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# ON THE IRREDUCIBLE REPRESENTATIONS OF THE ALTERNATING GROUP WHICH REMAIN IRREDUCIBLE IN CHARACTERISTIC $p$

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Abstract. We consider the problem of which ordinary irreducible representations of the alternating group $\mathfrak{A}_n$ remain irreducible modulo a prime $p$. We solve this problem for $p = 2$, and present a conjecture for odd $p$, which we prove in one direction.

## 1. Introduction

In the modular representation theory of finite groups, it is interesting to know which ordinary irreducible representations remain irreducible modulo a prime $p$. For the symmetric groups $\mathfrak{S}_n$, this amounts to classifying the irreducible Specht modules over any field, and this problem has been solved through the combined efforts of James, Mathas, Lyle and the author. In this paper, we address the same problem for the alternating groups $\mathfrak{A}_n$. For these groups we do not have anything like the very rich Specht module theory of the symmetric groups, but we can exploit the very close relationship between the symmetric groups and the alternating groups. Accordingly, the majority of this paper concerns the representation theory of the symmetric group.

Since $\mathfrak{A}_n$ has index 2 in $\mathfrak{S}_n$, the case $p = 2$ behaves very differently from the case of odd characteristic; this case is actually straightforward to deal with, using Benson's work on the representation theory of the alternating group in characteristic 2. For odd characteristic, we reduce the problem to one concerning the symmetric groups, and conjecture a solution to this problem. We then prove this conjecture in one direction (i.e. we prove that the ordinary characters which we claim remain irreducible in characteristic $p$ really do remain irreducible).

In the next section, we summarise the background theory we shall require, most of which concerns the symmetric group. In Section 3, we prove our main result in

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characteristic 2. In Section 4, we give our conjecture for the case of odd characteristic, and explore the combinatorics of the partitions involved in the conjecture. In Section 5 we give our partial proof of this conjecture.

## 2. Background

Almost all of this paper is concerned with the representation theory of the symmetric group, with the consequences for the alternating group being easily deduced. The classic reference for the modular representation theory of the symmetric group is James's book [J2]. Here we summarise the important points, together with some newer material.

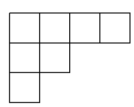
For this paper, let $\mathbb{F}$ be any field. By [J2, Theorem 11.5], every field is a splitting field for $\mathfrak{S}_n$; when necessary, we shall assume that $\mathbb{F}$ is a splitting field for $\mathfrak{A}_n$. We use the convention that the characteristic of a field is the order of its prime subfield.

If $N$ and $M$ are modules for some group algebra and $m$ is a non-negative integer, we may write $N \sim M^m$ to indicate that $N$ has the same composition factors (with multiplicities) as $M^{\oplus m}$.

2.1. **Partitions and Specht modules.** A *partition* of $n$ is defined to be a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers whose sum is $n$. When writing partitions, we frequently group equal parts, and omit zeroes. We write $\varnothing$ for the unique partition of 0. We often identify a partition with its *Young diagram*

$$[\lambda] = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \leqslant \lambda_i\},$$

whose elements are called *nodes*. We frequently draw the Young diagram by means of boxes in the plane, so that, for example, the Young diagram of $(4,2,1)$ is as follows.



If $\lambda$ is a partition, the *conjugate partition* $\lambda'$ is given by

$$\lambda'_i = |\{j \mid \lambda_j \leqslant i\}|.$$

Given a positive integer $p$, a partition is called *$p$-regular* if it does not have $p$ equal positive parts, and *$p$-restricted* if its conjugate is $p$-regular.

To each partition $\lambda$ of $n$ is associated a *Specht module* $S^\lambda$ for $\mathbb{F}\mathfrak{S}_n$. If $\mathbb{F}$ has infinite characteristic, then the modules $S^\lambda$ are irreducible and pairwise non-isomorphic, and afford all the irreducible representations of $\mathbb{F}\mathfrak{S}_n$ as $\lambda$ ranges over the set of partitions of $n$. We write $\chi^\lambda$ for the character of the representation afforded by $S^\lambda$.

If $\mathbb{F}$ has finite characteristic $p$, then $S^\lambda$ affords the $p$-modular reduction of the character $\chi^\lambda$, and is no longer necessarily irreducible. If $\lambda$ is $p$-regular, then $S^\lambda$ has an irreducible cosocle $D^\lambda$; the modules $D^\lambda$ are pairwsie non-isomorphic and afford all the irreducible representations of $\mathbb{F}\mathfrak{S}_n$ as $\lambda$ ranges over the set of $p$-regular partitions of $n$.

We also need to address briefly the *Schur algebra* $\mathcal{S}_\mathbb{F}(n,n)$ over $\mathbb{F}$. This is a finite-dimensional algebra whose module category is equivalent to the category of polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{F})$ of degree $n$; the book by Green [G] is the essential reference for the Schur algebra. For each partition $\lambda$ of $n$, one defines a *Weyl module* $\Delta^\lambda$ for $\mathcal{S}_\mathbb{F}(n,n)$. This has an irreducible cosocle $L^\lambda$, and the modules $L^\lambda$ give all the irreducible representations of $\mathcal{S}_\mathbb{F}(n,n)$ as $\lambda$ varies.

2.2. **Hooks and rim hooks.** The combinatorial representation theory of the symmetric group is controlled by *hooks*. If $\lambda$ is a partition and $(i,j)$ is a node of $\lambda$, we define the $(i,j)$-*hook length* $h_\lambda(i,j)$ to be the number of nodes in the Young diagram directly to the right of or directly below $(i,j)$, including $(i,j)$ itself. That is, $h_\lambda(i,j) = \lambda_i - i + \lambda'_j - j + 1$.

Now suppose $e$ is a positive integer. The *rim* of (the Young diagram of) $\lambda$ is the set of nodes $(i,j)$ of $\lambda$ such that $(i+1,j+1)$ is not a node of $\lambda$. A *rim $e$-hook* of $\lambda$ is a connected portion $\hbar$ of the rim consisting of exactly $e$ nodes, such that $\lambda \setminus \hbar$ is the Young diagram of a partition.

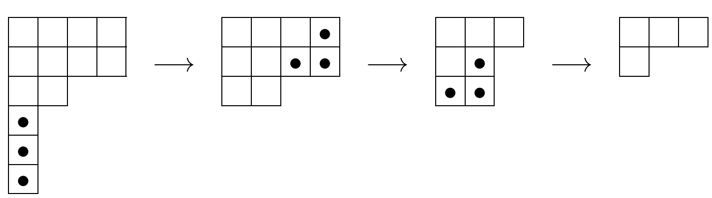
Suppose $\hbar$ is a rim $e$-hook of a partition $\lambda$. The top-rightmost node in $\hbar$ is called the *hand node* of $\hbar$, and the bottom-leftmost node of $\hbar$ is the *foot node*. If we write these nodes as $(i,k)$ and $(l,j)$ respectively, then we have the following:

- $(i,k)$ is the last node in its row, i.e. $k = \lambda_i$;
- $(l,j)$ is the last node in its column, i.e. $l = \lambda'_j$;
- the $(i,j)$-hook length $h_\lambda(i,j)$ equals $e$.

Thus there is a correspondence between rim hooks of length $e$, and nodes whose hook length is $e$. We say that $(i,j)$ is the node *corresponding to* $\hbar$.

A partition $\lambda$ is an *$e$-core* if it has no rim $e$-hooks, or equivalently if none of its hook lengths equals $e$. Given any partition $\lambda$, we can obtain an $e$-core by repeatedly removing rim $e$-hooks from $\lambda$. The $e$-core we obtain is independent of the choice of rim hooks removed at each stage, and is called the $e$-core of $\lambda$. The number of rim hooks removed to reach the $e$-core is the *$e$-weight* of $\lambda$.

**Example.** Suppose $\mathbb{F}$ has characteristic 3, and let $\lambda = (4^2, 2, 1^3)$. Then $\lambda$ has 3-core $(3,1)$ and 3-weight 3, as can be seen from the following diagram.



2.3. **Blocks and the abacus.** The most important application of hooks and cores is the classification of the blocks of $\mathfrak{S}_n$. If $\mathbb{F}$ has characteristic $p$ and $\lambda$ and $\mu$ are partitions of $n$, then $S^\lambda$ and $S^\mu$ lie in the same block of $\mathbb{F}\mathfrak{S}_n$ if and only if $\lambda$ and $\mu$ have the same $p$-core. In this case, we will abuse terminology by saying that $\lambda$ and $\mu$ lie in the same block. Two partitions of $n$ having the same $p$-core necessarily have the same $p$-weight, and so we may talk about the ($p$-)weight and ($p$-)core of a block of $\mathbb{F}\mathfrak{S}_n$.

Partitions and blocks are more easily visualised using the *abacus*, which is a combinatorial gadget introduced by James. Given $p$, take an abacus with $p$ vertical runners, and mark positions $0, 1, \dots$ on these runners reading from from left to right and top to bottom, so that the $j$th runner from the left contains the positions $j-1, j+p-1, j+2p-1, \dots$ from the top down. For example, if $p = 5$, then the abacus is marked as follows.

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ | & | & | & | & | \\ 5 & 6 & 7 & 8 & 9 \\ | & | & | & | & | \\ 10 & 11 & 12 & 13 & 14 \\ | & | & | & | & | \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

If $x < y$, then we may say that position $y$ is *lower* than position $x$, or that position $y$ *occurs after* position $x$.

Now suppose $\lambda$ is a partition, and choose an integer $r \geqslant \lambda'_1$. For $i = 1, \dots, r$, define $\beta_i = \lambda_i + r - i$. Place a bead on the abacus at position $\beta_i$, for each $i$. The resulting configuration is called the *abacus display for* $\lambda$ *with* $r$ *beads*.

The important feature of an abacus display for a partition is that moving a bead into an empty space $a$ positions above it on the same runner corresponds to removing a rim $ap$-hook from $\lambda$. Thus, an abacus display for the $p$-core of $\lambda$ is obtained by moving all the beads up their runners as far as they will go. Two partitions of $n$ therefore lie in the same block if and only if they have the same numbers of beads on corresponding runners. Accordingly, we may talk of the abacus display for a block by specifying the number of beads on each runner and the weight of the block.

Following Richards [R], we define a numbering of the runners of the abacus for a given block $B$, and an array of integers called the *pyramid* for $B$, as follows. First move all the beads up as far as possible to obtain an abacus display for the $p$-core of the block. Now for each runner $R$, let $q_{(R)}$ be the position of the first empty space on that runner. Arrange the resulting $p$ integers in ascending order as $q_0 < \cdots < q_{p-1}$, and number the runners from 0 to $p-1$ so that position $q_i$ appears on runner $i$ for each $i$. We use this numbering (rather than the left-to-right numbering employed elsewhere) exclusively in this paper.

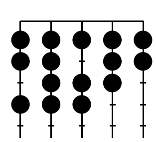
Now for each $0 \leqslant i < j \leqslant p-1$, define

$$_iB_j = \left\lfloor \frac{q_j - q_i}{p} \right\rfloor.$$

Then the array of integers $({}_iB_j)$ is the pyramid for $B$. (Note that this is not quite the definition given by Richards, but it is more convenient for our purposes.)

Now we can define the $p$-quotient of a partition $\lambda$. Take an abacus display for $\lambda$, and for $i \in \{0, \dots, p-1\}$ regard runner $i$ on its own as an abacus with one runner, and let $\lambda(i)$ be the corresponding partition. That is, $\lambda(i)_j$ is the number of empty spaces above the $j$th lowest bead on runner $i$. Then $(\lambda(0), \dots, \lambda(p-1))$ is called the $p$-quotient of $\lambda$, and together with the $p$-core of $\lambda$ it specifies $\lambda$. Our numbering of the runners of the abacus means that the $p$-quotient does not depend on the choice of the number of beads on the abacus. Moreover, a partition is uniquely specified by its $p$-quotient together with the pyramid of the block in which it lies.

**Example.** Suppose $p = 5$ and $\lambda = (3^3, 2^3, 1^2)$. This has the following abacus display with 15 beads.



Moving the beads up to the highest possible positions, we obtain an abacus display for the 5-core (2), and we have $(q_0, q_1, q_2, q_3, q_4) = (14, 15, 17, 18, 21)$. So the runners are numbered $1, 4, 2, 3, 0$ from left to right, and the 5-quotient of $\lambda$ is $(\varnothing, (1), (1^2), \varnothing, \varnothing)$. The pyramid of the block $B$ containing $\lambda$ has ${}_0B_4 = {}_1B_4 = 1$, and ${}_iB_j = 0$ for all other $i, j$.

In this paper we shall be concerned with self-conjugate partitions, so we now discuss the effect of conjugation on abacus displays. Given an abacus display for $\lambda$, an abacus display for $\lambda'$ is obtained by rotating the abacus display by $180°$, and replacing each empty space with a bead and each bead with an empty space. This rotation reverses the order $\prec$, so that runner $i$ becomes runner $p-1-i$. So if $\lambda$ has $p$-quotient $(\lambda(0), \ldots, \lambda(p-1))$, the $p$-quotient of $\lambda'$ is $(\lambda(p-1)', \ldots, \lambda(0)')$. As a consequence, we see that a self-conjugate partition $\lambda$ must have $\lambda(i) = \lambda(p-1-i)'$ for each $i$. We can also relate the pyramids of the blocks $B, B^*$ containing $\lambda, \lambda'$: we have ${}_iB_j = {}_{p-1-j}{B^*}_{p-1-i}$ for each $i, j$.

2.4. **Scopes pairs.** Scopes [S] discovered important relationships between symmetric group blocks of the same weight. Suppose $B$ is a block of $\mathbb{F}\mathfrak{S}_n$ of weight $w$, and take an abacus display for $B$. Suppose runners $i$ and $j$ are adjacent, with runner $i$ to the left of runner $j$, and that there are $\kappa$ more beads on runner $j$ than on runner $i$. Then there is a block $A$ of $\mathbb{F}\mathfrak{S}_{n-\kappa}$ of weight $w$, with an abacus obtained from the given abacus for $B$ by interchanging these two runners. We say that $(A, B)$ is a *$[w : \kappa]$-pair*.

The two blocks in a $[w : \kappa]$-pair have many features in common. These have been discussed at length elsewhere, so we state just the results we need. First we compare the pyramids of $A$ and $B$; the proof of the following lemma is immediate from the definition.

**Lemma 2.1.** *Suppose $(A, B)$ is a $[w : \kappa]$-pair as above, and $0 \leqslant k < l \leqslant p-1$. Then*

$$
{}_kA_l = \begin{cases} {}_kB_l - 1 & (k = i,\ l = j), \\ {}_kB_l & (\textit{otherwise}). \end{cases}
$$

Now we quote a result about induction and restriction in $[w : \kappa]$-pairs, and use it to provide an inductive step for our proofs. Given a module lying in $B$, we write $M{\downarrow_A}$ for the projection of $M{\downarrow_{\mathfrak{S}_{n-\kappa}}}$ onto $A$; we define the notation ${\uparrow^B}$ similarly.

**Proposition 2.2.** *Suppose $(A, B)$ is a $[w : \kappa]$-pair as above. If $S$ is any simple module in $A$ or $B$, then $S{\uparrow^B}$ (or $S{\downarrow_A}$, respectively) has at least $\kappa!$ composition factors.*

*Proof.* This may be proved using similar arguments to those found in [S], or using Kleshchev's modular branching rules [BK]. □

**Proposition 2.3.** *Suppose $(A, B)$ is a $[w : \kappa]$-pair as above, and that $\lambda$ is a partition in $B$. Define the partition $\Phi(\lambda)$ in $A$ by swapping runners $i$ and $j$ of the abacus display. Then $\lambda$ and $\Phi(\lambda)$ have the same $p$-quotient, and if there are no beads on runner $i$ in the abacus display for $\lambda$ which have an empty space immediately to the right (on runner $j$), then $S^\lambda$ and $S^{\Phi(\lambda)}$ have the same number of composition factors.*

*Proof.* The assertion about quotients is straightforward, since the numbering of runners in the abacus for $A$ is the same as the numbering of runners in the abacus display for $B$ but with $i$ and $j$ interchanged. Now suppose the given condition on the abacus display for $\lambda$ holds. Then there must be exactly $\kappa$ beads on runner $j$ of this abacus display which have no bead immediately to the left, and so by the Branching Rule [J2, Theorem 9.3] we have

$$S^{\lambda}{\downarrow_A}\sim (S^{\Phi(\lambda)})^{\kappa!}.$$

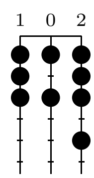
Similarly, we have

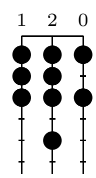
$$S^{\Phi(\lambda)}{\uparrow^B}\sim (S^{\lambda})^{\kappa!}.$$

Since induction and restriction and projection onto blocks are exact functors, Proposition 2.2 gives the result. $\square$

**Example.** Suppose $\mathbb{F}$ has characteristic 3, and $B$ is the weight 2 block of $\mathbb{F}\mathfrak{S}_{10}$ with 3-core $(3,1)$. Then $B$ has an abacus in which the numbers of beads on the runners are $3,2,4$ from left to right. Hence there is a block $A$ of $\mathbb{F}\mathfrak{S}_8$ possessing an abacus where the numbers of beads are $3,4,2$ from left to right; $A$ is the weight 2 block with 3-core $(2)$, and $(A,B)$ is a $[2:2]$-pair.

$B$ contains the partition $\lambda=(6,1^4)$, for which we have $\Phi(\lambda)=(5,1^3)$. We can see the relationship between these partitions from their abacus displays.

$(6,1^4) =$ 


$(5,1^3) =$ 


We have

$$S^{\lambda}{\downarrow_A}\sim (S^{\Phi(\lambda)})^2, \qquad S^{\Phi(\lambda)}{\uparrow^B}\sim (S^{\lambda})^2,$$

so that these two Specht modules have the same composition length.

2.5. **Rouquier blocks.** Suppose $B$ is a block of $\mathbb{F}\mathfrak{S}_n$ of weight $w$, and calculate the pyramid for $B$. We say that $B$ is *Rouquier* if ${}_iB_j\geqslant w-1$ for every $i<j$. Rouquier blocks have proved to be a useful tool in the study of the representation theory of $\mathfrak{S}_n$. One of the main advantages of working with Rouquier blocks is that their decomposition numbers are well understood. Suppose $B$ is a Rouquier block of $\mathbb{F}\mathfrak{S}_n$, and that $\mu$ is a partition in $B$. Then it is easy to check that $\mu$ is $p$-regular if and only if $\mu(0)=\varnothing$. Now suppose $\lambda$ and $\mu$ are partitions in $B$ with $\mu$ $p$-regular. Given any partitions $\alpha,\beta,\gamma$, let $c^{\alpha}_{\beta\gamma}$ be the corresponding Littlewood–Richardson coefficient, which we interpret as zero if $|\alpha|\neq|\beta|+|\gamma|$. Define

$$\delta_{\lambda\mu}=\sum\left(\prod_{i=0}^{p-1}c^{\lambda(i)}_{\tau(i)\sigma(i+1)}\prod_{j=1}^{p-1}c^{\mu(j)}_{\sigma(j)'\tau(j)}\right),$$

where the sum is over all choices of partitions $\sigma(1),\dots,\sigma(p-1),\tau(1),\dots,\tau(p-1)$, and we interpret $\tau(0)$ and $\sigma(p)$ as $\varnothing$.

Now recall the Weyl modules $\Delta^\lambda$ and simple modules $L^\lambda$ for the Schur algebra $\mathcal{S}_{\mathbb{F}}(n,n)$. If $\mu$ and $\nu$ are $p$-regular partitions in $B$, define

$$\epsilon_{\mu\nu} = \begin{cases} \prod_{i=1}^{p-1} \left[\Delta^{\mu(i)'} : L^{\nu(i)'}\right] & (\text{if } |\mu(i)| = |\nu(i)| \text{ for all } i) \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have the following result. This was proved by Turner [T], using earlier results for the decomposition numbers for Rouquier blocks of Iwahori–Hecke algebras due to Chuang and Tan [CT] and independently Leclerc and Miyachi [LM].

**Theorem 2.4.** *Suppose $B$ is a Rouquier block of $\mathbb{F}\mathfrak{S}_n$. If $\lambda$ and $\nu$ are partitions in $B$ with $\nu$ $p$-regular, then*

$$[S^\lambda : D^\nu] = \sum_\mu \delta_{\lambda\mu}\epsilon_{\mu\nu},$$

*summing over all $p$-regular partitions $\mu$ in $B$.*

In order to use this theorem, we shall need the following lemma; this was proved in [F2] with a rather narrower definition of Rouquier blocks.

**Lemma 2.5** ([F2, Lemma 3.1])**.** *Suppose $B$ is a block of $\mathbb{F}\mathfrak{S}_n$ of weight $w$. Then there is a sequence $n = n_0 < n_1 < \cdots < n_t$ of integers and a sequence $B = B_0, B_1, \dots, B_t$, where $B_i$ is a block of $\mathbb{F}\mathfrak{S}_{n_i}$ such that $(B_{i-1}, B_i)$ is a $[w : n_i - n_{i-1}]$-pair for each $i$, and $B_t$ is a Rouquier block.*

2.6. **$p$-regularisation.** In [J1], James proved a theorem which describes one composition factor of every Specht module $S^\lambda$, even when $\lambda$ is not $p$-regular. This is of particular interest in the study of irreducible Specht modules, since it tells us which simple module $D^\mu$ is isomorphic to a given irreducible Specht module.

Given a prime $p$ and $l > 0$, we define the $l$th *ladder* in $\mathbb{N} \times \mathbb{N}$ to be the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i + (p-1)j = l + p - 1\}.$$

Now given a partition $\lambda$, we define the *$p$-regularisation* $\lambda^{\mathrm{reg}}$ by finding the nodes of $\lambda$ contained in each ladder, and moving them to the highest possible positions in that ladder. It is a straightforward exercise to check that this gives the Young diagram of a $p$-regular partition.

**Example.** Suppose $\lambda = (6^2, 3, 2^3)$. Then the 2-regularisation of $\lambda$ is $(7, 6, 4, 3, 1)$. The Young diagrams of these two partitions, with nodes labelled according to the ladders containing them, are as follows.

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1&2&3&4&5&6\\ \hline
2&3&4&5&6&7\\ \hline
3&4&5\\ \cline{1-3}
4&5\\ \cline{1-2}
5&6\\ \cline{1-2}
6&7\\ \cline{1-2}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1&2&3&4&5&6&7\\ \hline
2&3&4&5&6&7\\ \cline{1-6}
3&4&5&6\\ \cline{1-4}
4&5&6\\ \cline{1-3}
5\\ \cline{1-1}
\end{array}$$

Now we have the following.

**Theorem 2.6** ([J1, Theorem A])**.** *Suppose $\mathbb{F}$ has prime characteristic $p$, and $\lambda$ is a partition. Then $[S^\lambda : D^{\lambda^{\mathrm{reg}}}] = 1$.*

2.7. **Dual modules and the sign representation.** If $M$ is an $\mathbb{F}\mathfrak{S}_n$-module, let $M^*$ denote the dual module. Let sgn denote the one-dimensional 'sign' representation of $\mathbb{F}\mathfrak{S}_n$.

**Theorem 2.7.**
(1) [J2, Theorem 11.5] *Every simple $\mathbb{F}\mathfrak{S}_n$-module is self-dual. In particular, if $\mathbb{F}$ has infinite characteristic then ${S^\lambda}^* \cong S^\lambda$ for every $\lambda$.*
(2) [J2, Theorem 8.15] *For any partition $\lambda$ and any field $\mathbb{F}$,*

$$S^\lambda \otimes \operatorname{sgn} \cong (S^{\lambda'})^*.$$

The functor $-\otimes \operatorname{sgn}$ is a self-equivalence of the category of $\mathbb{F}\mathfrak{S}_n$-modules, and this induces an involutory bijection on the set of blocks of $\mathbb{F}\mathfrak{S}_n$; let $B^*$ denote the image of $B$ under this map, and call $B^*$ the *conjugate* block to $B$. Theorem 2.7 implies that $B$ and $B^*$ have the same weight, and that the $p$-core of $B^*$ is the conjugate partition to the $p$-core of $B$.

2.8. **Irreducible Specht modules and Weyl modules.** The question of which ordinary irreducible representations of $\mathfrak{S}_n$ remain irreducible in characteristic $p$ amounts to classifying the irreducible Specht modules over a field of characteristic $p$. This was done for $p = 2$ by James and Mathas, and for odd $p$ by the author, building on work of James, Mathas and Lyle.

Suppose $\lambda$ is a partition and $p$ is a prime, and define the *$p$-power diagram* of $\lambda$ to be the diagram obtained by filling the $(i, j)$-box in the Young diagram of $\lambda$ with $\nu_p(h_\lambda(i, j))$, for each node $(i, j)$. (As usual, $\nu_p(h)$ denotes the largest power of $p$ dividing an integer $h$.)

For example, if $\lambda = (18, 11, 4, 1)$, then the 2-power diagram of $\lambda$ is as follows (here and throughout this paper we omit zeroes from $p$-power diagrams).

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | | | 1 | | 2 | | 1 | | | 1 | | 2 | | 1 | |
| | | 1 | | | 1 | | 2 | | 1 | | | | | | | | |
| | | 1 | | | | | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | | |

Now we say that $\lambda$ is a *$p$-JM partition* if the following property holds: whenever $(i, j)$ is a node of $\lambda$ with $\nu_p(h_\lambda(i, j)) > 0$, either all the entries in row $i$ of the $p$-power diagram are equal or all the entries in column $j$ are equal. It is easy to see that a $p$-JM partition is $p$-regular if and only if all the entries in any column of the $p$-power diagram are equal. Now we have the following theorem.

**Theorem 2.8** ([JM1, JM2, L, F1, F2])**.**
(1) *If $\mathbb{F}$ has characteristic 2, then the Specht module $S^\lambda$ is irreducible if and only if $\lambda$ is a 2-regular or 2-restricted 2-JM partition, or $\lambda = (2^2)$.*
(2) *If $\mathbb{F}$ has odd characteristic $p$, then the Specht module $S^\lambda$ is irreducible if and only if $\lambda$ is a $p$-JM partition.*

We also need the classification of irreducible Weyl modules for the Schur algebra $\mathcal{S}_\mathbb{F}(n, n)$.

**Theorem 2.9** ([JM1, Theorem 4.5])**.** *Let $\lambda$ be a partition of $n$, and suppose $\mathbb{F}$ has prime characteristic $p$. Then the Weyl module $\Delta^\lambda$ is irreducible if and only if $\lambda$ is a $p$-restricted $p$-JM partition.*

For example, the partition $\lambda = (18, 11, 4, 1)$ above is a 2-regular 2-JM partition, so if $\mathbb{F}$ has characteristic 2 the Specht modules $S^\lambda$ and $S^{\lambda'}$ and the Weyl module $\Delta^{\lambda'}$ are irreducible.

2.9. **Restriction from the symmetric group to the alternating group.** Finally, we address the alternating group $\mathfrak{A}_n$. Suppose $\mathbb{F}$ is a splitting field for $\mathfrak{A}_n$, and $M$ is an irreducible $\mathbb{F}\mathfrak{S}_n$-module. If the characteristic of $\mathbb{F}$ is not 2, then basic Clifford theory tells us that $M{\downarrow_{\mathfrak{A}_n}}$ is irreducible if and only if $M \otimes \mathrm{sgn} \ncong M$; otherwise $M{\downarrow_{\mathfrak{A}_n}}$ splits as a direct sum $M^+ \oplus M^-$ of two irreducible modules. Furthermore, all irreducible $\mathbb{F}\mathfrak{A}_n$-modules arise in this way. If the characteristic of $\mathbb{F}$ is infinite, then by Theorem 2.7, we have $S^\lambda \otimes \mathrm{sgn} \cong S^{\lambda'}$; so the irreducible character $\chi^\lambda$ restricts to an irreducible character $\psi^\lambda$ of the alternating group if $\lambda \neq \lambda'$ (and in this case $\chi^{\lambda'}$ also restricts to $\psi^\lambda$), while if $\lambda = \lambda'$, then $\chi^\lambda$ restricts to the sum of two irreducible characters $\psi^{\lambda+}, \psi^{\lambda-}$.

The situation in characteristic 2 is more complicated, and was settled by Benson [B]. If $\lambda$ is a 2-regular partition, then we call $\lambda$ an *S-partition* if for every $j \geqslant 1$ we have

- $\lambda_{2j-1} - \lambda_{2j} \leqslant 2$, and
- if $\lambda_{2j-1} - \lambda_{2j} = 2$, then $\lambda_{2j-1}$ and $\lambda_{2j}$ are odd.

Then we have the following.

**Theorem 2.10** ([B, Theorem 1.1] )**.** *Suppose $\mathbb{F}$ has characteristic 2 and is a splitting field for $\mathfrak{A}_n$, and let $\lambda$ be a 2-regular partition of $n$. Then $D^\lambda{\downarrow_{\mathfrak{A}_n}}$ splits as a direct sum of two non-isomorphic simple $\mathbb{F}\mathfrak{A}_n$-modules if $\lambda$ is an S-partition, and otherwise is irreducible.*

Armed with these results, we can reduce our main problem to a question purely concerning the representation theory of $\mathfrak{S}_n$. The following result is immediate from the fact that modular reduction and restriction are exact functors which commute.

**Proposition 2.11.** *Suppose $\mathbb{F}$ has prime characteristic $p$ and is a splitting field for $\mathfrak{A}_n$, and that $\psi$ is an irreducible character of the alternating group $\mathfrak{A}_n$, of the form $\psi^\lambda$ or $\psi^{\lambda\pm}$ for $\lambda$ a partition of $n$. Then the $p$-modular reduction of $\psi$ is irreducible if and only if one of the following happens:*

I. *$\lambda \neq \lambda'$, and $S^\lambda$ is an irreducible $\mathbb{F}\mathfrak{S}_n$-module which remains irreducible on restriction to $\mathbb{F}\mathfrak{A}_n$;*
II. *$\lambda = \lambda'$ and $S^\lambda$ is an irreducible $\mathbb{F}\mathfrak{S}_n$-module;*
III. *$\lambda = \lambda'$ and $S^\lambda$ has exactly two composition factors, which both remain irreducible on restriction to $\mathbb{F}\mathfrak{A}_n$.*

## 3. The main result in characteristic 2

Now we give our main result in characteristic 2.

**Theorem 3.1.** *Suppose $\psi = \psi^\lambda$ or $\psi^{\lambda\pm}$ is an irreducible character of the alternating group $\mathfrak{A}_n$. Then the 2-modular reduction of $\psi$ is irreducible if and only if the 2-modular reduction of $\chi^\lambda$ is irreducible, i.e. if and only if $\lambda$ is a 2-regular or 2-restricted 2-JM partition or $\lambda = (2^2)$.*

In order to prove this result, we need to examine conditions I–III of Proposition 2.11 in characteristic 2. This involves closer examination of S-partitions.

**Lemma 3.2.** *Suppose $\lambda$ is a 2-regular 2-JM partition, and $\lambda \neq \lambda'$. Then $\lambda$ is not an S-partition.*

*Proof.* Suppose $\lambda$ is a 2-regular 2-JM partition and an S-partition. We will show that $\lambda$ is self-conjugate.

We have $\lambda_{2j-1} - \lambda_{2j} \leqslant 2$ for each $j$. If $\lambda_{2j-1} - \lambda_{2j} = 2$ for some $j$, then $\lambda_{2j}$ is odd and hence at least 1, and we have

$$h_\lambda(2j-1, \lambda_{2j}) = 4, \qquad h_\lambda(2j, \lambda_{2j}) = 1,$$

contradicting the 2-JM condition. So in fact $\lambda_{2j-1} - \lambda_{2j} \leqslant 1$ for every $j$.

Now suppose $\lambda_{2j} - \lambda_{2j+1} \geqslant 2$ for some $j$. Then we have

$$h_\lambda(2j-1, \lambda_{2j}-1) = 4, \qquad h_\lambda(2j, \lambda_{2j}-1) = 2,$$

again contradicting the 2-JM condition.

We conclude that $\lambda_j - \lambda_{j+1} \leqslant 1$ for every $j$, i.e. $\lambda$ is 2-restricted. A partition which is both 2-regular and 2-restricted has the form $(r, r-1, \dots, 2, 1)$ for some $r$, and so is self-conjugate. □

**Lemma 3.3.** *Suppose $\lambda$ is a partition of $n$ with $\lambda = \lambda'$. Then the 2-regularisation of $\lambda$ is an S-partition.*

*Proof.* We prove this by induction on $n$, with the case $n = 0$ being trivial. If $n \geqslant 1$, then define a partition $\lambda^-$ as follows. Suppose $l$ is maximal such that ladder $l$ contains nodes of $\lambda$, and suppose in fact that ladder $l$ contains $i$ nodes of $\lambda$. If $i$ is odd, then the fact that $\lambda = \lambda'$ means that $l$ is odd and $\left(\frac{l+1}{2}, \frac{l+1}{2}\right)$ is a node of $\lambda$; in this case we define $\lambda^- = \lambda \setminus \left(\frac{l+1}{2}, \frac{l+1}{2}\right)$. If $i$ is even, then define $\lambda^-$ by removing the highest and lowest nodes of $\lambda$ lying in ladder $l$. In either case, $\lambda^-$ will be self-conjugate.

Let $\mu^-$ be the 2-regularisation of $\lambda^-$, and $\mu$ the 2-regularisation of $\lambda$. $\lambda^-$ is self-conjugate, so by induction $\mu^-$ is an S-partition. We consider how $\mu$ is obtained from $\mu^-$.

- If $i$ is odd, write $i = 2j-1$. $\mu$ is obtained from $\mu^-$ by adding the node $(2j-1, l+2-2j)$. In particular, $\mu^-_{2j-1} = l+1-2j$ is even, and so $\mu^-_{2j} \geqslant \mu^-_{2j-1} - 1$. So $\mu_{2j} \geqslant \mu_{2j-1} - 2$, with $\mu_{2j-1}$ being odd. The condition on $\mu_{2j'-1}$ and $\mu_{2j'}$ for any $j' \neq j$ follows from the corresponding conditions on $\mu^-$.
- If $i$ is even, write $i = 2j$. $\mu$ is obtained from $\mu^-$ by adding the nodes $(2j-1, l+2-2j)$ and $(2j, l+1-2j)$. Hence $\mu_{2j-1} - \mu_{2j} = 1$. The condition on $\mu_{2j'-1}$ and $\mu_{2j'}$ for any $j' \neq j$ follows from the corresponding conditions on $\mu^-$. □

Now we can prove our main result.

*Proof of Theorem* 3.1. Suppose $\mathbb{F}$ is a splitting field for $\mathfrak{A}_n$ of characteristic 2, and consider possibilities I–III of Proposition 2.11. Suppose that $\lambda$ is either $(2^2)$ or a 2-regular or 2-restricted 2-JM partition. If $\lambda = \lambda'$, then condition II of Proposition 2.11 is satisfied. If not, then $S^\lambda \cong D^{\lambda^{\mathrm{reg}}}$, and the 2-regularisation $\lambda^{\mathrm{reg}}$ is either $\lambda$ (if $\lambda$ is 2-regular) or $\lambda'$ (if $\lambda$ is 2-restricted). In either case, $\lambda^{\mathrm{reg}}$ is a non-self-conjugate 2-regular 2-JM partition; by Lemma 3.2 $\lambda$ is not an S-partition, and so by Theorem 2.10 condition I is satisfied.

Conversely, suppose one of the conditions I–III holds for $\lambda$. If condition I or condition II holds, then certainly $\lambda$ is as claimed, so all we need to do is show that condition III cannot hold. If $\lambda = \lambda'$, then by Lemma 3.3 $\lambda^{\mathrm{reg}}$ is an S-partition. So the composition factor $D^{\lambda^{\mathrm{reg}}}$ of $S^\lambda$ splits on restriction to the alternating group, and condition III fails. □

## 4. The case of odd characteristic

4.1. **The main conjecture.** Now we consider the case where $\mathbb{F}$ has odd characteristic. We begin by refining the conditions of Proposition 2.11.

**Theorem 4.1.** *Suppose $\mathbb{F}$ is a splitting field for $\mathfrak{A}_n$ of odd characteristic $p$, and $\psi = \psi^\lambda$ or $\psi^{\lambda\pm}$ is an irreducible character of the alternating group $\mathfrak{A}_n$. Then the $p$-modular reduction of $\psi$ is irreducible if and only if one of the following holds.*

A. *$\lambda$ is a $p$-JM partition.*
B. *$\lambda = \lambda'$ and the Specht module $S^\lambda$ has exactly two composition factors.*

*Proof.* We must relate conditions A and B to conditions I–III of Proposition 2.11. Suppose first that condition A holds. If $\lambda = \lambda'$, then condition II holds. If $\lambda \neq \lambda'$, then we must show that condition I holds, i.e. $S^\lambda \ncong S^\lambda \otimes \mathrm{sgn}$. Assuming otherwise, we have

$$S^{\lambda'} \cong (S^\lambda \otimes \mathrm{sgn})^* \cong (S^\lambda)^* \cong S^\lambda,$$

using Theorem 2.7 and the fact that $S^\lambda$ is irreducible. But it is well known that in odd characteristic there are no isomorphisms between distinct Specht modules (this is an easy consequence of [J2, Corollary 13.17]), and we have a contradiction. So condition I holds.

Obviously, conditions I and II each imply condition A, so condition A is equivalent to conditions I and II.

Clearly, condition III implies condition B, so we must show that the converse is true. If condition B holds, suppose that $D^\mu$ and $D^\xi$ are the composition factors of $S^\lambda$; we must show that these simple modules do not split on restriction to the alternating group, i.e. $D^\mu \ncong D^\mu \otimes \mathrm{sgn}$ and $D^\xi \ncong D^\xi \otimes \mathrm{sgn}$. Since $S^\lambda$ has exactly one composition factor isomorphic to $D^{\lambda^{\mathrm{reg}}}$, we know that $D^\mu \ncong D^\xi$. Furthermore, $S^\lambda$ is indecomposable [J2, Corollary 13.18], so is a non-split extension of (say) $D^\mu$ by $D^\xi$. Theorem 2.7 tells us that $S^\lambda \cong (S^\lambda \otimes \mathrm{sgn})^*$, and so $S^\lambda$ is also a non-split extension of $D^\xi \otimes \mathrm{sgn}$ by $D^\mu \otimes \mathrm{sgn}$. Hence

$$D^\xi \otimes \mathrm{sgn} \cong D^\mu \ncong D^\xi, \qquad D^\mu \otimes \mathrm{sgn} \cong D^\xi \ncong D^\mu.$$

So condition III is equivalent to condition B, and the theorem is proved. □

In order to solve our main problem, therefore, it remains to classify the self-conjugate partitions $\lambda$ such that $S^\lambda$ has exactly two composition factors, when $\mathbb{F}$ is a field of odd characteristic. We now describe a conjectured classification, analogous to the description of $p$-JM partitions in terms of $p$-power diagrams.

**Definition.** Suppose $p$ is an odd prime and $\lambda$ is a partition. We say that a partition $\lambda$ is an *R-partition* if

- $\lambda = \lambda'$,
- $\lambda$ is not a $p$-JM partition, and
- there is a node $\mathfrak{n}$ of $\lambda$ such that for any node $(i,j) \neq \mathfrak{n}$ of $\lambda$ for which $\nu_p(h_\lambda(i,j)) > 0$, either all the entries in row $i$ of the $p$-power diagram for $\lambda$ are equal, or all the entries in column $j$ are equal.

Informally, $\lambda$ is an R-partition if it self-conjugate and satisfies the conditions of being a $p$-JM partition except at one node. Since $\lambda$ is not a $p$-JM partition, this node is unique; we call it the *distinguished node* of $\lambda$.

**Example.** Suppose $p = 3$, and $\lambda = (9,7,4,3,2^3,1^2)$. Then $\lambda$ has the following 3-power diagram.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | | 1 | | | | |
| | | 1 | | | | | | |
| | | | | | | | | |
| 1 | 1 | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |

So $\lambda$ is an R-partition, with distinguished node $(3,3)$.

Although the definition of an R-partition depends on the prime $p$, the latter will always be clear from the context. Now we can state our conjecture.

**Conjecture 4.2.** *Suppose $\lambda$ is a self-conjugate partition.*

(1) *If* $\mathrm{char}(\mathbb{F}) = 3$*, then $S^\lambda$ has exactly two composition factors if and only if one of the following holds:*
- $\lambda = (r, 1^{r-1})$ *or* $(r, 2, 1^{r-2})$ *for some* $r \equiv 2 \pmod 3$*;*
- $\lambda$ *has weight* $1$*;*
- $\lambda = (3^3)$.

(2) *If* $\mathrm{char}(\mathbb{F}) = p \geqslant 5$*, then $S^\lambda$ has exactly two composition factors if and only if $\lambda$ is an R-partition.*

*Remarks.*

(1) Note that the partitions described in the first two cases for $p = 3$ are R-partitions. However, there are other R-partitions when $p = 3$ (for example, the partition $(9,7,4,3,2^3,1^2)$ above) which yield Specht modules with more than two composition factors in characteristic 3. The reason for the difference in characteristic 3 will be seen later in the discussion of Rouquier blocks, and is in some sense analogous to the way 2-JM partitions can yield reducible Specht modules in characteristic 2.

(2) We have been able to verify Conjecture 4.2 for blocks of weight at most 3 in all characteristics, and also for blocks of weight at most 5 when $p = 5$, and blocks of weight at most 6 when $p = 3$. We hope that the reader will find this evidence compelling. It is worth while noting that, according

to our conjecture and the analysis of R-partitions below, a Specht module corresponding to a self-conjugate partition can have composition length 2 only if it lies in a block of odd weight. In fact, we conjecture that a Specht module corresponding to a self-conjugate partition of even weight has a composition factor $D^\mu$ such that $D^\mu \otimes \mathrm{sgn} \cong D^\mu$; this statement (which is trivial for blocks of weight 0 and straightforward to verify for blocks of weight 2) would imply that Conjecture 4.2 holds for partitions with even weight.

The purpose of the rest of this paper is to prove the 'if' part of Conjecture 4.2.

4.2. **Analysis of R-partitions.** In order to prove our conjecture in one direction, we must examine R-partitions in greater detail. We assume from now on that $p$ is odd.

**Lemma 4.3.** *Suppose $\lambda$ is an R-partition with distinguished node $\mathfrak{n}$. Then:*

(1) *$\mathfrak{n}$ is of the form $(i,i)$ for some $i$;*
(2) *$\nu_p(h_\lambda(j,j)) = 0$ for all nodes $(j,j)$ of $\lambda$ with $j \neq i$;*
(3) *if $(j,k)$ is a node of $\lambda$ with $j < k$ and $\nu_p(h_\lambda(j,k)) > 0$, then all the entries in column $k$ of the $p$-power diagram are equal;*
(4) *if $(j,k)$ is a node of $\lambda$ with $j > k$ and $\nu_p(h_\lambda(j,k)) > 0$, then all the entries in row $j$ of the $p$-power diagram are equal.*

*Proof.*

(1) Write $\mathfrak{n} = (i,j)$. Since $\lambda = \lambda'$, the node $\mathfrak{n}' = (j,i)$ also satisfies the conditions of the definition. But $\mathfrak{n}$ is unique, so $i = j$.
(2) Suppose $\nu_p(h_\lambda(j,j)) > 0$. Then either all the entries in row $j$ of the $p$-power diagram are equal, or all the entries in column $j$ are equal. But since $\lambda$ is self-conjugate, these properties are equivalent, so they both hold. In particular, $p$ divides $h_\lambda(j,\lambda_j)$, so $h_\lambda(j,\lambda_j) \geqslant p$, which implies that $\lambda_j = \lambda_{j+1}$. But then $h_\lambda(j,j)$ and $h_\lambda(j+1,j)$ differ by 1, and so both cannot be divisible by $p$; a contradiction.
(3) We know that either the entries in column $k$ of the diagram are all equal, or the entries in row $j$ are all equal. If the latter holds, then we have $\nu_p(h_\lambda(j,j)) > 0$, and so by (2) we have $j = i$. But then all the entries in row $i$ are equal, and $\lambda$ is a $p$-JM partition; a contradiction.
(4) This follows from (3), given that $\lambda$ is self-conjugate. □

In order to give a better description of R-partitions, we examine what happens to an R-partition $\lambda$ when we remove the rim hook $\hbar$ corresponding to the distinguished node. $\hbar$ has length $ap$ for some $a$, and (since $\lambda$ is self-conjugate) is symmetric about the line $j = k$; that is, whenever $\hbar$ contains a node $(j,k)$ it also contains the node $(k,j)$. $\hbar$ contains a unique node of the form $(j,j)$ (namely, the unique such node in the rim of $\lambda$), and hence contains an odd number of nodes altogether. So $a$ is odd; $\lambda$ can take different forms according to whether $a > 1$ or $a = 1$, and we consider the case $a > 1$ first.

**Proposition 4.4.** *Suppose $\lambda$ is a R-partition, with distinguished node $\mathfrak{n}$ and corresponding rim hook $\hbar$ of length $ap$. If $a > 1$, then $\mathfrak{n} = (1,1)$ and $\lambda_2 \leqslant \dfrac{p+1}{2}$.*

*Proof.* We certainly have $\mathfrak{n} = (i,i)$ for some $i$. Write $a = 2b+1$, and number the nodes of $\hbar$ as $\mathfrak{n}_1, \dots, \mathfrak{n}_{ap}$ from top right to bottom left. Then $\mathfrak{n}_1 = (i, \lambda_i) = \left(i, i + bp + \frac{p-1}{2}\right)$. Write $\mathfrak{n}_{bp}$ as $(j,k)$; then $k - j = \frac{p+1}{2}$, and $j \geqslant i$.

Suppose that $(j+1,k)$ is a node of $\lambda$. Then $\lambda_{j+1} \geqslant k$; since $(j,k)$ is contained in the rim of $\lambda$, $(j+1,k+1)$ is not a node of $\lambda$, and so in fact $\lambda_{j+1} = k$. Hence

$$\begin{aligned} h_\lambda(j+1,j+1) &= \lambda_{j+1} - (j+1) + \lambda'_{j+1} - (j+1) + 1 \\ &= 2k - 2j - 1 \\ &= p. \end{aligned}$$

Since $j+1 > i$, this contradicts Lemma 4.3(2).

So $(j+1,k)$ is not a node of $\lambda$, and we therefore have $\lambda'_k = j$. This gives

$$\begin{aligned} h_\lambda(i,k) &= \lambda_i - i + \lambda'_k - k + 1 \\ &= \left(i + bp + \frac{p-1}{2}\right) - i + j - k + 1 \\ &= bp. \end{aligned}$$

So $\nu_p(h_\lambda(i,k)) > 0$, and hence by Lemma 4.3(3) we have $p \mid h_\lambda(l,k)$ for every $1 \leqslant l \leqslant j$. So we have

$$p \mid \lambda_l - l + \lambda'_k - k + 1 = \lambda_l - l + j - k + 1$$

for each such $l$, and hence

$$h_\lambda(l,l) \equiv 2(\lambda_l - l) + 1 \equiv 2(k - j - 1) + 1 \equiv 0 \pmod{p}.$$

Now Lemma 4.3(2) gives $l = i$ for every $1 \leqslant l \leqslant j$; in other words, $i = j = 1$. So $k = \frac{p+3}{2}$ and $\lambda'_k = j = 1$, which implies $\lambda_2 \leqslant \frac{p+1}{2}$. □

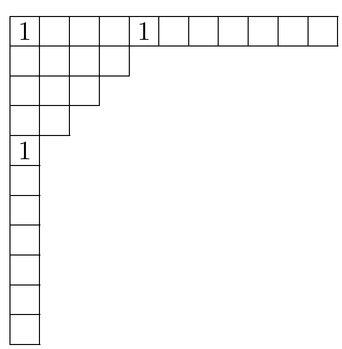
From Proposition 4.4, we see that if $a > 1$ and we remove the rim hook $\hbar$ from $\lambda$, we are left with the partition $\xi$ given by

$$\xi_j = \max\{\lambda_{j+1} - 1, 0\}$$

for all $j$. $\xi$ is clearly self-conjugate and satisfies $\xi_1 \leqslant \frac{p-1}{2}$. This in turn implies that $\xi$ is a $p$-core, since the largest hook length in $\xi$ is

$$h_\xi(1,1) = 2\xi_1 - 1 < p.$$

**Example.** Suppose $p = 7$ and $\lambda = (11,4,3,2,1^7)$. This has the following 7-power diagram.




So $\lambda$ is an R-partition. The rim hook corresponding to the distinguished node $(1,1)$ has length 21, and when we remove it, we obtain $\xi = (3,2,1)$, which is a 7-core.

Now we prove a converse to Proposition 4.4.

**Proposition 4.5.** *Suppose $\xi$ is a self-conjugate partition with $\xi_1 \leqslant \frac{p-1}{2}$. Suppose $\lambda$ is a self-conjugate partition obtained from $\xi$ by adding a rim ap-hook to $\xi$, for some integer $a$. Then $\lambda$ is an R-partition.*

*Proof.* Let $\hbar$ be the rim hook which is added to $\xi$ to obtain $\lambda$, and let $(j,k)$ be the hand node of $\hbar$. Since $\lambda$ and $\xi$ are both self-conjugate, the foot node of $\hbar$ must be $(k,j)$. So we have

$$k - j = j - k + (ap-1),$$

which gives

$$k - j = \frac{ap-1}{2}.$$

This implies that $a$ is odd, and it also gives $k \geqslant \frac{ap+1}{2}$. We claim that, in fact, $(j,k) = \left(1, \frac{ap+1}{2}\right)$. If not, then the node $(j-1,k)$ is a node of $\lambda$. $(j-1,k)$ cannot be a node of $\hbar$ since $(j,k)$ is the hand node, and so $(j-1,k) \in \xi$. This gives $\xi_{j-1} \geqslant k \geqslant \frac{ap+1}{2} \geqslant \frac{p+1}{2}$; but by assumption $\xi_{j-1} \leqslant \xi_1 \leqslant \frac{p-1}{2}$; a contradiction.

So the hand node of $\hbar$ is $\left(1, \frac{ap+1}{2}\right)$ and the foot node is $\left(\frac{ap+1}{2}, 1\right)$. We therefore have

$$\lambda_i = \begin{cases} \dfrac{ap+1}{2} & (i=1), \\ \xi_{i-1}+1 & \left(2 \leqslant i \leqslant \frac{p+1}{2}\right), \\ 1 & \left(\frac{p+3}{2} \leqslant i \leqslant \frac{ap+1}{2}\right), \\ 0 & \left(\frac{ap+3}{2} \leqslant i\right). \end{cases}$$

So if $i \geqslant \frac{p+3}{2}$, then the $i$th column of $\lambda$ and the $i$th row of $\lambda$ each have length at most 1, so the values in the $p$-power diagram are automatically constant down the $i$th column and along the $i$th row. We have $h_\lambda(1,1) = ap$, so in order to show that $\lambda$ is an R-partition with distinguished node $(1,1)$, it suffices to show that whenever $(i,l)$ is a node of $\lambda$ with $i, l \leqslant \frac{p+1}{2}$ and $(i,l) \neq (1,1)$, we have $p \nmid h_\lambda(i,l)$. We assume that $i \leqslant l$; the case $i > l$ will then follow from the fact that $\lambda$ is self-conjugate.

If $2 \leqslant i \leqslant l \leqslant \frac{p+1}{2}$, then

$$\begin{aligned} h_\lambda(i,l) &= \lambda_i - i + \lambda_l - l + 1 \\ &= \xi_{i-1} - (i-1) + \xi_{l-1} - (l-1) + 1 \\ &= h_\xi(i-1, l-1), \end{aligned}$$

and this is not divisible by $p$, since $\xi$ is a $p$-core.

If $i = 1$ and $2 \leqslant l \leqslant \frac{p+1}{2}$, then

$$h_\lambda(i,l) = \lambda_1 + \lambda_l - l = \frac{ap+1}{2} + \xi_{l-1} + 1 - l,$$

and the inequalities $2 \leqslant l \leqslant \frac{p+1}{2}$ and $0 \leqslant \xi_{l-1} \leqslant \frac{p-1}{2}$ imply that this lies strictly between $\frac{a-1}{2}p$ and $\frac{a+1}{2}p$, and therefore cannot be divisible by $p$. □

Now we examine the case where the rim hook corresponding to the distinguished node of an R-partition has length $p$.

**Proposition 4.6.** *Suppose $\lambda$ is an R-partition, and let $\xi$ be the partition obtained by removing the rim hook $\hbar$ corresponding to the distinguished node of $\lambda$. If $\hbar$ contains exactly $p$ nodes, then $\xi$ is a self-conjugate $p$-JM partition with the property that $\nu_p(h_\xi(j,k)) = 0$ whenever $(j,k)$ is a node of $\xi$ with $|j-k| \leqslant \frac{p-1}{2}$.*

*Proof.* Since $\hbar$ has length $p$, the hand and foot nodes of $\hbar$ must be $\big(i, i+\frac{p-1}{2}\big)$ and $\big(i+\frac{p-1}{2}, i\big)$ respectively, where $(i,i)$ is the distinguished node of $\lambda$. Hence $\xi_l = \lambda_l$ whenever $l \geqslant i+\frac{p+1}{2}$, and so $h_\xi(j,k) = h_\lambda(j,k)$ whenever $(j,k)$ is a node of $\xi$ with $j$ or $k$ at least $i+\frac{p+1}{2}$. So for $j \geqslant i+\frac{p+1}{2}$ the entries in column $j$ or row $j$ of the $p$-power diagram for $\xi$ are constant (since they are constant in the diagram for $\lambda$), and so to prove the proposition it suffices to show that $\nu_p(h_\xi(j,k)) = 0$ whenever $(j,k)$ is a node of $\xi$ with $j,k \leqslant i+\frac{p-3}{2}$; this automatically includes all nodes of $\xi$ with $|j-k| \leqslant \frac{p-1}{2}$, since for $k \geqslant i+\frac{p-1}{2}$ we have $\xi_k \leqslant i-1$.

So suppose $(j,k)$ is a node of $\xi$ with $j,k \leqslant i+\frac{p-3}{2}$. If $j,k < i$, then we have $h_\xi(j,k) = h_\lambda(j,k)$, and this is not divisible by $p$, since if it were, then either $h_\lambda(j,j)$ or $h_\lambda(k,k)$ would be also, contradicting Lemma 4.3(2). So we need only consider the case where at least one of $j$ and $k$ lies between $i$ and $i+\frac{p-3}{2}$. By replacing $(j,k)$ with $(k,j)$ if necessary, we may assume that $j \leqslant k$, and hence that $i \leqslant k \leqslant i+\frac{p-3}{2}$. Now we have $\lambda_{k+1} = \xi_k + 1$, which gives $h_\xi(j,k) = h_\lambda(j,k+1)$. If this is divisible by $p$, then by Lemma 4.3(3) the entries in column $k+1$ of the $p$-power diagram for $\lambda$ are constant; in particular, $\nu_p(h_\lambda(i,k+1)) > 0$. But the $(i,k+1)$ rim hook of $\lambda$ is a proper subset of the $(i,k)$ rim hook of $\lambda$, so has length strictly less than $p$; a contradiction. □

We also prove a converse to this result. First we state a simple lemma concerning self-conjugate $p$-JM partitions; this is analogous to Lemma 4.3, and is proved in exactly the same way.

**Lemma 4.7.** *Suppose $\xi$ is a self-conjugate $p$-JM partition.*

(1) *If $(j,j)$ is a node of $\xi$, then $\nu_p(h_\xi(j,j)) = 0$.*
(2) *If $(j,k)$ is a node of $\xi$ with $\nu_p(h_\xi(j,k)) > 0$ and $j \leqslant k$, then all the entries in the $k$th column of the $p$-power diagram for $\xi$ are equal.*
(3) *If $(j,k)$ is a node of $\xi$ with $\nu_p(h_\xi(j,k)) > 0$ and $j \geqslant k$, then all the entries in the $j$th row of the $p$-power diagram for $\xi$ are equal.*

**Proposition 4.8.** *Suppose $\xi$ is a self-conjugate $p$-JM partition with the property that $\nu_p(h_\xi(j,k)) = 0$ whenever $(j,k)$ is a node of $\xi$ with $|j-k| \leqslant \dfrac{p-1}{2}$. Suppose $\lambda$ is a self-conjugate partition obtained by adding a rim hook of length $p$ to $\xi$. Then $\lambda$ is an R-partition.*

*Proof.* Let $\hbar$ be the rim hook added to $\xi$ to obtain $\lambda$, and write $(i,k)$ for the hand node of $\hbar$. The foot node must then be $(k,i)$, and so we have

$$k - i = i - k + p - 1,$$

so that $k = i + \frac{p-1}{2}$.

So $\lambda_j = \xi_j$ whenever $j \geqslant i+\frac{p+1}{2}$, and hence $h_\lambda(j,l) = h_\xi(j,l)$ whenever $(j,l)$ is a node of $\lambda$ with $j$ or $l$ at least $i+\frac{p+1}{2}$. If $j \geqslant i+\frac{p+1}{2}$, then $\xi_j < j$, so the entries in row $j$ and column $j$ of the $p$-power diagram for $\xi$ are constant (by Lemma 4.7) and so the entries in row $j$ and column $j$ of the diagram for $\lambda$ are constant.

It remains to show that $\nu_p(h_\lambda(j,l)) = 0$ whenever $j,l \leqslant i+\frac{p-1}{2}$ and $(j,l) \neq (i,i)$; since $h_\lambda(i,i) = p$, this will imply that $\lambda$ is an R-partition with distinguished node $(i,i)$. We assume, without loss of generality, that $j \leqslant l$.

If $j \leqslant l < i$, then we have $\lambda_j = \xi_j$ and $\lambda_l = \xi_l$, so that $h_\lambda(j,l) = h_\xi(j,l)$. This cannot be divisible by $p$, since if it were, then either $h_\xi(j,j)$ or $h_\xi(l,l)$ would be as well, contradicting Lemma 4.7.

If $i \leqslant j \leqslant l \leqslant i + \frac{p-1}{2}$, then we have $\lambda_j, \lambda_l \leqslant \lambda_i = i + \frac{p-1}{2}$, so that

$$\begin{aligned} h_\lambda(j,l) &= \lambda_j + \lambda_l - (j+l) + 1 \\ &\leqslant 2\big(i + \frac{p-1}{2}\big) - (2i+1) + 1 \\ &= p - 1, \end{aligned}$$

so $h_\lambda(j,l)$ cannot be divisible by $p$.

Next, we suppose that $j < l = i$. Then we have $\lambda_j = \xi_j$ and $\lambda_l = \lambda_i = i + \frac{p-1}{2}$. So

$$h_\lambda(j,l) = \xi_j - j + \frac{p+1}{2};$$

if this is divisible by $p$, then so is

$$2(\xi_j - j) + 1 = h_\xi(j,j);$$

a contradiction.

Finally, we consider the case where $j < i < l \leqslant i + \frac{p-1}{2}$. In this case we have

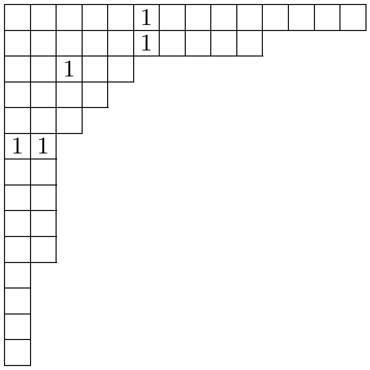
$$\begin{aligned} h_\lambda(j,l) &= \lambda_j - j + \lambda_l - l + 1 \\ &= \xi_j - j + (\xi_{l-1} + 1) - l + 1 \\ &= h_\xi(j, l-1). \end{aligned}$$

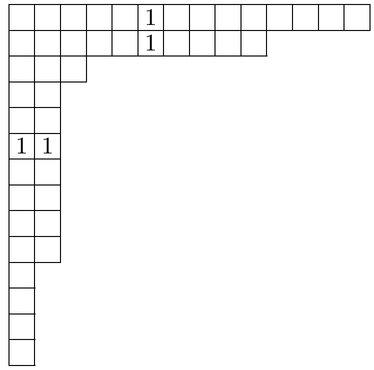
If this is divisible by $p$, then by Lemma 4.7 the entries in column $l-1$ of the $p$-power diagram for $\xi$ are constant, and, in particular, $h_\xi(i-1, l-1)$ is divisible by $p$. But $|(i-1) - (l-1)| \leqslant \frac{p-1}{2}$, and this contradicts the assumption on $\xi$. $\square$

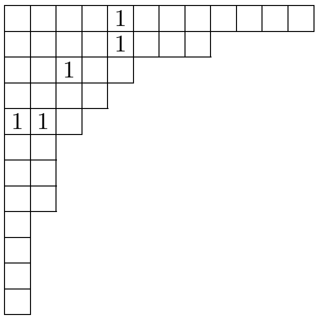
**Example.** Suppose $p = 5$, and consider the partitions

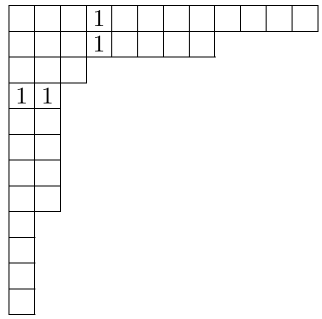
$$\lambda = (14, 10, 5, 4, 3, 2^5, 1^4), \qquad \mu = (12, 8, 5, 4, 3, 2^3, 1^4).$$

We have $h_\lambda(3,3) = h_\mu(3,3) = 5$, and when we remove the $(3,3)$-rim hook the resulting partition $\xi$ is a self-conjugate 5-JM partition, with $\nu_5(h_\xi(j,k)) = 0$ when $|j-k| \leqslant 2$. When we remove the $(3,3)$-rim hook from $\mu$, the resulting partition $\pi$ is a self-conjugate 5-JM partition, but has $\nu_5(h_\pi(2,4)) > 0$. As a consequence, $\lambda$ is an R-partition, but $\mu$ is not. We illustrate this with the 5-power diagrams of these partitions.

$\lambda =$ 

$\xi =$ 

$\mu =$ 

$\pi =$ 

We can summarise our analysis of R-partitions in the following way. R-partitions are of two distinct types:

I. self-conjugate partitions obtained by adding a rim $ap$-hook, for some odd $a$, to a self-conjugate partition $\xi$ with $\xi_1 \leqslant \frac{p-1}{2}$;

II. self-conjugate partitions obtained by adding a rim $p$-hook to a self-conjugate $p$-JM partition $\xi$ such that $\nu_p(h_\xi(j,k)) = 0$ whenever $|j-k| \leqslant \frac{p+1}{2}$.

Note that these types are not mutually exclusive: for Type I, we do not require $a > 1$, and clearly an R-partition of Type I with $a = 1$ is also an R-partition of Type II.

When $p = 3$, type I partitions are precisely those listed in the first case of Conjecture 4.2(1).

4.3. **R-partitions on the abacus.** As in [F2], we prove our main result by considering partitions on the abacus. We begin with a description of the abacus display for $p$-JM partitions, which was given in a slightly different form in [F2].

**Proposition 4.9** ([F2, Proposition 2.1])**.** *Let $p = \mathrm{char}(\mathbb{F})$. Suppose $\lambda$ is a partition lying in a block $B$ of $\mathbb{F}\mathfrak{S}_n$, and take an abacus display for $\lambda$. Then $\lambda$ is a $p$-JM partition if and only if:*

(1) $\lambda(k) = \varnothing$ *for* $k \neq 0, p-1$;
(2) $\lambda(p-1)$ *is a $p$-regular $p$-JM partition;*
(3) $\lambda(0)$ *is a $p$-restricted $p$-JM partition; and*
(4) $\lambda(k)_1 + \lambda(l)'_1 \leqslant {}_kB_l + 1$ *for all* $k < l$.

We seek a similar description for R-partitions. Recalling the discussion from §2.3 of the effect of conjugation on abacus displays, we see that if we take an abacus display for a self-conjugate partition in which the number of beads is a multiple of $p$, then

- runner $\frac{p-1}{2}$ is the middle runner, and contains exactly $c$ beads, and
- for any $i$, the $i$th runner from the left and the $i$th runner from the right have numbers summing to $p-1$, and contain exactly $2c$ beads between them.

In fact, it will be more convenient for us to use an abacus in which the number of beads is $cp+\frac{p+1}{2}$, for an integer $c$. Then the above statements imply the following lemma.

**Lemma 4.10.** *Suppose $\lambda$ is a self-conjugate partition, and take an abacus display for $\lambda$ with $cp+\dfrac{p+1}{2}$ beads, for an integer $c$. Then:*

- *runner $\frac{p-1}{2}$ is the leftmost runner, and contains exactly $c+1$ beads;*
- *for any $i$, the $i$th runner from the right and the $(i+1)$th runner from the left will have numbers summing to $p-1$, and contain exactly $2c+1$ beads between them.*

Now we can describe type I R-partitions on the abacus.

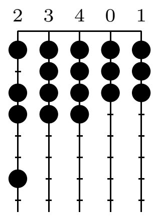
**Proposition 4.11.** *Suppose $\lambda$ is a self-conjugate partition, and take an abacus display for $\lambda$ with $cp+\dfrac{p+1}{2}$ beads. Then $\lambda$ is an R-partition of type I if and only if:*

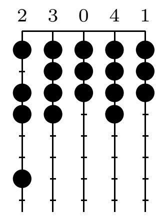
(1) *on any runner, the number of beads is either $c$ or $c+1$;*
(2) *$\lambda\left(\frac{p-1}{2}\right)$ has the form $(b+1,1^b)$ for some $b$; and*
(3) *$\lambda(i)=\varnothing$ for all $i\neq\frac{p-1}{2}$.*

*Proof.* We prove the 'only if' part; the 'if' part is very similar. $\lambda$ is obtained by adding a rim $ap$-hook to a self-conjugate partition $\xi$ with $\xi_1\leqslant\frac{p-1}{2}$. $\xi$ has the same $p$-core as $\lambda$ (in fact, $\xi$ is the $p$-core of $\lambda$), so property (1) will hold for $\lambda$ if and only if it holds for $\xi$. Consider the abacus display for $\xi$, which is obtained by sliding all the beads as far up their runners as they will go. The condition $\xi_1\leqslant\frac{p-1}{2}$ implies that the last bead in the abacus display occurs at or before position $cp+p-1$. In particular, the number of beads on any runner is at most $c+1$. Now Lemma 4.10 implies that the number of beads on any runner is either $c$ or $c+1$.

$\lambda$ is obtained from $\xi$ by adding a single $ap$-hook; this corresponds to moving a single bead to a lower position on the same runner. If this runner is runner $j$, then we have $\lambda(i)=\varnothing$ for $i\neq j$. The fact that $\lambda$ is self-conjugate implies that $\lambda(p-1-j)=\lambda(j)'$, and so we must have $j=\frac{p-1}{2}$, and $\lambda(j)=\lambda(j)'$. Viewing runner $j$ as an abacus with one runner, we see that $\lambda(j)$ is obtained from $\xi(j)=\varnothing$ by adding a single rim $a$-hook, which means that $\lambda(j)$ is a hook partition, and hence $\lambda(j)=\left(\frac{a+1}{2},1^{(a-1)/2}\right)$. □

**Example.** If $p = 5$ and $a$ is odd, then there are exactly four R-partitions of type I and weight $a$. For example, if $a = 5$, these are

$(13, 1^{12}) =$ 

, $(13, 2, 1^{11}) =$ 

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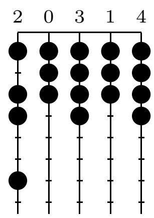
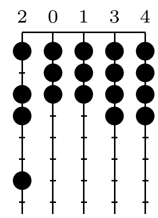
$(13, 3, 2, 1^{10}) =$ 

, $(13, 3^2, 1^{10}) =$ 

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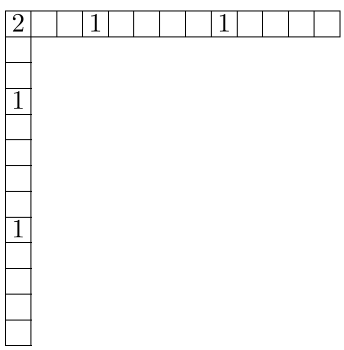
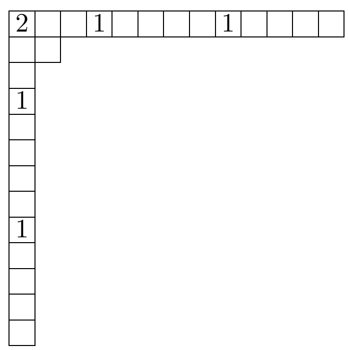
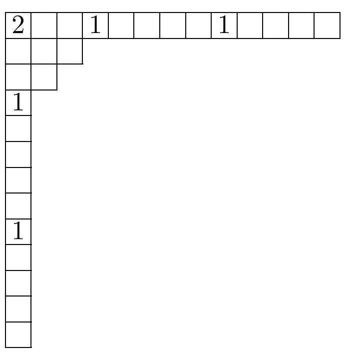
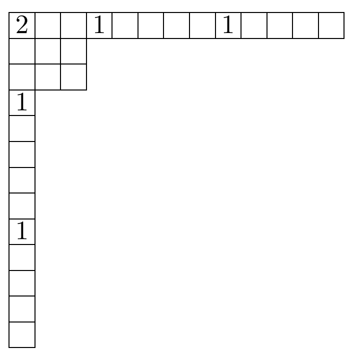
and the 5-power diagrams of these partitions are as follows.




Now we turn to type II partitions.

**Proposition 4.12.** *Suppose $\lambda$ is a self-conjugate partition lying in a block $B$. Then $\lambda$ is an R-partition of type II if and only if:*

(1) $\lambda(p-1)$ *is a $p$-regular $p$-JM partition;*
(2) $\lambda(0)$ *is a $p$-restricted $p$-JM partition;*

(3) $\lambda\left(\frac{p-1}{2}\right) = (1)$;
(4) $\lambda(k) = \varnothing$ *for* $k \notin \left\{0, \frac{p-1}{2}, p-1\right\}$;
(5) $\lambda(k)_1 + \lambda(l)'_1 \leqslant {}_kB_l + 1$ *for all* $k < l$.

*Proof.* We prove the 'only if' part; the converse is similar. $\lambda$ is obtained by adding a rim $p$-hook to a self-conjugate $p$-JM partition $\xi$. Thus $\lambda(k) = \xi(k)$ except for one value of $k$; since $\lambda$ is self-conjugate, this value must be $\frac{p-1}{2}$. So $\lambda\left(\frac{p-1}{2}\right)$ is obtained by adding a rim 1-hook to $\xi\left(\frac{p-1}{2}\right) = \varnothing$, i.e. $\lambda\left(\frac{p-1}{2}\right) = (1)$. By Proposition 4.9, properties (1), (2) and (4) hold for $\xi$, and so they hold for $\lambda$.

So it remains to show property (5). Since the block containing $\xi$ has the same pyramid as $B$, property (5) follows from the properties of $\xi$, except possibly in the case where $k$ or $l$ equals $\frac{p-1}{2}$. Assuming $k = \frac{p-1}{2}$ (the other case following by conjugation), the result is clearly true unless $l = p-1$. So it remains to show that ${}_{\frac{p-1}{2}}B_{p-1} \geqslant \lambda(p-1)'_1$.

Write $s = \lambda(p-1)'_1 = \xi(p-1)'_1$. Since $\xi$ is a $p$-JM partition, Proposition 4.9 tells us that ${}_{\frac{p-1}{2}}B_{p-1} \geqslant s-1$; so property (5) can only fail if ${}_{\frac{p-1}{2}}B_{p-1} = s-1$. Assuming this, we take an abacus display for $\xi$ with $cp + \frac{p+1}{2}$ beads. Then there are $c+1$ beads on runner $\frac{p-1}{2}$, and $c+s$ beads on runner $p-1$. Since $\xi(p-1)'_1 = s$, there must be beads in the top $c$ positions on runner $p-1$, followed by an empty space, at position $x$, say. We have

$$(*) \qquad cp + 1 \leqslant x \leqslant cp + p - 1.$$

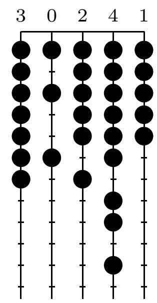
Suppose the first bead below position $x$ on runner $p-1$ occurs at position $x+dp$. Moving this bead up into the space at position $x$ corresponds to removing a rim $dp$-hook $\hbar$ from $\xi$. Let $(j,k)$ be the node of $\xi$ corresponding to $\hbar$; we calculate $j$ and $k$.

The fact that $\xi$ is a $p$-JM partition together with Proposition 4.9(4) implies that there are no beads on the abacus after position $x$ except on runner $p-1$, where there are $s$ such beads. So the bead at position $x+dp$ corresponds to $\xi_s$ (that is, $x + dp = \xi_s + cp + \frac{p+1}{2} - s$), and when it is moved up to position $x$, it is still the $s$th lowest bead, so corresponds to the $s$th part of the resulting partition. This means that removing $\hbar$ from $\xi$ only affects the $s$th part of $\xi$, so $\hbar$ must be a horizontal strip in row $s$. Hence $(j,k)$ is the leftmost node of this strip, i.e. $j = s$ and

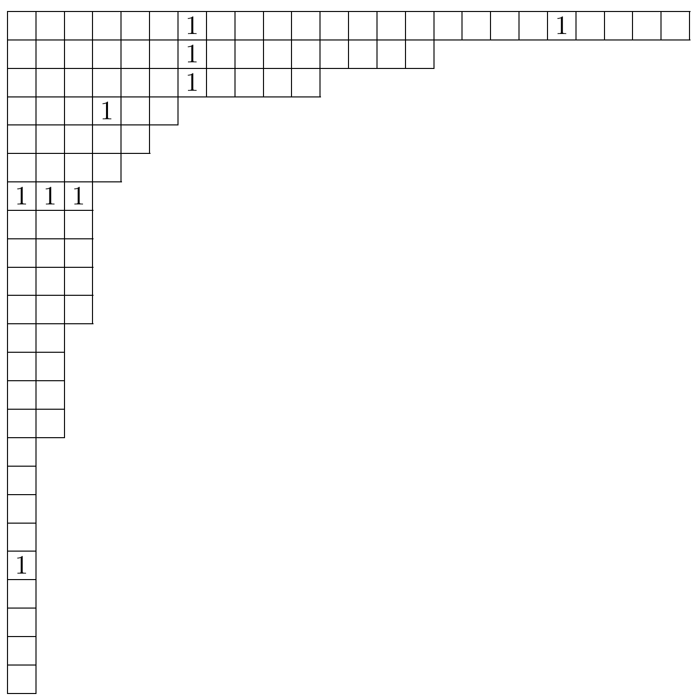
$$\begin{aligned} k &= \xi_s - dp + 1 \\ &= x + dp - cp - \frac{p+1}{2} + s - dp + 1 \\ &= x - cp - \frac{p+1}{2} + s + 1. \end{aligned}$$

So now $(*)$ yields $|k-j| \leqslant \frac{p-1}{2}$, but this contradicts the definition of a type II partition. The result follows. $\square$

**Example.** Let $p=5$ and consider the partition $(24,15,11,6,5,4,3^5,2^4,1^9)$, which has the following abacus display.




We can read off the 5-quotient of $\lambda$ as $\big((3,1),\varnothing,(1),\varnothing,(2,1^2)\big)$. $(2,1^2)$ is a 5-regular 5-JM partition, and the block $B$ containing $\lambda$ has ${}_0B_2 = {}_2B_4 = 3$, ${}_0B_4 = 6$. So $\lambda$ is an R-partition, as we can also see from its 5-power diagram.




## 5. Proof of the 'if' half of Conjecture 4.2

5.1. **The main result for type I partitions.** In this section we prove that a Specht module corresponding to an R-partition of type I has exactly two composition factors. In order to do this, we extend the set of partitions under consideration. Say that a partition $\lambda$ (not necessarily self-conjugate) is a *super-hook* if its $p$-core $\xi$ has $\xi_1, \xi_1' \leqslant \frac{p-1}{2}$, and if its $p$-quotient satisfies $\lambda\big(\frac{p-1}{2}\big) = (b+1,1^b)$ for some $b$, and $\lambda(i) = \varnothing$ for $i \neq \frac{p-1}{2}$. By Proposition 4.11, the super-hooks include the R-partitions of type I.

**Lemma 5.1.** *Suppose $\xi \neq \varnothing$ is a partition with $\xi_1, \xi'_1 \leqslant \frac{p-1}{2}$, and take an abacus display for $\xi$ with $cp + \frac{p+1}{2}$ beads. Then there are adjacent runners $i$ and $j$, with $i$ to the left of $j$, neither of which is runner $\frac{p-1}{2}$, such that runner $j$ contains one more bead than runner $i$.*

*Proof.* The fact that $\xi_1 \leqslant \frac{p-1}{2}$ means that the last bead on the abacus display is at or before position $cp+p-1$; on the other hand, $\xi'_1 \leqslant \frac{p-1}{2}$, so the first empty space is at or after position $cp+1$. So every runner has beads in the top $c$ positions, and no beads outside the top $c+1$ positions. Furthermore, the leftmost runner contains exactly $c+1$ beads, and among positions $cp+1, \dots, cp+p-1$ there are exactly $\frac{p-1}{2}$ beads. From this we see that runner $\frac{p-1}{2}$ is the leftmost runner.

Since $\xi \neq \varnothing$, it has a removable node. Hence for some $x$ there is a bead at position $x$ and an empty space at position $x-1$. If we let $i, j$ be the runners containing positions $x-1, x$ respectively, then the preceding paragraph implies that neither $i$ nor $j$ is the leftmost runner, and that there are exactly $c$ beads on runner $i$ and $c+1$ beads on runner $j$. □

Now we can prove the main result of this subsection.

**Proposition 5.2.** *Suppose* char($\mathbb{F}$) *is an odd prime $p$, and $\lambda$ is a super-hook. Then $S^\lambda$ has exactly two composition factors. In particular, every Specht module corresponding to an R-partition of type I has exactly two composition factors.*

*Proof.* We proceed by induction on $|\xi|$, where $\xi$ is the $p$-core of $\lambda$. If $\xi = \varnothing$, then $\lambda = \left(\left(bp+\frac{p+1}{2}\right), 1^{bp+(p-1)2}\right)$. This is a 'hook partition', and the composition length of such a partition is known thanks to Peel's theorem [P, Theorem 2]; in particular, $S^\lambda$ has exactly two composition factors. Now assume $\xi \neq \varnothing$, let $B$ be the block containing $\lambda$ and take an abacus display for $B$. By Lemma 5.1, there are adjacent runners $i$ and $j$, neither of which is runner $\frac{p-1}{2}$, such that runner $j$ has one more bead than runner $i$. Let $A$ be the block of $\mathbb{F}\mathfrak{S}_{n-1}$ obtained by moving a bead from runner $j$ to runner $i$. Then $(A, B)$ is a $[w:1]$-pair, and the $p$-core $\pi$ of $A$ is properly contained in $\xi$. In particular, $|\pi| < |\xi|$ and $\pi_1 \leqslant \xi_1 \leqslant \frac{p-1}{2}$, $\pi'_1 \leqslant \xi'_1 \leqslant \frac{p-1}{2}$.

In the abacus display for $\lambda$, the beads on runners $i$ and $j$ are as high up as possible, since neither $i$ nor $j$ equals $\frac{p-1}{2}$. So there cannot be any bead on runner $i$ with an empty space immediately to its right. So by Proposition 2.3 $S^\lambda$ has the same number of composition factors as $S^{\Phi(\lambda)}$, and $\Phi(\lambda)$ is a super-hook. By induction, $S^{\Phi(\lambda)}$ has exactly two composition factors, and the result follows. □

This enables us to prove the 'if' part of Conjecture 4.2 for $p = 3$.

*Proof of the 'if' half of Conjecture* 4.2(1). If $\lambda$ has the form $(r, 1^{r-1})$ or $(r, 2, 1^{r-2})$ with $r \equiv 2 \pmod 3$, then $\lambda$ is an R-partition of Type I, and the result follows from Proposition 5.2.

Blocks of weight 1 are very well understood; in particular, a 3-block of weight 1 has a decomposition matrix of the form

$$\begin{array}{|cc} \hline 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{array},$$

with the rows corresponding to partitions in decreasing dominance order. If $\lambda$ is one of these partitions, then it must be the middle one (since conjugation of partitions reverses the dominance order), and hence $S^\lambda$ has exactly two composition factors.

Finally, we look at the case $\lambda = (3^3)$. The decomposition matrix for $\mathfrak{S}_9$ in characteristic 3 appears in [J2], and shows that $S^{(3^3)}$ has two composition factors, namely $D^{(6,3)}$ and $D^{(4,3,2)}$. □

5.2. **The main result for type II partitions.** In this section, we address partitions of type II when $p \geqslant 5$. Again, we extend the class of partitions under consideration. Say that a partition (not necessarily self-conjugate) is *good* if it satisfies conditions (1)–(5) of Proposition 4.12. We shall prove the following.

**Proposition 5.3.** *If* $\mathrm{char}(\mathbb{F}) = p \geqslant 5$ *and* $\lambda$ *is a good partition, then* $S^\lambda$ *has exactly two composition factors.*

To prove this, we use a sort of downwards induction, with the Rouquier blocks as the initial case. First we show that property (5) of Proposition 4.12 allows us to induce Specht modules appropriately.

**Proposition 5.4.** *Suppose* $(B, C)$ *is a* $[w : \kappa]$*-pair. Suppose that* $\lambda$ *is a good partition in* $B$*. Then there is a good partition* $\Psi(\lambda)$ *in* $C$ *such that* $S^\lambda$ *and* $S^{\Psi(\lambda)}$ *have the same number of composition factors.*

*Proof.* Suppose an abacus for $B$ is obtained from an abacus for $C$ by moving beads from runner $j$ to runner $i$. Let $\Psi(\lambda)$ be the partition obtained by swapping runners $i$ and $j$ in the abacus display for $\lambda$. If we can show that there is no bead on runner $i$ in the abacus display for $\lambda$ with an empty space immediately to the left (equivalently, there is no bead on runner $i$ in the abacus display for $\Psi(\lambda)$ with an empty space immediately to the right), then the result will follow from Proposition 2.3 and Lemma 2.1.

Suppose there are $c$ beads on runner $i$ and $d$ beads on runner $j$ in the abacus display for $\lambda$. Since runner $j$ lies to the left of runner $i$, we have ${}_iB_j = d - c - 1$. Write $s = \lambda(i)_1$, $t = \lambda(j)'_1$. The given condition on $\lambda$ then says that $s + t \leqslant d - c$. The lowest bead on runner $i$ occurs in row $s + c$ of the abacus, while the highest empty space on runner $j$ occurs in row $d - t + 1$. So every empty space on runner $j$ occurs after every bead on runner $i$, and the result follows. □

**Corollary 5.5.** *Proposition* 5.3 *holds if and only if it holds for partitions in Rouquier blocks.*

*Proof.* This follows from Proposition 5.4 and Lemma 2.5. □

It remains to prove Proposition 5.3 for Specht modules in Rouquier blocks; this is easy, given the formula in §2.5 for the decomposition numbers in these blocks.

**Proposition 5.6.** *Proposition* 5.3 *holds for partitions in Rouquier blocks.*

*Proof.* Suppose $\lambda$ is a good partition lying in a Rouquier block $B$. Recall that we have $\lambda(i) = \varnothing$ unless $i \in \left\{0, \frac{p-1}{2}, p-1\right\}$, and $\lambda\left(\frac{p-1}{2}\right) = (1)$. Suppose we have partitions $\sigma(1), \dots, \sigma(p), \tau(0), \dots, \tau(p-1)$ with $\tau(0) = \sigma(p) = \varnothing$ such that $\prod_{i=0}^{p-1} c^{\lambda(i)}_{\tau(i)\sigma(i+1)} \neq 0$. Then we must have $\sigma(1) = \lambda(0)$ and $\tau(p-1) = \lambda(p-1)$; $\sigma\left(\frac{p+1}{2}\right)$ and $\tau\left(\frac{p-1}{2}\right)$ must equal $(1)$ and $\varnothing$ in some order, and $\sigma(i), \tau(j)$ must be $\varnothing$ for all remaining values of $i$ and $j$.

Now suppose $\mu$ is a $p$-regular partition in $B$, with $\prod_{j=1}^{p-1} c^{\mu(j)}_{\sigma(j)'\tau(j)} \neq 0$. Since $p \geqslant 5$, we have $1 \neq \frac{p-1}{2}$ and $\frac{p+1}{2} \neq p-1$, so given these values of $\sigma(i), \tau(j)$ we must have

$$\mu(1) = \sigma(1)', \quad \mu\big(\tfrac{p-1}{2}\big) = \tau\big(\tfrac{p-1}{2}\big), \quad \mu\big(\tfrac{p+1}{2}\big) = \sigma\big(\tfrac{p+1}{2}\big)', \quad \mu(p-1) = \tau(p-1),$$

and $\mu(i) = \varnothing$ for all remaining values of $i$. So $\delta_{\lambda\mu}$ is non-zero only when $\mu$ is one of the partitions $\mu^1, \mu^2$ with $p$-quotients

$$(\varnothing, \lambda(0)', \varnothing, \dots, \varnothing, (1), \varnothing, \varnothing, \dots, \lambda(p-1)),$$
$$(\varnothing, \lambda(0)', \varnothing, \dots, \varnothing, \varnothing, (1), \varnothing, \dots, \lambda(p-1))$$

(where the (1) occurs in position $\frac{p-1}{2}$ or $\frac{p+1}{2}$), and in fact we have $\delta_{\lambda\mu^1} = \delta_{\lambda\mu^2} = 1$. By Theorem 2.9 the Weyl modules $\Delta^{\lambda(0)}, \Delta^{(1)}, \Delta^{\lambda(p-1)'}$ are all irreducible, and so for $i = 1, 2$ and any $\nu$ we have

$$\epsilon_{\mu^i\nu} = \begin{cases} 1 & (\nu = \mu^i), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence by Theorem 2.4 the composition factors of $S^\lambda$ are precisely $D^{\mu^1}$ and $D^{\mu^2}$. $\square$

This completes the proof of Proposition 5.3, and hence the proof of the ‘if’ part of Conjecture 4.2.

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