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FROM CONJUGACY CLASSES IN THE WEYL GROUP TO UNIPOTENT CLASSES, II

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Abstract. Let G be a connected reductive group over an algebraically closed field of characteristic p. In an earlier paper we defined a surjective map Φ_p from the set $\underline{W}$ of conjugacy classes in the Weyl group W to the set of unipotent classes in G. Here we prove three results for Φ_p. First we show that Φ_p has a canonical one-sided inverse. Next we show that $\Phi_0 = r\Phi_p$ for a unique map r. Finally, we construct a natural surjective map from $\underline{W}$ to the set of special representations of W which is the composition of Φ_0 with another natural map and we show that this map depends only on the Coxeter group structure of W.

Introduction

0.1. Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Let $\underline{\underline{G}}$ be the set of unipotent conjugacy classes in G. Let $\underline{\mathbf{W}}$ be the set of conjugacy classes in the Weyl group $\mathbf{W}$ of G. Let $\Phi : \underline{\mathbf{W}} \to \underline{\underline{G}}$ be the (surjective) map defined in [L8]. For $C \in \underline{\mathbf{W}}$ we denote by m_C the dimension of the fixed point space of $w : \mathcal{V} \to \mathcal{V}$ where $w \in C$ and $\mathcal{V}$ is the reflection representation of the Coxeter group $\mathbf{W}$. The following result provides a one-sided inverse for Φ.

Theorem 0.2. *For any $\gamma \in \underline{\underline{G}}$ the function $\Phi^{-1}(\gamma) \to \mathbf{N}$, $C \mapsto m_C$ reaches its minimum at a unique element $C_0 \in \Phi^{-1}(\gamma)$. Thus we have a well-defined map $\Psi : \underline{\underline{G}} \to \underline{\mathbf{W}}$, $\gamma \mapsto C_0$ such that $\Phi\Psi : \underline{\underline{G}} \to \underline{\underline{G}}$ is the identity map.*

It is likely that when $\mathbf{k} = \mathbf{C}$, the map Ψ coincides with the map $\underline{\underline{G}} \to \underline{\mathbf{W}}$ defined in [KL, Section 9], which we will denote here by Ψ_0' (note that Ψ_0' has not been computed explicitly in all cases). It is enough to prove the theorem in the case where G is almost simple; moreover, in that case it is enough to consider one group in each isogeny class. When G has type A, the theorem is trivial (Φ is a bijection). For the remaining types the proof is given in Sections 1 and 2.

0.3. Let G_0 be a connected reductive algebraic group over $\mathbf{C}$ of the same type as G; we identify the Weyl group of G_0 with $\mathbf{W}$. Let $\underline{\underline{G}}_0$ be the set of unipotent classes of G_0. Let $\Phi_0 : \underline{\mathbf{W}} \to \underline{\underline{G}}_0$, $\Psi_0 : \underline{\underline{G}}_0 \to \underline{\mathbf{W}}$ be the maps defined like Φ and Ψ (for G_0 instead of G).

Theorem 0.4. (a) *There is a unique (necessarily surjective) map $\rho : \underline{\underline{G}} \to \underline{\underline{G}}_0$ such that $\Phi_0 = \rho\Phi$. We have $\rho = \Phi_0\Psi$.*

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(b) *There is a unique (necessarily injective) map* $\pi : \underline{\underline{G}}_0 \to \underline{\underline{G}}$ *such that* $\Psi_0 = \Psi\pi$. *We have* $\pi = \Phi\Psi_0$.

The second sentence in 0.4(a) (resp. 0.4(b)) follows from the first since $\Phi\Psi = 1$. The uniqueness of ρ (resp. π) follows from the surjectivity of Φ (resp. injectivity of Ψ). The surjectivity of ρ (if it exists) follows from the surjectivity of Φ_0. The injectivity of π (if it exists) follows from the injectivity of Ψ_0. The existence of ρ (resp. π), which is equivalent to the identity $\Phi_0 = \Phi_0\Psi\Phi$ (resp. $\Psi_0 = \Psi\Phi\Psi_0$) is proved in Section 3, where various other characterizations of ρ and π are given.

0.5. Let $\hat{\mathbf{W}}$ be the set of irreducible representations of $\mathbf{W}$ (over $\mathbf{Q}$) up to isomorphism. Let $\mathcal{S}_{\mathbf{W}}$ be the subset of $\hat{\mathbf{W}}$ introduced in [L1]; it consists of representations of $\mathbf{W}$ which were later called "special representations". For any $E \in \hat{\mathbf{W}}$ let $[E]$ be the unique object of $\mathcal{S}_{\mathbf{W}}$ such that $E, [E]$ are in the same two-sided cell of $\mathbf{W}$; thus $E \mapsto [E]$ is a surjective map $\hat{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$. For any $\gamma \in \underline{\underline{G}}_0$ let $E_\gamma \in \hat{\mathbf{W}}$ be the Springer representation corresponding to γ. We define $\tilde{\Phi}_0 : \underline{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$ by $C \mapsto [E_{\Phi_0(C)}]$. Let $\underline{\underline{G}}_0^{\spadesuit} = \{\gamma \in \underline{\underline{G}}_0; E_\gamma \in \mathcal{S}_{\mathbf{W}}\}$. It is known that $\gamma \mapsto E_\gamma$ is a bijection $\underline{\underline{G}}_0^{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$. This, together with the surjectivity of Φ_0, shows that $\tilde{\Phi}_0$ is surjective. We now define $\tilde{\Psi}_0 : \mathcal{S}_{\mathbf{W}} \to \underline{\mathbf{W}}$ by $\tilde{\Psi}_0(E_\gamma) = \Psi_0(\gamma)$, $\gamma \in \underline{\underline{G}}_0^{\spadesuit}$. Note that $\tilde{\Psi}_0$ is injective and $\tilde{\Phi}_0\tilde{\Psi}_0 = 1$. We have the following result.

Theorem 0.6. (a) *The (surjective) map* $\tilde{\Phi}_0 : \underline{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$ *depends only on the Coxeter group structure of* $\mathbf{W}$. *In particular,* $\tilde{\Phi}_0$ *for* G_0 *of type* B_n *coincides with* $\tilde{\Phi}_0$ *for* G_0 *of type* C_n $(n \geq 3)$.

(b) *The (injective) map* $\tilde{\Psi}_0 : \mathcal{S}_{\mathbf{W}} \to \underline{\mathbf{W}}$ *depends only on the Coxeter group structure of* $\mathbf{W}$. *In particular,* $\tilde{\Psi}_0$ *for* G_0 *of type* B_n *coincides with* $\tilde{\Psi}_0$ *for* G_0 *of type* C_n $(n \geq 3)$.

The proof is given in Section 4. Note that (a) was conjectured in [L7, 1.4].

Let $\underline{\mathbf{W}}_{\spadesuit}$ be the image of $\tilde{\Psi}_0$ (a subset of $\underline{\mathbf{W}}$). We say that $\underline{\mathbf{W}}_{\spadesuit}$ is the set of *special conjugacy classes* of $\mathbf{W}$. Note that $\tilde{\Phi}_0$ defines a bijection $\underline{\mathbf{W}}_{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$. Thus, the set of special conjugacy classes in $\mathbf{W}$ is in natural bijection with the set of special representations of $\mathbf{W}$. From Theorem 0.6 we see that there is a natural retraction $\underline{\mathbf{W}} \to \underline{\mathbf{W}}_{\spadesuit}$, $C \mapsto \tilde{\Psi}_0(\tilde{\Phi}_0(C))$ which depends only on the Coxeter group structure of $\mathbf{W}$.

0.7. The paper is organized as follows. In Section 1 (resp. Section 2) we describe explicitly the map Φ in the case where G is almost simple of classical (resp. exceptional) type and prove Theorem 0.2. In Section 3 we prove Theorem 0.4. In Section 4 we prove Theorem 0.6. In Section 5 we describe explicitly for each simple type the bijection $\underline{\mathbf{W}}_{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ defined by $\tilde{\Phi}_0$.

0.8. Notation. For $\gamma, \gamma' \in \underline{\underline{G}}$ (or $\gamma, \gamma' \in \underline{\underline{G}}_0$) we write $\gamma \leq \gamma'$ if γ is contained in the closure of γ'.

1. Isometry groups

1.1. Let $\mathcal{P}^1$ be the set of sequences $p_* = (p_1 \geq p_2 \geq \cdots \geq p_\sigma)$ in $\mathbf{Z}_{>0}$. For $p_* \in \mathcal{P}^1$ we set $|p_*| = p_1 + p_2 + \cdots + p_\sigma$, $\tau_{p_*} = \sigma$, $\mu_j(p_*) = |\{k \in [1, \sigma]; p_k = j\}|$ $(j \in \mathbf{Z}_{>0})$. Let $\mathcal{P}^0 = \{p_* \in \mathcal{P}^1; \tau_{p_*} = \text{ even}\}$. For $N \in \mathbf{N}, \kappa \in \{0, 1\}$ let $\mathcal{P}^\kappa_N = \{p_* \in \mathcal{P}^\kappa; |p_*| = N\}$.

Let $\tilde{\mathcal{P}} = \{p_* \in \mathcal{P}^0; p_1 = p_2, p_3 = p_4, \dots\}$. Let $\mathcal{S}^1$ be the set of $r_* \in \mathcal{P}^1$ such that r_k is even for all k. Let $\mathcal{S}^0 = \mathcal{S}^1 \cap \mathcal{P}^0$. For $N \in \mathbf{N}, \kappa \in \{0,1\}$ let $\mathcal{S}^\kappa_N = \mathcal{S}^\kappa \cap \mathcal{P}^\kappa_N$.

We fix $\mathbf{n} \in \mathbf{N}$ and we define $n \in \mathbf{N}$, $\kappa \in \{0,1\}$ by $\mathbf{n} = 2n + \kappa$. We set $\mathcal{A}^k_{2n} = \{(r_*, p_*) \in \mathcal{S}^\kappa \times \tilde{\mathcal{P}}; |r_*| + |p_*| = 2n\}$.

1.2. In the remainder of this section we fix V, a $\mathbf{k}$-vector space of dimension $\mathbf{n} = 2n + \kappa$ (see 1.1) with a fixed bilinear form $(,) : V \times V \to \mathbf{k}$ and a fixed quadratic form $Q : V \to \mathbf{k}$ such that (i) or (ii) below holds:

(i) $Q = 0$, $(x, x) = 0$ for all $x \in V$, $V^\perp = 0$;

(ii) $Q \neq 0$, $(x, y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in V$, $Q : V^\perp \to \mathbf{k}$ is injective.

Here, for any subspace V' of V we set $V'^\perp = \{x \in V; (x, V') = 0\}$. In case (ii) it follows that $V^\perp = 0$ unless $\kappa = 1$ and $p = 2$ in which case $\dim V^\perp = 1$. Let $Is(V)$ be the subgroup of $GL(V)$ consisting of all $g \in GL(V)$ such that $(gx, gy) = (x, y)$ for all $x, y \in V$ and $Q(gx) = Q(x)$ for all $x \in V$. In this section we assume that G is the identity component of $Is(V)$.

1.3. In this subsection we assume that $\mathbf{n} \geq 3$. Let W be the group of permutations of $[1, \mathbf{n}]$ that commute with the involution $\chi : i \mapsto \mathbf{n} - i + 1$. If $Q = 0$ or if $\kappa = 1$ we identify (as in [L8, 1.4, 1.5]) $\mathbf{W}$ with W. If $Q \neq 0$ and $\kappa = 0$ we identify (as in [L8, 1.4, 1.5]) $\mathbf{W}$ with the group W' of even permutations of $[1, \mathbf{n}]$ commuting with χ; in this case let $\underline{\mathbf{W}}_0$ (resp. $\underline{\mathbf{W}}_1$) be the set of conjugacy classes in $\mathbf{W}$ which are not conjugacy classes of W (resp. from a single conjugacy class in W) and we denote by $\underline{\tilde{\mathbf{W}}}$ the set of W-conjugacy classes in W'.

If $Q = 0$, we identify $\underline{\mathbf{W}}$ with $\mathcal{A}^1_{2n}$ by associating to the conjugacy class of $w \in W$ the pair (r_*, p_*) where r_* is the multiset consisting of the sizes of cycles of w which are χ-stable and p_* is the multiset consisting of the sizes of the cycles of w which are not χ-stable. If $Q \neq 0$, we identify $\underline{\mathbf{W}} = \mathcal{A}^1_{2n}$ (if $\kappa = 1$) and $\underline{\tilde{\mathbf{W}}} = \mathcal{A}^0_{2n}$ (if $\kappa = 0$) by associating to the conjugacy class of w in W the pair (r_*, p_*) where r_* is the multiset consisting of the sizes of cycles of w (other than fixed points) which are χ-stable and p_* is the multiset consisting of the sizes of cycles of w which are not χ-stable.

1.4. Let $\mathcal{T}_{2n}$ be the set of all $c_* \in \mathcal{P}^1_{2n}$ such that $\mu_j(c_*)$ is even for any odd j.

Let $\mathcal{T}^{(2)}_{2n}$ be the set of all pairs (c_*, ϵ) where $c_* \in \mathcal{T}_{2n}$ and $j \mapsto \epsilon(j) \in \{0,1\}$ is a function defined on the set $\{j \in \{2, 4, 6, \dots\}; \mu_j(c_*) \in \{2, 4, 6, \dots\}\}$.

Let $\tilde{\mathcal{T}}^{(2)}_{2n} = \{(c_*, \epsilon) \in \mathcal{T}^{(2)}_{2n}; \tau_{c_*} \text{ is even}\}$.

Let $\mathcal{Q}$ be the set of all $c_* \in \mathcal{P}^1$ such that $\mu_j(c_*)$ is even for any even j. For $N \in \mathbf{N}$ let $\mathcal{Q}_N = \mathcal{Q} \cap \mathcal{P}^1_N$.

If $Q = 0, p \neq 2$, we identify $\underline{\underline{G}} = \mathcal{T}_{2n}$ by associating to $\gamma \in \underline{\underline{G}}$ the multiset consisting of the sizes of the Jordan blocks of an element of γ.

If $Q = 0, p = 2$, we identify $\underline{\underline{G}} = \mathcal{T}^{(2)}_{2n}$ by associating to $\gamma \in \underline{\underline{G}}$ the pair (c_*, ϵ) where c_* is the multiset consisting of the sizes of the Jordan blocks of an element of γ; $\epsilon(j)$ is equal to 0 if $((g-1)^{j-1}(x), x) = 0$ for all $x \in \ker(g-1)^j$ $(g \in \gamma)$ and $\epsilon(j) = 1$ otherwise (see [S1]).

If $Q \neq 0, \kappa = 1, p \neq 2$, we identify $\underline{\underline{G}} = \mathcal{Q}_{\mathbf{n}}$ by associating to $\gamma \in \underline{\underline{G}}$ the multiset consisting of the sizes of the Jordan blocks of an element of γ.

If $Q \neq 0, \kappa = 1, p = 2$, we identify $\underline{\underline{G}} = \mathcal{T}^{(2)}_{2n}$ by associating to $\gamma \in \underline{\underline{G}}$ the pair (c_*, ϵ) corresponding as above to the image of γ under the obvious bijective homomorphism from G to a symplectic group of an $\mathbf{n}-1$-dimensional vector space.

If $Q \neq 0, \kappa = 0$, we denote by $\underline{\underline{G}}_{(0)}$ (resp. $\underline{\underline{G}}_{(1)}$) the set of unipotent classes in G which are not conjugacy classes in $Is(V)$ (resp. are also conjugacy classes in $Is(V)$); let $\tilde{\underline{\underline{G}}}$ be the set of $Is(V)$-conjugacy classes of unipotent elements of G. We have an obvious imbedding $\underline{\underline{G}}_{(1)} \subset \tilde{\underline{\underline{G}}}$.

If $Q \neq 0, \kappa = 0, p \neq 2$, we identify $\tilde{\underline{\underline{G}}} = \mathcal{Q}_{\mathbf{n}}$ by associating to $\gamma \in \tilde{\underline{\underline{G}}}$ the multiset consisting of the sizes of the Jordan blocks of an element of γ.

If $Q \neq 0, \kappa = 0, p = 2$, we identify $\tilde{\underline{\underline{G}}} = \tilde{\mathcal{T}}_{2n}^{(2)}$ by associating to $\gamma \in \tilde{\underline{\underline{G}}}$ the pair $(c_*, \epsilon) \in \mathcal{T}_{2n}^{(2)}$ corresponding as above to the image of γ under the obvious imbedding of $Is(V)$ into the symplectic group of $V, (,)$.

We define $\iota : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}$ by $(r_*, p_*) \mapsto c_*$ where the multiset of entries of c_* is the union of the multiset of entries of r_* with the multiset of entries of p_*. We define $\iota^{(2)} : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}^{(2)}$ by $(r_*, p_*) \mapsto (c_*, \epsilon)$ where $c_* = \iota(r_*, p_*)$ and for any $j \in \{2, 4, 6, \dots\}$ such that $\mu_j(c_*) \in \{2, 4, 6, \dots\}$, we have $\epsilon(j) = 1$ if $j = r_i$ for some i and $\epsilon(j) = 0$, otherwise. When $\kappa = 0$ we define $\tilde{\iota}^{(2)} : \mathcal{A}_{2n}^0 \to \tilde{\mathcal{T}}_{2n}^{(2)}$ to be the restriction of $\iota^{(2)} : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}^{(2)}$.

In the remainder of this section we assume that $\mathbf{n} \geq 3$. We define $\Xi : \mathcal{S}^\kappa \to \mathcal{P}^1$ by

$$(r_1 \geq r_2 \geq \cdots \geq r_\sigma) \mapsto (r_1 + \psi(1) \geq r_2 + \psi(2) \geq \cdots \geq r_\sigma + \psi(\sigma))$$

if $\sigma + \kappa$ is even,

$$(r_1 \geq r_2 \geq \cdots \geq r_\sigma) \mapsto (r_1 + \psi(1) \geq r_2 + \psi(2) \geq \cdots \geq r_\sigma + \psi(\sigma) \geq 1)$$

if $\sigma + \kappa$ is odd, where $\psi : [1, \sigma] \to \{-1, 0, 1\}$ is as follows:

if $t \in [1, \sigma]$ is odd and $r_t < r_x$ for any $x \in [1, t-1]$, then $\psi(t) = 1$;

if $t \in [1, \sigma]$ is even and $r_x < r_t$ for any $x \in [t+1, \sigma]$, then $\psi(t) = -1$;

for all other $t \in [1, \sigma]$ we have $\psi(t) = 0$.

We define $\iota' : \mathcal{A}_{2n}^\kappa \to \mathcal{Q}_{\mathbf{n}}$ by $(r_*, p_*) \mapsto c_*$ where the multiset of entries of c_* is the union of the multiset of entries of $\Xi(r_*)$ with the multiset of entries of p_*. (We will see below that ι' is well defined.)

With the identifications above and those in 1.3, we see that the map $\Phi : \underline{\mathbf{W}} \to \underline{\underline{G}}$ becomes:

(a) $\iota : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}$ if $Q = 0, p \neq 2$, see [L8, 3.7, 1.1];

(b) $\iota' : \mathcal{A}_{2n}^1 \to \mathcal{Q}_{\mathbf{n}}$ if $Q \neq 0, \kappa = 1, p \neq 2$, see [L8, 3.8, 1.1];

(c) $\iota^{(2)} : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}^{(2)}$ if $Q = 0, p = 2$, see [L8, 4.6, 1.1];

(d) $\iota^{(2)} : \mathcal{A}_{2n}^1 \to \mathcal{T}_{2n}^{(2)}$ if $Q \neq 0, \kappa = 1, p = 2$, see [L8, 4.6, 1.1];

and that (when $Q \neq 0, \kappa = 0$) the map $\tilde{\Phi} : \tilde{\underline{\mathbf{W}}} \to \tilde{\underline{\underline{G}}}$ induced by Φ becomes:

(e) $\iota' : \mathcal{A}_{2n}^0 \to \mathcal{Q}_{\mathbf{n}}$ if $Q \neq 0, \kappa = 0, p \neq 2$, see [L8, 3.9, 1.1];

(f) $\tilde{\iota}^{(2)} : \mathcal{A}_{2n}^0 \to \tilde{\mathcal{T}}_{2n}^{(2)}$ if $Q \neq 0, \kappa = 0, p = 2$, see [L8, 4.6, 1.1].

In particular, ι' is well defined.

We see that to prove 0.2 it is enough to prove the following statement:

(g) *In each of the cases (a)–(f), the function $(r_*, p_*) \mapsto \tau_{p_*}/2$ on any fibre of the map described in that case, reaches its minimum value at exactly one element in that fibre.*

We have used that in the case where $Q \neq 0, \kappa = 0$, the fibre of Φ over any element in $\underline{\underline{G}}_{(0)}$ has exactly one element (necessarily in $\underline{\mathbf{W}}_0$) and the fibre of Φ over any element in $\underline{\underline{G}}_{(1)}$ is contained in $\underline{\mathbf{W}}_1$ and is the same as the fibre of $\tilde{\Phi}$ over that element.

1.5. We prove 1.4(g) in the case 1.4(a). Let $c_* \in \mathcal{T}_{2n}$. Let $(r_*, p_*) \in \iota^{-1}(c_*)$. Let $M_e = \mu_e(r_*)$, $N_e = \mu_e(p_*)$, $Q_e = \mu_e(c_*)$ so that $M_e + N_e = Q_e$. If e is odd, then $M_e = 0$ hence $N_e = Q_e$. Thus $\sum_e N_e \geq \sum_{e \text{ odd}} Q_e$. We see that the minimum value of the function 1.4(g) on $\iota^{-1}(c_*)$ is reached when $M_e = Q_e, N_e = 0$ for e even and $M_e = 0, N_e = Q_e$ for e odd. This proves 1.4(g) in our case.

1.6. We prove 1.4(g) in the case 1.4(c). Let $(c_*, \epsilon) \in \mathcal{T}_{2n}^{(2)}$. Let $(r_*, p_*) \in (\iota^{(2)})^{-1}(c_*, \epsilon)$. Let $M_e = \mu_e(r_*)$, $N_e = \mu_e(p_*)$, $Q_e = \mu_e(c_*)$ so that $M_e + N_e = Q_e$. If e is odd, then $M_e = 0$ hence $N_e = Q_e$. If e is even, Q_e is even ≥ 2 and $\epsilon(e) = 0$, then $M_e = 0$. Thus

$$\sum_e N_e \geq \sum_{e \text{ odd}} Q_e + \sum_{e \text{ even}, Q_e \text{ even}, \geq 2, \epsilon(e)=0} Q_e.$$

We see that the minimum value of the function 1.4(g) on $(\iota^{(2)})^{-1}(c_*, \epsilon)$ is reached when $M_e = 0, N_e = Q_e$ for e odd and for e even with Q_e even, $\geq 2, \epsilon(e) = 0$ and $M_e = Q_e, N_e = 0$ for all other e. This proves 1.4(g) in our case. The same proof yields 1.4(g) in the case 1.4(d); a similar proof yields 1.4(g) in the case 1.4(f).

1.7. In this subsection we prove 1.4(g) in the cases 1.4(b) and 1.4(e).

Let $\mathcal{R}$ be the set of all $r_* = (r_1 \geq r_2 \geq \cdots \geq r_\tau) \in \mathcal{Q}$ such that the following conditions are satisfied. Let $J_{r_*} = \{k \in [1, \tau]; r_k \text{ is odd}\}$. We write the multiset $\{r_k; k \in J_{r_*}\}$ as a sequence $r^1 \geq r^2 \geq \cdots \geq r^s$. (We have necessarily $\tau = s \mod 2$.) We require that:

-if $\tau \neq 0$, then $1 \in J_{r_*}$;

-if $\tau \neq 0$ is even, then $\tau \in J_{r_*}$;

-if $u \in [1, s-1]$ is odd, then $r^u > r^{u+1}$;

-if $u \in [1, s-1]$ is even, then there is no $k' \in [1, \tau]$ such that $r^u > r_{k'} > r^{u+1}$.

For $N \in \mathbf{N}$ we set $\mathcal{R}_N = \mathcal{R} \cap \mathcal{Q}_N$.

Note that for any even $N \in \mathbf{N}$, Ξ in 1.1 restricts to a bijection $\mathcal{S}_N^\kappa \to \mathcal{R}_{N+\kappa}$ with inverse map $\mathcal{R}_{N+\kappa} \to \mathcal{S}_N^\kappa$ given by

$$(r_1 \geq r_2 \geq \cdots \geq r_\tau) \mapsto (r_1 + \zeta(1) \geq r_2 + \zeta(2) \geq \cdots \geq r_\tau + \zeta(\tau))$$

if $r_\tau > \kappa$,

$$(r_1 \geq r_2 \geq \cdots \geq r_\tau) \mapsto (r_1 + \zeta(1) \geq r_2 + \zeta(2) \geq \cdots \geq r_{\tau-1} + \zeta(\tau - 1))$$

if $r_\tau = \kappa$, where $\zeta : [1, \tau] \to \{-1, 0, 1\}$ is given by $\zeta(k) = (-1)^k(1 - (-1)^{r_k})/2$. (Thus, $\zeta(k) = (-1)^k$ if r_k is odd and $\zeta(k) = 0$ if r_k is even. We have $r_k + \zeta(k) \in 2\mathbf{N}$ for any k and $r_k + \zeta(k) \geq r_{k+1} + \zeta(k+1)$ for $k \in [1, \tau - 1]$.)

Let $A_{\mathbf{n}}$ be the set of all pairs $(r_*, p_*) \in \mathcal{R} \times \tilde{\mathcal{P}}$ such that $|r_*| + |p_*| = \mathbf{n}$. We define $\tilde{\iota} : A_{\mathbf{n}} \to \mathcal{Q}_{\mathbf{n}}$ by $(r_*, p_*) \mapsto c_*$ where the multiset of entries of c_* is the union of the multiset of entries of r_* with the multiset of entries of p_*. In view of the bijection $\mathcal{S}_N^\kappa \to \mathcal{R}_{N+\kappa}$ defined by the restriction of Ξ we see that to prove 1.4(g) in our case it is enough to prove the following statement.

(a) *For any $c_* \in \mathcal{Q}_{\mathbf{n}}$ there is exactly one element $(r_*, p_*) \in \tilde{\iota}^{-1}(c_*)$ such that the number of entries of p_* is minimal.*

Let $c_* = (c_1 \geq c_2 \geq \cdots \geq c_\tau) \in \mathcal{Q}_{\mathbf{n}}$. Let $K = \{k \in [1, \tau]; c_k \text{ is odd}\}$. We write the multiset $\{c_k; k \in K\}$ as a sequence $c^1 \geq c^2 \geq \cdots \geq c^t$. (We have necessarily $\tau = \mathbf{n} = t \mod 2$.) We associate to c_* an element $(r_*, p_*) \in \mathcal{Q} \times \tilde{\mathcal{P}}$ by specifying $M_e = \mu_e(r_*)$, $N_e = \mu_e(p_*)$ for $e \geq 1$. Let $Q_e = \mu_e(c_*)$.

(i) If $e \in 2\mathbf{N} + 1$ and $Q_e = 2g + 1$, then $M_e = 1$, $N_e = 2g$.

(ii) If $e \in 2\mathbf{N}+1$ and $Q_e = 2g$, so that $c^d = c^{d+1} = \cdots = c^{d+2g-1} = e$ with d even, then $M_e = 2$, $N_e = 2g-2$ (if $g > 0$) and $M_e = N_e = 0$ (if $g = 0$).

(iii) If $e \in 2\mathbf{N}+1$ and $Q_e = 2g$ so that $c^d = c^{d+1} = \cdots = c^{d+2g-1} = e$ with d odd, then $M_e = 0$, $N_e = 2g$.

Thus the odd entries of r_* are defined. We write them in a sequence $r^1 \geq r^2 \geq \cdots \geq r^s$.

(iv) If $e \in 2\mathbf{N}+2$, $Q_e = 2g$ and if

$(*)$ $r^{2v} > e > r^{2v+1}$ for some v, or $e > r^1$, or $r^s > e$ (with s even), then $M_e = 0, N_e = 2g$.

(v) If $e \in 2\mathbf{N}+2$, $Q_e = 2g$ and if $(*)$ does not hold, then $M_e = 2g, N_e = 0$.

Now $r_* \in \mathcal{Q}, p_* \in \tilde{\mathcal{P}}$ are defined and $|r_*| + |p_*| = \mathbf{n}$.

Assume that $|r_*| > 0$; then from (iv) we see that the largest entry of r_* is odd. Assume that $|r_*| > 0$ and $\mathbf{n}$ is even; then from (iv) we see that the smallest entry of r_* is odd. If $u \in [1, s-1]$ and $r^u = r^{u+1}$, then from (i), (ii), (iii) we see that u is even. If $u \in [1, s-1]$ and there is $k' \in [1, \tau]$ such that $r^u > r_{k'} > r^{u+1}$, then $r_{k'}$ is even and $e = r_{k'}$ is as in (v) and u must be odd. We see that $r_* \in \mathcal{R}$.

We see that $c_* \mapsto (r_*, p_*)$ is a well-defined map $\tilde{\iota}' : \mathcal{Q}_{\mathbf{n}} \to A_{\mathbf{n}}$; moreover, $\tilde{\iota}\tilde{\iota}' : \mathcal{Q}_{\mathbf{n}} \to \mathcal{Q}_{\mathbf{n}}$ is the identity map.

We preserve the notation for c_*, r_*, p_* as above (so that $(r_*, p_*) \in \tilde{\iota}^{-1}(c_*)$) and we assume that $(r'_*, p'_*) \in \tilde{\iota}^{-1}(c_*)$. We write the odd entries of r'_* in a sequence $r'^1 \geq r'^2 \geq \cdots \geq r'^{s'}$.

Let $M'_e = \mu_e(r'_*)$, $N'_e = \mu_e(p'_*)$ for $e \geq 1$. Note that $M'_e + N'_e = M_e + N_e$.

In the setup of (i) we have $M'_e = 1, N'_e = N_e$. (Indeed, $M'_e + N'_e$ is odd, N'_e is even hence M'_e is odd. Since M'_e is $0, 1$ or 2 we see that it is 1.)

In the setup of (ii) and assuming that $g > 0$ we have $M'_e = 2, N'_e = N_e$ or $M'_e = 0, N'_e = N_e + 2$. (Indeed, $M'_e + N'_e$ is even, N'_e is even hence M'_e is even. Since M'_e is $0, 1$ or 2 we see that it is 0 or 2.) If $g = 0$, we have $M'_e = N'_e = 0$.

In the setup of (iii) we have $M'_e = 0, N'_e = N_e$. (Indeed, $M'_e + N'_e$ is even, N'_e is even hence M'_e is even. Since M'_e is $0, 1$ or 2 we see that it is 0 or 2. Assume that $M'_e = 2$. Then $e = r'^u = r'^{u+1}$ with u even in $[1, s'-1]$. We have $c^d = c^{d+1} = \cdots = c^{d+2g-1} = e$ with d odd. From the definitions we see that $u = d \mod 2$ and we have a contradiction. Thus, $M'_e = 0$.)

Now the sequence $r'^1 \geq r'^2 \geq \cdots \geq r'^{s'}$ is obtained from the sequence $r^1 \geq r^2 \geq \cdots \geq r^s$ by deleting some pairs of the form $r^{2h} = r^{2h+1}$. Hence in the setup of (iv) we have $r^{2v} = r'^{2v'} > e > r'^{2v'+1} = r^{2v+1}$ for some v' or $e > r'^1$ or $r'^{s'} > e$ (with s, s' even) and we see that $M'_e = 0$ so that $N'_e = 2g = N_e$.

In the setup of (v) we have $N'_e \geq 0$.

We see that in all cases we have $N'_e \geq N_e$. It follows that $\sum_e N'_e \geq \sum_e N_e$ (and the equality implies that $N'_e = N_e$ for all e hence $(r'_*, p'_*) = (r_*, p_*)$). This proves (a) and completes the proof of 1.4(g) in all cases hence the proof of 0.2 for G almost simple of type B, C or D.

2. Exceptional groups

2.1. In 2.2–2.6 we describe explicitly the map $\Phi : \underline{\mathbf{W}} \to \underline{\underline{G}}$ in the case where G is a simple exceptional group in the form of tables. Each table consists of lines of the form $[a, b, \ldots, r] \mapsto \gamma$ where $\gamma \in \underline{\underline{G}}$ is specified by its name in [S2] and $a, b, \ldots, r$ are the elements of $\underline{\mathbf{W}}$ which are mapped by Φ to γ (here $a, b, \ldots, r$ are specified by their name in [Ca]); by inspection we see that 0.2 holds in each case and in fact

$\Psi(\gamma) = a$ is the first element of $\underline{\mathbf{W}}$ in the list $a, b, \ldots, r$. The tables are obtained from the results in [L8].

2.2. Type G_2. If $p \neq 3$, we have

$[A_0] \mapsto A_0$
$[A_1] \mapsto A_1$
$[A_1 + \tilde{A}_1, \tilde{A}_1] \mapsto \tilde{A}_1$
$[A_2] \mapsto G_2(a_1)$
$[G_2] \mapsto G_2$.

When $p = 3$ the line $[A_1 + \tilde{A}_1, \tilde{A}_1] \mapsto \tilde{A}_1$ should be replaced by $[A_1 + \tilde{A}_1] \mapsto \tilde{A}_1$, $[\tilde{A}_1] \mapsto (\tilde{A}_1)_3$.

2.3. Type F_4. If $p \neq 2$, we have

$[A_0] \mapsto A_0$
$[A_1] \mapsto A_1$
$[2A_1, \tilde{A}_1] \mapsto \tilde{A}_1$
$[4A_1, 3A_1, 2A_1 + \tilde{A}_1, A_1 + \tilde{A}_1] \mapsto A_1 + \tilde{A}_1$
$[A_2] \mapsto A_2$
$[\tilde{A}_2] \mapsto \tilde{A}_2$
$[A_2 + \tilde{A}_1] \mapsto A_2 + \tilde{A}_1$
$[A_2 + \tilde{A}_2, \tilde{A}_2 + A_1] \mapsto \tilde{A}_2 + A_1$
$[A_3, B_2] \mapsto B_2$
$[A_3 + \tilde{A}_1, B_2 + A_1] \mapsto C_3(a_1)$
$[D_4(a_1)] \mapsto F_4(a_3)$
$[D_4, B_3] \mapsto B_3$
$[C_3 + A_1, C_3] \mapsto C_3$
$[F_4(a_1)] \mapsto F_4(a_2)$
$[B_4] \mapsto F_4(a_1)$
$[F_4] \mapsto F_4$.

When $p = 2$ the lines $[2A_1, \tilde{A}_1] \mapsto \tilde{A}_1$, $[A_2 + \tilde{A}_2, \tilde{A}_2 + A_1] \mapsto \tilde{A}_2 + A_1$, $[A_3, B_2] \mapsto B_2$, $[A_3 + \tilde{A}_1, B_2 + A_1] \mapsto C_3(a_1)$, should be replaced by

$[2A_1] \mapsto \tilde{A}_1$, $[\tilde{A}_1] \mapsto (\tilde{A}_1)_2$
$[A_2 + \tilde{A}_2] \mapsto \tilde{A}_2 + A_1$, $[\tilde{A}_2 + A_1] \mapsto (\tilde{A}_2 + A_1)_2$
$[A_3] \mapsto B_2$, $[B_2] \mapsto (B_2)_2$
$[A_3 + \tilde{A}_1] \mapsto C_3(a_1)$, $[B_2 + A_1] \mapsto (C_3(a_1))_2$,

respectively.

2.4. Type E_6. We have

$[A_0] \mapsto A_0$
$[A_1] \mapsto A_1$
$[2A_1] \mapsto 2A_1$
$[4A_1, 3A_1] \mapsto 3A_1$
$[A_2] \mapsto A_2$
$[A_2 + A_1] \mapsto A_2 + A_1$
$[2A_2] \mapsto 2A_2$
$[A_2 + 2A_1] \mapsto A_2 + 2A_1$
$[A_3] \mapsto A_3$
$[3A_2, 2A_2 + A_1] \mapsto 2A_2 + A_1$
$[A_3 + 2A_1, A_3 + A_1] \mapsto A_3 + A_1$

$[D_4(a_1)] \mapsto D_4(a_1)$
$[A_4] \mapsto A_4$
$[D_4] \mapsto D_4$
$[A_4 + A_1] \mapsto A_4 + A_1$
$[A_5 + A_1, A_5] \mapsto A_5$
$[D_5(a_1)] \mapsto D_5(a_1)$
$[E_6(a_2)] \mapsto A_5 + A_1$
$[D_5] \mapsto D_5$
$[E_6(a_1)] \mapsto E_6(a_1)$
$[E_6] \mapsto E_6$.

2.5. Type E_7. If $p \neq 2$, we have
$[A_0] \mapsto A_0$
$[A_1] \mapsto A_1$
$[2A_1] \mapsto 2A_1$
$[(3A_1)'] \mapsto (3A_1)''$
$[(4A_1)'', (3A_1)''] \mapsto (3A_1)'$
$[A_2] \mapsto A_2$
$[7A_1, 6A_1, 5A_1, (4A_1)'] \mapsto 4A_1$
$[A_2 + A_1] \mapsto A_2 + A_1$
$[A_2 + 2A_1] \mapsto A_2 + 2A_1$
$[A_3] \mapsto A_3$
$[2A_2] \mapsto 2A_2$
$[A_2 + 3A_1] \mapsto A_2 + 3A_1$
$[(A_3 + A_1)'] \mapsto (A_3 + A_1)''$
$[3A_2, 2A_2 + A_1] \mapsto 2A_2 + A_1$
$[(A_3 + 2A_1)'', (A_3 + A_1)''] \mapsto (A_3 + A_1)'$
$[D_4(a_1)] \mapsto D_4(a_1)$
$[A_3 + 3A_1, (A_3 + 2A_1)'] \mapsto A_3 + 2A_1$
$[D_4] \mapsto D_4$
$[D_4(a_1) + A_1] \mapsto D_4(a_1) + A_1$
$[D_4(a_1) + 2A_1, A_3 + A_2] \mapsto A_3 + A_2$
$[2A_3 + A_1, A_3 + A_2 + A_1] \mapsto A_3 + A_2 + A_1$
$[A_4] \mapsto A_4$
$[D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] \mapsto D_4 + A_1$
$[A_5'] \mapsto A_5''$
$[A_4 + A_1] \mapsto A_4 + A_1$
$[D_5(a_1)] \mapsto D_5(a_1)$
$[A_4 + A_2] \mapsto A_4 + A_2$
$[(A_5 + A_1)'', A_5''] \mapsto A_5'$
$[A_5 + A_2, (A_5 + A_1)'] \mapsto (A_5 + A_1)''$
$[D_5(a_1) + A_1] \mapsto D_5(a_1) + A_1$
$[E_6(a_2)] \mapsto (A_5 + A_1)'$
$[D_6(a_2) + A_1, D_6(a_2)] \mapsto D_6(a_2)$
$[E_7(a_4)] \mapsto D_6(a_2) + A_1$
$[D_5] \mapsto D_5$
$[A_6] \mapsto A_6$
$[D_5 + A_1] \mapsto D_5 + A_1$
$[D_6(a_1)] \mapsto D_6(a_1)$

$[A_7] \mapsto D_6(a_1) + A_1$
$[E_6(a_1)] \mapsto E_6(a_1)$
$[D_6 + A_1, D_6] \mapsto D_6$
$[E_6] \mapsto E_6$
$[E_7(a_3)] \mapsto D_6 + A_1$
$[E_7(a_2)] \mapsto E_7(a_2)$
$[E_7(a_1)] \mapsto E_7(a_1)$
$[E_7] \mapsto E_7$.

If $p = 2$, the line $[D_4(a_1) + 2A_1, A_3 + A_2] \mapsto A_3 + A_2$ should be replaced by $[D_4(a_1) + 2A_1] \mapsto A_3 + A_2$, $[A_3 + A_2] \mapsto (A_3 + A_2)_2$.

2.6. Type E_8. If $p \neq 2, 3$, we have

$[A_0] \mapsto A_0$
$[A_1] \mapsto A_1$
$[2A_1] \mapsto 2A_1$
$[(4A_1)', 3A_1] \mapsto 3A_1$
$[A_2] \mapsto A_2$
$[8A_1, 7A_1, 6A_1, 5A_1, (4A_1)''] \mapsto 4A_1$
$[A_2 + A_1] \mapsto A_2 + A_1$
$[A_2 + 2A_1] \mapsto A_2 + 2A_1$
$[A_3] \mapsto A_3$
$[A_2 + 4A_1, A_2 + 3A_1] \mapsto A_2 + 3A_1$
$[2A_2] \mapsto 2A_2$
$[3A_2, 2A_2 + A_1] \mapsto 2A_2 + A_1$
$[(A_3 + 2A_1)', A_3 + A_1] \mapsto A_3 + A_1$
$[D_4(a_1)] \mapsto D_4(a_1)$
$[4A_2, 3A_2 + A_1, 2A_2 + 2A_1] \mapsto 2A_2 + 2A_1$
$[D_4] \mapsto D_4$
$[A_3 + 4A_1, A_3 + 3A_1, (A_3 + 2A_1)''] \mapsto A_3 + 2A_1$
$[D_4(a_1) + A_1] \mapsto D_4(a_1) + A_1$
$[(2A_3)', A_3 + A_2] \mapsto A_3 + A_2$
$[A_4] \mapsto A_4$
$[2A_3 + 2A_1, A_3 + A_2 + 2A_1, 2A_3 + A_1, A_3 + A_2 + A_1] \mapsto A_3 + A_2 + A_1$
$[D_4(a_1) + A_2] \mapsto D_4(a_1) + A_2$
$[D_4 + 4A_1, D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] \mapsto D_4 + A_1$
$[2D_4(a_1), D_4(a_1) + A_3, (2A_3)''] \mapsto 2A_3$
$[A_4 + A_1] \mapsto A_4 + A_1$
$[D_5(a_1)] \mapsto D_5(a_1)$
$[A_4 + 2A_1] \mapsto A_4 + 2A_1$
$[A_4 + A_2] \mapsto A_4 + A_2$
$[A_4 + A_2 + A_1] \mapsto A_4 + A_2 + A_1$
$[D_5(a_1) + A_1] \mapsto D_5(a_1) + A_1$
$[(A_5 + A_1)', A_5] \mapsto A_5$
$[D_4 + A_3, D_4 + A_2] \mapsto D_4 + A_2$
$[E_6(a_2)] \mapsto (A_5 + A_1)''$
$[2A_4, A_4 + A_3] \mapsto A_4 + A_3$
$[D_5] \mapsto D_5$
$[D_5(a_1) + A_3, D_5(a_1) + A_2] \mapsto D_5(a_1) + A_2$
$[A_5 + A_2 + A_1, A_5 + A_2, A_5 + 2A_1, (A_5 + A_1)''] \mapsto (A_5 + A_1)'$

$[E_6(a_2)+A_2, E_6(a_2)+A_1] \mapsto A_5+2A_1$
$[2D_4, D_6(a_2)+A_1, D_6(a_2)] \mapsto D_6(a_2)$
$[E_7(a_4)+A_1, E_7(a_4)] \mapsto A_5+A_2$
$[D_5+2A_1, D_5+A_1] \mapsto D_5+A_1$
$[E_8(a_8)] \mapsto 2A_4$
$[D_6(a_1) \mapsto D_6(a_1)]$
$[A_6] \mapsto A_6$
$[A_6+A_1] \mapsto A_6+A_1$
$[A_7'] \mapsto D_6(a_1)+A_1$
$[A_7+A_1, D_5+A_2] \mapsto D_5+A_2$
$[E_6(a_1)] \mapsto E_6(a_1)$
$[D_6+2A_1, D_6+A_1, D_6] \mapsto D_6$
$[D_7(a_2)] \mapsto D_7(a_2)$
$[E_6] \mapsto E_6$
$[D_8(a_3), A_7''] \mapsto A_7$
$[E_6(a_1)+A_1] \mapsto E_6(a_1)+A_1$
$[E_7(a_3)] \mapsto D_6+A_1$
$[A_8] \mapsto D_8(a_3)$
$[D_8(a_2), D_7(a_1)] \mapsto D_7(a_1)$
$[E_6+A_2, E_6+A_1] \mapsto E_6+A_1$
$[E_7(a_2)+A_1, E_7(a_2)] \mapsto E_7(a_2)$
$[E_8(a_6)] \mapsto A_8$
$[E_8(a_7)] \mapsto E_7(a_2)+A_1$
$[D_8(a_1), D_7] \mapsto D_7$
$[E_8(a_3)] \mapsto D_8(a_1)$
$[E_7(a_1)] \mapsto E_7(a_1)$
$[D_8] \mapsto E_7(a_1)+A_1$
$[E_8(a_5)] \mapsto D_8$
$[E_7+A_1, E_7] \mapsto E_7$
$[E_8(a_4)] \mapsto E_7+A_1$
$[E_8(a_2)] \mapsto E_8(a_2)$
$[E_8(a_1)] \mapsto E_8(a_1)$
$[E_8] \mapsto E_8$.

If $p=3$, the line $[D_8(a_3), A_7''] \mapsto A_7$ should be replaced by $[D_8(a_3)] \mapsto A_7$, $[A_7''] \mapsto (A_7)_3$. If $p=2$, the lines $[(2A_3)', A_3+A_2] \mapsto A_3+A_2$, $[D_4+A_3, D_4+A_2] \mapsto D_4+A_2$, $[A_7+A_1, D_5+A_2] \mapsto D_5+A_2$, $[D_8(a_2), D_7(a_1)] \mapsto D_7(a_1)$ should be replaced by

$[(2A_3)'] \mapsto A_3+A_2$, $[A_3+A_2] \mapsto (A_3+A_2)_2$
$[D_4+A_3] \mapsto D_4+A_2$, $[D_4+A_2] \mapsto (D_4+A_2)_2$
$[A_7+A_1] \mapsto D_5+A_2$, $[D_5+A_2] \mapsto (D_5+A_2)_2$
$[D_8(a_2)] \mapsto D_7(a_1)$, $[D_7(a_1)] \mapsto (D_7(a_1))_2$,

respectively.

2.7. From the tables above we see that (assuming that G is almost simple of exceptional type) the following holds.

(a) *If γ is a distinguished unipotent class in G, then $\Phi^{-1}(\gamma)$ consists of a single conjugacy class in* $\mathbf{W}$.

In fact (a) is also valid without any assumption on G. Indeed, assume that $C \in \Phi^{-1}(\gamma)$. From the arguments in [L8, 1.1] we see that if C is not elliptic, then $\Phi(C)$ is not distinguished. Thus C must be elliptic. Then the desired result follows

from the injectivity of the restriction of Φ to elliptic conjugacy classes in $\mathbf{W}$, see [L8, 0.6].

2.8. We have the following result (for general G).

(a) *If C is an elliptic conjugacy class in $\mathbf{W}$, then $C = \Psi(\Phi(C))$. In particular, C is in the image of $\Psi : \underline{\underline{G}} \to \underline{\mathbf{W}}$.*

Since $C \in \Phi^{-1}(\Phi(C))$, (a) follows immediately from Theorem 0.2.

3. Proof of Theorem 0.4

3.1. There is a well-defined (injective) map $\pi' : \underline{\underline{G}}_0 \to \underline{\underline{G}}$, $\gamma_0 \mapsto \pi'(\gamma_0)$, where $\pi'(\gamma_0)$ is the unique unipotent class in G which has the same Springer representation of $\mathbf{W}$ as γ_0. One can show that π' coincides with the order preserving and dimension preserving imbedding defined in [S1, III, 5.2].

To prove Theorem 0.4(a) we can assume that G, G_0 are almost simple. It is also enough to prove the theorem for a single G in each isogeny class of almost simple groups. We can assume that p is a bad prime for G (if p is not a bad prime, the result is obvious). Now G, G_0 cannot be of type A since p is a bad prime for G. In the case where G is of exceptional type, the theorem follows by inspection of the tables in Section 2. In the case where G is of type B, C or D so that $p = 2$, we define $\tilde{\rho} : \underline{\underline{G}} \to \underline{\underline{G}}_0$ by $\tilde{\rho}(\gamma) = \gamma_0$ where $\gamma \in \underline{\underline{G}}$ is in the "unipotent piece" of G indexed by $\gamma_0 \in \underline{\underline{G}}_0$ (in the sense of [L6]). It is enough to prove that

$$\text{(a)} \qquad \tilde{\rho}\Phi = \Phi_0.$$

(This would prove the existence of ρ in Theorem 0.4 and that $\rho = \tilde{\rho}$.) If G, G_0 are of type C_n ($n \geq 2$), then Φ_0, Φ may be identified with $\iota : \mathcal{A}^1_{2n} \to \mathcal{T}_{2n}$, $\iota^{(2)} : \mathcal{A}^1_{2n} \to \mathcal{T}^{(2)}_{2n}$ (see 1.4) and by [L4], $\tilde{\rho}$ may be identified with $\mathcal{T}^{(2)}_{2n} \to \mathcal{T}_{2n}$, $(c_*, \epsilon) \mapsto c_*$ (notation of 1.4). Then the identity (a) is obvious. The proof of (a) for G of type B and D is given in 3.5.

Similarly, to prove Theorem 0.4(b) it is enough to prove that

$$\text{(b)} \qquad \Psi\pi' = \Psi_0.$$

(This would prove the existence of π in Theorem 0.4 and that $\pi = \pi'$.) Again, it is enough to prove (b) in the case where G is of type B, C or D so that $p = 2$. The proof is given in 3.9.

3.2. In this subsection we assume that all simple factors of G are of type A, B, C or D. Then $\tilde{\rho} : \underline{\underline{G}} \to \underline{\underline{G}}_0$ is defined as in 3.1. We show:

(a) $\Phi_0(C) = \tilde{\rho}\Phi(C)$ *for any elliptic conjugacy class C in $\mathbf{W}$.*

From the definitions we see that

(b) $\tilde{\rho}\pi' = 1$.

Now for C as in (a) we have by definition $\Phi(C) = \pi'\Phi_0(C)$. To prove (a) it is enough to show that $\Phi_0(C) = \tilde{\rho}\pi\Phi_0(C)$. But this follows from (b).

3.3. In this subsection we assume that $V, Q, (,), \mathbf{n}, Is(V)$ are as in 1.2 with $p = 2$ and that either $Q \neq 0$ or $V = 0$. Let $SO(V)$ be the identity component of $Is(V)$. Let U_V be the set of unipotent elements in $SO(V)$.

We say that $u \in U_V$ is split if the corresponding pair (c_*, ϵ) (see 1.4) satisfies $\mu_j(c_*) =$even for all j and $\epsilon(j) = 0$ for all even j such that $\mu_j(c_*) > 0$ (μ_j as in 1.1).

For $u \in U_V$ let $e = e_u$ be the smallest integer ≥ 0 such that $(u-1)^e = 0$. (When $u = 1$ we have $e = 1$ if $\mathbf{n} > 0$ and $e = 0$ if $\mathbf{n} = 0$. When $u \neq 1$ we have $e \geq 2$.) If $u \neq 1$, we define $\lambda = \lambda_u : V \to \mathbf{k}$ by $\lambda(x) = \sqrt{(x, (u-1)^{e-1}x)}$, a linear form on V; in this case we define $L = L_u$ as follows:

$$L = (u-1)^{e-1}V \text{ if } \lambda = 0; \quad L = (\ker\lambda)^\perp \text{ if } \lambda \neq 0, \mathbf{n} = \text{even};$$
$$L = \{x \in (\ker\lambda)^\perp; Q(x) = 0 \text{ if } \lambda \neq 0, \mathbf{n} = \text{odd}.$$

Note that $L \neq 0$.

As in [L5, 2.5], for any $u \in U_V$ we define subspaces $V^a = V^a_u$ ($a \in \mathbf{Z}$) of V as follows. If $u = 1$ we set $V^a = V$ for $a \leq 0$, $V^a = 0$ for $a \geq 1$. If $u \neq 1$ so that $e = e_u \geq 2$, and $\lambda = \lambda_u$, $L = L_u$ are defined, we have $L \subset L^\perp$, $Q|L = 0$ (see [L5, 2.2(a), (e)]) and we set $\bar{V} = L^\perp/L$; now $\bar{V}$ has an induced nondegenerate quadratic form and there is an induced unipotent element $\bar{u} \in SO(\bar{V})$; let $r : L^\perp \to L^\perp/L = \bar{V}$ be the canonical map. Since $\dim\bar{V} < \dim V$, we can assume by induction that $\bar{V}^a = \bar{V}^a_{\bar{u}}$ are defined for $a \in \mathbf{Z}$. We set

$$V^a = V \text{ if } a \leq 1-e, \quad V^a = r^{-1}(\bar{V}^a) \text{ if } 2-e \leq a \leq e-1, \quad V^a = 0 \text{ if } a \geq e$$

when $\lambda = 0$ and

$$V^a = V \text{ if } a \leq -e, \quad V^a = r^{-1}(\bar{V}^a) \text{ if } 1-e \leq a \leq e, \quad V^a = 0 \text{ if } a \geq 1 = e$$

when $\lambda \neq 0$. This completes the inductive definition of $V^a = V^a_u$.

Note that if $u \in U_V$ is split and $u \neq 1$, then $\bar{u}$ (as above) is split.

Now assume that $V = V' \oplus V''$ where V', V'' are subspaces of V such that $(V', V'') = 0$ so that $Q|_{V'}$, $Q|_{V'}$ are nondegenerate and let $u \in U_V$ be such that V', V'' are u-stable; we set $u' = u|_{V'}$, $u'' = u|_{V''}$. We assume that $u' \in U_{V'}$, $u'' \in U_{V''}$. For $a \in \mathbf{Z}$ we set $V^a = V^a_u$, $V'^a = V'^a_{u'}$, $V''^a = V''^a_{u''}$. We show:

(a) *If u'' is split, then $V^a = V'^a \oplus V''^a$ for all a.*

If $u = 1$, then (a) is trivial. So we can assume that $u \neq 1$. We can also assume that (a) is true when V is replaced by a vector space of smaller dimension. Let $e = e_u, e' = e_{u'}, e'' = e_{u''}$. Let $\lambda = \lambda_u, L = L_u$. Let $r : L^\perp \to L^\perp/L = \bar{V}$ be as above. If $u' \neq 1$ (resp. $u'' \neq 1$), we define $\lambda', L', r', \bar{V}'$ (resp. $\lambda'', L'', r'', \bar{V}''$) in terms of $u', Q_{V'}$ (resp. $u'', Q_{V''}$) in the same way as $\lambda, L, r, \bar{V}$ were defined in terms of u, Q. Note that $\lambda'' = 0$ (when $u'' \neq 1$) since u'' is split.

Assume first that $e'' > e'$. We have $e = e''$ hence $u'' \neq 1$. Moreover, $L = 0 \oplus L''$, $\bar{V} = V' \oplus \bar{V}''$. By the induction hypothesis we have $\bar{V}^a = V'^a \oplus \bar{V}''^a$ for all a. If $a \leq 1-e$, then $V^a = V = V' \oplus V'' = V'^a \oplus V''^a$. (We use that $V' = V'^a$; if $e' \geq 2$ this follows from $a \leq -e'$; if $e' \leq 1$ this follows from $a \leq 0$). If $2-e \leq a \leq e-1$ we have

$$V^a = (0 \oplus r'')^{-1}(\bar{V}^a) = (0 \oplus r'')^{-1}(V'^a \oplus \bar{V}''^a) = V'^a \oplus V''^a$$

(we use that $e'' = e$); if $a \geq e$, then $V^a = 0 = V'^a \oplus V''^a$. (We use that $V'^a = 0$; if $e' \geq 2$ this follows from $a \geq e'+1$; if $e' \leq 1$ this follows from $a \geq 1$.)

Next we assume that $e' = e''$ (hence both are equal to $e \geq 2$) and that $\lambda = 0$ (hence $\lambda' = 0$). Then $L = L' \oplus L''$, $\bar{V} = \bar{V}' \oplus \bar{V}''$. By the induction hypothesis we have $\bar{V}^a = \bar{V}'^a \oplus \bar{V}''^a$ for all a. If $a \leq 1-e$, then $V^a = V = V' \oplus V'' = V'^a \oplus V''^a$. If $2-e \leq a \leq e-1$, we have

$$V^a = (r' \oplus r'')^{-1})(\bar{V}^a) = (r' \oplus r'')^{-1})(\bar{V}'^a \oplus \bar{V}''^a) = V'^a \oplus V''^a.$$

If $a \geq e$, then $V^a = 0 = V'^a \oplus V''^a$.

Next we assume that $e' > e''$ (hence $e' = e \geq 2$) and $\lambda = 0$ (hence $\lambda' = 0$). Then $L = L' \oplus 0$, $\bar{V} = \bar{V}' \oplus V''$. By the induction hypothesis we have $\bar{V}^a = \bar{V}'^a \oplus V''^a$ for all a. If $a \leq 1-e$, then $V^a = V = V' \oplus V'' = V'^a \oplus V''^a$. (We use that $a \leq 1-e''$.) If $2-e \leq a \leq e-1$, we have

$$V^a = (r' \oplus 0)^{-1}(\bar{V}^a) = (r' \oplus 0)(\bar{V}'^a \oplus V''^a) = V'^a \oplus V''^a.$$

If $a \geq e$, then $V^a = 0 = V'^a \oplus V''^a$. (We use that $a \geq e''$.)

Finally, we assume that $e' \geq e''$ (hence $e' = e \geq 2$) and $\lambda \neq 0$ (hence $\lambda' \neq 0$). Then $L = L' \oplus 0$, $\bar{V} = \bar{V}' \oplus V''$. By the induction hypothesis we have $\bar{V}^a = \bar{V}'^a \oplus V''^a$ for all a. If $a \leq -e$, then $V^a = V = V' \oplus V'' = V'^a \oplus V''^a$. (We use that $a \leq 1-e''$.) If $1-e \leq a \leq e$, we have

$$V^a = (r'+0)^{-1}(\bar{V}^a) = (r'+0)^{-1}(\bar{V}'^a \oplus V''^a) = V'^a \oplus V''^a.$$

If $a \geq e+1$, then $V^a = 0 = V'^a \oplus V''^a$. (We use that $a \geq e''$.)

This completes the proof of (a).

3.4. In this subsection we assume that $V, Q, (,), \mathbf{n}, Is(V)$ are as in 1.2 with $p = 2$ and $\mathbf{n} \geq 3$. Let $SO(V)$ be the identity component of $Is(V)$. Let V_0 be a $\mathbf{C}$-vector space of dimension $\mathbf{n}$ with a fixed nondegenerate symmetric bilinear form $(,)$. Let $SO(V_0)$ be the corresponding special orthogonal group. Let U_{V_0} be the set of unipotent elements in $SO(V_0)$. For any $u_0 \in U_{V_0}$ we define subspaces $V_0^a = (V_0)^a_{u_0}$ ($a \in \mathbf{Z}$) of V_0 in the same way as V_u^a were defined in 3.5 for $u \in U_V$ except that we now take λ to always be zero (compare [L5, 3.3]).

Assume that we are given a direct sum decomposition

$$V = V' \oplus V_1'' \oplus V_2'' \oplus \ldots \oplus V_{2k-1}'' \oplus V_{2k}''$$

where V', V_i'' are subspaces of V such that $Q|_{V_i''} = 0$, $(V', V_i'') = 0$ for $i \in [1, 2k]$, $(V_i'', V_j'') = 0$ if $i + j \neq 2k+1$. Let

$$V'' = V_1'' \oplus V_2'' \oplus \ldots \oplus V_{2k-1}'' \oplus V_{2k}''.$$

Let

$$V_0 = V_0' \oplus V_{10}'' \oplus V_{20}'' \oplus \ldots \oplus V_{2k-1,0}'' \oplus V_{2k,0}''$$

be a direct sum decomposition where V_0', V_{i0}'' are subspaces of V_0 such that $(,)$ is 0 on V_{i0}'', $(V_0', V_{i0}'') = 0$ for $i \in [1, 2k]$, $(V_{i0}'', V_{j0}'') = 0$ if $i + j \neq 2k+1$. Let

$$V_0'' = V_{10}'' \oplus V_{20}'' \oplus \ldots \oplus V_{2k-1,0}'' \oplus V_{2k,0}''.$$

Let L (resp. L_0) be the simultaneous stabilizer in $SO(V)$ (resp. $SO(V_0)$) of the subspaces V', V_i'' (resp. V_0', V_{i0}''), $i \in [1, 2k]$.

Let $u \in SO(V)$ be such that $uV' = V'$, $uV_i'' = V_i''$ for $i \in [1, 2k]$ and such that the $SO(V)$-conjugacy class γ of u is also an $Is(V)$-conjugacy class. Then $uV'' = V''$. Let $u' \in SO(V')$, $u'' \in SO(V'')$ be the restrictions of u. Note that u'' is split. Let γ_1 be the conjugacy class of u in L. Let $\gamma_1' = \tilde{\rho}_1(\gamma_1)$, a unipotent class in L_0; here $\tilde{\rho}_1 : \underline{\underline{L}} \to \underline{\underline{L}}_0$ is defined like $\tilde{\rho}$ in 3.1 but in terms of L, L_0 instead of G, G_0. Let $u_0 \in \gamma_1'$. Let u_0', u_0'' be the restrictions of u_0 to V_0', V_0''. By results in [L5, 2.9], we have

$$\dim {V'_{0,u_0'}}^a = \dim {V'_{u'}}^a, \quad \dim {V''_{0,u_0''}}^a = \dim {V''_{u''}}^a$$

for all a.

Let γ_0 be the conjugacy class of u_0 in $SO(V_0)$. From 3.3(a) and the analogous result for $V_0 = V_0' \oplus V_0''$ we see that $\dim {V_{0,u_0}}^a = \dim V_u^a$ for all a. From this we

deduce, using the definitions and the fact that γ is an $Is(V)$-conjugacy class, that $\gamma_0 = \tilde{\rho}(\gamma)$.

3.5. We now prove 3.1(a) assuming that $G = SO(V), G_0 = SO(V_0)$ are as in 3.4. Let $C \in \underline{\mathbf{W}}$ be such that C is a W-conjugacy class. We can find a standard parabolic subgroup $\mathbf{W}'$ of $\mathbf{W}$ and an elliptic conjugacy class C' of $\mathbf{W}'$ such that $C' \subset C$. Let P (resp. P_0) be a parabolic subgroup of G (resp. G_0) of the same type as $\mathbf{W}'$. Let L be a Levi subgroup of P and let L_0 be a Levi subgroup of P_0. We can assume that L, L_0 are as in 3.4. Let $\Phi_0^{L_0}, \Phi^L$ be the maps analogous to Φ_0, Φ with G_0, G replaced by L_0, L. Let $\gamma_1 = \Phi^L(C')$. By definition, $\gamma := \Phi(C)$ is the unipotent class in G that contains γ_1; note that γ is an $Is(V)$-conjugacy class. Let $\gamma_1' = \tilde{\rho}_1(\gamma_1)$ (notation of 3.4), a unipotent class in L_0. Using 3.2(a) for L, L_0, C' instead of G, G_0, C we see that $\gamma_1' = \tilde{\rho}_1\Phi^L(C') = \Phi_0^{L_0}(C')$. By definition, $\gamma_0 := \Phi_0(C)$ is the unipotent class in G_0 that contains γ_1'. From 3.4 we have $\gamma_0 = \tilde{\rho}(\gamma)$ that is $\Phi_0(C) = \tilde{\rho}\Phi(C)$. This completes the proof of 3.1(a) and that of Theorem 0.4.

3.6. The equality $\rho = \tilde{\rho}$ in 3.1 provides an explicit computation of the map $\tilde{\rho}$ for special orthogonal groups (since the maps Φ, Ψ are described in each case explicitly in Sections 1 and 2). The first explicit computation of $\tilde{\rho}$ in this case was given in [Xue] in terms of Springer representations instead of the maps Φ, Ψ.

3.7. According to [S1, III, 5.4(b)], there is a well-defined map $\rho' : \underline{\underline{G}} \to \underline{\underline{G}}_0, \gamma \mapsto \gamma_0$ such that: $\gamma \le \pi'(\gamma_0)$ ($\gamma_0 \in \underline{\underline{G}}_0$); if $\gamma \le \pi'(\gamma_0')$ ($\gamma_0' \in \underline{\underline{G}}_0$), then $\gamma_0 \le \gamma_0'$. We note the following result (see [Xue]):

(a) *If G is almost simple of type B, C or D, then $\tilde{\rho} = \rho'$.*

Next we note:

(b) *For any G we have $\rho = \rho'$.*

We can assume that G is almost simple and that p is a bad prime for G. If G is of exceptional type, then ρ can be computed from the tables in Section 2 and ρ' can be computed from the tables in [S1, IV]; the result follows. If G is of type B, C or D, then as we have seen earlier we have $\tilde{\rho} = \rho$ and the result follows from (a).

3.8. The results in this subsection are not used elsewhere in this paper. We define a map $\rho'' : \underline{\underline{G}} \to \underline{\underline{G}}_0$ as follows. Let $\gamma \in \underline{\underline{G}}$. We can find a Levi subgroup L of a parabolic subgroup P of G and $\gamma_1 \in \underline{\underline{L}}$ such that $\gamma_1 \subset \gamma$ and γ_1 is "distinguished" in L (that is, any torus in the centralizer in L of an element in γ_1 is contained in the centre of L). Let L_0 be a Levi subgroup of a parabolic subgroup P_0 of G_0 of the same type as L. We have $\gamma_1 = \pi_L(\gamma_1')$ for a well-defined $\gamma_1' \in \underline{\underline{L}}_0$ where π_L is the map analogous to π in 3.1 but for L, L_0 instead of G, G_0. Let $\rho''(\gamma)$ be the unique unipotent class in G_0 which contains γ_1'. This is independent of the choices and $\gamma \mapsto \rho''(\gamma)$ defines the map ρ''. We show:

(a) $\rho'' = \rho$.

Let $\gamma, L, L_0, P, P_0, \gamma_1, \gamma_1'$ be as above. Let $\mathbf{W}'$ be a standard parabolic subgroup of $\mathbf{W}$ of the same type as P, P_0. We can find an elliptic conjugacy class C' of $\mathbf{W}'$ such that $\gamma_1 = \Phi^L(C')$, $\gamma_1' = \Phi^{L_0}(C')$, where Φ^L, Φ^{L_0} are defined like Φ in terms of L, L_0 instead of G. Let C be the conjugacy class of $\mathbf{W}$ that contains $\mathbf{W}'$. From [L8, 1.1, 4.5] we see that $\Phi(C)$ is the unique unipotent class in G that contains γ_1 and $\Phi_0(C)$ is the unique unipotent class in G_0 that contains γ_1'. Thus $\Phi(C) = \gamma$

and $\Phi_0(C) = \rho(\gamma)$. We see that

$$\rho''(\gamma) = \rho''(\Phi(C)) = \Phi_0(C) = \rho\Phi(C) = \rho(\gamma).$$

This proves (a).

3.9. In this subsection we assume that G is of type B, C or D and $p = 2$. The identity $\Psi_0 = \Psi\pi'$ follows from the explicit combinatorial description of Ψ_0, Ψ given in Section 1 and the explicit combinatorial description of π' given in [S1, III, 6.1, 7.2, 8.2]. This completes the proof of 3.1(b) hence also that of Theorem 0.4(b).

The fact that Ψ'_0 (see Theorem 0.2) and π' might be described by the same combinatorics was noticed by the author in 1987 who proposed it as a problem to Spaltenstein; he proved it in [S3] (see [S3, p. 193]). Combining this with the (simple) description of Ψ given in Section 1, we deduce that $\Psi'_0 = \Psi\pi'$. In particular, we have $\Psi_0 = \Psi'_0$.

4. Proof of Theorem 0.6

4.1. Let $\underline{\underline{G}}^{\spadesuit}$ be the image of $\underline{\underline{G}}_0^{\spadesuit}$ (see 0.5) under the imbedding $\pi' : \underline{\underline{G}}_0 \to \underline{\underline{G}}$ (see 3.1). The unipotent classes in $\underline{\underline{G}}^{\spadesuit}$ are said to be special. The following result can be extracted from [S1, III].

(a) *There exists order preserving maps $e : \underline{\underline{G}} \to \underline{\underline{G}}$, $e_0 : \underline{\underline{G}}_0 \to \underline{\underline{G}}_0$ such that the image of e (resp. e_0) is equal to $\underline{\underline{G}}^{\spadesuit}$ (resp. $\underline{\underline{G}}_0^{\spadesuit}$) and such that for any $\gamma \in \underline{\underline{G}}$, (resp. $\gamma_0 \in \underline{\underline{G}}^0$), $\gamma \le e(\gamma)$ (resp. $\gamma_0 \le e_0(\gamma_0)$). The map e (resp. e_0) is unique. Moreover, we have $e^2 = e$, $e_0^2 = e_0$, $e = \pi' e_0 \rho'$ where ρ' is as in 3.7 .*

Strictly speaking (a) does not appear in [S1, III], in the form stated above since the notion of special representations from [L1] and the related notion of special unipotent class do not explicitly appear in [S1, III], (although they served as a motivation for Spaltenstein, see [S1, III, 9.4]). Actually (a) is a reformulation of results in [S1, III], taking into account developments in the theory of Springer representations which occurred after [S1, III] was written.

We now discuss (a) assuming that G is almost simple. Let $d_0 : \underline{\underline{G}}_0 \to \underline{\underline{G}}_0$ be a map as in [S1, III, 1.4]. If G is of exceptional type, we further require that the image of d_0 has a minimum number of elements (see [S1, III, 9.4]). Let $e_0 = d_0^2$. Then e_0 is order preserving and $\gamma_0 \le e_0(\gamma_0)$ for any $\gamma_0 \in \underline{\underline{G}}_0$. Moreover, if we set $e = \pi' e_0 \rho'$, then e is order preserving and $\gamma \le e(\gamma)$ for any $\gamma \in \underline{\underline{G}}$ (see [S1, III, 5.6]). If G is of type B, C or D, then the map e_0 is described combinatorially in [S1, III, 3.10]; hence its image is explicitly known; using the explicit description of the Springer correspondence given in these cases in [L2] we see that this image is exactly $\underline{\underline{G}}_0^{\spadesuit}$. (See also [S1, III, 3.11].) If G is of exceptional type, then the image of e_0 is described explicitly in the tables in [S1, pp. 247–250] and one can again check that it is exactly $\underline{\underline{G}}_0^{\spadesuit}$. Then the image of e is automatically $\underline{\underline{G}}^{\spadesuit}$. The uniqueness in (a) is discussed in 4.2.

4.2. Let $\gamma \in \underline{\underline{G}}$. We show that

(a) *there is a unique element $\tilde{e}(\gamma) \in \underline{\underline{G}}^{\spadesuit}$ such that:*

($*$) *$\gamma \le \tilde{e}(\gamma)$; if $\gamma \le \gamma'$, ($\gamma' \in \underline{\underline{G}}^{\spadesuit}$), then $\tilde{e}(\gamma) \le \gamma'$.*

Moreover, we have $\tilde{e}(\tilde{e}(\gamma)) = \tilde{e}(\gamma)$.

We show that $\tilde{e}(\gamma) = e(\gamma)$ satisfies ($*$). We have $\gamma \le e(\gamma)$. If $\gamma' \in \underline{\underline{G}}^{\spadesuit}$ and $\gamma \le \gamma'$, then $e(\gamma) \le e(\gamma') = \gamma'$ (since $e^2 = e$), as required. Conversely, let $\tilde{e}(\gamma)$ be

as in $(*)$. From $\gamma \le \tilde{e}(\gamma)$ we deduce $e(\gamma) \le e(\tilde{e}(\gamma))$ hence $e(\gamma) \le \tilde{e}(\gamma)$. On the other hand, taking $\gamma' = e(g)$ in $(*)$ (which satisfies $\gamma \le \gamma'$) we have $\tilde{e}(\gamma) \le e(\gamma)$ hence $\tilde{e}(\gamma) = e(\gamma)$. This proves (a) and that

(b) $\tilde{e}(\gamma) = e(\gamma)$.

Note that (a), (b) and the analogous statements for G_0 establish the uniqueness statement in 4.1(a) and the identity

(c) $\tilde{e} = \pi' \tilde{e}_0 \rho'$

where $\tilde{e}_0 : \underline{\underline{G}}_0 \to \underline{\underline{G}}_0$ is the map analogous to $\tilde{e} : \underline{\underline{G}} \to \underline{\underline{G}}$ (for G_0 instead of G); we have $\tilde{e}_0 = e_0$. Note that another proof of (c) which does not rely on the results of [S1, III], is given in [Xue].

According to [L3], for any $\gamma \in \underline{\underline{G}}_0$ we have

(d) $E_{\tilde{e}_0(\gamma)} = [E_\gamma]$.

4.3. In this subsection we assume that $p = 2$, G, G_0 are simple adjoint of type B_n $(n \ge 3)$ and let G^*, G_0^* be almost simple simply connected groups of type C_n defined over $\mathbf{k}, \mathbf{C}$, respectively. Note that G, G^* have the same Weyl group $\mathbf{W}$. Define $\xi_0 : \underline{\underline{G}}_0 \to \mathcal{S}_{\mathbf{W}}$ by $\gamma \mapsto [E_\gamma]$ (notation of 0.5). Define $\xi : \underline{\underline{G}} \to \mathcal{S}_{\mathbf{W}}$ by $\gamma \mapsto$ (Springer representation attached to $\tilde{e}(\gamma)$). We write

$$\underline{\underline{G}}^*, \underline{\underline{G}}_0^*, \Phi^*, \Phi_0^*, \rho^*, \rho'^*, \xi_0^*, \xi^*$$

for the analogues of $\underline{\underline{G}}, \underline{\underline{G}}_0, \Phi, \Phi_0, \rho, \rho', \xi_0, \xi$ with G, G_0 replaced by G^*, G_0^*. We show that

(a) $\xi_0 \rho'(\gamma) = \xi(\gamma)$

for any $\gamma \in \underline{\underline{G}}$. The right-hand side is the Springer representation attached to $\tilde{e}(\gamma)$. The left-hand side is $[E_{\rho'(\gamma)}]$ which by 4.2(d) is equal to $E_{\tilde{e}_0(\rho'(\gamma))}$; by the definition of π' this equals the Springer representation attached to $\pi' \tilde{e}_0 \rho'(\gamma) = \tilde{e}(\gamma)$ (see 4.2(c)), proving (a).

Let $\alpha : G^* \to G$ be the standard exceptional isogeny. Let $\alpha' : \mathbf{W} \to \mathbf{W}$, $\alpha'' : \mathcal{S}_{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$ be the induced bijections. (α', α'' are the identity in our case.) We consider the diagram

$$\begin{array}{ccccccc}
\underline{\mathbf{W}} & \xrightarrow{\Phi_0} & \underline{\underline{G}}_0 & \xrightarrow{=} & \underline{\underline{G}}_0 & \xrightarrow{\xi_0} & \mathcal{S}_{\mathbf{W}} \\
{\scriptstyle =}\uparrow & & {\scriptstyle \rho}\uparrow & & {\scriptstyle \rho'}\uparrow & & {\scriptstyle =}\uparrow \\
\underline{\mathbf{W}} & \xrightarrow{\Phi} & \underline{\underline{G}} & \xrightarrow{=} & \underline{\underline{G}} & \xrightarrow{\xi} & \mathcal{S}_{\mathbf{W}} \\
{\scriptstyle \alpha'}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \alpha''}\downarrow \\
\underline{\mathbf{W}} & \xrightarrow{\Phi^*} & \underline{\underline{G}}^* & \xrightarrow{=} & \underline{\underline{G}}^* & \xrightarrow{\xi^*} & \mathcal{S}_{\mathbf{W}} \\
{\scriptstyle =}\downarrow & & {\scriptstyle \rho^*}\downarrow & & {\scriptstyle \rho'^*}\downarrow & & {\scriptstyle =}\downarrow \\
\underline{\mathbf{W}} & \xrightarrow{\Phi_0^*} & \underline{\underline{G}}_0^* & \xrightarrow{=} & \underline{\underline{G}}_0^* & \xrightarrow{\xi_0^*} & \mathcal{S}_{\mathbf{W}}
\end{array}$$

The top three squares are commutative by Theorem 0.4(a), 3.7(b) and (a) (from left to right). The bottom three squares are commutative by Theorem 0.4(a), 3.7(b) and (a) (from left to right) with G replaced by G^*. The middle three squares are commutative by the definition of α. We see that the diagram above is commutative. It follows that $\xi_0 \Phi_0 = \xi \Phi = \xi^* \Phi^* = \xi_0^* \Phi_0^*$. In particular, $\xi_0 \Phi_0 = \xi_0^* \Phi_0^*$. From the definition we have $\tilde{\Phi}_0 = \xi_0 \Phi_0$; similarly, we have $\tilde{\Phi}_0^* = \xi_0^* \Phi_0^*$. Hence we have

$\tilde{\Phi}_0 = \tilde{\Phi}_0^*$. This proves the last sentence in Theorem 0.6(a). (Note that in the case where $C \in \underline{\mathbf{W}}$ is elliptic, the equality $\tilde{\Phi}_0(C) = \tilde{\Phi}_0^*(C)$ follows also from [L9, 3.6].)

4.4. We now repeat the arguments of 4.3 in the case where $G = G^*, G_0 = G_0^*$ are simple of type F_4 with $p = 2$ or of type G_2 with $p = 3$ or of type B_2 with $p = 2$ and $\alpha : G^* \to G$ is the standard exceptional isogeny. In these cases $\alpha' : \mathbf{W} \to \mathbf{W}$ is the nontrivial involution of the Coxeter group $\mathbf{W}$ and $\alpha'' : \mathcal{S}_{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$ is induced by α'. We have $\tilde{\Phi}_0 = \tilde{\Phi}_0^*$. As in 4.3 we obtain that $\tilde{\Phi}_0\alpha' = \alpha''\tilde{\Phi}_0$. This, together with 4.3, implies the validity of Theorem 0.6(a).

4.5. In the setup of 4.3 we define $\eta_0 : \mathcal{S}_{\mathbf{W}} \to \underline{\underline{G}}_0$ by $E_\gamma \mapsto \gamma$ where $\gamma \in \underline{\underline{G}}_0^{\spadesuit}$ (notation of 0.5). Define $\eta : \mathcal{S}_{\mathbf{W}} \to \underline{\underline{G}}$ by $E_\gamma \mapsto \pi'(\gamma)$ where $\gamma \in \underline{\underline{G}}_0^{\spadesuit}$. Let π'^*, η_0^*, η^* be the analogues of π', η_0, η with G, G_0 replaced by G^*, G_0^*. We consider the diagram

$$\begin{tikzcd}
\underline{\mathbf{W}} \arrow[d, "="'] & \underline{\underline{G}}_0 \arrow[l, "\Psi_0"'] \arrow[d, "\pi'"'] & \mathcal{S}_{\mathbf{W}} \arrow[l, "\eta_0"'] \\
\underline{\mathbf{W}} \arrow[d, "\alpha'"'] & \underline{\underline{G}} \arrow[l, "\Psi"'] \arrow[d, "\alpha"'] & \mathcal{S}_{\mathbf{W}} \arrow[l, "\eta"'] \arrow[u, "="'] \arrow[d, "\alpha''"'] \\
\underline{\mathbf{W}} & \underline{\underline{G}}^* \arrow[l, "\Psi^*"'] & \mathcal{S}_{\mathbf{W}} \arrow[l, "\eta^*"'] \arrow[d, "="'] \\
\underline{\mathbf{W}} \arrow[u, "="] & \underline{\underline{G}}_0^* \arrow[l, "\Phi_0^*"'] \arrow[u, "\pi'^*"'] & \mathcal{S}_{\mathbf{W}} \arrow[l, "\eta_0^*"']
\end{tikzcd}$$

The left top square is commutative by Theorem 0.4(b) and by the equality $\pi = \pi'$. Similarly, the left bottom square is commutative. The right top square and the right bottom square are commutative by definition. The middle two squares are commutative by the definition of α. We see that the diagram above is commutative. It follows that $\Psi_0\eta_0 = \Psi\eta = \Psi^*\eta^* = \Psi_0^*\eta_0^*$. In particular, $\Psi_0\eta_0 = \Psi_0^*\eta_0^*$. Hence we have $\tilde{\Psi}_0 = \tilde{\Psi}_0^*$. This proves the last sentence in Theorem 0.6(b).

4.6. We now repeat the arguments of 4.5 in the case where $G = G^*, G_0 = G_0^*$ are simple of type F_4 with $p = 2$ or of type G_2 with $p = 3$ or of type B_2 with $p = 2$ and $\alpha : G^* \to G$ is the standard exceptional isogeny. In these cases $\alpha' : \mathbf{W} \to \mathbf{W}$ is the nontrivial involution of the Coxeter group $\mathbf{W}$ and $\alpha'' : \mathcal{S}_{\mathbf{W}} \to \mathcal{S}_{\mathbf{W}}$ is induced by α'. We have $\tilde{\Psi}_0 = \tilde{\Psi}_0^*$. As in 4.5 we obtain that $\tilde{\Psi}_0\alpha'' = \alpha'\tilde{\Psi}_0$. This, together with 4.5, implies the validity of Theorem 0.6(b).

5. Explicit description of the set of special conjugacy classes in $\mathbf{W}$

5.1. In this section we assume that $\mathbf{k} = \mathbf{C}$. We will describe in each case, assuming that G is simple, the set $\underline{\mathbf{W}}_{\spadesuit}$ of special conjugacy classes in $\mathbf{W}$ and the bijection $\tau : \underline{\mathbf{W}}_{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ induced by $\tilde{\Phi}_0$ in 0.5.

We fix $n \geq 2$. Let A be the set of all $(x_1 \geq x_2 \geq \dots)$ where $x_i \in \mathbf{N}$ are zero for large i such that:

$x_1 + x_2 + \dots = 2n$,

for any $i \geq 1$ we have $x_{2i-1} = x_{2i} \mod 2$,

for any $i \geq 1$ such that x_{2i-1}, x_{2i} are odd we have $x_{2i-1} = x_{2i}$.

Let A' be the set of all $((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots))$ where $y_i, z_i \in \mathbf{N}$ are zero for large i such that:

$(y_1 + y_2 + \dots) + (z_1 + z_2 + \dots) = n$,

$y_{i+1} \leq z_i \leq y_i + 1$ for $i \geq 1$.

Define $h : A \to A'$ by

$(x_1 \geq x_2 \geq \dots) \mapsto ((y_1 \geq y_2 \geq), (z_1 \geq z_2 \geq))$

where

$y_i = x_{2i}/2, z_i = x_{2i-1}/2$ if x_{2i-1}, x_{2i} are even,

$y_i = (x_{2i} - 1)/2, z_i = (x_{2i-1} + 1)/2$ if $x_{2i-1} = x_{2i}$ are odd.

Define $h' : A' \to A$ by

$((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots)) \mapsto (x_1 \geq x_2 \geq \dots)$

where

$x_{2i} = 2y_i, x_{2i-1} = 2z_i$ if $z_i \leq y_i$,

$x_{2i} = 2y_i + 1, x_{2i-1} = 2z_i - 1$ if $z_i = y_i + 1$.

Note that h, h' are inverse bijections.

5.2. We preserve the setup of 5.1 and we assume that G is simple of type C_n. Let W be the group of permutations of $[1, 2n]$ which commute with the involution $\chi : i \mapsto 2n - i + 1$. We identify $\mathbf{W}$ with W as in [L8, 1.4]. To $(x_1 \geq x_2 \geq \dots) \in A$ we associate an element of $W = \mathbf{W}$ which is a product of disjoint cycles with sizes given by the nonzero numbers in $x_1, x_2, \dots$ where each cycle of even size is χ-stable and each cycle of odd size is (necessarily) not χ-stable. This identifies A with the subset $\underline{\mathbf{W}}_\spadesuit$ of $\underline{\mathbf{W}}$. We identify in the standard way $\hat{\mathbf{W}}$ with the set of all $((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots))$ where $y_i, z_i \in \mathbf{N}$ are zero for large i such that $(y_1 + y_2 + \dots) + (z_1 + z_2 + \dots) = n$. Under this identification A' becomes $\mathcal{S}_{\mathbf{W}}$ (the special representations of $\mathbf{W}$). The bijection $\tau : \underline{\mathbf{W}}_\spadesuit \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ becomes the bijection $h : A \xrightarrow{\sim} A'$ in 5.1.

5.3. Now assume that G is simple of type B_n $(n \geq 2)$. Let G_1 be a simple group of type C_n over $\mathbf{C}$. Then $\mathbf{W}$ can be viewed both as the Weyl group of G and that of G'. The bijection sets $\underline{\mathbf{W}}_\spadesuit, \mathcal{S}_{\mathbf{W}}$ and the bijection $\tau : \underline{\mathbf{W}}_\spadesuit \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ is the same from the point of view of G' as from that of G (see 0.6) hence it is described combinatorially as in 5.2.

In the case where G is simple of type A_n $(n \geq 1)$ then $\underline{\mathbf{W}}_\spadesuit = \underline{\mathbf{W}}, \mathcal{S}_{\mathbf{W}} = \hat{\mathbf{W}}$ are all naturally parametrized by partitions of n and the bijection τ is the identity map in these parametrizations.

5.4. We fix $n \geq 2$. Let C be the set of all $((x_1 \geq x_2 \geq \dots), (e_1, e_2, \dots))$ where $x_i \in \mathbf{N}$ are zero for large i and $e_i \in \{0, 1\}$ are such that

$x_1 + x_2 + \dots = 2n$,

for any $i \geq 1$ we have $x_{2i-1} = x_{2i} \mod 2$, $e_{2i-1} = e_{2i}$,

for any $i \geq 1$ such that x_{2i-1}, x_{2i} are odd we have $x_{2i-1} = x_{2i}, e_{2i-1} = e_{2i} = 0$,

for any $i \geq 1$ such that x_{2i-1}, x_{2i} are even and $e_{2i-1} = e_{2i} = 0$ we have $x_{2i-1} = x_{2i}$,

for any $i \geq 1$ such that $x_{2i} = 0$ we have $x_{2i-1} = 0$ and $e_{2i-1} = e_{2i} = 0$,

for any $i \geq 1$ such that $x_{2i} = x_{2i+1}$ are even we have $e_{2i} = e_{2i+1} = 0$.

Let C' be the set of all $((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots))$ where $y_i, z_i \in \mathbf{N}$ are zero for large i such that

$(y_1 + y_2 + \dots) + (z_1 + z_2 + \dots) = n$,

$y_{i+1} - 1 \leq z_i \leq y_i$ for $i \geq 1$.
Define $k : C \to C'$ by
$((x_1 \geq x_2 \geq \dots), (e_1, e_2, \dots)) \mapsto ((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots))$
where
$y_i = (x_{2i} + 1)/2, z_i = (x_{2i-1} - 1)/2$ if $x_{2i-1} = x_{2i}$ are odd,
$y_i = x_{2i}/2, z_i = x_{2i-1}/2$ if $x_{2i-1} = x_{2i}$ are even and $e_{2i-1} = e_{2i} = 0$,
$y_i = (x_{2i} + 2)/2, z_i = (x_{2i-1} - 2)/2$ if x_{2i-1}, x_{2i} are even and $e_{2i-1} = e_{2i} = 1$.
Define $k' : C' \to C$ by
$((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots)) \to ((x_1 \geq x_2 \geq \dots), (e_1, e_2, \dots))$
where
$x_{2i} = 2y_i, x_{2i-1} = 2z_i, e_{2i} = e_{2i-1} = 0$ if $y_i = z_i$,
$x_{2i} = 2y_i - 1, x_{2i-1} = 2z_i + 1, e_{2i} = e_{2i-1} = 0$ if $y_i = z_i + 1$,
$x_{2i} = 2y_i - 2, x_{2i-1} = 2z_i + 2, e_{2i} = e_{2i-1} = 1$, if $y_i \geq z_i + 2$.
Note that k, k' are inverse bijections.

Let C_0 be the set of all $((x_1 \geq x_2 \geq \dots), (e_1, e_2, \dots))$ where $x_i \in \mathbf{N}$ are zero for large i and $e_i \in \{0, 1\}$ are such that
$x_1 + x_2 + \dots = 2n$,
$x_{2i-1} = x_{2i}$ are even, $e_{2i-1} = e_{2i} = 0$ for $i \geq 1$.
Note that $C_0 \subset C$.

Let C'_0 be the set of all $((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots))$ where $y_i, z_i \in \mathbf{N}$ are zero for large i such that
$(y_1 + y_2 + \dots) + (z_1 + z_2 + \dots) = n$,
$y_i = z_i$ for $i \geq 1$.
Note that $C'_0 \subset C'$. Now $k : C \xrightarrow{\sim} C'$ restricts to a bijection $k_0 : C_0 \xrightarrow{\sim} C'_0$. It maps $((y_1 \geq y_2 \geq \dots), (y_1 \geq y_2 \geq \dots))$ to $((2y_1, 2y_1, 2y_2, 2y_2, \dots), (0, 0, 0, \dots))$.

5.5. We preserve the setup of 5.4 and we assume that $n \geq 3$ and that G is simple of type D_n. Let W be as in 5.2 and let W' be the subgroup of W consisting of even permutations of $[1, 2n]$. We identify $\mathbf{W} = W'$ as in [L8, 1.4, 1.5]. Let $\chi : [1, 2n] \to [1, 2n]$ be as in 5.2. Let $\underline{\mathbf{W}}_{\spadesuit,0}$ (resp. $\underline{\mathbf{W}}_{\spadesuit,1}$) be the set of special conjugay classes in $\mathbf{W}$ which are not conjugacy classes of W (resp. form a single conjugacy class in W). Let $\mathcal{S}_{\mathbf{W},0}$ (resp. $\mathcal{S}_{\mathbf{W},1}$) be the set of special representations of $\mathbf{W}$ which do not extend (resp. extend) to W-modules.

To $(x_1 \geq x_2 \geq \dots) \in C$ we associate an element of $W' = \mathbf{W}$ which is a product of disjoint cycles with sizes given by the nonzero numbers in $x_1, x_2, \dots$ where each cycle of even size is χ-stable and each cycle of odd size is (necessarily) not χ-stable. This identifies $C - C_0$ with $\underline{\mathbf{W}}_{\spadesuit,1}$ and C_0 with $\underline{\mathbf{W}}_{\spadesuit,0}$ modulo the fixed point free involution given by conjugation by an element in $W - W'$.

An element $((y_1 \geq y_2 \geq \dots), (z_1 \geq z_2 \geq \dots)) \in C'$ can be viewed as in 5.2 as an irreducible representations of W. This identifies $C' - C'_0$ with $\mathcal{S}_{\mathbf{W},1}$ and C'_0 with $\mathcal{S}_{\mathbf{W},0}$ modulo the fixed point free involution given by $E \mapsto E'$ where $E \oplus E'$ extends to a W-module. Under these identifications, the bijection $\tau : \underline{\mathbf{W}}_{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ becomes the bijection $(C - C_0) \sqcup C_0 \sqcup C_0 \xrightarrow{\sim} (C' - C'_0) \sqcup C'_0 \sqcup C'_0$, $(a, b, c) \mapsto (k(a), k_0(b), k_0(c))$.

5.6. In 5.7–5.11 we describe explicitly the bijection $\tau : \underline{\mathbf{W}}_{\spadesuit} \xrightarrow{\sim} \mathcal{S}_{\mathbf{W}}$ (in the case where G simple of exceptional type) in the form of a list of data $\alpha \mapsto \beta$ where α is a special conjugacy class and β is a special representation (we use notation of [Ca] for the conjugacy classes in $\mathbf{W}$ and the notation of [S1] for the objects of $\hat{\mathbf{W}}$).

5.7 Type G_2. We have:
$A_0 \mapsto \epsilon$
$A_2 \mapsto \theta'$
$G_2 \mapsto 1.$

5.8. Type F_4. We have:
$A_0 \mapsto \chi_{1,4}$
$2A_1 \mapsto \chi_{4,4}$
$4A_1 \mapsto \chi_{9,4}$
$A_2 \mapsto \chi_{8,4}$
$\tilde{A}_2 \mapsto \chi_{8,2}$
$D_4(a_1) \mapsto \chi_{12}$
$D_4 \mapsto \chi_{8,1}$
$C_3 + A_1 \mapsto \chi_{8,3}$
$F_4(a_1) \mapsto \chi_{9,1}$
$B_4 \mapsto \chi_{4,1}$
$F_4 \mapsto \chi_{1,1}.$

5.9. Type E_6. We have:
$A_0 \mapsto 1_{36}$
$A_1 \mapsto 6_{25}$
$2A_1 \mapsto 20_{20}$
$A_2 \mapsto 30_{15}$
$A_2 + A_1 \mapsto 64_{13}$
$2A_2 \mapsto 24_{12}$
$A_2 + 2A_1 \mapsto 60_{11}$
$A_3 \mapsto 81_{10}$
$D_4(a_1) \mapsto 80_7$
$A_4 \mapsto 81_6$
$D_4 \mapsto 24_6$
$A_4 + A_1 \mapsto 60_5$
$D_5(a_1) \mapsto 64_4$
$E_6(a_2) \mapsto 30_3$
$D_5 \mapsto 20_2$
$E_6(a_1) \mapsto 6_1$
$E_6 \mapsto 1_0.$

5.10. Type E_7. We have:
$A_0 \mapsto 1_{63}$
$A_1 \mapsto 7_{46}$
$2A_1 \mapsto 27_{37}$
$(3A_1)' \mapsto 21_{36}$
$A_2 \mapsto 56_{30}$
$A_2 + A_1 \mapsto 120_{25}$
$A_2 + 2A_1 \mapsto 189_{22}$
$A_3 \mapsto 210_{21}$
$2A_2 \mapsto 168_{21}$
$A_2 + 3A_1 \mapsto 105_{21}$
$(A_3 + A_1)' \mapsto 189_{20}$
$D_4(a_1) \mapsto 315_{16}$

$D_4 \mapsto 105_{15}$
$D_4(a_1) + A_1 \mapsto 405_{15}$
$D_4(a_1) + 2A_1 \mapsto 378_{14}$
$2A_3 + A_1 \mapsto 210_{13}$
$A_4 \mapsto 420_{13}$
$A_5' \mapsto 105_{12}$
$A_4 + A_1 \mapsto 512_{11}$
$D_5(a_1) \mapsto 420_{10}$
$A_4 + A_2 \mapsto 210_{10}$
$D_5(a_1) + A_1 \mapsto 378_9$
$E_6(a_2) \mapsto 405_8$
$E_7(a_4) \mapsto 315_7$
$D_5 \mapsto 189_7$
$A_6 \mapsto 105_6$
$D_5 + A_1 \mapsto 168_6$
$D_6(a_1) \mapsto 210_6$
$A_7 \mapsto 189_5$
$E_6(a_1) \mapsto 120_4$
$E_6 \mapsto 21_3$
$E_7(a_3) \mapsto 56_3$
$E_7(a_2) \mapsto 27_2$
$E_7(a_1) \mapsto 7_1$
$E_7 \mapsto 1_0$.

5.11. Type E_8. We have:
$A_0 \mapsto 1_{120}$
$A_1 \mapsto 8_{91}$
$2A_1 \mapsto 35_{74}$
$A_2 \mapsto 112_{63}$
$A_2 + A_1 \mapsto 210_{52}$
$A_2 + 2A_1 \mapsto 560_{47}$
$A_3 \mapsto 567_{46}$
$2A_2 \mapsto 700_{42}$
$D_4(a_1) \mapsto 1400_{37}$
$D_4(a_1) + A_1 \mapsto 1400_{32}$
$D_4 \mapsto 525_{36}$
$(2A_3)' \mapsto 3240_{31}$
$A_4 \mapsto 2268_{30}$
$D_4(a_1) + A_2 \mapsto 2240_{28}$
$A_4 + A_1 \mapsto 4096_{26}$
$A_4 + 2A_1 \mapsto 4200_{24}$
$D_5(a_1) \mapsto 2800_{25}$
$A_4 + A_2 \mapsto 4536_{23}$
$A_4 + A_2 + A_1 \mapsto 2835_{22}$
$D_5(a_1) + A_1 \mapsto 6075_{22}$
$D_4 + A_3 \mapsto 4200_{21}$
$E_6(a_2) \mapsto 5600_{21}$
$E_8(a_8) \mapsto 4480_{16}$
$D_5 \mapsto 2100_{20}$

$D_6(a_1) \mapsto 5600_{15}$
$A_6 \mapsto 4200_{15}$
$A_7' \mapsto 6075_{14}$
$A_6 + A_1 \mapsto 2835_{14}$
$A_7 + A_1 \mapsto 4536_{13}$
$E_6(a_1) \mapsto 2800_{13}$
$D_7(a_2) \mapsto 4200_{12}$
$E_6(a_1) + A_1 \mapsto 4096_{11}$
$E_6 \mapsto 525_{12}$
$E_7(a_3) \mapsto 2268_{10}$
$A_8 \mapsto 2240_{10}$
$D_8(a_2) \mapsto 3240_9$
$E_8(a_6) \mapsto 1400_8$
$E_8(a_7) \mapsto 1400_7$
$E_8(a_3) \mapsto 700_6$
$E_7(a_1) \mapsto 567_6$
$D_8 \mapsto 560_5$
$E_8(a_5) \mapsto 210_4$
$E_8(a_4) \mapsto 112_3$
$E_8(a_2) \mapsto 35_2$
$E_8(a_1) \mapsto 8_1$
$E_8 \mapsto 1_0$.

5.12. From the tables above we see that if $\mathbf{W}$ is of type G_2, F_4, E_8 and (c^k) is the conjugacy class of the k-th power of a Coxeter element of $\mathbf{W}$ then:
(type G_2) $\langle c^1\rangle, \langle c^2\rangle$ are special, $\langle c^3\rangle$ is not special;
(type F_4) $\langle c^1\rangle, \langle c^2\rangle, \langle c^3\rangle$ are special, $\langle c^4\rangle$ is not special;
(type E_8) $\langle c^1\rangle, \langle c^2\rangle, \langle c^3\rangle, \langle c^4\rangle, \langle c^5\rangle$ are special, $\langle c^6\rangle$ is not special.
The numbers $2, 3, 5$ above may be explained by the fact that the Coxeter number of $\mathbf{W}$ of type G_2, F_4, E_8 is $2 \times 3, 3 \times 4, 5 \times 6$, respectively.

5.13. Theorem 0.6 suggests that the set $\underline{\mathbf{W}}_\spadesuit$ should also make sense in the case where $\mathbf{W}$ is replaced by a finite Coxeter group Γ (not necessarily a Weyl group). Assume for example that Γ is a dihedral group of order $2m$, $m \geq 3$, with standard generators s_1, s_2. We expect that if m is odd, then $\underline{\Gamma}_\spadesuit$ consists of 1, the conjugacy class of s_1s_2 and the conjugacy class containing s_1 and s_2; if m is even then $\underline{\Gamma}_\spadesuit$ consists of 1, the conjugacy class of s_1s_2 and the conjugacy class of $s_1s_2s_1s_2$. This agrees with the already known cases when $m = 3, 4, 6$.

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