

LOCALIZATION IN QUIVER MODULI SPACES

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ABSTRACT. Torus fixed points of quiver moduli spaces are given by stable representations of the universal (abelian) covering quiver. As far as the Kronecker quiver is concerned they can be described by stable representations of certain bipartite quivers coming along with a stable colouring. By use of the glueing method it is possible to construct a huge class of such quivers implying a lower bound for the Euler characteristic. For certain roots it is even possible to construct all torus fixed points.

1. INTRODUCTION

A common method providing topological information of algebraic varieties is the consideration of fixed points under a torus action. For instance, the Euler characteristic is already given by the Euler characteristic of their fixed point components. If we consider moduli spaces of stable quiver representations, we also often obtain interesting objects as fixed point components like indecomposable tree modules in the case of the Kronecker quiver; see [29]. In general, torus fixed points of quiver moduli spaces are given by representations of the universal (abelian) covering quiver.

The main focus of this paper is on torus fixed points of Kronecker moduli spaces, denoted by $M_{d,e}^s(m)$, which parametrize isomorphism classes of stable representations of dimension (d, e) of the generalized Kronecker quiver with m arrows; see Sections 2 and 4.1 for a more detailed description. It is particularly interesting to note that by using the localization method we are able to prove parts of a conjecture concerning the Euler characteristic of these moduli spaces. The conjecture is based on ideas suggested by Michael Douglas [7]. It has its origin in the relationship between the Euler characteristic of moduli spaces of quiver representations and the Witten index in superstring theory and the entropy of black holes respectively; see for instance [6]. Douglas conjectures the existence of a formula expressing the entropy of a black hole as a continuous function of its electric and magnetic charge, in the limit of charge. In mathematical terms it says that for coprime dimension vectors (d, e) the logarithm of the Euler characteristic $M_{d,e}^s(m)$ continuously depends on the fraction $\frac{e}{d}$. More specifically, this means that there exists a continuous function f such that for every coprime dimension vector (d, e) there exists another dimension vector (d_s, e_s) such that

$$f\left(\frac{e}{d}\right) = \lim_{n \rightarrow \infty} \frac{\ln \chi(M_{d_s+nd, e_s+ne}^s(m))}{d_s + nd}.$$

In particular, the right hand side converges. In [28] a candidate for this function could be determined and it could be proved that under the assumption of continuity

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the function is already uniquely determined by $f(1)$; see also Section 6.1 for the exact statements.

Even if continuity is still an open question, by use of the localization method we are able to calculate the value at the point one. Actually, we are able to determine a formula for the Euler characteristic of the Kronecker moduli spaces for the dimension vectors $(d, d + 1)$. Moreover, we show that the Euler characteristic grows at least exponentially with the dimension vector which is an immediate consequence of the conjecture.

The paper is organized as follows: In the second section the notion of quivers and their representations is introduced. Moreover, we recall the definition of stability and general results concerning the representation spaces which are needed in the sequel.

In the third section we consider a torus action on quiver moduli spaces. We show that the fixed points of moduli spaces of quivers without oriented cycles are exactly the stable representations of the universal abelian covering quiver up to a certain translation. This means localizing the moduli spaces to torus fixed points induces new moduli spaces. Thus we can again consider a torus action on the resulting moduli spaces. It turns out that after localizing to torus fixed points suitably many times the remaining torus fixed points are representations of the universal covering quiver.

In the fourth section we apply the localization method to the generalized Kronecker quiver. The universal covering quiver is a regular m -tree coming along with a bipartite orientation. By the results of Section 3 every stable representation of a bipartite quiver which can be embedded into this m -tree defines a torus fixed point. Therefore, we investigate stable bipartite quivers in more detail, i.e., quivers with a fixed dimension vector allowing at least one stable representation. We construct stable bipartite quivers of dimension type $(d_s, e_s) + n(d, e)$ by glueing certain bipartite quivers of dimension types (d_s, e_s) and (d, e) . Thereby the dimension vector (d_s, e_s) is uniquely determined by (d, e) . The dimension type of a bipartite quiver is given by the sum of the dimensions of the sources and sinks respectively. In this way, for every coprime dimension type we can construct a huge class of such quivers.

In the fifth section we briefly treat combinatorics of trees. With the stated methods it is possible to count the number of stable bipartite quivers constructed in the preceding section.

In the last section several applications of the developed methods are treated. After investigating Douglas' conjecture in more detail, we study the function f at the point one. Since all localization data of dimension type $(d, (m - 1)d + 1)$ are known, we can determine a formula for the Euler characteristic in this case. By applying the reflection functor we can also determine $f(1)$.

Afterwards, by use of the methods of the fourth section and combinatorics of trees we can determine a lower bound for the Euler characteristic for every coprime dimension vector. In particular, we prove that the Euler characteristic grows at least exponentially with the dimension vector.

In the fourth subsection the case of the dimension vector $(3, 4)$ is considered as a detailed example. The fifth subsection deals with the dimension vector (d, d) . We prove that there does not exist any stable representation of the universal covering quiver if $d \geq 2$ because torus fixed points of this dimension type are always cyclic.

Thus it follows that the Euler characteristic vanishes in this case. In the sixth subsection we answer a question posed in [8]: When does there exist fixed point components containing infinitely many fixed points? Actually, there exist only finitely many torus fixed points for dimension vectors (d, e) such that $d = 1, 2$ or in the associated reflected cases.

2. RECOLLECTIONS AND NOTATION

Let k be an algebraically closed field. Let $Q = (Q_0, Q_1)$ be a quiver with vertices Q_0 and arrows Q_1 denoted by $\alpha : q \rightarrow q'$. There exist two maps $h, t : Q_1 \rightarrow Q_0$ which associate its tail $t(\alpha)$ and its head $h(\alpha)$ to an arrow $\alpha \in Q_1$. A vertex $q \in Q_0$ is called a sink (resp. a source) if $t^{-1}(q) = \emptyset$ (resp. $h^{-1}(q) = \emptyset$). In the following we only consider quivers without oriented cycles. A quiver is bipartite if $Q_0 = I \cup J$ such that every vertex $i \in I$ is a source and every vertex $j \in J$ is a sink. A subquiver $Q' \subseteq Q$ is called full if for all $q, q' \in Q'_0$ we have $\alpha \in Q'_1$ for all $\alpha \in Q_1$ with $h(\alpha) = q'$ and $t(\alpha) = q$ (resp. $h(\alpha) = q$ and $t(\alpha) = q'$).

Consider the free abelian group $\mathbb{Z}Q_0$ generated by Q_0 and the monoid of dimension vectors $\mathbb{N}Q_0 \subset \mathbb{Z}Q_0$. On $\mathbb{Z}Q_0$ we define a (non-symmetric) bilinear form, called the Euler form, by

$$\langle d, e \rangle := \sum_{q \in Q_0} d_q e_q - \sum_{\alpha \in Q_1} d_{t(\alpha)} e_{h(\alpha)}$$

for $d, e \in \mathbb{Z}Q_0$. Moreover, let $\{d, e\} := \langle d, e \rangle + \langle e, d \rangle$ be the symmetrized Euler form.

A finite-dimensional k -representation X of Q is given by finite-dimensional vector spaces X_q for every $q \in Q_0$ and linear maps $X_\alpha : X_q \rightarrow X_{q'}$ for every $\alpha : q \rightarrow q'$. By $\underline{\dim} X \in \mathbb{N}Q_0$ we denote the dimension vector of X . The support $\text{supp}(d)$ of a dimension vector $d \in \mathbb{N}Q_0$ is the full subquiver of Q defined by the vertices $\text{supp}(d)_0 = \{q \in Q_0 \mid d_q \neq 0\}$. In the following, we only consider dimension vectors which support is finite, i.e., the number of vertices and arrows is finite.

Let $d \in \mathbb{N}Q_0$ and consider the variety $R_d(Q) = \bigoplus_{\alpha \in Q_1} \text{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{h(\alpha)}})$ of k -representations of dimension d . The algebraic group $G_d = \prod_{q \in Q_0} \text{Gl}_{d_q}(k)$ acts on $R_d(Q)$ via simultaneous base change. The orbits are in bijection with the isomorphism classes of k -representations of Q with dimension vector d .

In the space of \mathbb{Z} -linear functions $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$ we consider the basis given by the elements q^* for $q \in Q_0$, i.e., $q^*(q') = \delta_{q,q'}$ for $q' \in Q_0$. Define $\text{dim} := \sum_{q \in Q_0} q^*$. After choosing $\Theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$, we define the slope function $\mu := \Theta / \text{dim} : \mathbb{N}Q_0 \setminus \{0\} \rightarrow \mathbb{Q}$.

The slope $\mu(\underline{\dim} X)$ of a representation $X \neq 0$ of Q is abbreviated to $\mu(X)$. A representation X of Q is called semistable (resp. stable) if for all proper subrepresentations $0 \neq U \subsetneq X$ we have

$$\mu(U) \leq \mu(X) \text{ (resp. } \mu(U) < \mu(X)\text{)}.$$

This definition is equivalent to that of A. King [16]: Let $\tilde{\Theta} \in \text{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$ be a linear form. A representation X is $\tilde{\Theta}$ -semistable (resp. $\tilde{\Theta}$ -stable) in the sense of King if $\tilde{\Theta}(\underline{\dim} X) = 0$ and

$$\tilde{\Theta}(\underline{\dim} U) \geq 0 \text{ (resp. } \tilde{\Theta}(\underline{\dim} U) > 0\text{)}$$

for all proper subrepresentations $0 \neq U \subsetneq X$. Now fixing a representation X define $\tilde{\Theta} := \mu(X) \cdot \dim -\Theta$. It is easy to check that the representation X is semistable (resp. stable) in the former sense if and only if it is $\tilde{\Theta}$ -semistable (resp. $\tilde{\Theta}$ -stable).

We call a dimension vector Θ -in divisible if $\mu(d) \neq \mu(e)$ for all $0 \neq e < d$. In this case semistability and stability coincide.

Denote the set of semistable (resp. stable) points by $R_d^{ss}(Q)$ (resp. $R_d^s(Q)$). In this situation we have the following theorem going back to Mumford's GIT and which was proved by King; see [21], [16]:

- Theorem 2.1.** (1) *The set of stable points $R_d^s(Q)$ is an open subset of the set of semistable points $R_d^{ss}(Q)$, which is an open subset of $R_d(Q)$.*
 (2) *There exists a categorical quotient $M_d^{ss}(Q) := R_d^{ss}(Q)/G_d$. Moreover, $M_d^{ss}(Q)$ is a projective variety.*
 (3) *There exists a geometric quotient $M_d^s(Q) := R_d^s(Q)/G_d$, which is a smooth open subvariety of $M_d^{ss}(Q)$.*

Note that the set of semistable (resp. stable) points of $R_d(Q)$ can be empty. For a detailed description of the theory of quotients see [20]. All statements about algebraic groups applied in this paper can, for instance, be found in [27] or [14].

Recall that we have $M_d^{ss}(Q) = \text{Proj}(k[X]_\chi^G)$, i.e., the moduli space is the projective spectrum of the ring of semi-invariants corresponding to the character χ of G_d defined by

$$\chi((g_q)_{q \in Q_0}) := \prod_{q \in Q_0} \det(g_q)^{\Theta(d) - \dim d \cdot \Theta_q},$$

where Θ is the linear form obtained from the previous consideration.

Remark 2.2.

- (1) Since Q is acyclic, there exists only one closed orbit and the affine quotient is just a point. Therefore, we get $k[R_d(Q)]^G = k$. Thus the Proj quotient has no affine component and is a projective variety.
- (2) Since $R_d(Q)$ is an affine space and thus smooth, we get that the open subset of stable points is smooth. Thus, since the moduli space $M_d^s(Q)$ is an orbit space associated to the group action restricted to the stable points, it is smooth as well. If semistability and stability coincide, $M_d^{ss}(Q)$ is a smooth projective variety. Obviously this is the case when d is Θ -in divisible.
- (3) The moduli space $M_d^{ss}(Q)$ does not parametrize the semistable representations, but the polystable ones. Polystable representations are such representations which can be decomposed into a direct sum of stable ones of the same slope.
- (4) For a stable representation X we have that its orbit is of maximal possible dimension. Since the scalar matrices act trivially on $R_d(Q)$, the isotropy group is at least one-dimensional. Thus, if the moduli space $M_d^s(Q)$ is not empty, for the dimension of the moduli space we have $\dim M_d^s(Q) = 1 - \langle d, d \rangle$.

Finally, we point out definitions and results which will be very useful at different points of this paper. For proofs of the next lemma see [12].

Lemma 2.3. *For a quiver Q let $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$ be a short exact sequence of representations.*

- (1) $\mu(M) \leq \mu(X)$ if and only if $\mu(X) \leq \mu(N)$ if and only if $\mu(M) \leq \mu(N)$.

- (2) *The following holds: $\min(\mu(M), \mu(N)) \leq \mu(X) \leq \max(\mu(M), \mu(N))$.*
- (3) *If $\mu(M) = \mu(X) = \mu(N)$, then X is semistable if and only if M and N are semistable.*

From the first property we immediately get that stable representations are indecomposable. For a quiver representation X we define by $\text{scss}(X)$ (strongly contracting semistability) the subrepresentation $Y \subset X$ for which the following holds:

- (1) $\mu(Y) = \max\{\mu(U) \mid U \subset X\}$.
- (2) $\dim(Y) = \max\{\dim(U) \mid U \subset X, \mu(U) = \mu(Y)\}$.

Thus Y is of maximal dimension among the subrepresentations with maximal slope. It is straightforward to check that the subrepresentation $\text{scss}(X)$ is uniquely determined; see e.g. [24].

Denote by E_q the simple representation corresponding to the vertex q defined by $(E_q)_q = k$ and $(E_q)_{q'} = 0$ for $q' \in Q_0$ with $q' \neq q$.

Define $e_q \in \mathbb{Z}Q_0$ by $(e_q)_{q'} := \delta_{q,q'}$ which is the dimension vector of E_q . For a quiver Q consider the matrix $A = (a_{q,q'})_{q,q' \in Q_0}$ defined by $a_{q,q'} = \{e_q, e_{q'}\}$ for $q, q' \in Q_0$. Fixing some $q \in Q_0$ define $r_q : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}Q_0$ by

$$r_q(e_{q'}) = e_{q'} - a_{q,q'} \cdot e_q.$$

Let Q_q be the quiver resulting from Q by reversing all arrows with head or tail q . We have the following theorem; see [1]:

Theorem 2.4. *Let Q be a quiver and $q \in Q_0$ a fixed vertex. Let q be a sink (resp. a source). Then there exists a functor R_q^+ (resp. R_q^-) : $\text{mod } kQ \rightarrow \text{mod } kQ_q$ with the following properties (if q is a source, replace $+$ by $-$):*

- (1) $R_q^+(U \oplus U') = R_q^+(U) \oplus R_q^+(U')$.
- (2) *Let U be an indecomposable representation of Q .*
 - (a) *If $U \cong E_q$, then $R_q^+(E_q) = 0$.*
 - (b) *If $U \not\cong E_q$, then $R_q^+(U)$ is indecomposable with $R_q^- R_q^+(U) \cong U$ and we have $\underline{\dim} R_q^+(U) = r_q(\underline{\dim}(U))$.*

Moreover, we have: $\text{End} U \cong \text{End} R_q^+(U)$.

3. LOCALIZATION IN QUIVER MODULI SPACES

Analogously to [24], in this section we introduce the localization in moduli spaces of stable representations. Some of the ideas are based on localization techniques in moduli spaces of simple representation provided by [22]. An explicit method to detect fixed points of these moduli spaces under a torus action is explained. These fixed points are stable representations of the universal abelian covering quiver. Note that, in contrast to [24], we do not restrict to the case of Θ -indivisible dimension vectors.

3.1. Torus fixed points. For the remaining part of the paper we fix $k = \mathbb{C}$. Let G be an algebraic group and $\chi : G \rightarrow \mathbb{C}^*$ be a character of G , i.e. a morphism of algebraic groups. Denote by $\Xi(G)$ the set of all characters of G with the group structure given in the obvious way. In the following the composition is written additively.

Let $G \subset Gl_n(\mathbb{C})$ be a linear algebraic group and V be a representation of G . For all characters $\chi \in \Xi(G)$ define the semi-invariants of weight χ by

$$V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \ \forall g \in G\}.$$

Note that if $\varphi : G \rightarrow Gl_n(\mathbb{C})$ is a rational representation, the definition can be transferred. If G is a torus, we obtain a decomposition into weight spaces $V = \bigoplus_{\chi \in \Xi(G)} V_\chi$.

Further, let $T := (\mathbb{C}^*)^{|Q_1|}$ be the $|Q_1|$ -dimensional torus. It acts on $R_d(Q)$ via

$$((t_\alpha)_{\alpha \in Q_1}) \cdot ((X_\alpha)_{\alpha \in Q_1}) = (t_\alpha \cdot X_\alpha)_{\alpha \in Q_1}.$$

Since U is a subrepresentation of a representation X if and only if $t \cdot U$ is a subrepresentation of $t \cdot X$ for every $t \in T$ and, moreover, the torus action preserves dimension vectors, the action also preserves (semi)-stability. Since the torus action commutes with the G_d -action, it induces a T -action on $M_d^s(Q)$.

Since the scalar matrices act trivially on $R_d(Q)$, the G_d -action factors through the quotient $PG_d := G_d/\mathbb{C}^*$. Let $X \in M_d^s(Q)$ be a fixed point under the torus action and let $Y \in R_d^s(Q)$ be a representative of X . Thus we have $t \cdot Y \cong Y$ for every $t \in T$. We consider the algebraic group

$$G := \{((g_q)_{q \in Q_0}, t) \in PG_d \times T \mid t \cdot Y = (g_q)_{q \in Q_0} * Y\}.$$

Then we get projections $p_1 : G \rightarrow PG_d$ and $p_2 : G \rightarrow T$ respectively with the following property:

Lemma 3.1. *Let $X \in M_d^s(Q)$ be a torus fixed point, $Y \in R_d^s(Q)$ a representative of X and let G be the corresponding algebraic group constructed above. The following holds:*

- (1) *The projection $p_2 : G \rightarrow T$ is an isomorphism.*
- (2) *In particular, the projection $p_1 : G \rightarrow PG_d$ induces a homomorphism of algebraic groups $\varphi := p_1 \circ p_2^{-1} : T \rightarrow PG_d$ such that $\varphi(t) * Y = t \cdot Y$.*

Proof. Since $X \in M_d^s(Q)$ is a fixed point, p_2 is surjective. Moreover, since Y is stable, its orbit is of maximal possible dimension. Thus the isotropy group of Y under the action of PG_d is trivial implying the injectivity. The second part immediately follows from this. □

The following lemma assures that we get a weight space decomposition of the vector spaces corresponding to some fixed point:

Lemma 3.2. *Let $T \cong (\mathbb{C}^*)^m$ with $m \geq 1$ be a torus. Every homomorphism of algebraic groups $\varphi : T \rightarrow PGL_d(\mathbb{C})$ can be lifted, i.e., there exists a homomorphism of algebraic groups $\psi : T \rightarrow GL_d(\mathbb{C})$ such that $\varphi = \pi \circ \psi$.*

Proof. In general, if G is a reductive algebraic group, the image of a morphism $\varphi : T \rightarrow G$ is again a torus and, therefore, contained in a maximal torus $T_0 \subset G$. Since all maximal tori are conjugate, we can assume that $T_0 = (\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$. Thus, in order to prove the statement, it suffices to prove that every morphism $\varphi : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^{n-1}$ can be lifted to a morphism $\psi : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^n$ where $\pi : (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^{n-1}$ is the projection induced by the projection $\pi : GL_d(\mathbb{C}) \rightarrow PGL_d(\mathbb{C})$. Note that, since π is surjective, every maximal torus is mapped to a maximal torus. Now if $\varphi(t) = (t^{r_2}, \dots, t^{r_n})$ and $\pi(t_1, \dots, t_n) = (\frac{t_2}{t_1}, \dots, \frac{t_n}{t_1})$, then we may set $\psi(t) = (1, t^{r_2}, \dots, t^{r_n})$. □

A lift $\psi : T \rightarrow G_d$ for φ can be decomposed in $|Q_0|$ morphisms of algebraic groups $\psi_q : T \rightarrow GL_{d_q}$. Thus, if X is a stable fixed point, Y a representative and $\varphi : T \rightarrow PG_d$ the corresponding morphism, we can fix a lift in order to get a weight space decomposition of each vector space, i.e., $Y_q = \bigoplus_{\chi \in \Xi(T)} Y_{q,\chi}$. For a

d -dimensional torus we have $\Xi(T) \simeq \mathbb{Z}^d$. As far as the torus $(\mathbb{C}^*)^{|Q_1|}$ is concerned we denote the canonical basis of $\Xi(T)$ by $(e_\alpha)_{\alpha \in Q_1}$.

If we choose another lift ψ' , one easily verifies that there exists a character $\chi \in \Xi(T)$ such that $\psi = \chi\psi'$. If we have two representatives Y and Y' of a fixed point, i.e., there exists some $g \in G_d$ such that $Y' = g * Y$, we can assume that the weight space decomposition does not change. Indeed, if φ is the morphism belonging to Y , for the morphism φ' belonging to Y' we have $\varphi' = \pi(g) \cdot \varphi \cdot \pi(g^{-1})$ where $\pi : G_d \rightarrow PG_d$ is the canonical projection. Thus, if ψ is a lift of φ , we have that $\psi' = g \cdot \psi \cdot g^{-1}$ is a lift of φ' . Thus we obtain that the dimensions of the weight spaces for both morphism ψ and ψ' coincide. Indeed, for $x \in Y_{q,\chi}$ we have $\chi(t)x = \psi(t)_q x$ if and only if $\chi(t)g_q x = \psi'(t)_q g_q x$. This also shows that g is compatible with the weight space decomposition. In abuse of notation, we also denote the lift by X in what follows.

The next lemma shows that these weight space decompositions are compatible with the linear maps corresponding to the fixed point:

Lemma 3.3. *Let $X = (X_\alpha)_{\alpha \in Q_1} \in M_d^s(Q)$ be a fixed point under the torus action. Let $\varphi : T \rightarrow PG_d$ be the corresponding morphism of algebraic groups and fix a lift $\psi : T \rightarrow G_d$. Then we have:*

$$X_\alpha(X_{q,\chi}) \subseteq X_{q',\chi+e_\alpha} \text{ for all } \chi \in \Xi(T), \alpha : q \rightarrow q'.$$

Proof. Let $t = (t_\alpha)_{\alpha \in Q_1} \in T$ and $x \in X_{q,\chi}$. Then we have

$$\psi_{q'}(t)X_\alpha(x) = \psi_{q'}(t)X_\alpha\psi_q(t)^{-1}\psi_q(t)(x) = t_\alpha X_\alpha \chi(t)(x) = (\chi + e_\alpha)(t)X_\alpha(x).$$

□

We want to define a quiver such that the components of the fixed point set $M_d^s(Q)^T$ correspond to moduli spaces of this quiver with compatible dimension vectors. Therefore, define the quiver \hat{Q} by the vertex set $\hat{Q}_0 = Q_0 \times \Xi(T)$ and for each arrow $\alpha : q \rightarrow q'$ and each character $\chi \in \Xi(T)$ we have an arrow

$$(\alpha, \chi) : (q, \chi) \rightarrow (q', \chi + e_\alpha)$$

in \hat{Q}_1 . This is the universal abelian covering quiver of Q .

Example 3.4.

- (1) Let $K(m)$ be the generalized Kronecker quiver having two vertices $Q_0 = \{q_1, q_2\}$ and m arrows $\alpha_i : q_1 \rightarrow q_2, i = 1, \dots, m$. Every connected component of the universal abelian covering quiver of the Kronecker quiver $K(m)$ is an infinite bipartite $(m - 1)$ -dimensional honeycomb lattice with an orientation such that every vertex is either a source or a sink.

Let X be a fixed point of the moduli space with respect to the torus action. Then define the corresponding dimension vector $\hat{d} \in \mathbb{N}\hat{Q}_0$ by $\hat{d}_{q,\chi} := \dim_{\mathbb{C}} X_{q,\chi}$.

The stability condition for representations of this quiver is induced from Θ , i.e., we define a linear form $\hat{\Theta} : \mathbb{Z}\hat{Q}_0 \rightarrow \mathbb{Z}$ such that $\hat{\Theta}_{q,\chi} = \Theta_q$ for all $q \in Q_0$ and all $\chi \in \Xi(T)$. Therefore, by the preceding considerations it follows that stable fixed points can be identified with stable representations of the just introduced quiver.

Next we show that such a representation corresponding to a fixed point X is unique in a certain way. By the preceding considerations choosing another lift ψ just changes the weights of the weight space decomposition by translation by some character μ . This corresponds to a group action of $\mathbb{Z}Q_1$ on \hat{Q}_0 defined by

$\mu \cdot (q, \chi) = (q, \chi + \mu)$. This induces a group action on the set of dimension vectors $\mathbb{N}\hat{Q}_0$. Two dimension vectors contained in the same orbit are said to be equivalent in the following. In the following, we consider the dimension vectors of \hat{Q} up to this equivalence. Thus we have in conclusion:

Theorem 3.5. *For all fixed points $X \in M_d^s(Q)^T$ there exists (up to equivalence) a unique dimension vector \hat{d} for \hat{Q} such that X corresponds to a stable representation of \hat{Q} with dimension vector \hat{d} .*

3.2. Description of fixed points. Converse to the last section we construct an embedding of stable representations of the quiver \hat{Q} into the fixed point set of the related moduli space. Therefore, fixing a representation of \hat{Q} we construct a representation of Q and show that the latter one is a stable fixed point.

Let \hat{d} be a dimension vector of \hat{Q} . Then we define d_q for every $q \in Q_0$ by

$$d_q = \sum_{\chi \in \Xi(T)} \hat{d}_{q,\chi}.$$

We call a dimension vector \hat{d} satisfying this property compatible with $d := (d_q)_{q \in Q_0}$.

Let $\hat{X} = ((\hat{X}_{q,\chi})_{q \in Q_0, \chi \in \Xi(T)}, (\hat{X}_{\alpha,\chi})_{\alpha \in Q_1, \chi \in \Xi(T)})$ be a representation of \hat{Q} . Define a representation X of Q by the vector spaces

$$X_q := \bigoplus_{\chi \in \Xi(T)} \hat{X}_{q,\chi}$$

and the linear maps

$$X_\alpha := \bigoplus_{\chi \in \Xi(T)} \left(\hat{X}_{\alpha,\chi} : \hat{X}_{q,\chi} \rightarrow \hat{X}_{q',\chi+e_\alpha} \right)$$

for all $\alpha : q \rightarrow q'$. This defines a linear map $P : R_{\hat{d}}(\hat{Q}) \rightarrow R_d(Q)$. Moreover, the decomposition of the vector spaces X_q for $q \in Q_0$ induces an embedding of $G_{\hat{d}}$ into G_d . Furthermore, define a morphism of algebraic groups $\psi = (\psi_q)_{q \in Q_0} : T \rightarrow G_d$ such that $\psi_q : T \rightarrow Gl(X_q)$ is defined by $\psi_q(t)x = \chi(t)x$ for all $t \in T$ and all $x \in \hat{X}_{q,\chi}$. Note that this makes ψ well-defined. Moreover, it follows by an easy calculation that

$$\psi_{q'}(t)X_\alpha\psi_q(t)^{-1}x = t_\alpha X_\alpha x$$

for all $x \in X_q$ which also shows that X is a torus fixed point.

We now prove that (semi-)stable representations are mapped to (semi-)stable representations. Therefore, we prove that it is enough to consider subspaces compatible with the weight space decomposition in order to check (semi-)stability. We say that a subrepresentation U of X is compatible with the weight space decomposition if we have $U_q = \bigoplus_{\chi \in \Xi(T)} U_{q,\chi}$ for all $q \in Q_0$ where $U_{q,\chi} \subset X_{q,\chi}$.

Lemma 3.6. *Let \hat{X} be a representation of \hat{Q} and let $X = P(\hat{X})$ such that*

$$X_q = \bigoplus_{\chi \in \Xi(T)} \hat{X}_{q,\chi}$$

is the weight space decomposition with respect to ψ as constructed above. If \hat{X} is semistable (resp. stable), i.e., for all proper subrepresentations U , which are compatible with the weight space decomposition, we have $\mu(U) \leq \mu(X) = \mu(\hat{X})$

(resp. $\mu(U) < \mu(X)$), then X is a semistable (resp. stable) representation of Q . In particular, if \hat{X} is stable, then X defines a stable torus fixed point.

Proof. Let \hat{X} be a semi-(stable) representation of \hat{Q} . We first show that X is semistable in both cases. Let $U = \text{scss}(X)$ and consider

$$\psi(t)U := (\psi_q((t_\alpha)_{\alpha \in Q_1})(U_q))_{q \in Q_0}$$

for each $(t_\alpha)_{\alpha \in Q_1} \in T$. Hence for each arrow $\alpha : q \rightarrow q'$ we obtain

$$X_\alpha \psi_q(t)U_q = \frac{1}{t_\alpha} \psi_{q'}(t)X_\alpha \psi_q(t)^{-1} \psi_q(t)U_q \subset \psi_{q'}(t)U_{q'}$$

because $X_\alpha U_q \subset U_{q'}$. Thus $\psi(t)U$ is a subrepresentation of X . Since $\psi_q(t)$ is invertible for every $q \in Q_0$, the dimension vectors of U and $\psi(t)U$ coincide. Because of the uniqueness of $\text{scss}(X)$ it follows that $\psi(t)U = U$ for all $t \in T$. This is equivalent to $\psi_q(t)U_q = U_q$ for all $t \in T$ and all $q \in Q_0$. This implies that $U = \text{scss}(X)$ is compatible with the weight space decomposition. Since the slope of $\text{scss}(X)$ is maximal among the set of subrepresentations of X and by assumption we get $U = X$. Hence X is semistable.

Thus it remains to show that X is stable if \hat{X} is stable. Assume that X is not stable. By the preceding considerations we know that X is semistable. Thus we may assume that there exists a subrepresentation U of X such that $\mu(U) = \mu(X)$. Consider again $\psi : T \rightarrow G_d$ from above. As above we obtain that $\{\psi(t)U \mid t \in T\}$ is a set of subrepresentations of X . Let $e := \underline{\dim}U$ and consider the quiver Grassmannian $\hat{\text{Gr}}_e(X)$ of subrepresentations of dimension e of X which is a projective variety because it is a closed subvariety of the product of the usual Grassmannians $\text{Gr}_{e_q}(X_q)$, $q \in Q_0$. It is also not empty because $U \in \hat{\text{Gr}}_e(X)$. From the considerations above we obtain a torus action on $\hat{\text{Gr}}_e(X)$ given by $(t, U) \mapsto \psi(t) \cdot U$. But since $\hat{\text{Gr}}_e(X)$ is projective it follows by Borel's Fixed Point Theorem that the fixed point set is not empty. Thus there exists a subrepresentation $U' \subset X$ with $\mu(U') = \mu(X)$ such that $\psi(t) \cdot U' = U'$. But this again means that U' is compatible with the weight space decomposition which is contradiction. It follows from the discussion from above that X is a stable torus fixed point in this case. \square

Since the linear map P is equivariant under the group action of $G_{\hat{d}}$ and sends semistable points to semistable points by the preceding lemma, it induces a map

$$P : M_{\hat{d}}^{ss}(\hat{Q}) \rightarrow M_d^{ss}(Q)$$

by use of the universal property of the quotient.

The action of $\mathbb{Z}Q_1$ on the dimension vectors $\mathbb{N}\hat{Q}_0$ introduced in the last section induces an action on the representations of \hat{Q} by translating the representation. We say that two representations X and X' of \hat{Q} are equivalent if there exists a $\mu \in \mathbb{Z}Q_1$ such that $\mu \cdot X \cong X'$. We also need the following lemma:

Lemma 3.7. *Let X and X' be stable representations of \hat{Q} such that $P(X)$ and $P(X')$ are isomorphic. Then X and X' are already equivalent.*

Proof. Let $Y = P(X)$ and $Y' = P(X')$, define $d := \underline{\dim}(Y)$ and let $g = (g_q)_{q \in Q_0} \in G_d$ be an isomorphism between Y and Y' . We have $Y'_\alpha g_q = g_{q'} Y_\alpha$ for all $\alpha : q \rightarrow q' \in Q_1$. We choose the morphism of algebraic groups $\psi = (\psi_q)_{q \in Q_0}$ corresponding to Y as above. Since Y' is a fixed point isomorphic to Y , by the considerations of

the last subsection we can choose the lift ψ' corresponding to Y' such that we have $\psi'_q = g_q \psi_q g_q^{-1}$ for all $q \in Q_0$. But as before for these lifts we have

$$y \in Y_{q,\chi}, \text{ i.e., } \psi(t)_q y = \chi(t)y \Leftrightarrow \psi'(t)_q(g_q y) = \chi(t)(g_q y).$$

Thus each g_q induces an isomorphism between the weight spaces $Y_{q,\chi}$ and $Y'_{q,\chi}$. Hence we may understand g as an isomorphism between X and X' because g is compatible with the weight space decomposition. \square

Every fixed point arises from such an embedding. Moreover, the images of these embeddings are pairwise disjoint so that we obtain the following concluding theorem:

Theorem 3.8. *The set of fixed points $M_d^s(Q)^T$ is isomorphic to the disjoint union of moduli spaces*

$$\bigcup_{\hat{d}} M_{\hat{d}}^s(\hat{Q}),$$

where \hat{d} ranges over all equivalence classes of dimension vectors being compatible with d .

3.3. Euler characteristic of moduli spaces. In this section we point out some basic properties of the Euler characteristic. For basics of algebraic topology see, for instance, [17].

Let X be a variety over the complex numbers of dimension n and let $H_c^i(X)$, $i \in \mathbb{N}_0$, be the i -th cohomology group considering cohomology with compact support with coefficients in \mathbb{C} which are \mathbb{C} -vector spaces satisfying $H_c^i(X) = 0$ if $i > 2n$ as is known. Define $h_c^i(X) = \dim_{\mathbb{C}} H_c^i(X)$. The Euler characteristic χ_c of X is defined by

$$\chi_c(X) = \sum_{k=0}^{2n} (-1)^k h_c^k(X).$$

If the variety X is a smooth quasi-projective complex variety, say of dimension n , we have $H_{2n-q}(X; \mathbb{C}) \cong H_c^q(X; \mathbb{C})$ where $H_q(X; \mathbb{C})$ denotes singular homology. Thus we have $\dim \text{Hom}_{\mathbb{C}}(H_c^q(X, \mathbb{C}), \mathbb{C}) = \dim H^{2n-q}(X; \mathbb{C})$. Thus, in particular, we have $\chi(X) = \chi_c(X)$ for the Euler characteristic. For more details see [18, Chapter IX.3-5]. Note that singular cohomology and cohomology with compact support clearly coincide if X is projective and thus compact.

By the following theorem, which is a consequence of [4, Chapter 2.5] (see also [2] for a treatment of etale cohomology with compact support), it follows that the localization method is suitable to calculate the Euler characteristic of varieties.

Theorem 3.9. *Let X be a complex variety with a torus T acting on it. Let X^T be the fixed point set of X under this action. Then we have*

$$\chi_c(X) = \chi_c(X^T).$$

Since we are mostly interested in moduli spaces of stable representations which are smooth quasi-projective varieties by Theorem 2.1, we will not distinguish between χ_c and χ in this case and denote it just by χ .

By Theorem 3.8 and because of the additivity of the Euler characteristic we obtain the following important result:

Theorem 3.10. *Let Q be a quiver with dimension vector d . Then for the Euler characteristic of the moduli space $M_d^s(Q)$ we have*

$$\chi(M_d^s(Q)) = \sum_{\hat{d}} \chi(M_{\hat{d}}^s(\hat{Q})),$$

where \hat{Q} is the universal abelian covering quiver and \hat{d} ranges over all equivalence classes being compatible with d .

Let Q be a quiver and d be a Θ -indivisible dimension vector. Consider the moduli space of stable representations $M_d^s(Q)$. From the formula for the Poincaré polynomials stated in [23] we obtain that the coefficients corresponding to the monomials in odd degree vanish so that the odd cohomology vanishes. Moreover, from the Hard Lefschetz Theorem (see for instance [11, Chapter 0.7]), we can conclude that

$$h^k(M_d^s(Q)) \leq h^{k+2}(M_d^s(Q))$$

for $k < n$ and

$$h^k(M_d^s(Q)) \geq h^{k+2}(M_d^s(Q))$$

for $k > n$ where n is the dimension of $M_d^s(Q)$. Since we also have

$$h^0(M_d^s(Q)) = h^{2n}(M_d^s(Q)) = 1,$$

we get the following result:

Corollary 3.11. *For moduli spaces of stable representations of a quiver Q with Θ -indivisible dimension vector d we have:*

$$\chi(M_d^s(Q)) \geq \dim M_d^s(Q) + 1.$$

3.4. Maps between universal quivers. In this subsection we introduce the universal covering quiver of a connected quiver Q . Moreover, we construct maps from this quiver to the universal abelian covering quivers which are obtained by applying the localization technique recursively. Since these maps become injective on finite subquivers after finitely many steps, the remaining torus fixed points do not have a cyclic support. The study of universal covering quivers in Representation Theory goes back into the 80's; see for instance [3, 5, 10].

Let $\Omega_1 = \{\alpha, \alpha^{-1} \mid \alpha \in Q_1\}$ where α^{-1} is the formal inverse of α . We will write $\alpha^{-1} : q' \rightarrow q$ for $\alpha : q \rightarrow q' \in Q_1$. A path p is a sequence $(q_1 \mid \alpha_1 \alpha_2 \dots \alpha_n \mid q_{n+1})$ such that $\alpha_i : q_i \rightarrow q_{i+1} \in \Omega_1$. Thereby, we have the equivalence generated by

$$(q \mid \alpha \alpha^{-1} \mid q) \sim (q \parallel q).$$

In what follows, we always consider paths up to this equivalence. The set of words in Q is generated by the arrows and their formal inverses, i.e., for a word w we have $w = \alpha_1 \dots \alpha_n$ where $\alpha_i \in \Omega_1$. Denote the set of words of Q by $W(Q)$. Note the difference between paths and words, i.e., a word may consist of any concatenation of arrows and their formal inverse whence two paths can only be concatenated if the head of one of the paths coincides with the tail of the other one. The universal covering quiver \tilde{Q} of Q is given by the vertex set

$$\tilde{Q}_0 = \{(q, w) \mid q \in Q_0, w \in W(Q)\}$$

and the arrow set

$$\tilde{Q}_1 = \{\alpha_{(q,w)} : (q, w) \rightarrow (q', w\alpha) \mid \alpha : q \rightarrow q' \in Q_1\}.$$

Note that the definition is slightly different from the usual definition for instance given in [10]. There a base point $q \in Q_0$ is fixed. The quiver, which is obtained, is the connected component which contains $(q, 1)$ in our definition. But due to the shape of the universal abelian covering quiver, which has infinitely many components and naturally appears by localization, we modified the definition. With this definition it is possible to obtain a surjective map between the two quivers.

For an $\alpha \in \Omega_1$ define

$$o(\alpha) = \begin{cases} 1 & \text{if } \alpha \in Q_1, \\ -1 & \text{if } \alpha^{-1} \in Q_1. \end{cases}$$

The universal abelian covering quiver \hat{Q} (see Section 3.1), is given by the vertex set

$$\hat{Q}_0 = Q_0 \times \mathbb{Z}Q_1 = \{(q, z_1) \mid q \in Q_0, z_1 \in \mathbb{Z}Q_1\}$$

and the arrow set

$$\hat{Q}_1 = \{(\alpha, z_1) : (q, z_1) \rightarrow (q', z_1 + e_\alpha) \mid \alpha : q \rightarrow q' \in Q_1, z_1 \in \mathbb{Z}Q_1\}.$$

The k -th universal abelian covering quiver is recursively defined by the vertex set

$$\hat{Q}_0^k = \hat{Q}_0^{k-1} \times \mathbb{Z}\hat{Q}_1^{k-1} = \{(q, z_1, \dots, z_k) \mid q \in Q_0, z_l \in \mathbb{Z}\hat{Q}_1^{l-1}\}$$

and the arrow set

$$\begin{aligned} \hat{Q}_1^k = & \{(\alpha, z_1, \dots, z_k) : (q, z_1, \dots, z_k) \rightarrow (q', z_1 + e_\alpha, \dots, z_k + e_{(\alpha, z_1, \dots, z_{k-1})}) \\ & \mid \alpha : q \rightarrow q' \in Q_1, z_l \in \mathbb{Z}\hat{Q}_1^{l-1} \text{ for } l = 1, \dots, k\}. \end{aligned}$$

where we define $\hat{Q}^0 = Q$. Note that $\hat{Q}_1^k = \hat{Q}_1^{k-1} \times \mathbb{Z}\hat{Q}_1^{k-1}$.

Fixing a vertex $q \in Q_0$ we consider those connected components of \tilde{Q} and \hat{Q}^k such that the vertices $(q, 1)$ and $(q, 0, \dots, 0)$ are contained in these components. By abuse of notation, we again denote these subquivers by \tilde{Q} and \hat{Q}^k . Fix some vertex $(q', w) \in \tilde{Q}_0$ in this connected component. This means that $w = (q \mid \alpha_1 \cdots \alpha_n \mid q')$ is a *path* in Q_1 and we may assume that $\alpha_i \neq \alpha_{i+1}^{-1}$ for all $i = 1, \dots, n - 1$. We call such a path reduced in what follows. By $l(w) = n$ we denote the length of the path w and, moreover, we define

$$h(i) := \frac{2i - 1 - o(\alpha_i)}{2}$$

and for $0 \leq l \leq l(w)$ we define

$$c_1^l(w) := \sum_{i=1}^l o(\alpha_i)e_{\alpha_i} \in \mathbb{Z}Q_1,$$

where $c_1^0(w) = 0$. Furthermore, we recursively define

$$c_k^l(w) := \sum_{i=1}^l o(\alpha_i)e_{(\alpha_i, c_1^{h(i)}(w), \dots, c_{k-1}^{h(i)}(w))} \in \mathbb{Z}\hat{Q}_1^{k-1},$$

where again $c_k^0(w) = 0$. Roughly speaking, $c_k^l(w)$ is the k -th coordinate of the vertex that we reach after l steps in some universal abelian covering quiver, when walking along the path w . We have the following lemma:

Lemma 3.12. (1) *In every connected component of the universal abelian covering quiver \hat{Q} of Q every vertex $(q, z) \in \hat{Q}_1$ is already uniquely determined by $z \in \mathbb{Z}Q_1$.*

(2) Let $((q, t_1) | (\alpha_1, z_1)^{o(\alpha_1)} \dots (\alpha_n, z_n)^{o(\alpha_n)} | (q', t_2))$ be a path such that

$$\sum_{i=1}^n o(\alpha_i) e_{\alpha_i} = 0.$$

Then we have $(q, t_1) = (q', t_2)$.

Proof. We may assume that $z = 0$. Assume that there exist two vertices $(q, 0)$ and $(q', 0)$ in one connected component. Thus there exists a reduced path

$$((q, 0) | (\alpha_1, z_1)^{o(\alpha_1)} \dots (\alpha_n, z_n)^{o(\alpha_n)} | (q', 0)),$$

where $z_i \in \mathbb{Z}Q_1$. We have

$$\sum_{i=1}^n o(\alpha_i) e_{\alpha_i} = 0.$$

Thus for every $1 \leq i \leq n$ there exists an $i' \neq i$ such that $\alpha_i = \alpha_{i'}^{-1}$. But this means that we walk along each arrow α_i in both directions and, therefore, we have $q = q'$.

In the second case, we obtain $q = q'$ as before. Moreover, we have $t_2 = t_1 + \sum_{i=1}^n o(\alpha_i) e_{\alpha_i}$. \square

Now we can define a map $f_k : \tilde{Q} \rightarrow \hat{Q}^k$ (between connected components) by

$$f_k((q, w)) = (q, c_1^{l(w)}(w), \dots, c_k^{l(w)}(w))$$

and for every arrow $\alpha_{(q,w)} : (q, w) \rightarrow (q', w\alpha)$ we define

$$f_k(\alpha) = (\alpha, (c_i^{l(w)}(w))_{i=1, \dots, k}) : (q, (c_i^{l(w)}(w))_{i=1, \dots, k}) \rightarrow (q', (c_i^{l(w\alpha)}(w\alpha))_{i=1, \dots, k}).$$

Roughly speaking a path w in \tilde{Q} starting in $(q, 1)$ is mapped to the same path in \hat{Q}^k . We get the image of such a path just by walking along “the same arrows” in \hat{Q}^k . But since the second quiver has cycles, different vertices and arrows can be mapped to the same vertices and arrows. But different paths are sent to different paths even if they have the same starting and terminating point. Recall that every arrow in both quivers corresponds to an arrow of the original quiver Q .

Note that we have

$$c_k^{l(w\alpha)}(w\alpha) = c_k^{l(w)}(w) + e_{(\alpha, c_1^{l(w)}(w), \dots, c_{k-1}^{l(w)}(w))}$$

for an arrow $\alpha \in Q_1$ and

$$c_k^{l(w\alpha)}(w\alpha) = c_k^{l(w)}(w) - e_{(\alpha, c_1^{l(w)+1}(w\alpha), \dots, c_{k-1}^{l(w)+1}(w\alpha))}$$

for an arrow $\alpha^{-1} \in Q_1$.

Proposition 3.13. (1) *The maps f_k are surjective for all $k \in \mathbb{N}$.*

(2) *For every finite connected subquiver $\mathcal{Q} \subseteq \tilde{Q}$ there exists an $k \in \mathbb{N}$ such that $f_{k'}|_{\mathcal{Q}}$ is injective for all $k' \geq k$.*

Proof. We first show that f_k is surjective. As already mentioned we consider the connected components such that $(q, 1) \in \tilde{Q}_0$ and $(q, 0, \dots, 0) \in \hat{Q}^k$. Thus let $(q', z_1, \dots, z_k) \in \hat{Q}_0^k$ such that there exists a reduced path

$$((q, 0, \dots, 0) | \alpha_1 \dots \alpha_n | (q', z_1, \dots, z_k))$$

in \hat{Q}^k which corresponds to a reduced path $w = (q | \alpha_1 \dots \alpha_n | q')$ in Q .

We have $z_t = c_t^{l(w)}(w)$ and thus $f_k(q', w) = (q', z_1, \dots, z_k)$.

Let $q \in Q_0$ and $w = \alpha_1 \dots \alpha_{n_k} \neq 1$ be a reduced path such that w starts and terminates at q . Moreover, assume that w and q are chosen such that n_k is minimal satisfying

$$f_k((q, 1)) = f_k((q, w)) = (q, 0, \dots, 0).$$

Then we claim that $f_{k+1}((q, w)) \neq f_{k+1}(q, 1) = (q, 0, \dots, 0)$. Assume that this is not the case. Then we have

$$f_{k+1}((q, w)) = (q, c_1^{l(w)}(w), \dots, c_{k+1}^{l(w)}(w)) = (q, 0, \dots, 0)$$

and, therefore,

$$c_t^{l(w)}(w) = \sum_{i=1}^{l(w)} o(\alpha_i) e_{(\alpha_i, c_1^{h(i)}(w), \dots, c_{t-1}^{h(i)}(w))} = 0$$

for all $t = 1, \dots, k + 1$. Thus there exist $i, i' \in \{1, \dots, n_k\}$ with $i < i'$ such that $\alpha_i = \alpha_{i'}$ and $c_t^{h(i)}(w) = c_t^{h(i')}(w)$ for all $t = 1, \dots, k$. Since the path is reduced, we also have $i' \neq i + 1$ and, therefore, $h(i') > h(i)$. But since

$$c_t^{h(i')}(w) - c_t^{h(i)}(w) = \sum_{j=h(i)+1}^{h(i')} o(\alpha_j) e_{(\alpha_j, c_1^{h(j)}(w), \dots, c_{t-1}^{h(j)}(w))} = 0,$$

applying Lemma 3.12, for the two vertices (q', w_1) and $(q', w_1 w_2)$ we have $f_k((q', w_1)) = f_k((q', w_1 w_2))$ with $l(w_2) < n_k$. But this contradicts the minimality of n_k . This already proves the statement. □

Remark 3.14.

- (1) We should note that the support of an indecomposable representation is clearly connected so that this proposition suffices for our purposes. But it is straightforward to check that the maps f_k extend to surjective maps on the whole quivers. In order to do so we can define $h(i)$ and c_k^l for words $w \in W(Q)$ similar to the definition for paths. This gives us a map f_k defined on the whole quiver.

Let $T_k := (\mathbb{C}^*)^{|\hat{Q}_1^{k-1}|}$. Define

$$M_d^s(Q)^{T, n} := (\dots (M_d^s(Q)^{T_1}) \dots)^{T_n}.$$

Note that for \tilde{Q} we also consider the stability condition induced by the one on Q . We obtain an equivalence relation of dimension vectors on \tilde{Q} analogously to the one on Q . Using Theorem 3.8 we get the following:

Theorem 3.15. *For all dimension vectors d there exists an $n \in \mathbb{N}_0$ such that we have*

$$M_d^s(Q)^{T, n'} \cong \bigcup_{\tilde{d}} M_{\tilde{d}}^s(\tilde{Q})$$

for all $n' \geq n$ where \tilde{d} ranges over all equivalence classes that are compatible with d .

Concerning the Euler characteristic of quiver moduli spaces we get the following corollary:

Corollary 3.16. *Let Q be a quiver with dimension vector d . Then for the Euler characteristic of the moduli space $M_d^s(Q)$ we have*

$$\chi(M_d^s(Q)) = \sum_{\tilde{d}} \chi(M_{\tilde{d}}^s(\tilde{Q})),$$

where \tilde{d} ranges over all equivalence classes being compatible with d .

Thus, if we are interested in the Euler characteristic, we may always assume that torus fixed points are given as representations of the universal covering quiver which has no cycles. For this quiver its representation theory often simplifies in comparison to the one of the universal abelian covering quiver.

Example 3.17.

- (1) Every connected component of the universal covering quiver of the Kronecker quiver $K(m)$ is an infinite bipartite regular m -tree. Note that in comparison to Example 3.4 all cycles split up.

We end this section with the following definition:

Definition 3.18. Let Q be a quiver and $\Theta \in \text{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$ a linear form. A tuple consisting of a finite subquiver \mathcal{Q} of \hat{Q} (resp. \tilde{Q}) and a dimension vector $d \in \mathbb{N}Q_0$ such that $M_d^s(\mathcal{Q}) \neq \emptyset$, where we consider the stability induced by Θ , is called localization data.

There exists an equivalence relation on the set of localization data obtained by translating the vertices by $\chi \in \mathbb{Z}Q_1$. In the following, we will always consider localization data up to this equivalence.

A localization data always comes along with an embedding into some covering quiver. This induces a colouring of the arrows $c : \mathcal{Q}_1 \rightarrow Q_1$. If we forget about this colouring we call such a data uncoloured localization data. Fixing an uncoloured localization data there can exist many colourings that induce different localization data.

4. LOCALIZATION IN KRONECKER MODULI SPACES

Since the main focus of the paper is on the generalized Kronecker quiver, in this section we specialize the introduced machinery to this case. We first recall some properties of Kronecker moduli spaces. Then we investigate stable bipartite quivers whose representations are torus fixed points of these moduli spaces when colouring the arrows appropriately. In particular, we construct a class of localization data which grows exponentially with the dimension vector.

4.1. Kronecker moduli spaces. Let $K(m)$ be the generalized Kronecker quiver having two vertices $Q_0 = \{q_1, q_2\}$ and m arrows $\alpha_i : q_1 \rightarrow q_2, i = 1, \dots, m$. A representation of this quiver with dimension vector (d, e) is given by two \mathbb{C} -vector spaces V and W of dimensions d and e and an m -tuple of linear maps

$$(X_1, \dots, X_m) \in \bigoplus_{i=1}^m \text{Hom}(V, W) = R_{d,e}(K(m)).$$

The group $(GL(V) \times GL(W))$ acts on $R_{d,e}(K(m))$ via simultaneous base change. Since the scalar matrices act trivially, the group action factors through the quotient

$(Gl(V) \times Gl(W))/\mathbb{C}^*$. For $\Theta = (1, 0)$ the slope function $\mu : \mathbb{N}^2 \setminus \{0\} \rightarrow \mathbb{Q}$ is given by

$$\mu(d, e) := \frac{d}{d + e}.$$

Thus we obtain the following criterion for the (semi-)stability of Kronecker representations:

Lemma 4.1. *A point $(X_1, \dots, X_m) \in R_{d,e}(K(m))$ is semistable (resp. stable) if and only if for all proper subspaces $0 \neq U \subsetneq V$ the following holds:*

$$\dim \sum_{k=1}^m X_k(U) \geq \frac{e}{d} \cdot \dim U \text{ (resp. } \dim \sum_{k=1}^m X_k(U) > \frac{e}{d} \cdot \dim U).$$

Thus a dimension vector is Θ -indivisible if and only if d and e are coprime. Note that the chosen stability is equivalent to the stability given by the linear form $\tilde{\Theta}_{(d,e)}((d', e')) := \langle (d', e'), (d, e) \rangle - \langle (d, e), (d', e') \rangle$ for $(d, e), (d', e') \in \mathbb{N}K(m)_0$ in King's sense. Actually the consideration of this stability goes back to Schofield's theorem [25, Theorem 6.1] because it follows that the moduli spaces for Schur roots are non-empty. In the following, we call the geometric quotient $M_{d,e}^s(m) := M_{d,e}^s(K(m))$ Kronecker moduli space. If (d, e) is a coprime root, by use of Theorem 2.1 we obtain that $M_{d,e}^s(m)$ is a non-empty smooth projective variety parametrizing the isomorphism classes of stable representations. In this case, $M_{d,e}^s(m)$ is thus a smooth complex manifold when considering the standard topology.

Remark 4.2.

- (1) Note that for $m = 1$ there exist only indecomposable (resp. stable) representations of dimensions $(1, 0)$, $(0, 1)$ and $(1, 1)$ and for $m = 2$ the only cases of interest are the dimension vectors (d, d) , $(d, d + 1)$ and $(d + 1, d)$ for $d \in \mathbb{N}$. The roots (d, d) for $d \geq 2$ are no Schur roots which means that the moduli spaces $M_{d,d}^s(2)$ are empty.

Furthermore, $(d, d + 1)$ is a real Schur root which means that the moduli space is a point. Thus we will assume that $m \geq 3$ if we do not explicitly say anything else.

We state some helpful properties of Kronecker moduli spaces:

- Proposition 4.3.** (1) *There exist isomorphisms of moduli spaces $M_{d,e}^s(m) \simeq M_{e,d}^s(m)$ and $M_{d,e}^s(m) \simeq M_{me-d,e}^s(m)$.*
- (2) *If $M_{d,e}^s(m) \neq \emptyset$, we have $\dim M_{d,e}^s(m) = 1 - d^2 - e^2 + dem$.*
 - (3) *We have $M_{d,e}^s(m) \neq \{pt\}$ if and only if*

$$\frac{m - \sqrt{m^2 - 4}}{2} < \frac{e}{d} < \frac{m + \sqrt{m^2 - 4}}{2}$$

holds.

Proof. We obtain the first isomorphism by considering the map

$$(X_1, \dots, X_m) \rightarrow (X_1^T, \dots, X_m^T)$$

using that subrepresentations become factor representations under this map. The second one is obtained via the reflection functor mapping representations of dimension (d, e) to representations of dimension $(e, me - d)$; see Theorem 2.4. Since we,

moreover, have that

$$\frac{e}{d} > \frac{e'}{d'} \Leftrightarrow \frac{e}{me-d} > \frac{e'}{me'-d'}$$

the statement follows.

The second part is a special case of the fourth part of Remark 2.2.

If $M_{d,e}^s(m) \neq \{pt\}$ holds, then

$$\frac{m - \sqrt{m^2 - 4}}{2} \leq \frac{e}{d} \leq \frac{m + \sqrt{m^2 - 4}}{2}$$

follows from the second part of the proposition. But $K(2)$ with dimension vector (d, d) is the only case such that equality holds.

If the inequalities are satisfied properly, (d, e) is an imaginary Schur root; see [15]. In particular, we have $\langle (d, e), (d, e) \rangle = d^2 + e^2 - dem \leq 0$. Following [25] there exists an open subset of $R_{d,e}(K(m))$ which contains those representations which are stable in the sense of King with $\Theta_{(d,e)}((d', e')) = \langle (d', e'), (d, e) \rangle - \langle (d, e), (d', e') \rangle$. \square

4.2. Localization data of the Kronecker quiver. In this subsection we investigate the support of the dimension vectors which arise from the localization method in detail. Moreover, we investigate stable bipartite quivers and the possibilities of colouring their arrows so that stable representations of such quivers become torus fixed points.

Let $(d, e) \in \mathbb{N}^2$ be a dimension vector of the Kronecker quiver and let

$$X = ((V, W), (X_1, \dots, X_m)) \in M_{d,e}^s(m)^T$$

be a fixed point. From the considerations of the third section we get a morphism of algebraic groups $\varphi : T \rightarrow (Gl(V) \times Gl(W))/\mathbb{C}^*$, for which we can choose a lift $\psi : T \rightarrow Gl(V) \times Gl(W)$. It can be decomposed into two morphisms $\psi_1 : T \rightarrow Gl(V)$ and $\psi_2 : T \rightarrow Gl(W)$. Let

$$V = \bigoplus_{\chi \in \Xi(T)} V_\chi \text{ and } W = \bigoplus_{\chi \in \Xi(T)} W_\chi$$

be the weight space decompositions with respect to ψ_1 and ψ_2 respectively. They satisfy

$$X_k(V_\chi) \subseteq W_{\chi + e_k}$$

for all $\chi \in \Xi(T) \cong \mathbb{Z}^m$ and $k = 1, \dots, m$.

The universal abelian covering quiver $\hat{K}(m)$ has vertices $(1, \chi)$ and $(2, \chi)$, where χ runs through all characters of $\Xi(T)$, and arrows

$$(1, \chi) \rightarrow (2, \chi + e_k)$$

for every $k \in \{1, \dots, m\}$ and every $\chi \in \mathbb{Z}^m$.

For every fixed point we obtain a dimension vector \hat{d} defined by

$$\hat{d}_{1,\chi} = \dim V_\chi \text{ and } \hat{d}_{2,\chi} = \dim W_\chi$$

for $(1, \chi), (2, \chi) \in \hat{K}(m)_0$.

The other way around consider $\hat{K}(m)$ and a dimension vector $\hat{d} \in \mathbb{N}\hat{K}(m)_0$. A stable representation of this quiver corresponds to a torus fixed point with dimension vector (d, e) where

$$d = \sum_{\chi \in \Xi(T)} \hat{d}_{1,\chi} \text{ and } e = \sum_{\chi \in \Xi(T)} \hat{d}_{2,\chi}$$

In what follows, we call (d, e) dimension type of the representation.

Definition 4.4. Let Q be a quiver with a fixed linear form $\Theta \in \text{Hom}(\mathbb{Z}Q_0, \mathbb{Z})$. A tuple consisting of the quiver Q and a dimension vector $d \in \mathbb{N}Q_0$ is called stable if $M_d^s(Q) \neq \emptyset$ where we consider the stability induced by Θ .

If it is clear which dimension vector we consider, we will simply call such a tuple stable quiver.

Remark 4.5.

- (1) The stability condition for representations of $\hat{K}(m)$ is induced by the original linear form $\Theta = (1, 0)$. It is given by

$$\mu(\hat{d}) = \frac{\sum_{\chi \in \Xi(T)} \hat{d}_{1,\chi}}{\sum_{\chi \in \Xi(T)} \hat{d}_{1,\chi} + \hat{d}_{2,\chi}}.$$

If we consider a general bipartite quiver with vertices $I \cup J$, in what follows, we always fix this stability condition, i.e., the one induced by the linear form $\Theta_i = 1$ for all sources $i \in I$ and $\Theta_j = 0$ for all sinks $j \in J$.

Let (\mathcal{Q}, d) with $\mathcal{Q} = (I \cup J, R)$ be a localization data. It comes along with an embedding of \mathcal{Q} into a connected component $\hat{K}(m)_c$ of $\hat{K}(m)$. Moreover, it defines a colouring $c : R \rightarrow \{1, \dots, m\}$. We can assume that \mathcal{Q} defines a full subquiver of $\hat{K}(m)_c$. Then we have that R and the colouring c are already uniquely determined by the vertex set $I \cup J$. Nevertheless, R and c play an important role because they will describe different localization data for a fixed uncoloured localization data or a fixed stable bipartite quiver.

This also means that a fixed point X determines a tuple (I, J, \hat{d}) which is unique up to translation by a vector $\mu \in \mathbb{Z}Q_1 \cong \mathbb{Z}^m$. In what follows we always consider such tuples up to translation.

For a bipartite quiver Q with vertex set $I \cup J$ and fixed dimension vector d define $A_i := \{j \in J \mid \exists \alpha : i \rightarrow j \in Q_1, d_j \geq 1\}$ and $A_j := \{i \in I \mid \exists \alpha : i \rightarrow j \in Q_1, d_i \geq 1\}$. Furthermore, define $A_I = \bigcup_{i \in I} A_i$ and $R_i = |A_i|$ and $R_j = |A_j|$.

Definition 4.6. A bipartite quiver is called m -bipartite if we have for all sources $i \in I$ and all sinks $j \in J$ that $R_i, R_j \leq m$. A tree quiver is a quiver whose underlying graph is a tree such that there exists at most one arrow between every two vertices.

Remark 4.7.

- (1) Consider a stable bipartite tree quiver (\mathcal{Q}, \hat{d}) of dimension type (d, e) with $\mathcal{Q} = (I \cup J, R)$. Choose a colouring of the arrows $c : R \rightarrow \{1, \dots, m\}$. Then we get a localization data of $K(m)$ if \mathcal{Q} and c satisfy the following conditions:
 - (a) The quiver \mathcal{Q} is m -bipartite.
 - (b) For all $(i, j), (i, j') \in R$ such that $j \neq j'$ we have $c(i, j') \neq c(i, j)$.
 - (c) Analogously, for $(i, j), (i', j) \in R$ such that $i \neq i'$ we have $c(i, j) \neq c(i', j)$.

We call a colouring satisfying these conditions stable. Fixing a vertex $q \in \mathcal{Q}_0$ and setting $c(q) = (q, 1)$ every colouring of the arrows induces

a colouring of the vertices $c : \mathcal{Q}_0 \rightarrow \tilde{K}(m)_0$ and, thus, we get an embedding of \mathcal{Q} into $\tilde{K}(m)$.

As far as $\hat{K}(m)$ is concerned we can also fix $q \in \mathcal{Q}_0$ and set $c(q) = (q, 0) \in \hat{K}(m)$. This also induces a colouring of all vertices, but it does not give an embedding of \mathcal{Q} into $\hat{K}(m)$ in general. But it induces a map by identifying vertices of the same colour. Thus it also identifies \mathcal{Q} with a subquiver of $\hat{K}(m)$.

In both cases, every stable representations of \mathcal{Q} defines a stable representation of $\tilde{K}(m)$ and $\hat{K}(m)$ respectively.

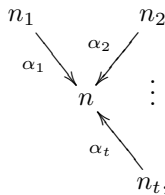
- (2) It is also easy to check that if $i \in I$ is a source such that $\hat{d}_i = 1$, then we have $m \geq R_i > \frac{e}{d}$.

Remark 4.8.

- (1) In order to test an m -bipartite quiver with a fixed dimension vector for stability, we do not need to consider an explicit representation. We can rather restrict to representations X of this dimension satisfying for all $j \in J$ and all subsets $A'_j \subseteq A_j$ with $R'_j := |A'_j|$ the following property:

$$\dim \bigcap_{i \in A'_j} X_\alpha(X_i) = \max\{0, \sum_{i \in A'_j} \dim X_\alpha(X_i) - (R'_j - 1) \dim X_j\}.$$

Roughly speaking, subrepresentations of such representations have the smallest possible slope. The existence of such representations (for general bipartite quivers) is already obtained when considering bipartite quivers of the form



where (n, n_1, \dots, n_t) denotes the dimension vector. If $n_i \leq n$ for all $1 \leq i \leq t$, there always exists a representation X of this quiver such that for all tuples of linear maps $X_{\alpha_{i_1}}, \dots, X_{\alpha_{i_k}}$ with $1 \leq k \leq t$ and $1 \leq i_1 < i_2 < \dots < i_k \leq t$ the intersections of the images is of smallest possible dimension. One verifies the existence and the above dimension formula by induction on the number of arrows.

Remark 4.9.

- (1) Fixing a stable m -bipartite tree quiver it may happen that different colourings of the arrows lead to different types of localization data. For instance, if we consider a colouring c such that this colouring induces a vertex with the colour $\chi \in \mathbb{Z}^m$ and one with $\chi - e_k$, we have an arrow $\alpha : \chi - e_k \rightarrow \chi$ and also a linear map

$$X_{\alpha: \chi - e_k \rightarrow \chi} : V_{\chi - e_k} \rightarrow V_\chi.$$

We call such an arrow induced. Obviously, the dimension of the corresponding moduli space of the universal abelian cover increases at least by

one in comparison to the dimension of the moduli space corresponding to the bipartite quiver; see Proposition 4.3.

Moreover, it can happen that two different vertices are identified when choosing a colouring. Thus a colouring of the arrows and vertices respectively can induce two types of cycles; see also Section 6.4 for a detailed example. Recall that we can apply Theorem 3.15 to quivers with unoriented cycles, and after suitably many localization steps the remaining torus fixed points are representations of the universal covering quiver which has no cycles.

Lemma 4.10. *Let (\mathcal{Q}, d) be a stable m -bipartite tree quiver and c, c' stable colourings of the arrows. Then we have:*

- (1) *By colouring the arrows with c we obtain a localization data.*
- (2) *Fix c and c' such that c induces no cycles and c' induces at least one cycle. Moreover, let $\dim(M_{\mathcal{Q}}, c)$ and $\dim(M_{\mathcal{Q}}, c')$ be the dimensions of the resulting moduli spaces. We have*

$$\dim(M_{\mathcal{Q}}, c) \leq \dim(M_{\mathcal{Q}}, c').$$

Proof. Fixing a stable m -bipartite quiver and a stable colouring of the arrows we obtain a localization data. Every stable representation of \mathcal{Q} induces a stable representation of $\hat{K}(m)$, no matter if the colouring leads to cycles or not. Induced arrows let the dimension of the moduli space increase. Thus it remains to prove that the dimension of the moduli space increases if a colouring induces a cycle which does not come from an induced cycle. Let (\mathcal{Q}, c) and (\mathcal{Q}, c') be the two resulting subquivers of $\hat{K}(m)$ and $d(c)$ and $d(c')$ be the resulting dimension vectors respectively. Assume $c'(j_1) = c'(j_2)$ and let $j_{1,2}$ be the corresponding vertex of (\mathcal{Q}, c') , i.e., $d(c')_{j_{1,2}} = d_{j_1} + d_{j_2}$. We have $R_{j_1}, R_{j_2} \geq 1$ in \mathcal{Q} . Define $\dim A_j = \sum_{i \in A_j} d_i$. Then we have for the colouring c' producing this cycle

$$\begin{aligned} \dim(M_{\mathcal{Q}}, c') &= \dim(M_{\mathcal{Q}}, c) + d_{j_1}^2 + d_{j_2}^2 - (d_{j_1} + d_{j_2})^2 - \dim A_{j_1} d_{j_1} \\ &\quad - \dim A_{j_2} d_{j_2} + (\dim A_{j_1} + \dim A_{j_2})(d_{j_1} + d_{j_2}) \\ &= \dim(M_{\mathcal{Q}}, c) - 2d_{j_1} d_{j_2} + \dim A_{j_1} d_{j_2} + \dim A_{j_2} d_{j_1} \geq \dim(M_{\mathcal{Q}}, c). \end{aligned}$$

Indeed, we have $\dim A_{j_k} \geq d_{j_k}$ for $k = 1, 2$ because of the stability of \mathcal{Q} . The case $c'(i_1) = c'(i_2)$ is proved in the same way. □

Definition 4.11. A localization data (\mathcal{Q}, \hat{d}) is called localization data of type one if $\hat{d}_q \in \{0, 1\}$ for all $q \in \mathcal{Q}_0$.

4.3. Stability of bipartite quivers. In this section we study how to construct new stable bipartite quivers by glueing certain bipartite quivers of smaller dimension types. In fact these smaller quivers are often obtained by modifying (semi)-stable quivers at the vertex at which the two quivers are glued. Since each stable colouring gives rise to some localization data, this gives a huge class of localization data for every fixed dimension type. Note that this glueing method is also used to construct stable tree modules of the Kronecker quiver; see [29].

Let $Q = (I \cup J, Q_1)$ and $Q' = (I' \cup J', Q'_1)$ be two bipartite quivers and let $j \in J, j' \in J'$. Define the bipartite quiver

$$Q_{j,j'}(Q, Q') = (I \cup I' \cup J \setminus \{j\} \cup J' \setminus \{j'\} \cup \{j''\}, Q''_1)$$

such that $\alpha : i \rightarrow j_1 \in Q_1''$ if and only if $\alpha : i \rightarrow j_1 \in Q_1$ or $\alpha : i \rightarrow j_1 \in Q_1'$ with $j_1 \neq j, j'$ and $\alpha : i \rightarrow j'' \in Q_1''$ if and only if $\alpha : i \rightarrow j \in Q_1$ or $\alpha : i \rightarrow j' \in Q_1'$.

Thus the new quiver is generated by the former ones by identifying two vertices of the set of sinks of these quivers.

Definition 4.12. The quiver $Q_{j,j'}(Q, Q')$ is called the glueing quiver of Q and Q' and the vertices $j, j' = j''$ the glueing vertices.

Definition 4.13. Let (Q, d) be a tuple consisting of a bipartite quiver with sources I and $d \in \mathbb{N}Q_0$ a dimension vector. A subquiver of Q with sources I' is called boundary quiver if there exists exactly one $i_0 \in I'$ such that $|A_{i_0} \cap A_{I \setminus I'}| = 1$ and $|A_i \cap A_{I \setminus I'}| = 0$ for all $i \in I'$ with $i \neq i_0$. A boundary quiver is called proper boundary quiver if it does not contain any other boundary quiver.

Note that if $d_q \geq 1$ for all $q \in Q_0$, this means that boundary quivers are such subquivers which only have one common sink j with the remainder of the quiver and, moreover, such that this sink is a neighbour of exactly one source of that subquiver.

Fixing a representation X of Q , we abbreviate the dimension of the image of a subspace $U = \bigoplus_{i \in I} U_i$ with $U_i \subset X_i$ to d_U . Explicitly, we define

$$d_U := \dim \sum_{\substack{i \in I \\ \alpha \in \{\beta | t(\beta) = i\}}} X_\alpha(U_i).$$

Fixing a coprime dimension vector (d, e) with $d \geq 1$ we now determine a dimension vector (d_s, e_s) such that we are able to construct new stable bipartite quivers of dimension type $(d_s + (k + l)d, e_s + (k + l)e)$ by glueing quivers of the types $(d_s + kd, e_s + ke)$ and $l(d, e)$.

Fixing some coprime dimension vector (d, e) , we first show that there exists a coprime dimension vector (d_s, e_s) such that $d_s \leq d$ and $e_s \leq e$ satisfy the conditions:

- (1) $\frac{e + e_s}{d + d_s} d > e$ if $d \neq 1$.
- (2) $\frac{e_s - 1}{d_s} < \frac{e}{d}$ if $d \neq 1$ and $(e_s - 1)d = ed_s$ if $d = 1$.
- (3) $\frac{e + e_s}{d + d_s} d' < \lceil \frac{e}{d} d' \rceil \quad \forall 1 \leq d' < d$.
- (4) $\gcd(d + d_s, e + e_s) = 1$.

These are conditions which should intuitively be satisfied in order to be able to glue stable quivers of dimension types (d_s, e_s) and (d, e) to get one of dimension type $(d_s + d, e_s + e)$. Indeed, the second and third condition make sure that the former representation of dimension (d, e) and its subrepresentations do not contradict the stability condition; see also the inequality below implied by the second condition. Moreover, the first inequality makes sure that the slope of such a representation is not too small so that the representation of dimension (d_s, e_s) and its subrepresentations does not contradict the stability condition. The last condition has clearly to be satisfied in order to avoid proper semistable representation. We refer to these conditions as glueing conditions. We will see that these conditions are also sufficient. If the second condition is satisfied, it is straightforward that we also have

$$(1) \quad \frac{e_s - 1}{d_s} < \frac{e + 1}{d} \Leftrightarrow \frac{e + e_s}{d + d_s} d < e + 1.$$

Moreover, the first condition is equivalent to $\frac{e_s}{d_s} > \frac{e}{d}$.

Lemma 4.14. *Let $(d, e) \in \mathbb{N}^2$ such that $d \leq e$ and d, e are coprime. There exists a coprime dimension vector (d_s, e_s) satisfying the glueing conditions. It is uniquely determined if we also assume that $0 \leq d_s < d$ and $1 \leq e_s < e$.*

Proof. We first consider the special case $d = 1$. It is easy to see that $(0, 1)$ satisfies these properties for $(d, e) = (1, n)$ with $n \in \mathbb{N}$.

If $d \geq 2$, we already have $e \geq 3$. Moreover, from the first two conditions it already follows that $d_s, e_s \neq 0$ has to be satisfied. Choose $d_s \in \mathbb{N}$ minimal such that $d \mid 1 + ed_s$. This is possible because $\gcd(d, e) = 1$ and, therefore, there exist $\lambda, \mu \in \mathbb{Z}$ such that $\lambda d = 1 - \mu e$. Define

$$e_s = \frac{1 + d_s e}{d}.$$

Because of the choice of d_s , we have $e_s \in \mathbb{N}$.

Moreover, we get

$$-e(d + d_s) + d(e + e_s) = -ed - ed_s + de + d_s e + 1 = 1.$$

It follows that $\gcd(d + d_s, e + e_s) = 1$.

Now we get

$$\frac{e_s}{d_s} = \frac{1 + d_s e}{d d_s} > \frac{e}{d}$$

and also

$$\frac{e_s - 1}{d_s} = \frac{d_s e - d + 1}{d d_s} < \frac{e}{d}.$$

Thus it remains to prove the third property. By an easy calculation we get

$$\frac{e + e_s}{d + d_s} = \frac{e}{d} \left(\frac{ed + ed_s + 1}{ed + ed_s} \right) = \frac{e}{d} \left(1 + \frac{1}{ed + ed_s} \right).$$

Moreover, since

$$\lceil \frac{e}{d} d' \rceil - \frac{e}{d} d' \geq \frac{1}{d}$$

and

$$\frac{d'}{d(d + d_s)} < \frac{1}{d + d_s}$$

for each $d' < d$, the existence of such a vector follows.

If (d'_s, e'_s) is another dimension vector satisfying the desired properties, it is straightforward that the glueing conditions imply

$$d'_s e_s - \frac{d'_s}{d} - 1 < d_s e'_s - 1 < d'_s e_s.$$

But since $\frac{d'_s}{d} < 1$ it follows that $d'_s e_s = d_s e'_s$. But since (d_s, e_s) and (d'_s, e'_s) are both coprime we already have $(d_s, e_s) = (d'_s, e'_s)$. □

In what follows, we call a vector (d_s, e_s) satisfying these properties starting vector for (d, e) . In the remainder of the section we assume that (d, e) is coprime and (d_s, e_s) is the corresponding starting vector as constructed in Lemma 4.14. Note that in this case $(d_s + kd, e_s + kd)$ also satisfies the glueing conditions for every $k \in \mathbb{N}$.

Remark 4.15.

- (1) If we want to decompose a coprime dimension vector (d, e) into $(d, e) = (d_s, e_s) + k(d', e')$ such that (d', e') and (d_s, e_s) satisfy the glueing conditions, we can proceed as follows: Let $e' \in \mathbb{N}$ minimal such that

$$e \mid 1 + de' \text{ and } d' = \frac{1 + e'd}{e}.$$

Now we compute d_s and e_s from (d', e') as before. Checking that

$$\frac{e - e_s}{e'} = \frac{d - d_s}{d'},$$

it follows that $e' \mid e - e_s$ and $d' \mid d - d_s$ because $\gcd(d', e') = 1$ and, trivially, $e - e_s, d - d_s \in \mathbb{N}$ hold. Now define $k = \frac{d - d_s}{d'}$.

We need other properties of these natural numbers. By use of $e_s d - ed_s = 1$ we get

$$(ke + e_s)(k'd + d_s) + k - k' = (kd + d_s)(k'e + e_s),$$

where $k, k' \in \mathbb{N}$. For a fixed $k \in \mathbb{N}$ and $d_1 = k'd + d' \in \mathbb{N}$ with $0 \leq d' < d$, $k' \in \mathbb{N}$ and $0 < d_1 \leq kd + d_s$ define a map $f : \mathbb{N} \cap (0, kd + d_s] \rightarrow \mathbb{N}$ by

$$f(d_1) = \min\{n \in \mathbb{N} \mid \frac{(ke + e_s)d_1 + n}{kd + d_s} \in \mathbb{N}\}.$$

Note that f is injective because $\gcd(kd + d_s, ke + e_s) = 1$. Then we get the following lemma:

Lemma 4.16. *Let d_s, e_s, d, e fulfill the glueing conditions. Then we have*

$$(ke + e_s)(k'd + d_s) + k - k' = 0 \pmod{(kd + d_s)}$$

for all $k' \leq k$.

Let $d_1 = k'd + d'$ with $0 \leq d' < d$. Then we have $f(d_1) = k - k'$ if $d' = d_s$ and thus $f(d_1) \geq k + 1$ if $d' \neq d_s$.

Now we show how to get a stable bipartite quiver of dimension type $(d_s + (k + l)d, e_s + (k + l)e)$ by glueing a stable bipartite quiver of type $(d_s + kd, e_s + ke)$ and certain quivers of type $(ld, le + 1)$. We again point out Remark 4.8. Thus we do not always consider specific representations, but those satisfying the properties mentioned in the remark.

In the following, if we fix a bipartite quiver \mathcal{Q} , we always additionally fix a dimension vector $\hat{d} \in \mathbb{N}\mathcal{Q}_0$. In abuse of notation we do not always mention it and, moreover, if we glue two bipartite quivers the dimension vector of the glueing quiver is denoted by \hat{d} again. We just additionally specify the dimension corresponding to the glueing vertex. The remaining vertices keep the dimension.

Let $\mathcal{S}_{ld, le+1}^m(k)$ be the set of tuples consisting of a connected m -bipartite quiver \mathcal{Q} of dimension type $(ld, le + 1)$ and a sink j with $\hat{d}_j \geq 1$ satisfying the following properties:

- (1) There exists a representation T (with the corresponding dimension vector) of the quiver such that for every d' -dimensional subspace U with $d' < ld$ which is concentrated on sources we have

$$d_U > \frac{(k + l)e + e_s}{(k + l)d + d_s} d'.$$

- (2) After decreasing the dimension of the sink j by one, the resulting quiver is connected and there exists a factor representation T/E_j of T which is semistable, i.e., the tuple consisting of the induced quiver and dimension vector is semistable.

Let $\mathcal{T}_{d,e}^m$ be the set of all stable m -bipartite quivers of dimension type (d, e) .

Theorem 4.17. *Let d and e be coprime, d, d_s, e, e_s fulfill the glueing conditions and let $k, l \in \mathbb{N}$ where $l \geq 1$. Let $T^0 \in \mathcal{T}_{d_s+kd, e_s+ke}^m$ and $(T^1, j_1) \in \mathcal{S}_{ld, le+1}^m(k)$. Moreover, let j_0 be a sink of T^0 of dimension $\hat{d}_{j_0} \geq 1$ such that $R_{j_0} + R_{j_1} \leq m$. Then $Q_{j_0, j_1}(T^0, T^1)$ with glueing vertex j_2 where $\hat{d}_{j_2} := \hat{d}_{j_0} + \hat{d}_{j_1} - 1$ is an element of $\mathcal{T}_{d_s+(k+l)d, e_s+(k+l)e}^m$.*

Proof. Given a stable representation S of T^0 and a representation T of T^1 satisfying the conditions from above, we consider the following representation of $Q_{j_0, j_1}(T^0, T^1)$: the corresponding semistable factor representation of T induces a one-dimensional subspace of T_{j_1} which we identify with an arbitrary one-dimensional subspace of S_{j_0} .

For some subspace U which is concentrated on sources of one of the two subquivers we denote by d_U the dimension of its image corresponding to its original quiver and by d'_U the dimension of its image corresponding to the glueing quiver.

First let U be a d' -dimensional subspace corresponding to T such that $d' < ld$. Then by definition we have

$$\frac{(k+l)e + e_s}{(k+l)d + d_s} d' < d_U = d'_U.$$

If $d' = ld$, the same inequality follows from $d_U = le + 1$ and the glueing conditions.

Since we also have

$$(2) \quad \frac{e_s + ke}{d_s + kd} > \frac{e_s + (k+l)e}{d_s + (k+l)d}$$

(see the properties of the dimension vectors), the same follows for subspaces of S .

It remains to prove that subspaces composed of subspaces of both subquivers fulfil the stability condition. Thus let U' and U'' be two subspaces of dimensions $1 \leq d' \leq ld$ and $1 \leq d'' \leq kd + d_s$, respectively, such that we have proper inequality at least once. Here U' corresponds to T and U'' to S . We assume that the image of U' contains the one-dimensional subspace which is factored out because otherwise the considerations from above apply.

Thus it suffices to prove that

$$d'_{U' \oplus U''} \geq d_{U'} + d_{U''} - 1 \geq \frac{le}{ld} d' + d_{U''} > \frac{(k+l)e + e_s}{(k+l)d + d_s} (d' + d''),$$

where the first inequality follows by construction and the second inequality follows from the semistability of the quiver obtained from T^1 after decreasing the dimension of the vertex j_1 by one. The last inequality is equivalent to

$$d_{U''} > \frac{(k+l)e + e_s}{(k+l)d + d_s} d'' + \frac{d'}{d((k+l)d + d_s)}$$

using $e_s d - d_s e = 1$.

By the preceding lemma together with the assumption we have

$$d_{U''} \geq \frac{(ke + e_s)d'' + f(d'')}{kd + d_s}.$$

First, let $d'' < kd + d_s$. Assuming, without loss of generality, that $d' = ld$, it remains to prove that

$$ld'' + ((k + l)d + d_s)f(d'') > l(kd + d_s).$$

But this is easily verified.

Finally, let $d'' = kd + d_s$ and $d' = l'd + d_1 < ld$ with $0 \leq d_1 < d$. We have

$$\frac{(k + l)e + e_s}{(k + l)d + d_s}(kd + d_s) = ke + e_s - \frac{l}{(k + l)d + d_s}$$

again using $e_s d - e d_s = 1$. Thus it remains to prove

$$\left\lceil \frac{e}{d}(l'd + d_1) \right\rceil = l'e + \left\lceil \frac{e d_1}{d} \right\rceil > \frac{(k + l)e + e_s}{(k + l)d + d_s}(l'd + d_1) - \frac{l}{(k + l)d + d_s}$$

which follows from the third glueing condition and inequality (2) together with $l > l'$. □

If T^0 and T^1 satisfy the condition of the theorem we call T^0 the starting quiver for T^1 . Now we apply the result to specific quivers. Therefore, let $T \in \mathcal{T}_{d,e}^m$. Starting with this quiver, we construct new quivers \hat{T} of dimension type $(d, e + 1)$ in one of the following ways:

- (1) Choose an $i \in I$ such that $R_i < m$ and define the new quiver by the vertex set $\hat{T}_0 = T_0 \cup \{j\}$ and the arrow set $\hat{T}_1 = T_1 \cup \{\alpha : i \rightarrow j\}$. Finally, let $\hat{d}_j = 1$.
- (2) Choose a vertex $j \in J$ with $1 < R_j < m$ and increase the dimension of the vertex by one.
- (3) Choose a vertex $j \in J$ such that $\hat{d}_j < \sum_{i \in A_j} \hat{d}_i$ and increase the dimension of the vertex j by one.

Denote the set of the resulting quivers by $\hat{\mathcal{T}}_{d,e}^m$ and refer to j as modified vertex. Given a representation X of $T \in \mathcal{T}_{d,e}^m$ we can modify it under consideration of Remark 4.8 in order to get a representation of \hat{T} . Thus we modify the linear maps corresponding to the arrows with sink j in such a way that the condition of Remark 4.8 is satisfied for the sink j .

Corollary 4.18. *Let d, d_s, e, e_s be as before and $k \in \mathbb{N}$. Moreover, let $T^0 \in \mathcal{T}_{d_s + kd, e_s + ke}^m$ and $T^1 \in \hat{\mathcal{T}}_{d,e}^m$ with modified vertex j_1 . Furthermore, let j_0 with $\hat{d}_{j_0} \geq 1$ be a sink of T^0 such that $R_{j_0} + R_{j_1} \leq m$. Then $Q_{j_0, j_1}(T^0, T^1)$ with glueing vertex j , where $\hat{d}_j := \hat{d}_{j_0} + \hat{d}_{j_1} - 1$, is an element of $\mathcal{T}_{d_s + (k+1)d, e_s + (k+1)e}$.*

Proof. We just have to check the two conditions stated before the preceding theorem. Thus let U be a d' -dimensional subspace of a modified representation \hat{X} of T^1 . Since T^1 results from a stable quiver we have $d_U > \frac{e}{d}d'$.

Moreover, by the third glueing condition it follows that

$$\frac{(k + 1)e + e_s}{(k + 1)d + d_s}d' < \left\lceil \frac{e}{d}d' \right\rceil \leq d_U.$$

If $d' = d$, the same inequality follows from inequality (1) together with $\hat{d}_{j_1} \leq \sum_{i \in A_{j_1}} \hat{d}_i$ and

$$d_U = e + 1 > \frac{(k + 1)e + e_s}{(k + 1)d + d_s}d. \quad \square$$

Fixing a coprime dimension vector (d, e) we now deal with the question how to construct a certain set of stable m -bipartite quivers which is countable. Therefore, we assign a set of stable m -bipartite quivers to tuple of natural numbers which is uniquely determined by the dimension vector; see also Example 4.20. These numbers correspond to the number of possible glueing vertices and possible colourings of the constructed quivers.

Fix a dimension vector (d, e) and the corresponding starting vector (d_s, e_s) . Denote by $\mathcal{T}_{n_1}^{(d,e)}$ the set of stable bipartite quivers of dimension type $(d_s, e_s) + n_1(d, e)$ with $n_1 \geq 1$. As before let $\hat{\mathcal{T}}_{n_1}^{(d,e)}$ be the set which results by modifying a vertex j_1 . Now we continue recursively: Let $S \in \mathcal{T}_{n_k-1, \dots, n_1}^{(d,e)}$ and $T \in \hat{\mathcal{T}}_{n_k, \dots, n_1}^{(d,e)}$. Now let $\mathcal{T}_{1, n_k, \dots, n_1}^{(d,e)}$ be the set consisting of all quivers $Q_{j_0, j_1}(S, T)$ such that $R_{j_0} + R_{j_1} \leq m$. Moreover, let the dimension of the glueing vertex j be given by $\hat{d}_j = \hat{d}_{j_0} + \hat{d}_{j_1} - 1$. In general let $\mathcal{T}_{n_{k+1}, \dots, n_1}^{(d,e)}$ be the set of glueing quivers resulting from glueing a quiver $S \in \mathcal{T}_{n_{k+1}-1, n_k, \dots, n_1}^{(d,e)}$ and a quiver $T \in \hat{\mathcal{T}}_{n_k, \dots, n_1}^{(d,e)}$ as described.

Corollary 4.19. *The sets $\mathcal{T}_{n_k, \dots, n_1}^{(d,e)}$ only contain stable quivers.*

Proof. It suffices to prove that these quivers satisfy the conditions of Corollary 4.18.

We assume that $\mathcal{T}_{n_k, \dots, n_1}^{(d,e)}$ only contains stable quivers. We have to prove that $\mathcal{T}_{n_{k+1}, \dots, n_1}^{(d,e)}$ just consists of stable quivers for all $n_{k+1} \geq 1$. Therefore we show that the quivers in $\mathcal{T}_{n_k-1, \dots, n_1}^{(d,e)}$ are starting quivers for all quivers in $\hat{\mathcal{T}}_{n_k, \dots, n_1}^{(d,e)}$.

Let (d^k, e^k) be the dimension type corresponding to $\hat{\mathcal{T}}_{n_k, \dots, n_1}^{(d,e)}$ and (d_s^k, e_s^k) the one belonging to $\mathcal{T}_{n_k-1, \dots, n_1}^{(d,e)}$. It suffices to prove that

$$(d_s^{k+1}, e_s^{k+1}) = (d_s^k, e_s^k) + (n_k - 1)(d^k, e^k)$$

is the starting vector for

$$(d^{k+1}, e^{k+1}) = (d_s^k, e_s^k) + n_k(d^k, e^k).$$

Indeed, the quivers in $\hat{\mathcal{T}}_{n_k, \dots, n_1}^{(d,e)}$ are obtained by the modification described in Corollary 4.18. But this is equivalent to

$$e_s^{k+1} = \frac{1 + d_s^{k+1} e^{k+1}}{d^{k+1}}$$

with the additional condition $d_s^{k+1} \leq d^{k+1}$; see Lemma 4.14. The second property follows immediately, the first one is equivalent to

$$e_s^k = \frac{1 + d_s^k e^k}{d^k},$$

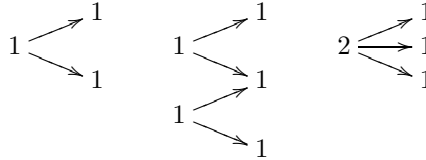
which follows by a direct calculation. Therefore, the claim follows by the induction hypothesis. \square

Example 4.20. Let $(d_s, e_s) = (0, 1)$ and $(d, e) = (1, n - 1)$. Then we always obtain a corresponding tuple of natural numbers (n_k, \dots, n_1) to a fixed coprime dimension vector by proceeding as mentioned in Remark 4.15. More detailed we have $(d^k, e^k) = (d_s^{k-1}, e_s^{k-1}) + n_k(d^{k-1}, e^{k-1})$ and in this way we recursively obtain the whole tuple. The recursion terminates if $(d_s, e_s) = (0, 1)$.

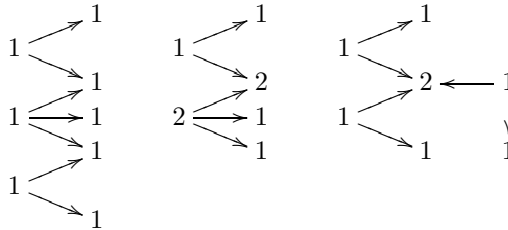
For instance, consider $(d', e') = (5, 8)$. The tuple of numbers is given by $(n_2, n_1) = (2, 2)$ with $n = 2$. In more detail we get

$$(d', e') = (1, 2) + 2(2, 3) = (0, 1) + (1, 1) + 2((0, 1) + 2(1, 1)).$$

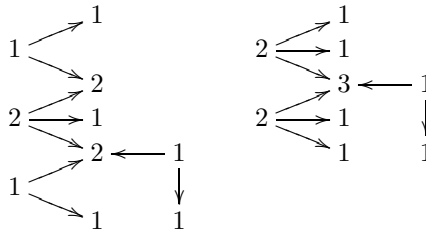
Initially, consider the localization data of the dimension types $(1, 2)$ and $(2, 3)$, i.e.,



where the numbers at the vertices indicate the dimension vector. By use of Corollary 4.18 we obtain the following localization data of dimension type $(3, 5)$ by glueing:



Next, for instance, we obtain the following localization data of type $(5, 8)$ by glueing:



5. ASYMPTOTICS AND COMBINATORICS OF TREES

The purpose of this section is to treat some aspects of combinatorics of trees. Fixing certain properties we count the number of trees satisfying these properties. This machinery will be used to count torus fixed points and fixed point components, respectively. This gives rise to a lower bound for the number of fixed points and thus for the Euler characteristic of moduli spaces of the Kronecker quiver.

Let $a(x) = \sum_{n \geq 0} a_n x^n$ be a power series. In the following denote by $[x^n]a(x) := a_n$ where $n \geq 0$ its n -th coefficient.

Definition 5.1. A tree is a connected acyclic graph. A rooted tree is a tree where a point is specified to be the root. A graph without cycles is called a forest, in particular, the components are trees.

When restricting to trees the points (resp. vertices) are often called knots. For further details according to trees and their combinatorics see, for example, [13] or [26].

5.1. Simply generated trees. We discuss simply generated trees, which we relate to localization data as constructed in the last section. Simply generated trees were introduced by Meir and Moon (see [19]), and are constructed as follows: Fix a formal power series

$$\phi(x) = \sum_{n \geq 0} \phi_n x^n$$

such that $\phi_n \geq 0$ for all $n \geq 0$, $\phi_0 > 0$ and $\phi_j > 0$ for at least one $j \geq 2$. Let \mathcal{T} be the family of finite rooted trees. Define the weight $\omega_\phi(T)$ of a tree $T \in \mathcal{T}$ by

$$\omega_\phi(T) = \prod_{j \geq 0} \phi_j^{D_j(T)},$$

where $D_j(T)$ is the number of knots with j successors. Denote by $|T|$ the number of knots of a tree T and set

$$y_n = \sum_{|T|=n} \omega_\phi(T).$$

Now the generating function $y(x) = \sum_{n \geq 1} y_n x^n$ satisfies the functional equation $y(x) = x\phi(y(x))$. Define $\mathcal{T}_\phi := \{T \in \mathcal{T} \mid \omega_\phi(T) \neq 0\}$. We call a tree $T \in \mathcal{T}_\phi$ simply-generated by ϕ .

For instance, if we define $\phi(x) = 1 + 2x + x^2$, we obtain the family of binary trees. Indeed, $y(x)$ satisfying $y(x) = x\phi(y(x))$ is its generating function, i.e., y_n is the number of binary trees with n knots. Here we take into account that we distinguish between left and right successors.

5.2. Lagrange inversion theorem asymptotic behaviour. In this section we briefly discuss the Lagrange inversion theorem, which will become an important tool later.

Theorem 5.2. *Let $\phi(x) = \sum_{n \geq 0} \phi_n x^n$ be a power series such that $\phi(0) \neq 0$ and let $y(x)$ be a power series satisfying the functional equation $y(x) = x\phi(y(x))$. Let $g(x)$ be another power series. Then $y(x)$ is invertible and for the coefficients of $g(y(x))$ we have*

$$[x^n]g(y(x)) = \frac{1}{n}[u^{n-1}]g'(u)\phi(u)^n$$

for all $n \geq 1$. Moreover, we have

$$[x^n](y(x))^m = \frac{m}{n}[u^{n-m}]\phi(u)^n.$$

Note that this theorem is equivalent to the formulation of the Lagrange inversion theorem as usually stated in literature. For proofs and further details see, for instance, [26] or [9].

By an easy calculation using Theorem 5.2 we obtain the following special case which is important when counting localization data:

Corollary 5.3. *Let $\phi(x) = 1 + ax^b$ and y such that $y(x) = x\phi(y(x))$. Then we have*

$$[x^n]y(x) = \frac{1}{n} \binom{n}{\frac{n-1}{b}} a^{\frac{n-1}{b}}$$

if $b|n - 1$ and $[x^n]y(x) = 0$ otherwise.

Corollary 5.4. *Let $\phi(x) = 1 + ax^b$ and y such that $y(x) = x\phi(y(x))$. Then we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln([x^{nb+1}]y(x)) = b \ln b - (b - 1) \ln(b - 1) + \ln a.$$

Proof. Recall the Stirling formula, i.e., for every $n \in \mathbb{N}$ we have

$$(3) \quad \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}.$$

Using

$$\lim_{n \rightarrow \infty} \ln \left(\frac{an + b}{cn + d} \right) = \ln(a) - \ln(c),$$

where $a, c \in \mathbb{R}^+$ and $b, d \in \mathbb{R}$ and

$$\lim_{n \rightarrow \infty} \frac{\ln n^t}{n} = 0$$

for all fixed $t \in \mathbb{Q}^+$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{1}{nb+1} \binom{nb+1}{n} a^n \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\left(\frac{(nb+1)^{nb+1}}{n^n (n(b-1)+1)^{n(b-1)+1}} \right) a^n \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\left(\frac{nb+1}{n} \right)^n \left(\frac{nb+1}{n(b-1)+1} \right)^{n(b-1)+1} a^n \right) \\ &= \ln b + (b-1) \ln \frac{b}{b-1} + \ln a \\ &= b \ln b - (b-1) \ln(b-1) + \ln a. \quad \square \end{aligned}$$

Corollary 5.5. *Let y and ϕ as in Corollary 5.3 and let $m \geq 1$. We have*

$$[x^n]y(x)^m = \frac{m}{n} \binom{n}{n-m} a^{\frac{n-m}{b}}$$

if $b|n - m$ and $n \geq m$ and $[x^n]y(x)^m = 0$ otherwise.

Let $a, b, m, n \in \mathbb{N}^+$. Define

$$\mathcal{A}_{a,b,m,n} := [x^n]y(x)^m$$

if $y(x)$ satisfies the functional equation $y(x) = x\phi(y(x))$ where $\phi(x) = 1 + ax^b$. Also define $\mathcal{A}_{a,b,n} := \mathcal{A}_{a,b,1,n}$

6. APPLICATIONS

In this section we discuss several consequences of the last sections and state several applications. First we discuss the asymptotic behaviour of the Euler characteristic of Kronecker moduli spaces. Then we consider some cases for which it is possible to calculate the Euler characteristic exactly.

6.1. Conjecture concerning the asymptotic behaviour of the Euler characteristic. In this subsection we discuss a conjecture, which is based on ideas of Michael Douglas, concerning the Euler characteristic of Kronecker moduli spaces. Several consequences are also discussed. Originally, in [7] Douglas suggested to fix $r \in \mathbb{R}_+$ and to consider $(d, e) \in \mathbb{N}_+^2$ with $\gcd(d, e) = 1$ and $\frac{e}{d} \approx r$ to obtain the following:

- (1) There exists a $C_r \in \mathbb{R}$ such that for $e, d \gg 0$ we have

$$\frac{\ln(\chi(M_{d,e}^s(m)))}{d} \approx C_r.$$

- (2) The function $r \mapsto C_r$ is continuous.

This means that Douglas supposed that $\frac{\ln(\chi(M_{d,e}^s(m)))}{d}$ and therefore the Euler characteristic is asymptotically already determined by the fraction $\frac{e}{d}$. Moreover, the Euler characteristic depends continuously on it. Let

$$m_1 := \frac{m - \sqrt{m^2 - 4}}{2} \text{ and } m_2 := \frac{m + \sqrt{m^2 - 4}}{2}.$$

Based on this, in [28] the following precise formulation was developed:

Conjecture 6.1. *There exists a continuous function $f : [m_1, m_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ such that the following holds: for all $r \in [m_1, m_2]$ and all $\varepsilon > 0$ there exists an $\delta > 0$ and an $n \in \mathbb{N}$ such that for all $(d, e) \in \mathbb{N}^2$ with $\gcd(d, e) = 1$, $|r - e/d| < \delta$ and $|d + e| > n$ we have*

$$\left| f(r) - \frac{\ln(\chi(M_{d,e}^s(m)))}{d} \right| < \varepsilon.$$

Remark 6.2.

- (1) We may also rephrase the conjecture as follows: there exists a continuous function f such that for every coprime dimension vector (d, e) there exists a dimension vector (d_s, e_s) such that

$$f\left(\frac{e}{d}\right) = \lim_{n \rightarrow \infty} \frac{\ln \chi(M_{d_s+nd, e_s+ne}^s(m))}{d_s + nd}.$$

In particular, the right hand side converges.

We discuss some consequences of the conjecture which are proved in [28]. For the remainder of this subsection we assume that the conjecture is true. Define

$$K := (m - 1)^2 \ln((m - 1)^2) - (m^2 - 2m) \ln(m^2 - 2m).$$

Theorem 6.3. *If Conjecture 6.1 holds, the function f is given by*

$$f(r) = \frac{K}{\sqrt{m-2}} \cdot \sqrt{r(m-r)-1}.$$

In particular, the constant K is its value at the point $r = 1$. Moreover, we have that the Euler characteristic asymptotically only depends on the dimension of the moduli space:

Corollary 6.4. *If Conjecture 6.1 holds, we have that the logarithm of the Euler characteristic $\ln(\chi(M_{d,e}^s(m)))$ is asymptotically proportional to*

$$\sqrt{dem - d^2 - e^2} = \sqrt{\dim M_{d,e}^s(m) - 1}.$$

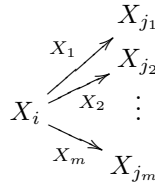
In this paper we prove that $f(1) = K$ and that the Euler characteristic grows at least exponentially. Note that, if a continuous function as conjectured exists, it follows from Theorem 6.3 that it is already uniquely determined by $f(1)$.

6.2. The case of the dimension vector $(d-1, d)$. In this section we investigate the value at 1 of the function treated in Section 6.1. This means investigating the dimension vector $(d-1, d)$. The Euler characteristic of the corresponding moduli space is, by applying the reflection functor, the same as the one corresponding to the dimension vector $(d, (m-1)d+1)$. The latter one is considered in the following. In particular, we show that the value at the point one is the one conjectured in Section 6.1.

By Theorem 3.15 it is enough to consider the universal covering quiver of the Kronecker quiver $K(m)$. As a consequence, for the remainder of this subsection we only consider localization data such that the corresponding quiver is a subquiver of the universal covering quiver. Under this assumption, we will see that each localization data (\mathcal{Q}, \tilde{d}) is of type one, i.e., $\tilde{d}_q \in \{0, 1\}$ for all $q \in \mathcal{Q}_0$. This already implies that every localization data consists of subdata of dimension type $(1, m)$.

Lemma 6.5. *Every localization data (\mathcal{Q}, \tilde{d}) of dimension type $(d, (m-1)d+1)$ is of type one. In particular, we have $\chi(M_{\tilde{d}}^s(\mathcal{Q})) = 1$.*

Proof. Let (\mathcal{Q}, \tilde{d}) be a localization data of dimension type $(d, (m-1)d+1)$ and let X be a stable representation of this data. Consider a subrepresentation



The stability condition implies

$$d_{X_i} > \frac{(m-1)d+1}{d} \dim X_i > (m-1) \dim X_i.$$

In particular, this holds if $\dim X_i = \dim X_{j_k} = 1$ for $k = 1, \dots, m$. Moreover, we have $\dim X_{j_k} \geq \dim X_i$ for all k . Indeed, if we had $\dim X_{j_k} = l$ such that $l < \dim X_i$, we could consider the (at least) $(\dim X_i - l)$ -subspace $\ker(X_k)$ which would (at most) have a $(\dim X_i - l)(m-1)$ -dimensional image. This contradicts the stability condition.

Therefore, the subrepresentation is of dimension type $(\dim X_i, e')$ with $e' \geq m \dim X_i$. Furthermore, because of the stability every k -dimensional subspace at least has an $((m-1)k+1)$ -dimensional image.

If we fix a proper boundary quiver, which exists because the original quiver has no cycles, this subquiver just has one common vertex with the remainder of the quiver and the corresponding subdata is of dimension type (d_1, md_1) . But for the dimension type $(d-d_1, b)$ of the remainder of the data we have

$$b \geq (m-1)(d-d_1) + 1.$$

Let $h \geq 1$ be the dimension of the intersection of (the vector spaces corresponding to the common vertex of) the two subrepresentations of X corresponding to the two subdata. Then we get

$$(m-1)d+1 = b + d_1 m - h \geq (m-1)(d-d_1) + 1 + d_1 m - h = (m-1)d + d_1 - h + 1.$$

Therefore, we have $h \geq d_1$ and thus $h = d_1$.

We now prove that after removing the subdata of dimension type $(d_1, (m - 1)d_1)$, i.e., the subdata of dimension type (d_1, md_1) except the common vertex, we get a localization data of dimension type $(d - d_1, (m - 1)(d - d_1) + 1)$. It suffices to prove stability because the original subdata has a d_1 -dimensional intersection with the remainder.

For an arbitrary subspace $U \subset \bigoplus_{i \in I} X_i$ with $\dim U < d - d_1$ we have

$$d_U > \frac{(m - 1)d + 1}{d} \dim U.$$

Since $\dim U < d - d_1$, we also have

$$d_U > \frac{(m - 1)(d - d_1) + 1}{(d - d_1)} \dim U$$

proving the claim in-between. Indeed, in this case we have

$$\left\lceil \frac{(m - 1)d + 1}{d} \dim U \right\rceil = \left\lceil \frac{(m - 1)(d - d_1) + 1}{(d - d_1)} \dim U \right\rceil.$$

Thus we can proceed by induction on the number of sources in order to show that all localization data are of type one.

Consider some data such that corresponding quiver has one source. Obviously, it is a stable quiver of type 1.

Assume that the quiver has $n + 1$ sources. We may remove a proper boundary quiver so that we again get a localization data, which is of the requested type by induction hypothesis. But since the original quiver has no cycles, there exist at least two proper boundary quivers. Thus the assertion follows by applying the induction hypothesis to the respective subquivers after removing a proper boundary quiver. The second statement, for instance, follows when considering the dimension formula mentioned in Remark 2.2. □

Theorem 6.6. *We have*

$$\chi(M_{d,d+1}^s(m)) = \frac{m}{(d + 1)((m - 1)d + m)} \binom{(m - 1)^2 d + (m - 1)m}{d}.$$

Moreover, we also have

$$f(1) = \lim_{d \rightarrow \infty} \frac{\ln(\chi(M_{d,d+1}^s(m)))}{d} = (m - 1)^2 \ln(m - 1)^2 - (m^2 - 2m) \ln(m^2 - 2m)$$

for f defined in Section 6.1.

Proof. As shown previously, we may assume that all subdata of a localization data with one source have vertex set

$$I \cup J = \{i, j_1, \dots, j_m\}$$

and arrow set

$$R = \{(i, j_1), \dots, (i, j_m)\}$$

with $\tilde{d}_i = \tilde{d}_{j_k} = 1$. In particular, the moduli spaces of the considered localization data are zero-dimensional yielding that the Euler characteristic is one.

Under consideration of Remark 4.7 there exists exactly one possibility to choose a stable colouring c taking into account the symmetries of the symmetric group S_m . Again by Remark 4.7 and by Corollary 4.18 we can glue k subquivers on each vertex j_l , $1 \leq l \leq m$, with $0 \leq k \leq (m - 1)$ in order to get a localization data. But

we have to take into account the symmetries of S_k . Assuming that there is only one starting knot let $y(x)$ the generating function of such quivers and consider

$$\begin{aligned} \phi(x) &= 1 + \frac{(m-1)}{|S_1|}x^{m-1} + \frac{(m-1)(m-2)}{|S_2|}x^{2(m-1)} \dots \\ &\quad + \frac{\prod_{i=1}^{m-1}(m-i)}{|S_{m-1}|}x^{(m-1)(m-1)} \\ &= \sum_{i=0}^{m-1} x^{i(m-1)} \binom{m-1}{i} = (1+x^{m-1})^{m-1}. \end{aligned}$$

By Section 5.1 we have that the generating function satisfies the functional equation $y(x) = x(\phi(y(x)))$. Now the generating function for all localization data is obtained as follows: We start with the unique localization data of dimension type $(1, m)$ having m knots. The resulting generating function is $y(x)^m$ and by applying the Lagrange inversion theorem and Corollary 5.5, respectively, we obtain that

$$[x^n]y(x)^m = \frac{m}{n}[u^{n-m}]\phi(u)^n = \frac{m}{n} \binom{n(m-1)}{\frac{n-m}{m-1}}.$$

If we assign the weight 0 to the sink of the starting quiver, every such quiver that has $(m-1)d + 1$ knots corresponds to a localization data of dimension type $(d, (m-1)d + 1)$. The other way around, we may assume that every localization data has some sink i with weight 0 what gives us d choices. This means for every localization data we exactly get d trees. Hence we get

$$\begin{aligned} \chi(M_{d,(m-1)d+1}^s(m)) &= \frac{m}{d((m-1)d+1)} \binom{(m-1)^2d+(m-1)}{d-1} \\ &= \frac{m}{d((m-1)(d-1)+m)} \binom{(m-1)^2(d-1)+(m-1)m}{d-1}. \end{aligned}$$

Since $\chi(M_{d-1,d}^s(m)) = \chi(M_{d,(m-1)d+1}^s(m))$, the assertion is proved.

The second statement is obtained when applying the same arguments as used in the proof of Corollary 5.4; then we get

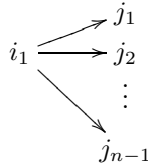
$$\begin{aligned} f(1) &= \lim_{d \rightarrow \infty} \frac{1}{d} \left(\frac{(m-1)^2d+(m-1)}{d-1} \right)^{d-1} \left(\frac{(m-1)^2d+(m-1)}{(m^2-2m)d+m} \right)^{(m^2-2m)d+m} \\ &= (m-1)^2 \ln(m-1)^2 - (m^2-2m) \ln(m^2-2m). \quad \square \end{aligned}$$

6.3. A lower bound. The aim of this section is to determine a lower bound for the Euler characteristic of Kronecker moduli spaces for coprime dimension vectors which also proves that the Euler characteristic grows at least exponentially as conjectured by Douglas. Therefore, we consider such coprime dimension vectors (d, e) of $K(m)$ satisfying $e > (m-1)d$. The remaining cases are obtained by the isomorphisms of the moduli spaces stated in Proposition 4.3. In the considered cases the moduli spaces are zero-dimensional. Moreover, we will see that the recursive construction of the localization data simplifies.

As a consequence of Theorem 3.15 we again assume that all torus fixed points are representations of the universal covering quiver.

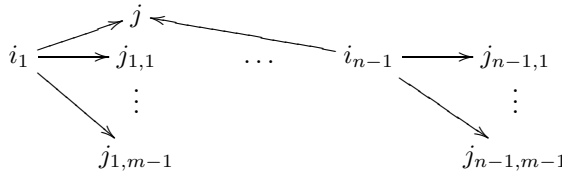
Initially, consider the dimension vectors $(1, n-1)$ and $(1, n)$ with $2 \leq n \leq m-1$ which correspond to the dimension vectors $(n-1, m(n-1)-1)$ and $(n, mn-1)$ by

the mentioned isomorphisms. For the dimension type $(1, n - 1)$ there exists only one localization data



where $\tilde{d}_{j_k} = \tilde{d}_{i_1} = 1$ for all $1 \leq k \leq n - 1$. Analogously, we obtain the unique localization data of dimension type $(1, n)$.

Consider the following localization data of dimension type $(n - 1, m(n - 1) - 1)$, where $\tilde{d}_j = n - 2$ and $\tilde{d}_{j_{k,l}} = \tilde{d}_{i_k} = 1$ otherwise:



Again we analogously obtain the data of type $(n, mn - 1)$.

Remark 6.7.

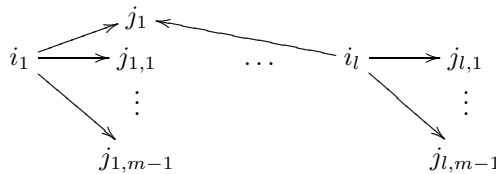
- (1) For the dimension vector $(n, mn - 1)$, $1 \leq n \leq m$, this is also the only localization data because obviously each one-dimensional subspace is forced to have an m -dimensional image. Moreover, because of the stability condition, we have for each subspace U of dimension $d' < n$ which corresponds to a stable representation of this data that

$$d_U > \frac{nm - 1}{n} d'.$$

Therefore, we have $d_U \geq md'$ for all $d' < n$. But, for any other data of this dimension type this condition is not satisfied.

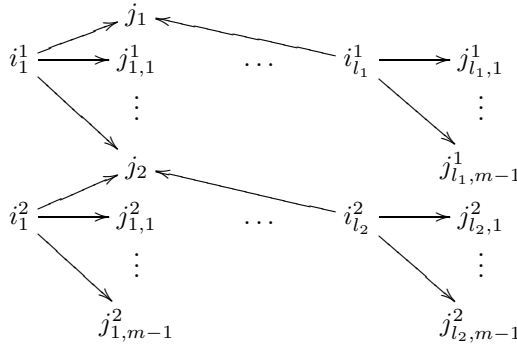
- (2) We also get this localization data by applying the reflection functor; see Theorem 2.4.

By use of the procedure introduced in Section 4.3 we can glue these quivers. Fix $m \in \mathbb{N}$ and define Q^l by



Let $I \cup J$ be the set of vertices and define $J'_1 := J \setminus \{j_1\}$. Let $\tilde{d}_{j_1} = l - 1$ and let $\tilde{d}_q = 1$ for the remaining vertices. Define the gluing quiver $Q^{l_1, l_2} := Q_{j, j_2}(Q^{l_1}, Q^{l_2})$ with $j \in J'_1$. For the resulting data define $\tilde{d}_{j_2} = l_2$ whereby the dimensions of the

other vertices remain constant. For instance, we obtain:



We again consider the construction of Corollary 4.19. Let $(d, e) = (n_1(n - 1) + 1, n_1(m(n - 1) - 1) + m) = (1, m) + n_1(n - 1, m(n - 1) - 1)$ with $n_1 \in \mathbb{N}$. Then we obtain the cases

$$(4) \quad \frac{m(n - 1) - 1}{n - 1}d \leq e \leq \frac{mn - 1}{n}d.$$

Note that for $n = 2$ we obtain $(d, e) = (d_s, e_s) + n_1(d', e')$ with $(d_s, e_s) = (1, m)$ and $(d', e') = (1, m - 1)$ and thus $d_s \geq d', e_s > e'$. But, nevertheless, the glueing conditions are satisfied because we have $(1, m) = (0, 1) + (1, m - 1)$ and $(0, 1)$ is the starting vector for $(1, m - 1)$ as constructed in Lemma 4.14.

Now the quivers are glued as explained in Section 4.3. Fixing $n_1 \geq 1$ we denote the resulting data by $\mathcal{Q}_{n_1}^n$. They obviously result if one successively glues n_1 -times some data of dimension type $(n - 1, m(n - 1) - 1)$ to some data of type $(1, m)$. Call the glueing vertex corresponding to the first glueing initial glueing vertex. If j_1 is the initial glueing vertex, denote by $\hat{\mathcal{Q}}_{n_1}^n$ the set of quivers obtained by increasing the dimension of j_1 by one.

We now recursively define

$$\mathcal{Q}_{n_{k+1}, \dots, n_1}^n = \{Q_{j, j_1}(S, T) \mid S \in \mathcal{Q}_{n_{k+1}-1, n_k, \dots, n_1}^n, T \in \hat{\mathcal{Q}}_{n_k, \dots, n_1}^n\},$$

where $j \in S_0$ such that $R_j = 1$ and where j_1 is the initial glueing vertex of $T \in \hat{\mathcal{Q}}_{n_k, \dots, n_1}^n$. Furthermore, let $\mathcal{Q}_{0, n_k, \dots, n_1}^n = \mathcal{Q}_{n_k-1, \dots, n_1}^n$. By Corollary 4.19 we know that every data $S \in \mathcal{Q}_{n_{k+1}-1, \dots, n_1}^n$ is a localization data and that each of them satisfies the properties of the starting quiver for each $T \in \hat{\mathcal{Q}}_{n_k, \dots, n_1}^n$. Thus it follows that every data which is obtained in such a way is a localization data.

Remark 6.8.

- (1) Let (d, e) be given such that (4) holds. Then we can determine the corresponding tuple (n_k, \dots, n_1) as described in Remark 4.15. Note that there is an easier method to get this tuple by simply solving linear equations; see [29].

Next we determine the cardinality of these sets in order to obtain a lower bound for the Euler characteristic. The moduli spaces of the considered localization data are zero-dimensional, i.e., a point. Furthermore, by Proposition 4.3 we can assume that $n \geq \frac{m+1}{2}$. This is another advantage simplifying combinatorics. Indeed, because of this assumption it is just possible to glue *one* quiver on each vertex of

dimension one. Otherwise, there is no suitable colouring to obtain a localization data from the produced quiver because it is no subquiver of the regular m -tree.

Initially, consider the set \mathcal{Q}_1 consisting of the localization data of dimension type $(n, mn - 1)$. After modifying a sink, considering the properties of Remark 4.7 and taking into account all symmetries and the fact that all quivers are glued as mentioned above, there exist

$$\binom{m-1}{n} \text{ possibilities}$$

to choose a colouring $c : R \mapsto \{1, \dots, m\}$ where R is the set of arrows. Note that this is the case $n_1 = 1$.

Each of the quivers has $n(m - 1)$ knots, i.e., vertices $j \in J$ such that $R_j = 1$. Denote by $a_{n_1}^n$ the cardinality of $\hat{\mathcal{Q}}_{n_1}^n$ in consideration of the different colourings. Furthermore, let $K_{n_1}^n$ be the number of knots of these quivers which coincide for all quivers in this set.

Using the notation of Section 5 we have

$$a_{n_1}^n = \binom{m-1}{n} \mathcal{A}_{(n-1), (n-1)(m-1), n(m-1), n(m-1) + (n_1-1)(n-1)(m-1)}.$$

Moreover, we have

$$K_{n_1}^n = n(m - 1) + (n_1 - 1)(n - 1)(m - 1) - (n_1 - 1).$$

Considering the construction we get the following lemma by an easy observation.

Lemma 6.9. *Let $(n_{k+1}, \dots, n_1) \in \mathbb{N}^{k+1}$.*

(1) *The number of knots of the quivers in $\hat{\mathcal{Q}}_{n_{k+1}, \dots, n_1}^n$ is given by*

$$K_{n_{k+1}, \dots, n_1}^n = K_{n_k-1, \dots, n_1}^n + n_{k+1} K_{n_k, \dots, n_1}^n - n_{k+1}.$$

(2) *Moreover, we have*

$$a_{n_{k+1}, \dots, n_1}^n = a_{n_k-1, \dots, n_1}^n \cdot \mathcal{A}_{a_{n_k, \dots, n_1}^n, K_{n_k, \dots, n_1}^n, K_{n_k-1, \dots, n_1}^n, K_{n_k-1, \dots, n_1}^n + n_{k+1} K_{n_k, \dots, n_1}^n}.$$

Fixing a dimension vector, it suffices to determine the corresponding tuple of natural numbers in order to get a lower bound for the Euler characteristic. Given a tuple as above define $K_{d,e}^m := K_{n_{k+1}, \dots, n_1}^m$ and $a_{d,e}^m := a_{n_{k+1}, \dots, n_1}^m$ and consider the function

$$\phi_{d,e,m}(x) = 1 + a_{d,e}^m x^{K_{d,e}^m}.$$

Then the generating function $y_{d,e,m}(x)$ satisfies the functional equation $y_{d,e,m}(x) = x\phi(y_{d,e,m}(x))$. Since we are interested in some asymptotic value, which is independent of the number of starting knots, we can assume that there exists just one starting knot. Even the starting quiver only gives us a constant, which we may ignore.

For every coloured tree constructed like this we obtain some localization data by assigning the weight 0 to the source of the starting quiver. Thus it may happen that different trees define the same localization data. But, if (d, e) is the considered dimension vector, the number of possible starting quivers is bounded by d .

Theorem 6.10. *Let $e > (m - 1)d$. We have*

$$\lim_{n \rightarrow \infty} \frac{\ln(\chi(M_{d_s+nd, e_s+nd}^s(m)))}{d_s + nd} \geq \frac{1}{d} (\ln a_{d,e}^m + K_{d,e}^m \ln K_{d,e}^m - (K_{d,e}^m - 1) \ln(K_{d,e}^m - 1)).$$

Proof. Using the arguments of this section, we have

$$\chi(M_{d_s+nd, e_s+nd}^s(m)) \geq \frac{[x^{nK_{d,e}^m+1}]y_{d,e,m}(x)}{d_s + nd}.$$

Thus by Corollary 5.4 we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln(\chi(M_{d_s+nd, e_s+nd}^s(m)))}{d_s + nd} &\geq \lim_{n \rightarrow \infty} \frac{1}{d_s + nd} \ln \left(\frac{[x^{nK_{d,e}^m+1}]y_{d,e,m}(x)}{d_s + nd} \right) \\ &= \frac{1}{d} (\ln a_{d,e}^m + K_{d,e}^m \ln K_{d,e}^m - (K_{d,e}^m - 1) \ln(K_{d,e}^m - 1)) \\ &=: L_{d,e}^m \end{aligned}$$

which proves the theorem. □

By use of the isomorphisms of the moduli spaces we also get a lower bound for arbitrary d and e .

Example 6.11. This example applies the introduced methods to the case $(d, e) = (5, 8)$ and $m = 3$. For the starting dimension vector we get $(d_s, e_s) = (3, 5)$, for the localization data of this type see Example 4.20.

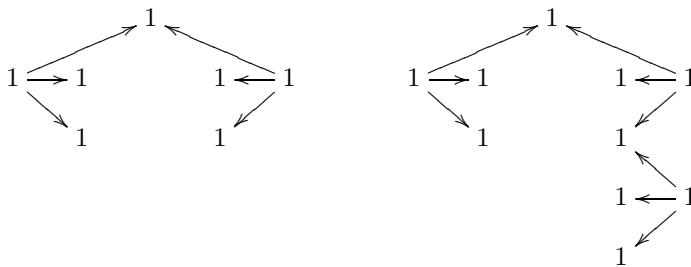
The reflected dimension vector is

$$(8, 19) = (2, 5) + 2(3, 7) = (2, 5) + 2((1, 3) + 2(1, 2))$$

and we get $n = n_1 = n_2 = 2$. We have $K_{2,5}^3 = 4$, $K_{3,7}^3 = 5$, $a_{2,5}^3 = 1$ and $a_{3,7}^3 = 8$. From this we obtain $K_{8,19}^3 = K_{5,8}^3 = 12$ and $a_{8,19}^3 = a_{5,8}^3 = 1664$. Thus in conclusion we have

$$L_{5,8}^3 = \frac{1}{5} \ln \left(1664 \cdot \frac{12^{12}}{11^{11}} \right).$$

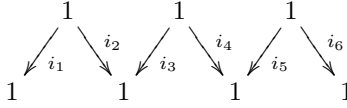
In order to get the discussed data of dimension $(8, 19)$ we glue quivers of dimension $(3, 7)$ twice to a quiver of dimension $(2, 5)$:



Note that the vertices which are glued are one of the sinks of the data on the left hand side having one neighbour and one of the sinks of the data on the right hand side having two neighbours. Note that one of those two vertices of the quiver on the right hand side is the initial glueing vertex which was defined for simplifying the calculation of the lower bound. Moreover, the glueing vertex of the resulting data has dimension two.

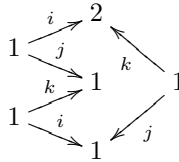
The resulting data can be glued successively to localization data of dimension $(5, 12)$ in order obtain localization data of dimension $(5, 12) + k(8, 19)$ for every $k \in \mathbb{N}$.

6.4. **The case of the dimension vector (3,4).** In this section we consider the case $d = 3$ and $e = 4$ with $m \geq 3$ in detail. Consider the stable bipartite quiver given by



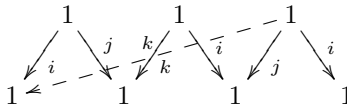
Therefore, by colouring the arrows in the colours $\{1, \dots, m\}$ satisfying the conditions of Remark 4.7 we obtain a localization data. In this case, the conditions are $c(i_l) \neq c(i_{l+1})$ for $1 \leq l \leq 5$. Each colouring is unique up to the symmetry of the symmetric group S_2 .

The colourings (i, j, k, i, j, k) and (i, j, k, i, j, i) , such that $i, j, k \in \{1, \dots, m\}$ are pairwise different, give rise to two cases, which we now consider in greater detail. In the first case we obtain



There is no new symmetry arising from this colouring. Furthermore, the moduli space is a point for this dimension vector. Note that the cycle breaks down after a second localization so that we get back the former quiver.

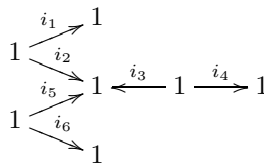
The second special case is



The colouring induces an extra arrow and therefore another symmetry. In particular, the localization data is already determined by the choice of the colour of the free arrow, i.e., the one that does not appear in the cycle. But because of the extra arrow the moduli space is \mathbb{P}^1 so that the Euler characteristic is two.

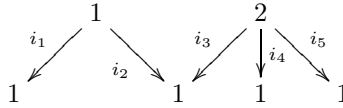
Note that $\chi(\mathbb{P}^1) = 2$ follows also from a second localization. Indeed, by considering the quiver without its colouring the fixed points are those representations satisfying $X_{i_5} = 0$ or $X_{i_7} = 0$ where i_7 is the extra arrow. Thus we again get back the original localization data by a second localization. In conclusion we obtain that there are $\frac{m(m-1)^5}{|S_2|}$ possibilities to choose a colouring.

Further localization data are given by colourings of the following stable bipartite quiver:



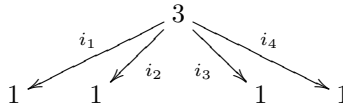
with the conditions $c(i_1) \neq c(i_2)$, $c(i_3) \neq c(i_4)$, $c(i_5) \neq c(i_6)$ and $c(i_2), c(i_3), c(i_5)$ pairwise different. In consideration of the symmetries of S_3 we obtain $\frac{m(m-1)^4(m-2)}{|S_3|}$

possibilities. We also get



with the conditions $c(i_3), c(i_4), c(i_5)$ pairwise different and $c(i_1) \neq c(i_2) \neq c(i_3)$. Thus we get $\frac{m(m-1)^3(m-2)}{|S_2|}$ possibilities.

If $m \geq 4$, we finally get the localization data coming from



with the condition that the colours of all arrows are pairwise different, hence $\binom{m}{4}$ possibilities.

Since all fixed point components may be understood as points, for the Euler characteristic we have

$$\chi(M_{3,4}^s(m)) = \binom{m}{4} + \frac{m(m-1)^3(m-2)}{2} + \frac{m(m-1)^4(m-2)}{6} + \frac{m(m-1)^5}{2}.$$

One easily verifies that this is the same result one obtains by the algorithm from [23], i.e.,

$$\chi(M_{3,4}^s(m)) = \frac{1}{24}m(m-1)(4m^2 - 7m + 2)(4m^2 - 7m + 1).$$

6.5. The case of the dimension vector (d, d) . The next application is to consider the Euler characteristic of Kronecker moduli spaces corresponding to the dimension vectors (d, d) , $d \in \mathbb{N}$. We will see that the Euler characteristic vanishes if $d \geq 2$. In this section we consider the Kronecker quiver $K(m)$ with $m \geq 1$.

Lemma 6.12. *We have that every stable torus fixed point*

$$X = ((V, W), (X_1, \dots, X_m)) \in M_{d, (m-1)d}^s(m)$$

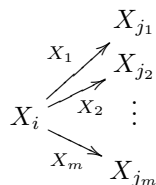
has a cycle. Thus there exists a subspace $U \subset W$ and maps $f_1, \dots, f_{2k} \in \{X_1, \dots, X_m\}$ with $f_i \neq f_{i+1}$ for $1 \leq i \leq 2l - 1$ such that

$$f_1 \circ f_2^{-1} \dots \circ f_{2k-1} \circ f_{2k}^{-1}(U) = U.$$

Remark 6.13.

- (1) From the proof we even get the stronger result that the quiver of some localization data with this dimension is forced to be cyclic. In particular, there exists no subquiver having just one common vertex with the remainder of the quiver.

Proof. Let (\mathcal{Q}, \hat{d}) be a localization data and let X be a stable representation of this data. Consider a subdata of the form



Because of the stability we have

$$d_{X_i} > \frac{(m-1)d}{d} \dim X_i = (m-1) \dim X_i.$$

We also have $\dim X_{j_k} \geq \dim X_i$ for all k . Indeed, if we had $\dim X_{j_k} = l$ such that $l < \dim X_i$, we could consider the $(\dim X_i - l)$ -subspace $\ker(X_k)$. It would have a $(\dim X_i - l)(m-1)$ -dimensional image, which obviously contradicts the stability condition.

Therefore, the subdata is of dimension type $(\dim X_i, e')$ with $e' \geq m \dim X_i$. Moreover, the stability implies that each k -dimensional subspace corresponding to X has at least an $((m-1)k + 1)$ -dimensional image.

Assume that the localization data would not have a cycle. Thus, in particular, it would have some proper boundary quiver which apparently would be of dimension type (d_1, md_1) . If we denote by b the sum of the dimensions corresponding to the sinks of the remainder of the data, we get

$$b \geq (m-1)(d - d_1) + 1.$$

Define $h := (m-1)(d - d_1) + b - (m-1)d$ which is the minimal possible dimension of the intersection of the two corresponding subrepresentations (at the common vertex) of a stable representation of the considered data. It follows that

$$(m-1)d = b + d_1m - h \geq (m-1)(d - d_1) + 1 + d_1m - h = (m-1)d + d_1 - h + 1.$$

It follows $h \geq d_1 + 1$ and thus $d_1 = 0$. □

Corollary 6.14. *The Euler characteristic of the Kronecker moduli spaces with dimension vector (d, d) vanishes if $d \geq 2$.*

Proof. By the previous lemma we know that each representation of a localization data of dimension type $(d, (m-1)d)$ has a cycle. But because of Theorem 3.15 we can assume that fixed points of each Kronecker moduli space do not have cycles. Hence there are no stable representations of the universal covering quiver of dimension type $(d, (m-1)d)$.

Because of the isomorphism between $M_{d, (m-1)d}^s(m)$ and $M_{d, d}^s(m)$, in conclusion we get

$$\chi(M_{d, (m-1)d}^s(m)) = \chi(M_{d, d}^s(m)) = 0. \quad \square$$

Note that the vanishing of the Euler characteristic does not imply that the moduli spaces are empty. There can even exist torus fixed point components after more than one localization step. But after suitably many localization steps there do not exist any torus fixed points.

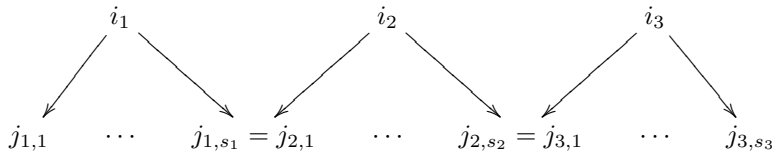
6.6. Finiteness of the fixed point set. In this section we investigate and answer a question posed in [8]. Namely, for which coprime dimension vectors is the set of fixed points finite and for which dimension vectors exists at least one n -dimensional fixed point component with $n \geq 1$.

Theorem 6.15. *Let $d \geq 3$, $e \geq 4$ and $m \geq 3$. Then there exist infinitely many torus fixed points.*

Proof. Since the torus action is compatible with the isomorphisms, we may assume

$$d \leq e \leq \frac{m}{2}d.$$

Furthermore, let $m' \leq \frac{m}{2} \in \mathbb{N}$ such that $(m' - 1)d < e < m'd$. By [29] there exists a stable bipartite quiver $s_{d,e}^m$ of type one which consists of subdata of dimension type $(1, m')$ and $(1, m' + 1)$ respectively. Since $d \geq 3$, there exists a subdata of the form

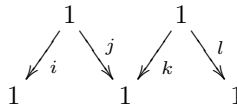


with $s_1, s_2, s_3 \in \{m', m' + 1\}$. Fix an arbitrary colouring c of the arrows which satisfies

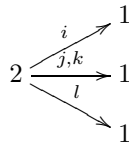
$$c(i_1, j_{1,1}) = c(i_2, j_{2,s_2}) = c(i_3, j_{3,s_3}) = 1, c(i_1, j_{1,s_1}) = c(i_3, j_{3,1}) = 2 \text{ and } c(i_2, j_{2,1}) = 3$$

and $c(i_3, j_{3,k}) \neq 3$ for every $k = 1, \dots, s_3$. This is possible because $s_3 < m$. This colouring induces an extra arrow $(i_3, j_{1,1})$ such that $c(i_3, j_{1,1}) = 3$. Hence the associated moduli space is at least one-dimensional implying that there are infinitely many torus fixed points. \square

6.7. Open questions. A fundamental question is how to determine all localization data and if it is perhaps enough to know all localization data of type one. Also, one could ask if it is possible to put all localization data down to the case of localization data of type one. For instance, when considering the localization data



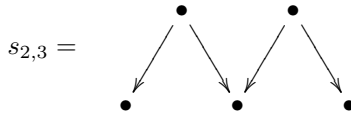
we always assumed $c(j) \neq c(k)$. But the localization data



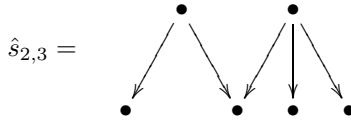
may, in a sense, be understood to correspond to “the missing case” $c(j) = c(k)$. But this raises another problem: we get additional conditions for $c(i)$ and $c(l)$ and moreover different symmetries. For instance, in the first case we have the symmetries of S_2 . But in the second one we have the symmetries of S_3 .

Another question is how to count or get all localization data (at least all of type one). Unfortunately, by use of the glueing method we do not get all localization data of type one. If it were possible to get all data of this type and if it could be shown that the other data come in a way from quivers of type one, one could probably prove the continuity. This would suffice to prove the existence of the conjectured function.

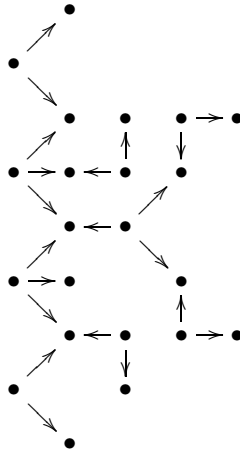
Finally, we give an example for a quiver of type one, which cannot be constructed by use of the glueing method. Let $(d, e) = (7, 10) = (2, 3) + (5, 7) = (2, 3) + (2, 3) + (3, 4)$. Then we have $(d_s, e_s) = (2, 3)$. Consider



and



where the dots represent vertices of dimension one. We get the data of dimension type $(3, 4)$ in the same way. But we do not get the following localization data of dimension type $(9, 13) = (2, 3) + (7, 10)$ by sticking together the ones above:



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