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COMPLEMENT TO THE APPENDIX OF: “ON THE HOWE DUALITY CONJECTURE”

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Abstract. Let $\mathbb{F}$ be a local field, nonarchimedean and of characteristic not 2. Let (V, Q) be a nondegenerate quadratic space over $\mathbb{F}$, of dimension n. Let M_r be the direct sum of r copies of V. We prove that, for $r < n$ there is no nonzero distribution on M_r which under the action of the orthogonal group transforms according to the character determinant.

Let $\mathbb{F}$ be a local field, nonarchimedean and of characteristic not 2. Let V be a vector space over $\mathbb{F}$ of finite dimension $n \geq 1$ equipped with a nondegenerate quadratic form Q. Let $G = O(Q)$ be the orthogonal group of Q and χ the character of G such that $\chi(g) = 1$(resp. $\chi(g) = -1$) if the determinant of g is 1 (resp. -1).

Let $r \geq 1$ be an integer and let M_r be the direct sum of r copies of V or equivalently the set of linear maps from $\mathbb{F}^r$ onto V or, choosing a basis of V, the space of $n \times r$ matrices with coefficients in $\mathbb{F}$. We let G act on M_r on the left and $\mathrm{GL}_r(\mathbb{F})$ act on the right. Finally, let $\mathcal{S}(M_r)$ be the space of locally constant functions on M_r with compact support and values in $\mathbb{C}$ and its dual $\mathcal{S}'(M_r)$ the space of distributions on M_r. It will be convenient to include the case $r = 0$ with $M_0 = (0)$ and the trivial action of G.

Theorem. *Suppose that $r < n$. If $T \in \mathcal{S}'(M_r)$ transforms according to the character χ under the action of $G = O(Q)$, then $T = 0$.*

This result was stated in Appendix 2 in [R] and used in the proof of the theorem in Chapter II. However, the proof given there was confusing to the reader due to incomplete detail. As the result itself is interesting we present here, in this short note, a full version of our proof.

Note that this is false if $r \geq n$. Indeed, suppose first that $r = n$ and let $x = (\xi_1, \dots, \xi_n)$ be an orthogonal basis of V. Then $Q(\xi_i) = \alpha_i \neq 0$ (the quadratic form is nondegenerate) and by Witt Theorem the orbit of x under G is the set of all orthogonal systems $(\eta_i, \dots, \eta_n)$ such that for all i, $Q(\eta_i) = \alpha_i$. In particular, it is a closed orbit. Now the isotropy subgroup of x in G is trivial so that the orbit is homeomorphic to G. Choose a Haar measure dg on G. The distribution $\chi(g)dg$ is a nontrivial distribution on G transforming according to the character χ. We may view this distribution as a distribution on Gx, hence on V. If $r > n$ we note that M_n is imbedded into M_r by $(\xi_1, \dots, \xi_n) \mapsto (\xi_1, \dots, \xi_n, 0, \dots, 0)$ as a closed invariant subset and, therefore, $\mathcal{S}'(M_n) \subseteq \mathcal{S}'(M_r)$.

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From now on we assume that $r \leq n-1$. We shall prove the theorem by induction on the dimension n of V. If $n = 1$, then $r = 0$ and the result is trivial.

The bilinear form associated to Q is the bilinear form $B(\xi, \eta) = \big(Q(\xi+\eta) - Q(\xi) - Q(\eta)\big)/2$; let ρ be the linear map from V to its dual V^* given by $B(\xi, \eta) = \langle \xi, \rho(\eta)\rangle$. An element $x \in M_r$ is a linear map from $\mathbb{F}^r$ to V. If we identify $\mathbb{F}^r$ with its dual in the usual way, then ${}^t x$ is a linear map from V^* into $\mathbb{F}^r$ and we may define $\mu(x) = {}^t x \rho x$, the so-called moment map. The fibers of this map are stable under G, so by Bernstein's localization principle [Ber84, AGRS] it is enough to prove that for every $\sigma \in \mathrm{Im}(\mu)$ the space $\mathcal{S}'(\mu^{-1}(\sigma))^\chi$ of distributions on $\mu^{-1}(\sigma)$ which transform according to χ is reduced to (0).

The group $\mathrm{GL}_r(\mathbb{F})$ acts on the right on M_r. We let this group act on $r \times r$ symmetric matrices as usual: $(u, y) \mapsto {}^t u y u$. Then μ is an equivariant map for $\mathrm{GL}_r(\mathbb{F})$ and it is enough to choose an element σ for each orbit of $\mathrm{GL}_r(\mathbb{F})$. In particular, we may assume that σ is diagonal:

$$\sigma = \mathrm{diag}(\alpha_1, \dots, \alpha_s, 0, \dots, 0), \quad \alpha_i \neq 0.$$

Note that $s \leq r < n$. Assume that $s \neq 0$. Let $x = (\xi_1, \dots, \xi_r) \in \mu^{-1}(\sigma)$. Then

$$B(\xi_i, \xi_j) = \alpha_i \delta_{i,j} \quad 1 \leq i, j \leq s.$$

Let W be the subspace of V generated by $\xi_1, \dots, \xi_s$; it has dimension s.

Let Ξ be the set of all s-tuples $(\eta_1, \dots, \eta_s) \in M_s$ such that $B(\eta_i, \eta_s) = \alpha_i \delta_{i,j}$, $1 \leq i, j \leq s$. It is a closed subset of M_s. By Witt's theorem, G acts transitively on Ξ. The map $\nu : (\eta_1, \dots, \eta_r) \mapsto (\eta_1, \dots, \eta_s)$ is a G-equivariant map from $\mu^{-1}(\sigma)$ onto Ξ. The fiber $\nu^{-1}(\nu(x))$ is homeomorphic to the space Γ of all $(r-s)$-tuples $(\zeta_1, \dots, \zeta_{r-s}) \in (W^\perp)^{r-s}$ such that $B(\zeta_i, \zeta_j) = 0$, $1 \leq i, j \leq r - s$. The isotropy group of x is isomorphic to $O(Q_{W^\perp})$ and it acts on the fiber Γ in the obvious way. By Frobenius descent [Ber84, AGRS] we get a bijection between the spaces $\mathcal{S}'(\mu^{-1}(\sigma))^\chi$ and $\mathcal{S}'(\Gamma)^\chi$. As $s \neq 0$, the dimension of $W^\perp$ is strictly smaller than n so that, by induction, $\mathcal{S}'(\Gamma)^\chi = (0)$.

We are left with the case $s = 0$. Put $\Gamma = \mu^{-1}(0)$. We already know that if $T \in \mathcal{S}'(M_r)^\chi$ then its support is contained in Γ. Suppose that $x = (\xi_1, \dots, \xi_r) \in \Gamma$. The subspace H_1 generated by the ξ_i is (a totally) isotropic subspace. The dimension t of H_1 is at most the $\mathbb{F}$-rank ℓ of G. For $0 \leq t \leq \ell$, let $\Gamma_t = \{x \mid \ \dim H_1 = t\}$. Then $\Gamma = \bigcup \Gamma_t$ and we have a stratification of Γ.

We want to find the orbits of G in Γ_t. Fix a decomposition $V = E_0 \oplus E_1 \oplus E_2$ with the following properties. The restriction of Q to E_0 is nondegenerate and anisotropic, $E_0^\perp = E_1 \oplus E_2$. There exists a basis $e_1, \dots, e_\ell$ of E_1 and a basis $f_1, \dots, f_\ell$ of E_2 such that $B(e_i, e_j) = B(f_i, f_j) = 0$ and $B(e_i, f_j) = \delta_{i,j}$. By Witt's theorem each orbit of G in Γ_t contains a point x_t such that $\xi_i = e_i$ for $1 \leq i \leq t$ (and therefore $H_1 = \bigoplus_1^t \mathbb{F} e_i$). If $E_0 \neq (0)$, then $O(Q_{E_0})$ viewed as a subgroup of $O(Q)$ contains an element of determinant -1 and such an element leaves x_t invariant. Then, by Frobenius reciprocity $\mathcal{S}'(G x_t)^\chi = (0)$. This being true for all the orbits of G in Γ we conclude that $\mathcal{S}'(\Gamma)^\chi = (0)$ and we are done.

Therefore we may assume that $E_0 = (0)$. Now, if $t < \ell$, consider the following element h of G:

$$h(e_i) = e_i, \ h(f_i) = f_i, \ i < \ell, \quad h(e_\ell) = f_\ell\,, \ h(f_\ell) = e_\ell.$$

Then $h(x_t) = x_t$ and as before this implies that $\mathcal{S}'(\Gamma \setminus \Gamma_\ell)^\chi = (0)$.

Finally, we must look at Γ_ℓ. Recall that $GL_r(\mathbb{F})$ acts on Γ_ℓ on the right. Let $\gamma = (e_1, \ldots, e_\ell, 0, \ldots, 0)$ with $r-\ell$ zeros be the $(2\ell \times r)$ matrix in which e_i are to be considered as column vectors of size 2ℓ with zeros at all places except for a 1 at the i-place . Then $\Gamma_\ell = G\gamma\,\mathrm{GL}_r(\mathbb{F})$. Write an element of G as a 2 by 2 matrix relative to the decomposition $V = E_1 \oplus E_2 \approx \mathbb{F}^\ell \oplus \mathbb{F}^\ell$ and an element of $GL_r(\mathbb{F})$ as a 2 by 2 matrix relative to the decomposition $\mathbb{F}^r = \mathbb{F}^\ell \oplus \mathbb{F}^{r-\ell}$. Note that the $2\ell \times r$ matrix γ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where 1 denotes the $\ell \times \ell$ identity matrix.

To find the isotropy subgroup of γ in $G \times GL_r(\mathbb{F})$ we solve the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We must have $a\alpha = 1$, $c = 0$, $\beta = 0$. This implies that $d = {}^t a^{-1}$ and $a^{-1}b$ is antisymmetric. Let P be the subgroup $\beta = 0$ of $\mathrm{GL}_r(\mathbb{F})$ and consider the map $g\gamma u \mapsto Pu$ of Γ_ℓ onto $P \setminus \mathrm{GL}_r(\mathbb{F})$. The above computation shows that it is well defined and that the fibers of this application are exactly the orbits of G in Γ_ℓ. Let N be the subgroup of G defined by $a = 1,\ c = 0,\ d = 1,\ b$ antisymmetric. Then N is the isotropy subgroup of γu in G. Therefore the orbit is homeomorphic to G/N and if dg is a nonzero invariant measure on G/N, then any distribution on the orbit which transforms according to χ is a multiple of the distribution

$$S_u : \varphi \mapsto \int_{G/N} \varphi(g)\chi(g)dg.$$

For $\lambda \in \mathbb{F}^*$, the matrix $\lambda \mathrm{Id}_r$ belongs to P which shows that the orbit of γu is stable by dilation. Furthermore,

$$g\gamma u(\lambda \mathrm{Id}_r) = g \begin{pmatrix} \lambda \mathrm{Id}_\ell & 0 \\ 0 & \lambda^{-1}\mathrm{Id}_\ell \end{pmatrix} \gamma u.$$

It follows that

$$\langle S_u, \varphi(\lambda\cdot)\rangle = |\lambda|^{-\ell(\ell-1)} \langle S_u, \varphi\rangle.$$

However, the Bernstein localization principle [Ber84, AGRS] implies that the space $\Sigma_u \mathbb{F} S_u$ is weakly dense in $\mathcal{S}'(\Gamma_\ell)^\chi$ so that any $T \in \mathcal{S}'(\Gamma_\ell)^\chi$ has this homogeneity property. Suppose that such a T extends to a distribution on Γ still transforming by χ; this extension is unique therefore it will have the same homogeneity property. View T as a distribution on the vector space M_r and let $\widehat{T}$ be its Fourier transform. Then

$$\langle \widehat{T}, \varphi(\lambda x)\rangle = |\lambda|^{2r\ell + \ell(\ell-1)} \langle \widehat{T}, \varphi\rangle.$$

On the other hand, $\widehat{T} \in \mathcal{S}'(M_r)^\chi$ and so its support is contained in Γ and it must satisfy

$$\langle \widehat{T}, \varphi(\lambda x)\rangle = |\lambda|^{-\ell(\ell-1)} \langle \widehat{T}, \varphi\rangle.$$

If $T \neq 0$, then $2r\ell + \ell(\ell-1) = -\ell(\ell-1)$, which is impossible. So $T = 0$ and we are done!

References

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