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CLOSURES OF K-ORBITS IN THE FLAG VARIETY FOR $SU^*(2n)$

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ABSTRACT. We characterize the Sp_{2n}-orbits in the flag variety for SL_{2n} with rationally smooth closure via a pattern avoidance criterion, also showing that the singular and rationally singular loci of such orbit closures coincide.

1. Introduction

Let G be a complex reductive group with Borel subgroup B and let $K = G^\theta$ be the fixed point subgroup of an involution of G. In this paper we continue the program begun in [M07] and continued in [MT08], using pattern avoidance to characterize the K-orbits in G/B with rationally smooth closure (as in [LS90]). Here we consider the case $G = SL(2n, \mathbb{C}), K = Sp(2n, \mathbb{C})$. We will adapt the techniques used in [B98] to study Schubert varieties for complex classical groups, focussing on the poset and graph structures of the set of orbits with closures contained in a given one.

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2. Preliminaries

Set $G = SL(2n, \mathbb{C}), K = Sp(2n, \mathbb{C})$. Let B be the subgroup of upper triangular matrices in G. The quotient G/B may be identified with the variety of complete flags $V_0 \subset V_1 \subset \cdots \subset V_{2n}$ in $\mathbb{C}^{2n}$. The group K acts on this variety with finitely many orbits; these are parametrized by the set I_{2n} of involutions in the symmetric group S_{2n} without fixed points [MO88, RS90]. In more detail, let $\langle \cdot, \cdot \rangle$ be the standard nondegenerate skew form on $\mathbb{C}^{2n}$ with isometry group K. Then a flag $V_0 \subset \cdots \subset V_{2n}$ lies in the orbit $\mathcal{O}_\pi$ corresponding to the involution π if and only if the rank of $\langle \cdot, \cdot \rangle$ on $V_i \times V_j$ equals the cardinality $\#\{k : 1 \leq k \leq i, \pi(k) \leq j\}$ for all $1 \leq i, j \leq 2n$.

We will be using a modified version of the usual notion of pattern avoidance for permutations. We say that $\pi = \pi_1 \dots \pi_{2n}$ (in one-line notation) includes the pattern $\mu = \mu_1 \dots \mu_{2m}$ if there are indices $i_1 < i_2 < \cdots < i_{2m}$ permuted by π such that $\pi_{i_j} > \pi_{i_k}$ if and only if $\mu_j > \mu_k$ (the usual definition would omit the condition that π permute the i_j). We say that π avoids μ if it does not include it. For example, the involution 47513826 includes the pattern 351624: the indices $1, 2, 4, 6, 7, 8$ are permuted by the involution and the first and third, fourth and sixth, and second and fifth of these are flipped. On the other hand, the involution

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78436512 does not include the pattern 4312, for although the indices $4, 3, 1, 2$ occur in that order in the involution they are not permuted by it.

There are well-known poset- and graph-theoretic criteria for rational smoothness of complex Schubert varieties due to Carrell and Peterson [C94]. These have been extended by Hultman to our setting (or more generally to K-orbits in any flag variety G/B where the symmetric pair (G, K) corresponds to a real form G_0 of the reductive group G whose Cartan subgroups form a single conjugacy class [H09]). To state them we first recall that the standard partial order on K-orbits, by containment of their closures, corresponds to the (restriction of the) reverse Bruhat order on I_{2n} [RS90]. The poset I_{2n} equipped with this order is then graded via the rank function

$$r(\pi) = n^2 - \sum_{i<\pi(i)} (\pi(i) - i - \#\{k \in \mathbb{N} : i < k < \pi(i), \pi(k) < i\}),$$

where this quantity equals the difference in dimension between $\mathcal{O}_\pi$ and $\mathcal{O}_c$, the unique closed orbit, corresponding to the involution $w_0 = 2n \ldots 1$ [RS90]. Let I_π be the interval consisting of all $\pi' \leq \pi$ in the reverse Bruhat order. Then Hultman has shown that $\bar{\mathcal{O}}_\pi$ is rationally smooth if and only if I_π is rank-symmetric in the sense of having the same number of elements of rank i as of rank $r - i$ for all i, where r is the rank of π; equivalently, if and only if the rank-generating function $P_\pi = \sum_{\pi' \leq \pi} q^{r(\pi')}$ is palindromic [H09, 5.9]. If we make I_π into a graph BG_π by decreeing that the vertices μ and ν are adjacent if and only if $\nu = t\mu t \neq \mu$ for some transposition t in S_{2n}, then $\bar{\mathcal{O}}_\pi$ is rationally smooth if and only BG_π is regular of degree r. If $\mu < \pi$ and we make the reverse Bruhat interval $[\mu, \pi]$ into a graph $BG_{\mu,\pi}$ by the same recipe, then $\bar{\mathcal{O}}_\pi$ is rationally smooth at $\mathcal{O}_\mu$ if and only if the degree of μ in $BG_{\mu,\pi}$ is $r(\pi) - r(\mu)$ [H09, 4.5,5.8,6.7] (but in general $BG_{\mu,\pi}$ need not be regular or rank-symmetric in this situation). In general the the degree of μ in $BG_{\mu,\pi}$ is always at least $r := r(\pi) - r(\mu)$. We call μ an irregular vertex if it has larger degree than r.

3. Main result

We begin with a lemma about the inductive behavior of vertex degrees in Bruhat graphs.

Lemma. *Let μ, π be two involutions in I_{2n} with $\mu \leq \pi$ in reverse Bruhat order. Let t be a transposition of two indices flipped by both μ and π and set $\pi = \tilde{\pi}t, \mu = \tilde{\mu}t$ (so that $\tilde{\pi}, \tilde{\mu}$ are* **not** *in I_{2n}). Let π', μ' be the unique involutions in I_{2n-2} such that $\tilde{\pi}, \tilde{\mu}$ include the patterns π', μ', respectively, in the indices fixed by t. Assume that the vertex μ' is irregular in $BG_{\mu',\pi'}$. Then μ is irregular in $BG_{\mu,\pi}$.*

Proof. Note first that the one-line notation of π', for example, is obtained from that of π by deleting the indices flipped by t and then replacing the ith smallest of the surviving indices by i. Thus if $\pi = 361542$ and t flips 1 and 3, then $\pi' = 4321$. We say that the transposition (a, d) flipping the indices a and d with $a < d$ encapsulates the transposition (b, c) with $b < c$ if $a < b < c < d$. Then the rank difference $r(\pi) - r(\mu)$ is given by $r(\pi') - r(\mu') + 2(n(\mu) - n(\pi))$, where $n(\mu), n(\pi)$ are the numbers of transpositions in μ, π, respectively, encapsulating t. Now every edge from μ' in $BG_{\mu',\pi'}$ corresponds to an edge from μ in $BG_{\mu,\pi}$ in an obvious way. For every transposition counted by $n(\mu)$ but not $n(\pi)$ one easily locates two additional edges from μ in $BG_{\mu,\pi}$, showing that μ is irregular whenever μ' is, as desired. $\square$

Now we can characterize the K-orbits with rationally smooth closure.

Theorem 1. *The orbit $\mathcal{O}_\pi$ has rationally smooth closure if and only if π avoids the 17 patterns:* 351624, 64827153, 57681324, 53281764, 43218765, 65872143, 21654387, 21563487, 34127856, 43217856, 34128765, 36154287, 21754836, 63287154, 54821763, 46513287, 21768435.

Proof. Note first that this list of bad patterns is stable under the automorphism of the Dynkin diagram: the first nine patterns are fixed by this automorphism while the next four pairs of patterns are interchanged. Suppose first that π coincides with one of the bad patterns. Then the bottom vertex in BG_π is irregular, as one sees from the following table. Here the rank of each bad pattern (regarded as an element of I_6 or I_8) is given in the middle column and the transpositions labelling the edges from the bottom vertex are given in the right column, abbreviating the flip of the ith and jth coordinates by ij.

vertex	rank	edges from bottom vertex
351624	4	12,13,14,23,24
64827153	5	12,13,23,24,25,34,35
57681324	5	12,13,14,23,24,34
53281764	7	12,13,14,23,24,25,34,35
43218765	8	12,13,14,15,23,24,25,26,34,35
65872143	4	12,13,23,24,34
21654387	10	12,13,14,15,16,17,23,24,25,26,34,35
21563487	11	12,13,14,15,16,17,23,24,25,26,34,35
34127856	10	12,13,14,15,16,23,24,25,26,34,35
43217856	9	12,13,14,15,23,24,25,26,34,35
34128765	9	12,13,14,15,16,23,24,25,26,34,35
36154287	9	12,13,14,15,16,23,24,25,26,34,35
21754836	9	12,13,14,15,16,23,24,25,26,34,35
63287154	6	12,13,23,24,25,34,35
54821763	6	12,13,23,24,25,34,35
46513287	8	12,13,14,15,23,24,25,34,35
21768435	8	12,13,14,15,23,24,25,34,35

Now if π contains a bad pattern, then repeated use of Lemma 1 shows that $BG_{\mu,\pi}$ is irregular at μ, where the one-line notation of μ is obtained from that of π by rewriting the indices in the bad pattern in decreasing order and leaving the other indices unchanged. Conversely, suppose that π avoids all patterns in the above list. We will show that the rank-generating polynomial P_π is palindromic, or more precisely, it is the product of sums of the form $1 + q + \cdots + q^t$ for various t. Let $\pi = \pi_1 \ldots \pi_{2n}$ and assume first that $2n - \pi_1 \leq \pi_{2n} - 1$ (i.e., 1 is closer to the end of $\pi_1 \ldots \pi_{2n}$ than $2n$ is to the beginning). Set $\pi^{(1)} = t\pi t$, where t is the transposition interchanging π_1 and $\pi_1 + 1$, so that 1 appears one place further to the right in $\pi^{(1)}$ than in π. Define $\pi^{(2)}, \ldots, \pi^{(2n-\pi_1)}$ similarly, so that 1 appears at the end of $\pi^{(2n-\pi_1)}$. If $\mu = \mu_1, \ldots, \mu_{2n} < \pi$, then Proctor's criterion for the Bruhat order [P82] shows that $\mu_1 \geq \pi_1$. If $\mu_1 = \pi_1$, then one checks that $\mu' < \pi'$, where μ', π' are obtained from μ, π by omitting the indices 1 and μ_1, replacing all indices i between 1 and μ_1 by $i-1$, and replacing all indices $j > \mu_1$ by $j-2$; moreover, π' continues to avoid all bad patterns. If instead $\mu_1 > \pi_1$, then we claim that $\mu \le \pi^{(1)}$ and that $\pi^{(1)}$ continues to avoid all bad patterns. If this holds, then induction shows that $\mu \le \pi^{(\mu_1 - \pi_1)}$; whence we may as above eliminate the indices 1 and μ_1 from μ and $\pi^{(\mu_1-\pi_1)}$ and repeat the above procedure. Using the formula for the rank function in I_{2n}, we deduce that P_π factors in the way claimed above, where the first factor is $1 + q + \cdots + q^{2n-\pi_1}$.

To prove the claim, set $\pi_1 = k, \pi_{k+1} = i$, and suppose that there is μ with $\mu < \pi, \mu \not\le \pi^{(1)}$, and $\mu_i > \pi_1$. Set $j := \mu_k$. There are two cases. If $i < k$, then the conditions $\mu < \pi, \mu \not\le \pi^{(1)}$ force $\pi_j > k+1$, whence π contains the pattern $p := 465132$ (since it avoids the pattern 351624). The assumption $2n - \pi_1 \le \pi_{2n} - 1$ implies that $\pi \ne p$, so that π is the product of three transpositions forming the pattern p and at least one more transposition. Now one checks that no matter how one chooses this transposition to force $2n - \pi_1 \le \pi_{2n} - 1$ we get a bad pattern in π, a contradiction; more precisely, one of the five patterns 46513287, 63287154, 65872143, 64827153, or 57681324 must occur in π. Similarly, if instead, $i > k+1$, then one must again have $\pi_j > k+1$. In this case π contains the pattern 361542, and once again the assumption $2n - \pi_1 \le \pi_{2n} - 1$ forces π to contain a bad pattern. Here the bad patterns that arise are 36154287 and 53281764.

If instead, $\pi_{2n} - 1 < 2n - \pi_1$, then one repeats the above argument, replacing 1 by $2n$ and moving $2n$ to the left instead of 1 to the right. Thus we define $\pi^{(1)}, \pi^{(2)}$, and so on, so that $2n$ appears one place to the left in $\pi^{(1)}$ than it does in π; if $\mu \le \pi$, then we must have $\mu_{2n} \le \pi_{2n}$, and if $\mu_{2n} < \pi_{2n}$, then we must have $\mu \le \pi^{(1)}$, lest π contain a bad pattern. Here the two "bad seeds" that must be ruled out are 546213 and 532614; these give rise to the bad patterns 21768435, 54821763, 65872143, 64827153, 57681324, 21754836, and 53281764.

Finally, we must ensure in both cases that $\pi^{(1)}$ avoids all bad patterns whenever π does. This requires that we rule out four more "bad seeds", namely 216543, 432165, 215634, and 341265; we achieve this by ruling out the bad patterns 21654387, 43218765, 34127856, 43217856, 21563487, and 34128765. Also excluding the bad pattern 351624 of length 6, we see that if π avoids all bad patterns, then P_π factors in the desired way and $\mathcal{O}_\pi$ has rationally smooth closure, as required. □

4. Smoothness and the bottom vertex

We now consider reverse Bruhat intervals $[\mu, \pi]$ and their graphs $BG_{\mu,\pi}$. We will find (as for Schubert varieties in type A) that it is only necessary to test one vertex in this graph to determine whether or not $\bar{\mathcal{O}}_\pi$ is (rationally) smooth at $\bar{\mathcal{O}}_\mu$.

Theorem 2. *If $\mu < \pi$ and the degree of μ in $BG_{\mu,\pi}$ equals $r(\pi) - r(\mu)$, then $\bar{\mathcal{O}}_\pi$ is smooth along $\bar{\mathcal{O}}$. In particular, the singular and rationally singular loci of $\bar{\mathcal{O}}$ coincide.*

Proof. Assume first that $\mathcal{O}_\mu = \mathcal{O}_c$, the closed orbit. Fix a basis (e_i) of $\mathbb{C}^{2n}$ such that

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i < j, i + j = 2n+1, \\ -1 & \text{if } i > j, i+j = 2n+1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle\cdot,\cdot\rangle$ is the skew form. Let (a_{ij}) be a family of complex parameters indexed by ordered pairs (i,j) satisfying either $i \le n < j$ or $n < i < j$. We assume that $a_{ij} = -a_{2n+1-j,2n+1-i}$ and $a_{i,2n+1-i} = 0$, if $i \le n$, but otherwise put no restrictions on the a_{ij}. Define a basis (b_i) of $\mathbb{C}^{2n}$ via

$$b_i = \begin{cases} e_i + \sum_{j=n+1}^{2n} a_{ij} e_j & \text{if } i \le n, \\ e_i + \sum_{j=i+1}^{2n} a_{ij} e_j & \text{otherwise.} \end{cases}$$

Then the Gram matrix $G := (g_{ij} = (\langle b_i, b_j\rangle))$ of the b_i relative to the form satisfies

$$g_{ij} = \begin{cases} 2a_{i,2n+1-j} & \text{if } i < j \le n, \\ -g_{ji} & \text{if } j < i \le n, \\ a_{ij} & \text{if } i < n < j < 2n+1-i, \\ 1 & \text{if } i < n < j = 2n+1-i, \\ -g_{ji} & \text{if } j < n < i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the matrix G is skew-symmetric and has zeroes below the antidiagonal from lower left to upper right. The antidiagonal entries are all ± 1. Now one checks that the set $\mathcal{F}$ of all flags $V_0 \subset \ldots \subset V_{2n}$ where (b_i) runs through all bases obtained as above from the a_{ij} and V_i is the span of $b_1, \ldots, b_i$ is a slice in the sense of Brion to $\mathcal{O}_c$ at the flag f_c corresponding to the basis (e_i) [Br99, 2.1]. Intersecting $\mathcal{F}$ with $\bar{\mathcal{O}}_\pi$ we get another slice to $\mathcal{O}_c$ at f_c.

By hypothesis there are $n^2 - n - r(\pi)$ distinct conjugates $c = tw_0t$ of w_0 by a transposition t such that $c \not\le \pi$. One computes that $d := tw_0$ also satisfies $d \not\le \pi$. Writing d as $d_1 \ldots d_{2n}$, let i be the smallest index such that $\pi_1 \ldots \pi_i \not\le d_1 \ldots d_i$ in the standard partial order on sequences used to characterize the Bruhat order [P82]. Thus if $\pi_1 \ldots \pi_i$ is rearranged in increasing order as $\pi'_1 \ldots \pi'_i$ and similarly $d_1 \ldots d_i$ is rearranged as $d'_1 \ldots d'_i$, then $\pi'_j > d'_j$ for some j. Then for some k there are more indices $\ell \le i$ with $d_\ell < k$ than indices $m \le i$ with $\pi_m < k$. Equating all minors of the appropriate size lying in the first i rows and columns $d_1, \ldots, d_i$ of the Gram matrix to 0, we arrive at $n^2 - n - r(\pi)$ polynomials vanishing on $\bar{\mathcal{O}}_\pi \cap \mathcal{F}$, each involving a distinct variable raised to the first power with coefficient ± 1. Then the Jacobian matrix of these polynomials has rank $n^2 - n - r(\pi)$; whence by the Jacobian criterion both $\mathcal{F} \cap \bar{\mathcal{O}}_\pi$ and $\bar{\mathcal{O}}_\pi$ are smooth at $\mathcal{O}_c$, as desired [Br99, 2.1].

If $\mathcal{O}_c$ is replaced by any orbit $\mathcal{O}_\mu \subset \bar{\mathcal{O}}_\pi$, then let G_μ be the matrix whose ij-entry is 1 if $j = \mu_i > i, -1$ if $j = \mu_i < i$, and 0 otherwise. This is the Gram matrix of a basis (b_i) obtained by suitably rearranging the basis (e_i); let f_μ be the corresponding flag. Now consider the set of all Gram matrices G whose ij-entries agree with those of G_μ if $j \ge \mu_i$ and whose other possibly nonzero entries are determined as follows. There are $n^2 - n - r(\mu)$ conjugates c of μ by transpositions with $c > \mu$. Write each c as $c_1 \ldots c_{2n}$ and let i be the smallest index with $c_i < \mu_i$. Then the other possibly nonzero entries of G appear in the positions (i, c_i) together with their transposes (c_i, i). Entries of G not in one of the positions specified above are 0. There are no further restrictions on these entries apart, of course, from being skew-symmetric. This set of Gram matrices stands in bijection to a set $\mathcal{F}_\mu$ of flags and which is a slice to $\bar{\mathcal{O}}_\mu$ at f_μ. Then one shows as above that if the degree of μ in $BG_{\mu,\pi}$ equals the difference $r(\pi) - r(\mu)$, then there are $n^2 - n - r(\pi)$ polynomials vanishing on $\bar{\mathcal{O}}_\pi \cap \mathcal{F}_\mu$ whose Jacobian matrix has rank $n^2 - n - r(\pi)$; whence again $\bar{\mathcal{O}}_\pi$ is smooth along $\mathcal{O}_\mu$, as desired. □

There are two other symmetric pairs (G, K) of complex reductive groups satisfying the hypothesis of [H09] (that all Cartan subgroups in the corresponding real form G_0 of G are conjugate), namely $(Spin(2n, \mathbb{C}), Spin(2n-1, \mathbb{C}))$ and (E_6, F_4). In the first case all K-orbits in G/B have smooth closure. In the second case, eleven out of the forty-five K-orbits have rationally singular closure. Hultman has checked in each case that the bottom vertex of the Bruhat graph detects the rational singularity. It is not known whether smoothness and rational smoothness are equivalent for these orbit closures.

References

[B98] S. Billey, *Pattern avoidance and rational smoothness for Schubert varieties*, Adv. in Math. **139** (1998), 141–156. MR1652522 (99i:14058)

[Br99] M. Brion, *Rational smoothness and fixed points of torus actions*, Transf. Groups **4** (1999), 127–156. MR1712861 (2000f:14072)

[C94] J. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, Proc. Symp. Pure Math. **56** (1994), 53–61. MR1278700 (95d:14051)

[H09] A. Hultman, *Criteria for rational smoothness of some symmetric orbit closures*, arXiv:math.RT/0907.0936v1.

[LS90] V. Lakshmibai and B. Sandhya, *Criterion for smoothness of Schubert varieties in $SL(n)/B$*, Proc. Indian Acad. Sci. **100** (1990), 45–52. MR1051089 (91c:14061)

[MO88] T. Matsuki and T. Oshima, *Embedding of discrete series into principal series*, in The Orbit Method in Representation Theory, Progress in Math. **82**, Birkhäuser, Boston, 1988, 147–175. MR1095345 (92d:22020)

[M07] W. McGovern, *Closures of K-orbits in the flag variety for $U(p,q)$*, J. Alg. **322** (2009), 2709–2712. MR2559857

[MT08] W. McGovern and P. Trapa, *Pattern avoidance and smoothness of closures for orbits of a symmetric subgroup in the flag variety*, J. Alg. **322** (2009), 2713–2730. MR2559858

[P82] R. Proctor, *Classical Bruhat orders are lexicographic shellable*, J. Alg. **77** (1982), 104–126. MR665167 (84j:20044)

[RS90] R. W. Richardson and T. A. Springer, *On the Bruhat order for symmetric varieties*, Geom. Dedicata **35** (1990), 389–436. MR1066573 (92e:20032)

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