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A GEOMETRIC CONSTRUCTION OF TYPES FOR THE SMOOTH REPRESENTATIONS OF PGL(2) OVER A LOCAL FIELD

PAUL BROUSSOUS

Dedicated to Guy Henniart on his 60th birthday

Abstract. We show that almost all (Bushnell and Kutzko) types of $\mathrm{PGL}(2, F)$, F a non-Archimedean locally compact field of odd residue characteristic, naturally appear in the cohomology of finite graphs.

1. Introduction

Let F be a non-Archimedean locally compact field and G the group $\mathrm{PGL}(2, F)$. We assume that *the residue characteristic of F is not* 2. In previous works ([2], [3]) we defined a tower of directed graphs $(\tilde{X}_n)_{n\geq 0}$ lying G-equivariantly over the Bruhat-Tits tree X of G. We proved the two following facts:

Theorem 1 ([3], Theorem (3.2.4), page 502)**.** *Let $(\pi, \mathcal{V})$ be a non-spherical generic smooth irreducible representation. Then $(\pi, \mathcal{V})$ is a quotient of the cohomology space with compact support $H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C})$, where $n(\pi)$ is the conductor of π.*

Theorem 2 ([3], Theorem (5.3.2), page 512)**.** *If $(\pi, \mathcal{V})$ is a supercuspidal smooth irreducible representation of G, then we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G\, [H^1_c(\tilde{X}_{n(\pi)}, \mathbb{C}), \mathcal{V}] = 1\ .$$

In this paper we make the G-module structure of $H^1_c(\tilde{X}_n, \mathbb{C})$ more explicit for all $n \geq 0$, and draw some interesting consequences.

Let us fix an edge $[s_0, s_1]$ of X and denote by $\mathcal{K}_0$ and $\mathcal{K}_1$ the stabilizers in G of s_0 and $[s_0, s_1]$, respectively. Then $\mathcal{K}_0$ and $\mathcal{K}_1$ form a set of representatives of the two conjugacy classes of maximal compact subgroups in G. If n is even, we have a G-equivariant mapping $p_n : \tilde{X}_n \longrightarrow X$ which respects the graph structures. We denote by Σ_n the subgraph $p_n^{-1}([s_0, s_1])$. If n is odd, then after passing to the first barycentric subdivisions, we have a G-equivariant mapping $p_n : \tilde{X}_n \longrightarrow X$ which respects the graph structures. We denote by Σ_n the subgraph $p_n^{-1}(S(s_0, 1/2))$, where $S(s_0, 1/2)$ denotes the set of points x in X such that $d(x, s_0) \leq 1/2$ (here d is the natural distance on the standard geometric realization of X, normalized in such a way that $d(s_0, s_1) = 1$).

Then for all n, Σ_n is a finite graph, equipped with a an action of $\mathcal{K}_1$ if n is even, and $\mathcal{K}_0$ if n is odd. So the cohomology spaces $H^1(\Sigma_n, \mathbb{C})$ provide finite dimensional smooth representations of $\mathcal{K}_1$ or $\mathcal{K}_0$, according to the parity of n.

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For each $n \geq 0$, we define an finite set $\mathcal{P}_n$ of pairs $(\mathcal{K}, \lambda)$ formed of a maximal compact subgroup $\mathcal{K} \in \{\mathcal{K}_0, \mathcal{K}_1\}$ and of an irreducible smooth representation of $\mathcal{K}$. By definition we have $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ if and only if there exists $k \in \{0, 1, ..., n\}$ such that $(\mathcal{K}, \lambda)$ is an irreducible constituent of the representation $H^1(\Sigma_k, \mathbb{C})$. For $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ and $k \leq n$, we denote by m_λ^k the multiplicity of λ in $H_c^1(\Sigma_k, \mathbb{C})$ and we set $m_{n,\lambda} = m_\lambda = m_\lambda^0 + \cdots + m_\lambda^n$. Note that m_λ depends on $(\mathcal{K}, \lambda)$ and n.

The main results of this article are the following.

Theorem A. *For all $n \geq 0$, we have the direct sum decomposition*

$$H_c^1(\tilde{X}_n, \mathbb{C}) = \mathbf{St}_G \oplus \bigoplus_{(\mathcal{K},\lambda)\in\mathcal{P}_n} (c\text{-ind}_{\mathcal{K}}^G\ \lambda)^{m_\lambda}\ .$$

(Here $\mathbf{St}_G$ denotes the Steinberg representation of G).

Theorem B. *For all $n \geq 0$, any element of $\mathcal{P}_n$ is:*

a) *either a type in the sense of Bushnell and Kutzko's type theory* [6], *which is not a type for the unramified principal series,*

b) *or a pair of the form $(\mathcal{K}_0, \chi \circ \det \otimes \mathbf{St}_{\mathcal{K}_0})$, where χ is a smooth character of $F^\times$ of order* 2, *trivial on the group of* 1*-units in $F^\times$, and $\mathbf{St}_{\mathcal{K}_0}$ is the representation inflated from the Steinberg representation of* PGL(2) *of the residue field of F,*

c) *or the pair $(\mathcal{K}_1, \mathbf{1}_{\mathcal{K}_1})$, where $\mathbf{1}$ denotes a trivial character.*

It is worth noting that if the pairs of cases a) and b) are not types, they are *typical* in the sense of [9].

Corollary C. *Let $n \geq 0$. If $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ is a cuspidal type, then $m_{n,\lambda} = 1$.*

Indeed this follows from Theorems 2 and A using Frobenius reciprocity for compact induction.

By Theorem 1, any Bernstein component of G, different from the unramified principal series component, must have a type in $\mathcal{P}_n$ for n large enough. *Hence the graphs $\tilde{X}_n$, $n \geq 0$, provide a geometric construction of types for almost all Bernstein components of G.*

We conjecture that if $(\mathcal{K}, \lambda) \in \mathcal{P}_n$ is a type of G, then $m_\lambda = 1$.

Finally, let us observe that this construction gives a new proof that the irreducible supercuspidal representations of G are obtained by compact induction. Our proof differs from Kutzko's original proof ([10], also see [4]) only at the exhaustion steps. Indeed our "supercuspidal" types are the same as Kutzko's, but we prove that any irreducible supercuspidal representation contains such a type by using an argument based on [2] and [3], that is mainly on the existence of the new vector.

The article is organized as follows. The proof of Theorem A relies first on combinatorial properties of the graphs $\tilde{X}_n$ that are stated and proved in §2. Using these combinatorial properties and some homological arguments, we show in §3 how to relate the cohomology of $\tilde{X}_n$ to that of $\tilde{X}_{n-1}$. The irreducible components of $H^1(\Sigma_n)$ are determined in §4 when n is even, and in §5 and §6 when n is odd. A synthesis of the arguments of paragraphs 2 to 6, leading to a proof of Theorems A and B, is given in §7.

We shall assume that the reader is familiar with the language of Bushnell and Kutzko's type theory [6] and with the language of strata ([5], [4]).

2. Notation

We shall denote by
F a non-Archimedean non-discrete locally compact field,
$\mathfrak{o}$ its valuation ring,
$\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$,
ϖ the choice of a uniformizer of $\mathfrak{o}$,
$\mathbf{k} = \mathfrak{o}/\mathfrak{p}$ the residue field of F,
p the characteristic of $\mathbf{k}$,
$q = p^f$ the cardinal of $\mathbf{k}$,
G the group $\mathrm{PGL}(2, F)$.
t_ϖ the image of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ in G.
The results of this article are obtained by the following.

Hypothesis. *The characteristic of* $\mathbf{k}$ *is not* 2

We shall often define an element, a subset, or a subgroup of G by giving a (set of) representative(s) in $\mathrm{GL}(2, F)$.

We write T for the diagonal torus of G and $B \supset T$ for the upper standard Borel subgroup. We denote by T^0 the maximal compact subgroup of T, i.e., the set of matrices with coefficients in $\mathfrak{o}^\times$, and by T^n the subgroup of matrices with coefficients in $1 + \mathfrak{p}^n$, $n > 0$.

Let k, l be integers satisfying $k + l \geq 0$. Then

$$\mathfrak{A}(k, l) = \begin{pmatrix} \mathfrak{o} & \mathfrak{p}^l \\ \mathfrak{p}^k & \mathfrak{o} \end{pmatrix}$$

is an $\mathfrak{o}$-order of $\mathrm{M}(2, F)$. We denote by $\Gamma_0(k, l)$ the image in G of its group of units. There are two conjugacy classes of maximal compact subgroups of G. The first one has representative $K = \Gamma_0(0, 0)$. A representative $\tilde{I}$ of the second one is the semidirect product of the cyclic group generated by $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)$ with the Iwahori subgroup $I = \Gamma_0(1, 0)$.

The group K is filtered by the normal compact open subgroups

$$K_n = \begin{pmatrix} 1 + \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1 + \mathfrak{p}^n \end{pmatrix}, \ n \geq 1 \ .$$

The group I is filtered by the normal compact subgroups I_n, $n \geq 1$, defined by

$$I_{2k+2} = \begin{pmatrix} 1 + \mathfrak{p}^{k+1} & \mathfrak{p}^{k+1} \\ \mathfrak{p}^{k+2} & 1 + \mathfrak{p}^{k+1} \end{pmatrix}, \ I_{2k+1} = \begin{pmatrix} 1 + \mathfrak{p}^{k+1} & \mathfrak{p}^{k} \\ \mathfrak{p}^{k+1} & 1 + \mathfrak{p}^{k+1} \end{pmatrix}, \ k \geq 0 \ .$$

The subgroups I_n, $n \geq 1$, are normalized by Π.

We denote by X the Bruhat-Tits building of G. This is a uniform tree with valency $q+1$. As a G-set and as a simplicial complex X identifies with the following complex. Its vertices are the homothety classes $[L]$ of full $\mathfrak{o}$-lattices L in the vector space $V = F^2$. Two vertices $[L]$ and $[M]$ define an edge if and only if there exists a basis (e_1, e_2) of V such that, up to homothety, we have $L = \mathfrak{o}e_1 \oplus \mathfrak{o}e_2$ and $M = \mathfrak{o}e_1 \oplus \mathfrak{p}e_2$.

The vertices of the standard apartment (i.e., the apartment stabilized by T) are the $s_k = [\mathfrak{o} \oplus \mathfrak{p}^k]$, $k \in \mathbb{Z}$. The element t_ϖ acts as $t_\varpi s_k = s_{k+1}$, $k \in \mathbb{Z}$. The maximal compact subgroup K is the stabilizer of s_0 and $\tilde{I}$ (resp. I) is the global stabilizer (resp. pointwise stabilizer) of the edge $[s_0, s_1]$. If $l \geq k$, the pointwise stabilizer of the segment $[s_k, s_l]$ is $\Gamma_0(l, -k)$.

3. Combinatorics of $\tilde{X}_n$

We recall the construction of the directed graphs $\tilde{X}_n$, $n \geq 1$.

For any integer $k \geq 1$, an *oriented k-path* in X is an injective sequence of vertices $(s_i)_{i=0,\dots,k}$ in X such that, for $k = 0, \dots, k-1$, $\{s_i, s_{i+1}\}$ is an edge in X. We shall allow the index i to run over any interval of integers of length $k+1$. Let us fix an integer $n \geq 1$. The directed graph $\tilde{X}_n$ is constructed as follows. Its edge set (resp. vertex set) is the set of oriented $(n+1)$-paths (resp. n-paths) in X. If $a = \{s_0, s_1, ..., s_{n+1}\}$ is an edge of $\tilde{X}_n$, its head (resp. its tail) is $a^+ = \{s_1, s_2, ..., s_{n+1}\}$ (resp. $a^- = \{s_0, s_1, ..., s_n\}$). The graphs we obtain in this way are actually simplicial complex. The group G acts on $\tilde{X}_n$ is an obvious way; the action preserves the structure of directed graph.

When $n = 2m$ is even, we have a natural simplicial projection $p = p_n : \tilde{X}_n \to X$ given on vertices by $p(s_{-m}, \dots, s_0, \dots, s_m) = s_0$. It is G-equivariant. Let $e = \{s_0, t_0\}$ be an edge of X. We are going to describe the finite simplicial complex $p^{-1}(e)$. An edge in $\tilde{X}_n$ above the edge e corresponds to an oriented $(2m+1)$-path of one of the following forms:

i) $(s_{-m}, s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1}, t_m)$,

ii) $(t_{-m}, t_{-m+1}, \dots, t_0, s_0, \dots, s_{m-1}, s_m)$.

Let $C_{2m-1}(e)$ be the set of $(2m-1)$-paths $c = (s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1})$. We say that $c \in C_{2m-1}(e)$ *lies above* e. Fix $c \in C_{2m-1}(e)$ and consider the simplicial subcomplex $\tilde{X}_{2m}[e, c]$ of $\tilde{X}_{2m}$ whose edges correspond to the $(2m+1)$-paths of the form

$$(a, s_{-m+1}, \dots, s_0, t_0, \dots, t_{m-1}, b)\,.$$

So a (resp. b) can be any neighbour of s_{-m+1} (resp. t_{m-1}) different from s_{-m+2} (resp. t_{m-1}), with the convention that $s_1 = t_0$ and $t_{-1} = s_0$. The simplicial complex $\tilde{X}_{2m}[e, c]$ is connected. It is indeed isomorphic to the complete bipartite graph with sets of vertices:

$$\{a\ ;\ a \text{ neighbour of } s_{-m+1},\ a \neq s_{-m+2}\} \text{ and } \{b\ ;\ b \text{ neighbour of } t_{m-1},\ b \neq t_{m-2}\}\,.$$

Lemma 3.1. *Let e and e' be two edges of X and $c \in C_{2m-1}(e)$, $c' \in C_{2m-1}(e')$. Then $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] \neq \emptyset$ if and only if we are in one of the following cases:*

i) $e = e'$ and $c = c'$ (so that $\tilde{X}_{2m}[e, c] = \tilde{X}_{2m}[e', c']$);

ii) $e \cap e'$ is reduced to one vertex of X and $c \cup c'$ is an oriented $2m$-path in X. In that case $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$ is reduced to the vertex of $\tilde{X}_{2m}$ corresponding to the $2m$-path $c \cup c'$.

Proof. If $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] \neq \emptyset$, then $e \cap e' = p(\tilde{X}_{2m}[e, c]) \cap p(\tilde{X}_{2m}[e', c']) \neq \emptyset$. Assume first that $e = e'$. Then $c = c'$, for if $c \neq c'$, then $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c'] = \emptyset$; indeed, if $\tilde{s}$ is a vertex of $\tilde{X}_{2m}[e, c]$, then it determines c uniquely. Now assume that $e \cap e'$ is a vertex. Let $\tilde{s} \in \tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$. Then $\tilde{s}$ contains c and c' as subsequences, with $c \neq c'$. So by a length argument $s = c \cup c'$. Conversely if $c \cup c'$ is an oriented $2n$-path, then $c \cup c'$ is a vertex of $\tilde{X}$ lying in $\tilde{X}_{2m}[e, c] \cap \tilde{X}_{2m}[e', c']$. $\square$

Corollary 3.2. *For any edge e of X, the connected components of $p^{-1}(e)$ are the $\tilde{X}_{2m}[e, c]$, where c runs over $C_{2m-1}(e)$.*

Define a 1-dimensional simplicial complex Y_{2m-1} in the following way. Its vertices are the connected components $\tilde{X}_{2m}[e, c]$, where e runs over the edges of X and c

over $C_{2m-1}(e)$, and two vertices $\tilde{X}_{2m}[e,c]$ and $\tilde{X}_{2m}[e',c']$ are linked by an edge if they intersect. Note that Y_{2m-1} is naturally a G-simplicial complex.

Corollary 3.3. *As a G-simplicial complex, Y_{2m-1} is canonically isomorphic to the complex $\tilde{X}_{2m-1}$.*

Lemma 3.4. *For all $n \geq 0$, the simplicial complex $\tilde{X}_n$ is connected.*

Proof. When n is even this is [2, Lemma (4.1)]. For odd n, the proof is similar. □

Assume that $m \geq 1$. We say that an edge of $\tilde{X}_{2m-1}$ *lies above* a vertex s_0 of X if as an oriented $2m$-path it has the form $(s_{-m}, \ldots, s_o, \ldots, s_m)$. For any vertex s_0 of X we write $\tilde{X}_{2m-1}[s_o]$ for the subsimplicial complex of $\tilde{X}_{2m-1}$ formed of the edges lying above s_0.

Lemma 3.5. *When $m = 1$ the simplicial complexes $\tilde{X}_{2m-1}[s_0] = \tilde{X}_1[s_0]$ are connected.*

Proof. We may identify the neighbour vertices of s_0 in X with the points of the projective line $\mathbb{P}^1(\bar{M}) \simeq \mathbb{P}^1(\mathbf{k})$, where $s_0 = [M]$ and $\bar{M} = M/\mathfrak{p}M$. The vertices of $\tilde{X}_1[s_0]$ are the oriented 1-paths (s_0, x), (y, s_0), x, $y \in \mathbb{P}^1(\bar{M})$. Two oriented 1-paths of the form (x, s_0) and (s_0, y) are linked by the edge (x, s_0, y). Let (x, s_0), (y, s_0) be two oriented 1-paths with $x \neq y$. Since $|\mathbb{P}^1(\mathbf{k})| \geq 3$, there exists $z \in \mathbb{P}^1(\bar{M})$ distinct from x and y. Then (x, s_0) is linked to (s_o, z) via the path (x, s_0, z) and (s_0, z) is linked to (y, s_0) via the path (y, s_0, z). For vertices of the form (s_0, x), (s_0, y) the proof is similar. □

We now assume that $m > 1$. We write $C_{2m-2}(s_0)$ for the set $(2m-2)$-paths of the form $(s_{-m+1}, \ldots, s_0, \ldots, s_{m-1})$. For any $c \in C_{2m-2}(s_0)$, we consider the subsimplicial complex $\tilde{X}_{2n-1}[s_0, c]$ of $\tilde{X}_{2m-1}$ whose edges corresponds to the $2m$-paths of the form $(a, s_{-n+1}, \ldots, s_0, s_{n-1}, b)$. We have results similar to Lemma 3.1 and Corollaries 3.2 and 3.3.

Lemma 3.6. i) *For any vertex s_0 of X and for $c \in C_{2m-2}(s_0)$, $\tilde{X}_{2m-1}[s_0, c]$ is connected. It is indeed isomorphic to a complete bipartite graph constructed on two sets of $q = |\mathbf{k}|$ elements.*

ii) *Let s and s' be vertices of X, $c \in C_{2m-2}(s)$ and $c' \in C_{2m-2}(s')$. Then $\tilde{X}_{2m-1}[s,c] \cap \tilde{X}_{2m-1}[s',c'] \neq \emptyset$ if and only if $s = s'$ and $c = c'$, or $\{s, s'\}$ is an edge in X and $c \cup c'$ is an oriented $2n-1$-path. In this last case $\tilde{X}_{2m-1}[s,c] \cap \tilde{X}_{2m-1}[s',c'] = \{\tilde{s}\}$, where the vertex $\tilde{s}$ of $\tilde{X}_{2m-1}$ corresponds to the $(2n-1)$-path $c \cup c'$.*

iii) *For any vertex s of X, the connected components of $\tilde{X}_{2m-1}[s]$ are $\tilde{X}_{2m-1}[s,c]$, c running over $C_{2m-2}(s)$.*

We can consider the 1-dimensional simplicial complex Z_{2m-2} whose vertices are the connected components $\tilde{X}_{2m-1}[s,c]$, s running over the vertices of X and c over $C_{2m-2}(s)$, and where two connected components define an edge if and only if they intersect. Note that Z_{2m-2} is naturally a G-simplicial complex.

Corollary 3.7. *As a G-simplicial complex Z_{2m-2} is isomorphic to X_{2n-2}.*

4. The cohomology of $\tilde{X}_n$: First reductions

If Σ is a locally finite 1-dimensional simplicial complex, we write Σ^0 (resp. $\Sigma^{(1)}$,Σ^1) for its set of vertices (resp. non-oriented edges, oriented edges). We

let $C_0(\Sigma)$ (resp. $C_1(\Sigma)$) denote the $\mathbb{C}$-vector space with basis Σ^0 (resp. Σ^1). We define the space $C_c^0(\Sigma,\mathbb{C}) = C_c^0(\Sigma)$ (resp. $C_c^1(\Sigma,\mathbb{C}) = C_c^1(\Sigma)$) of oriented simplicial 0-cochains (resp. 1-cochains) with compact support by:

$C_c^0(\Sigma)$ = space of all linear forms $f : C_0(\Sigma) \to \mathbb{C}$ such that $f(s) = 0$ except for a finite number of vertices s;

$C_c^1(\Sigma)$ = space of all linear forms $\omega : C_1(\Sigma) \to \mathbb{C}$ such that $\omega([a,b]) = 0$ except for a finite number of oriented edges $[a,b]$ and $\omega([a,b]) = -\omega([b,a])$.

We set $C_c^k(\Sigma) = 0$ for $k \in \mathbb{Z}\backslash\{0,1\}$ and define a coboundary map $d : C_c^0(\Sigma) \to C_c^1(\Sigma)$ by $df([a,b]) = f(b) - f(a)$. The cohomology of the cochain complex $(C_c^\bullet(\Sigma), d)$ computes the cohomology with compact support $H_c^\bullet(\Sigma,\mathbb{C}) = H_c^\bullet(\Sigma)$ of (the standard geometric realization of) Σ. If Σ is acted upon by a group H whose action is simplicial, then $(C_c^\bullet(\Sigma), d)$ is in a straightforward way a complex of H-modules, and its cohomology computes $H_c^1(\Sigma)$ as a H-module. When Σ is finite we drop the subscripts c.

Since the stabilizer of a finite number of vertices of X is open in G, we see that for $n \geq 1$, the G-modules $C_c^0(\tilde{X}_n)$, $C_c^1(\tilde{X}_n)$ and therefore $H_c^1(\tilde{X}_n)$ are smooth.

In the sequel we fix $m \geq 1$ and we abbreviate $\tilde{X}_{2m} = \tilde{X}$. The disjoint union $\tilde{X}^1 = \bigsqcup_{e\in X^{(1)}} \tilde{X}_e$, where $\tilde{X}_e = p^{-1}(e)$, induces an isomorphism:

$$\begin{array}{ccc} C_c^1(\tilde{X}) & \simeq & \bigoplus_{e\in X^{(1)}} C^1(\tilde{X}_e), \\ \omega & \mapsto & (\omega_{|C_1(\tilde{X}_e)})_{e\in X^{(1)}}. \end{array} \tag{4.1}$$

Similarly, the non-disjoint union $\tilde{X}^0 = \bigcup_{e\in X^{(1)}} \tilde{X}_e^0$ induces an injection:

$$\begin{array}{cccc} j: & C_c^0(\tilde{X}) & \hookrightarrow & \bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e), \\ & f & \mapsto & (f_{|C_0(\tilde{X}_e)})_{e\in X^{(1)}}. \end{array} \tag{4.2}$$

We have the following commutative diagram of G-modules:

$$\begin{tikzcd}
 & H_c^0(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \arrow[r, "\varphi"] \arrow[d] & \mathrm{coker} j \arrow[d] & \\
0 \arrow[r] & C_c^0(\tilde{X}) \arrow[r, "j"] \arrow[d, "d"'] & \bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e) \arrow[r] \arrow[d, "\oplus d_e"'] & \mathrm{coker} j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_c^1(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{e\in X^{(1)}} C^1(\tilde{X}_e) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & H_c^1(\tilde{X}) \arrow[r] & \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \arrow[r] & 0 &
\end{tikzcd}$$

Here, for $e \in X^{(1)}$, d_e denote the coboundary map $C^0(\tilde{X}_e) \to C^1(\tilde{X}_e)$. Since $\tilde{X}$ is connected ([2] Lemma 4.1) and non-compact, we have $H_c^0(\tilde{X}) = 0$. So *the snake lemma* gives the kernel-cokernel exact sequence:

$$0 \to \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \to \mathrm{coker} j \to H_c^1(\tilde{X}) \to \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0,$$

that is,

$$0 \to \mathrm{coker j}/\varphi\big(\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)\big) \to H_c^1(\tilde{X}) \to \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0. \tag{4.3}$$

Abbreviate $Y = Y_{2m-1}$.

Lemma 4.1. *We have a canonical isomorphism of G-modules*

$$\mathrm{coker} j/\varphi\big(\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)\big) \simeq H^1_c(Y)\,.$$

Proof. From Corollary 3.2 we have

$$\bigoplus_{e\in X^{(1)}} C^0(\tilde{X}_e) = \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C_{2m-1}(e)} C^0(\tilde{X}_{2m}[e,c])\,.$$

So the map j is given by $f \mapsto \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C_{2m-1}(e)} f_{e,c}$, where $f_{e,c} = f_{|C_0(\tilde{X}_{2m}[e,c])}$. Consider the G-equivariant morphism of vector spaces

$$\psi \;:\; \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C_{2m-1}(e)} C^0(\tilde{X}_{2m}[e,c]) \to C^1_c(Y)$$

given as follows. If σ is an oriented edge of Y, then their exist uniquely determined edges e_o, e'_o of X, $c_o \in C_{2m-1}(e_o)$, $c'_o \in C_{2m-1}(e'_o)$, such that σ corresponds to the intersection $\tilde{X}_{2m}[e_o,c_o] \cap \tilde{X}_{2m}[e'_o,c'_o] = \{s_o\}$, $s_o \in \tilde{X}^0$. We then set

$$\psi[(f_{e,c})_{e,c}](\sigma) = f_{e'_o,c'_o}(s_o) - f_{e_o,c_o}(s_o)\,.$$

Then ψ is surjective and its kernel is precisely $j(C^0_c(\tilde{X}))$. So we may identify $\mathrm{coker} j$ with $C^1_c(Y)$. From Corollary 3.2, we have

$$\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) = \bigoplus_{e\in X^{(1)}} \bigoplus_{c\in C_{2m1}(e)} H^0(\tilde{X}_{2m}[e,c])$$

so that we may identify $\bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e)$ with $C^0_c(\tilde{Y})$. Under our identifications the map $\varphi : \bigoplus_{e\in X^{(1)}} H^0(\tilde{X}_e) \to \mathrm{coker} j$ corresponds to the coboundary map $d : C^0_c(Y) \to C^1_c(Y)$, and we are done since all our identifications are G-equivariant. □

Proposition 4.2. *For $m \geq 1$, we have an isomorphism of G-modules:*

$$H^1_c(\tilde{X}_{2m}) \simeq H^1_c(\tilde{X}_{2m-1}) \oplus c\text{-}\mathrm{ind}^G_{\mathcal{K}_{e_o}} H^1(\tilde{X}_{e_o})$$

for any edge e_o of X and where $\mathcal{K}_{e_o}$ denotes the stabilizer of e_o in G.

Proof. From the short exact sequence (4.3) and Lemma 4.1, we have the exact sequence of G-modules:

$$0 \to H^1_c(Y) \to H^1_c(\tilde{X}) \to \bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e) \to 0. \tag{4.4}$$

Since G acts transitively on the edges of X, $\bigoplus_{e\in X^{(1)}} H^1(\tilde{X}_e)$ identifies with the compactly induced representation $c\text{-}\mathrm{ind}^G_{\mathcal{K}_{e_o}} H^1(\tilde{X}_{e_o})$. Moreover, this induced representation is projective in the category of smooth complex representations of G. This is classical and follows from Frobenius reciprocity for compact induction together with the fact that the category of smooth $\mathcal{K}_{e_o}$-modules is semisimple. So the sequence (4.4) splits. □

We assume that $m \geq 1$ and we abbreviate $\tilde{X} = \tilde{X}_{2m-1}$. The disjoint union $\tilde{X}^1 = \bigsqcup_{s\in X^0} \tilde{X}_{2m-1}[s]^1$ induces an isomorphism:

$$\begin{array}{ccc} C^1_c(\tilde{X}) & \simeq & \bigoplus_{s\in X^0} C^1(\tilde{X}_{2m-1}[s]), \\ \omega & \mapsto & (\omega_{|C_1(\tilde{X}_{2m-1}[s])})_{s\in X^0}. \end{array} \tag{4.5}$$

Similarly, the non-disjoint union $\tilde{X}^0 = \bigcup_{s\in X^0} \tilde{X}^0_s$ induces an injection:

$$\begin{array}{rccc} j: & C^0_c(\tilde{X}) & \hookrightarrow & \bigoplus_{s\in X^0} C^0(\tilde{X}_{2m-1}[s]), \\ & f & \mapsto & (f_{|C_0(\tilde{X}_{2m-1}[s])})_{s\in X^0}. \end{array} \tag{4.6}$$

We have the following commutative diagram of G-modules:

$$\begin{tikzcd}
 & H^0_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{s\in X^0} H^0(\tilde{X}_{2m-1}[s]) \arrow[r, "\varphi"] \arrow[d] & \mathrm{coker} j \arrow[d] & \\
0 \arrow[r] & C^0_c(\tilde{X}) \arrow[r, "j"] \arrow[d, "d"'] & \bigoplus_{s\in X^0} C^0(\tilde{X}_{2m-1}[s]) \arrow[r] \arrow[d, "\oplus d_s"'] & \mathrm{coker} j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_c(\tilde{X}) \arrow[r] \arrow[d] & \bigoplus_{s\in X^0} C^1(\tilde{X}_{2m-1}[s]) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & H^1_c(\tilde{X}) \arrow[r] & \bigoplus_{s\in X^0} H^1(\tilde{X}_{2m-1}[s]) \arrow[r] & 0 &
\end{tikzcd}$$

Here, for $s \in X^0$, d_s denote the coboundary map $C^0(\tilde{X}_{2m-1}[s]) \to C^1(\tilde{X}_{2m-1}[s])$. By Lemma 3.4, $\tilde{X}$ is connected. So we have $H^0_c(\tilde{X}) = 0$ since $\tilde{X}$ is non-compact. The *snake lemma* gives the kernel-cokernel exact sequence:

$$0 \to \mathrm{coker} j/\varphi\big(\bigoplus_{s\in X^0} H^0(\tilde{X}_{2m-1}[s])\big) \to H^1_c(\tilde{X}) \to \bigoplus_{s\in X^0} H^1(\tilde{X}_{2m-1}[s]) \to 0. \tag{4.7}$$

Lemma 4.3. *We have a canonical isomorphism of G-modules*

$$\mathrm{coker} j/\varphi\big(\bigoplus_{s\in X^0} H^0(\tilde{X}_{2m-1}[s])\big) \simeq H^1_c(\tilde{X}_{2m-2})\,.$$

Proof. It is similar to the proof of Lemma 4.1 and relies on Lemma 3.6 and Corollary 3.7. □

Proposition 4.4. *For $m \geq 1$, we have an isomorphism of G-modules:*

$$H^1_c(\tilde{X}_{2m-1}) \simeq H^1_c(\tilde{X}_{2m-2}) \oplus c\text{-ind}^G_{\mathcal{K}_{s_o}} H^1(\tilde{X}_{s_o})$$

for any vertex s_o and where $\mathcal{K}_{s_o}$ denotes the stabilizer of s_o in G.

Proof. Similar to the proof of Proposition 4.2. □

Recall [3] that $\tilde{X}_0$ is different from X. This is a directed graph whose set of vertices is isomorphic to X^0 as a G-set and whose set of edges is isomorphic to the G-set of oriented edges of X.

5. Determination of the inducing representations – I

Let $m \geq 0$ be a fixed integer and $e_0 = [s_0, s_1]$ the standard edge. The aim of this section is to determine the $\mathcal{K}_{e_0}$-module $H^1(\tilde{X}_{2m}[e_0])$. Here we have $\mathcal{K}_{e_0} = \tilde{I}$, the normalizer in G of the standard Iwahori subgroup. We have the semidirect products

$$\tilde{I} = \langle \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \rangle \ltimes I = E^\times I$$

for any totally ramified subfield extension $E/F \subset \mathrm{M}(2,F)$ such that $E^\times$ normalizes I.

We first assume that $m \geq 1$. By Corollary 3.2, we have the disjoint union

$$\tilde{X}_{2m}[e_0] = \coprod_{c \in C_{2m-1}(e_0)} \tilde{X}_{2m}[e_0, c] \ .$$

The group $\tilde{I}$ acts transitively on $C_{2m-1}(e_0)$. This comes form the standard fact that I, the pointwise stabilizer of e_0 acts transitively on the apartments of X containing e_0.

Let $c_0 \in C_{2m-1}(e_0)$ be the path

$$s_{-m+1}, ..., s_0, s_1, ..., s_m \ .$$

The global stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ in $\tilde{I}$ is the pointwise stabilizer of c_0 in $\tilde{I}$, that is,

$$\Gamma_0(m, m-1) = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{p}^{m-1} \\ \mathfrak{p}^m & \mathfrak{o}\times \end{pmatrix} = T^0 I_{2m-1} \ .$$

It follows that

$$H^1(\tilde{X}_{2m}[e_0]) = \mathrm{ind}_{T^0 I_{2m-1}}^{\tilde{I}} H^1(\tilde{X}_{2m}[e_0, c_0]) \ . \tag{5.1}$$

On the other hand, an easy calculation shows that the pointwise stabilizer of $\tilde{X}_{2m}[e_0, c_0]$ is $T^1 I_{2m}$, where T^1 is the congruence subgroup of T given by

$$T^1 = \begin{pmatrix} 1+\mathfrak{p} & 0 \\ 0 & 1+\mathfrak{p} \end{pmatrix} \ .$$

So the $T^0 I_{2m-1}$-module $H^1(\tilde{X}_{2m}[e_0, c_0])$ may be viewed as a representation of the finite group $T^0 I_{2m-1}/T^1 I_{2m}$, that is, a semidirect product of the cyclic group $\mathbf{k}^\times$ with the abelian group $I_{2m-1}/I_{2m} \simeq \mathbf{k} \oplus \mathbf{k}$.

Set $\Gamma = \tilde{X}_{2m}[e_0, c_0]$. This is a finite directed graph. Let Σ_{-m} (resp. Σ_{m+1}) denote the set of vertices of X that are neighbours of s_{-m+1} and different from s_{-m+2} (resp. neighbours of s_m and different from s_{m-1}). Then the vertex set of Γ is

$$\begin{aligned} \Gamma^0 &= \{(a, s_{-m+1}, ..., s_0, ..., s_m);\ a \in \Sigma_{-m}\} \coprod \{(s_{-m+1}, ..., s_0, ..., s_m, b);\ b \in \Sigma_{m+1}\} \\ &\simeq \Sigma_{-m} \coprod \Sigma_{m+1} \end{aligned}$$

and its edge set is

$$\Gamma^1 = \{(a, s_{-m+1}, ..., s_0, ..., s_m, b)\ ;\ a \in \Sigma_{-m},\ b \in \Sigma_{m+1}\} \simeq \Sigma_{-m} \times \Sigma_{m+1} \ .$$

In particular, Γ is a bipartite graph based on two sets of q elements. In particular, its Euler character is given by

$$\chi(\Gamma) = 1 - \dim_{\mathbb{C}} H^1(\Gamma) = 2q - q^2 \ ,$$

so that

$$\dim_{\mathbb{C}} H^1(\Gamma) = q^2 - 2q + 1 = (q-1)^2 \ . \tag{5.2}$$

Let $\mathbb{C}[\Gamma^1]$ be the space of complex function on Γ^1 and let $\mathcal{H}(\Gamma)$ be the space of harmonic 1-cochains on Γ:

$$\mathcal{H}(\Gamma) = \{f \in \mathbb{C}[\Gamma^1];\ \sum_{a \in \Gamma^1,\ s \in a} [a : s] f(a) = 0\ ,\ \text{all } s \in \Gamma^0\} \ .$$

Here $[a:s]$ denote an incidence number. In our case:

$$f \in \mathcal{H}(\Gamma) \text{ iff } \left\{ \begin{array}{rcll} \displaystyle\sum_{a\in\Sigma_{-m}} f(a, s_{-m+1}, ..., s_m, b) & = & 0, & \text{for all } b, \\ \displaystyle\sum_{b\in\Sigma_{m+1}} f(a, s_{-m+1}, ..., s_m, b) & = & 0, & \text{for all } a. \end{array} \right. \tag{5.3}$$

This is a standard result (see e.g. [3, Lemma (1.3.2)]) that, as a $T^0 I_{2m-1}/T^1 I_{2m}$-module, $H^1(\Gamma)$ is isomorphic to the contragredient module of $\mathcal{H}(\Gamma)$.

An easy computation shows that we may identify Γ^1 with $\mathbf{k}\times\mathbf{k}$ in such a way that:

1) an element of $I_{2m-1} = \begin{pmatrix} 1+\mathfrak{p}^m & \mathfrak{p}^{m-1} \\ \mathfrak{p}^m & 1+\mathfrak{p}^m \end{pmatrix}$ acts as

$$\big(1 + \begin{pmatrix} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{pmatrix}\big).(x,y) = (x+\bar{b}, y+\bar{c})$$

for $a,b,c,d \in \mathfrak{o}$, $x,y \in \mathbf{k}$, and

2) an element of T^0 acts as

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.(x,y) = (\bar{a}\bar{d}^{-1}x, \bar{d}\bar{a}^{-1}y)$$

and the condition (5.3) gives us:

$$f \in \mathcal{H}(\Gamma) \text{ iff } \left\{ \begin{array}{rcll} \displaystyle\sum_{x\in\mathbf{k}} f(x,y) & = & 0, & \text{for all } y \in \mathbf{k}, \\ \displaystyle\sum_{y\in\mathbf{k}} f(x,y) & = & 0, & \text{for all } x \in \mathbf{k}. \end{array} \right.$$

A basis of $\mathbb{C}[\Gamma]$ is formed of the functions $\chi_1 \otimes \chi_2(x,y) = \chi_1(x)\chi_2(y)$, where, for $i = 1,2$, χ_i runs over the characters of $(\mathbf{k},+)$. It is clear that the $(q-1)^2$-dimensional subspace of $\mathbb{C}[\Gamma]$ generated by the $\chi_1\otimes\chi_2$, $\chi_1 \not\equiv 1$, $\chi_2 \not\equiv 1$, is contained in $\mathcal{H}(\Gamma)$. So using (5.2), we obtain:

$$\mathcal{H}(\Gamma) = \mathrm{Span}\{\chi_1\otimes\chi_2\ ; \chi_i \in \widehat{\mathbf{k}}, \chi_i \not\equiv 1\ , i = 1,2\}\ . \tag{5.4}$$

It follows from (5.4) that, as an I_{2m-1}/I_{2m}-module, the space $\mathcal{H}(\Gamma)$ is the direct sum of 1-dimensional representations corresponding to the characters $\alpha = \alpha(\chi_1,\chi_2)$, $\chi_i \not\equiv 1$, $i = 1,2$, given by

$$\alpha\big(1 + \begin{pmatrix} \varpi^m a & \varpi^{m-1} b \\ \varpi^m c & \varpi^m d \end{pmatrix}\big) = \chi_1(b)\chi_2(c)\ .$$

In particular, $\mathcal{H}(\Gamma)$ is isomorphic to its contragredient and therefore isomorphic to $H^1(\Gamma)$ as an I_{2m-1}/I_{2m}-module. In the language of strata (the reader may refer to [4, §4]), for $\chi_i \not\equiv 1$, $i = 1,2$, the character $\alpha(\chi_1,\chi_2)$ corresponds to a stratum of the form $[\mathcal{I}, 2m, 2m-1, \beta]$, where $\mathcal{I}$ is the standard *Iwahori order* and $\beta \in \mathrm{M}(2,F)$ is an element of the form $\Pi^{2m-1}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $u,v \in \mathfrak{o}^\times$. In the terminology of [4, §4, page 98] this stratum is a *ramified simple stratum.*

We now have enough material to prove the following result.

Proposition 5.1. *Let λ be an irreducible constituent of*

$$H^1(\tilde{X}_{2m}[e_0]) = \mathrm{ind}_{T^0 I_{2m-1}}^{\tilde{I}} H^1(\tilde{X}_{2m}[e_0, c_0])\ .$$

Then the compactly induced representation $c\text{-ind}_{\tilde{I}}^{G}\lambda$ *is irreducible, whence supercuspidal.*

Proof. It is a standard result that an irreducible compactly induced representation is supercuspidal (see e.g. [8, page 194]).

The proof of the irreducibility is also standard by an argument due to Kutzko. But we repeat it for convenience. By Frobenius reciprocity, the restriction of λ to I_{2m-1} contains a character $\alpha(\chi_1,\chi_2)$ corresponding to a (ramified) simple stratum. Since λ is irreducible and since $\tilde{I}$ normalizes I_{2m-1}, the restriction $\lambda_{|I_{2m-1}}$ is a direct sum $\alpha_1\oplus\cdots\oplus\alpha_r$ of $\tilde{I}$-conjugates of $\alpha(\chi_1,\chi_2)$. They all correspond to simple strata. Let $g\in G$ be an element intertwining λ with itself. Then by restriction it intertwines a character α_i with a character α_j for some $j=1,...,r$. By [4, Lemma (16.1), page 111], such an element G must belong to $\tilde{I}$. It follows that the G-intertwining of λ is equal to $\tilde{I}$ and that the representation $c\text{-ind}_{\tilde{I}}^{G}\lambda$ is irreducible according to Mackey's irreducibility criterion ([8, Proposition (1.5), page 195]). $\square$

We finally consider the case $m=0$. The directed graph $\tilde{X}_0$ has X^0 as vertex set. An edge $\{t,s\}$ in X gives rise to two edges $[s,t]$ and $[t,s]$ in $\tilde{X}_0$. Since the action of G on $\tilde{X}_0$ preserves the structure of the digraph, the G-module $H_c^1(\tilde{X}_0)$ may be computed using the complex

$$0\longrightarrow C_c^0(\tilde{X}_0)\longrightarrow C_c^{(1)}(\tilde{X}_0),$$

where $C_c^{(1)}(\tilde{X}_0)$ is the space of (unoriented) 1-cochains, that is the space of maps from $\tilde{X}_0^{(1)}$ (unoriented edges) to $\mathbb{C}$ with finite support. The coboundary map is given here by $df(a)=f(a^+)-f(a^-)$. Consider the G-equivariant injection $j:C_c^1(X)\longrightarrow C_c^{(1)}(\tilde{X}_0)$ given by $j(\omega):[s,t]\mapsto\omega([s,t])$. We have the commutative diagram of G-modules:

$$\begin{tikzcd}
0 \arrow[r] & C_c^0(X) \arrow[r, "\mathrm{id}"] \arrow[d] & C_c^0(\tilde{X}_0) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_c^1(X) \arrow[r, "j"] & C^{(1)}(\tilde{X}_0) \arrow[r] & C_c^{(1)}(\tilde{X}_0)/\mathrm{Im}\, j \arrow[r] & 0
\end{tikzcd}$$

The quotient $C_c^{(1)}(\tilde{X}_0)/\mathrm{Im}j$ identifies with the subspace of $C_c^{(1)}(\tilde{X}_0)$ formed by those functions f satisfying $f([s,t])=f([t,s])$ for all edges $\{s,t\}$ of X. This subspace is nothing more than the compactly induced representation $\mathrm{c-Ind}_{\tilde{I}}^{G}\mathbf{1}_{\tilde{I}}$. The cokernel exact sequence gives us:

$$0\longrightarrow H_c^1(X)\longrightarrow H_c^1(\tilde{X}_0)\longrightarrow c\text{-ind}_{\tilde{I}}^{G}\mathbf{1}_{\tilde{I}}\longrightarrow 0.$$

Now we use the following two facts to obtain Proposition 5.2:

– the representation $c\text{-ind}_{\tilde{I}}^{G}\mathbf{1}_{\tilde{I}}$ is a projective object of the category of smooth representations of G,

– the G-module $H_c^1(X)$ is isomorphic to the Steinberg representation $\mathbf{St}_G$ of G ([7]).

Proposition 5.2. *The G-module $H_c^1(\tilde{X}_0)$ is isomorphic to $\mathbf{St}_G\oplus c\text{-ind}_{\tilde{I}}^{G}\mathbf{1}_{\tilde{I}}$.*

6. The inducing representations – II

We now determine the $\mathcal{K}_{s_0}$-module $H^1(\tilde{X}_{2m+1}[s_0])$. The arguments are very often similar to those of the previous section and we will not give all details. Since the case $m = 0$ requires slightly different techniques we postpone it to the end of the section and assume first that $m > 0$.

Recall that the stabilizer $\mathcal{K}_{s_0}$ of s_0 in G is the image K of $\mathrm{GL}(2, \mathfrak{o})$ in G.

Let $c_0 \in C_{2m}(s_0)$ be the path $(s_{-m}, ..., s_0, ..., s_m)$. Its pointwise stabilizer is $\Gamma_0(m, m) = T^0 K_m$. So as a K-module, $H^1(\tilde{X}_{2m+1}[s_0])$ is isomorphic to the induced representation $\mathrm{Ind}_{T^0 K_m}^K H^1(\tilde{X}_{2m+1}[s_0, c_0])$. Moreover, the pointwise stabilizer of $\tilde{X}_{2m+1}[s_0, c_0]$ is $T^1 K_{m+1}$ and $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ may be viewed as a representation of $T^0 K_m / T^1 K_{m+1}$.

As in the previous section, one may consider the bipartite graph Ω whose both vertice sets identify with $\mathbf{k}$, equipped with an action of K_m on Ω^1 given by

$$\big[I_2 + \varpi^m \begin{pmatrix} a & b \\ c & d \end{pmatrix}\big] \cdot (x, y) = (x + \bar{b}, y + \bar{c}) \ ,$$

the action of T^0 being given by

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot (x, y) = (\bar{a}\bar{d}^{-1} x, \bar{d}\bar{a}^{-1} y) \ .$$

Then the contragredient of the $T^0 K_m / T^1 K_{m+1}$-module $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ is isomorphic to the space $\mathcal{H}(\Omega)$ of harmonic cochains on Ω. As in the previous section the latter space is generated by the functions $\chi_1 \otimes \chi_2$, where χ_i, $i = 1, 2$, runs over the non-trivial characters of $(k, +)$. The line $\mathbb{C}\chi_1 \otimes \chi_2$ is acted upon by K_m via the character $\alpha(\chi_1, \chi_2)$ given by

$$\alpha(\chi_1, \chi_2)\big(I_2 + \varpi^m \begin{pmatrix} a & b \\ c & d \end{pmatrix}\big) = \chi_1(b)\chi_2(c) \ .$$

It follows that $\mathcal{H}(\Omega)$ is isomorphic to its contragredient and that $H^1(\tilde{X}_{2m+1}[s_0, c_0])$ is the direct sum of the characters $\alpha(\chi_1, \chi_2)$, $\chi_i \not\equiv 1$, $i = 1, 2$.

For $\chi_i \not\equiv 1$, $i = 1, 2$, the character $\alpha(\chi_1, \chi_2)$ corresponds to a stratum of the form $[\mathrm{M}(2, \mathfrak{o}), m, m-1, \beta]$, where $\beta \in \mathrm{M}(2, F)$ is given by $\varpi^{-m} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, u, $v \in \mathfrak{o}^\times$. This stratum is either simple and non-scalar or split fundamental according to whether uv mod $\mathfrak{p}$ is a square in $\mathbf{k}^\times$ or not (here we have used the fact that $\mathrm{Char}\,(\mathbf{k}) \neq 2$.

It is clear that T^0 leaves the set of characters corresponding to simple strata (resp. split fundamental strata) stable. So we may write

$$H^1(\tilde{X}_{2m+1}[s_0, c_0]) = H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{simple}} \oplus H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{split}},$$

where $H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{simple}}$ (resp. $H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{split}}$) is the $T^0 K_m$-submodule which decomposes as a K_m / K_{m+1}-module as a direct sum of (characters corresponding to) simple non-scalar strata (resp. split fundamental strata).

We have a result similar to Proposition 5.1, whose proof uses the same arguments.

Proposition 6.1. *Let λ be an irreducible constituent of*

$$\mathrm{Ind}_{T^0 K_m}^K H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{simple}} \subset H^1(\tilde{X}_{2m+1}[s_0]) \ .$$

Then the compactly induced representation $c\text{-}\mathrm{ind}_K^G \lambda$ is irreducible, whence supercuspidal.

The study of $\mathrm{Ind}_{T^0K_m}^{K} H^1(\tilde{X}_{2m+1}[s_0,c_0])_{\mathrm{split}}$ is the aim of the next section.

We are now going to determine the K-module structure of $H^1(\tilde{X}_1[s_0])$. Set $\mathbf{G} = \mathrm{PGL}(2,\mathbf{k}) \simeq K/K^1$ and write $\mathbf{B}$ and $\mathbf{T}$ for the upper Borel subgroup and diagonal torus of $\mathbf{G}$, respectively. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. As a K-set the set of neighbour vertices of s_0 is isomorpic to $\mathbb{P}^1(\mathbf{k}) = \mathbf{G}/\mathbf{B}$.

The graph $\Omega = \tilde{X}_1[s_0]$ has for the vertex set the set of paths of the form (s,s_0) or (s_0,s) where s runs over the neighbour vertices of s_0 in X. So the space $C^0(\Omega)$ of 0-cochains identifies with the space $\mathcal{F}(\mathbb{P}^1(\mathbf{k})\coprod\mathbb{P}^1(\mathbf{k}))$ of complex-valued functions on the disjoint union $\mathbb{P}^1(\mathbf{k})\coprod\mathbb{P}^1(\mathbf{k})$, thus has a $\mathbf{G}$-module $C^0(\Omega)$ that is isomorphic to $\mathbf{1_G}\oplus\mathbf{St_G}\oplus\mathbf{1_G}\oplus\mathbf{St_G}$, where $\mathbf{1}$ denotes a trivial representation and $\mathbf{St}$ a Steinberg representation.

The $\mathbf{G}$-set Ω^1 is the set of paths of the form (s,s_0,t), where s and t are two different neighbour vertices of s_0. This $\mathbf{G}$-set is isomorphic to the quotient $\mathbf{G}/\mathbf{T}$. The space $C^{(1)}(\Omega)$ of unoriented 1-cochains identifies as a G-module with the space $\mathcal{F}(\mathbf{G}/\mathbf{T})$.

Fix a non-trivial character ψ of $\mathbf{U}$. It is well known that the induced representation $\mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$ is multiplicity free. Its irreducible constituent form is by definition the *generic* (irreducible) representations of $\mathbf{G}$. Moreover, an irreducible representation is generic if and only if it is not a character.

We have a natural G-equivariant map $\Phi : \mathcal{F}(\mathbf{G}/\mathbf{T}) \longrightarrow \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$, given by

$$\Phi(f)(g) = \sum_{u\in\mathbf{U}} f(gu)\bar{\psi}(u)\ ,\ f\in\mathcal{F}(\mathbf{G}/\mathbf{T}),\ g\in\mathbf{G}\ .$$

If a function f lies in the kernel of Φ, then we have $\sum_{u\in\mathbf{U}} f(gu)\theta(u) = 0$, for all $g\in\mathbf{G}$ and all non-trivial character θ of $\mathbf{U}$. Indeed, it suffices to use the fact that the action of $\mathbf{T}$ on $\mathbf{U}$ by conjugation acts transitively on the non-trivial characters of $\mathbf{U}$ and the right invariance of f under the action of T. So the kernel of Φ consists of the function f such that $u\mapsto f(gu)$ is a constant function on U, for all $g\in G$. In other words, $\mathrm{Ker}\,\Phi = \mathcal{F}(G/B) \simeq \mathbf{1_G}\oplus\mathbf{St_G}$. By a dimension argument, we see that Φ is surjective. It follows that

$$C^{(1)}(\Omega) \simeq \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\,\psi \oplus \mathbf{1_G}\oplus\mathbf{St_G}\ .$$

We have the cochain complex of G-modules:

$$0 \longrightarrow C^0(\Omega) \longrightarrow C^{(1)}(\Omega) \longrightarrow 0.$$

Since Ω is connected the kernel of the coboundary operator is the trivial module $\mathbb{C}$. Hence in the Grothendieck groups of G-modules, we have $dC^0(\Omega) \simeq 2.\mathbf{1_G} + 2.\mathbf{St_G} - \mathbf{1_G} = \mathbf{1_G} + 2.\mathbf{St_G}$. Therefore,

$$H^1(\Omega) = C^1(\Omega)/dC^0(\Omega) \simeq \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi + \mathbf{1_G} + \mathbf{St_G} - \mathbf{1_G} - 2.\mathbf{St_G} = \mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi - \mathbf{St_G}\ .$$

Since $q = |\mathbf{k}|$ is odd, there exists a unique non-trivial character of $\mathbf{k}^\times/(\mathbf{k}^\times)^2$, that we denote by χ_0. The irreducible constituents of the Gelfand-Graev representation $\mathrm{Ind}_{\mathbf{U}}^{\mathbf{G}}\psi$ are the following:

– the irreducible cuspidal representations of $\mathbf{G}$,

– the principal series $\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}\chi\otimes\chi^{-1}$, where $\chi : \mathbf{k}^\times \longrightarrow \mathbb{C}^\times$ is a character such that $\chi^2 \not\equiv 1$ (i.e. $\chi\notin\{\mathbf{1},\chi_0\}$).

– the steinberg representation $\mathbf{St_G}$,

– (when q is odd) the twisted representation $\mathbf{St_G}\otimes\chi_0$.

If σ is a cuspidal representation of $\mathbf{G} = K/K^1$, then the induced representation $c\text{-ind}_K^G \sigma$ is irreducible and supercuspidal ([4, (11.5), page 81]). Such a representation of G is called a *level* 0 *supercuspidal* representation.

A principal series of $\mathbf{G} = K/K^1$ may be writen as $\mathrm{Ind}_I^K \rho$, where ρ is a character of I/I^1. The pair (I, ρ) is actually a type in the sense of Bushnell and Kutzko's type theory. For technical reasons we postpone definitions and references to the next section. Since the representation $\mathrm{Ind}_I^K \rho$ is irreducible, it is a type of the same constituent as (I, ρ).

To sum up, we have proved the following.

Proposition 6.2. *An irreducible constituent λ of $H^1(\tilde{X}_1[s_0])$ is of one of the following forms:*

(i) *the inflation of a cuspidal representation of $\mathbf{G}$; in that case $c\text{-ind}_K^G \lambda$ is a level 0 irreducible supercuspidal representation of G,*

(ii) *the inflation to K of the representation $\mathbf{St}_{\mathbf{G}} \otimes \chi_0$,*

(iii) *a type of the form $\mathrm{Ind}_I^K \rho$, where the ρ is inflated from a character of $I/I^1 \simeq (\mathbf{k}^\times \times \mathbf{k}^\times)/\mathbf{k}^\times$ of the form $\chi \otimes \chi^{-1}$, $\chi^2 \not\equiv 1$.*

Note that in (iii), the pair $(K, \mathrm{Ind}_I^K \rho)$ is a principal series type.

7. The inducing representations – III

We keep the notation as in the previous section. To determine the structure of $\mathrm{Ind}_{T^0 K_m}^K H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{split}}$, we first recall crucial facts on split strata and types for principal series representations. The basic reference for *type theory* is [6].

Let χ be a character of T, that we view as a character of T^0 by restriction. Assume that the conductor of χ is $n > 0 : T^n \subset \mathrm{Ker}\,\chi$ and n is minimal for this property. Set

$$J_\chi = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o}^\times \end{pmatrix} = \Gamma_0(\mathfrak{p}^n) \ .$$

If U and $\bar{U}$ denote the groups of upper and lower unipotent matrices respectively, then J_χ has an Iwahori decomposition,

$$J_\chi = (J_\chi \cap \bar{U}) \cdot (J_\chi \cap T) \cdot (J_\chi \cap U)$$

and one may define a character ρ_χ of J_χ by

$$\rho_\chi(\bar{u} t^0 u) = \chi(t^0) \ , \ \bar{u} \in J_\chi \cap \bar{U}, \ u \in J_\chi \cap U, \ t^0 \in T^0 \ .$$

Let $\mathcal{R}_{[T,\chi]}$ be the Bernstein component of the category of smooth representations of G whose objects are the representations $\mathcal{V}$ satisfying the following property: Any irreducible subquotient of $\mathcal{V}$ occurs in a parabolically induced representation $\mathrm{Ind}_{\mathcal{B}}^G(\chi \otimes \chi_0)$, where $\mathcal{B}$ is a Borel subgroup with Levi component T and χ_0 an unramified character of T. We then have:

Theorem 7.1 (A. Roche)**.** *The pair (J_χ, ρ_χ) is a type for $\mathcal{R}_{[T,\chi]}$.*

This is indeed Theorem (7.7) of [11]. Note that our J_χ is not exactly the same as Roche's, but a conjugate under an element of T (see [11, Example (3.5)]).

Proposition 7.2. *With the notation as before, assume that $\chi_{|T^0}$ is not of the form $\alpha \circ \mathrm{Det}$, where α is a character of $\mathfrak{o}^\times$ (necessarily of order 2). Then the induced representation $\mathrm{Ind}_{J_\chi}^K \rho_\chi$ is irreducible. In particular, it is a type for $\mathcal{R}_{[T,\chi]}$.*

Proof. Let W be the extended affine Weyl group of G w.r.t. T and set $W_\chi = \{w \in W\ ;\ w\chi = \chi\}$. Then by Theorem (4.14) of [11], the G-intertwining of ρ_χ is $J_\chi W_\chi J_\chi$. The hypothesis on χ forces $W_\chi = T/T^0$. So $(J_\chi W_\chi J_\chi) \cap K = J_\chi T^0 J_\chi = J_\chi$, and we may apply Mackey's criterion of irreducibility. □

For $n > 0$ and $q \in \{0, ..., n\}$, define compact open subgroups of G as follows:

$$ {}_q\mathfrak{h}_1 = \begin{pmatrix} 1+\mathfrak{p}^n & \mathfrak{p}^q \\ \mathfrak{p}^{n+1} & 1+\mathfrak{p}^n \end{pmatrix} \text{ and } {}_q\mathfrak{h}_2 = \begin{pmatrix} 1+\mathfrak{p}^{n+1} & \mathfrak{p}^q \\ \mathfrak{p}^{n+1} & 1+\mathfrak{p}^{n+1} \end{pmatrix}. $$

These groups are particular cases of groups considered in [1, §(2.3)]. The quotients ${}_q\mathfrak{h}_1/{}_q\mathfrak{h}_2$, $q = 0, ..., n$, are abelian, and for $\alpha \in \mathbf{k}^\times$, one may define a character ψ_α of ${}_q\mathfrak{h}_1/{}_q\mathfrak{h}_2$ by the formula

$$ \psi_\alpha\big(I_2 + \begin{pmatrix} \varpi^n a & \varpi^q b \\ \varpi^{n+1} c & \varpi^n d \end{pmatrix}\big) = \psi(\alpha(a-d)), $$

where ψ is a fixed non-trivial character of $(\mathbf{k}, +)$. We shall need the following result.

Lemma 7.3. *If a smooth representation of K contains $(\psi_\alpha)_{|{}_n\mathfrak{h}_1}$ by restriction, then it contains the character $(\psi_\alpha)_{|{}_0\mathfrak{h}_1}$.*

Proof. Since the characteristic of $\mathbf{k}$ is not 2, then $\alpha \neq -\alpha$ and $(\psi_\alpha)_{|{}_n\mathfrak{h}_1}$ is the restriction to ${}_n\mathfrak{h}_1$ of a split fundamental stratum of K_n/K_{n+1}. Our lemma is then a particular case of [1, Lemma (2.4.5)]. □

Proposition 7.4. *Let λ be an irreducible constituent of $\mathrm{Ind}_{T^0K_m}^K H^1(\tilde{X}_{2m+1}[s_0, c_0])_{\mathrm{split}}$. Then with the notation as above, λ is of the form $\mathrm{Ind}_{J_\chi}^K \rho_\chi$, for some principal series type (J_χ, ρ_χ) with χ of conductor $m+1$.*

Proof. We know that such a λ contains a split fundamental stratum of the form $[\mathrm{M}(2, \mathfrak{o}), m, m-1, b]$, where $b = \varpi^{-m}\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, $u, v \in \mathfrak{o}^\times$, and uv is a square *modulo* $\mathfrak{p}$. If $\alpha \in \mathfrak{o}$ is such that $\alpha^2 \equiv uv$ *mod* $\mathfrak{p}$, then the stratum is equivalent to a K-conjugate of $[\mathrm{M}(2, \mathfrak{o}), m, m-1, b']$, where $b' = \varpi^{-m}\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$. So we deduce that λ contains this latter stratum by restriction. Now consider the group ${}_n\mathfrak{h}_1$ for $n = m$. The representation λ contains the character $(\psi_\alpha)_{|{}_n\mathfrak{h}_1}$ by restriction. By applying Lemma 7.3 we obtain that it contains the character $(\psi_\alpha)_{|{}_0\mathfrak{h}_1}$. This character clearly extends to $T^0{}_0\mathfrak{h}_1 = \Gamma_0(m+1, 0)$ and the quotient $T^0{}_0\mathfrak{h}_1/{}_0\mathfrak{h}_1$ is abelian. It follows that λ contains an extension of ψ_α to $\Gamma_0(m+1, 0)$. Such an extension is of the form (J_χ, ρ_χ), for some character χ of T of conductor $m+1$. The fact that λ is induced from (J_χ, ρ_χ) follows from Proposition 7.2. □

8. Synthesis

We now prove Theorems A and B of the Introduction.

By Propositions 4.2 and 4.4, we have isomorphisms of G-modules:

$$ H_c^1(\tilde{X}_{2m}) \simeq H_c^1(\tilde{X}_{2m-1}) \oplus c\text{-}\mathrm{ind}_{\mathcal{K}_1}^G H^1(\Sigma_{2m}),\ m \geq 1, \tag{8.1} $$

$$ H_c^1(\tilde{X}_{2m+1}) \simeq H_c^1(\tilde{X}_{2m}) \oplus c\text{-}\mathrm{ind}_{\mathcal{K}_0}^G H^1(\Sigma_{2m+1}),\ m \geq 0. \tag{8.2} $$

Recall that with the notation of the introduction, we have:

– $\Sigma_{2m} = \tilde{X}_{2m}[e_0]$, $\Sigma_{2m+1} = \tilde{X}_{2m+1}[s_0]$,

– $\mathcal{K}_0 = \mathcal{K}_{s_0}$, $\mathcal{K}_1 = \mathcal{K}_{e_0}$.

Moreover, by Proposition 5.2, we have

$$H^1_c(\tilde{X}_0) \simeq \mathrm{St}_G \oplus c\text{-}\mathrm{ind}^G_{\mathcal{K}_1} H^1(\Sigma_0) \tag{8.3}$$

so that (1) holds for $m = 0$. Hence, Theorem A follows from (8.1) and (8.2) by a straightforward inductive argument.

Theorem B follows from the discription of the irreducible components of $H^1(\Sigma_n)$ given in Proposition 5.1 (n even and $n > 0$), Proposition 5.2 ($n = 0$), and Propositions 6.1 and 7.4 (n odd).

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Département de Mathématiques, Université de Poitiers, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France

E-mail address: paul.broussous@math.univ-poitiers.fr