

TEMPERED REPRESENTATIONS AND NILPOTENT ORBITS

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ABSTRACT. Given a nilpotent orbit \mathcal{O} of a real, reductive algebraic group, a necessary condition is given for the existence of a tempered representation π such that \mathcal{O} occurs in the wave front cycle of π . The coefficients of the wave front cycle of a tempered representation are expressed in terms of volumes of precompact submanifolds of an affine space.

1. INTRODUCTION

Let G be a real, reductive algebraic group with Lie algebra \mathfrak{g} , and let π be an irreducible, admissible representation of G . The wave front cycle of π , denoted $\text{WF}(\pi)$, is an integral linear combination of nilpotent coadjoint orbits defined in [2], [13].

Theorem 1.1. *Suppose \mathcal{O} is an orbit contained in $\text{WF}(\pi)$ for a tempered representation π , let $\nu \in \mathcal{O}$, and let L be a Levi factor of $Z_G(\nu)$. Then $L/Z(G)$ is compact.*

This theorem was conjectured by David Vogan. It makes precise the widely held intuition that orbits occurring in the wave front cycles of tempered representations are necessarily large.

In [1] Bala and Carter attempted a uniform understanding of nilpotent orbits in complex semisimple Lie algebras. They observed that for every nilpotent orbit \mathcal{O} , there is a unique (up to conjugacy) maximal complex Levi subalgebra \mathfrak{l} meeting \mathcal{O} . Moreover, $\mathcal{O} \cap \mathfrak{l}$ is a single orbit for the adjoint group of \mathfrak{l} . Bala and Carter defined a nilpotent orbit to be *distinguished* if it does not meet a proper Levi subalgebra, and they observed that to parametrize nilpotent orbits it is enough to parametrize distinguished nilpotent orbits in every Levi subalgebra.

Motivated by the work of Bala and Carter, Noël defined a nilpotent orbit for a real, reductive algebraic group to be *noticed* if it does not meet a proper Levi subalgebra [8]. By Levi subalgebra, we mean the Levi factor of a real parabolic subalgebra.

If \mathcal{O}_ν is a nilpotent orbit, then it is not hard to show that $Z_G(\nu)$ is compact modulo $Z(G)$ iff \mathcal{O}_ν is a noticed nilpotent orbit. Thus, we have an alternate way of stating Theorem 1.1.

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Corollary 1.2. *If $\mathcal{O} \subset \mathfrak{g}^*$ is a nilpotent coadjoint orbit occurring in the wave front cycle of a tempered representation, then \mathcal{O} does not meet \mathfrak{m}^* for any proper Levi subalgebra $\mathfrak{m} \subset \mathfrak{g}$.*

Note that this corollary relates a set of nilpotent orbits that arises naturally in representation theory with a set of nilpotent orbits that arises naturally in structure theory.

For p -adic groups, similar results were obtained much earlier. In [6], Mœglin shows that the p -adic analogue for supercuspidal representations follows from results of [7]. Similarly, Mœglin proves the result for tempered representations of classical p -adic groups in [6].

The purpose of this paper is to show that there is a simple proof of Theorem 1.1, which is in the spirit of the Kostant-Kirillov orbit method.

In [10] and [11], Rossmann associated to each irreducible, tempered representation π a finite union of regular coadjoint orbits we call \mathcal{O}_π . If π has regular infinitesimal character, then \mathcal{O}_π is a single coadjoint orbit. For each nilpotent coadjoint orbit, $\mathcal{O} \subset \mathfrak{g}^*$, fix $X \in \mathcal{O}$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G -equivariant isomorphism, and let $\{X, H, Y\}$ be an \mathfrak{sl}_2 -triple containing X . Put $S_X = X + Z_{\mathfrak{g}}(Y)$.

Theorem 1.3. *There exists a canonical measure on $S_X \cap \mathcal{O}_\pi$ such that*

$$WF(\pi) = \sum_{\mathcal{O}_\pi \cap S_X \text{ precompact}} \text{vol}(\mathcal{O}_\pi \cap S_X) \mathcal{O}_X.$$

The sum is over nilpotent coadjoint orbits \mathcal{O}_X such that $\mathcal{O}_\pi \cap S_X$ is precompact.

If π has regular infinitesimal character, then “precompact” may be replaced by “compact” in the above theorem. In the case where G is compact, the wave front cycle of π is $\dim(\pi) \cdot 0$ where 0 denotes the zero orbit. In this case, our formula reduces to the well-known observation of Kirillov that the symplectic volume of the coadjoint orbit associated to π is the dimension of π (see, for instance, page 173 of [5]).

The reader will see in section four that the compactness in Theorem 1.1 is a consequence of the compactness of the manifold $\mathcal{O}_\pi \cap S_X$ in the case of regular infinitesimal character.

All of these results should be true for an arbitrary reductive Lie group. We only state them in the case of real, reductive algebraic groups because we use results of [10], [11], and [12], which are only stated for real, reductive algebraic groups. However, the author believes that these results should be true for an arbitrary reductive Lie group.

2. RAO’S LEMMA AND THREE COROLLARIES OF BARBASCH AND VOGAN

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit. The Kostant-Kirillov symplectic form ω is defined on \mathcal{O} by the formula

$$\omega_\lambda(\text{ad}_X^* \lambda, \text{ad}_Y^* \lambda) = \lambda([X, Y]).$$

The top dimensional form

$$\frac{\omega^m}{m!(2\pi)^m}$$

on \mathcal{O} gives rise to the canonical measure on \mathcal{O} . Here $m = \frac{\dim \mathcal{O}}{2}$. We will often abuse notation and write \mathcal{O} for the orbit as well as the canonical measure on the orbit.

In this section, we recall an unpublished lemma of Rao and three corollaries of Barbasch and Vogan. All of this material can be found on pages 46, 47, and 48 of [2]. However, unlike the previous treatment, we need to carefully keep track of certain constants for our applications. Thus, we provide updated statements, and for the convenience of the reader we provide sketches of updated proofs.

Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G -equivariant isomorphism. Let \mathcal{O}_X be a nilpotent orbit in $\mathfrak{g}^* \cong \mathfrak{g}$, and let $\{X, H, Y\}$ be an \mathfrak{sl}_2 -triple with nilpositive element X . Put $S_X = X + Z_{\mathfrak{g}}(Y)$. The map

$$\phi : G \times S_X \rightarrow \mathfrak{g}^*$$

given by $\phi : (g, \xi) \mapsto g \cdot \xi$ is a submersion. In particular, every orbit $\mathcal{O}_{\nu} \subset G \cdot S_X$ is transverse to S_X , and $G \cdot S_X \subset \mathfrak{g}^*$ is open.

Fix a Haar measure on G . This choice determines a Lebesgue measure on $\mathfrak{g} \cong \mathfrak{g}^*$. If $\xi \in S_X$, then we have a direct sum decomposition

$$\mathfrak{g} = [\mathfrak{g}, X] \oplus Z_{\mathfrak{g}}(Y) \cong T_X \mathcal{O}_X \oplus T_{\xi} S_X.$$

We then obtain a Lebesgue measure on S_X as the “quotient” of the Lebesgue measure on \mathfrak{g} and the canonical measure on $\mathcal{O}_X \subset \mathfrak{g}^*$. Further, given $\nu \in \mathfrak{g}^*$, denote by \mathcal{F}_{ν} the fiber over ν under the map ϕ . If $g \cdot \xi = \nu$, then we have an exact sequence

$$0 \rightarrow T_{\nu}^*(G \cdot S_X) \rightarrow T_{(g, \xi)}^*(G \times S_X) \rightarrow T_{(g, \xi)}^* \mathcal{F}_{\nu} \rightarrow 0.$$

This exact sequence together with the above remarks and our choice of Haar measure on G determine a smooth measure on \mathcal{F}_{ν} . Moreover, integration against these measures on the fibers of ϕ yields a continuous surjective map

$$\phi_* : C_c^{\infty}(G \times S_X) \longrightarrow C_c^{\infty}(G \cdot S_X).$$

Dualizing, we get an injective pullback map on distributions

$$\phi^* : D(G \cdot S_X) \rightarrow D(G \times S_X).$$

Now, we are ready to state Rao’s lemma.

Lemma 2.1 (Rao). *If $\nu \in S_X$, then there exists a smooth measure $m_{\nu, X}$ on $\mathcal{O}_{\nu} \cap S_X$ such that*

$$\phi^*(\mathcal{O}_{\nu}) = m_G \otimes m_{\nu, X}.$$

Here m_G denotes the fixed choice of Haar measure on G . Although ϕ^ depends on this choice of Haar measure, $m_{\nu, X}$ does not.*

One can write down $m_{\nu, X}$ by giving a top dimensional form on $\mathcal{O}_{\nu} \cap S_X$, well-defined up to sign. Essentially, we just divide the canonical measure on \mathcal{O}_{ν} by the canonical measure on \mathcal{O}_X . More precisely, the composition of the inclusion $[\mathfrak{g}, \nu] \hookrightarrow \mathfrak{g}$ and the projection defined by the decomposition $\mathfrak{g} = [\mathfrak{g}, X] \oplus Z_{\mathfrak{g}}(Y)$ yields a map

$$T_{\nu} \mathcal{O}_{\nu} \cong [\mathfrak{g}, \nu] \rightarrow [\mathfrak{g}, X] \cong T_X \mathcal{O}_X.$$

This map is a surjection because \mathcal{O}_{ν} is transverse to S_X . It pulls back to an exact sequence

$$0 \rightarrow T_X^* \mathcal{O}_X \rightarrow T_{\nu}^* \mathcal{O}_{\nu} \rightarrow T_{\nu}^*(\mathcal{O}_{\nu} \cap S_X) \rightarrow 0.$$

The canonical measures on \mathcal{O}_{ν} and \mathcal{O}_X determine top dimensional alternating tensors up to sign on $T_{\nu}^* \mathcal{O}_{\nu}$ and $T_X^* \mathcal{O}_X$. Hence, our exact sequence gives a top dimensional, alternating tensor on $T_{\nu}^*(\mathcal{O}_{\nu} \cap S_X)$, well-defined up to sign.

We will need three corollaries of Barbasch and Vogan. Let $\mathcal{N} \subset \mathfrak{g}^*$ be the nilpotent cone. If $\nu \in \mathfrak{g}^*$, define

$$\mathcal{N}_\nu = \mathcal{N} \cap \overline{\bigcup_{t>0} \mathcal{O}_{t\nu}}.$$

Corollary 2.2 (Barbasch and Vogan). *We have four statements.*

- (a) *If \mathcal{O}_X is a nilpotent orbit, then $\mathcal{O}_X \subset \mathcal{N}_\nu$ if, and only if $\mathcal{O}_\nu \cap S_X \neq \emptyset$.*
- (b) *An orbit $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open if, and only if $\mathcal{O}_\nu \cap S_X$ is precompact.*
- (c) *If ν is semisimple, then $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open if, and only if $\mathcal{O}_\nu \cap S_X$ is compact.*
- (d) *Further, $\mathcal{O}_X \cap S_X = \{X\}$ for any nilpotent orbit \mathcal{O}_X .*

We sketch a proof. Note $G \cdot S_X \subset \mathfrak{g}^*$ is an open subset containing \mathcal{O}_X ; thus, $\mathcal{O}_X \subset \mathcal{N}_\nu$ iff $\mathcal{O}_{t\nu} \cap S_X \neq \emptyset$ for sufficiently small $t > 0$. However, if $\gamma_t = \exp(-\frac{1}{2}(\log(t))H)$, then

$$\mathcal{O}_{t\nu} \cap S_X = X + t\gamma_t(\mathcal{O}_\nu \cap S_X - X).$$

In particular, $\mathcal{O}_\nu \cap S_X \neq \emptyset$ iff $\mathcal{O}_{t\nu} \cap S_X \neq \emptyset$ for any $t > 0$. This verifies part (a).

For parts (b) and (c), one shows that $\mathcal{O}_\nu \cap S_X$ bounded implies that $S_X \cap \mathcal{N}_\nu = \{X\}$ and $\mathcal{O}_\nu \cap S_X$ unbounded implies that $S_X \cap \mathcal{N}_\nu$ is unbounded. This follows from a straightforward calculation utilizing the ad_H -decomposition of $Z_{\mathfrak{g}}(Y)$ into eigenspaces with nonpositive eigenvalues and the above relationship between $\mathcal{O}_\nu \cap S_X$, $\mathcal{O}_{t\nu} \cap S_X$, and γ_t . Using that $(G \cdot S_X) \cap \mathcal{N}$ is the union of nilpotent orbits containing \mathcal{O}_X in their closures, parts (b) and (c) follow.

If we let $\nu = X$ in the last paragraph, we arrive at part (d).

Corollary 2.3 (Barbasch and Vogan). *Let $n = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{N}_\nu)$. Then*

$$\lim_{t \rightarrow 0^+} t^{-n} \mathcal{O}_{t\nu} = \sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \text{vol}(\mathcal{O}_\nu \cap S_X) \mathcal{O}_X.$$

The volumes are computed with respect to the measures defined in Lemma 2.1. Moreover, the Fourier transform of the right-hand side is the first nonzero term in the asymptotic expansion of the generalized function $\widehat{\mathcal{O}}_\nu$.

Again, we sketch a proof. Fix X , a nilpotent element with $\mathcal{O}_X \subset \mathcal{N}_\nu$ open, and let $m = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{O}_X)$. We first show

$$\lim_{t \rightarrow 0^+} t^{-m} \mathcal{O}_{t\nu} = \text{vol}(\mathcal{O}_\nu \cap S_X) \mathcal{O}_X$$

on the open set $G \cdot S_X$. By Rao's lemma, $\phi^*(\mathcal{O}_\nu) = m_G \otimes m_{\nu, X}$. Thus, it is enough to show

$$\lim_{t \rightarrow 0^+} t^{-m} m_G \otimes m_{t\nu, X} = \text{vol}(\mathcal{O}_\nu \cap S_X) m_G \otimes \delta_X$$

if $\mathcal{O}_X \subset \mathcal{N}_\nu$ is an open orbit. Note that the support of the measure $m_{t\nu, X}$ is the precompact set $\mathcal{O}_{t\nu} \cap S_X$, and the precompact sets $\mathcal{O}_{t\nu} \cap S_X$ converge uniformly to δ_X by the above relationship between $\mathcal{O}_\nu \cap S_X$, $\mathcal{O}_{t\nu} \cap S_X$, and γ_t . Thus, it is enough to show

$$t^{-m} \text{vol}(\mathcal{O}_{t\nu} \cap S_X) = \text{vol}(\mathcal{O}_\nu \cap S_X).$$

This follows from a straightforward computation utilizing the above definition of the measures $m_{t\nu, X}$.

By Theorem 3.2 of [2], the distribution $\widehat{\mathcal{O}}_\nu$ has an asymptotic expansion at the origin

$$t^r \widehat{\mathcal{O}}_{t\nu} \sim t^l D_l + \dots$$

where D_l is the leading term and r is the number of positive roots of G . If we show $n = l - r$, then our limit will exist everywhere. If $n > l - r$, then the limit $\lim_{t \rightarrow 0^+} t^n \widehat{\mathcal{O}}_{t\nu}$ must be zero everywhere. However, we have seen that the Fourier transform of this limit is nonzero on \mathcal{O}_X whenever $\mathcal{O}_X \subset \mathcal{N}_\nu$ is of maximal dimension.

The limit $\lim_{t \rightarrow 0^+} t^{-l+r} \mathcal{O}_{t\nu}$ must exist and be nonzero. However, if $n < l - r$, then the homogeneity degree of this invariant distribution and Corollary 3.9 of [2] imply that such a distribution would have to be supported on orbits of dimension greater than $\dim \mathcal{N}_\nu$. But, this is impossible since the limit $\lim_{t \rightarrow 0^+} t^{-l+r} \mathcal{O}_{t\nu}$, if it exists, is clearly supported in \mathcal{N}_ν . Hence, $n = l - r$ and the limit $\lim_{t \rightarrow 0^+} t^{-n} \mathcal{O}_{t\nu}$ exists.

Now, let $k = \dim \mathcal{N}_\nu$, and let \mathcal{N}_k be the union of nilpotent orbits of dimension at least k . We have shown that our desired limit formula holds on \mathcal{N}_k . However, in theory the limit could differ from

$$\sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \text{vol}(\mathcal{O}_\nu \cap S_X) \mathcal{O}_X$$

by a distribution u supported on orbits of dimension less than \mathcal{N}_ν . However, we deduce $u = 0$ from Corollary 3.9 of [2] after checking the homogeneity degree of the terms in our limit formula.

Corollary 2.4 (Barbasch and Vogan). *Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan, and let $C \subset (\mathfrak{h}^*)'$ be a connected component of the regular set. If $\nu, \lambda \in C$, then $\mathcal{N}_\nu = \mathcal{N}_\lambda$.*

Suppose $\mathcal{O}_X \subset \mathcal{N}_\nu$ is an open orbit, let $n = \frac{1}{2}(\dim \mathcal{O}_\nu - \dim \mathcal{O}_X)$, and observe $\lim_{t \rightarrow 0^+} t^{-n} \mathcal{O}_{t\nu} = \lim_{t \rightarrow 0^+} \frac{1}{n!} \partial(\nu)^n|_{t\nu} \mathcal{O}_{t\nu}$ on $G \cdot S_X$ by the proof of Corollary 2.3. By Lemma 22 of [4],

$$\lim_{t \rightarrow 0^+} \partial(\nu)^n|_{t\nu} \mathcal{O}_{t\nu} = \lim_{t \rightarrow 0^+} \partial(\nu)^n|_{t\lambda} \mathcal{O}_{t\lambda}$$

on $G \cdot S_X$ if $\nu, \lambda \in C$. Clearly the support of $\lim_{t \rightarrow 0^+} \partial(\nu)^n|_{t\lambda} \mathcal{O}_{t\lambda}$ must be contained in \mathcal{N}_λ . Thus, the explicit formula for the limit on the left on $G \cdot S_X$ for open orbits $\mathcal{O}_X \subset \mathcal{N}_\nu$ in the proof of Corollary 2.3 then implies $\mathcal{O}_X \subset \mathcal{N}_\lambda$ whenever $\mathcal{O}_X \subset \mathcal{N}_\nu$ is open. Thus, we deduce $\mathcal{N}_\nu \subset \mathcal{N}_\lambda$. By symmetry we have equality.

3. ROSSMANN'S CHARACTER FORMULA

Let V be a finite dimensional, real vector space, and let μ be a smooth, rapidly decreasing measure on V . Then the Fourier transform of μ is defined to be

$$\widehat{\mu}(l) = \int_V e^{i\langle l, X \rangle} d\mu(X).$$

Note $\widehat{\mu}$ is a Schwartz function on V^* . Given a tempered distribution D on V^* , its Fourier transform \widehat{D} is a tempered, generalized function on V defined by

$$\langle \widehat{D}, \mu \rangle := \langle D, \widehat{\mu} \rangle.$$

Let G be a real, reductive algebraic group, and let π be an irreducible, tempered representation of G with character Θ_π . Let θ_π be the Lie algebra analogue of the character of π . Rossmann associated to π a finite union of regular, coadjoint orbits $\mathcal{O}_\pi \subset \mathfrak{g}^*$. He proved the following theorem [10], [11].

Theorem 3.1 (Rossmann). *As generalized functions, we have*

$$\widehat{\mathcal{O}}_\pi = \theta_\pi.$$

Here $\widehat{\mathcal{O}}_\pi$ denotes the Fourier transform of the canonical measure on \mathcal{O}_π .

If $\mathcal{O}_\pi = \mathcal{O}_\nu$ is a single orbit, then the leading term of the asymptotic expansion of $\widehat{\mathcal{O}}_\pi = \widehat{\mathcal{O}}_\nu$ is

$$\sum_{\substack{\mathcal{O}_X \subset \mathcal{N}_\nu \\ \dim \mathcal{O}_X = \dim \mathcal{N}_\nu}} \text{vol}(\mathcal{O}_\pi \cap S_X) \widehat{\mathcal{O}}_X$$

by Corollary 2.3. More generally, suppose $\mathcal{O}_\pi = \bigcup \mathcal{O}_{\nu_i}$, and define $\text{AS}(\pi) = \bigcup \mathcal{N}_{\nu_i}$. Then the leading term of $\widehat{\mathcal{O}}_\pi$ is a sum over the leading terms of the $\widehat{\mathcal{O}}_{\nu_i}$ of minimal degree. This is just the sum over $\text{vol}(\mathcal{O}_\pi \cap S_X) \widehat{\mathcal{O}}_X$ where $\mathcal{O}_X \subset \text{AS}(\pi)$ varies over orbits of maximal dimension. Hence, we have shown

$$\text{WF}(\pi) = \sum_{\substack{\mathcal{O}_X \subset \text{AS}(\pi) \\ \dim \mathcal{O}_X = \dim \text{AS}(\pi)}} \text{vol}(\mathcal{O}_\pi \cap S_X) \mathcal{O}_X.$$

Thus, to prove Theorem 1.3, we need only show that $\mathcal{O}_X \subset \text{AS}(\pi)$ is of maximal dimension iff $\mathcal{O}_\pi \cap S_X$ is precompact and nonempty. If $\mathcal{O}_X \subset \text{AS}(\pi)$ is of maximal dimension, then $\mathcal{O}_X \cap \mathcal{N}_{\nu_i}$ is open for every i and nonempty for at least one i , and by Corollary 2.2, $\mathcal{O}_{\nu_i} \cap S_X$ is precompact for every i and nonempty for at least one i . We conclude $\mathcal{O}_\pi \cap S_X$ is precompact and nonempty.

Conversely, suppose $\mathcal{O}_\pi \cap S_X$ is precompact and nonempty. Then $\mathcal{O}_{\nu_i} \cap S_X$ is precompact for every i and nonempty for some i . Corollary 2.2 tells us that $\mathcal{O}_X \subset \mathcal{N}_{\nu_i}$ is open for some i and $\mathcal{O}_X \cap \mathcal{N}_{\nu_i}$ is open for every i . Hence, we have that $\mathcal{O}_X \subset \text{AS}(\pi)$ is open. But, by Theorem D of [12], $\text{AS}(\pi)$ is the closure of the union of the orbits in $\text{AS}(\pi)$ of maximal dimension. Thus, $\mathcal{O}_X \subset \text{AS}(\pi)$ open implies that $\mathcal{O}_X \subset \text{AS}(\pi)$ is of maximal dimension.

4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. It is enough to prove the theorem when π is an irreducible, tempered representation. Let $\mathcal{O}_X \subset \text{AS}(\pi)$ be an open orbit, and identify $\mathfrak{g} \cong \mathfrak{g}^*$ via a G -equivariant isomorphism. Let $L \subset Z_G(X)$ be a Levi factor. Then there exists an \mathfrak{sl}_2 -triple $\{X, H, Y\}$ such that $L = Z_G\{X, H, Y\}$. To prove the theorem, we must show

$$(*) \quad Z_G\{X, H, Y\}/Z(G) \text{ is compact.}$$

Now, if $\mathcal{O}_\pi = \bigcup \mathcal{O}_{\nu_i}$, then $\text{AS}(\pi) = \bigcup \mathcal{N}_{\nu_i}$. Any open orbit in $\text{AS}(\pi)$ must be open in some \mathcal{N}_{ν_i} . Hence, it is enough to prove $(*)$ whenever \mathcal{O}_X is open in \mathcal{N}_ν for a regular element $\nu \in \mathfrak{g}^*$.

Next, supplement A and supplement C of [9] imply that every regular orbit \mathcal{O}_ν can be written as a limit of regular semisimple orbits in the following sense. Let $\xi \in \mathfrak{g}^* \cong \mathfrak{g}$ be a semisimple element in the closure of \mathcal{O}_ν , and let $\mathfrak{h} \subset \mathfrak{g}$ be a fundamental Cartan in $Z_{\mathfrak{g}}(\xi)$. Then there exists a connected component $C \subset (\mathfrak{h}^*)'$ such that

$$\lim_{\substack{\lambda \in C \\ \lambda \rightarrow \xi}} \mathcal{O}_\lambda = \mathcal{O}_\nu.$$

Hence, $\mathcal{N}_\nu = \mathcal{N}_\lambda$ for some regular semisimple element λ , and it is enough to prove $(*)$ for \mathcal{O}_X open in \mathcal{N}_λ with λ regular, semisimple.

Fix X and suppose $\mathcal{O}_X \subset \mathcal{N}_\lambda$ is open for some λ regular semisimple. By Corollary 2.2, $\mathcal{O}_\lambda \cap S_X$ is compact and nonempty. Now, $L = Z_G\{X, H, Y\}$ acts on this space, and L must have at least one closed orbit (for instance, one can take an orbit of minimal dimension). Without loss of generality, we make it $L \cdot \lambda \subset \mathcal{O}_\lambda \cap S_X$. Choose a Cartan \mathfrak{h} and a component of the regular set $C \subset (\mathfrak{h}^*)'$ such that $\lambda \in C$. If $\xi \in U_1 = G \cdot C$, then $\mathcal{O}_X \subset \mathcal{N}_\lambda = \mathcal{N}_\xi$ is open by Corollary 2.4. It then follows from Corollary 2.2 that $\mathcal{O}_\xi \cap S_X$ is compact for all ξ in the open set U_1 . Define $U = U_1 \cap S_X$, an open subset of S_X .

Now, $L/Z_G(\lambda) \cong L \cdot \lambda \subset \mathcal{O}_\lambda \cap S_X$ is a closed subset of a compact set; hence, $L/Z_G(\lambda)$ is compact. Note that $Z_G(\lambda) \subset G$ is a Cartan since λ is regular, semisimple. Thus, $Z_L(\lambda) \subset Z_G(\lambda)$ is abelian and consists of semisimple elements. Hence, the connected component of the identity $Z_L(\lambda)_0$ must be contained in a Cartan B of L . We have seen that $L/Z_L(\lambda)$ is compact; hence, $L/Z_L(\lambda)_0$ is compact because $Z_L(\lambda)$ has finitely many components. This implies L/B is compact and finally

$$L/Z(L)$$

is compact since semisimple groups are compact iff they are compact modulo a Cartan.

Because the fibers of the projection

$$L/Z(G) \rightarrow L/Z(L)$$

are homeomorphic to $Z(L)/Z(G)$, to show that $L/Z(G)$ is compact, it is enough to show $Z(L)/Z(G)$ is compact.

The following lemma is the key step in proving that $Z(L)/Z(G)$ is compact.

Lemma 4.1. *Let $Z(\mathfrak{l})$ denote the center of $\mathfrak{l} = \text{Lie}(L)$, and let $Z(\mathfrak{g})$ denote the center of \mathfrak{g} . Then*

$$\bigcap_{\xi \in U} (Z(\mathfrak{l}) \cap Z_{\mathfrak{l}}(\xi)) = Z(\mathfrak{g}).$$

Proof. Clearly the right-hand side is contained in the left-hand side. To show the other direction, suppose $W \in (Z(\mathfrak{l}) \cap Z_{\mathfrak{l}}(\xi))$ for all $\xi \in U$. We will show $W \in Z(\mathfrak{g})$. Since $W \in \mathfrak{l}$, we know $X, H, Y \in Z_{\mathfrak{g}}(W)$. Further, $Z_{\mathfrak{g}}(W) \cap Z_{\mathfrak{g}}(Y) \subset Z_{\mathfrak{g}}(Y)$ is a vector subspace containing $U - X$ since $W \in Z_{\mathfrak{l}}(\xi)$ for $\xi \in U$ and $X \in Z_{\mathfrak{g}}(W)$. Since $U - X \subset Z_{\mathfrak{g}}(Y)$ is an open subset, we must have

$$Z_{\mathfrak{g}}(W) \supset Z_{\mathfrak{g}}(Y).$$

Now, view \mathfrak{g} as a finite dimensional module for $\text{Span}_{\mathbb{R}}\{X, H, Y\} \cong \mathfrak{sl}_2\mathbb{R}$. Note $Z_{\mathfrak{g}}(W) \subset \mathfrak{g}$ is a subalgebra and a submodule for $\text{Span}_{\mathbb{R}}\{X, H, Y\}$ since $X, H, Y \in Z_{\mathfrak{g}}(W)$. But, the lowest weight vectors of each irreducible summand of \mathfrak{g} are in $Z_{\mathfrak{g}}(W)$ since $Z_{\mathfrak{g}}(W) \supset Z_{\mathfrak{g}}(Y)$, and the lowest weight vectors of any finite dimensional \mathfrak{sl}_2 module generate the entire module. Thus, $Z_{\mathfrak{g}}(W) = \mathfrak{g}$ and $W \in Z(\mathfrak{g})$ as desired. \square

Before we get back to showing that $Z(L)/Z(G)$ is compact, we need two general remarks. First, suppose A is an abelian, real algebraic group, suppose $\phi : A \rightarrow \text{Aut}(V)$ is a representation of A on a real vector space V , and suppose $S \subset V$ is a compact A -stable subset of V . Then A acts on S with compact orbits. This can be proved as follows. After complexifying the representation, we may diagonalize the action of the image $\phi(A)$ since it is abelian and consists of semisimple elements. Now, A must be isomorphic to a product of copies of S^1 , \mathbb{R}^\times , and \mathbb{C}^\times .

Using that every one-dimensional character of these groups has either compact or unbounded image in \mathbb{C} , we deduce that every orbit of A on $V \otimes \mathbb{C}$ is either compact or unbounded. In particular, A must act on a compact $S \subset V$ with compact orbits.

Second, if A is an abelian, real algebraic group and A_1, A_2 are cocompact, algebraic, closed subgroups, then $A_1 \cap A_2$ is cocompact in A . This is because the fibers of the map

$$A/(A_1 \cap A_2) \rightarrow A/A_1$$

are homeomorphic to $A_1/(A_1 \cap A_2) \cong A_1 A_2/A_2$, which is compact because it is a closed subset of A/A_2 . More generally, if A_1, \dots, A_n is a finite collection of cocompact, algebraic, closed subgroups of a real, abelian algebraic group A , then

$$A/\prod_{i=1}^n A_i$$

is compact.

Now, back to the proof that $Z(L)/Z(G)$ is compact. By the first remark, $Z(L)$ acts on $\mathcal{O}_\xi \cap S_X$ with compact orbits for every $\xi \in U$. In particular, $Z(L)/(Z(L) \cap Z_L(\xi))$ is compact for all $\xi \in U$. In the above lemma, we showed

$$\bigcap_{\xi \in U} (Z(L) \cap Z_L(\xi)) = Z(\mathfrak{g}).$$

However, one can clearly choose $\xi_1, \dots, \xi_k \in U$ such that the identity still holds when taking the intersection over this finite set. Then, by the second remark,

$$Z(L)/\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i))$$

is compact. Since $\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i))$ is a real algebraic group, it has a finite number of connected components and

$$Z(L)/\left(\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i))\right)_0$$

is also compact where $(\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i)))_0$ denotes the identity component. But, since $\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i))$ and $Z(G)$ share a Lie algebra, $Z(L)/Z(G)$ is a quotient of

$$Z(L)/\left(\prod_{i=1}^k (Z(L) \cap Z_L(\xi_i))\right)_0.$$

Thus, $Z(L)/Z(G)$ is compact. This completes the proof of Theorem 1.1.

5. TWO EXAMPLES

In this section, we give two examples. The author learned these examples from David Vogan.

Example 5.1. Nilpotent orbits in $U(2, 2)$ can be parametrized by signed Young diagrams of signature $(2, 2)$ (see page 140 of [3]). A nilpotent orbit of $U(2, 2)$ is noticed iff it corresponds to a signed Young diagram for which any two rows of equal length begin with the same sign. One can read this off of an explicit description of the Levi factor of the centralizer of a nilpotent element in the corresponding

nilpotent orbit, which in turn can be easily computed from Remark 5.1.18 on page 74 of [3] and the discussion on pages 139–140 of [3].

On the other hand, which nilpotent orbits of $U(2, 2)$ occur in the wave front cycle of a discrete series can be read off from Example 2.6.7 on page 363 of [14]. In Yamamoto’s table, the far left column is filled in by sequences of numbers and signs. The entry corresponds to a discrete series representation if the sequence contains only signs. One then observes from Yamamoto’s table that the nilpotent orbits corresponding to discrete series for $U(2, 2)$ are precisely the noticed nilpotent orbits of $U(2, 2)$.

Of course, Yamamoto is really computing the associated variety of a discrete series representation. That the associated variety of the representation corresponds to the wave front cycle of the representation under the Kostant-Sekiguchi correspondence is a deep result of [13].

Example 5.2. Nilpotent orbits in $\mathrm{Sp}(4, \mathbb{C})$ can be parametrized by partitions of four whose odd parts occur with even multiplicity (see page 70 of [3]). Of these orbits, the principal orbit (corresponding to the trivial partition 4) and the subregular orbit (corresponding to the partition 2^2) are the noticed ones as can be read off of page 88 of [3].

However, the wave front cycle of any tempered representations of $\mathrm{Sp}(4, \mathbb{C})$ contains only the principal orbit. Therefore, in this case, the set of nilpotent orbits corresponding to tempered representations is a proper subset of the set of noticed nilpotent orbits.

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REFERENCES

1. P. Bala, R. Carter, *Classes of unipotent elements in simple algebraic groups. I*, Math Proc. Cambridge. Philos. Soc., 79 (1976), 401–425. MR0417306 (54:5363a)
2. D. Barbasch, D. Vogan, *The local structure of characters*, J. Funct. Anal. 37 (1980), 27–55. MR576644 (82e:22024)
3. D. Collingwood, W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, (1993), New York, NY. MR1251060 (94j:17001)
4. Harish-Chandra, *Some results on an invariant integral on a semi-simple Lie algebra*, Ann. of Math. (2), 80 (1964), 551–593. MR0180629 (31:4862b)
5. Kirillov, *Lectures on the Orbit Method*, Graduate Studies in Mathematics, Volume 64, American Mathematical Society, (2004), Providence, RI. MR2069175 (2005c:22001)
6. C. Moeglin, *Front d’Onde des representations des groupes classiques p -adiques*, American Journal of Mathematics, 118, no. 6 (1996), 1313–1346. MR1420926 (98d:22015)
7. C. Moeglin, J. L. Waldspurger, *Modeles de Whittaker degeneres pour des groupes p -adiques*, Math Z. 196 (1987), 427–452. MR913667 (89f:22024)
8. A. Noël, *Nilpotent orbits and theta stable parabolic subalgebras*, Journal of Representation Theory, No. 2, (1998), 1–32. MR1600330 (99g:17023)
9. W. Rossmann, *Limit orbits in reductive Lie algebras*, Duke Math. J. 49 No. 1 (1982), 215–229. MR650378 (84e:22021)
10. W. Rossmann, *Limit characters of reductive Lie groups*, Invent. Math. 61 (1980), no. 1, 53–66. MR587333 (81m:22021)

11. W. Rossmann, *Tempered representations and orbits*, Duke Math. J. 49 No. 1 (1982), 231-247. MR650379 (84m:22024)
12. W. Rossmann, *Picard-Lefschetz theory and characters of a semisimple Lie group*, Invent. Math. 121 (1995), 579-611. MR1353309 (96j:22017)
13. W. Schmid, K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Ann. of Math. (2), 151 (2001), 1071-1118. MR1779564 (2001j:22017)
14. A. Yamamoto, *Orbits in the flag variety and images of the moment map for classical groups I*, Journal of Representation Theory, No. 1, (1997), 329-404. MR1479152 (98j:22024)

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