$ on $v_2(y)$ are induced from the ones on $Y_{\mathbf{i_2}}(y) \cdot l^{H'} := Y_n(y_n)\cdots Y_{n-1}(\overline{y}_{n-1}) \cdot l^{H'}$ $(\mathbf{i_2} = (n, \ldots, 1, 0, 1, \ldots, n-1))$ since $e_i v_{\overline{n}} = 0$ for $i = 0, 1, \ldots, n-1$. We also get $\gamma_i(v_2(y))$ and $\varepsilon_i(v_2(y))$ from the ones for $Y_{\mathbf{i_2}}(y) \cdot l^{H'}$ where $v_2(y) = \overline{\sigma}(v_1(x))$:

$$
\begin{aligned}
&e_0^c\colon y_0 \mapsto cy_0, \quad y_i \mapsto y_i, \quad \overline{y}_i \mapsto \overline{y}_i \ (i \ne 0),\\
&e_1^c\colon y_1 \mapsto y_1\frac{cy_1\overline{y}_1 + y_0}{y_1\overline{y}_1 + y_0}, \quad \overline{y}_1 \mapsto \overline{y}_1\frac{c(y_1\overline{y}_1 + y_0)}{cy_1\overline{y}_1 + y_0}, \quad y_i \mapsto y_i, \quad \overline{y}_i \mapsto \overline{y}_i \ (i \ne 1),\\
&e_i^c\colon y_i \mapsto y_i\frac{cy_i\overline{y}_i + y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i + y_{i-1}\overline{y}_{i-1}}, \quad \overline{y}_i \mapsto \overline{y}_i\frac{c(y_i\overline{y}_i + y_{i-1}\overline{y}_{i-1})}{cy_i\overline{y}_i + y_{i-1}\overline{y}_{i-1}},\\
&\qquad y_j \mapsto y_j, \quad \overline{y}_j \mapsto \overline{y}_j \ (j \ne i), \quad (i = 2, \ldots, n-1),\\
&\gamma_0(v_2(y)) = \frac{y_0^2}{(y_1\overline{y}_1)^2}, \quad \gamma_1(v_2(y)) = \frac{(y_1\overline{y}_1)^2}{y_0y_2\overline{y}_2},\\
&\gamma_i(v_2(y)) = \frac{(y_i\overline{y}_i)^2}{y_{i-1}\overline{y}_{i-1}y_{i+1}\overline{y}_{i+1}} \quad (i = 2, \ldots, n-2),\\
&\gamma_{n-1}(v_2(y)) = \frac{(y_{n-1}\overline{y}_{n-1})^2}{y_{n-2}\overline{y}_{n-2}y_n^2},\\
&\varepsilon_0(v_2(y)) = \frac{y_1^2}{y_0}, \ \varepsilon_1(v_2(y)) = \frac{y_2}{y_1}\left(1 + \frac{y_0}{y_1\overline{y}_1}\right),\\
&\varepsilon_{n-1}(v_2(y)) = \frac{y_n^2}{y_{n-1}}\left(1 + \frac{y_{n-2}\overline{y}_{n-2}}{y_{n-1}\overline{y}_{n-1}}\right),\\
&\varepsilon_i(v_2(y)) = \frac{y_{i+1}}{y_i}\left(1 + \frac{y_{i-1}\overline{y}_{i-1}}{y_i\overline{y}_i}\right) \quad (i = 2\cdots, n-1).
\end{aligned}
$$

The explicit forms of $\varepsilon_i(v_1(x))$and $\gamma_i(v_1(x))$ $(1\le i\le n)$ are also induced from the ones for $Y_{\mathbf{i}_1}(x)\cdot l^H := Y_0(x_0)\cdots Y_1(\overline{x}_1)\cdot l^H$ and, we define $\varepsilon_0(v_1(x)) := \varepsilon_0(v_2(y))$ and $\gamma_0(v_1(x)) := \gamma_0(v_2(y))$ $(v_2(y) := \overline{\sigma}(v_1(x)))$:

$$
\begin{aligned}
\varepsilon_0(v_1(x)) &= \frac{1}{x_0}\left(1+\frac{lx_1\overline{x}_1}{x_0}\right)^2,\\
\varepsilon_i(v_1(x)) &= \frac{x_{i-1}}{x_i}\left(1+\frac{x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i}\right) \quad (1\le i\le n-2),\\
\varepsilon_{n-1}(v_1(x)) &= \frac{x_{n-2}}{x_{n-1}}\left(1+\frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}\right), \quad \varepsilon_n(v_1(x)) = \frac{x_{n-1}}{x_n}.\\
\gamma_0(v_1(x)) &= \frac{x_0^2}{(lx_1\overline{x}_1)^2},\ \gamma_1(v_1(x)) = \frac{(lx_1\overline{x}_1)^2}{x_0x_2\overline{x}_2},\\
\gamma_i(v_1(x)) &= \frac{(x_i\overline{x}_i)^2}{x_{i-1}\overline{x}_{i-1}x_{i+1}\overline{x}_{i+1}}\ (2\le i\le n-2),\\
\gamma_{n-1}(v_1(x)) &= \frac{(x_{n-1}\overline{x}_{n-1})^2}{x_{n-2}\overline{x}_{n-2}x_n^2}, \quad \gamma_n(v_1(x)) = \frac{x_n^2}{x_{n-1}\overline{x}_{n-1}}.
\end{aligned}
$$

For $i=0$, we define $e_0^c(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c\circ\overline{\sigma}(v_1(x)) = \overline{\sigma}^{-1}\circ e_0^c(v_2(y))$. Then we get

$$
\begin{aligned}
e_0^c\colon \quad & x_0\mapsto \quad x_0\frac{(cx_0+lx_1\bar{x}_1)^2}{c(x_0+lx_1\bar{x}_1)^2}, \qquad x_i\mapsto x_i\frac{cx_0+lx_1\bar{x}_1}{c(x_0+lx_1\bar{x}_1)}\ (1\le i\le n),\\
& \bar{x}_i\mapsto \quad \bar{x}_i\frac{cx_0+lx_1\bar{x}_1}{c(x_0+lx_1\bar{x}_1)}\ (1\le i\le n-1),\\
e_i^c\colon \quad & x_i\mapsto \quad x_i\frac{cx_i\overline{x}_i+x_{i+1}\bar{x}_{i+1}}{x_i\overline{x}_i+x_{i+1}\overline{x}_{i+1}}, \quad \overline{x}_i\mapsto\overline{x}_i\frac{c(x_i\overline{x}_i+x_{i+1}\bar{x}_{i+1})}{cx_i\overline{x}_i+x_{i+1}\bar{x}_{i+1}},\\
& \qquad x_j\mapsto x_j, \quad \overline{x}_j\mapsto\overline{x}_j\ (j\ne i), \quad (1\le i<n-1),\\
e_{n-1}^c\colon \quad & x_{n-1}\mapsto \quad x_{n-1}\frac{cx_{n-1}\overline{x}_{n-1}+x_n^2}{x_{n-1}\overline{x}_{n-1}+x_n^2}, \qquad \overline{x}_{n-1}\mapsto\overline{x}_{n-1}\frac{c(x_{n-1}\overline{x}_{n-1}+x_n^2)}{cx_{n-1}\overline{x}_{n-1}+x_n^2},\\
& \qquad x_i\mapsto x_i, \quad \overline{x}_i\mapsto\overline{x}_i\ (i\ne n-1),\\
e_n^c\colon \quad & x_n\mapsto \quad cx_n, \quad x_i\mapsto x_i, \quad \overline{x}_i\mapsto\overline{x}_i\ (i\ne n).
\end{aligned}
$$

In order to prove Theorem 5.2, it suffices to show the following:

(5.33) $\qquad e_i^c = \overline{\sigma}^{-1}\circ e_i^c\circ\overline{\sigma},\ \gamma_i=\gamma_i\circ\overline{\sigma},\ \varepsilon_i=\varepsilon_i\circ\overline{\sigma}\ (i\ne 0,n),$

(5.34) $\qquad e_0^{c_1}e_n^{c_2} = e_n^{c_2}e_0^{c_1},$

(5.35) $\qquad \gamma_n(e_0^c(v_1(x))) = \gamma_n(v_1(x)), \quad \gamma_0(e_n^c(v_1(x))) = \gamma_0(v_1(x)),$

(5.36) $\qquad \varepsilon_0(e_0^c(v_1(x))) = c^{-1}\varepsilon_0(v_1(x)),$

which are immediate from the above formulae. Let us show (5.33). Set $v_2(y) := \overline{\sigma}(v_1(x))$ and $v_1(x') := \overline{\sigma}^{-1}(e_i^c(v_2(y)))$ for $i=2,\dots,n-2$. Then $x'_j = x_j$ and $\overline{x}'_j = \overline{x}_j$

for $j \neq i-1, i$, and we have

$$\begin{aligned}
&a(v_2(y)) = a(v_1(x)),\\
&x'_i = \frac{1}{a(v_2(y))}\left(\overline{y}_{i+1} + \frac{cy_i\overline{y}_i}{y_{i+1}}\right) = x_i\frac{cx_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}}{x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}},\\
&\overline{x}'_i = \frac{1}{a(v_2(y))}\frac{cy_i\overline{y}_i y_{i+1}}{cy_i\overline{y}_i + y_{i+1}\overline{y}_{i+1}} = \overline{x}_i\frac{c(x_i\overline{x}_i + x_{i+1}\overline{x}_{i+1})}{cx_i\overline{x}_i + x_{i+1}\overline{x}_{i+1}},\\
&x'_{i-1} = \frac{1}{a(v_2(y))}\left(\overline{y}_i + \frac{cy_{i-1}\overline{y}_{i-1}}{y_i}\right) = x_{i-1},\\
&\overline{x}'_{i-1} = \frac{1}{a(v_2(y))}\frac{cy_{i-1}\overline{y}_{i-1}y_i}{cy_i\overline{y}_i + y_{i-1}\overline{y}_{i-1}} = \overline{x}_{i-1},
\end{aligned}$$

where the formula $y_i\overline{y}_i = a(v_1(x))x_i\overline{x}_i$ is useful to obtain these results. Therefore we have $e_i^c = \overline{\sigma}^{-1} \circ e_i^c \circ \overline{\sigma}$ for $i = 2, \ldots, n-2$. Others are obtained similarly.

5.10. **Ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$.** Let us investigate the ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$.

By the explicit forms of the geometric crystal $\mathcal{V}(\mathfrak{g})_l$, if we assume that l is a positive real number, it is clear that it has a natural positive structure $\theta_l\colon (\mathbb{C}^\times)^m \to \mathcal{V}(\mathfrak{g})_l$ $(x \mapsto v(x))$ where $m = \dim \mathcal{V}(\mathfrak{g})_l$. Then we have the following theorem:

Theorem 5.4. *For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$, suppose that $l > 0$. Then the ultra-discretization $\mathcal{UD}_{\theta_l}(\mathcal{V}(\mathfrak{g})_l)$ associated with the positive structure θ_l is isomorphic to the crystal $B_\infty(\mathfrak{g}^L)$ (4.11).*

Proof. First we consider the case $\mathfrak{g} \neq A_{2n}^{(2)}$. Let $\mathcal{V}(\mathfrak{g})$ be the affine geometric crystal in [KNO]. It follows from its explicit form ([KNO]) that restricting $l = 1$, we have the isomorphism

$$\mathcal{V}(\mathfrak{g})_{l|l=1} \xrightarrow{\sim} \mathcal{V}(\mathfrak{g}). \tag{5.37}$$

The resulting crystal of the ultra-discretization $\mathcal{UD}_{\theta_l}$ does not depend on l. This and (5.37) imply

$$\mathcal{UD}_{\theta_l}(\mathcal{V}(\mathfrak{g})_l) \cong \mathcal{UD}_\theta(\mathcal{V}(\mathfrak{g}))$$

as crystals. We show in [KNO] that $\mathcal{UD}_\theta(\mathcal{V}(\mathfrak{g})) \cong B_\infty(\mathfrak{g}^L)$. Thus, we have $\mathcal{UD}_{\theta_l}(\mathcal{V}(\mathfrak{g})_l) \cong B_\infty(\mathfrak{g}^L)$.

As for the case $\mathfrak{g} = A_{2n}^{(2)}$, we consider as follows: For the geometric crystal $\mathcal{V}(A_{2n}^{(2)\dagger})_l$, we define $\overline{e}_i := e_{n-i}$, $\overline{\gamma}_i := \gamma_{n-i}$ and $\overline{\varepsilon}_i := \varepsilon_{n-i}$ and set $x_i \mapsto y_{n-i}$ $(i = 0, \ldots, n)$ and $\overline{x}_i \mapsto \overline{y}_{n-i}$ $(i = 1, \ldots, n-1)$. We denote by $\overline{\mathcal{V}}_l$ the $A_{2n}^{(2)}$-geometric crystal thus obtained. The explicit form of $\overline{\mathcal{V}}_l$ is e.g., $\overline{e}_0^c : y_0 \mapsto cy_0$, $\overline{\varepsilon}_0(y) = \frac{y_1}{y_0}$ and $\overline{\gamma}_0(y) = \frac{y_n^2}{y_1\overline{y}_1}$, etc. By using the birational map $\overline{\sigma}$ for $A_{2n}^{(2)}$ as in 5.8, we obtain the isomorphism of $A_{2n}^{(2)}$-geometric crystal

$$\overline{\sigma} : \mathcal{V}(A_{2n}^{(2)})_l \xrightarrow{\sim} \overline{\mathcal{V}}_{l^2}.$$

Here $\overline{\mathcal{V}}_l$ is isomorphic to $\mathcal{V}(A_{2n}^{(2)\dagger})_l$ as an $A_{2n}^{(2)\dagger}$-geometric crystal. Due to the results in [KNO] and the fact $\mathcal{V}(A_{2n}^{(2)\dagger})_l|_{l=1} = \mathcal{V}(A_{2n}^{(2)\dagger})$, we have

$$\mathcal{UD}_{\theta_{l^2}}(\overline{\mathcal{V}}_{l^2}) \cong \mathcal{UD}_{\theta_{l^2}}(\mathcal{V}(A_{2n}^{(2)\dagger})_{l^2}) = \mathcal{UD}_\theta(\mathcal{V}(A_{2n}^{(2)\dagger})) \cong B_\infty(A_{2n}^{(2)}).$$

Let us denote by $\overline{B}_l$ the perfect crystal $B_l(A_{2n}^{(2)})$ considered as an ${A_{2n}^{(2)}}^\dagger$-perfect crystal. Then the crystal $B_\infty(A_{2n}^{(2)})$ can be regarded as the limit of perfect ${A_{2n}^{(2)}}^\dagger$-crystals $\{\overline{B}_l\}$. □

6. Folding of $D_n^{(1)}$-geometric crystal

6.1. $D_n^{(1)}$-geometric crystal $\mathcal{B}_L(D_n^{(1)})$. We review the geometric crystal $\mathcal{B}_L(D_n^{(1)})$ $(n \geq 4)$ in [KOTY].

Taking $L \in \mathbb{C}^\times$, define the geometric crystal $\mathcal{B}_L(D_n^{(1)}) := (\mathcal{B}_L(D_n^{(1)}), \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ as follows:

$$\mathcal{B}_L(D_n^{(1)}) := \{l = (l_1, \ldots, l_n, \bar{l}_{n-1}, \ldots, \bar{l}_1) \in (\mathbb{C}^\times)^{2n-1} \mid l_1 \cdots l_n \bar{l}_{n-1} \cdots \bar{l}_1 = L\},$$

$$\varepsilon_0(l) = l_1\left(\frac{l_2}{\bar{l}_2}+1\right),\ \varepsilon_{n-1}(l) = l_n\bar{l}_{n-1},\ \varepsilon_n(l) = \bar{l}_{n-1},\ \varepsilon_i(l) = \bar{l}_i\left(\frac{l_{i+1}}{\bar{l}_{i+1}}+1\right),$$

$$\gamma_0(l) = \frac{\bar{l}_1\bar{l}_2}{l_1l_2},\ \gamma_{n-1}(l) = \frac{l_{n-1}}{l_n\bar{l}_{n-1}},\ \gamma_n(l) = \frac{l_{n-1}l_n}{\bar{l}_{n-1}},\ \gamma_i(l) = \frac{l_i\bar{l}_{i+1}}{\bar{l}_il_{i+1}},$$

$$e_0^c(l) = \left(\frac{l_1}{\xi_2}, \frac{\xi_2 l_2}{c}, \ldots, \xi_2\bar{l}_2, \frac{c\bar{l}_1}{\xi_2}\right),\ e_{n-1}^c(l) = \left(\ldots, cl_{n-1}, \frac{l_n}{c}, \ldots\right),$$

$$e_n^c(l) = \left(\ldots, cl_n, \frac{\bar{l}_{n-1}}{c}, \ldots\right),$$

$$e_i^c(l) = \left(\ldots, \frac{cl_i}{\xi_{i+1}}, \frac{\xi_{i+1}l_{i+1}}{c}, \ldots, \xi_{i+1}\bar{l}_{i+1}, \frac{\bar{l}_i}{\xi_{i+1}}, \ldots\right),$$

$$\left(\xi_i := \frac{c\bar{l}_i + l_i}{\bar{l}_i + l_i}\right)\ (i = 1, \ldots, n-2).$$

This is isomorphic to the geometric crystal $\mathcal{V}(D_n^{(1)})_L$.

Proposition 6.1. *We have the following isomorphism of $D_n^{(1)}$-geometric crystals:*

$$\mathcal{B}_L(D_n^{(1)}) \xrightarrow{\sim} \mathcal{V}(D_n^{(1)})_L. \tag{6.1}$$

Proof. Define $\Xi : \mathcal{B}_L(D_n^{(1)}) \to \mathcal{V}(D_n^{(1)})_L$ $(l \mapsto x)$ to be

$$x_i := \frac{1}{\bar{l}_1\bar{l}_2\cdots\bar{l}_i},\ \overline{x}_i := \frac{l_1\cdots l_i}{L}\ (i = 1\ldots, n-2),$$
$$x_{n-1} := \frac{1}{\bar{l}_1\bar{l}_2\cdots\bar{l}_{n-1}l_n},\ x_n := \frac{1}{\bar{l}_1\bar{l}_2\cdots\bar{l}_{n-1}}.$$

The inverse Ξ^{-1} is given by

$$l_1 = L\overline{x}_1,\ \bar{l}_1 = \frac{1}{x_1},\ l_i = \frac{\overline{x}_i}{\overline{x}_{i-1}},\ \bar{l}_i = \frac{x_{i-1}}{x_i}\ (i = 2, \ldots, n-2),$$
$$l_{n-1} = \frac{x_{n-1}}{\overline{x}_{n-2}},\ l_n = \frac{x_n}{x_{n-1}},\ \bar{l}_{n-1} = \frac{x_{n-2}}{x_n}.$$

Then it is easy to see that it commutes with the action of e_i and preserves γ_i and ε_i. For example, for $x = \Xi(l)$, we have

$$\xi_2 = \frac{c\bar{l}_2 + l_2}{\bar{l}_2 + l_2} = \frac{c\frac{x_1}{x_2} + \frac{\overline{x}_2}{\overline{x}_1}}{\frac{x_1}{x_2} + \frac{\overline{x}_2}{\overline{x}_1}} = \frac{cx_1\overline{x}_1 + x_2\overline{x}_2}{x_1\overline{x}_1 + x_2\overline{x}_2} =: \Gamma_1$$

Then we obtain

$$\Xi(e_0^c(l)) = \Xi(\frac{l_1}{\xi_2}, \frac{\xi_2 l_2}{c}, \ldots, \xi_2 \bar{l}_2, \frac{c\bar{l}_1}{\xi_2}) = (\frac{cx_1}{\Gamma_1}, \frac{x_2}{c}, \ldots, \frac{\overline{x}_2}{c}, \Gamma_1 \overline{x}_1) = e_0^c \Xi(l),$$

$$\varepsilon_0(\Xi(l)) = \varepsilon_0(x) = L\frac{x_1\overline{x}_1 + x_2\overline{x}_2}{x_1} = L\frac{\frac{l_1}{L\bar{l}_1} + \frac{l_1 l_2}{L\bar{l}_1\bar{l}_2}}{\frac{1}{\bar{l}_1}} = l_1\left(\frac{l_2}{\bar{l}_2} + 1\right) = \varepsilon_0(l).$$

Other cases are shown similarly. □

6.2. **Folding.** We introduce certain involutions on $\mathcal{B}_L(D_n^{(1)})$ corresponding to the Dynkin diagram automorphisms for $D_n^{(1)}$. Let us consider the following Dynkin diagram automorphisms for $D_N^{(1)}$:

$$\begin{aligned} N = n+1 \qquad & \sigma_0^{(n)} : \alpha_0 \leftrightarrow \alpha_1, \quad \alpha_i \mapsto \alpha_i \quad (i \neq 0, 1), \\ N = n+1 \qquad & \sigma_1^{(n)} : \alpha_n \leftrightarrow \alpha_{n+1}, \quad \alpha_i \mapsto \alpha_i \quad (i \neq n, n+1), \\ N = 2n \qquad & \sigma_2^{(n)} : \alpha_i \leftrightarrow \alpha_{2n-i} \quad (i = 0, 1, \ldots, 2n). \end{aligned}$$

For each involution $\sigma_j^{(n)}$ on $D_N^{(1)} (j = 0, 1, 2)$, we define the involution $\Sigma_j^{(n)}$ on $\mathcal{B}_L(D_N^{(1)})$ $(j = 0, 1, 2)$ as a unique solution l' of the equations for a given l:

$$\gamma_{\sigma_j^{(n)}(i)}(l') = \gamma_i(l), \qquad \varepsilon_{\sigma_j^{(n)}(i)}(l') = \varepsilon_i(l) \quad (i = 0, 1, \ldots, N). \tag{6.2}$$

Then we get $(l' := \Sigma_j^{(n)}(l))$

$\Sigma_0^{(n)} \colon \mathcal{B}_L(D_{n+1}^{(1)}) \longrightarrow \mathcal{B}_L(D_{n+1}^{(1)})$,

$$l_1' = \bar{l}_1, \quad \bar{l}_1' = l_1, \quad l_i' = l_i, \quad \bar{l}_i' = \bar{l}_i \ (i \neq 1).$$

$\Sigma_1^{(n)} \colon \mathcal{B}_L(D_{n+1}^{(1)}) \longrightarrow \mathcal{B}_L(D_{n+1}^{(1)})$,

$$l_n' = l_n l_{n+1}, \quad l_{n+1}' = \frac{1}{l_{n+1}}, \quad \bar{l}_n' = l_{n+1}\bar{l}_n, \quad l_i' = l_i, \quad \bar{l}_i' = \bar{l}_i \quad (i \neq, n, n+1).$$

$\Sigma_2^{(n)} \colon \mathcal{B}_L(D_{2n}^{(1)}) \longrightarrow \mathcal{B}_L(D_{2n}^{(1)})$

$$l_1' = \frac{l_{2n-1}\bar{l}_{2n-1}}{l_{2n-1} + \bar{l}_{2n-1}}, \quad \bar{l}_1' = \frac{l_{2n} l_{2n-1}\bar{l}_{2n-1}}{l_{2n-1} + \bar{l}_{2n-1}}, \quad l_{2n}' = \frac{\bar{l}_1}{l_1},$$

$$l_{2n-1}' = \frac{l_1\bar{l}_2}{l_2}\left(\frac{l_2}{\bar{l}_2} + 1\right), \quad \bar{l}_{2n-1}' = l_1\left(\frac{l_2}{\bar{l}_2} + 1\right),$$

$$l_{2n-i}' = \frac{l_i\bar{l}_i}{l_{i+1}}\left(\frac{l_{i+1} + \bar{l}_{i+1}}{l_i + \bar{l}_i}\right), \quad \bar{l}_{2n-i}' = \frac{l_i\bar{l}_i}{\bar{l}_{i+1}}\left(\frac{l_{i+1} + \bar{l}_{i+1}}{l_i + \bar{l}_i}\right) \quad (2 \leq i \leq 2n-2).$$

Lemma 6.2. *Let $(X, \{e_i\}_{i=1,2,3}, \{\gamma_i\}_{i=1,2,3}, \{\varepsilon_i\}_{i=1,2,3})$ be a A_3-geometric crystal $(a_{ij} = -1$ if $|i-j| = 1)$. and set $E_1^c := e_1^c \circ e_3^c (= e_3^c \circ e_1^c)$. Then we have*

$$E_1^c e_2^{c^2 d} E_1^{cd} e_2^d = e_2^d E_1^{cd} e_2^{c^2 d} E_1^c \quad (c, d \in \mathbb{C}^\times), \quad \gamma_2(E_1^c(x)) = c^{-2}\gamma_2(x).$$

Proof. The second equation is easily obtained:

$$\gamma_2(E_1^c(x)) = \gamma_2(e_1^c \circ e_3^c(x)) = c^{-1}\gamma_2(e_3^c(x)) = c^{-2}\gamma_2(x) \quad (x \in X).$$

The first one is derived as follows:

$$\begin{aligned} E_1^c e_2^{c^2 d} E_1^{cd} e_2^d &= (e_1^c e_3^c) e_2^{c^2 d} (e_3^{cd} e_1^{cd}) e_2^d = e_1^c e_2^{cd} e_3^{c^2 d} e_2^c e_1^{cd} e_2^d = e_1^c e_2^{cd} e_3^{c^2 d} e_1^d e_2^{cd} e_1^c \\ &= e_1^c e_2^{cd} e_1^d e_3^{c^2 d} e_2^{cd} e_1^c = e_2^d e_1^{cd} e_2^c e_3^{c^2 d} e_2^{cd} e_1^c = e_2^d (e_1^{cd} e_3^{cd}) e_2^{c^2 d} (e_3^c e_1^c) = e_2^d E_1^{cd} e_2^{c^2 d} E_1^c. \end{aligned}$$

□

6.3. **Fixed-point variety-$B_n^{(1)}$.** As for the involution $\Sigma_1^{(n)}$ we consider the fixed-point variety $X_1^{(n,L)} \subset \mathcal{B}_L(D_{n+1}^{(1)})$:

$$X_1^{(n,L)} := \{l = (l_1, \ldots, l_{n+1}, \bar{l}_n, \ldots, \bar{l}_1) | \Sigma_1^{(n)}(l) = l (\Leftrightarrow l_{n+1} = 1)\}.$$

Proposition 6.3. *Let $\mathcal{B}_L(D_{n+1}^{(1)}) = (\mathcal{B}_L, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be the $D_{n+1}^{(1)}$-geometric crystal as above. Then we have a $B_n^{(1)}$-geometric crystal structure on $X_1^{(n,L)}$ as follows:*

$$(e_i^{(B_n^{(1)})})^c := \begin{cases} e_i^c & \text{for } i \neq n, \\ e_n^c \circ e_{n+1}^c & \text{for } i = n, \end{cases}, \quad \gamma_i^{(B_n^{(1)})} := \gamma_i, \qquad \varepsilon_i^{(B_n^{(1)})} := \varepsilon_i \quad (i = 0, \ldots, n).$$

Proof. It suffices to check the conditions for a geometric crystal. But, most are trivial except the cases related to $i = n$. Namely, we have to see the Verma relation between e_{n-1} and e_n, $\gamma_i^{(B_n^{(1)})}((e_n^{B_n^{(1)}})^c(x))$, $\gamma_n^{(B_n^{(1)})}((e_i^{B_n^{(1)}})^c(x))$ and $\varepsilon_n^{(B_n^{(1)})}((e_n^{B_n^{(1)}})^c(x))$, which are immediate from Lemma 6.2. □

Moreover, the geometric crystal $X_1^{(n,L)}$ induces another $B_n^{(1)}$- geometric crystal $\mathcal{B}_L(B_n^{(1)})$:

$$\mathcal{B}_L(B_n^{(1)}) := \{m = (m_1, \ldots, m_n, \overline{m}_n, \ldots, \overline{m}_1)(\mathbb{C}^\times)^{2n} \,|\, m_1 \cdots m_n \overline{m}_n \cdots \overline{m}_1 = L\},$$

$$\varepsilon_0(m) = m_1 \left(\frac{m_2}{\overline{m}_2} + 1\right), \varepsilon_n(m) = \overline{m}_n, \varepsilon_i(m) = \overline{m}_i \left(\frac{m_{i+1}}{\overline{m}_{i+1}} + 1\right) (i = 1, \ldots, n-1),$$

$$\gamma_0(m) = \frac{\overline{m}_1 \overline{m}_2}{m_1 m_2}, \gamma_n(m) = \frac{m_n}{\overline{m}_n}, \gamma_i(m) = \frac{m_i \overline{m}_{i+1}}{\overline{m}_i m_{i+1}} (i = 1, \ldots, n-1),$$

$$e_0^c(m) = (\frac{m_1}{\xi_2}, \frac{\xi_2 m_2}{c}, \ldots, \xi_2 \overline{m}_2, \frac{c\overline{m}_1}{\xi_2}), \; e_n^c(m) = (\cdot\cdot, c m_n, \frac{\overline{m}_n}{c}, \cdot\cdot),$$

$$e_i^c(m) = (\cdots, \frac{c m_i}{\xi_{i+1}}, \frac{\xi_{i+1} m_{i+1}}{c}, \ldots, \xi_{i+1} \overline{m}_{i+1}, \frac{\overline{m}_i}{\xi_{i+1}}, \ldots)$$

$$(i = 1, \ldots, n-1) \quad (\xi_i := \frac{c\overline{m}_i + m_i}{\overline{m}_i + m_i}).$$

Let $\eta \colon \mathcal{B}_L(B_n^{(1)}) \longrightarrow X_1^{(n,L)}$ $(m \mapsto l)$ be the morphism defined by

$$l_i = m_i, \quad \bar{l}_i = \overline{m}_i \quad (i = 1, \cdots, n),$$

where $(l_1, \ldots, l_n, 1, \bar{l}_n, \ldots, \bar{l}_1) \in X_1^{(n,L)}$ and $m = (m_1, \ldots, m_n, \overline{m}_n, \overline{m}_1) \in \mathcal{B}_L(B_n^{(1)})$. It is trivial that η commutes with the actions e_i and preserves γ_i and ε_i. Then η is an isomorphism of $B_n^{(1)}$-geometric crystals.

Proposition 6.4. *We have the following isomorphisms of $B_n^{(1)}$-geometric crystals:*

$$\mathcal{V}(B_n^{(1)})_L \xleftarrow{\sim} \mathcal{B}_L(B_n^{(1)}) \xrightarrow{\sim} X_1^{(n,L)}. \tag{6.3}$$

Proof. The second isomorphism in (6.3) is given by η. Then we see the first one. Define $\Xi \colon \mathcal{B}_L(B_n^{(1)}) \to \mathcal{V}(B_n^{(1)})_L$ $(m \mapsto x)$ to be

$$x_i := \frac{1}{\overline{m}_1 \overline{m}_2 \cdots \overline{m}_i} \; (i = 1, \ldots, n), \quad \overline{x}_i = \frac{m_1 \cdots m_i}{L} \; (i = 1, \ldots, n-1),$$

and the inverse is

$$m_1 = L\overline{x}_1, \quad m_i = \frac{\overline{x}_i}{\overline{x}_{i-1}} \ (2 \le i \le n-1),$$
$$m_n = \frac{x_n}{\overline{x}_{n-1}}, \quad \overline{m}_1 = \frac{1}{x_1}, \quad \overline{m}_i = \frac{x_{i-1}}{x_i} \ (2 \le i \le n).$$

Then, by a direct calculation, we can check that Ξ commutes with any e_i^c and preserves γ_i and ε_i. Thus, Ξ is an isomorphism of $B_n^{(1)}$-geometric crystals. □

6.4. **Fixed-point variety-$D^{(2)}_{n+1}$.** As for the involution

$$\Sigma_3^{(n)} := \Sigma_0^{(n+1)} \circ \Sigma_1^{(n+1)} (= \Sigma_1^{(n+1)} \circ \Sigma_0^{(n+1)})$$

on $\mathcal{B}_L(D^{(1)}_{n+2})$, we consider the fixed-point variety $X_3^{(n,L)} \subset \mathcal{B}_L(D^{(1)}_{n+2})$:

$$X_3^{(n,L)} := \{l = (l_1, \dots, l_{n+2}, \overline{l}_{n+1}, \dots, \overline{l}_1) | \Sigma_3^{(n,L)}(l) = l (\Leftrightarrow l_{n+2} = 1,\ l_1 = \overline{l}_1)\}.$$

Proposition 6.5. *Let $\mathcal{B}_L(D^{(1)}_{n+2}) = (\mathcal{B}_L, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be the $D^{(1)}_{n+2}$-geometric crystal as above. Then we have a $D^{(2)}_{n+1}$-geometric crystal structure on $X_3^{(n,L)}$ as follows:*

$$(e_i^{(D^{(2)}_{n+1})})^c := \begin{cases} e^c_{i+1} & \text{for } i \ne 0, n, \\ e^c_0 \circ e^c_1 & \text{for } i = 0, \\ e^c_{n+1} \circ e^c_{n+2} & \text{for } i = n, \end{cases} \quad \gamma_i^{(D^{(2)}_{n+1})} := \gamma_{i+1},\ \varepsilon_i^{(D^{(2)}_{n+1})} := \varepsilon_{i+1} \quad (i = 0, \dots, n).$$

Proof. We can prove it by the similar argument for $X_1^{(n,L)}$ using Lemma 6.2. □

The geometric crystal $X_3^{(n,L)}$ induces another $D^{(2)}_{n+1}$-geometric crystal $\mathcal{B}_L(D^{(2)}_{n+1})$:

$$\mathcal{B}_L(D^{(2)}_{n+1}) := \{m = (m_0, m_1, \dots, m_n, \overline{m}_n, \dots, \overline{m}_1)(\mathbb{C}^\times)^{2n} \mid m_0^2 m_1 \cdots m_n \overline{m}_n \cdots \overline{m}_1 = L\},$$

$$\varepsilon_0(m) = m_0 \left(\frac{m_1}{\overline{m}_1} + 1\right), \ \varepsilon_n(m) = \overline{m}_n, \ \varepsilon_i(m) = \overline{m}_i \left(\frac{m_{i+1}}{\overline{m}_{i+1}} + 1\right) \ (i = 1, \dots, n-1),$$

$$\gamma_0(m) = \frac{\overline{m}_1}{m_1}, \ \gamma_{n-1}(m) = \frac{m_{n-1}}{m_n \overline{m}_{n-1}}, \ \gamma_n(m) = \frac{m_n}{\overline{m}_n},$$

$$\gamma_i(m) = \frac{m_i \overline{m}_{i+1}}{\overline{m}_i m_{i+1}} \ (i = 1, \dots, n-2),$$

$$e^c_0(m) = (\frac{cm_0}{\xi_1}, \frac{\xi_1 m_1}{c^2}, \dots, \xi_1 \overline{m}_1) \ (\xi_1(m) := \frac{c^2 \overline{m}_1 + m_1}{\overline{m}_1 + m_1}),$$

$$e^c_n(m) = (\dots, cm_n, \frac{\overline{m}_n}{c}, \dots),$$

$$e^c_i(m) = (\dots, \frac{cm_i}{\xi_{i+1}}, \frac{\xi_{i+1} m_{i+1}}{c}, \dots, \xi_{i+1} \overline{m}_{i+1}, \frac{\overline{m}_i}{\xi_{i+1}}, \dots) \quad (i = 1, \dots, n-1)$$
$$(\xi_i := \frac{c\overline{m}_i + m_i}{\overline{m}_i + m_i}).$$

Let $\eta \colon \mathcal{B}_L(D^{(2)}_{n+1}) \longrightarrow X_3^{(n,L)}$ $(m \mapsto l)$ be the morphism defined by

$$l_{i+1} = m_i \quad (i = 0, 1, \cdots, n), \quad \overline{l}_{i+1} = \overline{m}_i \quad (i = 1, \cdots, n),$$

where $(l_1,\dots,l_{n+1},1,\bar{l}_{n+1},\dots,\bar{l}_2,l_1)\in X_3^{(n,L)}$ and $m=(m_0,m_1,\dots,m_n,\overline{m}_n,\overline{m}_1)$ $\in\mathcal{B}_L(D^{(2)}_{n+1})$. Then, η is an isomorphism of $D^{(2)}_{n+1}$-geometric crystal.

Proposition 6.6. *We have the following isomorphisms of $D^{(2)}_{n+1}$-geometric crystals:*

$$\mathcal{V}(D^{(2)}_{n+1})_L\xleftarrow{\sim}\mathcal{B}_{L^2}(D^{(2)}_{n+1})\xrightarrow{\sim}X_3^{(n,L^2)}. \tag{6.4}$$

Proof. The second isomorphism in (6.4) is given by η. Then we see the first one. Define $\Xi\colon\mathcal{B}_{L^2}(D^{(2)}_{n+1})\to\mathcal{V}(D^{(2)}_{n+1})_L$ $(m\mapsto x)$ to be

$$x_0:=\frac{1}{m_0},\quad x_i:=\frac{1}{m_0^2\overline{m}_1\overline{m}_2\cdots\overline{m}_i}\ (i=1,\dots,n),\quad \overline{x}_i=\frac{m_1\cdots m_i}{L^2}\ (i=1,\dots,n-1).$$

and the inverse is

$$m_0=\frac{1}{x_0},\ m_1=L^2\overline{x}_1,\ m_i=\frac{\overline{x}_i}{\overline{x}_{i-1}}\ (2\le i\le n-1),$$
$$m_n=\frac{x_n}{\overline{x}_{n-1}},\ \overline{m}_1=\frac{x_0^2}{x_1},\ \overline{m}_i=\frac{x_{i-1}}{x_i}\ (2\le i\le n).$$

Then, calculating directly, we can check that Ξ is an isomorphism of geometric crystals. □

6.5. **Fixed-point variety-$A^{(2)}_{2n-1}$.** As for the involution $\Sigma_2^{(n)}$ we consider the fixed-point variety $X_2^{(n,L)}\subset\mathcal{B}_L(D^{(1)}_{2n})$:

$$X_2^{(n,L)}:=\{l=(l_1,\dots,l_{2n},\bar{l}_{2n-1},\cdots,\bar{l}_1)|\Sigma_2^{(n)}(l)=l\},$$

where the condition $\Sigma_2^{(n)}(l)=l$ is equivalent to

(6.5)
$$l_1=\tfrac{l_{2n-1}\bar{l}_{2n-1}}{l_{2n-1}+\bar{l}_{2n-1}},\quad \bar{l}_1=\tfrac{l_{2n}l_{2n-1}\bar{l}_{2n-1}}{l_{2n-1}+\bar{l}_{2n-1}},$$
$$l_{2n-1}=\tfrac{l_1\bar{l}_2}{l_2}\left(\tfrac{l_2}{\bar{l}_2}+1\right),\quad \bar{l}_{2n-1}=l_1\left(\tfrac{\bar{l}_2}{l_2}+1\right),$$
$$l_{2n-i}=\tfrac{l_i\bar{l}_i}{l_{i+1}}\tfrac{l_{i+1}+\bar{l}_{i+1}}{l_i+\bar{l}_i},\quad \bar{l}_{2n-i}=\tfrac{l_i\bar{l}_i}{\bar{l}_{i+1}}\tfrac{l_{i+1}+\bar{l}_{i+1}}{l_i+\bar{l}_i}\quad (i=2,\dots,2n-2),\quad l_{2n}=\tfrac{\bar{l}_1}{l_1}.$$

Note that all of the equations in (6.5) are not necessarily independent. Indeed, we only need $2n$-equations: $l_1=\cdots,l_n=\cdots$ and $\bar{l}_1=\cdots,\bar{l}_n=\cdots$ in (6.5).

Proposition 6.7. *Let $\mathcal{B}_L(D^{(1)}_{2n})=(\mathcal{B}_L,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ be the $D^{(1)}_{2n}$-geometric crystal as above. Then we have a $A^{(2)}_{2n-1}$-geometric crystal structure on $X_2^{(n,L)}$ as follows:*

$$(e_i^{(A^{(2)}_{2n-1})})^c:=\begin{cases}e_i^c\circ e_{2n-i}^c & \text{for } 0\le i\le n-1,\\ e_n^c & \text{for } i=n,\end{cases}\quad \begin{aligned}&\gamma_i^{(A^{(2)}_{2n-1})}:=\gamma_i(=\gamma_{2n-i}),\\ &\varepsilon_i^{(A^{(2)}_{2n-1})}:=\varepsilon_i(=\varepsilon_{2n-i})\quad (i=0,\dots,n).\end{aligned}$$

The proof is similar to the previous cases.

The geometric crystal $X_2^{(n,L)}$ induces another $A^{(2)}_{2n-1}$-geometric crystal $\mathcal{B}_L(A^{(2)}_{2n-1})$:

$$\begin{aligned}\mathcal{B}_L(A^{(2)}_{2n-1}) := \{ & m=(m_1,\dots,m_n,\overline{m}_n,\dots,\overline{m}_1)\\ & \in (\mathbb{C}^\times)^{2n} \mid m_n\overline{m}_n(m_1\cdots m_{n-1}\overline{m}_{n-1}\cdots\overline{m}_1)^2 = L^2\}.\end{aligned}$$

For $m \in \mathcal{B}_L(A^{(2)}_{2n-1})$ set

$$\begin{aligned}
&\mu(m) := \frac{L}{m_1\cdots m_{n-1}\overline{m}_n\cdots\overline{m}_1}, \qquad \overline{\mu}(m) := \frac{L}{m_1\cdots m_n\overline{m}_{n-1}\cdots\overline{m}_1}\left(=\frac{1}{\mu(m)}\right),\\
&\varepsilon_0(m) = m_1\left(\frac{m_2}{\overline{m}_2}+1\right),\ \varepsilon_{n-1}(m) = \overline{m}_{n-1}(\mu(m)+1),\\
&\varepsilon_n(m) = \overline{m}_n,\ \varepsilon_i(m) = \overline{m}_i\left(\frac{m_{i+1}}{\overline{m}_{i+1}}+1\right),\\
&\gamma_0(m) = \frac{\overline{m}_1\overline{m}_2}{m_1m_2},\ \gamma_{n-1}(m) = \frac{m_{n-1}}{\overline{m}_{n-1}}\overline{\mu}(m),\ \gamma_n(m) = \frac{m_n}{\overline{m}_n},\\
&\gamma_i(m) = \frac{m_i\overline{m}_{i+1}}{\overline{m}_i m_{i+1}}\ \ (1\le i\le n-2),\\
&e_0^c(m) = \left(\frac{m_1}{\xi_1},\frac{\xi_1 m_2}{c},\dots,\xi_1\overline{m}_2,\frac{c\overline{m}_1}{\xi_1}\right),\ e_n^c(m) = \left(\dots,cm_n,\frac{\overline{m}_n}{c},\dots\right),\\
&e_{n-1}^c(m) = \left(\dots,m_{n-1}\frac{c}{\xi_{n-1}},m_n\frac{\xi_{n-1}^2}{c^2},\overline{m}_n\xi_{n-1}^2,\frac{\overline{m}_{n-1}}{\xi_{n-1}},\dots\right),\\
&e_i^c(m) = \left(\dots,\frac{cm_i}{\xi_i},\frac{\xi_i m_{i+1}}{c},\dots,\xi_i\overline{m}_{i+1},\frac{\overline{m}_i}{\xi_i},\dots\right)\quad (i=1,\dots,n-1),
\end{aligned}$$

where

$$\xi_i := \begin{cases}\frac{c\overline{m}_{i+1}+m_{i+1}}{\overline{m}_{i+1}+m_{i+1}} & \text{for } i\ne n-1,\\ \frac{c+\mu(m)}{1+\mu(m)} & \text{for } i=n-1.\end{cases}$$

Let $\eta\colon \mathcal{B}_L(A^{(2)}_{2n-1}) \longrightarrow X_2^{(n,L^2)}$ $(m\mapsto l)$ be the morphism defined by

$$l_i = m_i,\quad \overline{l}_i = \overline{m}_i\quad (i=1,\dots,n-1),\quad l_n = \frac{m_n}{1+\mu(m)},\quad \overline{l}_n = \frac{\overline{m}_n}{1+\overline{\mu}(m)},$$

where $l_i, \overline{l}_i$ $(i=n+1,\dots,2n)$ are uniquely determined by (6.5) and then, η is an isomorphism of $A^{(2)}_{2n-1}$-geometric crystal.

Proposition 6.8. *We have the following isomorphisms of $A^{(2)}_{2n-1}$-geometric crystals:*

$$\mathcal{V}(A^{(2)}_{2n-1})_L \xleftarrow{\sim} \mathcal{B}_L(A^{(2)}_{2n-1}) \xrightarrow{\sim} X_2^{(n,L^2)}. \tag{6.6}$$

Proof. The second isomorphism in (6.6) is given by η. Then let us check the first one. Define $\Xi\colon \mathcal{B}_L(A^{(2)}_{2n-1}) \to \mathcal{V}(A^{(2)}_{2n-1})_L$ $(m \mapsto x)$ to be

$$x_i := \frac{1}{\overline{m}_1\overline{m}_2\cdots\overline{m}_i},\quad \overline{x}_i = \frac{m_1\cdots m_i}{L^2}\ (i=1,\dots,n-1),\quad x_n = \frac{1}{\overline{m}_n(\overline{m}_1\cdots\overline{m}_{n-1})^2},$$

and then the inverse is

$$m_1 = L^2\overline{x}_1,\quad \overline{m}_1 = \frac{1}{x_1},\quad m_i = \frac{\overline{x}_i}{\overline{x}_{i-1}},\quad \overline{m}_i = \frac{x_{i-1}}{x_i}\ (2\le i\le n-1),$$
$$m_n = \frac{x_n}{L^2\overline{x}_{n-1}^2},\quad \overline{m}_n = \frac{x_{n-1}^2}{x_n}.$$

Then, by direct calculations, we can check that Ξ is an isomorphism of geometric crystals. □

6.6. **Fixed-point variety-$A^{(2)}_{2n}$.** As for the involution $\Sigma_4^{(n)} := \Sigma_2^{(n+1)}\Sigma_1^{(2n+1)}\Sigma_0^{(2n+1)}$ we consider a fixed-point variety $X_4^{(n,L)} \subset \mathcal{B}_L(D^{(1)}_{2n+2})$:

$$X_4^{(n,L)} := \{l = (l_1,\dots,l_{2n+2},\overline{l}_{2n+1},\cdots,\overline{l}_1) \mid \Sigma_4^{(n)}(l) = l\},$$

where the condition $\Sigma_4^{(n)}(l) = l$ is equivalent to

$$l_1 = \overline{l}_1 = \tfrac{l_{2n+1}\overline{l}_{2n+1}}{l_{2n+1}+\overline{l}_{2n+1}},\quad l_{2n+1} = \tfrac{l_1\overline{l}_2}{l_2}\left(\tfrac{l_2}{\overline{l}_2}+1\right),\quad \overline{l}_{2n+1} = l_1\left(\tfrac{l_2}{\overline{l}_2}+1\right),\quad l_{2n+2}=1,$$
$$l_{2n+2-i} = \tfrac{l_i\overline{l}_i}{l_{i+1}}\tfrac{l_{i+1}+\overline{l}_{i+1}}{l_i+\overline{l}_i},\quad \overline{l}_{2n+2-i} = \tfrac{l_i\overline{l}_i}{\overline{l}_{i+1}}\tfrac{l_{i+1}+\overline{l}_{i+1}}{l_i+\overline{l}_i}\quad (i=2,\dots,2n). \tag{6.7}$$

Note that all the equations in (6.7) are not necessarily independent. Indeed, we only need $2n$-equations: $l_1 = \cdots, l_n = \cdots$ and $\overline{l}_1 = \cdots, \overline{l}_n = \cdots$ in (6.7).

Proposition 6.9. *Let $\mathcal{B}_L(D^{(1)}_{2n+2}) = (\mathcal{B}_L,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ be the $D^{(1)}_{2n+2}$-geometric crystal as above. Then we have a $A^{(2)}_{2n}$-geometric crystal structure on $X_4^{(n,L)}$ as follows:*

$$(e_i^{(A^{(2)}_{2n})})^c := \begin{cases} e_0^c\circ e_1^c\circ e_{2n+1}^c\circ e_{2n+2}^c & \text{for } i=0,\\ e_{i+1}^c\circ e_{2n-i+1}^c & \text{for } 1\le i\le n-1,\\ e_{n+1}^c & \text{for } i=n,\end{cases}$$
$$\gamma_i^{(A^{(2)}_{2n})} := \begin{cases} \gamma_0(=\gamma_1=\gamma_{2n+1}=\gamma_{2n+2}) & \text{for } i=0,\\ \gamma_{i+1}(=\gamma_{2n-i+1}) & \text{for } i\ne 0,\end{cases}$$
$$\varepsilon_i^{(A^{(2)}_{2n})} := \begin{cases} \varepsilon_0(=\varepsilon_1=\varepsilon_{2n+1}=\varepsilon_{2n+2}) & \text{for } i=0,\\ \varepsilon_{i+1}(=\varepsilon_{2n-i+1}) & \text{for } i\ne 0.\end{cases}$$

We can prove it by arguments similar to those of the previous cases.

The geometric crystal $X_4^{(n,L)}$ induces another $A^{(2)}_{2n}$-geometric crystal $\mathcal{B}_L(A^{(2)}_{2n})$:

$$\mathcal{B}_L(A^{(2)}_{2n}) := \{m = (m_0,m_1,\dots,m_n,\overline{m}_n,\dots,\overline{m}_1) \in (\mathbb{C}^\times)^{2n} \mid m_n\overline{m}_n(m_0^2m_1\cdots m_{n-1}\overline{m}_{n-1}\cdots\overline{m}_1)^2 = L^2\}.$$

For $m \in \mathcal{B}_L(A^{(2)}_{2n})$ set

$$\mu(m) := \frac{L}{m_0^2 m_1 \cdots m_{n-1}\overline{m}_n \cdots \overline{m}_1}, \quad \overline{\mu}(m) := \frac{L}{m_0^2 m_1 \cdots m_n \overline{m}_{n-1} \cdots \overline{m}_1} (= \frac{1}{\mu(m)}).$$

$$\varepsilon_0(m) = m_0\left(\frac{m_1}{\overline{m}_1}+1\right), \ \varepsilon_{n-1}(m) = \overline{m}_{n-1}(\mu(m)+1),$$

$$\varepsilon_n(m) = \overline{m}_n, \ \varepsilon_i(m) = \overline{m}_i\left(\frac{m_{i+1}}{\overline{m}_{i+1}}+1\right),$$

$$\gamma_0(m) = \frac{\overline{m}_1}{m_1}, \ \gamma_{n-1}(m) = \frac{m_{n-1}}{\overline{m}_{n-1}}\overline{\mu}(m), \ \gamma_n(m) = \frac{m_n}{\overline{m}_n},$$

$$\gamma_i(m) = \frac{m_i\overline{m}_{i+1}}{\overline{m}_i m_{i+1}} \ (1 \le i \le n-2),$$

$$e_0^c(m) = e_0^c(m) = (\frac{cm_0}{\xi_0}, \frac{\xi_0 m_1}{c^2}, \ldots, \xi_0\overline{m}_1), \ e_n^c(m) = (\ldots, cm_n, \frac{\overline{m}_n}{c}, \ldots),$$

$$e_{n-1}^c(m) = (\ldots, m_{n-1}\frac{c}{\xi_{n-1}}, m_n\frac{\xi_{n-1}^2}{c^2}, \overline{m}_n\xi_{n-1}^2, \frac{\overline{m}_{n-1}}{\xi_{n-1}}, \ldots),$$

$$e_i^c(m) = (\ldots, \frac{cm_i}{\xi_i}, \frac{\xi_i m_{i+1}}{c}, \ldots, \xi_i\overline{m}_{i+1}, \frac{\overline{m}_i}{\xi_i}, \ldots) \quad (1 \le i \le n-2),$$

where

$$\xi_i := \begin{cases} \frac{c^2\overline{m}_1+m_1}{\overline{m}_1+m_1} & \text{for } i=0, \\ \frac{c\overline{m}_{i+1}+m_{i+1}}{\overline{m}_{i+1}+m_{i+1}} & \text{for } i \neq 0, n-1 \\ \frac{c+\mu(m)}{1+\mu(m)} & \text{for } i=n-1. \end{cases}$$

Let $\eta\colon \mathcal{B}_L(A^{(2)}_{2n}) \longrightarrow X_4^{(n,L^2)}$ $(m \mapsto l)$ be the morphism defined by

$$l_1 = \bar{l}_1 = m_0, \quad l_i = m_{i-1}, \quad \bar{l}_i = \overline{m}_{i-1} (i=2,\ldots,n),$$
$$l_{n+1} = \frac{m_n}{1+\mu(m)}, \quad \bar{l}_{n+1} = \frac{\overline{m}_n}{1+\overline{\mu}(m)},$$

where $l_i, \bar{l}_i$ $(i = n+2, \ldots, 2n+2)$ are uniquely determined by (6.7) and then, η is an isomorphism of $A^{(2)}_{2n}$-geometric crystal.

Proposition 6.10. *We have the following isomorphisms of $A^{(2)}_{2n}$-geometric crystals:*

$$\mathcal{V}(A^{(2)}_{2n})_L \xleftarrow{\sim} \mathcal{B}_L(A^{(2)}_{2n}) \xrightarrow{\sim} X_4^{(n,L^2)}. \tag{6.8}$$

Proof. The second isomorphism in (6.8) is given by η. Then we check the first one. Define $\Xi\colon \mathcal{B}_L(A^{(2)}_{2n}) \to \mathcal{V}(A^{(2)}_{2n-1})_L$ $(m \mapsto x)$ to be

$$x_0 = \frac{1}{m_0}, \quad x_i := \frac{1}{m_0^2\overline{m}_1\overline{m}_2\cdots\overline{m}_i}, \quad \overline{x}_i = \frac{m_1\cdots m_i}{L^2} \ (i=1,\ldots,n-1),$$
$$x_n = \frac{1}{\overline{m}_n(m_0^2\overline{m}_1\cdots\overline{m}_{n-1})^2},$$

and the inverse Ξ^{-1} is

$$m_0 = \frac{1}{x_0},\ m_1 = L^2\overline{x}_1,\ \overline{m}_1 = \frac{x_0^2}{x_1},\ m_i = \frac{\overline{x}_i}{\overline{x}_{i-1}},\ \overline{m}_i = \frac{x_{i-1}}{x_i}\ (2 \le i \le n-1),$$
$$m_n = \frac{x_n}{\overline{x}_{n-1}^2},\ \overline{m}_n = \frac{x_{n-1}^2}{x_n}.$$

Then, by direct calculations, we can check that Ξ is an isomorphism of geometric crystals. $\square$

7. Product structure on affine geometric crystals

In general, if $\mathbb{X}_1$ and $\mathbb{X}_2$ are geometric crystals induced from unipotent crystals, the product $\mathbb{X}_1 \times \mathbb{X}_2$ possesses a geometric crystal structure ([BK], [N1]). More precisely, let $\mathbb{X}_1 = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ and $\mathbb{X}_2 = (Y, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ be geometric crystals. For $x \in X$ and $y \in Y$ set

(7.1) $$\gamma_i(x,y) := \gamma_i(x)\gamma_i(y),$$

(7.2) $$\varepsilon_i(x,y) := \varepsilon_i(x) + \frac{\varepsilon_i(x)\varepsilon_i(y)}{\varphi_i(x)} \quad (\varphi_i(x) = \gamma(x)\varepsilon_i(x)),$$

(7.3) $$e_i^c(x,y) := (e_i^{c_1}(x), e_i^{c_2}(y)) \qquad \text{where } c_1 := \frac{c\varphi_i(x)+\varepsilon_i(y)}{\varphi_i(x)+\varepsilon_i(y)}, \quad c_2 := \frac{c}{c_1}.$$

Theorem 7.1 ([BK, N1])**.** *Suppose that the geometric crystals $\mathbb{X}_1$ and $\mathbb{X}_2$ are induced from unipotent crystals. Then,* (7.1)–(7.3) *endow the product $X \times Y$ with a structure of a geometric crystal. Moreover, if $\mathbb{X}_1$ and $\mathbb{X}_2$ are positive, then $\mathbb{X}_1 \times \mathbb{X}_2$ is positive and we have the isomorphism of crystals:*

(7.4) $$\mathcal{UD}(\mathbb{X}_1 \times \mathbb{X}_2) \cong \mathcal{UD}(\mathbb{X}_1) \otimes \mathcal{UD}(\mathbb{X}_2).$$

As for the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l$ in Section 5, its data $e_i, \gamma_i, \varepsilon_i$ $(i \in I\setminus\{0\})$ are obtained from the ones of the geometric crystal $B_{\mathbf{i}}^- \cdot l^H$ as in 2.3 which is induced from the unipotent crystal on some $X_w \times l^H$ where $\mathbf{i}$ is a reduced word for w and X_w is the Schubert cell associated with $w \in W$. Thus, by Theorem 7.1 we have a $\mathfrak{g}_0$-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}$.

Theorem 7.2. *For any $k \in \mathbb{Z}_{\ge 0}$ and $L_1, \cdots, L_k \in \mathbb{C}^\times$, the product $\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}$ possesses an affine geometric crystal structure.*

Proof. By the argument above and Theorem 7.1, it is enough to check the conditions in Definition 2.1 related to $i = 0$. First, let us check $\gamma_j(e_i^c(x_1, \cdots, x_k)) = c^{a_{ij}}\gamma_j(x_1, \cdots, x_k)$ for $(i,j) = (i,0),\ (0,i)$. For $x_j \in \mathcal{V}(\mathfrak{g})_{L_j}$ and $i \in I$, we have

$$\begin{aligned}
\gamma_0(e_i^c(x_1,\cdots,x_k)) &= \gamma_0(e_i^{c_1}(x_1))\cdots\gamma_0(e_i^{c_k}(x_k)) \\
&= c_1^{a_{i0}}\gamma_0(x_1)\cdots c_k^{a_{i0}}\gamma_0(x_k) = c^{a_{i0}}\gamma_0(x_1,\cdots,x_k), \\
\gamma_i(e_0^c(x_1,\cdots,x_k)) &= \gamma_i(e_0^{c_1}(x_1))\cdots\gamma_i(e_0^{c_k}(x_k)) \\
&= c_1^{a_{0i}}\gamma_i(x_1)\cdots c_k^{a_{0i}}\gamma_i(x_k) = c^{a_{0i}}\gamma_i(x_1,\cdots,x_k),
\end{aligned}$$

where $c_1, \cdots, c_k$ are obtained by using (7.3) repeatedly.

Next, let us check the relation $\varepsilon_0(e_0^c(x_1,\cdots,x_k)) = c^{-1}\varepsilon_0(x_1,\cdots,x_k)$ by the induction on k. Assume $\varepsilon_0(e_0^c(x')) = c^{-1}\varepsilon_0((x'))$ and $\varphi_0(e_0^c(x')) = c\varphi_0((x'))$ where $x' = (x_1,\cdots,x_{k-1})$.

$$
\begin{aligned}
\varepsilon_0(e_0^c(x_1,\cdots,x_k)) &= \varepsilon_0(e_0^{c_1}(x'), e_0^{c_2}(x_k)) = \varepsilon_0(e_0^{c_1}(x')) + \frac{\varepsilon_0(e_0^{c_1}(x'))\varepsilon_0(e_0^{c_2}(x_k))}{\varphi_0(e_0^{c_1}(x'))} \\
&= c_1^{-1}\varepsilon_0(x') + \frac{\varepsilon_0(x')\varepsilon_0(x_k)}{cc_1\varphi_0(x')} = c_1^{-1}\left(\varepsilon_0(x') + \frac{\varepsilon_0(x')\varepsilon_0(x_k)}{c\varphi_0(x')}\right) \\
&= \frac{\varphi_0(x') + \varepsilon_0(x_k)}{c\varphi_0(x') + \varepsilon_0(x_k)} \cdot \frac{c\varphi_0(x') + \varepsilon_0(x_k)}{c\varphi_0(x')} \cdot \varepsilon_0(x') \\
&= c^{-1}\left(\varepsilon_0(x') + \frac{\varepsilon_0(x')\varepsilon_0(x_k)}{\varphi_0(x')}\right) \\
&= c^{-1}\varepsilon_0(x_1,\cdots,x_k).
\end{aligned}
$$

Finally, let us check the Verma relations. We need to see the following cases:

(i) The case $\mathfrak{g} = A_n^{(1)}$. $\begin{cases} e_0^a e_i^{ab} e_0^b = e_i^b e_0^{ab} e_i^a & \text{if } i = 1, n, \\ e_0^c e_i^d = e_i^d e_0^c & \text{if } i \neq 1, n. \end{cases}$

(ii) The case $\mathfrak{g} = B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$. $\begin{cases} e_0^a e_i^{ab} e_0^b = e_i^b e_0^{ab} e_i^a & \text{if } i = 2, \\ e_0^c e_i^d = e_i^d e_0^c & \text{if } i \neq 2. \end{cases}$

(iii) The case $\mathfrak{g} = D_{n+1}^{(2)}$, $A_{2n}^{(2)}$. $\begin{cases} e_0^a e_i^{a^2b} e_0^{ab} e_i^b = e_i^b e_0^{ab} e_i^{a^2b} e_0^a & \text{if } i = 1, \\ e_0^c e_i^d = e_i^d e_0^c & \text{if } i \neq 1. \end{cases}$

By the result in [KOTY], we have the product structure on $\{\mathcal{B}_l(A_n^{(1)})\}$ and $\{\mathcal{B}_l(D_N^{(1)})\}$. Since $\mathcal{V}(A_n^{(1)})_l \cong \mathcal{B}_l(A_n^{(1)})$ and $\mathcal{V}(D_n^{(1)})_l \cong \mathcal{B}_l(D_n^{(1)})$, we also have the product structure on $\{\mathcal{V}(A_n^{(1)})_l\}$ and $\{\mathcal{V}(D_n^{(1)})_l\}$. Hence, we have the Verma relations on their product. Thus, we obtain the case (i) and (ii) $\mathfrak{g} = D_n^{(1)}$.

For the case $\sigma(i) \neq 0$, it is easy to check the relation. Indeed, e.g., for the case (iii) $i = 1$, we have:

$$
\begin{aligned}
e_0^a e_1^{a^2b} e_0^{ab} e_1^b &= (\overline{\sigma}^{-1} e_n^a \overline{\sigma})(\overline{\sigma}^{-1} e_{n-1}^{a^2b} \overline{\sigma})(\overline{\sigma}^{-1} e_n^{ab} \overline{\sigma})(\overline{\sigma}^{-1} e_{n-1}^b \overline{\sigma}) \\
&= \overline{\sigma}^{-1}(e_n^a e_{n-1}^{a^2b} e_n^{ab} e_{n-1}^b)\overline{\sigma} = \overline{\sigma}^{-1}(e_{n-1}^b e_n^{ab} e_{n-1}^{a^2b} e_n^a)\overline{\sigma} = e_1^b e_0^{ab} e_1^{a^2b} e_0^a.
\end{aligned}
$$

Similarly, the other cases with $\sigma(i) \neq 0$ can be shown.

To complete (ii) and (iii), it suffices to check the cases $\sigma(i) = 0$; i.e., (ii) $i = 1$, (iii) $i = n$.

In the previous section we see that for $\mathfrak{g} \neq A_n^{(1)}$ the geometric crystal $\mathcal{V}(\mathfrak{g})_l$ is obtained from the geometric crystal $\mathcal{B}(D_N^{(1)})_{l'}$ ($l' = l$ or l^2) by the method of foldings. Thus, in the case (ii) we have $(e_0^{\mathfrak{g}})^c = (e_0^{D_N^{(1)}})^c$ and $(e_1^{\mathfrak{g}})^c = (e_1^{D_N^{(1)}})^c$ for $\mathfrak{g} = B_n^{(1)}, A_{2n-1}^{(2)}$. Since $(e_0^{D_N^{(1)}})^c(e_1^{D_N^{(1)}})^d = (e_1^{D_N^{(1)}})^d(e_0^{D_N^{(1)}})^c$, we have $(e_0^{\mathfrak{g}})^c(e_1^{\mathfrak{g}})^d = (e_1^{\mathfrak{g}})^d(e_0^{\mathfrak{g}})^c$ and then we completed (ii). In the case (iii), we have $(e_0^{\mathfrak{g}})^c = (e_0^{D_N^{(1)}})^c$ and

$$
(e_n^{\mathfrak{g}})^c = \begin{cases} (e_{n+1}^{D_{n+2}^{(1)}})^c \circ (e_{n+2}^{D_{n+2}^{(1)}})^c & \mathfrak{g} = D_{n+1}^{(2)}, \\ (e_{n+1}^{D_{2n+2}^{(1)}})^c & \mathfrak{g} = A_{2n}^{(2)}. \end{cases}
$$

Then $(e_0^{\mathfrak{g}})^c(e_n^{\mathfrak{g}})^d = (e_n^{\mathfrak{g}})^d(e_0^{\mathfrak{g}})^c$, which completes (iii). □

By Theorem 5.4, Theorem 7.1 and Theorem 7.2, we obtain

Corollary 7.3. *For positive real numbers L_j $(j = 1, \cdots, k)$, let $\theta_{L_j} : T' \to \mathcal{V}(\mathfrak{g})_{L_j}$ be the positive structure as in the previous section. Then*

$$\Theta := (\theta_{L_1}, \cdots, \theta_{L_k}) : {T'}^{\times k} \to \mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}$$

defines a positive structure on $\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}$ and we have the isomorphism of geometric crystals:

$$\mathcal{UD}_{\Theta}(\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}) \cong B_\infty(\mathfrak{g}^L)^{\otimes k}. \tag{7.5}$$

8. M-matrices and automorphisms

8.1. **Definition of M-matrices.** An M-matrix is an important object to realize tropical R map from geometric crystals, though its exact definition is not yet fixed. In this paper, we take the following as a temporary definition of M-matrices.

Definition 8.1. Let $\mathfrak{g}$ be an affine Lie algebra and $\mathcal{B}_L \subset (\mathbb{C}^\times)^k$ be a certain $\mathfrak{g}$-geometric crystal depending on $L \in \mathbb{C}^\times$ with a product structure. For $x \in \mathcal{B}_L$ and an indeterminate z, a square matrix $M_L(x, z)$ with entries in $\mathbb{C}(x, z)$ is an M-*matrix* if:

(i) For $x \in \mathcal{B}_L$, $i \in I$ and $c \in \mathbb{C}^\times$, there exist non-singular matrices $X_i(x, c, z)$ and $Y_i(x, c, z)$ whose entries are in $\mathbb{C}(x, c, z)$ and satisfying

$$M_L(e_i^c(x), z) = X_i(x, c, z) M_L(x, z) Y_i(x, c, z).$$

(ii) For a given $(x, y) \in \mathcal{B}_L \times \mathcal{B}_K$ and $L, K \in \mathbb{C}^\times$, the solution $(x', y') \in \mathcal{B}_K \times \mathcal{B}_L$ of the equation

$$M_L(x, z) M_K(y, z) = M_K(x', z) M_L(y', z),$$

uniquely exists for any z and the correspondence $(x, y) \mapsto (x', y')$ defines a birational map $R : \mathcal{B}_L \times \mathcal{B}_K \to \mathcal{B}_K \times \mathcal{B}_M$.

(iii) Let x_j, y_j be in $\mathcal{B}_{L_j}$ $(j = 1, 2, \ldots, m)$. Suppose $L_i \neq L_j$ $(i \neq j)$ and

$$M_{L_1}(x_1, z) \cdots M_{L_m}(x_m, z) = M_{L_1}(y_1, z) \cdots M_{L_m}(y_m, z).$$

Then $x_j = y_j$ $(j = 1, \ldots, m)$.

Example 8.2 ([KOTY]). *$A_n^{(1)}$-case: Let $\mathcal{B}_L(A_n^{(1)})$ be the geometric crystal as in Section 5.3. For $l = (l_1, \ldots, l_{n+1}) \in \mathcal{B}_L(A_n^{(1)})$, set*

$$M_L(l, z) := \begin{pmatrix} l_1^{-1} & & & & & -z \\ -1 & l_2^{-1} & & & & \\ & & \cdots & & & \\ & & & -1 & l_n^{-1} & \\ & & & & -1 & l_{n+1}^{-1} \end{pmatrix}^{-1}.$$

Then the matrix $M_L(l, z)$ is an M-matrix.

8.2. **Explicit form of the M-matrix of type $D_n^{(1)}$.** An explicit form of the M-matrix $M_L(l,z)$ for $\mathcal{B}_L(D_n^{(1)})$ is described [KOTY] where the geometric crystal $\mathcal{B}_L(D_n^{(1)})$ is as in Section 6.

For $l=(l_1,\dots,l_n,\bar{l}_{n-1},\dots,\bar{l}_1)\in\mathcal{B}_L(D_n^{(1)})$, the M-matrix $M_L(l,z)$ is given as follows [KOTY]: $M(l,z)$ is a $2n\times 2n$ matrix in the form $M_L(l,z)=A(l)+zB(l)+z^2C(l)$ where each matrix $A(l),B(l)$ and $C(l)$ are given by $C(l)=E_{1,2n}$,

$$A(l)_{i,i}=\begin{cases}\frac{l_i}{\bar{l}_i} & 1\le i\le n-1,\\ l_n & i=n,\\ \frac{1}{l_n} & i=n+1,\\ \frac{\bar{l}_{2n+1-i}}{l_{2n+1-i}} & n+2\le i\le 2n,\end{cases}\tag{8.1}$$

$$A(l)_{i,j}=l_j\cdots l_{i-1}\left(1+\frac{l_i}{\bar{l}_i}\right)\quad 1\le j<i\le n-1,\tag{8.2}$$

$$A(l)_{2n+1-j,2n+1-i}=\bar{l}_j\cdots\bar{l}_{i-1}\left(1+\frac{\bar{l}_i}{l_i}\right)\quad 1\le j<i\le n-1,\tag{8.3}$$

$$A(l)_{n,j}=l_j\cdots l_n\quad 1\le j\le n-1,\tag{8.4}$$

$$A(l)_{2n+1-j,n}=\bar{l}_j\cdots\bar{l}_{n-1}l_n\quad 1\le j\le n-1,\tag{8.5}$$

$$A(l)_{2n+1-j,n+1}=\bar{l}_j\cdots\bar{l}_{n-1}\quad 1\le j\le n-1,\tag{8.6}$$

$$A(l)_{2n+1-i,j}=(l_j\cdots l_n)(\bar{l}_i\cdots\bar{l}_{n-1})\qquad 1\le i,j\le n-1,\tag{8.7}$$

$$A(l)_{i,j}=0\qquad\text{otherwise.}\tag{8.8}$$

The matrix element $B(l)_{i,j}$ is given by $B(l)_{i,j}=A(l)_{i,1}A(l)_{2n,j}-LA(l)_{i,j}-\delta_{i,1}\delta_{j,2n}$ ([KOTY]).

8.3. **Matrix realization.** In terms of M-matrices, the involutions $\Sigma_0^{(n)},\Sigma_1^{(n)},\Sigma_2^{(n)}$ in Section 6 are realized by the adjoint action of certain matrices.

Proposition 8.3. *For each involution $\Sigma_i^{(n)}$ $(i=0,1,2)$, there exists a non-singular matrix $J_i^{(n)}=J_i^{(n)}(z)$ such that*

$$M_L(\Sigma_i^{(n)}(l),z)=J_i^{(n)}(z)M_L(l,z){J_i^{(n)}(z)}^{-1},$$

where the matrix $J_i^{(n)}$ does not depend on l. Each $J_i^{(n)}$ is as follows:

$$J_0^{(n)}=\left(\begin{array}{c|c|c}0&0&z\\\hline 0&E_{2n}&0\\\hline z^{-1}&0&0\end{array}\right),\quad J_1^{(n)}=\left(\begin{array}{c|c|c}E_n&0&0\\\hline 0&\begin{matrix}0&1\\1&0\end{matrix}&0\\\hline 0&0&E_n\end{array}\right),$$

$$J_2^{(n)}=\left(\begin{array}{c|c}0&z\cdot E_{2n}\\\hline E_{2n}&0\end{array}\right),$$

where E_m is the identity matrix of size m. The size of each matrix $J_i^{(n)}$ and $M_L(l,z)$ is $2n+2$ for $i=0$ and $i=1$, and $4n$ for $i=2$. Setting $J_3^{(n)}=J_0^{(n+1)}J_1^{(n+1)}$ $(=J_1^{(n+1)}J_0^{(n+1)})$ and $J_4^{(n)}=(J_0^{(2n+1)}J_1^{(2n+1)})J_2^{(n+1)}(=J_2^{(n+1)}(J_0^{(2n+1)}J_1^{(2n+1)}))$, we also have

$$M_L(\Sigma_k^{(n)}(l),z)=J_k^{(n)}M_L(l,z){J_k^{(n)}}^{-1}\quad (k=3,4).\tag{8.9}$$

Proof. Since the involution $\Sigma_0^{(n)}$ (resp. $\Sigma_1^{(n)}$) coincides with the involution σ_1 (resp. σ_n) in [KOTY] and the matrix $J_0^{(n)}$ (resp. $J_1^{(n)}$) is identified with the matrix $J_1(z)$ (resp. $J_n(z)$) as in [KOTY]. Thus, Lemma 3.10 in [KOTY] shows that our assertions for $\Sigma_k^{(n)}$ $(k = 0, 1)$ are right. Then let us show the case $k = 2$. Since we have the explicit form of $M_L(l, z)$ as in the last subsection, it is carried out by case-by-case calculations. For example, let us see the diagonal entries:

$$M_L(\Sigma_2^{(n)}(l), z)_{i,i} = \begin{cases} 1/l_{2n} & i = 1, \\ \frac{\bar{l}_{2n+1-i}}{l_{2n+1-i}} + zL & 2 \le i \le 2n-1, \\ \bar{l}_1/l_1 & i = 2n, \\ l_1/\bar{l}_1 & i = 2n+1, \\ \frac{l_{i-2n}}{\bar{l}_{i-2n}} + zL & 2n+2 \le i \le 4n-1, \\ l_{2n} & i = 4n. \end{cases}$$

Here note that the matrix $M_L(l, z)$ is a $4n \times 4n$-matrix.

On the other hand, the diagonal entries of the matrix

$$M' := J_n^{(2)} M_L(l, z) {J_n^{(2)}}^{-1} = J_n^{(2)} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} {J_n^{(2)}}^{-1} = \begin{pmatrix} M_4 & zM_3 \\ z^{-1}M_2 & M_1 \end{pmatrix}$$

are given by:

For $i = 1, \ldots, 2n$,

$$M'_{i,i} = M_L(l, z)_{i+2n, i+2n} = \begin{cases} 1/l_{2n} & i = 1, \\ \frac{\bar{l}_{2n+1-i}}{l_{2n+1-i}} + zL & 2 \le i \le 2n-1, \\ \bar{l}_1/l_1 & i = 2n, \end{cases}$$

For $i = 2n+1, \ldots, 4n$,

$$M'_{i,i} = M_L(l, z)_{i-2n, i-2n} = \begin{cases} l_1/\bar{l}_1 & i = 2n+1, \\ \frac{l_{i-2n}}{\bar{l}_{i-2n}} + zL & 2n+2 \le i \le 4n-1, \\ l_{2n} & i = 4n. \end{cases}$$

Then, we have $M_L(\Sigma_n^{(2)}(l), z)_{i,i} = M'_{i,i} (= (J_n^{(2)} M_L(l, z) {J_n^{(2)}}^{-1})_{i,i})$. The other cases are also obtained similarly. □

8.4. **Birational maps on fixed point varieties.** Let $\mathcal{R}_{LK}$ be the birational map defined by

$$\mathcal{R}_{LK} : \mathcal{B}_L(D_N^{(1)}) \times \mathcal{B}_K(D_N^{(1)}) \to \mathcal{B}_K(D_N^{(1)}) \times \mathcal{B}_L(D_N^{(1)}) (L, K \in \mathbb{C}^\times)$$
$$(l, m) \mapsto (l', m'),$$

where (l', m') be the unique solution of the equation

$$M_L(l, z) M_K(m, z) = M_K(l', z) M_L(m', z).$$

Let $X_i^{(n,L)}$ be one of the fixed point subvarieties in $\mathcal{B}_L(D_N^{(1)})$.

Theorem 8.4. *Let us denote $\mathcal{R}_{LK}^{(i)}$ as the restriction of $\mathcal{R}_{LK}$ on $X_i^{(n,L)} \times X_i^{(n,K)}$. Then $\mathcal{R}_{LK}^{(i)}$ is a well-defined birational map $X_i^{(n,L)} \times X_i^{(n,K)} \to X_i^{(n,K)} \times X_i^{(n,L)}$ $(L, K \in \mathbb{C}^\times)$.*

Proof. For $(l,m) \in X_i^{(n,L)} \times X_i^{(n,K)}$, set $(l',m') := \mathcal{R}(l,m)$, i.e.,

$$M_L(l,z)M_K(m,z) = M_K(l',z)M_L(m',z). \tag{8.10}$$

We have

$$\begin{aligned} M_L(l,z)M_K(m,z) &= M_L(\Sigma_i^{(n)}(l),z)M_K(\Sigma_i^{(n)}(m),z) \\ &= J_i^{(n)}M_L(l,z)M_K(m,z){J_i^{(n)}}^{-1} \\ &= J_i^{(n)}M_K(l',z)M_L(m',z){J_i^{(n)}}^{-1} \\ &= M_K(\Sigma_i^{(n)}(l'),z)M_L(\Sigma_i^{(n)}(m'),z). \end{aligned}$$

Then, we have

$$M_K(l',z)M_L(m',z) = M_K(\Sigma_i^{(n)}(l'),z)M_L(\Sigma_i^{(n)}(m'),z).$$

It follows from the uniqueness of the solution for (8.10) that

$$(l',m') = (\Sigma_i^{(n)}(l'),\Sigma_i^{(n)}(m'))$$

and then we have $(l',m') \in X_i^{(n,K)} \times X_i^{(n,L)}$. □

9. Tropical R maps

In this section, we define the notion of tropical R map and give explicit forms of the affine tropical R maps on the geometric crystals constructed above.

9.1. Definition of tropical R map.

Definition 9.1. Let $\{(X_\lambda,\{e_i^\lambda\},\{\gamma_i^\lambda\},\{\varepsilon_i^\lambda\})\}_{\lambda\in\Lambda}$ be a family of geometric crystals equipped with the product structures, where Λ is a certain index set and its element is called a *spectral parameter*. A birational map $\mathcal{R}_{\lambda\mu} : X_\lambda \times X_\mu \longrightarrow X_\mu \times X_\lambda$ $(\lambda,\mu \in \Lambda)$ is said to be a *tropical R map* (or shortly, tropical R) if it satisfies the following conditions:

$$(e_i^{X_\mu\times X_\lambda})^c \circ \mathcal{R}_{\lambda\mu} = \mathcal{R}_{\lambda\mu} \circ (e_i^{X_\lambda\times X_\mu})^c, \tag{9.1}$$

$$\varepsilon_i^{X_\lambda\times X_\mu} = \varepsilon_i^{X_\mu\times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{9.2}$$

$$\gamma_i^{X_\lambda\times X_\mu} = \gamma_i^{X_\mu\times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{9.3}$$

$$\mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)} = \mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)} \quad \text{on } X_\lambda \times X_\mu \times X_\nu,, \tag{9.4}$$

$$\mathcal{R}_{\mu\lambda}\mathcal{R}_{\lambda\mu} = \mathrm{id}_{\lambda\mu}, \tag{9.5}$$

for any $i \in I$ and any $\lambda,\mu,\nu \in \Lambda$. Here $\mathcal{R}^{(ij)}$ means that it acts on i-th and j-th components of the product.

In the rest of this section, we give the explicit forms of the tropical $\mathcal{R}$ for the affine geometric crystals.

9.2. $D_n^{(1)}$ **case** $(n \geq 4)$**.** Let $\mathcal{R}_{LK} : \mathcal{B}_L(D_N^{(1)}) \times \mathcal{B}_K(D_N^{(1)}) \to \mathcal{B}_K(D_N^{(1)}) \times \mathcal{B}_L(D_N^{(1)})$ $(L,K \in \mathbb{C}^\times)$ be the birational map as in 8.4. In [KOTY], it is shown that the morphism $\mathcal{R}_{LM}$ is a tropical R map for $\{\mathcal{B}_L(D_n^{(1)})\}_{L\in\mathbb{C}^\times}$. We shall describe the explicit form of $\mathcal{R}_{LM}$. Let $\sharp : \mathcal{B}_L(D_n^{(1)}) \to \mathcal{B}_L(D_n^{(1)})$ be an involution defined by

$$\sharp(l_1,l_2,\ldots,l_n,\bar{l}_{n-1},\ldots,\bar{l}_2,\bar{l}_1) = (\bar{l}_1,l_2,\ldots,l_n,\bar{l}_{n-1},\ldots,\bar{l}_2,l_1), \tag{9.6}$$

that is, $\sharp : l_1 \leftrightarrow \bar{l}_1$ and $* : \mathcal{B}_L(D_n^{(1)}) \times \mathcal{B}_M(D_n^{(1)}) \to \mathcal{B}_M(D_n^{(1)}) \times \mathcal{B}_L(D_n^{(1)})$ an involution defined by

$$((l_1, l_2, \ldots, \bar{l}_2, \bar{l}_1), (m_1, m_2, \ldots, \overline{m}_2, \overline{m}_1))^* = ((\overline{m}_1, \overline{m}_2, \ldots, m_2, m_1), (\bar{l}_1, \bar{l}_2, \ldots, l_2, l_1))$$

that is, $* : l_i \leftrightarrow \overline{m}_i, \quad \bar{l}_i \leftrightarrow m_i \ (1 \le i \le n-1), \quad l_n \leftrightarrow m_n$.

Following [KOTY], we define the rational functions V_i $(i = 0, 1, \ldots, n-1)$ and W_i $(i = 1, \ldots, n-1)$ on $\mathcal{B}_L(D_n^{(1)}) \times \mathcal{B}_M(D_n^{(1)})$ $(L, M \in \mathbb{C}^\times)$ by

$$W_i := V_i {V_i}^* + (M-L){V_i}^* + (L-M)V_i \quad (1 \le i \le n-2), \quad W_{n-1} := V_{n-1}{V_{n-1}}^*.$$

$$V_i = \sum_{j=1}^{n-2} (\theta_{i,j}(l,m) + \theta'_{i,j}(l,m)) + \sum_{j=1}^{n} (\eta_{i,j}(l,m) + \eta'_{i,j}(l,m)), \tag{9.7}$$

where we set $L = l_1 l_2 \cdots \bar{l}_2 \bar{l}_1$, $M = m_1 m_2 \cdots \overline{m}_2 \overline{m}_1$,

$$\theta_{i,j}(l,m) = \begin{cases} L \displaystyle\prod_{k=j+1}^{i} \frac{\overline{m}_k}{\bar{l}_k} & \text{for } 1 \le j \le i, \\ M \displaystyle\prod_{k=i+1}^{j} \frac{\bar{l}_k}{\overline{m}_k} & \text{for } i+1 \le j \le n-2, \end{cases} \tag{9.8}$$

$$\theta'_{i,j}(l,m) = L \left(\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\prod_{k=1}^{j} \frac{m_k}{l_k} \right) \quad \text{for} \quad j = 1, \ldots, n-2, \tag{9.9}$$

(9.10)

$$\eta_{i,j}(l,m) = \begin{cases} L \left(\displaystyle\prod_{k=j+1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\frac{\overline{m}_j}{l_j} \right) & \text{for} \quad 1 \le j \le i, \\ M \left(\displaystyle\prod_{k=i+1}^{j} \frac{\bar{l}_k}{\overline{m}_k} \right) \left(\frac{\overline{m}_j}{l_j} \right) & \text{for} \quad i+1 \le j \le n-1, \\ M \left(\displaystyle\prod_{k=i+1}^{n-1} \frac{\bar{l}_k}{\overline{m}_k} \right) l_n & \text{for} \quad j = n, \end{cases}$$

(9.11)

$$\eta'_{i,j}(l,m) = \begin{cases} L \left(\displaystyle\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\prod_{k=1}^{j} \frac{m_k}{l_k} \right) \left(\frac{l_j}{\overline{m}_j} \right) & \text{for} \quad 1 \le j \le n-1, \\ L \left(\frac{L}{M} \right)^{\delta_{i,n-1}} \left(\displaystyle\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\prod_{k=1}^{n-1} \frac{m_k}{l_k} \right) \left(\frac{1}{l_n} \right) & \text{for} \quad j = n. \end{cases}$$

Remark. Though W_i seems not to be a positive rational function, it is, indeed, positive by the formula ([KOTY] Lemma 4.14):

$$\left(\frac{1}{l_i} + \frac{1}{\overline{m}_i} \right) W_i = \frac{1}{m_i} V_i V_{i-1}^* + \frac{1}{\bar{l}_i} V_{i-1} V_i^*.$$

Now, we introduce the following rational transformation $\mathcal{R}$ on $(\mathbb{C}^\times)^{2n-1}\times(\mathbb{C}^\times)^{2n-1}$:

$$\mathcal{R}(l,m)=(l',m'),$$

where

(9.12)
$$l'_1=m_1\frac{V_0^\sharp}{V_1},\quad \overline{l}'_1=\overline{m}_1\frac{V_0}{V_1},\quad l'_i=m_i\frac{V_{i-1}W_i}{V_iW_{i-1}},\quad \overline{l}'_i=\overline{m}_i\frac{V_{i-1}}{V_i}\quad (2\le i\le n-1),$$
$$l'_n=m_n\frac{V_{n-1}}{V^*_{n-1}},\quad m'_1=l_1\frac{V_0}{V^*_1},\quad \overline{m}'_1=\overline{l}_1\frac{V_0^\sharp}{V^*_1},$$
$$m'_i=l_i\frac{V^*_{i-1}}{V^*_i},\quad \overline{m}'_i=\overline{l}_i\frac{V^*_{i-1}W_i}{V^*_iW_{i-1}}\quad (2\le i\le n-1),\quad m'_n=l_n\frac{V^*_{n-1}}{V_{n-1}}.$$

Here note that for $(l',m')=\mathcal{R}(l,m)$ we have $l'_1l'_2\cdots\overline{l}'_2\overline{l}'_1=M$ and $m'_1m'_2\cdots\overline{m}'_2\overline{m}'_1=L$. Then $\mathcal{R}$ defines a rational map $\mathcal{B}_L(D_n^{(1)})\times\mathcal{B}_M(D_n^{(1)})\longrightarrow\mathcal{B}_M(D_n^{(1)})\times\mathcal{B}_L(D_n^{(1)})$. The following is one of the main results in [KOTY]:

Theorem 9.2 ([KOTY]). *The rational map $\mathcal{R}=\mathcal{R}_{LM}:\mathcal{B}_L(D_n^{(1)})\times\mathcal{B}_M(D_n^{(1)})\longrightarrow\mathcal{B}_M(D_n^{(1)})\times\mathcal{B}_L(D_n^{(1)})$ gives a tropical R map of type $D_n^{(1)}$ on the family of affine geometric crystals, which will be denoted by $\mathcal{R}(D_n^{(1)})$ in the sequel.*

Here we describe the tropical R on $\mathcal{V}(D_n^{(1)})_L\times\mathcal{V}(D_n^{(1)})_M$: $\overline{\mathcal{R}}(x,y):=(\Xi,\Xi)\circ\mathcal{R}\circ(\Xi^{-1},\Xi^{-1})(x,y)$ $(x\in\mathcal{V}(D_n^{(1)})_L,\ y\in\mathcal{V}(D_n^{(1)})_M)$. We define the rational functions $\overline{V}_i$ $(i=0,1,\ldots,n-1)$ and $\overline{W}_i$ $(i=1,\ldots,n-1)$ on $\mathcal{V}(D_n^{(1)})_L\times\mathcal{V}(D_n^{(1)})_M$ $(L,M\in\mathbb{C}^\times)$ by

$$\overline{W}_i:=\overline{V}_i\overline{V}^*_i+(M-L)\overline{V}^*_i+(L-M)\overline{V}_i\quad(1\le i\le n-2),\quad \overline{W}_{n-1}:=\overline{V}_{n-1}\overline{V}^*_{n-1}.$$

(9.13)
$$\overline{V}_i=\sum_{j=1}^{n-2}(\overline{\theta}_{i,j}(x,y)+\overline{\theta}'_{i,j}(x,y))+\sum_{j=1}^{n}(\overline{\eta}_{i,j}(x,y)+\overline{\eta}'_{i,j}(x,y)),$$

where $\sharp$ is the involution on $\mathcal{V}(D_n^{(1)})_L$ and $*$ is the involution $\mathcal{V}(D_n^{(1)})_L\times\mathcal{V}(D_n^{(1)})_M\to\mathcal{V}(D_n^{(1)})_M\times\mathcal{V}(D_n^{(1)})_L$ defined by

$$\sharp\ :\ x_i\mapsto\frac{x_i}{Lx_1\overline{x}_1}\quad(i=1\cdots,n)\quad,\overline{x}_i\mapsto\frac{\overline{x}_i}{Lx_1\overline{x}_1}\quad(i=1,\cdots,n-2),$$
$$*\ :\ x_i\mapsto\frac{1}{M\overline{y}_i},\quad \overline{x}_i\mapsto\frac{1}{My_i}\quad(i=1,\cdots,n-2),$$
$$x_{n-1}\mapsto\frac{1}{My_n},\quad x_n\mapsto\frac{1}{My_{n-1}},$$
$$y_i\mapsto\frac{1}{L\overline{x}_i},\quad \overline{y}_i\mapsto\frac{1}{Lx_i}\quad(i=1,\cdots,n-2),\quad y_{n-1}\mapsto\frac{1}{Lx_n},\quad y_n\mapsto\frac{1}{Lx_{n-1}},$$

and

$$\overline{\theta}_{i,j}(x,y) := L\frac{x_iy_j}{x_jy_i} \quad (1 \le j \le i < n-1),$$
$$\overline{\theta}_{n-1,j}(x,y) := L\frac{x_ny_j}{x_jy_n} \quad (1 \le j < n-1),$$
$$\overline{\theta}_{i,j}(x,y) := M\frac{x_iy_j}{x_jy_i} \quad (i+1 \le j \le n-2),$$
$$\overline{\theta}'_{i,j}(x,y) := M\frac{x_i\overline{y}_j}{\overline{x}_jy_i} \ (0 \le i \le n-2),\ \overline{\theta}'_{n-1,j}(x,y) := M\frac{x_n\overline{y}_j}{\overline{x}_jy_n},$$
$$\overline{\eta}_{i,j}(x,y) := L^{1-\delta_{j,1}}\frac{x_i\overline{x}_{j-1}y_{j-1}}{x_j\overline{x}_jy_i} \quad (1 \le j \le i < n-1),$$
$$\overline{\eta}_{n-1,j}(x,y) := L^{1-\delta_{j,1}}\frac{x_n\overline{x}_{j-1}y_{j-1}}{x_j\overline{x}_jy_n} \quad (1 \le j \le n-1),$$
$$\overline{\eta}_{i,j}(x,y) := \frac{M}{L^{\delta_{j,1}}}\frac{x_i\overline{x}_{j-1}y_{j-1}}{x_j\overline{x}_jy_i} \quad (0 \le i < j \le n-2),$$
$$\overline{\eta}_{i,n-1}(x,y) = M\frac{x_i\overline{x}_{n-2}y_{n-2}}{x_{n-1}x_ny_i} \quad (i+1 \le n-1),$$
$$\overline{\eta}_{i,n}(x,y) := M\frac{x_iy_n}{x_{n-1}y_i} \quad (i \le n-2), \quad \overline{\eta}_{n-1,n}(x,y) := M\frac{x_n}{x_{n-1}},$$
$$\overline{\eta}'_{i,j}(x,y) := L^{\delta_{j,1}}M\frac{x_iy_j\overline{y}_j}{\overline{x}_{j-1}y_{j-1}y_i} \quad (1 \le j \le n-2,\ i < n-1),$$
$$\overline{\eta}'_{n-1,j}(x,y) := L^{\delta_{j,1}}M\frac{x_ny_j\overline{y}_j}{\overline{x}_{j-1}y_{j-1}y_n} \quad (1 \le j \le n-2),$$
$$\overline{\eta}'_{i,n-1}(x,y) := M\frac{x_iy_{n-1}y_n}{\overline{x}_{n-2}y_{n-2}y_i} \quad (0 \le i < n-1), \quad \overline{\eta}'_{n-1,n-1}(x,y) := M\frac{x_ny_{n-1}}{\overline{x}_{n-2}y_{n-2}},$$
$$\overline{\eta}'_{i,n}(x,y) := M\frac{x_iy_{n-1}}{y_ix_n} \quad (0 \le i < n-1), \quad \overline{\eta}'_{n-1,n}(x,y) := L\frac{y_{n-1}}{y_n},$$

where we understand $x_0 = y_0 = \overline{x}_0 = \overline{y}_0 = 1$. Note that in the rest of this section if we write $\overline{\theta}_{ij} = \overline{\theta}^{LM}_{i,j}$, $\overline{\theta}'_{ij} = \overline{\theta}'^{LM}_{i,j}$, $\overline{\eta}_{ij} = \overline{\eta}^{LM}_{i,j}$ and $\overline{\eta}'_{ij} = \overline{\eta}'^{LM}_{i,j}$, we understand that $\overline{\theta}^{LM}_{i,j}(x,y)^* = \overline{\theta}^{ML}_{i,j}((x,y)^*)$, $\overline{\theta}'^{LM}_{i,j}(x,y)^* = \overline{\theta}'^{ML}_{i,j}((x,y)^*)$, $\overline{\eta}^{LM}_{i,j}(x,y)^* = \overline{\eta}^{ML}_{i,j}((x,y)^*)$ and $\overline{\eta}'^{LM}_{i,j}(x,y)^* = \overline{\eta}'^{ML}_{i,j}((x,y)^*)$.

Set $\overline{\mathcal{R}} = \overline{\mathcal{R}}(D_n^{(1)})$: $\overline{\mathcal{R}}(x,y) = (x',y')$ where

$$x'_i := y_i\frac{\overline{V}_i}{\overline{V}_0}, \quad \overline{x}'_i := \overline{y}_i\frac{\overline{V}^\sharp_0\overline{W}_i}{\overline{V}_i\overline{W}_1}, \ (1 \le i \le n-2)$$
$$x'_{n-1} := y_{n-1}\frac{\overline{V}^*_{n-1}}{\overline{V}_0}, \quad x'_n := y_n\frac{\overline{V}_{n-1}}{\overline{V}_0},$$
$$y'_i := x_i\frac{\overline{V}^*_i\overline{W}_1}{\overline{V}^\sharp_0\overline{W}_i}, \quad \overline{y}'_i := \overline{x}_i\frac{\overline{V}_0}{\overline{V}^*_i} \ \ (1 \le i \le n-2),$$
$$y'_{n-1} := x_{n-1}\frac{\overline{V}_{n-1}\overline{W}_1}{\overline{V}^\sharp_0\overline{W}_{n-1}},\ y'_n := x_n\frac{\overline{V}^*_{n-1}\overline{W}_1}{\overline{V}^\sharp_0\overline{W}_{n-1}}.$$

9.3. **Tropical R for $B_n^{(1)}$ $(n \geq 2)$.** By applying the method of folding, we shall obtain an explicit form of the tropical R for $\{\mathcal{B}_L(B_n^{(1)})\}_{L\in\mathbb{C}^\times}$ and $\{\mathcal{V}(B_n^{(1)})_L\}_{L\in\mathbb{C}^\times}$. Indeed, it follows from Theorem 8.4 that we have the birational map $R(B_n^{(1)}) = R_{LK}(B_n^{(1)}) : X_1^{(n,L)} \times X_1^{(n,K)} \to X_1^{(n,K)} \times X_1^{(n,L)}$. Let us see that $R_{LK}(B_n^{(1)})$ is a tropical R.

Lemma 9.3. *The birational map $R(B_n^{(1)})$ is a tropical R on the $B_n^{(1)}$-geometric crystals $\{X_1^{(n,L)}\}_L$.*

Proof. It suffices to show (9.1)–(9.5) in Definition 9.1. The relations (9.4) and (9.5) for $\mathcal{R}(B_n^{(1)})$ are obtained from the one for $\mathcal{R}(D_{n+1}^{(1)})$. The others are easily shown since $\mathcal{R}(D_{n+1}^{(1)})$ commutes with the actions of e_i^c and preserves γ_i and ε_i $(i \in I)$, and the data $(e_i^{B_n^{(1)}})^c$, $\gamma_i^{B_n^{(1)}}$ and $\varepsilon_i^{B_n^{(1)}}$ on $X_1^{((n,L))}$ are defined as in Proposition 6.3. □

Now let us describe the explicit form of tropical R on $\mathcal{B}_L(B_n^{(1)})$. Set $\mathcal{R}(B_n^{(1)}) := (\eta^{-1},\eta^{-1}) \circ R(B_n^{(1)}) \circ (\eta,\eta)$ where η is as in 6.3. Let $\sharp : \mathcal{B}_L(B_n^{(1)}) \to \mathcal{B}_L(B_n^{(1)})$ be an involution defined by

$$\sharp(l_1,l_2,\dots,l_n,\bar{l}_n,\dots,\bar{l}_2,\bar{l}_1) = (\bar{l}_1,l_2,\dots,l_n,\bar{l}_n,\dots,\bar{l}_2,l_1), \tag{9.14}$$

that is, $\sharp : l_1 \leftrightarrow \bar{l}_1$. Let $* : \mathcal{B}_L(B_n^{(1)}) \times \mathcal{B}_M(B_n^{(1)}) \to \mathcal{B}_M(B_n^{(1)}) \times \mathcal{B}_L(B_n^{(1)})$ be an involution defined by

$$\begin{aligned}&((l_1,l_2,\dots,\bar{l}_2,\bar{l}_1),(m_1,m_2,\dots,\overline{m}_2,\overline{m}_1))^* \\ &\quad = ((\overline{m}_1,\overline{m}_2,\dots,m_2,m_1),(\bar{l}_1,\bar{l}_2,\dots,l_2,l_1)),\end{aligned}$$

that is, $* : l_i \leftrightarrow \overline{m}_i, \quad \bar{l}_i \leftrightarrow m_i$ $(1 \leq i \leq n)$.

Restricting the functions V_i and W_i for $D_{n+1}^{(1)}$ to $l_{n+1} = m_{n+1} = 1$, we define the rational functions V_i $(i = 0,1,\dots,n)$ and W_i $(i = 1,\dots,n)$ on $\mathcal{B}_L(B_n^{(1)}) \times \mathcal{B}_M(B_n^{(1)})$ $(L, M \in \mathbb{C}^\times)$ as

$$W_i := V_iV_i^* + (M-L)V_i^* + (L-M)V_i \quad (1 \leq i \leq n-1), \quad W_n := V_nV_n^*,$$

$$V_i = \sum_{j=1}^{n-1}(\theta_{i,j}(l,m) + \theta'_{i,j}(l,m)) + \sum_{j=1}^{n+1}(\eta_{i,j}(l,m) + \eta'_{i,j}(l,m)), \tag{9.15}$$

where $L = l_1 l_2 \cdots \bar{l}_2 \bar{l}_1$, $M = m_1 m_2 \cdots \overline{m}_2 \overline{m}_1$,

$$\theta_{i,j}(l,m) = \begin{cases} L \displaystyle\prod_{k=j+1}^{i} \frac{\overline{m}_k}{\bar{l}_k} & \text{for } 1 \le j \le i, \\ M \displaystyle\prod_{k=i+1}^{j} \frac{\bar{l}_k}{\overline{m}_k} & \text{for } i+1 \le j \le n-1, \end{cases} \tag{9.16}$$

$$\theta'_{i,j}(l,m) = L \left(\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\prod_{k=1}^{j} \frac{m_k}{l_k} \right) \quad \text{for} \quad j = 1, \dots, n-1, \tag{9.17}$$

$$\eta_{i,j}(l,m) = \begin{cases} L \left(\displaystyle\prod_{k=j+1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\frac{\overline{m}_j}{l_j} \right) & \text{for} \quad 1 \le j \le i, \\ M \left(\displaystyle\prod_{k=i+1}^{j} \frac{\bar{l}_k}{\overline{m}_k} \right) \left(\frac{\overline{m}_j}{l_j} \right) & \text{for} \quad i+1 \le j \le n, \\ M \left(\displaystyle\prod_{k=i+1}^{n} \frac{\bar{l}_k}{\overline{m}_k} \right) & \text{for} \quad j = n+1, \end{cases} \tag{9.18}$$

$$\eta'_{i,j}(l,m) = \begin{cases} L \left(\displaystyle\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\displaystyle\prod_{k=1}^{j} \frac{m_k}{l_k} \right) \left(\frac{l_j}{\overline{m}_j} \right) & \text{for} \quad 1 \le j \le n, \\ L \left(\frac{L}{M} \right)^{\delta_{i,n}} \left(\displaystyle\prod_{k=1}^{i} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\displaystyle\prod_{k=1}^{n} \frac{m_k}{l_k} \right) & \text{for} \quad j = n+1. \end{cases} \tag{9.19}$$

Now, we define the tropical R map $\mathcal{R}(B_n^{(1)})$ on $\mathcal{B}_L(B_n^{(1)}) \times \mathcal{B}_M(B_n^{(1)})$ by

$$\mathcal{R}(B_n^{(1)})(l,m) = (l',m')$$

where

(9.20)

$$l'_1 = m_1 \frac{V_0^\sharp}{V_1}, \quad \bar{l}'_1 = \overline{m}_1 \frac{V_0}{V_1}, \quad l'_i = m_i \frac{V_{i-1} W_i}{V_i W_{i-1}}, \quad \bar{l}'_i = \overline{m}_i \frac{V_{i-1}}{V_i} \quad (2 \le i \le n),$$

$$m'_1 = l_1 \frac{V_0}{V_1^*}, \quad \overline{m}'_1 = \bar{l}_1 \frac{V_0^\sharp}{V_1^*}, \quad m'_i = l_i \frac{V_{i-1}^*}{V_i^*}, \quad \overline{m}'_i = \bar{l}_i \frac{V_{i-1}^* W_i}{V_i^* W_{i-1}} \quad (2 \le i \le n).$$

Here note that for $(l',m') = \mathcal{R}(l,m)$ we have $l'_1 l'_2 \cdots \bar{l}'_2 \bar{l}'_1 = M$ and $m'_1 m'_2 \cdots \overline{m}'_2 \overline{m}'_1 = L$.

We shall describe the tropical R map on $\mathcal{V}(B_n^{(1)})_L \times \mathcal{V}(B_n^{(1)})_M$ defined by $\overline{\mathcal{R}}(x,y) := (\Xi,\Xi) \circ \mathcal{R}(B_n^{(1)}) \circ (\Xi^{-1},\Xi^{-1})(x,y)$ ($x \in \mathcal{V}(B_n^{(1)})_L$, $y \in \mathcal{V}(B_n^{(1)})_M$). Restricting the rational functions $\overline{V}_i$ and $\overline{W}_i$ for $D_{n+1}^{(1)}$ to $x_n = x_{n+1}$ and $y_n = y_{n+1}$, we define the rational functions $\overline{V}_i$ $(i = 0, 1, \dots, n)$ and $\overline{W}_i$ $(i = 1, \dots, n)$ on $\mathcal{V}(B_n^{(1)})_L \times \mathcal{V}(B_n^{(1)})_M$ $(L, M \in \mathbb{C}^\times)$ by

$$\overline{W}_i := \overline{V}_i \overline{V}_i^* + (M-L)\overline{V}_i^* + (L-M)\overline{V}_i \quad (1 \le i \le n-1), \quad \overline{W}_n := \overline{V}_n \overline{V}_n^*.$$

$$\overline{V}_i = \sum_{j=1}^{n-1} (\overline{\theta}_{i,j}(x,y) + \overline{\theta}'_{i,j}(x,y)) + \sum_{j=1}^{n+1} (\overline{\eta}_{i,j}(x,y) + \overline{\eta}'_{i,j}(x,y)), \tag{9.21}$$

where $\sharp$ is the involution on $\mathcal{V}(B_n^{(1)})_L$ and $*$ is the involution $\mathcal{V}(B_n^{(1)})_L\times\mathcal{V}(B_n^{(1)})_M\to\mathcal{V}(B_n^{(1)})_M\times\mathcal{V}(B_n^{(1)})_L$ defined by

$$\begin{aligned}
\sharp\quad &:\quad x_i\mapsto\frac{x_i}{Lx_1\overline{x}_1}\quad(i=1,\cdots,n),\quad \overline{x}_i\mapsto\frac{\overline{x}_i}{Lx_1\overline{x}_1}\quad(i=1,\cdots,n-1),\\
*\quad &:\quad x_i\mapsto\frac{1}{M\overline{y}_i},\quad \overline{x}_i\mapsto\frac{1}{My_i}\quad(i=1,\cdots,n-1),\quad x_n\mapsto\frac{1}{My_n},\\
&\qquad y_i\mapsto\frac{1}{L\overline{x}_i},\quad \overline{y}_i\mapsto\frac{1}{Lx_i}\quad(i=1,\cdots,n-1),\quad y_n\mapsto\frac{1}{Lx_n},
\end{aligned}$$

and

$$\begin{aligned}
&\overline{\theta}_{i,j}(x,y):=L\frac{x_iy_j}{x_jy_i}\quad(j\le i\le n, j<n),\quad \overline{\theta}_{i,j}(x,y):=M\frac{x_iy_j}{x_jy_i}\quad(i<j<n),\\
&\overline{\theta}'_{i,j}(x,y):=M\frac{x_i\overline{y}_j}{\overline{x}_jy_i}\quad(1\le j<n),\\
&\overline{\eta}_{i,j}(x,y):=L^{1-\delta_{j,1}}\frac{x_i\overline{x}_{j-1}y_{j-1}}{x_j\overline{x}_jy_i}\quad(1\le j\le i\le n, j<n),\\
&\overline{\eta}_{i,j}(x,y):=\frac{M}{L^{\delta_{j,1}}}\frac{x_i\overline{x}_{j-1}y_{j-1}}{x_j\overline{x}_jy_i}\quad(0\le i<j<n),\\
&\overline{\eta}_{i,n}(x,y)=M\frac{x_i\overline{x}_{n-1}y_{n-1}}{x_n^2y_i}\quad(0\le i\le n),\\
&\overline{\eta}_{i,n+1}(x,y):=M\frac{x_iy_n}{x_ny_i}\quad(i\le n),\\
&\overline{\eta}'_{i,j}(x,y):=L^{\delta_{j,1}}M\frac{x_iy_j\overline{y}_j}{\overline{x}_{j-1}y_{j-1}y_i}\quad(1\le j<n,\ i\le n),\quad \overline{\eta}'_{n,n+1}(x,y):=L,\\
&\overline{\eta}'_{i,n}(x,y):=M\frac{x_iy_n^2}{\overline{x}_{n-1}y_{n-1}y_i}\quad(0\le i\le n),\\
&\overline{\eta}'_{i,n+1}(x,y):=M\frac{x_iy_n}{y_ix_n}\quad(0\le i<n),
\end{aligned}$$

where we understand $x_0=\overline{x}_0=y_0=\overline{y}_0=1$. Note that as above e.g.,

$$\overline{\theta}_{i,j}(x,y)^*:=\overline{\theta}^{LM}_{i,j}(x,y)^*=\overline{\theta}^{ML}_{i,j}((x,y)^*).$$

Here we define $\overline{\mathcal{R}}(x,y)=\overline{\mathcal{R}}(B_n^{(1)})(x,y)=(x',y')$ where

$$\begin{aligned}
&x'_i:=y_i\frac{\overline{V}_i}{\overline{V}_0}\ (1\le i\le n-1)\quad x'_n:=y_n\frac{\overline{V}_n}{\overline{V}_0},\quad \overline{x}'_1:=\overline{y}_1\frac{\overline{V}_0^\sharp}{\overline{V}_1},\quad \overline{x}'_i:=\overline{y}_i\frac{\overline{V}_0^\sharp\overline{W}_i}{\overline{V}_i\overline{W}_1}\ \ (i\ge2),\\
&y'_1:=x_1\frac{\overline{V}_1^*}{\overline{V}_0^\sharp},\ y'_i:=x_i\frac{\overline{V}_i^*\overline{W}_1}{\overline{V}_0^\sharp\overline{W}_i}\ \ (i>1),\ \overline{y}'_i:=\overline{x}_i\frac{\overline{V}_0}{\overline{V}_i^*}\ \ (1\le i\le n-1),\quad y'_n:=x_n\frac{\overline{V}_n\overline{W}_1}{\overline{V}_0^\sharp\overline{W}_n}.
\end{aligned}$$

9.4. **Tropical R for $D_{n+1}^{(2)}$ ($n\ge2$).** As in the previous subsection, we shall describe tropical R maps of type $D_{n+1}^{(2)}$. We see the following lemma for $\{X_3^{(n,L)}\}_{L\in\mathbb{C}^\times}$.

Lemma 9.4. *The birational map $R(D_{n+1}^{(2)})$ is a tropical R map on $\{X_3^{(n,L)}\}_L$.*

The proof is the same as the one for Lemma 9.3.

Let us describe the explicit form of tropical R on $\mathcal{B}_L(D_{n+1}^{(2)})$ as in the previous subsection. Set $\mathcal{R}(D_{n+1}^{(2)}):=(\eta^{-1},\eta^{-1})\circ R(D_{n+1}^{(2)})\circ(\eta,\eta)$ where η is as in 6.4. Let

$* : \mathcal{B}_L(D^{(2)}_{n+1}) \times \mathcal{B}_M(D^{(2)}_{n+1}) \to \mathcal{B}_M(D^{(2)}_{n+1}) \times \mathcal{B}_L(D^{(2)}_{n+1})$ be an involution defined by

$$\begin{aligned}&((l_0, l_1, \dots, \bar{l}_2, \bar{l}_1), (m_0, m_1, \dots, \overline{m}_2, \overline{m}_1))^* \\ &\qquad = ((m_0, \overline{m}_1, \overline{m}_2, \dots, m_2, m_1), (l_0, \bar{l}_1, \bar{l}_2, \dots, l_2, l_1)),\end{aligned}$$

that is, $* : l_0 \leftrightarrow m_0$ and $l_i \leftrightarrow \overline{m}_i, \quad \bar{l}_i \leftrightarrow m_i \ (1 \le i \le n)$.

Restricting the functions V_i and W_i for $B^{(1)}_{n+1}$ to $l_1 = \bar{l}_1$ and $m_1 = \overline{m}_1$, and replacing X_i with X_{i-1} where $X = l, \bar{l}, m$, and $\overline{m}$, we define the rational functions V_i $(i = 0, 1, \dots, n{+}1)$ and W_i $(i = 1, \dots, n{+}1)$ on $\mathcal{B}_L(D^{(2)}_{n+1}) \times \mathcal{B}_M(D^{(2)}_{n+1})$ $(L, M \in \mathbb{C}^\times)$ as

$$W_i := V_i V_i^* + (M - L)V_i^* + (L - M)V_i \quad (1 \le i \le n), \quad W_{n+1} := V_{n+1}V_{n+1}^*.$$

$$V_i = \sum_{j=1}^{n} (\theta_{i,j}(l, m) + \theta'_{i,j}(l, m)) + \sum_{j=1}^{n+2} (\eta_{i,j}(l, m) + \eta'_{i,j}(l, m)), \tag{9.22}$$

where $L = l_0^2 l_1 l_2 \cdots \bar{l}_2 \bar{l}_1$, $M = m_0^2 m_1 m_2 \cdots \overline{m}_2 \overline{m}_1$,

$$\theta_{i,j}(l, m) = \begin{cases} L \displaystyle\prod_{k=j}^{i-1} \frac{\overline{m}_k}{\bar{l}_k} & \text{for } 1 \le j \le i \le n + 1, \\ M \displaystyle\prod_{k=i}^{j-1} \frac{\bar{l}_k}{\overline{m}_k} & \text{for } 0 \le i < j \le n, \end{cases} \tag{9.23}$$

$$\theta'_{i,j}(l, m) = L \left(\prod_{k=0}^{i-1} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\prod_{k=0}^{j-1} \frac{m_k}{l_k} \right) \quad \text{for} \quad j = 1, \dots, n, \tag{9.24}$$

$$\eta_{i,j}(l, m) = \begin{cases} L \left(\displaystyle\prod_{k=j}^{i-1} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\frac{\overline{m}_{j-1}}{l_{j-1}} \right) & \text{for} \quad 1 \le j \le i \le n + 1, \\ M \left(\displaystyle\prod_{k=i}^{j-1} \frac{\bar{l}_k}{\overline{m}_k} \right) \left(\frac{\overline{m}_{j-1}}{l_{j-1}} \right) & \text{for} \quad i + 1 \le j \le n + 1, \\ M \left(\displaystyle\prod_{k=i}^{n} \frac{\bar{l}_k}{\overline{m}_k} \right) & \text{for} \quad j = n + 2, \end{cases} \tag{9.25}$$

$$\eta'_{i,j}(l, m) = \begin{cases} L \left(\displaystyle\prod_{k=0}^{i-1} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\displaystyle\prod_{k=0}^{j-1} \frac{m_k}{l_k} \right) \left(\frac{l_{j-1}}{\overline{m}_{j-1}} \right) & \text{for} \quad 1 \le j \le n + 1, \\ L \left(\frac{L}{M} \right)^{\delta_{i,n+1}} \left(\displaystyle\prod_{k=0}^{i-1} \frac{\overline{m}_k}{\bar{l}_k} \right) \left(\displaystyle\prod_{k=0}^{n} \frac{m_k}{l_k} \right) & \text{for} \quad j = n + 2, \end{cases} \tag{9.26}$$

where we understand $\bar{l}_0 = l_0, \overline{m}_0 = m_0$. Now, we define the tropical R map $\mathcal{R}(D^{(2)}_{n+1})$ on $\mathcal{B}_L(D^{(2)}_{n+1}) \times \mathcal{B}_M(D^{(2)}_{n+1})$ by

$$\begin{aligned} &\mathcal{R}(D^{(2)}_{n+1})(l, m) = (l', m') \quad \text{where} \\ &l'_0 = m_0 \frac{V_0}{V_1}, \quad l'_i = m_i \frac{V_i W_{i+1}}{V_{i+1} W_i}, \quad \bar{l}'_i = \overline{m}_i \frac{V_i}{V_{i+1}} \quad (1 \le i \le n), \\ &m'_0 = l_0 \frac{V_0}{V_1^*}, \quad m'_i = l_i \frac{V_i^*}{V_{i+1}^*}, \quad \overline{m}'_i = \bar{l}_i \frac{V_i^* W_{i+1}}{V_{i+1}^* W_i} \quad (1 \le i \le n). \end{aligned} \tag{9.27}$$

Here note that for $(l',m') = \mathcal{R}(D^{(2)}_{n+1})(l,m)$ we have ${l'_0}^2 l'_1 l'_2 \cdots \overline{l}'_2 \overline{l}'_1 = M$ and ${m'_0}^2 m'_1 m'_2 \cdots \overline{m}'_2 \overline{m}'_1 = L$.

Next, we shall describe the tropical R on $\mathcal{V}(D^{(2)}_{n+1})_L \times \mathcal{V}(D^{(2)}_{n+1})_M$ defined by $\overline{\mathcal{R}}(x,y) := (\Xi,\Xi) \circ \mathcal{R}(D^{(2)}_{n+1}) \circ (\Xi^{-1},\Xi^{-1})(x,y)$ $(x \in \mathcal{V}(D^{(2)}_{n+1})_L,\ y \in \mathcal{V}(D^{(2)}_{n+1})_M)$.

In this case, we replace L (resp. M) for $\mathcal{B}_L(D^{(2)}_{n+1})$ (resp. $\mathcal{B}_M(D^{(2)}_{n+1})$) with L^2 (resp. M^2) since $\mathcal{V}(D^{(2)}_{n+1})_L \cong \mathcal{B}_{L^2}(D^{(2)}_{n+1})$ as in 6.4.

We define the rational functions $\overline{V}_i(x,y)$ $(i = 0,1,\dots,n+1)$ and $\overline{W}_i(x,y)$ $(i = 1,\dots,n+1)$ on $\mathcal{V}(D^{(2)}_{n+1})_L \times \mathcal{V}(D^{(2)}_{n+1})_M$ $(L,M \in \mathbb{C}^\times)$ by

$$\overline{W}_i := \overline{V}_i\overline{V}^*_i + (M^2-L^2)\overline{V}^*_i + (L^2-M^2)\overline{V}_i \quad (1 \le i \le n), \quad \overline{W}_{n+1} := \overline{V}_{n+1}\overline{V}^*_{n+1}.$$

$$\overline{V}_i = \sum_{j=1}^{n}(\overline{\theta}_{i,j}(x,y) + \overline{\theta}'_{i,j}(x,y)) + \sum_{j=1}^{n+2}(\overline{\eta}_{i,j}(x,y) + \overline{\eta}'_{i,j}(x,y)), \tag{9.28}$$

where $*$ is the involution $\mathcal{V}(D^{(2)}_{n+1})_L \times \mathcal{V}(D^{(2)}_{n+1})_M \to \mathcal{V}(D^{(2)}_{n+1})_M \times \mathcal{V}(D^{(2)}_{n+1})_L$ defined by

$$\begin{aligned} * \ :\ & x_0 \mapsto y_0,\ x_i \mapsto \frac{y_0^2}{M^2\overline{y}_i},\ \overline{x}_i \mapsto \frac{y_0^2}{M^2 y_i}\ (i=1,\cdots,n-1),\ x_n \mapsto \frac{y_0^2}{M^2 y_n}, \\ & y_0 \mapsto x_0,\ y_i \mapsto \frac{x_0^2}{L^2\overline{x}_i},\ \overline{y}_i \mapsto \frac{x_0^2}{L^2 x_i}\ (i=1,\cdots,n-1),\ y_n \mapsto \frac{x_0^2}{L^2 x_n}. \end{aligned}$$

and

$$\overline{\theta}_{i,j}(x,y) := \begin{cases} L^2\left(\frac{x_{i-1}}{y_{i-1}}\right)^{1+\delta_{i,1}}\left(\frac{y_{j-1}}{x_{j-1}}\right)^{1+\delta_{j,1}} & (1 \le j \le i \le n+1,\ j \le n), \\ M^2\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}-\delta_{j,1}} \frac{x_{i-1}y_{j-1}}{x_{j-1}y_{i-1}} & (0 \le i < j \le n), \end{cases}$$

$$\overline{\theta}'_{i,j}(x,y) := M^2\left(\frac{L^2}{M^2}\right)^{\delta_{j,1}}\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \frac{x_{i-1}\overline{y}_{j-1}}{\overline{x}_{j-1}y_{i-1}} \quad (1 \le j \le n),$$

$$\overline{\eta}_{i,j}(x,y) := \begin{cases} L^{2(1-\delta_{j,2})} y_0^{\delta_{j,1}+\delta_{j,2}} \left(\frac{x_0}{y_0}\right)^{\delta_{i,1}\delta_{j,1}} \frac{x_{i-1}\overline{x}_{j-2}y_{j-2}}{x_{j-1}\overline{x}_{j-1}y_{i-1}} & (1 \le j \le i \le n+1), \\ \frac{M^2}{L^{2\delta_{j,2}}}\left(\frac{x_0}{y_0^{1-\delta_{j,1}-\delta_{j,2}}}\right)^{\delta_{i,0}+\delta_{i,1}} \frac{x_{i-1}\overline{x}_{j-2}y_{j-2}}{x_{j-1}\overline{x}_{j-1}y_{i-1}} & (i+1 \le j \le n+1), \\ M^2\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \frac{x_{i-1}y_n}{x_n y_{i-1}} & (j = n+2), \end{cases}$$

$$\overline{\eta}'_{i,j}(x,y) := \begin{cases} \overline{\theta}'_{i,j}(x,y) \times \left(\frac{L^2}{y_0}\right)^{\delta_{j,2}} \frac{\overline{x}_{j-1}y_{j-1}}{\overline{x}_{j-2}y_{j-2}} & (1 \le j \le n), \\ M^2\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \frac{x_{i-1}y_n^2}{y_{i-1}\overline{x}_{n-1}y_{n-1}} & (j = n+1), \\ M^2\left(\frac{L^2}{M^2}\right)^{\delta_{i,n+1}}\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \frac{x_{i-1}y_n}{y_{i-1}x_n} & (j = n+2), \end{cases}$$

where we understand $x_{-1} = \overline{x}_0 = y_{-1} = \overline{y}_0 = 1$ and $\overline{x}_n = x_n$, $\overline{y}_n = y_n$.

Here we define $\overline{\mathcal{R}}(x,y)=\overline{\mathcal{R}}(D^{(2)}_{n+1})(x,y)=(x',y')$ by

$$
\begin{aligned}
&x'_0:=y_0\frac{\overline{V}_1}{\overline{V}_0}\ x'_i:=y_i\frac{\overline{V}_1\overline{V}_{i+1}}{\overline{V}_0{}^2}\ (1\le i\le n),\\
&\overline{x}'_i:=\overline{y}_i\frac{\overline{V}_1\overline{W}_{i+1}}{\overline{V}_{i+1}\overline{W}_1}\ \ (1\le i\le n-1),\\
&y'_0:=x_0\frac{\overline{V}^*_1}{\overline{V}_0},\ \ y'_i:=x_i\frac{\overline{V}^*_1\overline{V}^*_{i+1}\overline{W}_1}{\overline{V}_0{}^2\overline{W}_{i+1}}\ \ (1\le i\le n),\\
&\overline{y}'_i:=\overline{x}_i\frac{\overline{V}^*_1}{\overline{V}^*_{i+1}}\ \ (1\le i\le n-1).
\end{aligned}
$$

9.5. **Tropical R for $A^{(2)}_{2n-1}$ $(n\ge 3)$.** We shall describe tropical R's of type $A^{(2)}_{2n-1}$. We see the following lemma for $\{X_2^{(n,L)}\}_{L\in\mathbb{C}^\times}$.

Lemma 9.5. *The birational map $R(A^{(2)}_{2n-1})$ is a tropical R map on $\{X_2^{(n,L)}\}_L$.*

The proof is the same as the one for Lemma 9.3.

Let us describe the explicit form of tropical R on $\mathcal{B}_L(A^{(2)}_{2n-1})$. Set $\mathcal{R}(A^{(2)}_{2n-1}):=(\eta^{-1},\eta^{-1})\circ R(A^{(2)}_{2n-1})\circ(\eta,\eta)$ where η is as in 6.5.

Let $\sharp$ be the involution on $\mathcal{B}_L(A^{(2)}_{2n-1})$ defined by $\sharp: l_1\leftrightarrow \bar{l}_1$ for $l=(l_1,\cdots,\bar{l}_1)\in\mathcal{B}_L(A^{(2)}_{2n-1})$ and $*:\mathcal{B}_L(A^{(2)}_{2n-1})\times\mathcal{B}_M(A^{(2)}_{2n-1})\to\mathcal{B}_M(A^{(2)}_{2n-1})\times\mathcal{B}_L(A^{(2)}_{2n-1})$ an involution defined by

$$
\begin{aligned}
&((l_1,l_2,\dots,\bar{l}_2,\bar{l}_1),(m_1,m_2,\dots,\overline{m}_2,\overline{m}_1))^*\\
&\qquad\qquad=((\overline{m}_1,\overline{m}_2,\dots,m_2,m_1),(\bar{l}_1,\bar{l}_2,\dots,l_2,l_1))
\end{aligned}
$$

that is, $*:l_i\leftrightarrow\overline{m}_i,\quad \bar{l}_i\leftrightarrow m_i\ (1\le i\le n)$.

Restricting the functions V_i and W_i for $D^{(1)}_{2n}$ to $X_2^{(n,L^2)}\times X_2^{(n,M^2)}$, we define the rational functions V_i $(i=0,1,\dots,n)$ and W_i $(i=1,\dots,n)$ on $\mathcal{B}_L(A^{(2)}_{2n-1})\times\mathcal{B}_M(A^{(2)}_{2n-1})$ $(L,M\in\mathbb{C}^\times)$ as

$$
\begin{aligned}
&W_i:=V_iV_i^*+(M^2-L^2)V_i^*+(L^2-M^2)V_i\quad(1\le i\le n),\\
&V_i=\sum_{j=1}^{2n-2}(\theta_{i,j}(l,m)+\theta'_{i,j}(l,m))+\sum_{j=1}^{2n}(\eta_{i,j}(l,m)+\eta'_{i,j}(l,m)),
\end{aligned}
\tag{9.29}
$$

where $L^2=l_n\bar{l}_n(\prod_{i=1}^{n-1}l_i\bar{l}_i)^2$, $M^2=m_n\overline{m}_n(\prod_{i=1}^{n-1}m_i\overline{m}_i)^2$, $\Delta:=\frac{1+\overline{\mu}(l)}{1+\overline{\mu}(m)}$, and

$$\theta_{i,j}(l,m)=\begin{cases} L^2\displaystyle\prod_{k=j+1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\Delta^{\delta_{i,n}-\delta_{j,n}} & 1\le j\le i\le n,\\ M^2\displaystyle\prod_{k=i+1}^{j}\frac{\overline{l}_k}{\overline{m}_k}\Delta^{-\delta_{j,n}} & 0\le i<j\le n,\\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{2n-j-1}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n}}\dfrac{\overline{l}_{2n-j}(m_{2n-j}+\overline{m}_{2n-j})}{(l_{2n-j}+\overline{l}_{2n-j})\overline{m}_{2n-j}} & \\ \qquad\qquad 0\le i\le n<j\le 2n-2. \end{cases}$$

$$\theta'_{i,j}(l,m)=\begin{cases} L^2\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{j}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n}}\left(\dfrac{1+\mu(l)}{1+\mu(m)}\right)^{\delta_{j,n}} & j\le n,\\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{2n-j-1}\frac{\overline{l}_k}{\overline{m}_k}\right)\Delta^{\delta_{i,n}}\dfrac{m_{2n-j}(l_{2n-j}+\overline{l}_{2n-j})}{l_{2n-j}(m_{2n-j}+\overline{m}_{2n-j})} & n<j. \end{cases}$$

$$\eta_{i,j}(l,m)=\begin{cases} L^2\left(\displaystyle\prod_{k=j+1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\dfrac{\overline{m}_j}{l_j}\right)\Delta^{\delta_{i,n}}\mu(l)^{\delta_{j,n}} & 1\le j\le i\le n,\\ M^2\left(\displaystyle\prod_{k=i+1}^{j}\frac{\overline{l}_k}{\overline{m}_k}\right)\left(\dfrac{\overline{m}_j}{l_j}\right)\mu(l)^{\delta_{j,n}} & i<j\le n,\\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{n-1}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n}}\dfrac{1+\mu(m)}{1+\overline{\mu}(l)} & i\le n,\ j=n+1\\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{2n-j}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n}}\dfrac{(m_{2n-j+1}+\overline{m}_{2n-j+1})l_{2n-j+1}}{(l_{2n-j+1}+\overline{l}_{2n-j+1})\overline{m}_{2n-j+1}} & \\ \qquad\qquad i<n+1<j\le 2n-1, & \\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\dfrac{\overline{l}_1}{l_1}\Delta^{\delta_{i,n}} & i<j=2n. \end{cases}$$

$$\eta'_{i,j}(l,m)=\begin{cases} L^2\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{j}\frac{m_k}{l_k}\right)\left(\dfrac{l_j}{\overline{m}_j}\right)\Delta^{\delta_{i,n}}\overline{\mu}(m)^{\delta j,n} & 1\le j\le n,\\ L^2\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{n}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n}}\dfrac{1+\mu(l)}{(1+\mu(m))\mu(m)} & j=n+1,\\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\displaystyle\prod_{k=1}^{2n-j}\frac{\overline{l}_k}{\overline{m}_k}\right)\Delta^{\delta_{i,n}}\dfrac{(l_{2n-j+1}+\overline{l}_{2n-j+1})\overline{m}_{2n-j+1}}{(m_{2n-j+1}+\overline{m}_{2n-j+1})l_{2n-j+1}} & \\ \qquad\qquad n+1<j\le 2n-1, & \\ LM\left(\displaystyle\prod_{k=1}^{i}\frac{\overline{m}_k}{\overline{l}_k}\right)\Delta^{\delta_{i,n}}\dfrac{m_1}{\overline{m}_1} & j=2n. \end{cases}$$

Now, we define the tropical R

$$\mathcal{R}(A^{(2)}_{2n-1})\colon \mathcal{B}_L(A^{(2)}_{2n-1}) \times \mathcal{B}_M(A^{(2)}_{2n-1}) \to \mathcal{B}_M(A^{(2)}_{2n-1}) \times \mathcal{B}_L(A^{(2)}_{2n-1})$$

by

$$\mathcal{R}(A^{(2)}_{2n-1})(l,m) = (l',m')$$

where

$$\begin{aligned}
&l'_1 = m_1\frac{V_0^\sharp}{V_1}, \quad l'_i = m_i\frac{V_{i-1}W_i}{V_iW_{i-1}}, \quad \bar{l}'_i = \overline{m}_i\frac{V_{i-1}}{V_i} \quad (1 \le i \le n-1),\\
&l'_n = \frac{m_nV_{n-1}W_n}{(1+\mu(m))V_nW_{n-1}}\left(1+\frac{m_nW_n}{\overline{m}_nW_{n-1}}\overline{\mu}(m)\right),\\
&\bar{l}'_n = \frac{\overline{m}_nV_{n-1}}{(1+\overline{\mu}(m))V_n}\left(1+\frac{\overline{m}_nW_{n-1}}{m_nW_n}\mu(m)\right),\\
&m'_1 = l_1\frac{V_0}{V_1^*}, \quad \overline{m}'_1 = \bar{l}_1\frac{V_0^\sharp}{V_1^*}, \quad m'_i = l_i\frac{V_{i-1}^*}{V_i^*}, \quad \overline{m}'_i = \bar{l}_i\frac{V_{i-1}^*W_i^{(n)}}{V_i^*W_{i-1}^{(n)}} \quad (2 \le i \le n-1),\\
&m'_n = \frac{l_nV_{n-1}^*}{(1+\mu(l))V_n^*}\left(1+\frac{l_nW_{n-1}}{\bar{l}_nW_n}\overline{\mu}(l)\right),\\
&\overline{m}'_n = \frac{\bar{l}_nV_{n-1}^*W_n}{(1+\overline{\mu}(l))V_n^*W_{n-1}}\left(1+\frac{\bar{l}_nW_n}{l_nW_{n-1}}\mu(l)\right).
\end{aligned}$$

Here note that for $(l',m') = \mathcal{R}(A^{(2)}_{2n-1})(l,m)$ we have $(l'_1l'_2\cdots l'_{n-1}\bar{l}'_{n-1}\cdots\bar{l}'_2\bar{l}'_1)^2l'_n\bar{l}'_n = M^2$ and $(m'_1m'_2\cdots m'_{n-1}\overline{m}'_{n-1}\cdots\overline{m}'_2\overline{m}'_1)^2m'_n\overline{m}'_n = L^2$.

Next, we shall describe tropical R on $\mathcal{V}(A^{(2)}_{2n-1})_L \times \mathcal{V}(A^{(2)}_{2n-1})_M$. Let $*$ be the involution $\mathcal{V}(A^{(2)}_{2n-1})_L \times \mathcal{V}(A^{(2)}_{2n-1})_M \to \mathcal{V}(A^{(2)}_{2n-1})_M \times \mathcal{V}(A^{(2)}_{2n-1})_L$ defined by

$$* \quad : \quad x_i \mapsto \frac{1}{M^2\overline{y}_i},\ \overline{x}_i \mapsto \frac{1}{M^2y_i},\ y_i \mapsto \frac{1}{L^2\overline{x}_i},\ \overline{y}_i \mapsto \frac{1}{L^2x_i}\ (i=1,\cdots,n)$$

and $\sharp$ the involution on $\mathcal{V}(A^{(2)}_{2n-1})_L$ defined by

$$\sharp : x_i \mapsto \frac{x_i}{L^2x_1\overline{x}_1}, \quad \overline{x}_i \mapsto \frac{\overline{x}_i}{L^2x_1\overline{x}_1} \quad (i=1,\cdots,n-1), \quad x_n \mapsto \frac{x_n}{(L^2x_1\overline{x}_1)^2}.$$

We define the rational functions $\overline{V}_i(x,y)$ $(i=0,1,\dots,n)$ and $\overline{W}_i(x,y)$ $(i=1,\dots,n)$ on $\mathcal{V}(A^{(2)}_{2n-1})_L \times \mathcal{V}(A^{(2)}_{2n-1})_M$ $(L, M \in \mathbb{C}^\times)$ by

$$
\begin{aligned}
&\overline{V}_i = \sum_{j=1}^{2n-2}(\overline{\theta}_{i,j}(x,y)+\overline{\theta}'_{i,j}(x,y)) + \sum_{j=1}^{2n}(\overline{\eta}_{i,j}(x,y)+\overline{\eta}'_{i,j}(x,y)),\\
&\overline{W}_i := \overline{V}_i\overline{V}_i^* + (M^2-L^2)\overline{V}_i^* + (L^2-M^2)\overline{V}_i \quad (1\le i\le n),\\
&\mu(x) := \frac{x_n}{Lx_{n-1}\overline{x}_{n-1}} = \overline{\mu}(x)^{-1},\ \mu(y) := \frac{y_n}{My_{n-1}\overline{y}_{n-1}} = \overline{\mu}(y)^{-1},\\
&\Delta := \frac{1+\overline{\mu}(x)}{1+\overline{\mu}(y)}, \quad \nabla := \frac{1+\mu(x)}{1+\mu(y)}, \quad \Box := \frac{1+\mu(y)}{1+\overline{\mu}(x)},\\
&\overline{\theta}_{i,j}(x,y) := \begin{cases}
L^2\frac{x_iy_j}{x_jy_i}\left(\frac{y_{n-1}}{x_{n-1}}\right)^{\delta_{i,n}(1-\delta_{i,j})}(\Delta)^{\delta_{i,n}-\delta_{j,n}} & (1\le j\le i\le n),\\
M^2\frac{x_iy_j}{x_jy_i}\left(\frac{x_{n-1}}{y_{n-1}}\Delta^{-1}\right)^{\delta_{j,n}} & (0\le i<j\le n),\\
\frac{M^3}{L}\frac{x_ix_{2n-j-1}(y_{2n-j-1}\overline{y}_{2n-j-1}+y_{2n-j}\overline{y}_{2n-j})}{y_iy_{2n-j-1}(x_{2n-j-1}\overline{x}_{2n-j-1}+x_{2n-j}\overline{x}_{2n-j})}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & \\
\qquad (i\le n<j\le 2n-2), &
\end{cases}\\
&\overline{\theta}'_{i,j}(x,y) := \begin{cases}
M^2\frac{x_i\overline{y}_j}{y_i\overline{x}_j}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}}\left(\frac{L^2\overline{x}_{n-1}}{M^2\overline{y}_{n-1}}\nabla\right)^{\delta_{j,n}} & (1\le j\le n),\\
LM\frac{x_iy_{2n-j-1}(1+\frac{x_{2n-j-1}\overline{x}_{2n-j-1}}{x_{2n-j}\overline{x}_{2n-j}})}{y_ix_{2n-j-1}(1+\frac{y_{2n-j-1}\overline{y}_{2n-j-1}}{y_{2n-j}\overline{y}_{2n-j}})}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & (n<j\le 2n-2),
\end{cases}\\
&\overline{\eta}_{i,j}(x,y) := \begin{cases}
L^{2-2\delta_{j,1}}\frac{x_iy_{j-1}\overline{x}_{j-1}}{y_ix_j\overline{x}_j}\left(\frac{y_{n-1}}{x_{n-1}}\right)^{\delta_{i,n}(1-\delta_{i,j})}\Delta^{\delta_{i,n}}\left(\frac{Lx_ny_{n-1}}{x_{n-1}}\right)^{\delta_{j,n}} & \\
\qquad (1\le j\le i\le n), & \\
L^{-2\delta_{j,1}}M^2\frac{x_iy_{j-1}\overline{x}_{j-1}}{y_ix_j\overline{x}_j}(Lx_n)^{\delta_{j,n}} & (i<j\le n),\\
\frac{M^3}{L}\frac{x_i\overline{y}_{n-1}}{y_i\overline{x}_{n-1}}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}}\Box & (0\le i\le n, j=n+1),\\
\frac{M^3}{L}\frac{x_i\overline{y}_{2n-j}}{y_i\overline{x}_{2n-j}}\frac{1+\frac{y_{2n-j+1}\overline{y}_{2n-j+1}}{y_{2n-j}\overline{y}_{2n-j}}}{1+\frac{x_{2n-j}\overline{x}_{2n-j}}{x_{2n-j+1}\overline{x}_{2n-j+1}}}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & \\
\qquad (0\le i\le n<j\le 2n-1), & \\
\frac{M}{L}\frac{x_i}{y_ix_1\overline{x}_1}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & (0\le i\le n<j=2n),
\end{cases}\\
&\overline{\eta}'_{i,j}(x,y) := \begin{cases}
L^{2\delta_{j,1}}M^2\frac{x_iy_j\overline{y}_j}{y_iy_{j-1}\overline{x}_{j-1}}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}}\left(\frac{1}{My_n}\right)^{\delta_{j,n}} & (1\le j\le n),\\
L^2M\frac{x_iy_{n-1}\overline{x}_{n-1}}{y_ix_n}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}}\nabla & (j=n+1),\\
LM\frac{x_iy_{2n-j}}{y_ix_{2n-j}}\frac{\left(1+\frac{x_{2n-j}\overline{x}_{2n-j}}{x_{2n-j+1}\overline{x}_{2n-j+1}}\right)}{\left(1+\frac{y_{2n-j+1}\overline{y}_{2n-j+1}}{y_{2n-j}\overline{y}_{2n-j}}\right)}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & (n+1<j\le 2n-1),\\
LM^3\frac{x_iy_1\overline{y}_1}{y_i}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n}} & (j=2n),
\end{cases}
\end{aligned}
$$

where we understand $x_0=\overline{x}_0=y_0=\overline{y}_0=1$, $\overline{x}_n=x_n$ and $\overline{y}_n=y_n$.

Here we define $\overline{\mathcal{R}}(x,y)=\overline{\mathcal{R}}(A^{(2)}_{2n-1})(x,y)=(x',y')$ by

$$x'_i := y_i\frac{\overline{V}_i}{\overline{V}_0}\ ,\quad \overline{x}'_i := \overline{y}_i\frac{\overline{V}^\sharp_0\overline{W}_i}{\overline{V}_i\overline{W}_1}\ \ (1\le i\le n-1),$$
$$x'_n = y_n\frac{{\overline{V}^\sharp_0}^2\overline{W}_{n-1}\overline{W}_n}{\overline{V}_{n-1}\overline{V}_n\overline{W}_1^2}\frac{1+\frac{\overline{W}_n}{\overline{W}_{n-1}}\mu(y)}{1+\mu(y)},$$
$$y'_i := x_i\frac{\overline{V}^*_i\overline{W}_1}{\overline{V}^\sharp_0 W_i}\ \ \overline{y}'_i := \overline{x}_i\frac{\overline{V}_0}{\overline{V}^*_i}\ \ (1\le i\le n-1),\quad y'_n = x_n\frac{{\overline{V}_0}^2}{\overline{V}^*_{n-1}\overline{V}^*_n}\frac{1+\frac{\overline{W}_{n-1}}{\overline{W}_n}\mu(x)}{1+\mu(x)}.$$

9.6. **Tropical R for $A^{(2)}_{2n}$ $(n\ge 2)$.** We shall describe tropical R's of type $A^{(2)}_{2n}$. We see the following lemma for $\{X^{(n,L)}_2\}_{L\in\mathbb{C}^\times}$.

Lemma 9.6. *The birational map $R(A^{(2)}_{2n})$ is a tropical R map on the $A^{(2)}_{2n}$-geometric crystal $\{X^{(n,L)}_4\}_L$.*

The proof is the same as the one for Lemma 9.3.

Let us describe the explicit form of tropical R on $\mathcal{B}_L(A^{(2)}_{2n})$. Set $\mathcal{R}(A^{(2)}_{2n}) := (\eta^{-1},\eta^{-1})\circ R(A^{(2)}_{2n})\circ(\eta,\eta)$ where η is as in 6.6.

Let $*:\mathcal{B}_L(A^{(2)}_{2n})\times\mathcal{B}_M(A^{(2)}_{2n})\to\mathcal{B}_M(A^{(2)}_{2n})\times\mathcal{B}_L(A^{(2)}_{2n})$ be the involution defined by

$$\begin{aligned}((l_0,l_1,l_2,\ldots,\bar{l}_2,\bar{l}_1),&(m_0,m_1,m_2,\ldots,\overline{m}_2,\overline{m}_1))^*\\ &=((m_0,\overline{m}_1,\overline{m}_2,\ldots,m_2,m_1),(l_0,\bar{l}_1,\bar{l}_2,\ldots,l_2,l_1))\end{aligned}$$

that is, $*: l_0\leftrightarrow m_0,\quad l_i\leftrightarrow\overline{m}_i,\quad \bar{l}_i\leftrightarrow m_i\ (1\le i\le n)$.

Restricting the functions V_i and W_i for $A^{(2)}_{2n+1}$ to $X^{(n,L)}_4\times X^{(n,M)}_4$, we define the rational functions V_i $(i=0,1,\ldots,n+1)$ and W_i $(i=1,\ldots,n+1)$ on $\mathcal{B}_L(A^{(2)}_{2n})\times\mathcal{B}_M(A^{(2)}_{2n})$ $(L,M\in\mathbb{C}^\times)$ as

$$W_i := V_iV_i^* + (M^2-L^2)V_i^* + (L^2-M^2)V_i\quad(1\le i\le n+1),$$
$$V_i = \sum_{j=1}^{2n}(\theta_{i,j}(l,m)+\theta'_{i,j}(l,m)) + \sum_{j=1}^{2n+2}(\eta_{i,j}(l,m)+\eta'_{i,j}(l,m)),$$

where $L^2=l_n\bar{l}_n(l_0^2\prod_{i=1}^{n-1}l_i\bar{l}_i)^2$, $M^2=m_n\overline{m}_n(m_0^2\prod_{i=1}^{n-1}m_i\overline{m}_i)^2$, $\Delta:=\frac{1+\overline{\mu}(l)}{1+\overline{\mu}(m)}$, and

$$\theta_{i,j}(l,m) = \begin{cases} L^2\displaystyle\prod_{k=j}^{i-1}\frac{\overline{m}_k}{\bar{l}_k}\Delta^{\delta_{i,n+1}-\delta_{j,n+1}} & 1\le j\le i\le n+1,\\ M^2\displaystyle\prod_{k=i}^{j-1}\frac{\bar{l}_k}{\overline{m}_k}\Delta^{-\delta_{j,n+1}} & 0\le i<j\le n+1,\\ LM\left(\displaystyle\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\bar{l}_k}\right)\left(\displaystyle\prod_{k=0}^{2n-j}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n+1}}\frac{\bar{l}_{2n-j+1}(m_{2n-j+1}+\overline{m}_{2n-j+1})}{(l_{2n-j+1}+\bar{l}_{2n-j+1})\overline{m}_{2n-j+1}} & \\ \qquad 0\le i\le n+1<j. & \end{cases}$$

$$\theta'_{i,j}(l,m)=\begin{cases} L^2\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{j-1}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n+1}}\left(\frac{1+\mu(l)}{1+\mu(m)}\right)^{\delta_{j,n+1}} & j\le n+1,\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{2n-j}\frac{\overline{l}_k}{\overline{m}_k}\right)\Delta^{\delta_{i,n+1}}\frac{m_{2n-j+1}(l_{2n-j+1}+\overline{l}_{2n-j+1})}{l_{2n-j+1}(m_{2n-j+1}+\overline{m}_{2n-j+1})} & n+1<j.\end{cases}$$

$$\eta_{i,j}(l,m)=\begin{cases} L^2\left(\prod_{k=j}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\frac{\overline{m}_{j-1}}{l_{j-1}}\right)\Delta^{\delta_{i,n+1}}\mu(l)^{\delta_{j,n+1}} & 1\le j\le i,\\ M^2\left(\prod_{k=i}^{j-1}\frac{\overline{l}_k}{\overline{m}_k}\right)\left(\frac{\overline{m}_{j-1}}{l_{j-1}}\right)\mu(l)^{\delta_{j,n+1}} & 0\le i<j\le n+1,\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{n-1}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n+1}}\frac{1+\mu(m)}{1+\overline{\mu}(l)} & 0\le i\le n+1, j=n+2\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{2n-j+1}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n+1}}\frac{\frac{m_{2n-j+2}}{\overline{m}_{2n-j+2}}+1}{\frac{\overline{l}_{2n-j+2}}{l_{2n-j+2}}+1} & 0\le i<n+2<j\le 2n+1,\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\Delta^{\delta_{i,n+1}} & i\le n+1, j=2n+2.\end{cases}$$

$$\eta'_{i,j}(l,m)=\begin{cases} L^2\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{j-1}\frac{m_k}{l_k}\right)\left(\frac{l_{j-1}}{\overline{m}_{j-1}}\right)\Delta^{\delta_{i,n+1}}\overline{\mu}(m)^{\delta_{j,n+1}} & 1\le j\le n+1,\\ L^2\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{n}\frac{m_k}{l_k}\right)\Delta^{\delta_{i,n+1}}\frac{1+\mu(l)}{(1+\mu(m))\mu(m)} & j=n+2,\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\left(\prod_{k=0}^{2n-j+1}\frac{\overline{l}_k}{\overline{m}_k}\right)\Delta^{\delta_{i,n+1}}\frac{(l_{2n-j+2}+\overline{l}_{2n-j+2})\overline{m}_{2n-j+2}}{(m_{2n-j+2}+\overline{m}_{2n-j+2})l_{2n-j+2}} & n+2<j\le 2n+1,\\ LM\left(\prod_{k=0}^{i-1}\frac{\overline{m}_k}{\overline{l}_k}\right)\Delta^{\delta_{i,n+1}} & j=2n+2,\end{cases}$$

where we understand $\overline{l}_0=l_0$ and $\overline{m}_0=m_0$.

Now, we define the tropical R $\mathcal{R}(A^{(2)}_{2n})$: $\mathcal{B}_L(A^{(2)}_{2n})\times\mathcal{B}_M(A^{(2)}_{2n})\to\mathcal{B}_M(A^{(2)}_{2n})\times\mathcal{B}_L(A^{(2)}_{2n})$ by

$$\mathcal{R}(A^{(2)}_{2n})(l,m)=(l',m')$$

where

$$\begin{aligned}
&l'_0 = m_0\frac{V_0}{V_1}, \quad l'_i = m_i\frac{V_iW_{i+1}}{V_{i+1}W_i}, \quad \overline{l}'_i = \overline{m}_i\frac{V_i}{V_{i+1}} \quad (1 \le i \le n-1),\\
&l'_n = \frac{m_nV_nW_{n+1}}{(1+\mu(m))V_{n+1}W_n}\left(1+\frac{m_nW_{n+1}}{\overline{m}_nW_n}\overline{\mu}(m)\right),\\
&\overline{l}'_n = \frac{\overline{m}_nV_n}{(1+\overline{\mu}(m))V_{n+1}}\left(1+\frac{\overline{m}_nW_n}{m_nW_{n+1}}\mu(m)\right),\\
&m'_0 = l_0\frac{V_0}{V_1^*}, \quad m'_i = l_i\frac{V_i^*}{V_{i+1}^*}, \quad \overline{m}'_i = \overline{l}_i\frac{V_i^*W_{i+1}^{(n)}}{V_{i+1}^*W_i^{(n)}} \quad (1 \le i \le n-1),\\
&m'_n = \frac{l_nV_n^*}{(1+\mu(l))V_{n+1}^*}\left(1+\frac{l_nW_n}{\overline{l}_nW_{n+1}}\overline{\mu}(l)\right),\\
&\overline{m}'_n = \frac{\overline{l}_nV_n^*W_{n+1}}{(1+\overline{\mu}(l))V_{n+1}^*W_n}\left(1+\frac{\overline{l}_nW_{n+1}}{l_nW_n}\mu(l)\right).
\end{aligned}$$

Here note that for $(l',m') = \mathcal{R}(A_{2n}^{(2)})(l,m)$ we have $(l'_1l'_2\cdots l'_{n-1}\overline{l}'_{n-1}\cdots\overline{l}'_2\overline{l}'_1)^2l'_n\overline{l}'_n = M^2$ and $(m'_1m'_2\cdots m'_{n-1}\overline{m}'_{n-1}\cdots\overline{m}'_2\overline{m}'_1)^2m'_n\overline{m}'_n = L^2$.

Next, we shall describe tropical R on $\mathcal{V}(A_{2n}^{(2)})_L \times \mathcal{V}(A_{2n}^{(2)})_M$. Let $*$ be the involution $\mathcal{V}(A_{2n}^{(2)})_L \times \mathcal{V}(A_{2n}^{(2)})_M \to \mathcal{V}(A_{2n}^{(2)})_M \times \mathcal{V}(A_{2n}^{(2)})_L$ defined by $x_0 \leftrightarrow y_0$ and

$$x_i \mapsto \frac{y_0^2}{M^2\overline{y}_i},\ \overline{x}_i \mapsto \frac{y_0^2}{M^2y_i},\ x_n \mapsto \frac{y_0^4}{M^2y_n},\ y_i \mapsto \frac{x_0^2}{L^2\overline{x}_i},$$
$$\overline{y}_i \mapsto \frac{x_0^2}{L^2x_i}\ y_n \mapsto \frac{x_0^4}{L^2x_n},\ (i = 1,\cdots,n-1).$$

We define the rational functions $\overline{V}_i(x,y)$ $(i = 0,1,\dots,n+1)$ and $\overline{W}_i(x,y)$ $(i = 1,\dots,n+1)$ on $\mathcal{V}(A_{2n}^{(2)})_L \times \mathcal{V}(A_{2n}^{(2)})_M$ $(L,M \in \mathbb{C}^\times)$ by

$$\begin{aligned}
&\overline{V}_i = \sum_{j=1}^{2n}(\overline{\theta}_{i,j}(x,y)+\overline{\theta}'_{i,j}(x,y)) + \sum_{j=1}^{2n+2}(\overline{\eta}_{i,j}(x,y)+\overline{\eta}'_{i,j}(x,y)),\\
&\overline{W}_i := \overline{V}_i\overline{V}*_i + (M^2-L^2)\overline{V}_i^* + (L^2-M^2)\overline{V}_i \quad (1 \le i \le n+1),\\
&\mu(x) := \frac{x_n}{Lx_{n-1}\overline{x}_{n-1}} = \overline{\mu}(x)^{-1},\ \mu(y) := \frac{y_n}{My_{n-1}\overline{y}_{n-1}} = \overline{\mu}(y)^{-1},\\
&\Delta := \frac{1+\overline{\mu}(x)}{1+\overline{\mu}(y)}, \quad \nabla := \frac{1+\mu(x)}{1+\mu(y)}, \quad \Box := \frac{1+\mu(y)}{1+\overline{\mu}(x)}.
\end{aligned}$$

$$\overline{\theta}_{i,j}(x,y) :=
\begin{cases}
L^2\left(\dfrac{x_{i-1}}{y_{i-1}}\right)^{1+\delta_{i,1}}\left(\dfrac{y_{j-1}}{x_{j-1}}\right)^{1+\delta_{j,1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}-\delta_{j,n+1}} & (1 \le j \le i \le n+1),\\
M^2\dfrac{x_{i-1}y_{j-1}}{x_{j-1}y_{i-1}}\left(\dfrac{x_{n-1}}{y_{n-1}}\Delta^{-1}\right)^{\delta_{j,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}-\delta_{j,1}} & (0 \le i < j \le n+1),\\
LM\dfrac{x_{i-1}}{y_{i-1}}\left(\dfrac{M^2\overline{y}_{2n-j}}{L^2\overline{x}_{2n-j}}\right)^{1-\delta_{j,2n}}\dfrac{1+M^{2\delta_{j,2n}}\frac{y_{2n+1-j}\overline{y}_{2n+1-j}}{y_{2n-j}\overline{y}_{2n-j}}}{1+L^{2\delta_{j,2n}}\frac{x_{2n+1-j}\overline{x}_{2n+1-j}}{x_{2n-j}\overline{x}_{2n-j}}} & \\
\qquad\qquad \times\left(\frac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}}\left(\frac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}} & (i \le n+1 < j < 2n),
\end{cases}$$

$$\overline{\theta}'_{i,j}(x,y) := \begin{cases} L^2\dfrac{x_{i-1}}{y_{i-1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{M^2\overline{y}_{j-1}}{L^2\overline{x}_{j-1}}\right)^{1-\delta_{j,1}}\left(\dfrac{L^2\overline{x}_{n-1}}{M^2\overline{y}_{n-1}}\nabla\right)^{\delta_{j,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \\ \qquad (1\le j\le n+1), \\ LM\dfrac{x_{i-1}y_{2n-j}}{y_{i-1}x_{2n-j}}\dfrac{1+\frac{x_{2n-j}\overline{x}_{2n-j}}{L^{2\delta_{j,2n}}x_{2n+1-j}\overline{x}_{2n+1-j}}}{1+\frac{y_{2n-j}\overline{y}_{2n-j}}{M^{2\delta_{j,2n}}y_{2n+1-j}\overline{y}_{2n+1-j}}}\left(\dfrac{y_n}{x_n}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{y_0}{x_0}\right)^{\delta_{i,0}+\delta_{i,1}-\delta_{j,2n}} \\ \qquad (n+1<j\le 2n), \end{cases}$$

$$\overline{\eta}_{i,j}(x,y) := \begin{cases} L^{2+\delta_{j,n+1}-2\delta_{j,2}}\dfrac{x_{i-1}y_{j-2}\overline{x}_{j-2}}{y_{i-1}x_{j-1}\overline{x}_{j-1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\dfrac{x_0^{\delta_{i,1}+\delta_{j,1}-\delta_{j,2}}}{y_0^{\delta_{i,1}-\delta_{j,1}-\delta_{j,2}}}x_n^{\delta_{j,n+1}} \\ \qquad (1\le j\le i\le n+1), \\ M^2L^{\delta_{j,n+1}-2\delta_{j,2}}\dfrac{x_{i-1}y_{j-2}\overline{x}_{j-2}}{y_{i-1}x_{j-1}\overline{x}_{j-1}}\dfrac{x_0^{\delta_{i,0}+\delta_{i,1}+\delta_{j,1}-\delta_{j,2}}}{y_0^{\delta_{i,0}+\delta_{i,1}-\delta_{j,1}-\delta_{j,2}}}x_n^{\delta_{j,n+1}} \quad (0\le i<j\le n+1), \\ \dfrac{M^3}{L}\dfrac{x_{i-1}\overline{y}_{n-1}}{y_{i-1}\overline{x}_{n-1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}}\square \quad (0\le i<j=n+2), \\ \dfrac{M^3}{L}\dfrac{x_{i-1}\overline{y}_{2n-j+1}}{y_{i-1}\overline{x}_{2n-j+1}}\dfrac{1+\frac{y_{2n-j+2}\overline{y}_{2n-j+2}}{y_{2n-j+1}\overline{y}_{2n-j+1}}}{1+\frac{x_{2n-j+1}\overline{x}_{2n-j+1}}{x_{2n-j+2}\overline{x}_{2n-j+2}}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \\ \qquad (i\le n+1, n+2<j\le 2n) \\ LM\dfrac{x_{i-1}}{y_{i-1}}\dfrac{1+\frac{M^2y_1\overline{y}_1}{y_0^2}}{1+\frac{x_0^2}{L^2x_1\overline{x}_1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \quad (i\le n+1, j=2n+1), \\ LM\dfrac{x_{i-1}}{y_{i-1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{-1+\delta_{i,0}+\delta_{i,1}} \quad (i\le n+1, j=2n+2), \end{cases}$$

$$\overline{\eta}'_{i,j}(x,y) := \begin{cases} \overline{\theta}'_{i,j}(x,y)\dfrac{\overline{x}_{j-1}y_{j-1}}{\overline{x}_{j-2}y_{j-2}}\dfrac{L^{2\delta_{j,2}}x_0^{\delta_{j,2}}}{L^{2\delta_{j,n+1}}x_0^{2\delta_{j,1}}y_0^{\delta_{j,2}}}\left(\dfrac{M\overline{y}_{n-1}}{x_{n-1}y_n\nabla}\right)^{\delta_{j,n+1}} \quad (1\le j\le n+1), \\ L^2M\dfrac{x_{i-1}y_{n-1}\overline{x}_{n-1}}{y_{i-1}x_n}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}}\nabla \quad (j=n+2) \\ LM\dfrac{x_{i-1}y_{2n-j+1}}{y_{i-1}x_{2n-j+1}}\dfrac{\left(1+\frac{x_{2n-j+1}\overline{x}_{2n-j+1}}{x_{2n-j+2}\overline{x}_{2n-j+2}}\right)}{\left(1+\frac{y_{2n-j+2}\overline{y}_{2n-j+2}}{y_{2n-j+1}\overline{y}_{2n-j+1}}\right)}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{\delta_{i,0}+\delta_{i,1}} \\ \qquad (n+1<j\le 2n), \\ LM\dfrac{x_{i-1}}{y_{i-1}}\dfrac{1+\frac{x_0^2}{L^2x_1\overline{x}_1}}{1+\frac{M^2y_1\overline{y}_1}{y_0^2}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{-1+\delta_{i,0}+\delta_{i,1}} \quad (j=2n+1), \\ LM\dfrac{x_{i-1}}{y_{i-1}}\left(\dfrac{y_{n-1}}{x_{n-1}}\Delta\right)^{\delta_{i,n+1}}\left(\dfrac{x_0}{y_0}\right)^{-1+\delta_{i,0}+\delta_{i,1}} \quad (j=2n+2), \end{cases}$$

where we understand that $\overline{x}_0 = x_0$, $\overline{y}_0 = y_0$ and $x_{-1} = \overline{x}_{-1} = y_{-1} = \overline{y}_{-1} = 1$. Here we define $\overline{\mathcal{R}}(x,y) = \overline{\mathcal{R}}(A^{(2)}_{2n})(x,y) = (x', y')$ by

$$x'_0 := y_0 \frac{\overline{V}_1}{\overline{V}_0}, \quad x'_i := y_i \frac{\overline{V}_1 \overline{V}_{i+1}}{\overline{V}_0{}^2}, \quad \overline{x}'_i := \overline{y}_i \frac{\overline{V}_1 \overline{W}_{i+1}}{\overline{V}_{i+1} \overline{W}_1} \ (1 \le i \le n-1),$$

$$x'_n = y_n \frac{\overline{V}_1{}^2 \overline{W}_n \overline{W}_{n+1}}{\overline{V}_n \overline{V}_{n+1} \overline{W}_1{}^2} \left(1 + \frac{\mu(y)\overline{W}_{n+1}}{\overline{W}_n}\right)(1+\mu(y))^{-1},$$

$$y'_0 := x_0 \frac{\overline{V}^*_1}{\overline{V}_0}, \quad y'_i := x_i \frac{\overline{V}^*_1 \overline{V}^*_{i+1} \overline{W}_1}{\overline{V}_0{}^2 W_{i+1}}, \quad \overline{y}'_i := \overline{x}_i \frac{\overline{V}^*_1}{\overline{V}^*_{i+1}} \ (1 \le i \le n-1),$$

$$y'_n = x_n \frac{(\overline{V}^*_1)^2}{\overline{V}^*_n \overline{V}^*_{n+1}} \left(1 + \frac{\mu(x)\overline{W}_n}{\overline{W}_{n+1}}\right)(1+\mu(x))^{-1}.$$

9.7. **Uniqueness of the tropical R maps.**

Theorem 9.7. *Let $\overline{\mathcal{R}}$ be the tropical R as introduced in this section. Set $z_0 := \mathcal{R}(\mathbf{1},\mathbf{1})$. Let $\mathcal{R}'$ be a tropical R such that $\mathcal{R}'(\mathbf{1},\mathbf{1}) = \mathbf{z_0}$. Then we have $\overline{\mathcal{R}} = \mathcal{R}'$ as birational maps.*

Proof. Let $\mathcal{V}_l, \mathcal{V}_m$ $(l, m \in \mathbb{C}^\times)$ be the affine geometric crystals constructed in Section 5. By Theorem 5.4 and Theorem 7.1, we have that $\mathcal{UD}_{(\theta_l,\theta_m)}(\mathcal{V}_l(\mathfrak{g}) \times \mathcal{V}_m(\mathfrak{g}))$ is isomorphic to the crystal $B_\infty(\mathfrak{g}^L) \otimes B_\infty(\mathfrak{g}^L)$, where θ_l is the positive structure as in 5.10. Since $B_\infty(\mathfrak{g}^L) \otimes B_\infty(\mathfrak{g}^L)$ is connected, $\mathcal{V}_l(\mathfrak{g}) \times \mathcal{V}_m(\mathfrak{g})$ is prehomogeneous by Theorem 3.3. Therefore, by Lemma 3.2 we obtain $\overline{\mathcal{R}} = \mathcal{R}'$, which completes the proof of Theorem 9.7. □

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