

COHOMOLOGY OF DELIGNE-LUSZTIG VARIETIES FOR UNIPOTENT BLOCKS OF $GL_n(q)$

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ABSTRACT. We study the cohomology of parabolic Deligne-Lusztig varieties associated to unipotent blocks of $GL_n(q)$. We show that the geometric version of Broué's conjecture over $\overline{\mathbb{Q}}_\ell$, together with Craven's formula, holds for any unipotent block whenever it holds for the principal Φ_1 -block.

INTRODUCTION

Let \mathbf{G} be a connected reductive algebraic group over $\mathbb{F} = \overline{\mathbb{F}}_p$ with an \mathbb{F}_q -structure associated to a Frobenius endomorphism F . Let ℓ be a prime number different from p and let b be a unipotent ℓ -block of $\mathbf{G}^F = \mathbf{G}(\mathbb{F}_q)$. When ℓ is large, the defect group of b is abelian, and the geometric version of Broué's conjecture predicts that the cohomology of some Deligne-Lusztig variety should induce a derived equivalence between b and its Brauer correspondent [2].

When ℓ is assumed to be large (e.g., in the split case, when ℓ does not divide the order of the Weyl group of \mathbf{G}), it divides a unique cyclotomic polynomial $\Phi_d(q)$. In that situation many aspects of unipotent ℓ -blocks (such as the ordinary characters, the number of simple modules, the defect...) depend only on d . This suggests to study a generic version of block theory for finite reductive groups. Following this approach, Broué, Malle and Michel developed in [3] the theory of unipotent Φ_d -blocks and established a classification in terms of so-called d -cuspidal pairs. More precisely, to a Φ_d -block b they associate a pair (\mathbf{L}, χ) where \mathbf{L} is the centralizer of a Φ_d -torus and χ is a d -cuspidal character, such that the unipotent characters in the Φ_d -block are exactly the irreducible constituents of the Deligne-Lusztig induction $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$ of χ . This generalizes the case $d = 1$ where the Φ_1 -blocks correspond to Harish-Chandra series.

Since the Deligne-Lusztig induction $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$ of χ comes from the ℓ -adic cohomology of a Deligne-Lusztig variety, Broué conjectured in [2] that not only the characters of the block but also its whole structure should be encoded in the cohomology complex of a Deligne-Lusztig variety attached to \mathbf{L} . However, unlike the virtual character $R_{\mathbf{L}}^{\mathbf{G}}(\chi)$, this complex depends on a choice of a parabolic subgroup with Levi complement \mathbf{L} , which in turn depends on a choice of an element in a given conjugacy class of the Weyl group of \mathbf{G} . When \mathbf{L} is a torus, Broué and Michel identified in [4] which elements should be considered. They correspond to good d -regular elements or equivalently to elements of the Weyl group which lift in the braid monoid to certain d -roots of $\pi = \mathbf{w}_0^2$, where \mathbf{w}_0 is the lift of the longest

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element of the Weyl group. In a recent work [9], Digne and Michel introduced the notion of d -periodic elements (see §1.1 for the definition) to generalise this to the parabolic setting, when \mathbf{L} is no longer a torus. If b is a unipotent Φ_d -block, then it is to be expected that there exists a d -periodic element (\mathbf{I}, \mathbf{w}) such that the corresponding parabolic Deligne-Lusztig variety $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ is a good candidate for inducing the derived equivalence predicted by Broué’s conjecture. Furthermore, Chuang and Rouquier conjectured in [6] that this equivalence is perverse, with a perversity function that has recently been conjectured by Craven in [7]. Surprisingly, it can be expressed by a function C_d depending only on the generic degrees of the corresponding characters.

If we restrict our attention to the characteristic zero, then we obtain a conjectural explicit description of the unipotent part of the cohomology of $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ by means of the function C_d defined by Craven. The fundamental property that one derives from Broué’s conjecture is that the cohomology groups of the variety $X(\mathbf{I}, \mathbf{w}F) = \tilde{X}(\mathbf{I}, \mathbf{w}F)/\mathbf{L}_T^{\mathbf{w}F}$ are mutually disjoint. More precisely, it can be formulated as follows:

Conjecture 1. *Any d -cuspidal pair is conjugate to a pair $(\mathbf{L}_T^{\mathbf{w}F}, \chi)$ where (\mathbf{I}, \mathbf{w}) is a d -periodic element and χ is a unipotent character of $\mathbf{L}_T^{\mathbf{w}F}$. Moreover, if \mathcal{F}_χ is the corresponding $\overline{\mathbb{Q}}_\ell$ -local system on $X(\mathbf{I}, \mathbf{w}F)$, then (\mathbf{I}, \mathbf{w}) can be chosen such that:*

- (i) *The $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$ -modules $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$ are mutually disjoint.*
- (ii) *If ρ is an irreducible unipotent constituent of $H^i(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi)$, then ρ lies in the Φ_d -block associated to χ and $i = C_d(\deg \rho / \deg \chi)$.*

In addition, the endomorphism algebra $\text{End}_{\mathbf{G}^F}(H^\bullet(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_\chi))$ should be endowed with a natural structure of d -cyclotomic Hecke algebra. Let us note the following important consequence of this property: the eigenvalue of any sufficiently divisible power F^m of F on ρ should be $q^{m(a_\rho + A_\rho - a_\chi - A_\chi)/d}$.

The choice of a specific d -periodic element in this conjecture is not very relevant: it is conjectured that any other d -periodic element $(\mathbf{I}, \mathbf{w}')$ can be obtained from (\mathbf{I}, \mathbf{w}) by cyclic shifts, so that the cohomology of the corresponding varieties are isomorphic. This has already been proven when $\mathbf{I} = \emptyset$ and F acts trivially on W (see [9, Remark 7.4]).

When $d = 1$, the unipotent blocks correspond to the usual Harish-Chandra series. In particular, when $\mathbf{G}^F = \text{GL}_n(q)$, there is a unique unipotent block and it contains all the unipotent characters. The purpose of this paper is to show that from the cohomology of $X(\boldsymbol{\pi})$ one can actually deduce all the other interesting cases (see Corollary 3.2):

Theorem. *For $\text{GL}_n(q)$, Conjecture 1 holds whenever it holds for $d = 1$, that is, for the variety $X(\boldsymbol{\pi})$.*

Let us emphasize that Conjecture 1 is known to be true only in a very small number of cases, namely when $d = h$ is the Coxeter number by Lusztig [12], for groups of rank 2 by Digne, Michel and Rouquier [10] and when $d = n - 1$ for $\text{GL}_n(q)$ and $d = 4$ for groups of type D_4 by Digne and Michel [8]. Therefore, this theorem represents a very important step towards a proof of the geometric version of Broué’s conjecture.

Even though this result depends on the conjectural description of $X(\boldsymbol{\pi})$, one can give an effective proof of Conjecture 1 for principal Φ_d -blocks of $\text{GL}_n(q)$ when

$d > n/2$. In that case the defect group is cyclic, and the modular representation theory of the block is fully understood. We will address this problem in a subsequent paper, where we will compute the cohomology over $\overline{\mathbb{Z}}_\ell$ of the corresponding Deligne-Lusztig variety.

To give a flavour of the proof of the main theorem, recall that for $\mathrm{GL}_n(q)$, we have by [3] a combinatorial description of the Deligne-Lusztig induction associated to $\tilde{X}(\mathbf{I}, \mathbf{w}F)$. In terms of partitions, it corresponds to adding a certain number of d -hooks. By transitivity, one can decompose $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ by means of simpler varieties $\tilde{X}_{n,d}$, each of which corresponds to adding a single d -hook to a partition. Now, using the methods developed in [11] one can compute the cohomology of (some quotient of) $\tilde{X}_{n,d}$ in terms of $\tilde{X}_{n-1,d}$ and $\tilde{X}_{n-1,d-1}$ (see Theorem 2.1), providing an inductive argument to tackle Conjecture 1.

Finally, note that Theorem 2.1 can be generalised to many other situations in type B , C and D . However, two main problems arise: first, the limit case is either $X(w_0)$ or $X(\pi)$ and does not contain all the unipotent characters. Second, the methods in [11] work obviously for non-cuspidal unipotent characters only. We can obtain partial results on the principal series in that situation, which we believe are too coarse to be mentioned in this paper.

1. PARABOLIC VARIETIES FOR $\mathrm{GL}_n(q)$

The purpose of this section is to give the precise form that Conjecture 1 takes for $\mathrm{GL}_n(q)$. We shall specify which elements of the braid monoid should be considered and give explicit values for Craven's formula (see Conjecture 1.5).

Throughout this paper, \mathbf{G} will denote the general linear group $\mathrm{GL}_n(\mathbb{F})$ over $\mathbb{F} = \overline{\mathbb{F}}_p$. We will consider the split \mathbb{F}_q -structure on \mathbf{G} defined by the Frobenius endomorphism $F : (a_{i,j}) \in \mathbf{G} \mapsto (a_{i,j}^q) \in \mathbf{G}$ so that $\mathbf{G}^F = \mathrm{GL}_n(q)$. Note that for us only the isogeny class of (\mathbf{G}, F) will be relevant since we will be working with unipotent characters only. If \mathbf{H} is any F -stable subgroup of \mathbf{G} we will denote by $H = \mathbf{H}^F$ the associated finite group.

1.1. Braid group and Deligne-Lusztig varieties. The Weyl group W of \mathbf{G} is the symmetric group \mathfrak{S}_n and its Braid monoid B^+ is the usual Artin monoid. By convention we shall use bold letters to denote elements of B^+ . The simple reflections of W corresponding to the transpositions $(i, i+1)$ will be denoted by s_1, \dots, s_{n-1} . Their lift to the braid monoid, denoted by $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_{n-1}\}$, form a set of generators of B^+ .

Given a subset I of simple reflections, we denote by \mathbf{P}_I the standard parabolic subgroup containing the Borel subgroup of non-singular upper-triangular matrices, and by \mathbf{U}_I its unipotent radical. The group \mathbf{P}_I has a unique Levi complement \mathbf{L}_I containing the group of non-singular diagonal matrices. The parabolic subgroup of W generated by I , denoted by W_I , is the Weyl group of \mathbf{L}_I . Let $w \in W$. Assume that w is I -reduced and that ${}^w I = I$. Then the parabolic Deligne-Lusztig variety associated with the pair (I, w) is

$$X(I, wF) = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{P}_I w \mathbf{P}_I\} / \mathbf{P}_I.$$

This definition can be extended to pairs (\mathbf{I}, \mathbf{b}) where \mathbf{I} is a subset of \mathbf{S} and $\mathbf{b} \in B^+$ satisfy some properties that we shall not give since they will be automatic for d -periodic elements. The corresponding Deligne-Lusztig variety will be denoted by

$X(\mathbf{I}, \mathbf{b}F)$. In the particular case where $\mathbf{b} = \mathbf{w}$ is the lift of an element $w \in W$, then $X(\mathbf{I}, \mathbf{b}F)$ is canonically isomorphic to $X(I, wF)$. Following [9], we shall also fix an F -stable Tits homomorphism $t : B^+ \rightarrow N_{\mathbf{G}}(\mathbf{L}_{\emptyset})$. Then the variety $X(\mathbf{I}, \mathbf{b}F)$ has an étale covering $\tilde{X}(\mathbf{I}, \mathbf{b}F)$ with Galois group $\mathbf{I}_I^{t(\mathbf{b})F}$.

Let w_0 (resp. w_I) be the longest element of W (resp. W_I). The corresponding lift in the braid monoid will be denoted by \mathbf{w}_0 (resp. \mathbf{w}_I) and we set $\boldsymbol{\pi} = \mathbf{w}_0^2$ (resp. $\boldsymbol{\pi}_I = \mathbf{w}_I^2$). The following definition is given in [9].

Definition 1.1. *A d -periodic element is a pair (\mathbf{I}, \mathbf{b}) with $\mathbf{I} \subset \mathbf{S}$ and $\mathbf{b} \in B^+$ such that $\mathbf{b}F(\mathbf{b}) \cdots F^{d-1}(\mathbf{b}) = \boldsymbol{\pi}/\boldsymbol{\pi}_I$.*

Note that in our case (\mathbf{G}, F) is split so that the condition becomes $\mathbf{b}^d = \boldsymbol{\pi}/\boldsymbol{\pi}_I$.

Let $1 \leq d \leq n$. For studying Φ_d -blocks of $\text{GL}_n(q)$ we shall consider the following pairs, defined in [9]:

$$\mathbf{v}_d = \mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_{n-1-\lfloor \frac{d}{2} \rfloor} \mathbf{s}_{n-1} \mathbf{s}_{n-2} \cdots \mathbf{s}_{\lfloor \frac{d+1}{2} \rfloor} \in B^+,$$

$$\mathbf{J}_d = \{ \mathbf{s}_i \mid \lfloor \frac{d+1}{2} \rfloor + 1 \leq i \leq n - 1 - \lfloor \frac{d}{2} \rfloor \} \subset \mathbf{S}.$$

We are interested in computing the cohomology of the variety

$$X_{n,d} = X(\mathbf{J}_d, \mathbf{v}_d F)$$

with coefficients in any unipotent local system. Note that for $d > 1$, the element \mathbf{v}_d is reduced. Therefore, it is the lift of an element $v_d \in W$ and we can work with the variety $X(J_d, v_d F)$ instead. By [9, Lemma 11.7 and 11.8], the pair $(\mathbf{J}_d, \mathbf{v}_d)$ is d -periodic so that it makes sense to study the cohomology of $X_{n,d}$. The Galois covering $\tilde{X}(\mathbf{J}_d, \mathbf{v}_d F)$ of $X_{n,d}$ will be denoted by $\tilde{X}_{n,d}$.

When $d \leq n/2$, \mathbf{v}_d is not maximal in the sense that it is extendable to a d th root of $\boldsymbol{\pi}/\boldsymbol{\pi}_I$ for a proper subset \mathbf{I} of \mathbf{J}_d . However, it can still be used to associate to any unipotent block a “good” parabolic Deligne-Lusztig variety. Before making any precise statement, we shall briefly recall the combinatorial objects that we will use.

1.2. Φ_d -blocks of $\text{GL}_n(q)$. The unipotent characters of $\text{GL}_n(q)$ are labeled by the partitions of n . If λ is such a partition, we will denote by χ_λ the corresponding character, with the convention that $\chi_{(1,1,\dots,1)} = \text{St}_G$ is the Steinberg character of $G = \text{GL}_n(q)$. We shall also fix a representation V_λ over $\overline{\mathbb{Q}}_\ell$ of character χ_λ . For $1 \leq d \leq n$, the pair $(\mathbf{L}_{J_d}, v_d F)$ represents a d -Levi subgroup of \mathbf{G} . From [3], we know how to express the d -Harish-Chandra induction in terms of combinatorics of partitions. To fix the notation, let μ be a partition of $n - d$ and let $X = \{x_1 < x_2 < \cdots < x_s\}$ be a β -set associated to μ . We may and we will assume that X is big enough, so that it contains $\{0, 1, \dots, d - 1\}$. Let X' be the subset of X defined by $X' = \{x \in X \mid x + d \notin X\}$. It represents the possible d -hooks that can be added to μ . For $x \in X'$ we will denote by $\mu * x$ the partition of n which has $(X \setminus \{x\}) \cup \{x + d\}$ as a β -set.

The pair $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$ is a split Levi subgroup of type A_{n-1-d} . Consequently, the partition μ defines a unipotent local system \mathcal{F}_μ on $X_{n,d}$ such that $\mathbf{H}_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ and $\mathbf{H}_c^\bullet(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu}$ are isomorphic. Then we deduce from [3, Section 3.4] that there exist signs $\varepsilon_x = \pm 1$ such that the d -Harish-Chandra induction of χ_μ is given by

$$R\Gamma_{\mathbf{L}_{J_d}}^{\mathbf{G}}(\chi_{\mu}) = \sum (-1)^i H_c^i(X_{n,d}, \mathcal{F}_{\mu}) = \sum_{x \in X'} \varepsilon_x \chi_{\mu * x}.$$

In particular, the d -Harish-Chandra restriction of $\chi_{\lambda} \in \text{Irr GL}_n(q)$ is non-zero until we reach the d -core ν of λ , which corresponds to a d -cuspidal character χ_{ν} . The unipotent characters in the Φ_d -block of $\text{GL}_n(q)$ containing χ_{λ} are all the characters that can be obtained by successive d -inductions from χ_{ν} . They correspond to partitions of n that have ν as a d -core.

1.3. A parabolic variety associated to a Φ_d -block. The cohomology of the variety $X_{n,d}$ corresponds to a minimal d -induction since there is no d -split Levi subgroup between $(\mathbf{L}_{J_d}, t(\mathbf{v}_d)F)$ and (\mathbf{G}, F) . By transitivity, one can form a Deligne-Lusztig variety $X(\mathbf{I}, \mathbf{w}F)$ associated to the d -cuspidal character χ_{ν} . Let $n - ad$ be the size of ν and consider for $i = 1, \dots, a$ the pairs $(\mathbf{J}_d^{(i)}, \mathbf{v}_d^{(i)})$ where $(\mathbf{J}_d^{(1)}, \mathbf{v}_d^{(1)}) = (\mathbf{J}_d, \mathbf{v}_d)$ and $(\mathbf{J}_d^{(i+1)}, \mathbf{v}_d^{(i+1)})$ is the analogue of the pair $(\mathbf{J}_d, \mathbf{v}_d)$ for the split group $(\mathbf{L}_{J_d^{(i)}}, t(\mathbf{v}_d^{(i)} \cdots \mathbf{v}_d^{(1)})F)$ of type A_{n-1-id} . Then one can readily check that the pair $(\mathbf{I}, \mathbf{w}) = (\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)} \cdots \mathbf{v}_d^{(1)})$ is d -periodic.

By [9, Proposition 8.26] the associated Deligne-Lusztig variety $\tilde{X}(\mathbf{I}, \mathbf{w}F)$ is isomorphic to the following amalgamated product

$$\tilde{X}(\mathbf{J}_d^{(1)}, \mathbf{v}_d^{(1)}F) \times_{\mathbf{L}_{J_d^{(1)}}}^{t(\mathbf{v}_d^{(1)})F} \cdots \times_{\mathbf{L}_{J_d^{(a-1)}}}^{t(\mathbf{v}_d^{(a-1)} \cdots \mathbf{v}_d^{(1)})F} \tilde{X}_{\mathbf{L}_{J_d^{(a-1)}}}(\mathbf{J}_d^{(a)}, \mathbf{v}_d^{(a)}t(\mathbf{v}_d^{(a-1)} \cdots \mathbf{v}_d^{(1)})F).$$

Now each variety in this decomposition corresponds to a variety $\tilde{X}_{n-(i-1)d,d}$ for some $i = 1, \dots, a$. Since the cohomology of the latter with coefficients in a unipotent local system depends only on the isogeny class of the group (here, the split type A_{n-1-id}) we obtain the following description of the cohomology complex of $X(\mathbf{I}, \mathbf{w}F)$

$$(1.2) \quad R\Gamma_c(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_{\nu}) \simeq R\Gamma_c(\tilde{X}_{n,d}, \overline{\mathbb{Q}}_{\ell}) \otimes_{A_{n-1-d}} \cdots \otimes_{A_{n-1-(a-1)d}} R\Gamma_c(X_{n-(a-1)d,d}, \mathcal{F}_{\nu}).$$

Consequently, if we believe that the vanishing property in Conjecture 1 holds for the cohomology of $X(\mathbf{I}, \mathbf{w}F)$, then for any partition μ of $n - d$, the graded G -module $H_c^{\bullet}(X_{n,d}, \mathcal{F}_{\mu})$ should be multiplicity-free.

1.4. Craven’s formula for $\text{GL}_n(q)$. Let λ be a partition of n and ν its d -core. Conjecturally, the unipotent character χ_{λ} is a constituent of only one cohomology group of the complex $R\Gamma(X(\mathbf{I}, \mathbf{w}F), \mathcal{F}_{\nu})$. Craven proposed in [7] a formula which gives the degree of this cohomology group in terms of d and the generic degrees of χ_{λ} and χ_{ν} . More precisely, he considers a function C_d on some set of enhanced cyclotomic polynomials and conjectured that

$$(1.3) \quad \langle \chi_{\lambda}; H^i(X(\mathbf{I}, \mathbf{w}), \mathcal{F}_{\nu}) \rangle_G \neq 0 \iff i = C_d(\deg \chi_{\lambda}) - C_d(\deg \chi_{\nu}).$$

Let us recall the definition of C_d : Assume that $P \subset \mathbb{Q}[x]$ is a polynomial such that the non-zero roots z_1, \dots, z_m (written with multiplicity) of P are all roots of unity. Let us denote by $d^{\circ}(P)$ the degree of P and by $v(P)$ its valuation, that is, the

degree of $x^{d^\circ(P)}P(x^{-1})$. Then Craven’s function C_d is defined by

$$C_d(P) = \frac{1}{d} (d^\circ(P) + v(P)) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < 2\pi/d\} - \frac{1}{2} \#\{i = 1, \dots, m \mid z_i = 1\}.$$

Here, the argument $\text{Arg } z$ of a non-zero complex number z is taken in $[0; 2\pi)$. More generally, if $\zeta = \exp(2i\pi k/d)$ is a primitive d -root of unity, one can define a function C_ζ by replacing d by d/k and 1.3 should hold for d th roots of $(\pi/\pi_I)^k$. Note also that Craven’s function is additive: it satisfies $C_\zeta(PQ) = C_\zeta(P) + C_\zeta(Q)$.

For $\text{GL}_n(q)$, the degree $\text{deg } \chi_\lambda$ of the unipotent character χ_λ is explicitly known (see for example [5, Section 13]). It is a polynomial in q , whose valuation and degree are given by Lusztig’s functions a_λ and A_λ . Explicit formulae for these functions can be found for example in [13, §4.4]. Moreover, no factors of the form $(q - 1)$ can appear. In particular, Craven’s function can be written

$$C_\zeta(\text{deg } \chi_\lambda) = \frac{2\pi}{\text{Arg } \zeta} (a_\lambda + A_\lambda) + \#\{i = 1, \dots, m \mid \text{Arg } z_i < \text{Arg } \zeta\},$$

where z_1, \dots, z_m are the roots with multiplicity of the polynomial $\text{deg } \chi_\lambda$. Note that with this description it is already not obvious that the rational number on the right-hand side of (1.3) is actually an integer.

Since C_d is additive, formula (1.3) together with the quasi-isomorphism 1.2 suggests that the partition ν does not necessarily need to be a d -core in order for Craven’s formula to hold. In the case of a minimal d -induction (when $a = 1$) we can write everything explicitly using [7, Proposition 9.1]; the second equality follows from an easy calculation:

Lemma 1.4. *Let μ be a partition with corresponding β -set X , which we assume to be large enough. Let $s = |X|$ and $X' = \{x \in X \mid x + d \notin X\}$. For $x \in X'$, we have*

$$C_d(\text{deg } \chi_{\mu*x}) - C_d(\text{deg } \chi_\mu) = 2(n - d - x + \#\{y \in X \mid y < x\}) + \#\{y \in X \mid x < y < x + d\}$$

and

$$a_{\mu*x} + A_{\mu*x} - a_\mu - A_\mu = d(n - 1 - d + s - x).$$

These integers give conjecturally the degree of the cohomology group of $X_{n,d}$ in which $\chi_{\mu*x}$ will appear, as well as the corresponding eigenvalue of the Frobenius. Since we will work with the cohomology with compact support, we shall work with the integers

$$\pi_d(X, x) = 2(n - 1 + x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x + d\}$$

and

$$\gamma_d(X, x) = n + x - s.$$

They are readily deduced from the previous ones by taking into account the dimension of $X_{n,d}$ (which is equal to $\ell(v_d) = 2n - 1 - d$). Now Conjecture 1 can be deduced from the following:

Conjecture 1.5. *Let $n \geq 1$ be a positive integer and $1 \leq d \leq n$. Let μ be a partition of $n - d$ and let X be its β -set, assumed to be large enough. Then*

$$\text{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu) \simeq \bigoplus_{x \in X'} V_{\mu*x} [-\pi_d(X, x)] \otimes \overline{\mathbb{Q}}_\ell(\gamma_d(X, x))$$

as a complex of $G \times \langle F \rangle$ -modules, where $\overline{\mathbb{Q}}_\ell(m)$ denotes the one-dimensional $\langle F \rangle$ -module on which F acts by q^m .

The purpose of this paper is to prove that this conjecture holds for any d whenever it holds for $d = 1$. As a byproduct, we shall deduce the cohomology of parabolic Deligne-Lusztig varieties associated to any unipotent block from the knowledge of the cohomology of $X(\mathbf{w}_0^2)$.

2. DECOMPOSITION OF THE QUOTIENT

The group $(\mathbf{L}_{J_d}, \dot{v}_d F)$ is split for type A_{n-1-d} ; therefore, the unipotent representations of the corresponding finite group are labelled by partitions μ of $n-d$. To such a partition one can associate a unipotent $\overline{\mathbb{Q}}_\ell$ -local system \mathcal{F}_μ on $X_{n,d}$. From [3] we know that the irreducible constituents of the virtual character $\sum (-1)^i H_c^i(X_{n,d}, \mathcal{F}_\mu)$ correspond to the partitions of n obtained by adding a d -hook to μ . The restriction to \mathfrak{S}_n of the corresponding irreducible representation corresponds to a partition obtained by

- either restricting the hook (usually in two different ways),
- or restricting μ .

The main result of this section gives a geometric interpretation of this phenomenon.

Theorem 2.1. *Assume that $d \geq 2$. Let $I = \{s_j \mid 1 \leq j \leq n-2\}$. Let μ be a partition of $n-d$ and $\{\mu^{(j)}\}_j$ the set of partitions of $n-1-d$ obtained by restricting μ . Then there is a distinguished triangle in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$:*

$$\begin{aligned} \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) &\longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \\ &\longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus_j \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow . \end{aligned}$$

Remark 2.2. From [1, Proposition 1.1] we can deduce that the cohomology of a Deligne-Lusztig variety with coefficients in a unipotent local system depends only on the type of (\mathbf{G}, F) . Therefore, there is no ambiguity in the statement of the theorem.

We will use the results in [11] to compute the quotient of $\widetilde{X}(J_d, \dot{v}_d F)$ by the finite group U_I . Recall that $X_{n,d} = X(J_d, v_d F)$ can be decomposed into locally closed P_I -subvarieties X_z , where z is an I -reduced- J_d element of W . In our situation, at most two pieces will appear:

Lemma 2.3. *Assume that $2 \leq d \leq n-1$. The variety X_z is non-empty if and only if z is one of the two following elements:*

- (1) $z_0 = s_{n-1} s_{n-2} \cdots s_1$,
- (2) $z_1 = s_{n-1} s_{n-2} \cdots s_{n-\lfloor \frac{d}{2} \rfloor}$.

Proof. For simplicity, we shall denote $a = \lfloor \frac{d+1}{2} \rfloor$ and $b = n-1 - \lfloor \frac{d}{2} \rfloor$ so that $w = v_d = s_1 \cdots s_b s_{n-1} \cdots s_a$ and $J = J_d = \{a+1, \dots, b\}$. If z is a I -reduced- J element, then $z = s_{n-1} s_{n-2} \cdots s_i$ with $b+1 \leq i \leq n$ or $1 \leq i \leq a$. Recall from [11] that the variety X_z is non-empty if and only if there exists $y = y_1 \cdots y_r \in W_J$ and a z -distinguished subexpression γ of yw such that the products of the elements of γ lie in $(W_I)^z$. We first observe that for $i \notin \{1, n\}$ we have

$$(W_I)^z = \langle s_1, \dots, s_{i-2}, s_i s_{i-1} s_i, s_{i+1}, \dots, s_{n-1} \rangle.$$

Now, since z is reduced- J , the subexpression γ is the concatenation of (y_1, \dots, y_r) and an yz -distinguished subexpression $\tilde{\gamma}$ of w . If $i > b + 1$ or $2 \leq i \leq a$, the group W_J is included in $(W_I)^z$. Therefore, the product of the elements of $\tilde{\gamma}$ must lie in $(W_I)^z$. We shall distinguish two cases:

Case (1). We assume that $i > b + 1$. In that case z commutes with any element of W_J , so that $\tilde{\gamma}$ is a yz -distinguished subexpression of w . Then

- if z is trivial (that is, if $i = n$), then any y -distinguished subexpression of w contains necessarily s_{n-1} and hence cannot produce any element of $(W_I)^z = W_I$;
- if z is non-trivial, then $i \leq n - 1$, and a subexpression of w lies in $(W_I)^z$ if and only if it does not contain s_i or s_{i-1} . However, such a subexpression will never be yz -distinguished since for all v in W_I we have $vz s_{i-1} > vz$.

We deduce that the variety X_z is empty in this case.

Case (2). We assume that $2 \leq i \leq a$. The subexpression $\tilde{\gamma}$ is ${}^z yz$ -distinguished. Since $i \leq a$, we have ${}^z W_J = W_{a, \dots, b-1}$. For $j < i - 1$, we have $z s_j = s_j z$; moreover, $z s_{i-1}$ is I -reduced, so that $\tilde{\gamma}$ should start with (s_1, \dots, s_{i-1}) . In that case, the product of the elements of $\tilde{\gamma}$ cannot belong to $(W_I)^z$. Indeed, a subexpression of $s_{i-1} s_i \cdots s_b s_{n-1} \cdots s_a$ starting with s_{i-1} will never give an element of $(W_I)^z$, the only non-trivial situation being the case $a = i$:

- with the notation in [10, Section 2.1.2] we have $s_{a-1} \underline{s_{a+1} \cdots s_b s_{n-1} \cdots s_a} = \underline{s_{a+1} \cdots s_b s_{n-1} \cdots s_{a+1} s_{a-1} s_a}$ and neither s_{a-1} nor $s_{a-1} s_a$ belongs to $(W_I)^z$;
- $s_{a-1} s_a \underline{s_{a+1} \cdots s_b s_{n-1} \cdots s_a} = (s_{a-1} s_a s_{a-1}) s_{a-1} \underline{s_{a+1} \cdots s_b s_{n-1} \cdots s_a}$ and we are back to the previous case.

This forces the variety X_z to be empty. □

Proof of the Theorem. From the previous lemma we deduce that $\tilde{X}(J_d, v_d F)$ decomposes as a disjoint union $\tilde{X}(J_d, v_d F) = \tilde{X}_{z_0} \cup \tilde{X}_{z_1}$ with \tilde{X}_{z_0} being open. Using [11] we shall now determine the cohomology of the quotient of each of these varieties by U_I . Throughout the proof, we will denote $\tilde{I} = \{s_2, \dots, s_{n-1}\} \subset S$ the conjugate of I by w_0 .

When $z = z_0 = w_I w_0$ and $d > 2$ we are in the situation of [11, Proposition 3.4]. Indeed, $v_d = s_1 w'$ with $w' \in W_{2, \dots, n-1}$ and s_1 commutes with $W_{J_d} \subset W_{3, \dots, n-2}$ so that we obtain

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{z_0} / N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(K_{z_0}, \dot{v}F) / {}^{z_0} N', \overline{\mathbb{Q}}_\ell)$$

with $v = {}^{z_0} w'$ and $K_{z_0} = I \cap {}^{z_0} \Phi_{J_d} = {}^{z_0} J_d$. For simplicity, we shall consider the conjugate by z_0 of the right-hand side.

Recall that N and N' are normal subgroup of \mathbf{L}_{J_d} and are both contained in \mathbf{T} . In particular, any unipotent character of $\mathbf{L}_{J_d}^{wF}$ (resp. of $\mathbf{L}_{J_d}^{w'F}$) is trivial on N (resp. N'). Consequently, for any unipotent character χ of $\mathbf{L}_{J_d}^{wF}$ we obtain the following quasi-isomorphism of complexes of $L_I \times \langle F \rangle$ -modules:

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{z_0}, \overline{\mathbb{Q}}_\ell)_\chi \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}({}^{z_0} J_d, \dot{v}F), \overline{\mathbb{Q}}_\ell)_{z_0 \chi}.$$

Finally, we observe that the varieties $\tilde{X}_{\mathbf{L}_I}({}^{z_0} J_d, \dot{v}F)$ and $X_{n-1, d-1}$ have the same cohomology with coefficients in any unipotent local system. Indeed, if we denote (s_1, \dots, s_{n-2}) by (t_1, \dots, t_{n-2}) if d is odd or by (t_{n-2}, \dots, t_1) if d is even, then we have

$$v = t_1 t_2 \cdots t_{n-2-\lfloor \frac{d-1}{2} \rfloor} t_{n-2} t_{n-3} \cdots t_{\lfloor \frac{d}{2} \rfloor}$$

which corresponds to the element v_{d-1} in the Weyl group $W_I = \langle t_1, \dots, t_{n-2} \rangle$ of type A_{n-2} .

When $z = z_0$ and $d = 2$, we can write $v_2 = ww'$ with $w = s_{n-1}s_{n-2} \cdots s_2$ and $w' = s_1s_2 \cdots s_{n-1} = s_1w''$ so that $X_{n,2} \simeq X(\{s_2, \dots, s_{n-2}\}, \mathbf{w}w'F)$. Moreover, via this isomorphism we have

$$X_{z_0} \simeq \bigcup_{y \in W} X_{(z_0,y)}.$$

We claim that $X_{(z_0,y)}$ is empty unless $y \in W_I w_0 W_{J'_2}$ where $J'_2 = J_2^w = \{s_3, s_4, \dots, s_{n-1}\}$. The piece $X_{(z_0,y)}$ consists of pairs $(pz_0 \mathbf{P}_{J_2}, p'y \mathbf{P}_{J'_2})$ with $p, p' \in \mathbf{P}_I$ such that $p^{-1}p' \in z_0 \mathbf{P}_{J_2} w \mathbf{P}_{J'_2} y^{-1}$ and $p'^{-1}Fp \in y \mathbf{P}_{J'_2} w' \mathbf{P}_{J_2} z_0^{-1}$. In particular, if $X_{(z_0,y)}$ is non-empty, then the double coset $\mathbf{P}_I y \mathbf{P}_{J'_2}$ has a non-trivial intersection with $z_0 \mathbf{B}w$. But $z_0 = w_I w_0$ is reduced- \tilde{I} so that $\ell(z_0 w) = \ell(z_0) + \ell(w)$ and $z_0 \mathbf{B}w \subset \mathbf{P}_I w_0 \mathbf{P}_{J'_2}$. This forces y to lie in $W_I w_0 W_{J'_2}$. Note that $w_0(J'_2) \subset I$ so that the minimal element in this coset is z_0 and we have $X_{z_0} \simeq X_{(z_0,z_0)}$.

Now s_1 commutes with J'_2 and we can apply [11, Proposition 3.4] to obtain, after conjugation by z_0 ,

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{z_0} / N, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times \tilde{X}_{\mathbf{L}_I}(\mathbf{K}_{z_0}, \mathbf{v}v'F) / {}^{z_0}N', \overline{\mathbb{Q}}_\ell)$$

with $K_{z_0} = {}^{z_0}J_2$, $v = {}^{z_0}w$ and $v' = {}^{z_0}w''$. If we denote (s_1, \dots, s_{n-2}) by (t_{n-2}, \dots, t_1) we obtain ${}^{z_0}J_2 = \{t_2, \dots, t_{n-2}\}$ and $\mathbf{v}v' = \mathbf{t}_1 \cdots \mathbf{t}_{n-2} \mathbf{t}_{n-2} \cdots \mathbf{t}_1$ so that the pair $(\mathbf{K}_{z_0}, \mathbf{v}v')$ corresponds to $(\mathbf{J}_1, \mathbf{v}_1)$ in the Weyl group W_I of type A_{n-2} . As before, N and N' do not play any role if we consider the unipotent part of the previous quasi-isomorphism.

When $z = z_1$ we use [11, Proposition 3.2]: the conjugate of v_d by z_1 is

$$v = z_1 v_d z_1^{-1} = s_1 s_2 \cdots s_{n-2-\lfloor \frac{d}{2} \rfloor} s_{n-2} s_{n-3} \cdots s_{\lfloor \frac{d+1}{2} \rfloor}$$

which corresponds exactly to the element v_d in W_I . We can therefore identify the cohomology of the varieties $X_{\mathbf{L}_I}(K_{z_1}, vF)$ and $X_{n-1,d}$ with coefficients in any unipotent local system (see Remark 2.2). The group $\mathbf{P}_I \cap {}^{z_1} \mathbf{L}_{J_d}$ is a $\dot{v}F$ -stable parabolic subgroup of ${}^{z_1} \mathbf{L}_{J_d}$ and $\mathbf{L}_{K_{z_1}} = \mathbf{L}_I \cap {}^{z_1} \mathbf{L}_{J_d}$ is a stable Levi complement. Therefore, it makes sense to consider the Harish-Chandra restriction $*\mathrm{R}_K^J \chi$ of any unipotent character χ of $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{z_1} \mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{z_1}}^{\dot{v}F}$. From [11, Proposition 3.2] (see also [11, Remark 3.12]) we deduce the following quasi-isomorphism

$$\mathrm{R}\Gamma_c(U_I \backslash \tilde{X}_{z_1}, \overline{\mathbb{Q}}_\ell)_\chi[2](-1) \simeq \mathrm{R}\Gamma_c(\tilde{X}_{\mathbf{L}_I}(K_{z_1}, \dot{v}F), \overline{\mathbb{Q}}_\ell)_{*\mathrm{R}_K^J \chi}.$$

Let μ be a partition of $n-d$. The cohomology of the variety $X_{n,d}$ with coefficients in the local system \mathcal{F}_μ is given by

$$\mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu) \simeq \mathrm{R}\Gamma_c(\tilde{X}(\mathbf{J}_d, \mathbf{v}_d F), \overline{\mathbb{Q}}_\ell)_{\chi_\mu},$$

where χ_μ is the unipotent character of $\mathbf{L}_{J_d}^{\dot{v}F}$ corresponding to the partition μ . Since $(\mathbf{L}_{K_{z_1}}, \dot{v}F)$ is a split group of type A_{n-d-2} , the Harish-Chandra restriction of χ_μ from $\mathbf{L}_{J_d}^{\dot{v}F} \simeq ({}^{z_1} \mathbf{L}_{J_d})^{\dot{v}F}$ to $\mathbf{L}_{K_{z_1}}^{\dot{v}F}$ is the sum of the χ_{μ_i} 's where the μ_i 's are the partitions of $n-d-1$ obtained by restricting μ . With this description, we get the

following isomorphisms in $D^b(\overline{\mathbb{Q}}_\ell L_I \times \langle F \rangle\text{-mod})$:

$$\begin{aligned} \mathrm{R}\Gamma_c(U_I \backslash \widetilde{X}_{z_0}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} &\simeq \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu), \\ \mathrm{R}\Gamma_c(U_I \backslash \widetilde{X}_{z_1}, \overline{\mathbb{Q}}_\ell)_{\chi_\mu} &\simeq \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu_i})[-2](1). \end{aligned}$$

We conclude using the distinguished triangle associated to the decomposition $\widetilde{X}_{n,d} = \widetilde{X}_{z_0} \cup \widetilde{X}_{z_1}$ in which \widetilde{X}_{z_0} is open. □

3. COHOMOLOGY OVER $\overline{\mathbb{Q}}_\ell$

We have just seen how to relate the Harish-Chandra restriction of the cohomology of $X_{n,d}$ to the cohomology of smaller parabolic Deligne-Lusztig varieties. We shall now explain how this strategy provides an inductive method for a thorough determination of the cohomology of $X_{n,d}$ with coefficients in any unipotent local system. The main result in this section gives an inductive strategy towards a proof of Conjecture 1:

Theorem 3.1. *Let $n \geq 1$ and $2 \leq d \leq n - 1$. If Conjecture 1.5 holds for $(n, d + 1)$, $(n - 1, d - 1)$ and $(n - 1, d)$, then it holds for (n, d) .*

Note that we already know from [12] that Conjecture 1.5 holds in the Coxeter case, corresponding to (n, n) . Therefore, $d = 1$ is the only limit case. But $\pi = \mathbf{w}_0^2$ is a maximal 1-periodic element in the sense of [9] and in that specific case, a general conjecture for the cohomology has been already formulated in [4]: a unipotent character χ_λ can appear in $H_c^i(X(\pi))$ for $i = 4\nu_{\mathbf{G}} - 2A_\lambda$ only, where $\nu_{\mathbf{G}}$ is the number of positive roots. An important consequence of Theorem 3.1 is that knowing the cohomology of $X(\pi)$ is sufficient for determining all the other interesting cases:

Corollary 3.2. *For $\mathrm{GL}_n(q)$, Conjecture 1 holds for any $d \geq 1$ as soon as it holds for $d = 1$.*

Proof. Assume that Conjecture 1 holds for $d = 1$, that is, for the variety $X(\pi)$. Let $I = J_1 = \{s_2, \dots, s_{n-1}\}$ and $\mathbf{b} = \mathbf{v}_1 = \mathbf{s}_1 \cdots \mathbf{s}_{n-1} \mathbf{s}_{n-1} \cdots \mathbf{s}_1$. By [9, Proposition 8.26] we have

$$\mathrm{R}\Gamma_c(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c(\widetilde{X}(\mathbf{I}, \mathbf{b}F), \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}_I^{t(\mathbf{b})F}} \mathrm{R}\Gamma_c(X(\pi_I), \overline{\mathbb{Q}}_\ell).$$

Since the cohomology of $X(\pi_I)$ contains all the unipotent characters of $\mathbf{L}_I^{t(\mathbf{b})F}$, we deduce that for any partition μ of $n - 1$, the groups $H_c^i(X_{n,1}, \mathcal{F}_\mu)$ are submodules of the cohomology groups of $X(\pi)$. Consequently, they are disjoint as soon as Conjecture 1 holds for $X(\pi)$. Since we have assumed that it holds also for $X(\pi_I)$ we actually have

$$(3.3) \quad H_c^i(X(\pi), \overline{\mathbb{Q}}_\ell) \simeq \bigoplus_{\mu \vdash n-1} H_c^{i-4\nu_{\mathbf{L}_I} + 2A_\mu}(X_{n,1}, \mathcal{F}_\mu)(2\nu_{\mathbf{L}_I} - a_\mu - A_\mu)$$

as a $G \times \langle F \rangle$ -module. Now, the alternating sum of the cohomology groups of $X_{n,1}$ represents the Deligne-Lusztig induction from $\mathbf{L}_I^{t(\mathbf{b})F} \simeq L_I$ to G . Therefore, a character χ_λ appears in $H_c^\bullet(X_{n,1}, \mathcal{F}_\mu)$ if and only if μ is the restriction of λ , or equivalently, if λ is obtained from μ by adding a 1-hook. This, together with (3.3) and Lemma 1.4 proves that Conjecture 1.5 holds for $X_{n,1}$ and, therefore, for any variety $X_{n,d}$ by Theorem 3.1. We use 1.2 to conclude. □

Proof of the Theorem. Let X be a β -set associated to the partition μ of $n - d$. We can always assume that it contains $\{0, 1, \dots, d - 1\}$. The partitions $\mu^{(j)}$ of $n - d - 1$ which are obtained by restricting μ can be associated to the following β -set:

$$X^{(j)} = \{x_1^{(j)} < \dots < x_s^{(j)}\} \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_j - 1 & \text{if } i = j, \\ x_i & \text{otherwise.} \end{cases}$$

Let $I = \{s_1, \dots, s_{n-2}\}$. By Theorem 2.1, the Harish-Chandra restriction of the cohomology of $X_{n,d}$ can be inserted into the following distinguished triangle:

$$\begin{aligned} \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) &\longrightarrow \mathrm{R}\Gamma_c(X_{n,d}, \mathcal{F}_\mu)^{U_I} \\ &\longrightarrow \mathrm{R}\Gamma_c(X_{n-1,d}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow . \end{aligned}$$

Now, if we assume that Conjecture 1.5 holds for both $(n - 1, d - 1)$ and $(n - 1, d)$, the complexes on the left- and right-hand side are completely determined. Let us examine the different eigenvalues of F that can appear:

- (a) On $\mathcal{C} = \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n-1,d-1}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$, the eigenvalues of F are $q^{n-1+x-s}$ and q^{n+x-s} with $x \in X$ such that $x + d - 1 \notin X$. The character of the corresponding eigenspace is χ_λ where λ is the partition of $n - 1$ obtained by adding to μ a $(d - 1)$ -hook represented by x .
- (b) On $\mathcal{D}^{(j)} = \mathrm{R}\Gamma_c(X_{n-1,d}, \mathcal{F}_{\mu^{(j)}})[-2](1)$, the eigenvalues of F are q^{n+x-s} where $x \in X^{(j)}$ is such that $x + d \notin X^{(j)}$. The character of the corresponding eigenspace is χ_λ where λ is the partition of $n - 1$ obtained by adding to $\mu^{(j)}$ a d -hook represented by x .

We shall now determine $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ by studying each eigenspace of F separately. For x a positive integer, we can separate the following cases:

Case (1). Assume first that $x \in X$ and $x + d \notin X$. Let $\lambda = \mu * x$ be the partition of n obtained by adding to μ a d -hook from x . We want to prove that the q^{n-s+x} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is non-zero in degree $\pi_d(X, x)$ only and that its character is the Harish-Chandra restriction of χ_λ .

By (a), the q^{n+x-s} -eigenspace of F on \mathcal{C} will produce non-zero representations in the following two cases:

- If $x + d - 1 \notin X$, then one obtains a character associated to the β -set $(X \setminus \{x\}) \cup \{x + d - 1\}$ and it is concentrated in degree:

$$\begin{aligned} 2 + \pi_{d-1}(X, x) &= 2 + 2(n - 2 + x - \#\{y \in X \mid y < x\}) \\ &\quad - \#\{y \in X \mid x < y < x + d - 1\} \\ &= 2(n - 1 + x - \#\{y \in X \mid y < x\}) \\ &\quad - \#\{y \in X \mid x < y < x + d - 1\} \\ &= \pi_d(X, x). \end{aligned}$$

- If $x + 1 \in X$, then the corresponding β -set is $(X \setminus \{x + 1\}) \cup \{x + d\}$ and the associated character appears in degree $1 + \pi_{d-1}(X, x + 1)$ only.

But we have

$$\begin{aligned} \pi_{d-1}(X, x + 1) &= 2(n - 1 + x - \#\{y \in X \mid y < x + 1\}) \\ &\quad - \#\{y \in X \mid x + 1 < y < x + d\} \\ &= 2(n - 2 + x - \#\{y \in X \mid y < x\}) \\ &\quad - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

On the other hand, the q^{n+x-s} -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. This happens if and only if x and $x + d + 1$ are different from x_j . In that case, the β -set corresponding to the character of the eigenspace will be $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$. Furthermore, the degree in which this character will appear is $2 + \pi_d(X^{(j)}, x)$, which is clearly equal to $\pi_d(X, x)$ in that case.

Now, the β -set $Y = (X \setminus \{x\}) \cup \{x + d\}$ is associated to the partition $\lambda = \mu * x$. As mentioned in the beginning of Section 2, the restriction of λ is obtained by restricting the hook (usually in two different ways) or by restricting μ . In the framework of β -sets, it corresponds to decreasing specific elements of Y :

- If $x + d - 1 \notin X$, one can replace $x + d$ by $x + d - 1$ in Y and we obtain the β -set $(X \setminus \{x\}) \cup \{x + d - 1\}$.
- If $x + 1 \in X$, one can replace $x + 1$ by x in Y and we obtain the β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$.
- If $x_j \in X$ is different from x or from $x + d + 1$, and if $x_j - 1 \notin X$, then one can replace x_j by $x_j - 1$ in Y to obtain the β -set $(X^{(j)} \setminus \{x\}) \cup \{x + d\}$.

This shows that the character of the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is the Harish-Chandra restriction of $\chi_{\mu*x}$.

Lemma 3.4. *Let λ be a partition of n , with $n \geq 4$ and let χ be a (non-necessarily irreducible) unipotent character of $GL_n(q)$. Then the Harish-Chandra restriction of χ and χ_λ to $GL_{n-1}(q)$ are equal if and only if $\chi = \chi_\lambda$.*

Proof of the Lemma. Assume that there exists a partition $\nu = \{\nu_1 \leq \nu_2 \leq \dots \leq \nu_r\}$ of n with $\nu_1 \neq 0$ such that the difference between the Harish-Chandra restriction of χ_λ and χ_ν to $GL_{n-1}(q)$ is still a unipotent character. This means that in the Young diagram of ν , any box that can be removed can be replaced to form the Young diagram of λ . If $\nu \neq \lambda$, this is possible only if $\nu_1 = \nu_2 = \dots = \nu_r$.

Let χ be a unipotent character of $GL_n(q)$ which has the same Harish-Chandra restriction to $GL_{n-1}(q)$ as χ_λ . If $\chi \neq \chi_\lambda$, we deduce from the previous argument that all the irreducible constituents of χ are of the form χ_ν with $\nu = (a, a, \dots, a)$. This can happen if and only if $n = 3$ and $\lambda = (1, 2)$. □

When $n \geq 4$, we deduce that the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ is actually $\chi_{\mu*x}$. If $n = 3$, then the only ambiguity concerns $\chi_{\mu*x}$ when $\mu * x = (1, 2)$. In that case, the q^{n+x-s} -eigenspace can be either $\chi_{\mu*x}$ or $1_G + \text{St}_G$. But by [9, Corollary 8.28], the trivial character and the Steinberg character cannot occur in the same cohomology group of $X_{n,d}$ as soon as the dimension of this variety is non-zero.

Case (2). Assume now that $x \notin X$. The q^{n+x-s} -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. Since $x \notin X$, this forces $x = x_j^{(j)} = x_j - 1$ and $x_j + d - 1 \notin X$. In that case, its character corresponds to a partition with β -set

$(X \setminus \{x + 1\}) \cup \{x + d\}$ and it appears in degree $2 + \pi_d(X^{(j)}, x)$ only. On the other hand, the q^{n+x-s} -eigenspace of F on \mathcal{C} is non-zero if and only if $x + 1 \in X$ and $x + 1 + d - 1 = x + d \notin X$. By (a), the character of this eigenspace corresponds to a partition with β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$. Furthermore, it appears in degree $1 + \pi_{d-1}(X, x + 1)$ only. So in that case we have

$$\begin{aligned} \pi_{d-1}(X, x + 1) &= 2(n - 1 + x - \#\{y \in X \mid y < x + 1\}) \\ &\quad - \#\{y \in X \mid x + 1 < y < x + d\} \\ &= 2(n - 1 + x - \#\{y \in X \mid y < x\}) \\ &\quad - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) + 1 \end{aligned}$$

and

$$\begin{aligned} \pi_d(X^{(j)}, x) &= 2(n - 2 + x - \#\{y \in X^{(j)} \mid y < x\}) - \#\{y \in X^{(j)} \mid x < y < x + d\} \\ &= 2(n - 2 + x - \#\{y \in X \mid y < x\}) - (\#\{y \in X \mid x < y < x + d\} - 1) \\ &= \pi_d(X, x) - 1. \end{aligned}$$

We deduce that the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is isotypic and concentrated in two consecutive degrees. However, there are only a few unipotent characters that can have an isotypic Harish-Chandra restriction: they correspond to partitions of the form (a, a, \dots, a) . Among them we can find the Steinberg character St_G (with $a = 1$) and the trivial character 1_G (with $a = n$). But by [9, Corollary 8.28] they have respective eigenvalues 1 and q^{2n-1-d} . Let us write the β -set of μ as $X = \{0, 1, \dots, k - 1, \mu_1 + k, \mu_2 + k + 1, \dots, \mu_r + s - 1\}$ with $k \geq d$. Since $x \notin X$, one must have $k - 1 < x < \mu_r + s - 1$ and hence

$$(3.5) \quad d - 1 \leq n - 1 + k - s < n + x - s < n + \mu_r - 1 \leq 2n - 1 - d.$$

We deduce that the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ is either zero, or consists of two copies of the character χ_λ in two consecutive degrees, where $\lambda = (a, a, \dots, a)$ with $1 < a < n$. We shall actually prove that it is always zero, but before that we need to study the last case.

Remark 3.6. The Harish-Chandra restriction of χ_λ corresponds to the partition $(a - 1, a, \dots, a)$. Therefore, if the associated character appears in the cohomology of \mathcal{C} , then the β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$ must correspond to the partition $(a - 1, a, \dots, a)$. This gives a rather strong condition on X : we will have either

$$X = \{0, 1, \dots, k - 1, x + 1, b, b + 2, b + 3, \dots, \widehat{x + d}, \dots, b + r\}$$

with $b + 2 \leq x + d \leq b + r$, or

$$X = \{0, 1, \dots, k - 1, x + 1, x + d + 2, x + d + 3, \dots, x + d + r\}.$$

Case (3). Finally, assume that $x \in X$ and $x + d \in X$. The q^{n+x-s} -eigenspace of F on $\mathcal{D}^{(j)}$ is non-zero if and only if $x \in X^{(j)}$ and $x + d \notin X^{(j)}$. Since $x + d \in X$, this forces $x + d = x_j$ (and therefore $x_j - 1 \notin X$). In that case, the character of the eigenspace corresponds to a partition with β -set $(X^{(j)} \setminus \{x\}) \cup \{x + d\} = (X \setminus \{x\}) \cup \{x + d - 1\}$. On \mathcal{C} , the Frobenius has a non-zero q^{n+x-s} -eigenspace if and only if $x + d - 1 \notin X$ and its character is again associated to the β -set $(X \setminus \{x\}) \cup \{x + d - 1\}$. This ensures that the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)^{U_I}$ is isotypic. Using $x + d - 1$ instead of x in the inequalities (3.5) yields $0 < n + x - s < 2n - 2d$ and

therefore the previous argument applies. We deduce that the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ is again either zero or consists of two copies of the character χ_λ in two consecutive degrees, namely $\pi_d(X, x) - 1$ and $\pi_d(X, x)$, where $\lambda = (a, a, \dots, a)$ and $1 < a < n$.

To conclude, we need to prove that the q^{n+x-s} -eigenspaces of F are actually zero whenever $x \notin X$ or $x + d \in X$. Let us first summarize what we have proven so far:

- (1) If $x \in X$ and $x + d \notin X$, then the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ is $\chi_{\mu * x}$ and it appears in degree $\pi_d(X, x)$ only.
- (2) If $x \notin X$, the q^{n+x-s} -eigenspace of F is zero unless $x + 1 \in X$ and $x + d \notin X$. In that case, it may consist of two copies of χ_λ , one in degree $\pi_d(X, x) + 1$ and one in degree $\pi_d(X, x) + 2$, where $\lambda = (a, a, \dots, a)$ with $1 < a < n$. Moreover, the β -set $(X \setminus \{x + 1\}) \cup \{x + d\}$ must correspond to the partition $(a - 1, a, \dots, a)$ (see Remark 3.6).
- (3) If $x \in X$ and $x + d \in X$, the q^{n+x-s} -eigenspace of F is zero unless $x + d - 1 \notin X$. In that case, it can only be χ_λ -isotypic with $\lambda = (a, a, \dots, a)$ and $1 < a < n$. Moreover, it is non-zero in degrees $\pi_d(X, x) - 1$ and $\pi_d(X, x)$ only, and $(X \setminus \{x\}) \cup \{x + d - 1\}$ must be a β -set of the partition $(a - 1, a, \dots, a)$.

Now, if we assume that Conjecture 1.5 holds for the variety $X_{n,d+1}$, then we can use the distinguished triangle

$$\begin{aligned} \mathrm{R}\Gamma_c(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu) &\longrightarrow \mathrm{R}\Gamma_c(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I} \\ &\longrightarrow \mathrm{R}\Gamma_c(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1) \rightsquigarrow \end{aligned}$$

from Theorem 2.1 to prove that the eigenspaces of F on $H_c^\bullet(X_{n,d}, \mathcal{F}_\mu)$ in cases (2) and (3) are indeed zero.

Assume that $x \notin X$ and that there exists $1 < a < n$ such that the character $\chi_\lambda = \chi_{(a,a,\dots,a)}$ appears twice in the q^{n+x-s} -eigenspace of F on the cohomology of $X_{n,d}$; that is, in degrees $\pi_d(X, x) + 1$ and $\pi_d(X, x) + 2$. Then:

- If $x - 1 \notin X$, the q^{n+x-s} -eigenspace of F on $H_c^\bullet(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ is χ_λ -isotypic by (2) (we have $x - 1 + 1 \notin X$). Moreover, the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$ is zero since Conjecture 1.5 holds for the variety $X_{n,d+1}$. We deduce that the eigenspace on $H_c^\bullet(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I}$ is χ_λ -isotypic, which is impossible since no unipotent character can have χ_λ as a Harish-Chandra restriction when $1 < a < n$.
- If $x - 1 \in X$, then since $x - 1 + d + 1 \notin X$, the q^{n+x-s} -eigenspace of F on $H_c^\bullet(\mathbb{G}_m \times X_{n,d}, \overline{\mathbb{Q}}_\ell \otimes \mathcal{F}_\mu)$ and $H_c^\bullet(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$ can be determined as in case (1). It corresponds to the Harish-Chandra restriction of the partition $\mu * (x - 1)$ obtained from μ by adding a $(d + 1)$ -hook from $x - 1$. Furthermore, they will appear in degree $\pi_{d+1}(X, x - 1)$ only, which is equal to

$$\begin{aligned} \pi_{d+1}(X, x - 1) &= 2(n - 1 + x - \#\{y \in X \mid y < x - 1\}) \\ &\quad - \#\{y \in X \mid x - 1 < y < x + d\} \\ &= 2(n + x - \#\{y \in X \mid y < x\}) - \#\{y \in X \mid x < y < x + d\} \\ &= \pi_d(X, x) + 2. \end{aligned}$$

To these characters we have to add the contribution of χ_λ and possibly of another character $\chi_{\lambda'}$ corresponding to $\lambda = (a', a', \dots, a')$ (when case (3) applies to $x - 1$). Now, we claim that neither χ_λ nor $\chi_{\lambda'}$ can appear in the q^{n+x-s} -eigenspace of F on $H_c^\bullet(X_{n,d+1}, \bigoplus \mathcal{F}_{\mu^{(j)}})[-2](1)$. Indeed the assumptions on x force X (see Remark 3.6) to be either

$$X = \{0, 1, \dots, k-1, k+1, b, b+2, b+3, \dots, \widehat{k+d}, \dots, b+r\}$$

with $b+2 \leq k+d \leq b+r$, or

$$X = \{0, 1, \dots, k-1, k+1, k+d+2, k+d+3, \dots, k+d+r\}.$$

Therefore, a β -set corresponding to the partition $\mu^*(x-1)$ of $n+1$ is either

$$\{0, 1, \dots, k-2, k+1, b, b+2, b+3, \dots, b+r\}$$

with $b+2 \leq k+d \leq b+r$, or

$$\{0, 1, \dots, k-2, k+1, k+d, k+d+2, k+d+3, \dots, k+d+r\}.$$

We deduce that the restriction of $\mu^*(x-1)$ will never produce λ or λ' unless $r = 2$, $b = k+2$ and $d = 4$ in the first case, or $r = 2$ and $d = 2$ in the second case. In these very specific cases, we have either $X = \{0, \dots, k-1, k+1, k+2\}$, which corresponds to the partition $\mu = (1, 1)$ or $X = \{0, \dots, k-1, k+1, k+4\}$, which corresponds to $\mu = (1, 3)$. In this situation, we get $\lambda = (3, 3)$ and $\mu^*(x-1) = (2, 2, 3)$. But $(3, 3)$ cannot be obtained by restricting $\mu^*(x-1)$. This proves that the q^{n+x-s} -eigenspace of F on $H_c^{\pi_d(X,x)+2}(X_{n+1,d+1}, \mathcal{F}_\mu)^{U_I}$ is just χ_λ (plus possibly $\chi_{\lambda'}$), which is impossible by the properties of the Harish-Chandra restriction.

The same argument can be adapted to deal with case (3), if we look at the $q^{n+1+x-s}$ -eigenspace and distinguish whether $x+1+d$ is an element of X or not. The details are left to the reader. \square

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REFERENCES

- [1] Cédric Bonnafé and Raphaël Rouquier, *Coxeter orbits and modular representations*, Nagoya Math. J. **183** (2006), 1–34. MR2253885 (2007j:20066)
- [2] Michel Broué and Gunter Malle, *Zyklotomische Heckealgebren*, Astérisque **212** (1993), 119–189 (German). Représentations unipotentes génériques et blocs des groupes réductifs finis. MR1235834 (94m:20095)
- [3] Michel Broué, Gunter Malle, and Jean Michel, *Generic blocks of finite reductive groups*, Astérisque **212** (1993), 7–92. Représentations unipotentes génériques et blocs des groupes réductifs finis. MR1235832 (95d:20072)
- [4] Michel Broué and Jean Michel, *Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées*, Finite reductive groups (Luminy, 1994), Progr. Math., vol. 141, Birkhäuser Boston, Boston, MA, 1997, pp. 73–139 (French). MR1429870 (98h:20077)
- [5] Roger W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters; Reprint of the 1985 original; A Wiley-Interscience Publication. MR1266626 (94k:20020)

- [6] J. Chuang and R. Rouquier, *Calabi-Yau algebras and perverse Morita equivalences*, In preparation.
- [7] David Craven, *On the cohomology of Deligne-Lusztig varieties*, *arXiv:math/1107.1871*, Preprint (2011).
- [8] F. Digne and J. Michel, *Endomorphisms of Deligne-Lusztig varieties*, *Nagoya Math. J.* **183** (2006), 35–103. MR2253886 (2007e:20026)
- [9] F. Digne and J. Michel, *Parabolic Deligne-Lusztig varieties*, *arXiv:math/1110.4863*, preprint (2011).
- [10] François Digne, Jean Michel, and Raphaël Rouquier, *Cohomologie des variétés de Deligne-Lusztig*, *Adv. Math.* **209** (2007), no. 2, 749–822, DOI 10.1016/j.aim.2006.06.001 (French, with English summary). MR2296313 (2008k:20009)
- [11] Olivier Dudas, *Quotient of Deligne-Lusztig varieties*, *J. Algebra* **381** (2013), 1–20, DOI 10.1016/j.jalgebra.2013.01.032. MR3030506
- [12] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius*, *Invent. Math.* **38** (1976/77), no. 2, 101–159. MR0453885 (56 #12138)
- [13] George Lusztig, *Characters of reductive groups over a finite field*, *Annals of Mathematics Studies*, vol. 107, Princeton University Press, Princeton, NJ, 1984. MR742472 (86j:20038)

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