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ON THE THETA CORRESPONDENCE FOR $(\mathrm{GSp}(4), \mathrm{GSO}(4,2))$ AND SHALIKA PERIODS

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Abstract. We consider both local and global theta correspondences for GSp_4 and $\mathrm{GSO}_{4,2}$. Because of the accidental isomorphism $\mathrm{PGSO}_{4,2} \simeq \mathrm{PGU}_{2,2}$, these correspondences give rise to those between GSp_4 and $\mathrm{GU}_{2,2}$ for representations with trivial central characters. In the global case, using this relation, we characterize representations with trivial central character, which have Shalika period on $\mathrm{GU}(2,2)$ by theta correspondences. Moreover, in the local case, we consider a similar relationship for irreducible admissible representations without an assumption on the central character.

1. Introduction

Theta correspondence is a powerful method used to study representations of symplectic and orthogonal groups. In low rank cases, these groups are often related to general linear groups thanks to the accidental isomorphisms, and such relations have been proved to be very useful. For example, in a remarkable paper [12], Gan and Takeda established the local Langlands correspondence for GSp_4 over any nonarchimedean local field of characteristic zero by using an acute study in [13] on the local theta correspondences between GSp_4 and the similitude orthogonal groups $\mathrm{GSO}_{3,3}$, $\mathrm{GSO}_{4,0}$ and $\mathrm{GSO}_{2,2}$. Indeed, by the accidental isomorphisms, the latter orthogonal groups are related to GL_4, $D^\times \times D^\times$ where D is a quaternion algebra, and $\mathrm{GL}_2 \times \mathrm{GL}_2$, respectively, and they deduced the local Langlands correspondence for GSp_4 from that for general linear groups.

In another instance, theta correspondence has been used to realize functorial lifts and to characterize their images. A well-known example is the functorial lift from generic irreducible cuspidal automorphic representations of GSp_4 to GL_4 studied by Jacquet, Piatetski, Shapiro, and Shalika, in their unpublished paper, utilizing the close relationship between $\mathrm{GSO}_{3,3}$ and GL_4. Their proof is based on the computations of the pullbacks of the Whittaker period under the theta correspondence between GSp_4 and $\mathrm{GSO}_{3,3}$ in both directions. We may find their computations in Soudry [39]. They gave a characterization of the image of the theta correspondence in terms of Shalika period (cf. [39, Theorem in p. 264]). On the other hand, Jacquet and Shalika [16] showed an equivalence between the existence of a pole at $s = 1$ for the exterior square L-functions and the nonvanishing of Shalika period for irreducible cuspidal automorphic representation of GL_{2n}, and thus the above characterization by theta lift is related to the existence of a pole of the L-function.

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Similarly, Ginzburg, Rallis, and Soudry [14] studied a relationship between an existence of a pole of standard L-functions, a nonvanishing of certain periods, and a global theta correspondence for the pairs $(\mathrm{Sp}_{2n}, \mathrm{SO}_{n,n})$ and $(\mathrm{Sp}_{2n-2}, \mathrm{SO}_{n,n})$.

In [10], Gan and Takeda studied, in a similar way, both global and local theta correspondences for the pair $(\mathrm{GSp}_4, \mathrm{GSO}_{5,1})$. Again by the accidental isomorphism, $\mathrm{GSO}_{5,1}$ is closely related to $\mathrm{GL}_2(D)$ where D is a quaternion algebra. They studied the Shalika period on $\mathrm{GL}_2(D)$ and its relationship to the Shalika period on GL_4 under the Jacquet–Langlands correspondence when D is split at each archimedean place. Moreover, in the local case, for irreducible admissible representations of $\mathrm{GL}_2(D)$, they obtained a condition in terms of the local constant of the associated local Langlands parameter, which is equivalent to the nonvanishing of local Shalika period. Their computations may work even if D is not split at some archimedean place; however, in this case, the theta correspondence gives a correspondence between irreducible representations of $\mathrm{GL}_2(D)$ and those of $\{g \in \mathrm{GSp}_4 \mid \lambda(g) \in \mathrm{Nrm}(D^\times)\}$ where λ denotes the similitude character of GSp_4 and Nrm is the reduced norm Nrm of D. We note that as in the case of Gan and Takeda [10], if D is split at each archimedean place, this subgroup of GSp_4 coincides with GSp_4.

In the present paper, we shall study both global and local theta correspondences for the pair $\left(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2}\right)$. Here for an algebra A over the base field, let

$$\mathrm{GSp}_4(A)^+ = \{g \in \mathrm{GSp}_4(A) \mid \lambda(g) = \nu(h) \text{ for some } h \in \mathrm{GSO}_{4,2}(A)\}$$

or we simply denote it by GSp_4^+, where λ and ν denotes the similitude characters of GSp_4 of $\mathrm{GSO}_{4,2}$, respectively. Unlike the above cases, for any $\mathrm{GSO}_{4,2}$, GSp_4 and GSp_4^+ do not coincide. Hence, we should study theta correspondence for the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$. In our case, the group $\mathrm{GSO}_{4,2}$ is closely related to $\mathrm{GU}_{2,2}$. Indeed, we have

$$\mathrm{PGSO}_{4,2} \simeq \mathrm{PGU}_{2,2}.$$

As in the case of the dual pair $(\mathrm{GSp}_4, \mathrm{GSO}_{3,3})$, a study of the theta correspondence for this dual pair is motivated by the following analogue of the Jacquet–Shalika theorem [16].

Theorem A (Theorem 4.1 in [6])**.** *Let ξ be an idele class character of $\mathbb{A}_F^\times / F^\times$. Let (π, V_π) be an irreducible cuspidal unitary automorphic representation of $\mathrm{GU}(2,2)(\mathbb{A}_F)$ with the central character ω_π satisfying $\omega_\pi \mid_{\mathbb{A}_F^\times} = \xi^{-2}$. Then the following two conditions are equivalent:*

1. *The Shalika period with respect to ξ does not vanish on the space V_π of π.*
2. *π is globally generic and the partial twisted exterior square L-function $L^S\left(s, \pi, \wedge_t^2 \otimes \xi\right)$ has a simple pole at $s = 1$.*

Here we recall the definition of Shalika period with respect to ξ (see also (3.4) and (3.5)). Let E/F be a quadratic extension to define $\mathrm{GU}(2,2)$ and $P = MU$ the Siegel parabolic subgroup of $\mathrm{GU}(2,2)$ with $U = \{u(x) = \left(\begin{smallmatrix} 1_2 & x \\ & 1_2 \end{smallmatrix}\right) \mid {}^t\bar{x} = x \in \mathrm{Mat}_2(E)\}$ where $x \mapsto \bar{x}$ denotes the action by the nontrivial element in $\mathrm{Gal}(E/F)$. For a nontrivial additive character ψ of $\mathbb{A}_F/F$, we define a character ψ_0 of $U(\mathbb{A}_F)$ by $\psi_0(u(x)) = \psi(\mathrm{tr}(\left(\begin{smallmatrix} 0 & \eta \\ -\eta & 0 \end{smallmatrix}\right) \cdot x))$ for $\eta \in E^\times$ such that $\bar{\eta} = -\eta$. Then for $\varphi \in V_\pi$, Shalika period with respect to ξ is defined by

$$\int_{U(F)\backslash U(\mathbb{A}_F)} \int_{Z(\mathbb{A}_F)\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \varphi\left(\begin{pmatrix} g & \\ & \det g \cdot {}^t g^{-1} \end{pmatrix} u\right) \xi(\det g)\psi_0(u)\, dg\, du.$$

Our aim in this paper is not only to study the theta correspondence itself, but also to give a characterization of representations of $\mathrm{GU}_{2,2}$ or $\mathrm{GSO}_{4,2}$ which have Shalika period in terms of the theta correspondence. As we noted above, a similar consideration has been done for the dual pairs $(\mathrm{GSp}_4, \mathrm{GSO}_{3,3})$ and $(\mathrm{GSp}_4, \mathrm{GSO}_{5,1})$. Thus, our investigation here should complement the study of theta correspondence between GSp_4 and similitude orthogonal groups of degree six, and, its relationship to the Shalika periods on these groups. Further, unlike to the cases $(\mathrm{GSp}_4, \mathrm{GSO}_{3,3})$ and $(\mathrm{GSp}_4, \mathrm{GSO}_{5,1})$, both groups in our case have corresponding Shimura varieties when the base field is totally real. Hence, our case might be of some arithmetic interest.

Our main global result is a characterization of irreducible cuspidal automorphic representations of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ which have Shalika period.

Theorem B (Theorem 3.6)**.** *Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ with trivial central character. Then σ has Shalika period if and only if $\sigma = \theta^*(\Pi)$ for a generic irreducible cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$ with trivial central character. Here we write $\sigma = \theta^*(\Pi)$ when σ is given by the global theta lift $\Theta(\Pi^+)$ of Π^+ to $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ for an irreducible cuspidal automorphic representation Π^+ which appears as a subrepresentation of $\Pi|_{\mathrm{GSp}_4(\mathbb{A}_F)^+}$.*

From Theorem A and Theorem B, as in the case of GL_4, we obtain an equivalence of a nonvanishing of Shalika periods, an existence of the pole of twisted exterior square L-functions at $s = 1$ and a nonvanishing of theta lifts.

An essential step for the proof of this theorem is a computation of pull-backs of Whittaker periods in both directions as in [39]. Indeed, we shall show that Whittaker period of the theta lift from $\mathrm{GSp}_4(\mathbb{A}_F)^+$ to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ is described by Whittaker period of $\mathrm{GSp}_4(\mathbb{A}_F)^+$, and conversely, Whittaker period of the theta lift from $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ is described by Shalika period on $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. Then the "only if" part readily follows from this computation and a similar computation for a lower rank case. On the other hand, if the global theta lift is not zero, then we can show the existence of a pole of the twisted exterior square L-functions by an explicit computation of local theta correspondences for unramified principal series representations. Further, a computation of the pullback proves that an automorphic representation of $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ is generic. Then the characterization of Shalika period by Theorem A shows the "if" part.

Here we note that Takeo Okazaki, motivated by a conjecture of van Geeman and van Straten, also studied independently the global aspect of this theta correspondence and gave a sketch of proof for the relationship between the nonvanishing of theta lift and the existence of Shalika period in [27]. Though he did not discuss any local theory, as far as the author knows.

Our first aim in the local situation is to investigate the local theta correspondence itself. Indeed, we shall explicitly determine local theta lifts from GSp_4^+ to $\mathrm{GSO}_{4,2}$ following [13] (see Theorem 6.21). In particular, we have the following fundamental result in the theory of local theta correspondence.

Theorem C (Corollary 6.22)**.** *For the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$, the Howe duality holds over any nonarchimedean local field of characteristic zero, namely for any irreducible admissible representation of π of GSp_4^+ or $\mathrm{GSO}_{4,2}$, we have*

- *$\theta(\pi)$ is irreducible whenever $\Theta(\pi)$ is nonzero.*

- *the map $\pi \mapsto \theta(\pi)$ is injective on its domain.*

(See Section 6.1 *for definitions of* $\Theta(\pi)$ *and* $\theta(\pi)$*.)*

We note that for a proof of this theorem and an explicit computation of the theta correspondence, we need classifications of non-supercuspidal irreducible admissible representations of GSp_4^+ and $\mathrm{GSO}_{4,2}$. The classification for GSp_4^+ is deduced from that of GSp_4 and the study of restrictions of irreducible representations of GSp_4 to GSp_4^+ (cf. [11]). On the other hand, the classification for $\mathrm{GSO}_{4,2}$ is essentially new. We shall give the classification using a method in Sally and Tadic [32] and Konno [19].

Then, using the above observation on the local theta correspondence, we shall consider a local analogue of Theorem B, namely we shall characterize representations which have local analogue of Shalika period (see Definition 6.5) in terms of the local theta correspondence.

As a local analogue of pullbacks of Whittaker periods, we shall compute some twisted Jacquet modules of the induced Weil representations. Indeed, we shall show that Whittaker model on GSp_4^+ is related to Shalika period on $\mathrm{GSO}_{4,2}$ through the theta lift from $\mathrm{GSO}_{4,2}$ to GSp_4^+. Then we obtain a characterization for generic irreducible representations of $\mathrm{GSO}_{4,2}$, which have Shalika period.

Theorem D (Theorem 6.9)**.** *Let σ be a generic irreducible representation of* $\mathrm{GSO}_{4,2}$*. Then the following conditions are equivalent:*

(1) *σ has Shalika period.*
(2) *The small theta lift $\theta(\sigma)$ of σ to* GSp_4^+ *is nonzero.*
(3) *The small theta lift $\theta(\sigma)$ of σ to* GSp_4^+ *is generic.*

This is an analogue of [10, Theorem 7.6] in the case of $\mathrm{GSO}_{4,2}$, where they showed these equivalence relations for the dual pair $(\mathrm{GSp}_4, \mathrm{GSO}_{3,3})$. They also showed an equivalence between a nonvanishing of Shalika period and a genericity of big theta lifts for the dual pair $(\mathrm{GSp}_4, \mathrm{GSO}_{5,1})$ (see [10, Corollary 7.5]).

In the global setting, a nonvanishing of Shalika period implies that of Whittaker period, and thus we do not need any assumption on genericity for representations of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ up front in Theorem A, as opposed to Theorem D. We shall prove the following assertion in the local setting for essentially tempered representations by an explicit computation of local theta correspondences, especially by a comparison of big theta lifts and small theta lifts.

Theorem E (Theorem 6.26)**.** *Let σ be an essentially tempered irreducible representation of* $\mathrm{GSO}_{4,2}$*. If σ has Shalika period, then σ is generic.*

From this theorem, we obtain the following local analogue of Theorem B for essentially tempered irreducible representations of $\mathrm{GSO}_{4,2}$.

Theorem F (Theorem 6.27)**.** *Let σ be an essentially tempered irreducible representation of* $\mathrm{GSO}_{4,2}$*. Then σ has Shalika period if and only if $\sigma = \theta(\pi)$ for some generic irreducible representation π of* GSp_4^+*.*

Here we remark that from an accurate study on theta correspondences for the dual pairs $(\mathrm{GSp}_4, \mathrm{GSO}_{3,3})$ and $(\mathrm{GSp}_4, \mathrm{GSO}_{5,1})$ in [10] and [13], we can expect that similar results as Theorem E and Theorem F hold for these dual pairs. However, due to their purposes in [10] and [13], they do not pursue the above properties.

Let us explain the structure of this paper. In Section 2, we recall briefly the global theta lift for similitude groups. In Section 3, we compute the pull-back of the Whittaker period of the theta lift for the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$ in both directions. Using these computations, we prove Theorem B. Moreover, we give an example of an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}$ such that it does not have Shalika period and its twisted exterior square L-function has a pole at $s = 1$. In Section 4, we give a classification of non-supercuspidal irreducible representations of GSp_4^+ using the result in [11] on the restriction of admissible representations of GSp_4 to Sp_4. In Section 5, we give a classification of non-supercuspidal irreducible representations of $\mathrm{GSO}_{4,2}$ following [32] and [19]. In Section 6, we compute the local theta correspondence for $\left(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2}\right)$ following a similar computation in [13]. In Section 7, for the sake of completeness and for the sake of convenience for the reader, we recall the computation of local theta lifts from $\mathrm{Sp}_4(\mathbb{R})$ to $\mathrm{SO}_{4,2}(\mathbb{R})$ for discrete series representations by Paul [28].

2. Global theta lift

2.1. **Preliminaries.** Let F be a number field and E a quadratic extension of F. We denote by $x \mapsto \bar{x}$ the action of the nontrivial element of $\mathrm{Gal}(E/F)$ on E. Let us denote by $N_{E/F}$ the norm map from E to F. We choose $d \in F^\times \setminus (F^\times)^2$ such that $E = F(\eta)$ and $\bar{\eta} = -\eta$ with $\eta = \sqrt{d}$. Then we define the similitude unitary group $\mathrm{GU}_{2,2}$ by

$$\mathrm{GU}_{2,2}(F) = \left\{ g \in \mathrm{GL}_4(E) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \lambda(g) \in F^\times \right\}$$

and the similitude symplectic group GSp_4 by

$$\mathrm{GSp}_4(F) = \mathrm{GL}_4(F) \cap \mathrm{GU}_{2,2}(F).$$

Let us define the similitude orthogonal groups $\mathrm{GO}_{2+i,i}$ for $i = 0, 1, 2$ by

$$\mathrm{GO}_{2+i,i} = \left\{ g \in \mathrm{GL}_{2+i} \mid {}^t g S_i g = \nu(g) S_i,\ \nu(g) \in \mathrm{GL}_1 \right\}$$

where

$$S_0 = \begin{pmatrix} 2 & \\ & -2d \end{pmatrix},\ S_1 = \begin{pmatrix} & & 1 \\ & S_0 & \\ 1 & & \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} & & 1 \\ & S_1 & \\ 1 & & \end{pmatrix}.$$

We also define subgroups $\mathrm{GSO}_{2+i,i}$ of $\mathrm{GO}_{2+i,i}$ by

$$\mathrm{GSO}_{2+i,i} = \left\{ g \in \mathrm{GO}_{2+i,i} \mid \det(g) = \nu(g)^{1+i} \right\}.$$

Then we note that $\mathrm{GSO}_{2,0}(F) \simeq E^\times$, and we shall identify these groups. Under this identification, the similitude character ν is given by the norm map $N_{E/F}$.

As we noted in Section 1, $\mathrm{GU}_{2,2}$ is closely related to $\mathrm{GSO}_{4,2}$. Indeed, we can construct an isomorphism between $\mathrm{PGU}_{2,2}$ and $\mathrm{PGSO}_{4,2}$ as follows. Let us define a subgroup $\mathrm{GSU}_{2,2}$ of $\mathrm{GU}_{2,2}$ by

$$\mathrm{GSU}_{2,2} = \left\{ g \in \mathrm{GU}_{2,2} \mid \det(g) = \lambda(g)^2 \right\}.$$

We set the six-dimensional vector space over F by

$$\mathcal{V} = \left\{ \begin{pmatrix} 0 & x_1 & x_3+\eta x_4 & x_2 \\ -x_1 & 0 & x_5 & -x_3+\eta x_4 \\ -x_3-\eta x_4 & -x_5 & 0 & x_6 \\ -x_2 & x_3-\eta x_4 & -x_6 & 0 \end{pmatrix} \mid x_i \in F\,(1 \le i \le 6) \right\}$$

and let us define a mapping $\Psi : \mathcal{V} \to F$ by

$$\Psi(B) = \mathrm{Tr}(BJ^t\bar{B}J).$$

Indeed, we have

$$\Psi\left(\begin{pmatrix} 0 & x_1 & x_3+\eta x_4 & x_2 \\ -x_1 & 0 & x_5 & -x_3+\eta x_4 \\ -x_3-\eta x_4 & -x_5 & 0 & x_6 \\ -x_2 & x_3-\eta x_4 & -x_6 & 0 \end{pmatrix}\right)$$
$$= -4\{x_1x_6 + x_2x_5 + (x_3^2 - dx_4^2)\}.$$

Then we define a homomorphism $\phi : \mathrm{GSU}_{2,2} \to \mathrm{GO}(\mathcal{V})$ by

$$\mathcal{V} \ni B \mapsto gB^tg \in \mathcal{V}.$$

Here we note that we have

$$\nu(\phi(g)) = \lambda(g)^2. \tag{2.1}$$

As a basis for $\mathcal{V}$, we may take

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \\ -\eta & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then with respect to the basis $\{e_i \mid 1 \le i \le 6\}$, we may identify $\mathrm{GO}(\mathcal{V})$ as $\mathrm{GO}_{4,2}$. Moreover, from (2.1), ϕ gives a homomorphism

$$\Phi : \mathrm{GSU}_{2,2} \to \mathrm{GSO}_{4,2}.$$

For $\alpha \in E^\times$, let

$$r_\alpha = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\alpha}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GU}_{2,2}(F),$$

and we define an action of $E^\times$ on the space $\mathcal{V}$ by

$$\alpha \cdot v = \bar{\alpha} r_\alpha v r_\alpha \quad v \in \mathcal{V}.$$

We consider the semidirect product group $\mathrm{GSU}_{2,2} \rtimes E^\times$ with the action of $E^\times$ on $\mathrm{GSU}_{2,2}$ given by $\alpha \cdot g = r_\alpha g r_\alpha^{-1}$ for $\alpha \in E^\times$ and $g \in \mathrm{GSU}_{2,2}$. Then from the definition of the action of $E^\times$ on $\mathcal{V}$, we obtain a homomorphism

$$\Gamma : \mathrm{GSU}_{2,2} \rtimes E^\times \to \mathrm{PGSO}_{4,2}.$$

It is easy to see that this homomorphism is surjective. On the other hand, we have a surjective homomorphism

$$\Xi : \mathrm{GSU}_{2,2} \rtimes E^\times \to \mathrm{PGU}_{2,2} : (g, \alpha) \mapsto gr_\alpha.$$

Then we can show that $\mathrm{Ker}\ \Xi = \mathrm{Ker}\ \Gamma$, and thus we obtain the following isomorphism

$$\Gamma^* \circ (\Xi^*)^{-1} : \mathrm{PGU}_{2,2} \simeq \mathrm{PGSO}_{4,2} \tag{2.2}$$

where Γ^* (resp. Ξ^*) is the isomorphism of $\mathrm{GSU}_{2,2} \rtimes E^\times/\mathrm{Ker}\ \Gamma \xrightarrow{\sim} \mathrm{PGSO}_{4,2}$ $\left(\text{resp.}\mathrm{GSU}_{2,2} \rtimes E^\times/\mathrm{Ker}\ \Xi \xrightarrow{\sim} \mathrm{PGU}_{2,2}\right)$ induced by Γ (resp. Ξ).

Let us denote by B the Borel subgroup of GSp_4, which has the Levi decomposition $B = TN$ with

$$T(F) = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & \lambda a^{-1} & \\ & & & b\lambda^{-1} \end{pmatrix} \mid a, b, \lambda \in F^\times \right\}$$

and

$$N(F) = \left\{ w(a)v(A) \mid a \in F, A = {}^tA \in \mathrm{Mat}_{2\times 2}(F) \right\}$$

where

$$v(A) = \begin{pmatrix} 1_2 & A \\ & 1_2 \end{pmatrix} \text{ and } w(a) = \begin{pmatrix} 1 & a & & \\ & 1 & & \\ & & 1 & \\ & & -a & 1 \end{pmatrix}.$$

Moreover, we denote by P and Q the Siegel parabolic subgroup and the Klingen parabolic subgroup of GSp_4, respectively, which have the Levi decompositions $P = M_P N_P$ and $Q = M_Q N_Q$ with

$$M_P(F) = \left\{ \begin{pmatrix} g & \\ & \lambda {}^tg^{-1} \end{pmatrix} \mid g \in \mathrm{GL}_2(F), \lambda \in F^\times \right\},$$

$$N_P(F) = \left\{ \begin{pmatrix} 1_2 & X \\ & 1_2 \end{pmatrix} \mid X = {}^tX \in \mathrm{Mat}_{2\times 2}(F) \right\},$$

$$M_Q(F) = \left\{ \begin{pmatrix} a & & & \\ & b & & c \\ & & \lambda a^{-1} & \\ & d & & e \end{pmatrix} \mid \det \begin{pmatrix} b & c \\ d & e \end{pmatrix} = \lambda \in F^\times \right\},$$

and

$$N_Q(F) = \left\{ \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{pmatrix} \begin{pmatrix} 1 & & y & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid x, y, z \in F \right\}.$$

Also, we denote by B_H the Borel subgroup of $\mathrm{GSO}_{4,2}$. Let $B_H = T_H U$ be the Levi decomposition of B_H with

(2.3)

$$T_H(F) = \left\{ \mathrm{diag}(a, b; x) = \begin{pmatrix} a & & & & \\ & b & & & \\ & & x & & \\ & & & \lambda b^{-1} & \\ & & & & \lambda a^{-1} \end{pmatrix} \mid \begin{array}{l} a, b \in F^\times, x \in E^\times, \\ \lambda = N_{E/F}(x) \end{array} \right\}$$

and

$$U(F) = U_0(F)U_1(F)U_2(F)\tilde{U}(F) \tag{2.4}$$

where

$$U_0(F) = \left\{ u_0(x) = \begin{pmatrix} 1 & -{}^tX_0S_1 & 0 \\ & 1_4 & X_0 \\ & & 1 \end{pmatrix} \mid X_0 = \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$U_1(F) = \left\{ u_1(s_1,t_1) = \begin{pmatrix} 1 & -{}^tX_1S_1 & -\frac{1}{2}{}^tX_1S_1X_1 \\ & 1_4 & X_1 \\ & & 1 \end{pmatrix} \mid X_1 = \begin{pmatrix} 0 \\ s_1 \\ t_1 \\ 0 \end{pmatrix} \right\},$$

$$U_2(F) = \left\{ u_2(s_2,t_2) = \begin{pmatrix} 1 & & & & \\ & 1 & -{}^tX_2S_0 & -\frac{1}{2}{}^tX_2S_2X_2 & \\ & & 1_2 & X_2 & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \mid X_2 = \begin{pmatrix} s_2 \\ t_2 \end{pmatrix} \right\},$$

and

$$\tilde{U}(F) = \left\{ \tilde{u}(b) = \begin{pmatrix} 1 & -{}^t\widetilde{X}S_1 & 0 \\ & 1_4 & \widetilde{X} \\ & & 1 \end{pmatrix} \mid \widetilde{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -b \end{pmatrix} \right\}.$$

Finally, we define maximal parabolic subgroups P_H and Q_H of $\mathrm{GSO}_{4,2}$ with the Levi decompositions $P_H = M_{P_H}N_{P_H}$ and $Q_H = M_{Q_H}N_{Q_H}$ given by

$$M_{P_H}(F) = \left\{ \begin{pmatrix} a & & \\ & h & \\ & & a^{-1} \end{pmatrix} \mid a \in F^\times, h \in \mathrm{GSO}_{3,1} \right\},$$

$$M_{Q_H}(F) = \left\{ \begin{pmatrix} g & & \\ & x & \\ & & N_{E/F}(x)\cdot g^* \end{pmatrix} \mid g \in \mathrm{GL}_2, x \in E^\times \right\},$$

$$N_{P_H}(F) = \{u_0(x)u_1(y,z)\tilde{u}(w) \mid x,y,z,w \in F\},$$

and

$$N_{Q_H}(F) = \{u_0(p)u_1(q,r)u_2(s,t) \mid p,q,r,s,t \in F\},$$

where we identify $\mathrm{GSO}_{2,0}$ and $E^\times$, and we put

$$g^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tg^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.2. **Global theta lift for similitude groups.** Let X (resp. Y) be a finite dimensional vector space over F equipped with an alternating (resp. nondegenerate symmetric) bilinear form. Then the space $Z = X \otimes Y$ has a natural nondegenerate alternating form $\langle\, , \rangle$, and we have an embedding $\mathrm{Sp}(X) \times \mathrm{SO}(Y) \to \mathrm{Sp}(Z)$ defined by

$$(x \otimes y)(g,h) = xg \otimes h^{-1}y, \quad \text{for } x \in X, y \in Y, h \in \mathrm{SO}(Y), g \in \mathrm{Sp}(X). \tag{2.5}$$

Fix a nontrivial additive character ψ of $\mathbb{A}_F$ which is trivial on F and the Witt decomposition $Z = Z_+ \oplus Z_-$. Let us denote by $(\omega_\psi, \mathcal{S}(Z_+))$ the Schrödinger model

of $\widetilde{\mathrm{Sp}}(Z)$ corresponding to this polarization. We write a typical element of $\mathrm{Sp}(Z)$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$A \in \mathrm{Hom}(Z_+, Z_+), B \in \mathrm{Hom}(Z_+, Z_-), C \in \mathrm{Hom}(Z_-, Z_+), D \in \mathrm{Hom}(Z_-, Z_-).$$

Then the action of ω_ψ on $\mathcal{S}(Z_+)$ is given by the following formulas:

$$\omega_\psi\left(\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}\right)\phi(z_+) = \varepsilon\frac{\gamma_\psi(1)}{\gamma_\psi(\det A)}|\det(A)|^{\frac{1}{2}}\psi(\frac{1}{2}\langle z_+A, z_+B\rangle)\phi(z_+A), \tag{2.6}$$

$$\omega_\psi\left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right)\phi(z_+) = \varepsilon(\gamma_\psi(1))^{-\dim Z_+}\int_{Z_+}\psi\left(\langle z', z\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\rangle\right)\phi(z')\,dz', \tag{2.7}$$

where $\gamma_\psi(t)$ is a certain eighth root of unity called Weil factor and $\phi \in \mathcal{S}(Z_+)$.

Let $X = F^4$ be the space of row vectors with the symplectic form

$$\langle w_1, w_2\rangle = w_1\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}{}^tw_2.$$

Then we have $\mathrm{GSp}(X) = \mathrm{GSp}_4(F)$, and we let $\mathrm{GSp}_4(F)$ act on X from the right.

Let $Y = F^6$ be the space of column vectors with the nondegenerate symmetric bilinear form

$$(v_1, v_2) = {}^tv_1S_2v_2.$$

Then we have $\mathrm{GSO}(Y) = \mathrm{GSO}_{4,2}(F)$. Let us consider $Z = X \otimes_F Y$ as a symplectic space as above. Then $\mathrm{GSp}(Z)$ acts on Z from the right and we have a homomorphism

$$i : \mathrm{GSp}_4(F) \times \mathrm{GSO}_{4,2}(F) \to \mathrm{GSp}(Z)$$

given by the action (2.5). Then we note that $\lambda(i(g,h)) = \lambda(g)\nu(h)^{-1}$. Let

$$R = \{(g,h) \in \mathrm{GSp}_4 \times \mathrm{GSO}_{4,2} \mid \lambda(g) = \nu(h)\}.$$

Then the restriction of i to $R(F)$ gives a natural homomorphism $i : R(F) \to \mathrm{Sp}(Z)$. Moreover, we have $\mathrm{Sp}_4 \times \mathrm{SO}_{4,2} \subset R$.

Let us fix the polarization $Z = Z_+ + Z_-$ as above. For $\phi \in \mathcal{S}(Z_+(\mathbb{A}_F))$, we define the theta kernel by

$$\theta^\phi(g,h) = \sum_{z_+\in Z_+(F),} \omega_\psi \circ i(g,h)\phi(z_+) \quad (g,h) \in R(\mathbb{A}_F).$$

Let

$$\mathrm{GSp}_4(\mathbb{A}_F)^+ = \{g \in \mathrm{GSp}_4(\mathbb{A}_F) \mid \lambda(g) = \nu(h) \text{ for some } h \in \mathrm{GSO}_{4,2}(\mathbb{A}_F)\}. \tag{2.8}$$

and $\mathrm{GSp}_4(F)^+ = \mathrm{GSp}_4(\mathbb{A}_F)^+ \cap \mathrm{GSp}_4(F)$. We note that by the Bruhat decomposition, we can show that $\nu(\mathrm{GSO}_{4,2}(\mathbb{A}_F)) = \nu(B_H(\mathbb{A}_F))$, and thus we obtain $\nu(\mathrm{GSO}_{4,2}(\mathbb{A}_F)) = N_{E/F}(\mathbb{A}_E^\times)$. Hence, $g \in \mathrm{GSp}_4(\mathbb{A}_F)$ is in $\mathrm{GSp}_4(\mathbb{A}_F)^+$ if and only if $\lambda(g) \in N_{E/F}(\mathbb{A}_E^\times)$.

We call a function on $\mathrm{GSp}_4(\mathbb{A}_F)^+$ an automorphic form on $\mathrm{GSp}_4(\mathbb{A}_F)^+$ when it is left $\mathrm{GSp}_4(F)^+$-invariant, smooth, moderate growth, K-finite and $\mathfrak{Z}$-finite where we denote by K a maximal compact subgroup of $\mathrm{GSp}_4(\mathbb{A}_F)^+$ and by $\mathfrak{Z}$ the center of the universal enveloping algebra of $\mathrm{GSp}_4(\mathbb{A}_F)^+ \cap \mathrm{GSp}_4(F_\infty)$ with the archimedean part F_∞ of $\mathbb{A}_F$. Further, we call an automorphic on $\mathrm{GSp}_4(\mathbb{A}_F)^+$ a cusp form on

$\mathrm{GSp}_4(\mathbb{A}_F)^+$ if its restriction to $\mathrm{Sp}_4(\mathbb{A}_F)$ is a cusp form. Then for a cusp form f on $\mathrm{GSp}_4(\mathbb{A}_F)^+$, as in [15, Section 5.1], we define the theta lift of f to be $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ by

$$\theta(f,\phi)(h) = \int_{\mathrm{Sp}_4(F)\backslash \mathrm{Sp}_4(\mathbb{A}_F)} \theta^{\phi}(g_1 g, h)\overline{f(g_1 g)}\, dg_1$$

where $g \in \mathrm{GSp}_4(\mathbb{A}_F)^+$ is chosen such that $\lambda(g) = \nu(h)$. It defines an automorphic form on $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. Indeed, the analytic properties are proved as in the case of isometry groups, and it is $\mathrm{GSO}_{4,2}(F)$-invariant by [15, Lemma 5.1]. Similarly, for a cusp form f' on $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$, we define the theta lift to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ by

$$\theta(f',\phi)(g) = \int_{\mathrm{SO}_{4,2}(F)\backslash \mathrm{SO}_{4,2}(\mathbb{A}_F)} \theta^{\phi}(g, h_1 h)\overline{f'(h_1 h)}\, dh_1$$

where $h \in \mathrm{GSO}_{4,2}(\mathbb{A}_F)$ is chosen such that $\lambda(g) = \nu(h)$. Then as in the above case, it gives an automorphic form on $\mathrm{GSp}_4(\mathbb{A}_F)^+$.

3. Pullbacks of Whittaker period

3.1. **Theta lift from $\mathrm{GSp}_4(\mathbb{A}_F)^+$ to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$.** Our aim in this section is to compute the Whittaker period of the theta lift from $\mathrm{GSp}_4(\mathbb{A}_F)^+$ to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)^+$ and V_π its space of cusp forms on $\mathrm{GSp}_4(\mathbb{A}_F)^+$. Then we define the theta lift of π to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ by

$$\Theta(\pi) = \langle \theta(f,\phi) \mid f \in V_\pi, \phi \in \mathcal{S}(Z_+(\mathbb{A}_F))\rangle.$$

This is an automorphic representation of $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. Suppose that $\Theta(\pi)$ consists of cusp forms. Then we know that if the Howe duality holds for this dual pair, $\Theta(\pi)$ is irreducible and it is compatible with the local theta theta correspondence, i.e., $\Theta(\pi) \simeq \otimes_v \theta(\pi_v)$ (see Section 6.1 for the definition of $\theta(\pi_v)$). For a proof, see [7, Proposition 2.12] in the case of isometry groups, and it can be easily modified for our case. Further, the Howe duality is proved in Section 6 (see Corollary 6.22), and thus we have the irreducibility and the compatibility when $\Theta(\pi)$ is cuspidal.

Let us define a nondegenerate character ψ_U of $U(\mathbb{A}_F)$ by

$$\psi_U(u_0(x)u_1(s_1,t_1)u_2(s_2,t_2)\tilde{u}(b)) = \psi(2dt_2 + b). \tag{3.1}$$

Then for an automorphic form φ on $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$, we define the Whittaker period $W(\varphi)$ by

$$W(\varphi) = \int_{U(F)\backslash U(\mathbb{A}_F)} \varphi(u)\psi_U^{-1}(u)\, du.$$

Since we know

$$U_0 \lhd U_0U_1 \lhd U_0U_1U_2 \lhd U_0U_1U_2\tilde{U} = U,$$

we may write this integral as

$$W(\varphi) = \int_{F\backslash\mathbb{A}_F}\int_{(F\backslash\mathbb{A}_F)^2}\int_{(F\backslash\mathbb{A}_F)^2}\int_{F\backslash\mathbb{A}_F} \varphi(u_0(x)u_1(s_1,t_1)u_2(s_2,t_2)\tilde{u}(b)) \cdot \psi(-2dt_2 - b)\, dx\, ds_1\, dt_1\, ds_2\, dt_2\, db.$$

Then we shall compute $W(\theta(f,\phi))$ for $f \in V_\pi$ and $\phi \in \mathcal{S}(Z_+(\mathbb{A}_F))$.

Let us denote the standard basis of $Y = F^6$ by

$$\begin{aligned} y_{-2} &= {}^t(1,0,0,0,0,0), \quad & y_{-1} &= {}^t(0,1,0,0,0,0), \\ e_1 &= {}^t(0,0,1,0,0,0), \quad & e_2 &= {}^t(0,0,0,1,0,0), \\ y_1 &= {}^t(0,0,0,0,1,0), \quad & y_2 &= {}^t(0,0,0,0,0,1). \end{aligned}$$

Then we have $(y_i, y_j) = \delta_{ij}, (e_1, e_1) = 2$ and $(e_2, e_2) = -2d$. Moreover, we define maximal isotropic subspaces of Y by

$$Y_+ = F \cdot y_1 + F \cdot y_2 \quad \text{and} \quad Y_- = F \cdot y_{-1} + F \cdot y_{-2}$$

and an anisotropic subspace by

$$Y_0 = F \cdot e_1 + F \cdot e_2.$$

Clearly, we can write $Y = Y_+ + Y_- + Y_0$.

Let us denote the standard basis of $X = F^4$ by

$$x_1 = (1,0,0,0),\ x_2 = (0,1,0,0),\ x_{-1} = (0,0,1,0),\ x_{-2} = (0,0,0,1).$$

Then we define maximal isotropic subspaces

$$X_+ = F \cdot x_1 + F \cdot x_2 \quad \text{and} \quad X_- = F \cdot x_{-1} + F \cdot x_{-2}.$$

As a polarization $Z = Z_+ + Z_-$ of the symplectic space $Z = X \otimes Y$, we use

$$Z_\pm = X \otimes Y_\pm + X_\pm \otimes Y_0.$$

According to this polarization, we denote $z_+ \in Z_+$ by

$$z_+ = a_1 \otimes y_1 + a_2 \otimes y_2 + b_1 \otimes e_1 + b_2 \otimes e_2, \quad a_i \in X,\ b_i \in X_+,$$

and we denote $\phi \in \mathcal{S}(Z_+(\mathbb{A}_F))$ by

$$\phi(z_+) = \phi(a_1, a_2; b_1, b_2).$$

For $h_1 \in \mathrm{SO}_{4,2}(\mathbb{A}_F)$, let us define

$$W_0(\theta(f,\phi))(h_1) = \int_{F\backslash\mathbb{A}_F} \theta(f,\phi)(u_0(x)h_1)\,dx.$$

From the definition of the theta lift, we have

$$\begin{aligned} &W_0(\theta(f,\phi))(h_1) \\ &= \int_{F\backslash\mathbb{A}_F} \int_{\mathrm{Sp}_4(F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)} \sum_{a_i\in X, b_i\in X_+} \omega(g_1, u_0(x)h_1)\phi(a_1,a_2;b_1,b_2)\overline{f(g_1)}\,dg_1\,dx. \end{aligned}$$

Since $Z_-(1, u_0(x)) = Z_-$ and we have

$$z_+(1, u_0(x)) = z_+ + (x \cdot a_1 \otimes y_{-2} - x \cdot a_2 \otimes y_{-1}),$$

we find that

$$\omega(1, u_0(x))\phi(z_+) = \psi\left(-x\langle a_1, a_2\rangle\right)\phi(z_+).$$

Thus, in the above summation, only a_i such that $\langle a_1, a_2\rangle = 0$ contributes to the period $W_0(\theta(f,\phi))$, and we obtain

$$W_0(\theta(f,\phi))(h_1) = \int_{\mathrm{Sp}_4(F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)} \sum_{\substack{a_i\in X, \langle a_1,a_2\rangle=0,\\ b_i\in X_+}} \omega(g_1,h_1)\phi(a_1,a_2;b_1,b_2)\overline{f(g_1)}\,dg_1.$$

Furthermore, since the space spanned by a_1 and a_2 is isotropic, there exists $\gamma \in \mathrm{Sp}_4(F)$ such that $a_1\gamma^{-1}, a_2\gamma^{-1} \in X_-$.

Let us define an equivalence relation $\sim$ on $(X_-)^2$. For a_i and $a_i' \in X_-$, $(a_1, a_2) \sim (a_1', a_2')$ if there exists $\gamma \in \mathrm{Sp}_4(F)$ such that $a_i' = a_i\gamma$. We denote this equivalence class by $\mathcal{X}_-$ and its representative by $\overline{(a_1, a_2)}$. Then we may write $W_0(\theta(f,\phi))(h_1)$ by

$$\int_{\mathrm{Sp}_4(F)\backslash \mathrm{Sp}_4(\mathbb{A}_F)} \sum_{\overline{(a_1,a_2)}\in\mathcal{X}_-} \sum_{\gamma\in V(a_1,a_2)\backslash \mathrm{Sp}_4(F)} \sum_{b_i\in X_+} \omega(g_1,h_1)\phi(a_1\gamma, a_2\gamma, b_1, b_2)\overline{f(g_1)}\, dg_1.$$

Here

$$V(a_1,a_2) = \{g \in \mathrm{Sp}_4(F) \mid a_i g = a_i\}.$$

Lemma 3.1.

$$\sum_{b_i\in X_+} \omega(g_1,h_1)\phi(a_1\gamma, a_2\gamma, b_1, b_2) = \sum_{b_i\in X_+} \omega(\gamma g_1, h_1)\phi(a_1, a_2, b_1, b_2).$$

Proof. This is proved similarly as [5, Lemma 2]. □

Moreover, the following lemma can be proved similarly as the above computation of $W_0(\theta(f,\phi))$.

Lemma 3.2.

$$\int_{(F\backslash\mathbb{A}_F)^2} \omega(g, u_1(s_1,t_1)h)\phi(a_1,a_2,b_1,b_2)\, ds_1\, dt_1\, ds_2\, dt_2$$
$$= \begin{cases} \omega(g,h)\phi(a_1,a_2,b_1,b_2) & \textit{if } \langle a_2, b_1\rangle = 0 \textit{ and } \langle a_2, b_2\rangle = 0, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\int_{(F\backslash\mathbb{A}_F)^2} \psi(-2dt_2)\omega(g, u_2(s_2,t_2)h)\phi(a_1,a_2,b_1,b_2)\, ds_2\, dt_2$$
$$= \begin{cases} \omega(g,h)\phi(a_1,a_2,b_1,b_2) & \textit{if } \langle a_1, b_1\rangle = 0 \textit{ and } \langle a_1, b_2\rangle = -1, \\ 0 & \textit{otherwise.} \end{cases}$$

This lemma shows that

$$W(\theta(f,\phi)) = \int_{F\backslash\mathbb{A}_F}\int_{\mathrm{Sp}_4(F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)} \sum_{\overline{(a_1,a_2)}\in\mathcal{X}_-} \sum_{\gamma\in V(a_1,a_2)\backslash\mathrm{Sp}_4(F)} \sum_{\substack{\langle a_i,b_1\rangle=\langle a_2,b_2\rangle=0,\\ \langle a_1,b_2\rangle=-1, b_i\in X_+}}$$
$$\omega(\gamma g, \tilde{u}(b)h)\phi(a_1,a_2,b_1,b_2)\overline{f(g)}\, dg\, db.$$

It is easy to see that in the above integral, only terms such that a_1 and a_2 are linearly independent contribute to the function $W(\theta(f,\phi))(h)$. Therefore, we may take

$$a_1 = x_{-2} \quad \text{and} \quad a_2 = x_{-1}.$$

Then we should have

$$b_1 = 0 \quad \text{and} \quad b_2 = x_2.$$

Hence,

$$\begin{aligned}W(\theta(f,\phi)) &= \int_{F\backslash\mathbb{A}_F}\int_{\mathrm{Sp}_4(F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)}\psi(-b)\\ &\qquad\sum_{\gamma\in N_P(F)\backslash\mathrm{Sp}_4(F)}\omega(\gamma g_1,\tilde{u}(b))\phi(x_{-2},x_{-1},0,x_2)\overline{f(g_1)}\,dg_1\,db\\ &= \int_{F\backslash\mathbb{A}_F}\int_{N_P(\mathbb{A}_F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)}\int_{N_P(F)\backslash N_P(\mathbb{A}_F)}\\ &\qquad\psi(-b)\omega(vg_1,\tilde{u}(b))\phi(x_{-2},x_{-1},0,x_2)\overline{f(vg_1)}\,dv\,dg_1\,db.\end{aligned}$$

Since $Z_-(\mathbb{A}_F)(v(A),1)=Z_-(\mathbb{A}_F)$ and

$$\begin{aligned}&(x_{-2}\otimes y_1+x_{-1}\otimes y_2+x_2\otimes e_2)(v(A),1)\\ =&x_{-2}\otimes y_1+x_{-1}\otimes y_2+(x_2+a_{21}x_{-1}+a_{22}x_{-2})\otimes e_2\end{aligned}$$

for $A=\left(\begin{smallmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{smallmatrix}\right)$, we have

$$\omega(vg_1,\tilde{u}(b)h)\phi(x_{-2},x_{-1},0,x_2)=\psi(-da_{22})\omega(g_1,\tilde{u}(b)h)\phi(x_{-2},x_{-1},0,x_2).$$

Thus, we obtain

$$\begin{aligned}W(\theta(f,\phi))(h) = &\int_{N_P(\mathbb{A}_F)\backslash\mathrm{Sp}_4(\mathbb{A}_F)}\int_{F\backslash\mathbb{A}_F}\int_{N_P(F)\backslash N_P(\mathbb{A}_F)}\\ &\psi_N(w(b)v)\omega(g_1,\tilde{u}(b)h)\phi(x_{-2},x_{-1},0,x_2)\overline{f(vg_1)}\,dv\,dg_1\,db.\end{aligned}$$

Here we define a character ψ_N on $N(F)\backslash N(\mathbb{A}_F)$ by

$$\psi_N\left[w(a)v(A)\right]=\psi(-a-d\cdot a_{22}). \tag{3.2}$$

On the other hand, we have

$$(x_{-2}\otimes y_1+x_{-1}\otimes y_2+x_2\otimes e_2)(1,\tilde{u}(b))=(x_{-2}\otimes y_1+x_{-1}\otimes y_2+x_2\otimes e_2)(w(b),1).$$

Therefore,

$$\begin{aligned}&W(\theta(f,\phi))(h)\\ =&\int_{N_P(\mathbb{A}_F)\backslash G_1(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\psi_N(n)\omega(g_1,h)\phi(x_{-2},x_{-1},0,x_2)\overline{f(ng_1)}\,dv\,dg_1\,db\\ =&\int_{N_P(\mathbb{A}_F)\backslash G_1(\mathbb{A}_F)}\omega(g_1,h)\phi(x_{-2},x_{-1},0,x_2)\overline{W'(\pi(g_1)f)}dg_1\end{aligned}$$

where

$$W'(\pi(g_1)f)=\int_{N(F)\backslash N(\mathbb{A}_F)}f(ng_1)\psi_N^{-1}(n)\,dn, \tag{3.3}$$

which is a Whittaker period on $\mathrm{GSp}_4(\mathbb{A}_F)^+$. We shall say that π is generic with respect to ψ_N if this period is not zero on the space V_π of π. Then as in [9, pp. 2718–2719], we can show the following equivalence.

Proposition 3.3. *For an irreducible cuspidal automorphic representation (π,V_π) of $\mathrm{GSp}_4(\mathbb{A}_F)^+$, (π,V_π) is generic with respect to ψ_N if and only if the theta lift $\Theta(\pi)$ of (π,V_π) to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ is generic.*

3.2. **Theta lift from** $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ **to** $\mathrm{GSp}_4(\mathbb{A}_F)^+$**.** For an automorphic form φ on $\mathrm{GSp}_4(\mathbb{A}_F)^+$, we define Whittaker period of φ by (3.3), namely,

$$W'(\varphi) = \int_{N(F)\backslash N(\mathbb{A}_F)} \varphi(n)\psi_N^{-1}(n)\, dn.$$

Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. We define the theta lift of σ to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ by

$$\Theta(\sigma) = \langle \theta(f,\phi) \mid f \in V_\sigma, \phi \in \mathcal{S}(Z_+(\mathbb{A}_F))\rangle$$

which is an automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)^+$. As in the previous section, we know that if it consists of cusp forms, $\Theta(\sigma)$ is irreducible and compatible with the local theta correspondence (see [7, Proposition 2.12]).

Our aim in this subsection is to compute $W'(\theta(f,\phi))$ for $f \in V_\sigma$ and $\phi \in \mathcal{S}(Z_+(\mathbb{A}_F))$. As a polarization, we use

$$Z_\pm = X_\pm \otimes Y,$$

and we shall identify Z_+ with $Y \oplus Y$ by $x \otimes y \mapsto (\langle x, x_{-1}\rangle y, \langle x, x_{-2}\rangle y)$. Thus, we may consider the lift for $\phi \in \mathcal{S}[(Y \oplus Y)(\mathbb{A}_F)]$.

Let $N_1 = \{w(x) \mid x \in \mathbb{G}_a\}$. Then for $n_1 \in N_1(\mathbb{A}_F)$, we let

$$J(n_1) = \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} \theta(f,\phi)(un_1)\psi_N^{-1}(u)\, du.$$

Then as in the previous section, we have

$$J(n_1) = \int_{\mathrm{SO}_{4,2}(F)\backslash \mathrm{SO}_{4,2}(\mathbb{A}_F)} \sum_{(w_1,w_2)\in V_1} \omega(n_1,h_1)\phi(w_1,w_2)\overline{f(h_1)}\, dh_1$$

where

$$V_1 = \left\{ (v_1,v_2) \in (Y\oplus Y)(F) \mid \begin{pmatrix} (v_1,v_1) & (v_1,v_2) \\ (v_1,v_2) & (v_2,v_2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2d \end{pmatrix} \right\}.$$

Here we note that

$$\begin{aligned} &\int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \psi_N^{-1}(n_1)\omega(n_1,h_1)\phi(0,v_2)\overline{f(h_1)}\, dh_1 \\ =&\int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \psi_N^{-1}(n_1)\omega(1,h_1)\phi(0,v_2)\overline{f(h_1)}\, dh_1 = 0, \end{aligned}$$

since

$$\omega(n_1,h_1)\phi(0,v_2) = \omega(1,h_1)\phi\left[(0,v_2)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] = \omega(1,h_1)\phi(0,v_2).$$

Thus, it is enough for us to consider the contribution from

$$V_2 = \{(v_1,v_2) \in V_1 \mid v_1 \neq 0\}.$$

Since $(y_{-2}, e_2) \in V_2$, by Witt's theorem, we have

$$\begin{aligned}
&W'(\theta(f,\phi)) \\
&= \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \int_{H_1(F)\backslash H_1(\mathbb{A}_F)} \\
&\qquad \psi_N^{-1}(n_1) \sum_{\gamma \in H_2(F)\backslash H_1(F)} \omega(n_1,h_1)\phi(\gamma^{-1}y_{-2}, \gamma^{-1}e_2)\overline{f(h_1)}\, dh_1\, dn_1 \\
&= \int_{N_1(F)\backslash N_1(\mathbb{A}_F)} \psi_N^{-1}(n_1) \int_{H_2(F)\backslash H_1(\mathbb{A}_F)} \omega(n_1,h_1)\phi(y_{-2},e_2)\overline{f(h_1)}\, dh_1\, dn_1.
\end{aligned}$$

Here H_2 is the stabilizer of e_2 and y_{-2} in H_1. Then we have $H_2 = RU'$ where

$$R = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid r \in \mathrm{SO}_{3,1},\ re_2 = e_2 \right\}$$

and

$$U' = \{u_0(a)u_1(b,0)\tilde{u}(c) \mid a,b,c \in F\}.$$

We note that

$$\begin{aligned}
\omega\left[w(x), h_1\right]\phi(y_{-2},e_2) =& \omega(1,h_1)\phi\left[(y_{-2},e_2)\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] \\
=& \omega(1, u_1(0,-x/2d)h_1)\phi(y_{-2},e_2).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
W'(\theta(f,\phi)) =& \int_{F\backslash\mathbb{A}_F} \psi(x) \int_{H_2(\mathbb{A}_F)\backslash H_1(\mathbb{A}_F)} \int_{H_2(F)\backslash H_2(\mathbb{A}_F)} \\
&\omega(1, u_1(0,-x/2d)h_1)\phi(y_{-2},e_2)\overline{f(h_2h_1)}\, dh_2\, dh_1\, dx \\
=& \int_{H_2(\mathbb{A}_F)\backslash H_1(\mathbb{A}_F)} \omega(1,h_1)\phi(y_{-2},e_2) \int_{F\backslash\mathbb{A}_F} \psi(-2dx) \\
&\cdot \int_{U'(F)\backslash U'(\mathbb{A}_F)} \int_{R(F)\backslash R(\mathbb{A}_F)} \overline{f(ru'u_1(0,x)h_1)}\, dr\, du_0\, dx\, dh_1.
\end{aligned}$$

Let us define a subgroup

$$S = R \cdot U_{P_H}$$

and a character ψ' of $S(F)\backslash S(\mathbb{A}_F)$ by

$$\psi'\left(r \cdot u_0(x)u_1(y,z)\tilde{u}(w)\right) = \psi(2dz)$$

for $r \in R(\mathbb{A}_F)$ and $x,y,z,w \in \mathbb{A}_F$. Then we have

$$W'(\theta(f,\phi)) = \int_{H_2(\mathbb{A}_F)\backslash H_1(\mathbb{A}_F)} \omega(1,h_1)\phi(y_{-2},e_2)\overline{Sh(\pi(h_1)f)}\, dh_1$$

where a linear form $Sh : V_\sigma \to \mathbb{C}$ is defined by

$$Sh(f) = \int_{S(F)\backslash S(\mathbb{A}_F)} \psi'(s)f(s)\, dr\, du. \tag{3.4}$$

As in the previous section, we obtain the following equivalence.

Proposition 3.4. *For an irreducible cuspidal automorphic representation (σ, V_σ) of $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$, $Sh(f)$ is not identically zero on V_σ if and only if the theta lift $\Theta(\sigma)$ of (σ, V_σ) to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ is generic with respect to ψ_N.*

3.3. **Shalika period and global theta lift.** Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$ with trivial central character. Then we can consider (σ, V_σ) as an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ via (2.2). Therefore, we may consider the theta correspondence for $\mathrm{GSp}_4(\mathbb{A}_F)^+$ and $\mathrm{GU}_{2,2}(\mathbb{A}_F)$. Under this identification, it is easy to see that

(3.5)
$$Sh(f) = \int_{\mathbb{A}_F^\times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} f\left(\begin{pmatrix} g & \\ & \det g \cdot {}^t g^{-1}\end{pmatrix} n\right) \psi_0(n)\, dg\, dn$$

where ψ_0 is the character of $N(F)\backslash N(\mathbb{A}_F)$ defined by

$$\psi_0\begin{pmatrix} 1_2 & X \\ & 1_2\end{pmatrix} = \psi\left(\mathrm{tr}\left[\begin{pmatrix} & \eta \\ -\eta & \end{pmatrix} X\right]\right)$$

for $X = {}^t\overline{X} \in \mathrm{Mat}_{2\times 2}(\mathbb{A}_E)$. In [6], we called the period on the right-hand side Shalika period, and we say that σ has Shalika period if $Sh(f)$ is not identically zero on V_σ. Moreover, we have an equivalent condition that σ has Shalika period.

Theorem 3.5 (Theorem in [6])**.** *Let (σ, V_σ) be as above. Let S be a finite set of places of F containing all infinite places such that at every $v \notin S$, E_v/F_v is not a ramified extension and σ_v is unramified. Then the following conditions are equivalent.*

(1) *σ has Shalika period.*
(2) *σ is generic and the partial twisted exterior square L-function $L^S\left(s, \sigma, \wedge_t^2\right)$ has a simple pole at $s = 1$.*

From this theorem and the computation of pullbacks of Whittaker periods in the previous sections, we can give a characterization of irreducible cuspidal automorphic representations of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ which have Shalika period, in the language of the theta lift.

Theorem 3.6. *Let (σ, V_σ) be an irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ with trivial central character. Then σ has Shalika period if and only if $\sigma = \theta^*(\Pi)$ for a generic irreducible cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$ with trivial central character.*

Here we write $\sigma = \theta^*(\Pi)$ when $\sigma = \Theta(\Pi^+)$ for an irreducible cuspidal automorphic representation Π^+ of $\mathrm{GSp}_4(\mathbb{A}_F)^+$ which appears as a subrepresentation of $\Pi|_{\mathrm{GSp}_4(\mathbb{A}_F)^+}$.

Remark 3.7. Let Π' be another irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ such that $\sigma = \theta^*(\Pi')$. Then at each place v of F where $E_v \simeq F_v \oplus F_v$, we have $\Pi'_v \simeq \Pi_v$. On the other hand, at each place v of F where E_v/F_v is a quadratic extension, we have $\Pi'_v \simeq \Pi_v \otimes \chi_{E_v/F_v}$ or Π_v for the quadratic character of $F_v^\times$ corresponding to the quadratic extension E_v/F_v. This implies that $\Pi|_{\mathrm{GSp}_4(\mathbb{A}_F)^+} \simeq \Pi'|_{\mathrm{GSp}_4(\mathbb{A}_F)^+}$. Hence, our assertion does not depend on a choice of Π.

Proof. Suppose that σ has Shalika period. Then by Proposition 3.4, the theta lift $\Theta(\sigma)$ to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ is not zero and generic. Assume that $\Theta(\sigma)$ is not contained in the space of cusp forms. Then from the Rallis tower property, σ is given by the theta lift from an irreducible cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_2(\mathbb{A}_F)^+$ where

$$\mathrm{GL}_2(\mathbb{A}_F)^+ = \{g \in \mathrm{GL}_2(\mathbb{A}_F) \mid \det g = \nu(h) \text{ for } h \in \mathrm{GSO}_{4,2}(\mathbb{A}_F)\}.$$

Lemma 3.8. *The theta lift of* (π, V_π) *to* $\mathrm{GSO}_{3,1}(\mathbb{A}_F)$ *is not zero.*

Proof. First, we shall compute the Shalika period for the theta lift of π to $\mathrm{GSO}_{4,2}(\mathbb{A}_F)$. Let $X' = \mathrm{span}\{x_1, x_{-1}\}$ and $Y' = \mathrm{span}\{y_{\pm 1}, e_1, e_2\}$. Then $\mathrm{GL}(X') = \mathrm{GL}_2$ and $\mathrm{GSO}(Y') = \mathrm{GSO}_{3,1}$. As a polarization $Z' = Z'_+ + Z'_-$ of $Z' = X' \otimes Y$, we use

$$Z'_+ = X' \otimes (F \cdot y_2) + F \cdot x_1 \otimes Y' \text{ and } Z'_- = X' \otimes (F \cdot y_{-2}) + F \cdot x_{-1} \otimes Y'.$$

According to this polarization, we may write $z_+ = x \otimes y_2 + x_1 \otimes v$ and $\phi(z_+) = \phi(x, v)$ for $z_+ \in Z'_+(\mathbb{A}_F)$ and $\phi \in S(Z'_+(\mathbb{A}_F))$. Then it is easy to see that

$$\begin{aligned}
&\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{x, \in X, v \in Y'} \omega(g_1, ru)\phi(x, v)\overline{f(g_1)} dg_1\, dr\, du \\
&= \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{\gamma \in N_2(F)\backslash \mathrm{SL}_2(F), v \in Y'} \omega(\gamma g_1, ru)\phi(x_{-1}, v)\overline{f(g_1)} dg_1\, dr\, du \\
&= \int_{N_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \sum_{v \in Y'} \omega(g_1, ru)\phi(x_{-1}, v)\overline{f(g_1)} dg_1\, dr\, du
\end{aligned}$$

where N_2 is the upper unipotent subgroup of SL_2. Thus, we have

$$\begin{aligned}
Sh(\theta(f, \phi)) = &\int_{N_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \int_{R(F)\backslash R(\mathbb{A}_F)} \sum_{v \in Y'} \omega(g_1, r)\phi(x_{-1}, v)\overline{f(g_1)} \\
&\cdot \left(\int_{U_{P_H}(F)\backslash U_{P_H}(\mathbb{A}_F)} \psi^{-1}(y_2, u^{-1}v)\psi'(u)\, du \right) dr\, dg_1 \\
&= \int_{N_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \int_{R(F)\backslash R(\mathbb{A}_F)} \omega(g_1, r)\phi(x_{-1}, e_2)\overline{f(g_1)}\, dr\, dg_1.
\end{aligned}$$

Moreover, since $R(F)$ stabilizes e_2, we obtain

$$\begin{aligned}
Sh(\theta(f, \phi)) &= \int_{N_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \omega(g_1, 1)\phi(x_{-1}, e_2)\overline{f(g_1)}\, dr\, dg_1 \\
&= \int_{N_2(\mathbb{A}_F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \omega(g_1, 1)\phi(x_{-1}, e_2)\overline{W_{\psi,2}(f)(g_1)}\, dr\, dg_1
\end{aligned}$$

where we define

$$W_{\psi,2}(f) = \int_{F\backslash \mathbb{A}_F} \psi(dx) f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right) dx.$$

Here we have used measures which are normalized so that

$$\int_{U_{P_H}(F)\backslash U_{P_H}(\mathbb{A}_F)} du = \int_{R(F)\backslash R(\mathbb{A}_F)} dr = 1.$$

Since we assume that the Shalika period is not identically zero on V_σ, $W_{\psi,2}(f)$ is not identically zero on V_π. On the other hand, we consider the theta lift of π to $\mathrm{GSO}_{3,1}(\mathbb{A}_F)$. Then by a similar computation as in Section 3.1, we see that

$$\begin{aligned}
&\int_{(F\backslash \mathbb{A}_F)^2} \psi(-2dt)\theta(f, \phi)(u_2(s, t))\, ds\, dt \\
&= \int_{N_2(\mathbb{A}_F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \omega(g_1, 1)\phi(x_{-1}, e_2)\overline{W_{\psi,2}(f)(g_1)}\, dg_1.
\end{aligned}$$

Since $W_{\psi,2}(f)$ is not identically zero on V_π, the theta lift of π to $\mathrm{GSO}_{3,1}(\mathbb{A}_F)$ does not vanish. $\square$

By this lemma and the Rallis tower property, σ should not be cuspidal . This is a contradiction, so that the theta lift $\Theta(\sigma)$ of σ to $\mathrm{GSp}_4(\mathbb{A}_F)^+$ is cuspidal.

For an automorphic form $\varphi \in V_{\Theta(\sigma)}$, we define an automorphic form φ^* on $\mathrm{GSp}_4(\mathbb{A}_F)$ by

$$\varphi^*(g) = \begin{cases} \varphi(g) & \text{if } g \in \mathrm{GSp}_4(F)\mathrm{GSp}_4(\mathbb{A}_F)^+, \\ 0 & \text{otherwise.} \end{cases}$$

Let Π^* be the cuspidal generic automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_F)$ generated by φ^* for $\varphi \in V_{\Theta(\pi)}$. Then from the definition of Π^*, there exists an irreducible constituent Π of Π^* such that $\sigma = \theta^*(\Pi)$.

On the other hand, we suppose that $\sigma = \theta^*(\Pi)$ with some irreducible generic cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$. Then by Corollary 6.23, we see that

$$L^S(s, \sigma, \wedge_t^2) = L^S(s, \Pi, std \otimes \chi_{E/F})\zeta_F^S(s), \tag{3.6}$$

where S is the finite set of places such that at $v \notin S$, Π_v and σ_v are unramified. Since Π is generic, $L^S(s, \Pi, std \otimes \chi_{E/F})$ does not vanish at $s = 1$ by Shahidi [33, Theorem 5.1]. Thus, $L^S(s, \sigma, \wedge_t^2)$ has a pole at $s = 1$. Moreover, σ is generic by Proposition 3.3. Therefore, σ should have Shalika period by Theorem 3.5. □

3.4. **A remark on some nongeneric representations of** $\mathrm{GU}_{2,2}$**.** Since any irreducible cuspidal automorphic representation of GL_{2n} is generic (cf. [35]), for all irreducible cuspidal automorphic representation of GL_{2n}, the nonvanishing of Shalika period is equivalent to the existence of a pole of exterior square L-functions at $s = 1$. However, all irreducible cuspidal automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$ are not generic. Thus, for nongeneric irreducible cuspidal automorphic representations of $\mathrm{GU}_{2,2}(\mathbb{A}_F)$, the nonvanishing of Shalika period may not be equivalent to the existence of the pole of twisted exterior square L-functions at $s = 1$. Indeed, we can construct an irreducible cuspidal automorphic representation σ of $\mathrm{GU}_{2,2}(\mathbb{A}_\mathbb{Q})$ such that $L^S(s, \sigma, \wedge_t^2)$ has a pole at $s = 1$, but it does not have Shalika period.

Lemma 3.9. *There exists a quadratic extension $E/\mathbb{Q}$ and an irreducible cuspidal automorphic representation σ of $\mathrm{GU}_{2,2}(\mathbb{A}_\mathbb{Q})$ such that $\omega_\sigma = 1$ and the partial twisted exterior square L-function $L^S(s, \sigma, \wedge_t^2)$ has a simple pole at $s = 1$ for a finite set of places S given in Theorem 3.5, but σ does not have Shalika period.*

Proof. Let Φ be a Siegel cusp form of weight $\ell > 6$ and of full level. Suppose that Φ is a Hecke eigenform and not Saito-Kurokawa lifting. Then the cuspidal automorphic representation Π_Φ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ attached to Φ is irreducible [26, Corollary 3.4]. Note that the central character of Π_Φ is trivial.

From [21, Theorem 7.2], there exists an orthogonal space V of dimension 6 with Witt index 2 such that the theta lift of Π_Φ to $\mathrm{GSO}(V, \mathbb{A}_\mathbb{Q})$ is not zero. Moreover, from the result by Paul (see Section 7), the local theta lift of Π_∞ for $\mathrm{SO}_{3,1}$ is zero. Thus, the global theta lift of Π_Φ to $\mathrm{GSO}_{3,1}(\mathbb{A}_\mathbb{Q})$ is zero, and the global theta lift $\theta^*(\Pi_\Phi)$ of Π_Φ to $\mathrm{GSO}(V, \mathbb{A}_\mathbb{Q})$ is contained in the space of cusp forms by the Rallis tower property. Let σ be an irreducible constituent of $\theta^*(\Pi_\Phi)$.

As in Section 2.1, for some quadratic extension E of $\mathbb{Q}$, we have $\mathrm{PGU}_{2,2} \simeq \mathrm{PGSO}(V)$. Then we consider σ as an automorphic representation of $\mathrm{GU}_{2,2}(\mathbb{A}_\mathbb{Q})$. Since Π_Φ is not generic, from the computation in Section 3.1, σ is also nongeneric. In particular, σ does not have Shalika period by Theorem 3.5. On the other hand,

from Pitale, Saha, and Schmidt [29], the partial L-function $L^S(s, \Pi_\Phi, std \otimes \chi_{E/\mathbb{Q}})$ does not vanish at $s = 1$. Thus, we see that $L^S(s, \sigma, \wedge_t^2)$ has a pole at $s = 1$ from (3.6) and σ satisfies our required conditions. □

4. Non-supercuspidal representations of $\mathrm{GSp}_4(F)^+$

Let F be a nonarchimedean local field of characteristic zero and E a quadratic extension of F. We denote by $x \mapsto \bar{x}$ the action of the nontrivial element of $\mathrm{Gal}(E/F)$ on E. We choose $d \in F^\times \setminus (F^\times)^2$ such that $E = F(\eta)$ and $\bar{\eta} = -\eta$ with $\eta = \sqrt{d}$. Denote by $N_{E/F}$ the norm map from E to F. Let $\chi_{E/F}$ be the quadratic character of $F^\times$ corresponding to E via the local class field theory. Let ψ be a nontrivial additive character of F.

Let

$$G = \mathrm{GSp}_4(F)^+ = \left\{ g \in \mathrm{GSp}_4(F) \mid \lambda(g) \in N_{E/F}(E^\times) \right\}.$$

We note that this definition is consistent with global one from the remark after global definition (2.8).

From now on, we shall consider only admissible representations, so that we frequently simply call it representation. Then our aim in this section is to give a classification of non-supercuspidal irreducible representations of G using a classification of non-supercuspidal irreducible representations of GSp_4 by Sally and Tadic [32] and a study of the restriction of irreducible representations of GSp_4 to Sp_4 by Gan and Takeda [11]. We note that for a structure of parabolic inductions, [13, Lemma 5.1, Lemma 5.2] is convenient.

For an algebraic group X defined over F, we simply denote by X the set of F-rational points of X. For each positive integer n, we define

$$\mathrm{GL}_n^+ = \left\{ g \in \mathrm{GL}_n \mid \det g \in N_{E/F}(E^\times) \right\}.$$

Let P and Q be parabolic subgroups of GSp_4 defined as in Section 2.1. Let us denote by $B_G = B \cap G$, $P_G = P \cap G$ and $Q_G = Q \cap G$ the parabolic subgroups of G. Then we have the Levi decompositions $B_G = T_G N$, $P_G = M_{P_G} N_{P_G}$ and $P_G = M_{P_G} N_{P_G}$ where $N_{P_G} = N_P$, $N_{Q_G} = N_Q$, $T_G \simeq \mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1^+$, $M_{P_G} \simeq \mathrm{GL}_2 \times \mathrm{GL}_1^+$ and $M_{Q_G} \simeq \mathrm{GL}_2^+ \times \mathrm{GL}_1$.

For any irreducible representation π of GSp_4, GL_2 or GL_1, we denote by π_+ the restriction to G, GL_2^+ or GL_1^+, respectively. Moreover, for an irreducible representation π^+ of G or GL_2^+, we sometimes simply denote by $\mathrm{Ind}(\pi^+)$ for $\mathrm{Ind}_G^{\mathrm{GSp}_4}(\pi^+)$ and $\mathrm{Ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2}(\pi^+)$.

Any irreducible representation of T_G is of the form $\chi_1 \boxtimes \chi_2 \boxtimes \chi_+$ for characters χ_1, χ_2, and χ of $F^\times$, and we denote by $I_{B_G}(\chi_1, \chi_2; \chi_+)$ the parabolic induction of G corresponding to this representation of T_G. Moreover, any irreducible representation of M_{P_G} (resp. M_{Q_G}) is of the form $\tau \boxtimes \mu$ (resp. $\mu \boxtimes \tau$) with an irreducible representation τ of GL_2 (resp. GL_2^+) and a character μ of GL_1^+ (resp. GL_1), and we denote by $I_{P_G}(\tau, \mu)$ (resp. $I_{Q_G}(\mu, \tau)$) the corresponding parabolic induction of G.

Similarly, for representations τ of GL_2 and χ of GL_1, we denote by $I_P(\tau, \chi)$ (resp. $I_Q(\chi, \tau)$) the parabolic induction of GSp_4 for $\tau \boxtimes \chi$ (resp. $\chi \boxtimes \tau$) with respect to P (resp. Q).

For a character χ of GL_1, we denote by $\chi \cdot \pi_+$ the twist of π_+ by χ_+. Then we note that

$$(\chi \cdot \omega_{E/F}) \cdot \pi_+ = \chi \cdot \pi_+.$$

Let us denote by st the Steinberg representation of GL_2. We note that st_+ is an irreducible representation of GL_2^+.

Further, we define a character ψ_N of the unipotent radical N of the Borel subgroup B_G by

$$\psi_N\left[w(a)v(A)\right] = \psi(-a - d \cdot a_{22})$$

for $a \in F$ and $A = (a_{ij}) \in \mathrm{Sym}_2(F)$. This is a nondegenerate character of N, and there are two orbits of nondegenerate characters of N under the action of T_G. We fix a representative of the orbit not containing ψ_N, and we denote this character by $\psi_{N,-}$. When a representation of G is generic with respect to ψ_N, we simply say that it is generic.

4.1. **Discrete series representations.** In this section, we shall give a classification of non-supercuspidal essentially discrete series representations of G. Indeed, we prove the following lemma.

Lemma 4.1. (1) *Let χ be a character of $F^\times$. Then $I_{Q_G}(|-|^2, (\chi|-|^{-1}) \cdot st_+)$ contains an essentially discrete series representation $\chi\mathrm{St}_G$ as the unique irreducible submodule.*

(2) *Let τ^+ be an irreducible supercuspidal representation of GL_2^+. Suppose that χ is a nontrivial quadratic character of $F^\times$ such that $\tau \otimes \chi = \tau$ for any irreducible representation τ of GL_2 such that $\tau^+ \subset \tau_+$. Then $I_{Q_G}(\chi| - |, |-|^{-1/2} \cdot \tau^+)$ contains a generic essentially discrete series representation $St(\chi, \tau^+)$ as the unique irreducible submodule.*

(3) *Let τ be an irreducible supercuspidal representation of GL_2 with trivial central character, or $\tau = \chi \cdot st$ with nontrivial quadratic character $\chi \neq \chi_{E/F}$ of $F^\times$. Let μ be a character of $F^\times$. Then $I_{P_G}(\tau| - |^{1/2}, \mu_+| - |_+^{-1/2})$ contains an essentially discrete series representation $St(\tau, \mu_+)$ as the unique irreducible submodule.*

(4) *Let μ be a character of $F^\times$. Then $I_{P_G}(\chi_{E/F} \cdot st| - |^{-1/2}, \mu_+| - |_+^{1/2})$ contains two inequivalent essentially discrete series representation of G. The exactly one of them is generic with respect to ψ_N and we denote this irreducible representation by $St(\chi_{E/F} \cdot st, \mu_+)^+$. On the other hand, the other one is generic with respect to $\psi_{N,-}$ and we denote it by $St(\chi_{E/F} \cdot st, \mu_+)^-$.*

Moreover, any non-supercuspidal essentially discrete series representation of G appears in the above list.

Proof. First, we note that for an irreducible representation π of GSp_4, π_+ is irreducible if and only if $\pi \otimes \chi_{E/F} \simeq \pi$ (i.e., $\chi_{E/F} \in I(\pi)$ if we use the notation in [11]).

Let us consider case (1). Recall that the induced representation $I_Q(| - |^2, (\chi| - |^{-1}) \cdot st)$ contains the twisted Steinberg representation $\chi\mathrm{St}_{\mathrm{GSp}_4}$ as the unique irreducible submodule [13, Lemma 5.1]. From [11, Proposition 6.8], the restriction of $\chi\mathrm{St}_{\mathrm{GSp}_4}$ to Sp_4 is irreducible. In particular, its restriction to G is irreducible. We denote this irreducible representation of G by $\chi\mathrm{St}_G$. Moreover, we note that

$$I_{Q_G}(| - |^2, (\chi| - |^{-1}) \cdot st_+) = I_Q(| - |^2, (\chi| - |^{-1}) \cdot st)_+. \tag{4.1}$$

Thus, $\chi\mathrm{St}_G$ is an irreducible submodule of $I_{Q_G}(|-|^2, (\chi|-|^{-1})\cdot st_+)$, and it is obviously an essentially discrete series.

Let π^+ be an irreducible submodule of $I_{Q_G}(|-|^2, (\chi|-|^{-1})\cdot st_+)$. Then by the Frobenius reciprocity and (4.1), we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_G\left(\pi^+, I_{Q_G}(|-|^2, (\chi|-|^{-1})\cdot st_+)\right) \\ &\simeq \mathrm{Hom}_{\mathrm{GSp}_4}\left(\mathrm{Ind}(\pi^+), I_Q(|-|^2, (\chi|-|^{-1})\cdot st)\right). \end{aligned}$$

We note that $\mathrm{Ind}(\pi^+)$ is irreducible or isomorphic to $\pi\oplus(\pi\otimes\chi_{E/F})$ with an extension π of π^+ to GSp_4. If $\mathrm{Ind}(\pi^+)$ is irreducible, it should be isomorphic to $\chi\mathrm{St}_G$. However, since $(\mathrm{Ind}(\pi^+))|_G$ is reducible, this case does not occur. Thus, we may suppose $\mathrm{Ind}(\pi^+) = \pi\oplus(\pi\otimes\chi_{E/F})$. Then we should have $\chi\mathrm{St}_{\mathrm{GSp}_4} = \pi$ or $\pi\otimes\chi_{E/F}$. In any case, we have

$$\pi^+ = \pi_+ = \chi\mathrm{St}_{\mathrm{GSp}_4}|_G = \chi\mathrm{St}_G.$$

Therefore, $\chi\mathrm{St}_G$ is a unique irreducible submodule.

Let us consider the case (2). Then we have

$$I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_+) = I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)_+,$$

and $I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)$ contains the essentially discrete series $St(\chi,\tau)$ as the unique irreducible submodule [13, Lemma 5.1]. If τ_+ is irreducible, as in the previous case, we can show that $I_{Q(Z)^+}(\chi|-|, |-|^{-1/2}\cdot\tau_+)$ contains an essentially discrete series representation $St(\chi,\tau^+)$ as a unique irreducible submodule.

We suppose that $\tau_+ = \tau_1\oplus\tau_2$ and $\tau_1 = \tau^+$. Then we have

$$I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)|_G = I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_1)\oplus I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_2).$$

We note that $\mathrm{Ind}(\tau_1)$ is irreducible since τ_+ is reducible, and thus it is isomorphic to τ. Therefore, we see that

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GSp}_4}(St(\chi,\tau), I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)) \\ =&\mathrm{Hom}_{\mathrm{GSp}_4}(St(\chi,\tau), \mathrm{Ind}_G^{\mathrm{GSp}_4}(I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_1))) \\ =&\mathrm{Hom}_G(St(\chi,\tau)|_G, I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_1)). \end{aligned}$$

Since $St(\chi,\tau)$ is a unique irreducible submodule of $I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)$, the above space is one-dimensional. Furthermore, from [11, Proposition 6.8], we see that $St(\chi,\tau)|_G$ is reducible, and it decomposes into a direct sum of two inequivalent irreducible representations. Therefore, exactly one constituent of them appears in $I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_1)$. Then we denote this representation by $St(\chi,\tau^+)$.

Let π^+ be an irreducible submodule of $I_{Q_G}(\chi|-|, |-|^{-1/2}\cdot\tau_1)$. Then we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_G(\pi^+, I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)|_G) \\ &= \mathrm{Hom}_{\mathrm{GSp}_4}(\mathrm{Ind}(\pi^+), I_Q(\chi|-|, |-|^{-1/2}\cdot\tau)). \end{aligned}$$

Assume that $\mathrm{Ind}(\pi^+)$ is reducible. Then some irreducible constituent of $\mathrm{Ind}(\pi^+)$ is isomorphic to $St(\chi,\tau)$. However, its restriction to G is irreducible. This contradicts to the reducibility of $St(\chi,\tau)|_G$. Therefore, we may suppose that $\mathrm{Ind}(\pi^+)$ is irreducible. Then it should be isomorphic to $St(\chi,\tau)$, and we see that π^+ is an irreducible constituent of $St(\chi,\tau)|_G$. Thus, from the definition of $St(\chi,\tau^+)$, we should have $\pi^+ = St(\chi,\tau^+)$.

The case (3) can be proved in a similar way as in the case of (1) using [13, Lemma 5.2] and [13, Proposition 6.8].

Finally, we consider the case (4). Then we know that

$$I_{P_G}(st_\chi|-|^{-1/2}, \mu_+|-|_+^{1/2}) = I_P(st_\chi|-|^{-1/2}, \mu|-|^{1/2})_+.$$

From [13, Proposition 6.8], we see that $St(\chi \cdot st, \mu)_+$ is completely reducible. Since $St(\chi \cdot st, \mu)$ is generic with respect to ψ_N, exactly one irreducible constituent of $St(\chi \cdot st, \mu)_+$ is generic with respect to ψ_N. Then we denote this irreducible constituent by $St(\chi \cdot st, \mu_+)^+$ and the other one by $St(\chi \cdot st, \mu_+)^-$.

Since we know

$$(\psi_N)^t = \psi_{N,-} \quad \text{and} \quad (St(\chi \cdot st, \mu_+)^+)^t \simeq St(\chi \cdot st, \mu_+)^-$$

with some $t \in T \setminus T_G$, $St(\chi \cdot st, \mu_+)^-$ is generic with respect to $\psi_{N,-}$. Moreover, as in the previous cases, we can show that $I_{P_G}(st_\chi|-|^{-1/2}, \mu_+|-|_+^{1/2})$ does not have any other irreducible submodule.

Any irreducible constituent of restrictions of essentially discrete series representations of GSp_4 to G appear in the above list, so that the above representations give all essentially discrete series representations of G. □

4.2. **Nondiscrete series representations.** Let us classify irreducible nondiscrete series representations of G.

Let us define a character ψ_2 of upper unipotent subgroup N_2 of GL_2 by

$$\psi_2 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(-dx). \tag{4.2}$$

There are two orbits of nondegenerate characters of N_2 under the action of diagonal matrices in GL_2^+. Then we fix a representative of the orbit not containing ψ_2 and denote it by $\psi_{2,-}$

Any irreducible non-supercuspidal representation of GL_2 whose restriction to GL_2^+ is reducible should be of the form

$$\chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)$$

with a character χ of $F^\times$ where B_2 is the upper triangular subgroup of GL_2. We denote by $I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^+$ (resp. $I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^-$) the irreducible constituent of $I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)_+$ which is generic with respect to ψ_2 (resp. $\psi_{2,-}$).

Lemma 4.2. *The irreducible nondiscrete series representations of G fall into the following disjoint families:*

(1) $\pi \hookrightarrow I_{Q_G}(\chi|-|^{-s}, \tau^+)$ *with χ a unitary character, $s \geq 0$ and τ^+ an essentially discrete series representation of* GL_2^+. *In fact, π is the unique irreducible submodule, except in the case where $\chi|-|^{-s} = 1_{F^\times}$, or $\chi|-|^{-s} = \chi_{E/F}$ and* $\mathrm{Ind}(\tau^+)$ *is reducible. In those cases, this parabolic induction decomposes into a direst sum of two inequivalent irreducible representations.*

(2) *Let τ be a discrete series representation of* GL_2 *and χ a character of $F^\times$.*
   (a) *Suppose that $s = 0$, and $\tau \simeq \tau \otimes \omega_\tau$ with $\omega_\tau = \chi_{E/F}$. Then $I_{P_G}(\tau, \chi_+)$ decomposes into a direct sum of two inequivalent irreducible representations. The exactly one constituent is generic, and the other constituent is nongeneric.*
   (b) *Otherwise, $I_{P_G}(\tau|-|^{-s}, \chi_+)$ contains a unique irreducible submodule.*

(3) $\pi \hookrightarrow I_{B_G}(\chi_1|-|^{-s_1}, \chi_2|-|^{-s_2}; \chi_+)$ *where* χ_1 *and* χ_2 *are unitary characters,* χ *is an arbitrary character of* $F^\times$ *and* $s_1 \geq s_2 \geq 0$*. By induction in stages, we see that*

$$I_{B_G}(\chi_1|-|^{-s_1}, \chi_2|-|^{-s_2}; \chi_+) = I_{Q_G}(\chi_1|-|^{-s_1}, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_2|-|^{-s_2} \boxtimes \chi)_+).$$

(a) *If* $\chi_2|-|^{-s_2} = \chi_{E/F}$*, then* $I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^{\pm})$ *contains the unique irreducible submodule, respectively.*

(b) *For a nontrivial quadratic character* χ_0*,* $I_{Q_G}(\chi_0, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_0\chi_{E/F} \boxtimes \chi)_+)$ *decomposes into a direct sum of two inequivalent irreducible representations. One constituent is generic with respect to* ψ_N*, and the other one is generic with respect to* $\psi_{N,-}$*.*

(c) *If* $\chi_2|-|^{-s_2} = |-|^{-1}$*, then* $I_{B_2}^{\mathrm{GL}_2}(\chi\chi_2|-|^{-s_2} \boxtimes \chi)$ *is reducible and*

$$\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, (\chi \cdot |-|^{-1/2}) \circ \det)$$

as the unique irreducible submodule.

(d) *Otherwise,* $I_{Q_G}(\chi_1|-|^{-s_1}, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_2|-|^{-s_2} \boxtimes \chi)_+)$ *contains the unique irreducible submodule. In particular, if* $s_1 = s_2 = 0$*, this parabolic induction is irreducible.*

Proof. This lemma is proved in a similar way as Lemma 4.1 using [13, Proposition 5.3] and the restriction of irreducible representations of GSp_4 to G, so that we leave the details to the reader. □

5. Non-supercuspidal representations of $\mathrm{GSO}_{4,2}$

We keep the notation in Section 4. The aim in this section is to give a classification of non-supercuspidal irreducible representations of $H = \mathrm{GSO}_{4,2}(F)$ following Sally and Tadic [32] and Konno [19].

5.1. **Preliminaries.** As in the previous section, for an algebraic group X defined over F, we frequently denote by X the set of its F-rational points. Let B_H be the Borel subgroup of H with the Levi decomposition $B_H = T_H U$ where T_H and U are defined as in (2.3) and (2.4), respectively. Let A_0 be the split component of T_H given by

$$A_0 = \left\{ \mathrm{diag}(t_1, t_2; t) \mid t, t_i \in F^\times \right\}.$$

We denote by $\triangle_0$ the set of simple roots of A_0. Indeed, we have $\Delta_0 = \{\alpha_1, \alpha_2\}$ with

$$\alpha_1(\mathrm{diag}(t_1, t_2; t)) = t_1 t_2^{-1}, \quad \alpha_2(\mathrm{diag}(t_1, t_2; t)) = t_2 t^{-1}$$

for $\mathrm{diag}(t_1, t_2; t) \in A_0$. Moreover, we denote by Σ the set of B_H-positive relative roots. Then we have

$$\Sigma = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}.$$

Let us denote by W^H the Weyl group of A_0 in H. Moreover, we define parabolic subgroups P_H and Q_H of H as in Section 2.1. Then we note that P_H (resp. Q_H) is the maximal parabolic subgroup of H corresponding to the simple root α_2 (resp. α_1).

For $\alpha \in \triangle_0$, we denote by r_α the simple reflection corresponding to α. We set $w_1 = r_{\alpha_1}$ and $w_2 = r_{\alpha_2}$. Indeed, these reflections are represented by

$$w_1 = \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} -1 & & & & & \\ & & & & & 1 \\ & & -1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ & & & & & -1 \end{pmatrix}.$$

In order to study the reducibility of parabolic inductions, we need intertwining operators. For each standard Levi subgroup M, write W_M for the set of $w \in W^H$ of the minimal length in the coset wW^M such that $w(M)$ is again a standard Levi subgroup. Also, we set

$$\mathfrak{a}_M = \mathrm{Hom}(X^*(M), \mathbb{R}), \text{ and } \mathfrak{a}_M^* = X^*(M) \otimes \mathbb{R},$$

where $X^*(M)$ is the group of F-rational characters of M. For $w \in W_M$, $P_w = M_w N_w$ denotes the standard parabolic subgroup with the Levi component $M_w = w(M)$. For a standard parabolic subgroup P, set $\Sigma_P = \{\alpha|_{\mathfrak{a}_M} \mid \alpha \in \Sigma \setminus \Sigma^M\}$ and write Σ_P^r the set of reduced elements in it. Define

$$\mathrm{inv}_P(w) = \{\alpha \in \Sigma_P^r \mid w(\alpha) \notin \Sigma_{P_w}\}.$$

Let π be an irreducible representation of M and $\nu \in \mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes \mathbb{C}$. We regard ν as a quasi-character of M by $M \ni m \mapsto q^{\langle \nu, H_M(m)\rangle}$ (see [34, Section 1] for the map H_M), and consider a representation $\pi \otimes \nu$. Then the integral

$$[M(w, \pi \otimes \nu)\phi](g) = \int_{N_w \cap w(N) \backslash N_w} \phi(w^{-1}ng)\, dn, \quad \phi \in I_P^G(\pi \otimes \nu)$$

converges absolutely if $\alpha^\vee(\nu) \gg 0$ for every $\alpha \in \mathrm{inv}_P(w)$. It extends to a meromorphic function on $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$. Outside its pole it defines an intertwining operator

$$M(w, \pi \otimes \nu) : I_P^G(\pi \otimes \delta_P^s) \to I_{P_w}^G(w(\pi \otimes \nu))$$

(cf. [33] and [36, Section 2]).

For a character χ of $F^\times$, a character τ of $E^\times$, a representation π of GL_2, and π' of $\mathrm{GL}_2(E)$, we define

$$\chi[s] = \chi| - |_F^s, \tau[s] = \tau| - |_E^s, \pi[s] = \pi \otimes |\det|_F^s \text{ and } \pi'[s] = \pi' \otimes |\det|_E^s$$

for $s \in \mathbb{C}$. Also, we define

$$\chi_E = \chi \circ N_{E/F}, \quad \tau_F = \tau|_{F^\times} \quad \text{and} \quad \tau^\sigma(a) = \tau(\bar{a})$$

for $a \in E^\times$. Finally, we note that we have an accidental isomorphism

$$\mathrm{GSO}_{3,1} \simeq \{\mathrm{GL}_2(E) \times F^\times\}/\{(a^{-1}, N_{E/F}(a) \mid a \in E^\times)\},$$

and thus we may denote an irreducible representation of $\mathrm{GSO}_{3,1}$ by $\pi(\rho, \chi)$ for an irreducible representation ρ of $\mathrm{GL}_2(E)$ and a character χ of $F^\times$ such that $\omega_\rho = \chi_E$. In particular, we have

$$\pi\left(I_{B_2(E)}^{\mathrm{GL}_2(E)}(\tau \boxtimes \tau^\sigma \chi_E), \chi\tau_F\right) = I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi \boxtimes \tau)$$

for characters χ of $F^\times$ and τ of $E^\times$. Here we have denoted by B_2 and $B_{3,1}$ the Borel subgroups of GL_2 and $\mathrm{GSO}_{3,1}$, respectively.

Each irreducible representation of T_H is of the form $\chi_1 \boxtimes \chi_2 \boxtimes \tau$ with characters χ_1 and χ_2 of $F^\times$ and a character τ of $E^\times$, and we denote the corresponding parabolic induction of H by $I_{B_H}(\chi_1, \chi_2; \tau)$. When it has a unique irreducible quotient, we shall denote this quotient by $J_{B_H}(\chi_1, \chi_2; \tau)$.

Moreover, any irreducible representation of M_{P_H} (resp. M_{Q_H}) is of the form $\chi \boxtimes \pi$ (resp. $\pi \boxtimes \chi$) with a character χ of $F^\times$ (resp. $E^\times$) and an irreducible representation π of $\mathrm{GSO}_{3,1}$ (resp. GL_2), and we denote the corresponding parabolic induction by $I_{P_H}(\chi, \pi)$ (resp. $I_{Q_H}(\pi, \chi)$). When it has a unique irreducible quotient, we shall denote this quotient by $J_{P_H}(\chi, \pi)$ (resp. $J_{Q_H}(\pi, \chi)$).

5.2. **Parabolically induced representations for B_H.** In this section, we shall classify irreducible representation which appear as an irreducible constituent of the parabolic induction for minimal parabolic subgroup B_H.

5.2.1. *Parabolic inductions for unitary characters.* Let χ_1 and χ_2 be unitary characters of $F^\times$ and τ a unitary character of $E^\times$, and we define a character of T_H by $\rho = \chi_1 \boxtimes \chi_2 \boxtimes \tau$.

For $\beta \in \Sigma$, let $\mu_\beta(\rho)$ be the Plancherel measure of ρ with respect to β. Let $\Sigma' = \{\beta \in \Sigma \mid \mu_\beta(\rho) = 0\}$. Then $\pm\Sigma'$ is a root system of Σ. Thus, the group W' generated by the reflections associated to the elements of Σ' is a subgroup of $W(\rho) = \{w \in W^H \mid w\rho = \rho\}$. Let $R = R(\rho) = \{w \in W(\rho) \mid w\beta > 0, \beta \in \Sigma'\}$.

Theorem 5.1 (Knapp–Stein [18], Silberger [37], [38])**.** *Let ρ be as above. Then we have $W(\rho) = R \ltimes W'$, and the commutator algebra of $I_{B_H}(\rho)$ has dimension $|R|$.*

Then we have the following criterion for a reducibility of $I_{B_H}(\rho)$.

Lemma 5.2. *The parabolic induction $I_{B_H}(\rho)$ is reducible if and only if $W(\rho) = R = \{1, w_1w_2w_1w_2\}$, namely*

$$\chi_1^2 = \chi_2^2 = 1, (\chi_1\chi_2)_E\tau^\sigma = \tau, \chi_1 \neq \chi_2$$

and for $i = 1, 2$,

$$\chi_{i,E}\tau^\sigma \neq \tau.$$

In particular, if $\tau^\sigma = \tau$, $I_{B_H}(\rho)$ is reducible if and only if

$$\chi_1 = \chi_0, \chi_2 = \chi_0\chi_{E/F}$$

with a nontrivial quadratic character $\chi_0 \neq \chi_{E/F}$. When it is reducible, it decomposes into a direct sum of two tempered irreducible representations.

Proof. If R is given as above, this parabolic induction is reducible by Theorem 5.1.

We suppose that $I_{B_H}(\rho)$ is reducible. Then we note that R is not trivial. For $\alpha \in \Sigma$, we denote by A_α the subtorus of A_0 corresponding to α, and we put $M_\alpha = Z_H(A_\alpha)$. Then by [36], $\alpha \in \Sigma$ is in Σ', i.e., $\mu_\alpha(\rho) = 0$ if and only if $I_{B_H}^{M_\alpha}(\rho)$ is irreducible and $W(M_\alpha, A_0) \cap W(\rho) \neq \emptyset$. It is easy to see that $I_{B_H}^{M_\alpha}(\rho)$ is irreducible for any $\alpha \in \Sigma$, so that α is in Σ' if and only if $W(M_\alpha, A_0) \cap W(\rho) \neq \emptyset$.

If $w_1 \in W(\rho)$, $W(M_{\alpha_1}, A_0) \cap W(\rho) \neq \emptyset$ since $w_1 \in M_{\alpha_1}$. Therefore, $\alpha_1 \in \Sigma'$, so that we have $w_1 \in W'$ and $w_1 \notin R$ by Theorem 5.1. A similar assertion holds for $w_2, w_1w_2w_1$, and $w_2w_1w_2$, since these Weyl elements are contained in $M_{\alpha_2}, M_{\alpha_1+\alpha_2}$, and $M_{\alpha_1+2\alpha_2}$, respectively. Thus, R should not contain these Weyl elements.

On the other hand, suppose that R contains w_1w_2 or w_2w_1. By a direct computation, we see that

$$\begin{aligned} w_1 \in W(\rho) &\Longleftrightarrow \chi_1 = \chi_2, \\ w_2 \in W(\rho) &\Longleftrightarrow \chi_{2,E}\tau^\sigma = \tau, \\ w_2w_1 \in W(\rho) &\Longleftrightarrow \chi_1 = \chi_2, \chi_{1,E}\tau^\sigma = \tau, \\ w_1w_2 \in W(\rho) &\Longleftrightarrow \chi_1 = \chi_2, \chi_{1,E}\tau^\sigma = \tau, \\ w_2w_1w_2 \in W(\rho) &\Longleftrightarrow \chi_1 = \chi_2^{-1}, \\ w_1w_2w_1 \in W(\rho) &\Longleftrightarrow \chi_{1,E}\tau^\sigma = \tau, \\ w_2w_1w_2w_1 \in W(\rho) &\Longleftrightarrow \chi_1^2 = \chi_2^2 = 1, (\chi_1\chi_2)_E\tau^\sigma = \tau. \end{aligned} \tag{5.1}$$

Thus, we see that R should contain w_1. This contradicts the above observation. Hence, w_1w_2 and w_2w_1 are not contained in R, and we obtain

$$R = \{1, w = w_1w_2w_1w_2\}.$$

Moreover, since $w\alpha_1 < 0$ and $w\alpha_2 < 0$, $w_1, w_2, w_2w_1w_2$, and $w_1w_2w_1$ are not contained in $W(\rho)$, and so are w_1w_2 and w_2w_1. Hence, $W(\rho) = R$ and we get our required condition of ρ from (5.1).

Finally, we note that when this parabolic induction is reducible, it decomposes into a direct sum of two inequivalent tempered irreducible representations by [17, Theorem 3.6]. □

Remark 5.3. The above parabolic induction is generic by Rodier [31]. Moreover, when it is reducible, then exactly one constituent is generic and the other one is nongeneric.

5.2.2. *Reducible points.* Let χ_1, χ_2 and τ be as in the previous section. Then for real numbers ν_1, ν_2 and ν, we define a character $\underline{\chi}$ of T_G by $\underline{\chi} = \chi_1[\nu_1]\boxtimes\chi_2[\nu_2]\boxtimes\tau[\nu]$.

For $\alpha \in \triangle_0$, we write

$$P_\alpha = M_\alpha U_\alpha$$

the standard parabolic subgroup satisfying $\triangle_0^{M_\alpha} = \{\alpha\}$. Let G_α be the derived group of M_α. Then we have

$$G_{\alpha_1} = \mathrm{SL}_2, \quad G_{\alpha_2} = \mathrm{Res}_{E/F}\mathrm{SL}_2.$$

We recall that the study of $M^{M_\alpha}(r_\alpha, \underline{\chi})$ amounts to that of $M^{G_\alpha}(r_\alpha, \underline{\chi}|T_\alpha)$ where $T_\alpha = T \cap G_\alpha$. As we noted above, in our case, G_α is either SL_2 or $\mathrm{Res}_{E/F}\mathrm{SL}_2$. In the latter case, we extend $\alpha^\vee$ to $\alpha_E^\vee : \mathrm{Res}_{E/F}\mathbb{G}_m \to T_\alpha$. It is well-known that $M^{G_\alpha}(r_\alpha, \underline{\chi}|T_\alpha)$ has its only zero at

$$\begin{cases} \underline{\chi} \circ \alpha^\vee = |\,|_F & \text{if } G_\alpha = \mathrm{SL}_2, \\ \underline{\chi} \circ \alpha_E^\vee = |\,|_E & \text{if } G_\alpha = \mathrm{Res}_{E/F}\mathrm{SL}_2. \end{cases}$$

Here $\alpha_1^\vee(t) = \mathrm{diag}(t, t^{-1}; 1_2)$ for $t \in F^\times$ and $\alpha_{2,E}^\vee(t) = \mathrm{diag}(1, N_{E/F}(t); t/t^\sigma)$ for $t \in E^\times$

As in [19, Section 2], the reducible points of $I_B(\chi_1[\nu_1], \chi_2[\nu_2]; \tau[\nu])$ are studied by the observation of zeros of the intertwining operators $M(r_\alpha, \chi_1[\nu_1] \boxtimes \chi_2[\nu_2] \boxtimes \tau[\nu])$ for $\alpha \in \triangle_0$. Indeed, the reducible points are contained in

$$\mathfrak{r}_{\alpha_1} \cup \mathfrak{r}_{-\alpha_1} \cup \mathfrak{r}_{\alpha_2} \cup \mathfrak{r}_{-\alpha_2},$$

where $\mathfrak{r}_{\pm\alpha_i}$ are the set of reducible points given by the zeros of the intertwining operator $M(r_{\pm\alpha_i}, \chi_1[\nu_1] \boxtimes \chi_2[\nu_2] \boxtimes \tau[\nu])$, where

$$\mathfrak{r}_{\pm\alpha_1} = \{\chi_1[\nu_1] \boxtimes \chi_2[\nu_2] \boxtimes \tau[\nu] \mid \chi_1 = \chi_2,\, \nu_1 - \nu_2 = \pm 1\}$$

and

$$\mathfrak{r}_{\pm\alpha_2} = \{\chi_1[\nu_1] \boxtimes \chi_2[\nu_2] \boxtimes \tau[\nu] \mid \tau \cdot \chi_{2,E} = \tau^\sigma,\, \nu_2 = \pm 1\}.$$

5.2.3. *A formula of Jacquet module.* We shall recall a formula of Jacquet modules of parabolic inductions. Let G be the set of F-rational points of an algebraic group defined over F. For a parabolic subgroup P of G with the Levi decomposition $P = MN$, we denote by $r_M^G : K\Pi(G) \to K\Pi(M)$ the map induced by the normalized Jacquet functor where $K\Pi(G)$ (resp. $K\Pi(M)$) is the Grothendieck group of admissible representations of G (resp. M) of finite length. Then we have the following formula [3, Lemma 2.12]: for parabolic subgroups P_i and P_j with the Levi decomposition $P_i = M_i N_i$ and $P_j = M_j N_j$, we have

$$r_{M_i}^G \circ I_{M_j} = \sum_{w \in W^{P_j,P_i}} I_{(M_i)_w}^{M_i} \circ w \circ r_{(M_j)_w}^{M_j} \tag{5.2}$$

where

$$W^{P_j,P_i} = \left\{w \in W \mid w(B_H \cap M_j) \subset B_H, w^{-1}(B_H \cap M_i) \subset B_H\right\}$$

and

$$(M_j)_w = M_j \cap w^{-1}(M_i), (M_i)_w = w((M_j)_w) = w(M_j) \cap M_i.$$

In our case, it is easy to see that

$$\{w \in W^H \mid w(B_H \cap M_{P_H}) \subset B_H\} = \{1_6, w_1, w_2w_1, w_1w_2w_1\}$$

and

$$\{w \in W^H \mid w(B_H \cap M_{Q_H}) \subset B_H\} = \{1_6, w_2, w_1w_2, w_2w_1w_2\}.$$

Thus, we obtain

$$\begin{gathered} W^{B_H,B_H} = W^H, W^{P_H,P_H} = \{1_6, w_1, w_1w_2w_1\}, W^{Q_H,Q_H} = \{1_6, w_2, w_2w_1w_2\}, \\ W^{P_H,Q_H} = \{1_6, w_2w_1\}, W^{B_H,Q_H} = \{1_6, w_2, w_2w_1, w_2w_1w_2\}, \\ W^{Q_H,P_H} = \{1_6, w_1w_2\}, W^{Q_H,B_H} = \{1_6, w_2, w_1w_2, w_2w_1w_2\}, \\ W^{B_H,P_H} = \{1_6, w_1, w_1w_2, w_1w_2w_1\}, W^{P_H,B_H} = \{1_6, w_1, w_2w_1, w_1w_2w_1\}. \end{gathered}$$

5.2.4. *Reducibility: the case of* $\mathfrak{r}_{\alpha_1}$. We shall find a possible reducible point in $\mathfrak{r}_{\alpha_1}$. In this case, we have

$$\chi_1 = \chi_2, \quad \nu_1 - \nu_2 = 1.$$

Thus, we shall consider the following parabolic induction:

$$I_{B_H}(\chi[\nu+1], \chi[\nu]; \tau[\nu']).$$

We note that

$$I_{B_H}(\chi[\nu+1], \chi[\nu]; \tau[\nu']) = |-|^{\nu'} \otimes I_{B_H}(\chi[\nu+1], \chi[\nu]; \tau),$$

so that we may assume $\nu' = 0$. Moreover, we have

$$I_{B_H}(\chi[\nu+1], \chi[\nu]; \tau) = I_{Q_H}(\chi I_{B_2}^{\mathrm{GL}_2}(\delta_{B_2}^{1/2})[\nu+1/2], \tau)$$

where δ_{B_2} is the modulus function of B_2. Then there exists an exact sequence

$$0 \to I_{Q_H}(st_\chi[\nu+1/2], \tau) \to I_{B_H}(\chi[\nu+1], \chi[\nu]; \tau) \to I_{Q_H}(\chi(\det))[\nu+1/2], \tau) \to 0,$$

and we shall give possible reducible points of

$$I_{Q_H}(\chi(\det))[\nu],\tau) \text{ and } I_{Q_H}(st_\chi[\nu],\tau).$$

Lemma 5.4. *Both $I_{Q_H}(st_\chi[\lambda],\tau)$ and $I_{Q_H}(\chi(\det))[\lambda],\tau)$ are irreducible outside the points which are W_{M_Q}-conjugate to one of the following:*

$$\{\chi[2]\boxtimes\chi[1]\boxtimes\tau,\chi[1]\boxtimes\chi\boxtimes\tau \mid \chi_E\cdot\tau^\sigma=\tau\}\cup\{\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau \mid \chi^2=1\}$$

where $W_{M_Q}=W\cap M_{Q_H}$.

Proof. Note that we may suppose that $\lambda\geq 0$.

Let us consider the case $\lambda=0$. Write π for st_χ or $\chi(\det)$. Since π is unitarizable, $I_{Q_H}(\pi,\tau)$ is also unitarizable. Thus, its reducibility is equivalent to

$$\dim \mathrm{Hom}_H(I_{Q_H}(\pi,\tau),I_{Q_H}(\pi,\tau))\neq 1.$$

By the Frobenius reciprocity [3, Proposition 1.9], we have

$$\mathrm{Hom}_H(I_{Q_H}(\pi,\tau),I_{Q_H}(\pi,\tau))=\mathrm{Hom}_{M_{Q_H}}(r^H_{Q_H}\circ I_{Q_H}(\pi,\tau),\pi\boxtimes\tau).$$

Further, by the formula (5.2), we obtain

$$r^H_{Q_H}\circ I_{Q_H}(\pi,\tau)=\pi\boxtimes\tau+\pi^\vee\boxtimes(\omega_\pi\circ N_{E/F}\cdot\tau)+I^{M_{Q_H}}_{T_H}\circ w_2\circ r^{M_Q}_{T_H}(\pi\boxtimes\tau).$$

Recall that

$$r^{\mathrm{GL}_2}_{B_2}(\pi)=\begin{cases}\chi[-1/2]\boxtimes\chi[1/2] & \text{if } \pi=\chi(\det),\\ \chi[1/2]\boxtimes\chi[-1/2] & \text{if } \pi=st_\chi.\end{cases}$$

Hence, we obtain

$$\begin{aligned}&I^{M_{Q_H}}_{T_H}\circ w_2\circ r^{M_{Q_H}}_{T_H}(\pi\boxtimes\tau)\\ &=\begin{cases}I^{\mathrm{GL}_2}_{B_2}(\chi[-1/2]\boxtimes\chi[-1/2])\boxtimes\tau[1/2] & \text{if } \pi=\chi(\det),\\ I^{\mathrm{GL}_2}_{B_2}(\chi[1/2]\boxtimes\chi[1/2])\boxtimes\tau[-1/2] & \text{if } \pi=st_\chi\end{cases}\end{aligned}$$

and

$$\mathrm{Hom}_{M_{Q_H}}(I^{M_{Q_H}}_{T_H}\circ w_2\circ r^{M_{Q_H}}_{T_H}(\pi\boxtimes\tau),\pi\boxtimes\tau)=0.$$

This shows that the only possible reducible point with $\lambda=0$ is the case where $\pi^\vee\simeq\pi$, i.e., $\chi^2=1$.

Next, suppose $\lambda>0$ and consider $I_{Q_H}(st_\chi[\lambda],\tau)$. Then this is a standard module. Thus, by the Langlands classification, this is reducible if and only if the intertwining operator

$$M(w_2w_1w_2,st_\chi[\nu]\boxtimes\tau):I_{Q_H}(st_\chi[\nu],\tau)\to I_{Q_H}(st_{\chi^{-1}}[-\nu],\chi_E\cdot\tau)$$

has a zero at $\nu=\lambda$. The zeros of this operator are among those of

$$M(w_2w_1w_2,\chi[\nu+1/2]\boxtimes\chi[\nu-1/2]\boxtimes\tau),$$

or by the functional equation, of

$$\begin{aligned}&M(w_2,\chi^{-1}[-\nu+1/2]\boxtimes\chi[\nu+1/2]\boxtimes(\chi_E\cdot\tau)[\nu-1/2])\\ &\quad\circ M(w_1,\chi[\nu+1/2]\boxtimes\chi^{-1}[-\nu+1/2]\boxtimes(\chi_E\cdot\tau)[\nu-1/2])\\ &\quad\circ M(w_2,\chi[\nu+1/2]\boxtimes\chi[\nu-1/2]\boxtimes\tau).\end{aligned}$$

Indeed,

$$M(w_2,\chi[\nu+1/2]\boxtimes\chi[\nu-1/2]\boxtimes\tau)$$

has a zero at $\nu=3/2$ when $\tau\cdot\chi_E=\tau^\sigma$,

$$M(w_1,\chi[\nu+1/2]\boxtimes\chi^{-1}[-\nu+1/2]\boxtimes(\chi_E\cdot\tau)[\nu-1/2])$$

has a zero at $\nu = 0$ when $\chi^2 = 1$, and

$$M(w_2, \chi^{-1}[-\nu + 1/2] \boxtimes \chi[\nu + 1/2] \boxtimes (\chi_E \cdot \tau)[\nu - 1/2])$$

has a zero at $\nu = 1/2$ when $\tau \cdot \chi_E = \tau^\sigma$. Thus, the possible reducible points when $\nu > 0$ are

$$\{\chi[2] \boxtimes \chi[1] \boxtimes \tau, \chi[1] \boxtimes \chi \boxtimes \tau \mid \tau \cdot \chi_E = \tau^\sigma\}.$$

Finally, we consider $I_{Q_H}(\chi(\det))[\lambda], \tau)$ with $\lambda > 0$. We note that the image of the intertwining operator

$$M(w_1, \chi[\lambda + 1/2] \boxtimes \chi[\lambda - 1/2] \boxtimes \tau)$$

is $I_{Q_H}(\chi(\det))[\lambda], \tau)$. Suppose that $\lambda > 1/2$. Since $I_{B_H}(\chi[\lambda + 1/2], \chi[\lambda - 1/2]; \tau)$ is a standard module, then

$$\begin{aligned} &\mathrm{Im} M(w_2 w_1 w_2 w_1, \chi[\lambda + 1/2] \boxtimes \chi[\lambda - 1/2] \boxtimes \tau) \\ &= \mathrm{Im}\,(M(w_2 w_1 w_2, \chi[\lambda - 1/2] \boxtimes \chi[\lambda + 1/2] \boxtimes \tau) \\ &\qquad \circ M(w_1, \chi[\lambda + 1/2] \boxtimes \chi[\lambda - 1/2] \boxtimes \tau)) \end{aligned}$$

is irreducible. Thus, $I_{Q_H}(\chi(\det))[\lambda], \tau)$ is reducible only if

$$M(w_2 w_1 w_2, \chi[\nu - 1/2] \boxtimes \chi[\nu + 1/2] \boxtimes \tau)$$

has a zero at $\nu = \lambda$. Since this operator is equal to

$$\begin{aligned} &M(w_2, \chi^{-1}[-\nu - 1/2] \boxtimes \chi[\nu - 1/2] \boxtimes \tau[\nu + 1/2]) \\ &\quad \circ\ M(w_1, \chi[\nu - 1/2] \boxtimes \chi^{-1}[-\nu - 1/2] \boxtimes \tau[\nu + 1/2]) \\ &\quad \circ\ M(w_2, \chi[\nu - 1/2] \boxtimes \chi[\nu + 1/2] \boxtimes \tau), \end{aligned}$$

as in the above case, we see that the possible reducible point is

$$\{\chi[2] \boxtimes \chi[1] \boxtimes \tau \mid \tau \chi_E = \tau^\sigma\}.$$

On the other hand, we suppose that $0 < \lambda < 1/2$. Noting the operator

$$M(w_2, \chi[\lambda + 1/2] \boxtimes \chi^{-1}[-\lambda + 1/2] \boxtimes (\chi_E \cdot \tau^\sigma)[\lambda - 1/2])$$

is an isomorphism, we have

$$\begin{aligned} &I_{Q_H}(\chi(\det))[\lambda], \tau) \\ =&\mathrm{Im} M(w_1, \chi[\lambda + 1/2] \boxtimes \chi[\lambda - 1/2] \boxtimes \tau) \\ =&\mathrm{Im} M(w_1 w_2, \chi[\lambda + 1/2] \boxtimes \chi^{-1}[-\lambda + 1/2] \boxtimes (\chi_E \cdot \tau^\sigma)[\lambda - 1/2]) \end{aligned}$$

and

$$\begin{aligned} &\mathrm{Im}(M(w_1 w_2 w_1 w_2, \chi[\lambda + 1/2] \boxtimes \chi^{-1}[-\lambda + 1/2] \boxtimes (\chi_E \cdot \tau^\sigma)[\lambda - 1/2])) \\ =&\mathrm{Im}(M(w_1 w_2, \chi[\lambda - 1/2] \boxtimes \chi[\lambda + 1/2] \boxtimes \tau) | I_{Q_H}(\chi(\det))[\lambda], \tau)). \end{aligned}$$

Since $I_{B_H}(\chi[\lambda + 1/2] \boxtimes \chi[-\lambda + 1/2] \boxtimes (\chi_E \cdot \tau^\sigma)[\lambda - 1/2])$ is standard module, $I_{Q_H}(\chi(\det))[\lambda], \tau)$ is reducible only if

$$\begin{aligned} &M(w_1 w_2, \chi[\nu - 1/2] \boxtimes \chi[\nu + 1/2] \boxtimes \tau) \\ &\quad = M(w_1, \chi[\nu - 1/2] \boxtimes \chi^{-1}[-\nu - 1/2] \boxtimes (\chi_E \cdot \tau)[\nu + 1/2]) \\ &\qquad\qquad \circ\, M(w_2, \chi[\nu - 1/2] \boxtimes \chi[\nu + 1/2] \boxtimes \tau) \end{aligned}$$

has a zero at $\nu = \lambda$. However, it is easy to see that these operators have no zero on this region.

Finally, we suppose that $\lambda = 1/2$. Then we have

$$I_{B_H}(\chi[1], \chi; \tau) = I_{P_H}(\chi[1], I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi \boxtimes \tau)).$$

Since $I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi \boxtimes \tau)$ is irreducible and tempered, the above parabolic induction is a standard module. Thus, the image of the operator

$$\begin{aligned} &M(w_1 w_2 w_1, \chi[1] \boxtimes \chi \boxtimes \tau) \\ &= M(w_1, \chi \boxtimes \chi^{-1}[-1] \boxtimes (\chi_E \cdot \tau^\sigma)[1]) \circ M(w_2, \chi \boxtimes \chi[1] \boxtimes \tau) \circ M(w_1, \chi[1] \boxtimes \chi \boxtimes \tau) \end{aligned}$$

is irreducible. Therefore, the image of $M(w_1, \chi[1] \boxtimes \chi \boxtimes \tau)$ is reducible only if

$$M(w_1, \chi \boxtimes \chi^{-1}[-\nu] \boxtimes (\chi_E \cdot \tau^\sigma)[\nu]) \circ M(w_2, \chi \boxtimes \chi[\nu] \boxtimes \tau)$$

has a zero at $\nu = 1$. Thus, the possible reducible point is

$$\{\chi[1] \boxtimes \chi \boxtimes \tau \mid \tau^\sigma \cdot \chi_E = \tau\}. \qquad \square$$

5.2.5. *Reducibility: the case of* $\mathfrak{r}_{\alpha_2}$. Let us consider the case of $\mathfrak{r}_{\alpha_2}$. In this case, we have

$$\tau \cdot \chi_{2,E} = \tau^\sigma, \nu_2 = 1.$$

Note that from the first condition, we have $\chi_2^2 = 1$. Then we consider the representation

$$I_{B_H}(\chi_1[\nu], \chi_2[1]; \tau[\nu']),$$

and we have a short exact sequence

$$\begin{aligned} 0 \to I_{P_H}(\chi_1[\nu], \pi(\tau^\sigma St, \chi_2 \cdot \tau_F)[\nu' + 1/2]) &\to I_{B_H}(\chi_1[\nu], \chi_2[1]; \tau[\nu']) \\ &\to I_{P_H}(\chi_1[\nu], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F)[\nu' + 1/2]) \to 0 \end{aligned}$$

where we denote by St the Steinberg representation of $\mathrm{GL}_2(E)$. As in the previous case, we may suppose $\nu' = -1/2$. Then we shall prove the following lemma.

Lemma 5.5. *Both* $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma St, \chi_2 \cdot \tau_F))$ *and* $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))$ *are irreducible outside the points which are* W_{P_H}*-conjugate to one of the following:*

$$\{\chi_1 \boxtimes \chi_2[1] \boxtimes \tau[-1/2] \mid \tau \cdot \chi_{i,E} = \tau^\sigma\}$$

or

$$\{\chi[2] \boxtimes \chi[1] \boxtimes \tau[-1/2], \eta[1] \boxtimes \chi[1] \boxtimes \tau[-1/2] \mid \tau \cdot \chi_E = \tau^\sigma, \eta_E = \chi_E\}.$$

Proof. First, we consider the case of $\lambda = 0$. As in a proof of the previous lemma, we shall compute

$$\mathrm{End}_H(I_{P_H}(\chi_1, \pi(\tau^\sigma St, \chi_2 \cdot \tau_F))) \quad \text{and} \quad \mathrm{End}_H(I_{P_H}(\chi_1, \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))).$$

Write ρ for $\tau^\sigma St$ or $\tau^\sigma(\det)$. Then we have

$$\begin{aligned} &r_{P_H}^H \circ I_{P_H}(\chi_1, \pi(\rho, \chi_2 \cdot \tau_F)) \\ &\quad = \chi_1 \boxtimes \pi(\rho, \chi_2 \tau_F) + \chi_1^{-1} \boxtimes \pi(\chi_{1,E} \cdot \rho^\sigma, \chi_1^2 \chi_2 \tau_F) \\ &\qquad + I_T^{M_P} \circ w_1 \circ r_T^{M_P}(\chi_1 \boxtimes \pi(\rho, \chi_2 \cdot \tau_F)) \end{aligned}$$

and

$$\begin{aligned} &I_T^{M_P} \circ w_1 \circ r_T^{M_P}(\chi_1 \boxtimes \pi(\rho, \chi_2 \cdot \tau_F)) \\ &= \begin{cases} \chi_2[1] \boxtimes I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi_1 \boxtimes \tau[-1/2]) & \text{if } \rho = \tau^\sigma St, \\ \chi_2^{-1}[-1] \boxtimes I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi_1 \boxtimes \chi_{2,E} \cdot \tau[1/2]) & \text{if } \rho = \tau^\sigma(\det). \end{cases} \end{aligned}$$

In both cases, we have

$$\mathrm{Hom}_{M_P}(I_T^{M_P} \circ w_1 \circ r_T^{M_P}(\chi_1 \boxtimes \pi(\rho, \chi_2 \cdot \tau_F)), \chi_1 \boxtimes \pi(\rho, \chi_2 \cdot \tau_F)) = 0.$$

Therefore, the possible reducible point for $I_{P_H}(\chi_1, \pi(\rho, \chi_2 \cdot \tau_F))$ is

$$\{\chi_1 \boxtimes \chi_2[1] \boxtimes \tau[-1/2] \mid \tau \cdot \chi_{i,E} = \tau^\sigma\}$$

Next, we consider $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma St, \chi_2 \cdot \tau_F))$ with $\lambda > 0$. By the Langlands classification, this is reducible if and only if the intertwining operator

$$M(w_1 w_2 w_1, (\chi_1[\nu] \boxtimes \pi(\tau^\sigma St, \chi_2 \cdot \tau_F))$$

has a pole at $\nu = \lambda$. The zero of this operator are among those of $M(w_1 w_2 w_1, \chi_1[\nu] \boxtimes \chi_2[1] \boxtimes \tau[-1/2])$, or by the functional equation, of

$$\begin{aligned} &M(w_1, \chi_2[1] \boxtimes \chi_1^{-1}[-\nu] \boxtimes (\chi_{1,E}) \cdot \tau^\sigma)[\nu - 1/2]) \\ &\qquad \circ M(w_2, \chi_2[1] \boxtimes \chi_1[\nu] \boxtimes \tau[-1/2]) \circ M(w_1, \chi_1[\nu] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]). \end{aligned}$$

Indeed,

$$M(w_1, \chi_2[1] \boxtimes \chi_1^{-1}[-\nu] \boxtimes (\chi_{1,E}) \cdot \tau^\sigma)[\nu - 1/2])$$

has no zero on $\nu > 0$,

$$M(w_2, \chi_2[1] \boxtimes \chi_1[\nu] \boxtimes \tau[-1/2])$$

has a zero when $\nu = 1$ and $\chi_{1,E} = \chi_{2,E}$, and

$$M(w_1, \chi_1[\nu] \boxtimes \chi_2[1] \boxtimes \tau[-1/2])$$

has a zero when $\chi_1 = \chi_2$ and $\nu = 2$. Thus, the only possible reducible points are

$$\{\chi_1[2] \boxtimes \chi_1[1] \boxtimes \tau[-1/2], \chi_1[1] \boxtimes \eta[1] \boxtimes \tau[-1/2] \mid \eta_E = \chi_{1,E}, \chi_{i,E}\tau^\sigma = \tau\}.$$

Finally, we consider $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))$ with $\lambda > 0$. We note that the image of the intertwining operator

$$M(w_2, \chi_1[\lambda] \boxtimes \chi_2[1] \boxtimes \tau[-1/2])$$

is $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))$.

Suppose that $0 < \lambda < 1$. Since $M(w_1, \chi_2[1] \boxtimes \chi_1[\lambda] \boxtimes \tau[-1/2])$ is an isomorphism, we have

$$I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F)) = \mathrm{Im}(M(w_2 w_1, \chi_2[1] \boxtimes \chi_1[\lambda] \boxtimes \tau[-1/2])).$$

Thus, $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))$ is reducible only if

$$M(w_2 w_1, \chi_1[\nu] \boxtimes \chi_2[-1] \boxtimes \tau[1/2])$$

has a zero at $\nu = \lambda$. It is easily seen that this operator has no zero in this region.

Suppose that $\lambda > 1$. Then $I_{B_H}(\chi_1[\lambda] \boxtimes \chi_2[1] \boxtimes \tau[-1/2])$ is a standard module, and the image of

$$\begin{aligned} &M(w_1 w_2 w_1 w_2, \chi_1[\lambda] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]) \\ =&M(w_1 w_2 w_1, \chi_1[\lambda] \boxtimes \chi_2[-1] \boxtimes \tau[1/2]) \circ M(w_2, \chi_1[\lambda] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]) \end{aligned}$$

is irreducible. Thus, $I_{P_H}(\chi_1[\lambda], \pi(\tau^\sigma(\det), \chi_2 \cdot \tau_F))$ is reducible only if $M(w_1 w_2 w_1, \chi_1[\nu] \boxtimes \chi_2[-1] \boxtimes \tau[1/2])$ has zero at $\nu = \lambda$. Since it is equal to

$$\begin{aligned} &M(w_1, \chi_2[-1] \boxtimes \chi_1^{-1}[-\nu] \boxtimes (\chi_{1,E} \cdot \tau^\sigma)[\nu + 1/2]) \\ &\qquad \circ M(w_2, \chi_2[-1] \boxtimes \chi_1[\nu] \boxtimes \tau) \circ M(w_1, , \chi_1[\nu] \boxtimes \chi_2[-1] \boxtimes \tau[1/2]), \end{aligned}$$

the only possible reducible point is

$$\{\chi_2[2] \boxtimes \chi_2[1] \boxtimes \tau \mid \tau^\sigma \cdot \chi_{2,E} = \tau\}.$$

Finally, suppose $\lambda = 1$. Since the image of the operator

$$\begin{aligned} &M(w_2 w_1 w_2, \chi_1[1] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]) \\ &\quad = M(w_2, \chi_2[-1] \boxtimes \chi_1[1] \boxtimes \tau[1/2]) \circ M(w_1, \chi_1[1] \boxtimes \chi_2[-1] \boxtimes \tau[1/2]) \\ &\qquad\qquad \circ M(w_2, \chi_1[1] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]) \end{aligned}$$

is irreducible, $\mathrm{Im}(M(w_2, \chi_1[1] \boxtimes \chi_2[1] \boxtimes \tau[-1/2]))$ is reducible only if

$$M(w_2, \chi_2[-\nu] \boxtimes \chi_1[\nu] \boxtimes \tau[1/2]) \circ M(w_1, \chi_1[\nu] \boxtimes \chi_2[-\nu] \boxtimes \tau[1/2])$$

has a zero at $\nu = 1$. Thus, the possible reducible points are

$$\{\eta[1] \boxtimes \chi_2[1] \boxtimes \tau[-1/2] \mid \eta_E = \chi_{2,E}, \chi_{2,E}\tau^\sigma = \tau\}. \qquad \square$$

5.3. **Irreducible constituents.** In the previous sections, we gave all possible reducible points. Our aim in this section is to give irreducible constituents of parabolic inductions at those points. Let χ be a unitary character of $F^\times$ and τ a unitary character of $E^\times$. Twisting parabolic inductions by a character $x \mapsto |x|^s$ with $s \in \mathbb{R}$, it suffices to consider the case where the central character is unitary. Thus, we consider the following points:

$$\{\chi[2] \boxtimes \chi[1] \boxtimes \tau[-3/2] \mid \tau \cdot \chi_E = \tau^\sigma\}, \{\chi[1/2] \boxtimes \chi[-1/2] \boxtimes \tau \mid \chi^2 = 1\}$$

and

$$\{\eta[1] \boxtimes \chi \boxtimes \tau[-1/2], \eta[1] \boxtimes \chi[1] \boxtimes \tau[-1] \mid \tau \cdot \chi_E = \tau^\sigma, \eta = \chi \text{ or } \chi\chi_{E/F}\}.$$

Proposition 5.6. *At $\xi = \chi[2] \boxtimes \chi[1] \boxtimes \tau[-3/2]$ with $\tau \cdot \chi_E = \tau^\sigma$, we have*

$$\begin{aligned} I_{B_H}(\xi) &= I_{Q_H}(st_\chi[3/2], \tau[-3/2]) + I_{Q_H}(\chi(\det))[3/2], \tau[-3/2]) \\ &= I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]) + I_{P_H}(\chi[2], \pi(\tau^\sigma(\det), \chi \cdot \tau_F)[-1]), \end{aligned}$$

and irreducible constituents of representations on the right-hand side are given by

$$I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]) = \delta(\chi, \tau) + J_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]),$$

$$I_{P_H}(\chi[2], \pi(\tau^\sigma, \chi \cdot \tau_F)[-1]) = J_{B_H}(\chi[2], \chi[1]; \tau[-3/2]) + J_{Q_H}(St_\chi[3/2], \tau[-3/2]),$$

$$I_{Q_H}(st_\chi[3/2], \tau[-3/2]) = \delta(\chi, \tau) + J_{Q_H}(st_\chi[3/2], \tau[-3/2]),$$

$$\begin{aligned} I_{Q_H}(\chi(\det))[3/2], \tau[-3/2]) = {} & J_{B_H}(\chi[2], \chi[1]; \tau[-3/2]) \\ & + J_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]) \end{aligned}$$

where $\delta(\chi, \tau)$ is an essentially discrete series representation.

Proof. First, we note that

$$\begin{aligned} &I_{B_H}(\chi[2], \chi[1]; \tau[-3/2]) \\ =&I_{Q_H}(st_\chi[3/2], \tau[-3/2]) + I_{Q_H}(\chi(\det))[3/2], \tau[-3/2]) \\ =&I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]) + I_{P_H}(\chi[2], \pi(\tau^\sigma(\det), \chi \cdot \tau_F)[-1]). \end{aligned}$$

Let us compute the Jacquet module of the above parabolic inductions with respect to the Borel subgroup B_H. We recall that the Jacquet module of $I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi[1] \boxtimes \tau[-3/2])$ is given by

$$\chi[1] \boxtimes \tau[-3/2] + \chi[-1] \boxtimes \tau[-1/2].$$

Furthermore, $\chi[1] \boxtimes \tau[-3/2]$ is the Jacquet module of $\pi(\tau^\sigma St, \chi \cdot \tau_F)$ and $\chi[-1] \boxtimes \tau[-1/2]$ is the Jacquet module of $\pi(\tau^\sigma(\det), \chi \cdot \tau_F)$. Similarly, we know that the Jacquet module of $I_{B_2}^{\mathrm{GL}_2}(\chi[2] \boxtimes \chi[1])$ is

$$\chi[2] \boxtimes \chi[1] + \chi[1] \boxtimes \chi[2],$$

and $\chi[2] \boxtimes \chi[1]$ (resp. $\chi[1] \boxtimes \chi[2]$) is the Jacquet module of $st_\chi[3/2]$ (respectively $\chi(\det))[3/2]$). Then by a direct computation, we find that

$$\begin{aligned} (5.3) \quad & r_{B_H}^H \circ I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]) \\ &= \chi[2] \boxtimes \chi[1] \boxtimes \tau[-3/2] + \chi[1] \boxtimes \chi[2] \boxtimes \tau[-3/2] \\ &\qquad + \chi[1] \boxtimes \chi[-2] \boxtimes \tau[1/2] + \chi[-2] \boxtimes \chi[1] \boxtimes \tau[1/2] \end{aligned}$$

and

$$\begin{aligned} (5.4) \quad & r_{B_H}^H \circ I_{Q_H}(st_\chi[3/2], \tau[-3/2]) \\ &= \chi[2] \boxtimes \chi[1] \boxtimes \tau[-3/2] + \chi[2] \boxtimes \chi[-1] \boxtimes \tau[-1/2] \\ &\qquad + \chi[-1] \boxtimes \chi[2] \boxtimes \tau[-1/2] + \chi[-1] \boxtimes \chi[-2] \boxtimes \tau[3/2]. \end{aligned}$$

From (5.3) and (5.4),

$$\delta(\chi, \tau) := I_{Q_H}(st_\chi[3/2], \tau[-3/2]) \sqcap I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1])$$

is irreducible. Moreover, from Casselman's criterion on the square integrability, it should be square integrable. Moreover, this criterion shows that the induced representation $I_{B_H}(\chi[2], \chi[1]; \tau[-3/2])$ contains at most one square integrable representation.

Now, we note that the number of constituents of $I_{B_H}(\chi[2], \chi[1]; \tau[-3/2])$ is the same as that of standard modules contained in it. Indeed, they are $I_{B_H}(\chi[2], \chi[1]; \tau[-3/2])$, $I_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1])$, $I_{Q_H}(st_\chi[3/2], \tau[-3/2])$ and standard modules of H, i.e., tempered representations.

Any parabolic induction for B_H containing a tempered representation is given by the following three families: $I_{B_H}(\chi[1/2], \chi[-1/2]; \tau)$ and $I_{B_H}(\chi, \chi[1]; \tau[-1/2])$ and $I_{B_H}(\chi_1, \chi_2; \tau)$. Thus, $I_{B_H}(\chi[2], \chi[1]; \tau[-3/2])$ does not contain any tempered representation, but $\delta(\chi, \tau)$. Taking the above discussion into account, this implies that the number of irreducible constituents is four, and these constituents are given by

$$J_{B_H}(\chi[2], \chi[1]; \tau[-3/2]), J_{P_H}(\chi[2], \pi(\tau^\sigma St, \chi \cdot \tau_F)[-1]),$$
$$J_{Q_H}(st_\chi[3/2], \tau[-3/2]), \delta(\chi, \tau).$$

Then our assertion follows immediately. □

Proposition 5.7. *At $\xi = \chi[1] \boxtimes \chi \boxtimes \tau[-1/2]$ with $\tau \cdot \chi_E = \tau^\sigma$,*

$$\begin{aligned} I_{B_H}(\xi) &= I_{P_H}(\chi, \pi(\tau^\sigma St, \chi\tau_F)) + I_{P_H}(\chi, \pi(\tau^\sigma(\det), \chi\tau_F)) \\ &= I_{Q_H}(st_\chi[1/2], \tau[-1/2]) + I_{Q_H}(\chi(\det)[1/2], \tau[-1/2]), \end{aligned}$$

and we have

$$I_{Q_H}(st_\chi[1/2], \tau[-1/2]) = J_+(\chi, \tau) + \tau_+(\chi, \tau)$$

and

$$I_{Q_H}(\chi(\det)[1/2], \tau[-1/2]) = J_-(\chi, \tau) + \tau_-(\chi, \tau),$$

where $\tau_\pm(\chi,\tau)$ (resp. $J_\pm(\chi,\tau)$) are essentially tempered (resp. nontempered) representations such that

$$I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\tau_F)) = \tau_+(\chi,\tau)\oplus\tau_-(\chi,\tau)$$
$$(\textit{resp. } I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F)) = J_+(\chi,\tau)\oplus J_-(\chi,\tau))\,.$$

Proof. We note that

$$I_{B_H}(\chi,\chi[1];\tau[-1/2]) = I_{B_H}(\chi[1],\chi;\tau[-1/2]),$$

and thus we obtain

$$I_{B_H}(\chi[1],\chi;\tau[-1/2]) = I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\tau_F)) + I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F)).$$

From the computation in Lemma 5.4,

$$\dim \mathrm{End}_H(I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\tau_F))) \quad\text{and}\quad \dim \mathrm{End}_H(I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F)))$$

are at most two, so that they are of length at most two. The Jacquet modules of them, with respect to B_H, are given by

$$r^H_{B_H}\circ I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F)) = 2(\chi\boxtimes\chi[-1]\boxtimes\tau[1/2]+\chi[-1]\boxtimes\chi\boxtimes\tau[1/2])$$

and

$$r^H_{B_H}\circ I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\tau_F)) = 2(\chi\boxtimes\chi[1]\boxtimes\tau[-1/2]+\chi[1]\boxtimes\chi\boxtimes\tau[1/2]).$$

On the other hand, we have

$$\begin{aligned} I_{B_H}(\chi[1],\chi;\tau[-1/2]) &= I_{Q_H}(I^{\mathrm{GL}_2}_{B_2}(\chi[1]\boxtimes\chi),\tau[-1/2])\\ &= I_{Q_H}(st_\chi[1/2],\tau[-1/2]) + I_{Q_H}(\chi(\det)[1/2],\tau[-1/2]). \end{aligned}$$

Moreover, the Jacquet modules of them, with respect to B_H, are given by

$$\begin{aligned} &r^H_{B_H}\circ I_{Q_H}(st_\chi[1/2],\tau[-1/2])\\ &\qquad = 2(\chi[1]\boxtimes\chi\boxtimes\tau[-1/2]) + \chi\boxtimes\chi[1]\boxtimes\tau[-1/2] + \chi\boxtimes\chi[-1]\boxtimes\tau[1/2] \end{aligned}$$

and

$$\begin{aligned} &r^H_{B_H}\circ I_{Q_H}(\chi(\det)[1/2],\tau[-1/2])\\ &\qquad = 2(\chi[-1]\boxtimes\chi\boxtimes\tau[1/2]) + \chi\boxtimes\chi[1]\boxtimes\tau[-1/2]\chi\boxtimes\chi[-1]\boxtimes\tau[1/2]. \end{aligned}$$

Therefore, $J_+(\chi,\tau) := I_{Q_H}(st_\chi[1/2],\tau[-1/2])\sqcap I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F))$ has the Jacquet module $\chi[1]\boxtimes\chi\boxtimes\tau[-1/2]$, and hence it is irreducible. Since the length of $I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F))$ is at most two, we have another irreducible constituent $J_-(\chi,\tau)$ such that

$$I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\tau_F)) = J_+(\chi,\tau)\oplus J_-(\chi,\tau).$$

Similarly, we have two irreducible representations $\tau_\pm(\chi,\tau)$ such that

$$I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\tau_F)) = \tau_+(\chi,\tau)\oplus\tau_-(\chi,\tau).$$

Finally, we note that exactly one constituent of $I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\cdot\tau_F))$ is generic and all constituents of $I_{P_H}(\chi,\pi(\tau^\sigma,\chi\cdot\tau_F))$ are nongeneric by Rodier [31]. □

Proposition 5.8. *At $\xi\in\{\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau \mid \chi^2=1\}$, we have*

$$I_{B_H}(\xi) = I_{Q_H}(st_\chi,\tau) + I_{Q_H}(\chi(\det),\tau)$$

where each representation on the right-hand side is irreducible.

Proof. From (5.2), we have

$$(5.5)\quad r^H_{B_H}\circ I_{Q_H}(st_\chi,\tau) = 2(\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau)+2(\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]),$$

$$(5.6)\quad r^H_{P_H}\circ I_{B_H}(\chi[1/2],\chi[-1/2];\tau)$$
$$=\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[-1/2]\boxtimes\tau)+\chi[-1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[-1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[1/2])$$
$$+\chi[-1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[1/2]\boxtimes\tau)+\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2])$$

and

$$(5.7)\quad r^H_{Q_H}\circ I_{B_H}(\chi[1/2],\chi[-1/2];\tau)$$
$$=2(I^{\mathrm{GL}_2}_{B_2}(\chi[1/2]\boxtimes\chi[-1/2])\boxtimes\tau)+I^{\mathrm{GL}_2}_{B_2}(\chi[1/2]\boxtimes\chi[1/2])\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$$
$$+I^{\mathrm{GL}_2}_{B_2}(\chi[-1/2]\boxtimes\chi[-1/2])\boxtimes(\chi_E\cdot\tau^\sigma)[1/2].$$

Let π be an irreducible constituent of $I_{Q_H}(st_\chi,\tau)$ such that $r^H_{B_H}(\pi)$ contains $\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$. On the right-hand side of (5.6), a representation whose Jacquet module contains $\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$ is

$$\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[-1/2]\boxtimes\tau)\quad\text{or}\quad\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]).$$

In any case, $r_{B_H}(\pi)$ should contain $\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau$.

On the other hand, in (5.7), $I^{\mathrm{GL}_2}_{B_2}(\chi[1/2]\boxtimes\chi[1/2])\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$ is the only irreducible constituent whose Jacquet module contains $\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$. Thus, $r^H_{B_H}(\pi)$ should contain $2(\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2])$.

Let π' be another irreducible constituent of $I_{Q_H}(st_\chi,\tau)$. Then from the above discussion, we should have $r^H_{B_H}(\pi')=\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau$. However, an irreducible constituent on the right-hand side of (5.6) whose Jacquet module contains $\chi[1/2]\boxtimes\chi[-1/2]\boxtimes\tau$ is

$$\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[-1/2]\boxtimes\tau)\quad\text{or}\quad\chi[1/2]\boxtimes I^{\mathrm{GSO}_{3,1}}_{B_{3,1}}(\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]).$$

Thus, $r^H_{B_H}(\pi')$ should contain $\chi[1/2]\boxtimes\chi[1/2]\boxtimes(\chi_E\cdot\tau^\sigma)[-1/2]$. This is a contradiction, and thus we obtain the irreducibility of π.

Similarly, we can prove the irreducibility of $I_{Q_H}(\chi(\det),\tau)$. □

By a similar way as in the proof of Proposition 5.8, we can prove the following propositions.

Proposition 5.9. *At $\xi=\chi\boxtimes\chi\chi_{E/F}[1]\boxtimes\tau[-1/2]$ with $\tau^\sigma\cdot\chi_E=\tau$, we have*

$$I_{B_H}(\xi)=I_{P_H}(\chi,\pi(\tau^\sigma St,\chi\chi_{E/F}\tau_F))+I_{P_H}(\chi,\pi(\tau^\sigma(\det),\chi\chi_{E/F}\tau_F))$$

where each representation on the right-hand side is irreducible.

Proposition 5.10. *At $\xi\in\{\eta[1]\boxtimes\chi[1]\boxtimes\tau[-1/2]\mid\tau^\sigma\cdot\chi_E=\tau,\eta_E=\chi_E\}$, we have*

$$I_{B_H}(\xi)=I_{P_H}(\eta[1],\pi(\tau^\sigma St,\chi\cdot\tau_F))+I_P(\eta[1],\pi(\tau^\sigma(\det),\chi\cdot\tau_F))$$

where each representation on the right-hand side is irreducible.

5.4. **Parabolically induced representations for P_H.** We shall consider the parabolic induction $I_{P_H}(\chi[s], \pi[-s/4])$ with an irreducible unitary supercuspidal representation π of $\mathrm{GSO}_{3,1}$, a unitary character χ of $F^\times$, and $s \in \mathbb{R}$. Recall that we can write

$$\pi = \pi(\tau, \mu)$$

for an irreducible representation τ of $\mathrm{GL}_2(E)$ and a character μ of $F^\times$ such that $\mu \circ N_{E/F} = \omega_\tau$. Then the representation $\chi \boxtimes \pi(\tau, \mu)$ is ramified, i.e.,

$$w_1 w_2 w_1 \left(\chi \boxtimes \pi(\tau, \mu)\right) = \chi \boxtimes \pi(\tau, \mu),$$

if and only if

$$\chi^2 = 1 \quad \text{and} \quad \chi_E \otimes \tau^\sigma = \tau.$$

Let us denote by r the adjoint action of ${}^L M_{P_H}$ on $\mathrm{Lie}({}^L N_{Q_H})$. Then r is irreducible, and from the characterization of Shahidi's local factor [34, Theorem 3.5], we have

$$L(s, \pi \boxtimes \chi, r) = L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi)$$

where the L-factor on the right-hand side is the twisted Asai L-function.

Let us denote by ρ the Langlands parameter of τ, and let $M_{W'_E}^{W'_F}(\rho)$ be the multiplicative induction of ρ where W'_F (resp. W'_E) is the Weil-Deligne group of F (resp. E). Then we have [24, Theorem 3.3],

$$L(s, \tau, \mathrm{Asai}) = L(s, M_{W'_E}^{W'_F}(\rho)).$$

Suppose that ρ is of the form $\mathrm{Ind}_{W'_B}^{W'_K}(\omega)$ where B is a biquadratic extension of F and ω is a one-dimensional representation of W'_B. In this case, we have

$$M_{W'_K}^{W'_F}(\rho) = \mathrm{Ind}_{W'_{K_1}}^{W'_F}(M_{W'_B}^{W'_{K_1}}(\omega)) \oplus \mathrm{Ind}_{W'_{K_2}}^{W'_F}(M_{W'_B}^{W'_{K_2}}(\omega))$$

where K_1 and K_2 are distinct extensions of F between B and F. We note that $M_{W'_B}^{W'_{K_i}}(\omega)$ are one-dimensional representations. Then we have

$$\begin{aligned} L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi) &= L(s, M_{W'_K}^{W'_F}(\rho^\vee) \otimes \mu\chi) \\ &= L(s, \omega|_{K_1}^{-1} \cdot (\mu\chi) \circ N_{K_1/F}) L(s, \omega|_{K_2}^{-1} \cdot (\mu\chi) \circ N_{K_2/F}). \end{aligned}$$

It has a pole $s = 0$ if and only if

$$(\mu\chi) \circ N_{K_1/F} = \omega|_{W_{K_1}} \text{ or } (\mu\chi) \circ N_{K_2/F} = \omega|_{W_{K_2}}.$$

On the other hand, if ρ is not of the above form, we have (cf. [24, p. 9])

$$L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi) = 1.$$

For a unitary character ξ of $F^\times$, we define a subset of isomorphic classes of irreducible unitary supercuspidal representations of $\mathrm{GL}_2(E)$ by

$$\Phi_\xi = \left\{ \pi(\omega) \;\middle|\; \begin{array}{c} \omega\text{: a character of biquadratic extension } B^\times (\supset E) \text{ of } F \\ \text{such that } \omega|_K = \xi \circ N_{K/F} \text{ and } \omega \text{ does not factor through } N_{E/F} \end{array} \right\}.$$

Then we see that

$$L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi) \text{ has a pole at } s = 0 \Longleftrightarrow \tau \in \Phi_{\mu\chi}.$$

Moreover, we give another equivalent condition that $L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi)$ has a pole at $s = 0$. Then from the characterization of Shahidi's local factor, we have

$$L(s, \tau^\vee, \mathrm{Asai} \otimes \mu\chi) = L(s, \tau^\vee \otimes \widetilde{\mu\chi}, \mathrm{Asai})$$

for any extension $\widetilde{\mu\chi}$ of $\mu\chi$ to $E^\times$. By [24, Theorem 2.2], we know that $L(s, \tau^\vee, \text{Asai} \otimes \mu\chi)$ has a pole at $s = 0$ if and only if $\tau^\vee \otimes \widetilde{\mu\chi}$ is $\mathrm{GL}_2(F)$-distinguished. Since we have

$$\mathrm{Hom}_{\mathrm{GL}_2}(\tau^\vee \otimes \widetilde{\mu\chi}, 1) \simeq \mathrm{Hom}_{\mathrm{GL}_2}(\tau, \mu\chi \circ \det),$$

$L(s, \tau^\vee, \text{Asai}\otimes\mu\chi)$ has a pole at $s = 0$ if and only if τ is distinguished by a character $\mu\chi \circ \det$, namely,

$$\tau \text{ is distinguished by a character } \mu\chi \circ \det \Longleftrightarrow \tau \in \Phi_{\mu\chi}.$$

Then from [34, Theorem 8.1], we obtain the following classification.

Proposition 5.11. *The representation $I_{P_H}(\chi[s], \pi[-s/4])$ is irreducible unless $\chi^2 = 1$ and $\chi_E \otimes \tau^\sigma = \tau$ where $\pi = \pi(\tau, \mu)$.*

(1) *Suppose $\tau \in \Phi_{\mu\chi}$. Then $I_{P_H}(\chi[s], \pi[-s/4])$ is irreducible unless $s = 1$, and we have*

$$I_{P_H}(\chi[1], \pi[-1/4]) = \delta_{P_H}(\chi, \pi) + J_{P_H}(\chi[1], \pi[-1/4])$$

where $\delta_{P_H,2}(\chi, \pi)$ is a generic discrete series which is the unique irreducible submodule.

(2) *Suppose $\tau \notin \Phi_{\mu\chi}$. Then $I_{P_H}(\chi[s], \pi[-s/4])$ is irreducible unless $s = 0$. Moreover, $I_{P_H}(\chi, \pi)$ is a direct sum of two irreducible tempered representations.*

5.5. **Parabolically induced representations for Q_H.** We shall consider the parabolic induction $I_{Q_H}(\pi[s], \chi[-s])$ with an irreducible unitary supercuspidal representation π of GL_2 and a unitary character χ of $\mathrm{GSO}_2 \simeq E^\times$. Then the representation $\pi \boxtimes \chi$ is ramified, i.e.,

$$w_2 w_1 w_2 (\pi \boxtimes \chi) = \pi \boxtimes \chi,$$

if and only if

$$\pi \simeq \pi^\vee \quad \text{and} \quad \omega_\pi \circ N_{E/F} = 1.$$

Let us denote by r the adjoint action of ${}^L M_{Q_H}$ on $\mathrm{Lie}({}^L N_{Q_H})$. Then it decomposes $r = r_1 \oplus r_2$, and we have

$$L(s, \pi \boxtimes \chi, r_1) = L(s, \pi \otimes \theta(\chi(\chi^\sigma)^{-1})^\vee, \mathrm{std} \boxtimes \mathrm{std}) \quad \text{and} \quad L(s, \pi \boxtimes \chi, r_2) = L(s, \omega_\pi)$$

from the characterization of Shahidi's local factor [34, Theorem 3.5], where $\theta(\chi(\chi^\sigma)^{-1})$ is the irreducible representation of GL_2 associated to a character $\chi(\chi^\sigma)^{-1}$ of $E^\times$. Then the local factor $L(s, \pi \boxtimes \chi, r_1)$ has a pole at $s = 0$ if and only if

$$\pi \simeq \theta(\chi(\chi^\sigma)^{-1}),$$

and $L(2s, \pi \boxtimes \chi, r_2)$ has a pole at $s = 0$ if and only if

$$\omega_\pi = 1_{F^\times}.$$

Thus, from [34, Theorem 8.1], we obtain the following classification.

Proposition 5.12. *The representation $I_{Q_H}(\pi[s], \chi[-s])$ is irreducible unless $\pi \simeq \pi^\vee$ and $\omega_\pi \circ N_{E/F} = 1$. In that case, we have the following:*

(1) *Suppose that $\pi \simeq \theta(\chi(\chi^\sigma)^{-1})$. Then $I_{Q_H}(\pi[s], \chi[-s])$ is irreducible unless $s = 1$, and we have*

$$I_{Q_H}(\pi[1], \chi[-1]) = \delta_{Q_H,1}(\pi, \chi) + J_{Q_H}(\pi[1], \chi[-1])$$

where $\delta_{Q_H,1}(\pi, \chi)$ is an essentially discrete series representation.

(2) *Suppose* $\omega_\pi = 1_{F^\times}$. *Then* $I_{Q_H}(\pi[s], \chi[-s])$ *is irreducible unless* $s = \frac{1}{2}$, *and we have*

$$I_{Q_H}(\pi[1/2], \chi[-1/2]) = \delta_{Q_H,2}(\pi, \chi) + J_{Q_H}(\pi[1/2], \chi[-1/2])$$

where $\delta_{Q_H,2}(\pi, \chi)$ *is an essentially discrete series representation.*

(3) *Otherwise,* $I_{Q_H}(\pi[s], \chi[-s])$ *is irreducible unless* $s = 0$. *Then* $I_{Q_H}(\pi, \chi)$ *is a direct sum of two irreducible tempered representations.*

Remark 5.13. We give a remark on a genericity for irreducible representations of H. Let τ be an irreducible representation of $\mathrm{GSO}_{3,1}$ and χ a character of GL_1. From the study on a reducibility of parabolic inductions in this section, there are the following three cases:

- $I_{P_H}(\chi, \tau)$ is irreducible,
- $I_{P_H}(\chi, \tau)$ has a unique irreducible quotient and a unique submodule, or
- $I_{P_H}(\chi, \tau)$ decomposes into a direct sum of two irreducible representations.

Suppose that τ is an infinite-dimensional representation and that $I_{P_H}(\chi, \tau)$ is reducible. Since $I_{P_H}(\chi, \tau)$ has a unique generic constituent by [31, Theorem 2], it also has a unique nongeneric constituent. We often denote this generic (resp. nongeneric) constituent by $I_{P_H}(\chi, \tau)_{gen}$ (resp. $I_{P_H}(\chi, \tau)_{ng}$). The same is true for $I_{Q_H}(\rho, \xi)$ with an infinite-dimensional irreducible representation ρ of GL_2 and a character ξ of $E^\times$, and hence we often denote a unique generic (resp. nongeneric) constituent by $I_{Q_H}(\rho, \xi)_{gen}$ (resp. $I_{Q_H}(\rho, \xi)_{gn}$).

On the other hand, let π be an irreducible discrete series representation of H. Suppose that π is a submodule of the standard module $I_{P_H}(\chi, \tau)$. In particular, this standard module is reducible. Recall that the standard module conjecture for $\mathrm{SO}_{4,2}$ holds [25, Theorem 1.1], and so does $\mathrm{GSO}_{4,2}$ by Asgari–Shahidi [2, Corollary 3.5]. Thus, the Langlands quotient of $I_{P_H}(\chi, \tau)$ is not generic, and hence π should be generic. Similarly, when π is an irreducible submodule of a standard module for Q_H, we obtain a genericity of π. Hence, any discrete series representation is generic.

6. Local theta correspondence

We keep the notation in Section 4 and Section 5. In this section, we shall give an explicit computation of the local theta correspondence for the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$ (see Theorem 6.21). Further, we shall study a relationship between the local theta correspondence and Shalika period (see Theorem 6.9 and Theorem 6.26).

6.1. **Local theta correspondence for similitude groups.** Let us recall briefly the local theta correspondence for similitude groups (cf. [30] and [13, Section 2]). Let W be a symplectic space over F and V an even dimensional nondegenerate symmetric bilinear space over F. Then we denote by ω_ψ the Weil representation of $\mathrm{Sp}(W) \times \mathrm{O}(V)$ corresponding to ψ. We note that ω_ψ depends on V. Recall that ω_ψ is described as follows.

Fix a Witt decomposition $W = Y_+ \oplus Y_-$ and let $P(Y_+) = \mathrm{GL}(Y_+) \cdot N(Y_+)$ be the parabolic subgroup stabilizing the maximal isotropic subspace Y_+. The Weil representation ω_ψ can be realized on $\mathcal{S}(Y_- \otimes V)$ and the action of $P(Y_+) \times \mathrm{O}(V)$

is given by the usual formulas:

$$\begin{cases} \omega_\psi(h)\phi(x) = \phi(h^{-1}x), \quad \text{for } h \in \mathrm{O}(V), \\ \omega_\psi(a)\phi(x) = \chi_V(\det_Y(a)) \cdot |\det_Y(a)|^{\frac{1}{2}\dim V} \cdot \phi(a^{-1} \cdot x), \quad \text{for } a \in \mathrm{GL}(Y), \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x\rangle) \cdot \phi(x) \quad \text{for } b \in N(Y_+), \end{cases}$$

where χ_V is the quadratic character associated to $\operatorname{disc} V \in F^\times/F^{\times 2}$ and $\langle -, -\rangle$ is the natural symplectic form on $W \otimes V$. To describe the full action of $\mathrm{Sp}(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $\mathrm{O}(V)$ (respectively $\mathrm{Sp}(W)$), the maximal π-isotypic quotient is, by definition, the quotient space

$$\omega_\psi / \bigcap_{f \in \mathrm{Hom}(\omega_\psi, \pi)} \mathrm{Ker} f$$

where $\mathrm{Hom}(\omega_\psi, \pi)$ refers to the space of $\mathrm{O}(V)$-equivalent (resp. $\mathrm{Sp}(W)$-equivalent) homomorphisms. This quotient space is naturally a representation of $\mathrm{O}(V) \times \mathrm{Sp}(W)$ and has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation $\Theta_\psi(\pi)$ of $\mathrm{Sp}(W)$ (resp. $\mathrm{O}(V)$). We call $\Theta_\psi(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal quotient of $\Theta_\psi(\pi)$ which is semisimple (i.e., completely reducible); we call it the small theta lift of π. Then it was a conjecture of Howe that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on the domain.

Let us consider the theta correspondence for similitude groups. Let

$$R = \mathrm{GO}(V) \times \mathrm{GSp}(W)^+$$

where

$$\mathrm{GSp}(W)^+ = \{g \in \mathrm{GSp}(W) \mid \lambda_W(g) \in \mathrm{Im}\lambda_V\}.$$

The group R contains the subgroup

$$R_0 = \{(h, g) \in \mathrm{GO}(X) \times \mathrm{GSp}(W)^+ \mid \lambda_V(h) \cdot \lambda_W(g) = 1\}.$$

Then the Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(h, g)\phi = |\lambda_V(h)|^{-\frac{1}{8}\dim V \cdot \dim W} \omega(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Sp}(W).$$

Note that we used the normalization in [13]. Thus, the central elements $(t, t^{-1}) \in R_0$ act by the quadratic character $\chi_V(t)^{\frac{\dim W}{2}}$.

Now consider the (compactly) induced representation

$$\Omega = \mathrm{ind}_{R_0}^R \omega_\psi.$$

As a representation of R, Ω depends only on the orbit of ψ under the evident action of $\mathrm{Im}\lambda_V \subset F^\times$. For any irreducible admissible representation π of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$), the maximal π-isotypic quotient of Ω has the form

$$\pi \boxtimes \Theta(\pi)$$

where $\Theta(\pi)$ is some smooth representation of $\mathrm{GSp}(W)^+$ (resp. $\mathrm{GO}(V)$). Further, we let $\theta(\pi)$ be the maximal semisimple quotient of $\Theta(\pi)$. Note that though $\Theta(\pi)$ may be reducible, it has a central character $\omega_{\Theta(\pi)}$ given by

$$\omega_{\Theta(\pi)} = \chi_V^{\frac{\dim W}{2}} \cdot \omega_\pi.$$

The extended Howe conjecture for similitudes says that $\theta(\pi)$ is irreducible whenever $\Theta(\pi)$ is nonzero, and the map $\pi \mapsto \theta(\pi)$ is injective on its domain. It was shown by Roberts [30] that this follows from the Howe conjecture for isometry groups, and thus holds if the residual characteristic is not 2.

The following proposition is valid even when the residual characteristic is 2.

Proposition 6.1 (Proposition 2.3 in [12])**.** *Suppose that π is a supercuspidal representation of $\mathrm{GSp}(W)^+$ (resp. $\mathrm{GO}(V)$). Then we have*

(1) *$\Theta(\pi)$ is either zero or is an irreducible representation of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}(W)^+$).*
(2) *If π' is another supercuspidal representation such that $\Theta(\pi) = \Theta(\pi') \neq 0$, then $\pi = \pi'$.*

In one of the main theorems in this section, we shall show similar statements for all irreducible representations when $\dim W = 4$, $\dim V = 6$ and the Witt index of V is 2. Indeed, we shall compute explicitly the small theta lift to $\mathrm{GSO}(V)$ for all irreducible representations of $\mathrm{GSp}(W)^+$, and the other direction follows from the following general lemma.

Lemma 6.2. *Suppose that for any irreducible representation π of $\mathrm{GSp}(W)^+$:*

- *the small theta lift $\theta(\pi)$ to $\mathrm{GO}(V)$ is irreducible whenever $\Theta(\pi)$ is not zero,*
- *$\pi \mapsto \theta(\pi)$ is injective on its domain.*

Then for any irreducible representation σ of $\mathrm{GO}(V)$, the small theta lift $\theta(\sigma)$ to $\mathrm{GSp}(W)^+$ is irreducible whenever $\Theta(\sigma)$ is nonzero, and $\sigma \mapsto \theta(\sigma)$ is injective on its domain.

Proof. Let σ be an irreducible representation of $\mathrm{GO}(V)$ such that $\Theta(\sigma) \neq 0$. Suppose that $\theta(\sigma)$ contains two irreducible representations π_1 and π_2. Then we have

$$\mathrm{Hom}_{\mathrm{GSp}(W)^+ \times \mathrm{GO}(V)}(\Omega, \pi_i \boxtimes \sigma) \neq 0,$$

and we obtain surjective $\mathrm{GO}(V)$-homomorphisms

$$\Theta(\pi_i) \twoheadrightarrow \sigma.$$

Thus, we see that $\theta(\pi_i) \supset \sigma$. From our assumption on irreducibility of the small theta lifts, we see that

$$\theta(\pi_1) \simeq \sigma \simeq \theta(\pi_2),$$

and the injectivity shows that $\pi_1 \simeq \pi_2$. This implies that $\theta(\sigma)$ has a unique irreducible constituent with finite multiplicity.

Li, Sun, and Tian [22, Theorem A] show that the small theta lift for isometry groups is multiplicity free. By [1, Theorem 1.4] and [30, Theorem 4.4 (3)], this multiplicity one theorem implies that $\theta(\sigma)$ is multiplicity free. Thus, $\theta(\sigma)$ should be irreducible whenever $\Theta(\sigma)$ is nonzero. In addition, from the above proof, we see that $\sigma \mapsto \theta(\sigma)$ is injective on its domain. $\square$

We specialize to the case $\dim W = 4$, and we realize $\mathrm{GSp}(W)^+$ by GSp_4^+. Moreover, we consider the orthogonal spaces attached to a quadratic extension E over F, and they are given as follows. Let us denote by $\mathbb{H}$ the hyperbolic plane over F. Let $V_2^+ = E$ be the nondegenerate symmetric space with the pairing $(x, y) = tr(x\bar{y})$. Let $V_2^- = E$ be the nondegenerate symmetric space with the pairing $(x, y) = \kappa \cdot tr(x\bar{y})$ where $\chi_{E/F}(\kappa) = -1$. Then we let $V_{2r}^{\pm} = V_2^{\pm} + \mathbb{H}^{r-1}$, and we realize $\mathrm{GO}_{4,2} = \mathrm{GO}(V_6^{\pm})$ and $\mathrm{GO}_{3,1} = \mathrm{GO}(V_4^{\pm})$. As we noted above, ω_ψ and also Ω_ψ depends on a choice of $V_{2r}^{\pm}$. Hereafter, we denote by Ω_+ (resp. Ω_-) the Weil representation corresponding to V_{2r}^+ (resp. V_{2r}^-). Then we shall consider the theta correspondence for $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$. We note that there is no significant loss in restricting to $\mathrm{GSO}_{4,2}$ because of the following lemma.

Lemma 6.3. *Let π (resp. σ) be an irreducible representation of GSp_4^+ (resp. $\mathrm{GO}_{4,2}$) and suppose that*

$$\mathrm{Hom}_{\mathrm{GSp}_4^+ \times \mathrm{GO}_{4,2}}(\Omega_\pm, \pi \boxtimes \sigma) \neq 0.$$

Then $\sigma|_{\mathrm{GSO}_{4,2}}$ is irreducible as a representation of $\mathrm{GSO}_{4,2}$. If $\nu_0 = \lambda_{V_6^\pm}^{-3} \cdot \det$ is unique nontrivial quadratic character of $\mathrm{GO}_{4,2}/\mathrm{GSO}_{4,2}$, then $\sigma \otimes \nu_0$ does not participate in the theta correspondence with GSp_4^+.

Proof. This is proved similarly as [13, Lemma 2.4]. □

Let us denote by $\Theta_{\dim V}^{\pm}(\pi)$ and $\theta_{\dim V}^{\pm}(\pi)$ the big theta lift and small theta lift of irreducible representation π of GSp_4^+ to $\mathrm{GSO}(V_{2r}^{\pm})$ with respect to $\Omega_\pm$, respectively.

Further, we recall the result on the conservation relations by Kudla and Rallis [20, Corollary 3] for supercuspidal irreducible representations, and by Sun and Zhu [40, Theorem B] for all irreducible representations.

Theorem 6.4. *Let π be an irreducible representation of GSp_4^+. Then we have*

$$n^+(\pi) + n^-(\pi) = 12.$$

Here we define

$$n^{\pm}(\pi) = \min\{\dim V_{2r}^{\pm} \mid \theta_{\dim V_{2r}^{\pm}}^{\pm}(\pi) \neq 0, sgn(V_{2r}^{\pm}) = \pm\}.$$

Since we only consider the theta lift with respect to Ω_+, we shall simply denote the quadratic space, the induced Weil representation, the big theta lift the small theta lift by $V_{2r}^+ = V_{2r}$, $\Omega_+ = \Omega$, $\Theta_{\dim V^+}^+(\pi) = \Theta_{\dim V}(\pi)$ and $\theta_{\dim V^+}^+(\pi) = \theta_{\dim V}(\pi)$, respectively. Also, when $\dim V$ is clear, we simply write $\Theta_{\dim V}(\pi) = \Theta(\pi)$ and $\theta_{\dim V}(\pi) = \theta(\pi)$. For an irreducible representation σ of $\mathrm{GSO}_{4,2}$, we denote by $\Theta(\sigma)$ and $\theta(\sigma)$ the big theta lift and the small theta lift to GSp_4^+ with respect to $\Omega = \Omega_+$, respectively.

Finally, we give a definition of local analogue of Shalika period.

Definition 6.5. We define a subgroup S of $\mathrm{GSO}_{4,2}$ by

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid r \in \mathrm{SO}_{3,1},\, re_2 = e_2 \right\} \cdot U_{P_H},$$

and a character ψ' of S by

$$\psi'(r \cdot u_0(x) u_1(y, z) \tilde{u}(w)) = \psi(2dz).$$

For an irreducible representation σ of $\mathrm{GSO}_{4,2}$, we say that σ has Shalika period if

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\sigma, \mathrm{Ind}_S^{\mathrm{GSO}_{4,2}}(\psi')) \neq 0.$$

6.2. **Twisted Jacquet modules of induced Weil representations.** As in the global case, we can compute twisted Jacquet modules following [23].

Proposition 6.6. (1) *As a representation of* GSp_4^+,

$$\Omega_{\psi_U} \simeq \mathrm{ind}_{N_G}^{\mathrm{GSp}_4^+} \psi_N.$$

(2) *As a representation of* $\mathrm{GSO}_{4,2}$,

$$\Omega_{\psi_N} \simeq \mathrm{ind}_S^{\mathrm{GSO}_{4,2}} \psi'.$$

Here nondegenerate characters ψ_U *of* U *and* ψ_N *of* N *are defined as in* (3.1) *and* (3.2)*, respectively.*

Corollary 6.7. *Let* π *be an irreducible representation of* GSp_4^+*. Then* π *is generic with respect to* ψ_N *if and only if* $\Theta(\pi)$ *contains a generic constituent.*

Corollary 6.8. *Let* σ *be an irreducible representation of* $\mathrm{GSO}_{4,2}$*. The* σ *has Shalika period if and only if* $\Theta(\sigma)$ *contains a constituent which is generic with respect to* ψ_N*. In particular, if* σ *is generic and has Shalika period, then* $\theta(\sigma)$ *is generic.*

From these corollaries, we obtain the following equivalent conditions that characterize generic representations which have Shalika period.

Theorem 6.9. *Let* σ *be a generic irreducible representation of* $\mathrm{GSO}_{4,2}$*. Then the following conditions are equivalent:*

(1) σ *has Shalika period.*
(2) *The small theta lift* $\theta(\sigma)$ *of* σ *to* GSp_4^+ *is nonzero.*
(3) *The small theta lift* $\theta(\sigma)$ *of* σ *to* GSp_4^+ *is generic with respect to* ψ_N*.*

For the computation of theta lifts, we need a result on the theta correspondence for the dual pairs $\left(\mathrm{GL}_2^+, \mathrm{GSO}_{3,1}\right)$. As in Proposition 6.6, we can compute the twisted Jacquet module for this dual pair following [23], which is a local analogue of a proof of Lemma 3.8.

Lemma 6.10. *Let us regard the unipotent subgroup* U_2 *defined in Section* 2.1 *as a subgroup of* $\mathrm{GSO}_{3,1}$*, which is a maximal unipotent radical. Then as a representation of* GL_2^+*,*

$$\Omega_{\psi_{U_2}} = \mathrm{ind}_{N_2}^{\mathrm{GL}_2^+}(\psi_2).$$

Here we denote by ψ_{U_2} *a nondegenerate character of* U_2*, and* ψ_2 *is the nondegenerate character of* N_2 *defined by* (4.2)*.*

Corollary 6.11. *Let* τ^+ *be an irreducible representation of* GL_2^+*. Then* τ^+ *is generic with respect to* ψ_2 *if and only if* $\Theta(\pi)$ *contains a generic constituent. In particular, when* τ^+ *is an infinite-dimensional representation,* $\theta(\tau^+)$ *is not zero if and only if* τ^+ *is generic with respect to* ψ_2*.*

Proof. The first assertion follows from the above lemma. We suppose that τ^+ is infinite-dimensional. If $\theta(\tau^+)$ is not zero, this representation should be an infinite-dimensional representation. Thus, $\theta(\tau^+)$ is generic, in particular, $\Theta(\tau^+)$ contains a generic constituent, and our assertion follows. □

Moreover, we can compute explicitly theta lifts from GL_2^+ to $\mathrm{GSO}_{3,1}$, which is given in Cognet [4] when the residual characteristic of F is not 2. Here we note that for any irreducible representation τ^+ of GL_2^+, $\theta(\tau^+)|_{\mathrm{GSO}_{3,1}}$ is irreducible whenever $\theta(\tau^+)$ is irreducible, which is proved as in [13, Lemma 2.4], and thus we can consider a theta lift from GL_2^+ to $\mathrm{GSO}_{3,1}$.

Lemma 6.12. *Let τ^+ be an irreducible representation of GL_2^+. Suppose that the small theta lift $\theta(\tau^+)$ of τ^+ to $\mathrm{GSO}_{3,1}$ is not zero. Then $\theta(\tau^+)$ is given by*

$$\pi(BC(\tau), \chi_{E/F}\omega_\tau)$$

where τ is an irreducible representation of GL_2 such that $\tau^+ \subset \tau_+$, ω_τ is the central charter of τ, and $BC(\tau)$ is the base change lift of τ to $\mathrm{GL}_2(E)$, which does not depend on a choice of τ.

Proof. From [4], it suffices to show that $\theta(\tau^+)$ is irreducible. If τ^+ is supercuspidal, our assertion follows from Proposition 6.1.

Let χ_1 and χ_2 be characters of $F^\times$, and we suppose that τ^+ is an irreducible submodule of $I_{B_2}^{\mathrm{GL}_2}(\chi_1 \boxtimes \chi_2)_+$. Then the lift $\theta(\tau^+)$ is computed in a similar way as for the pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$, so that we give only a sketchy proof (cf. Section 6.4).

If necessary, replacing χ_1 and χ_2, we may assume that $\chi_1\chi_2^{-1} \neq \chi_{E/F}|-|$. Then from [13, Theorem A.2], we can show that (cf. [13, p.24])

$$\Theta(\tau^+)^* \subset \mathrm{Hom}_{\mathrm{GL}_2^+}(\Omega, I_{B_2}^{\mathrm{GL}_2}(\chi_1 \boxtimes \chi_2)_+) = I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi_{E/F}\chi_1^{-1}\chi_2 \boxtimes \chi_{1,E})^*,$$

and we have

$$I_{B_{3,1}}^{\mathrm{GSO}_{3,1}}(\chi_{E/F}\chi_1^{-1}\chi_2 \boxtimes \chi_{1,E}) \twoheadrightarrow \Theta(\tau^+).$$

Since this parabolic induction has a unique irreducible quotient, our assertion readily follows. □

6.3. **Normalized Jacquet module of induced Weil representations.** Let us recall the result in [13, Appendix] for the normalized Jacquet module of the induced Weil representation for the dual pair $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$.

Applying [11, Theorem A.2] for $n = 2$, $m = 6$ and $r = 2$ with [11, Remark A.5] taken into account, we obtain the following computation of normalized Jacquet modules with respect to maximal parabolic subgroups of GSp_4^+.

Lemma 6.13. *The normalized Jacquet module $R_{Q_G}(\Omega)$ of the Weil representation Ω along the parabolic Q_G has a $\mathrm{GSO}_{4,2} \times M_{Q_G}$ invariant filtration*

$$0 \to A \to R_{Q_G}(\Omega) \to B \to 0$$

with

$$A \simeq I_{P_H}(S(F^\times) \otimes \Omega_{4,1})$$

and

$$B \simeq \Omega_{6,1} \boxtimes |\det g|^{-1/2}|b|$$

where $\Omega_{4,1}$ (resp. $\Omega_{6,1}$) is the induced Weil representation for $\mathrm{GSO}(V_4^+) \times \mathrm{GL}_2^+$ (resp. $\mathrm{GSO}(V_6^+) \times \mathrm{GL}_2^+$), and the action of $M_{P_H} \times M_{Q_G}$ on $S(F^\times)$ is given as follows: Let $\varphi(A) \in S(F^\times)$. For $(a, h) \in \mathrm{GL}_1 \times \mathrm{GSO}_{3,1}$ and $(b, g) \in \mathrm{GL}_1 \times \mathrm{GL}_2$, we have

$$((a, h), (b, g))\varphi(A) = \chi_{E/F}(b)\varphi(a^{-1} \cdot \det g^{-1} \cdot A \cdot b).$$

Lemma 6.14. *Let $R_{P_G}(\Omega)$ denote the normalized Jacquet module of Ω along P_G. Then as a representation of $M_{P_G} \times \mathrm{GSO}_{4,2}$, $R_{P_G}(\Omega)$ has a 3-step filtration whose successive quotients are given as follows:*

(1) *The top piece of the filtration is*

$$A'' = S(F^\times) \otimes \chi_{E/F}(\det a)|\det a|^{3/2} \cdot |\lambda|^{-3/2},$$

where $(a, \lambda, h) \in (\mathrm{GL}_2 \times \mathrm{GL}_1^+) \times \mathrm{GSO}_{4,2}$ acts on $S(F^\times)$ by

$$(a, \lambda, h)\phi(t) = \phi(t \cdot \lambda \cdot \lambda(h)).$$

(2) *The second piece in the filtration is*

$$B'' = I_{B_G \times P_H}(S(F^\times \times F^\times))$$

where the action of the diagonal torus $\mathrm{diag}(a, d; 1)$ in B_G on $S(F^\times \times F^\times)$ is given by

$$(\mathrm{diag}(a, d; 1) \cdot \phi)(\lambda, t) = \chi_{E/F}(ad)|a| \cdot \phi(\lambda, td).$$

(3) *The bottom piece of the filtration is*

$$C'' = I_{Q_H}(S(F^\times) \otimes S(\mathrm{GL}_2)),$$

where the action of $(\mathrm{GL}_2 \times \mathrm{GL}_1^+) \times (\mathrm{GL}_2 \times \mathrm{GSO}_2)$ on $S(F^\times) \otimes S(\mathrm{GL}_2)$ is given by

$$(a, \lambda; b, h)\phi(t, g) = \chi_{E/F}(\det a)\phi(t \cdot \lambda \cdot \lambda(h), b^{-1}ga \cdot \lambda(h)).$$

Applying the [11, Theorem A.1] for $n = 2$, $m = 6$ and $r = 2$ with [11, Remark A.5] taken into account, we obtain the following computation of normalized Jacquet modules with respect to maximal parabolic subgroups of $\mathrm{GSO}_{4,2}$.

Lemma 6.15. *Let $R_{P_H}(\Omega)$ denote the normalized Jacquet module of Ω along P_H. Then we have a short exact sequence of $M_{P_H} \times \mathrm{GSp}_4^+$-modules:*

$$0 \to A_0 \to R_{P_H}(\Omega) \to B_0 \to 0.$$

Here,

$$B_0 \simeq \Omega_{4,2},$$

where $\Omega_{4,2}$ is the induced Weil representation for $\mathrm{GSp}_4^+ \times \mathrm{GSO}(V_4^+)$, and

$$A_0 \simeq I_{Q_G}(S(F^\times) \otimes \Omega_{4,1}),$$

where the action of $(\mathrm{GL}_1 \times \mathrm{GSO}_{3,1}) \times (\mathrm{GL}_1 \times \mathrm{GL}_2^+))$ on $S(F^\times)$ is given by

$$((a, h), (b, g)) \cdot f(x) = \chi_{E/F}(a)f(b^{-1} \cdot x \cdot a \cdot \lambda(h)^{-1})$$

and $\Omega_{4,1}$ denotes the induced Weil representation of $\mathrm{GL}_2^+ \times \mathrm{GSO}(V_4^+)$.

Lemma 6.16. *Let $R_{Q_H}(\Omega)$ denote the normalized Jacquet module of Ω along Q_H. Then as a representation of $M_{Q_H} \times \mathrm{GSp}_4^+$, $R_{Q_H}(\Omega)$ has a 3-step filtration whose successive quotients are given as follows:*

(1) *The top piece of the filtration is*

$$A' = \Omega_{2,2} \boxtimes |\lambda(h)|^{-1/2}|\det a|^{1/2}$$

where $(a, h; g) \in (\mathrm{GL}_2 \times \mathrm{GSO}_2)$ and $\Omega_{2,2}$ is the induced Weil representation for $\mathrm{GSO}(V_2^+) \times \mathrm{GSp}_4^+$.

(2) *The second piece in the filtration is*

$$B' = I_{B_H \times Q_G}(S(F^\times) \otimes \Omega_{2,1})$$

where $\Omega_{2,1}$ *is the induced Weil representation for* $\mathrm{GSO}(V_2^+) \times \mathrm{GL}_2^+$*, and the action of maximal torus* T_H *of* B_H *on* $S(F^\times)$ *is given by*

$$(\mathrm{diag}(a, b; h), 1)\phi(A) = \chi_{E/F}(b)|\lambda(h)|^{-1}\phi(Ab).$$

(3) *The bottom piece of the filtration is*

$$C' = I_{P_G}(S(F^\times) \otimes S(\mathrm{GL}_2))$$

where the action of $(a, h : b, \lambda) \in (\mathrm{GL}_2 \times \mathrm{GSO}_2) \times (\mathrm{GL}_2 \times \mathrm{GL}_1^+)$ *on* $S(F^\times) \otimes S(\mathrm{GL}_2)$ *is given by*

$$(a, h : b, \lambda)\varphi(t, A) = \chi_{E/F}(\det a)\varphi(\lambda(h)\lambda t, \lambda b^{-1}Aa).$$

Using the above lemmas and Lemma 6.12, we can compute some spaces of homomorphisms in a similar way as [13, Proposition 11.5].

Proposition 6.17. *Assume that* $\chi \neq |-|$*. Then as a representation of* $\mathrm{GSO}_{4,2}$*,*

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q_G}(\chi, \tau^+)) = I_{P_H}(\chi^{-1} \cdot \chi_{E/F}, \pi(BC(\chi \cdot \tau^+), \chi_{E/F} \cdot \chi^2\omega_{\tau^+}))^*.$$

if τ^+ *is generic with respect to* ψ_2*. Otherwise we have*

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q_G}(\chi, \tau^+)) = 0.$$

Proposition 6.18. *Suppose that* τ *is a discrete series representation of* GL_2 *and* $\omega_\tau \neq |-|^3$*. Then as a representation of* $\mathrm{GSO}_{4,2}$*,*

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{P_G}(\tau, \chi_+)) \hookrightarrow I_{Q_H}(\chi_{E/F} \cdot \tau^\vee, (\chi\omega_\tau)_E)^*.$$

Further, if τ *is supercuspidal, then*

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{P_G}(\tau, \chi_+)) = I_{Q_H}(\chi_{E/F} \cdot \tau^\vee, (\chi\omega_\tau)_E)^*.$$

Proposition 6.19. *Consider the space*

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi, \pi(\sigma, \xi)))$$

as a representation of GSp_4^+*. Then we have:*

(1) *If* $\chi \neq \chi_{E/F}$*, then*

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi, \pi(\sigma, \xi))) = 0$$

unless

$$\pi(\sigma, \xi) = \theta(\sigma_0)$$

with some irreducible representation σ_0 *of* GL_2^+*, in which case, we have*

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi, \pi(\sigma, \xi))) = I_{Q_G}(\chi^{-1} \cdot \chi_{E/F}, \sigma_0 \otimes \chi\chi_{E/F})^*.$$

(2) *if* $\chi = \chi_{E/F}$ *but* $\pi(\sigma, \xi)$ *does not participate in the theta correspondence with* GL_2^+*, then*

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi_{E/F}, \pi(\sigma, \xi))) = \Theta_{W,V_2}(\pi(\sigma, \xi))^*$$

where $\Theta_{W,V_2}(\pi(\sigma, \xi))$ *denotes the big theta lift of* $\pi(\sigma, \xi)$ *from* $\mathrm{GSO}_{3,1}$ *to* GSp_4^+*.*

(3) *if* $\chi = \chi_{E/F}$ *and* $\pi(\sigma, \xi) = \theta(\sigma_0)$*, then we have an exact sequence:*

$$0 \to \Theta_{W,V_2}(\pi(\sigma, \xi))^* \to \mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi, \pi(\sigma, \xi))) \to I_{Q_G}(1_{F^\times}, \sigma_0)^*.$$

Proposition 6.20. *Suppose that τ and χ are unitary representation of GL_2 and GSO_2, respectively. Then as a representation of GSp_4^+, we have*

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{Q_H}(\tau,\chi)) \hookrightarrow I_{P_G}(\chi_{E/F}\cdot\tau^\vee, \omega_\tau\chi_F^2)^*.$$

6.4. **Computation of theta lifts.**

Theorem 6.21. *Let π be an irreducible representation of GSp_4^+, and we consider the theta lift to $\mathrm{GSO}_{4,2}$ with respect to Ω_+. Then $\Theta(\pi)$ has a unique irreducible quotient $\theta(\pi)$ if it is nonzero. Moreover, the map $\pi \mapsto \theta(\pi)$ is injective on its domain, and $\theta(\pi)$ is given explicitly as follows.*

(1) **(Supercuspidal)** *Suppose that π is supercuspidal representation.*
- *Suppose $n^+(\pi) < 6$, and we denote by σ the irreducible representation of $\mathrm{GO}_{3,1}$ such that $\pi = \theta(\sigma)$. Let σ_0 be an irreducible constituent of $\sigma|_{\mathrm{GSO}_{3,1}}$. If σ_0 is invariant but not distinguished, then*

$$\theta(\pi) = \begin{cases} I_{P_H}(1,\sigma_0)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{P_H}(1,\sigma_0)_{ng} & \text{if } \pi \text{ is nongeneric.} \end{cases}$$

  On the other hand, if σ_0 is not invariant or distinguished, then

$$\theta(\pi) = I_{P_H}(1,\sigma_0).$$

  Here we say that $\sigma_0 = \pi(\tau,\mu)$ is invariant (resp. distinguished) if τ is Galois invariant (resp. $\tau \in \Phi_\mu$).
- *If $n^+(\pi) = 6$, $\theta(\pi)$ is supercuspidal.*
- *If $n^+(\pi) > 6$, $\theta(\pi) = 0$.*

(2) **(Discrete series representations)** *Suppose that π is an essentially discrete series representation.*
- *Suppose $\pi = St(\chi,\tau^+)$.*
  (a) *If τ^+ participate in the theta correspondence with $\mathrm{GSO}_{2,0}$ and $\tau^+ = \theta(\xi)$ with a character ξ of $E^\times$, then*

$$\theta(\pi) = \begin{cases} I_{B_H}(\chi\chi_{E/F}[-1], \chi_{E/F}; \chi_E\xi[-1/2]) & \text{if } \chi \neq \chi_{E/F}, \\ I_{P_H}(\chi_{E/F}, \pi(\chi_E\xi^\sigma\cdot St, \chi\chi_{E/F}\xi_F)) & \text{if } \chi = \chi_{E/F}. \end{cases}$$

  (b) *Otherwise, we have*

$$\theta(\pi) = \begin{cases} \delta_{P_H,2}(\chi\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F}\omega_{\tau^+}))^\vee & \text{if } BC(\tau^+) \in \Phi_{\chi\chi_{E/F}}, \\ I_{P_H}(\chi\chi_{E/F}[-1], \pi(BC(\tau^+)[1/2], \chi_{E/F}\omega_{\tau^+}[1])) & \text{if } BC(\tau^+) \notin \Phi_{\chi\chi_{E/F}}. \end{cases}$$

- *Suppose $\pi = St(\tau,\mu_+)$.*
  (a) *If τ is supercuspidal,*

$$\theta(\pi) = \delta_{Q_H,2}(\tau\otimes\chi_{E/F}, \mu_E^{-1})^\vee.$$

  (b) *Otherwise,*

$$\theta(\pi) = I_{Q_H}((\tau\otimes\chi_{E/F})^\vee[-1/2], (\omega_\tau\mu)_E[1/2]).$$

- *If $\pi = St(st_{\chi_{E/F}},\mu_+)^-$, $\theta(\pi) = 0$.*
- *If $\pi = St(st_{\chi_{E/F}},\mu_+)^+$, $\theta(\pi) = \tau_+(1_{F^\times}, \mu_E^{-1})^\vee$.*
- *If $\pi = \chi\mathrm{St}_G$, $\theta(\pi) = \delta(\chi_{E/F},\chi_E)$.*

(3) **(Nondiscrete series representations)**

- *Suppose* $\pi \hookrightarrow I_{Q_G}(\chi, \tau^+)$ *with an essentially discrete series representation* τ^+ *of* GL_2^+ *and a character* χ *of* $F^\times$ *such that* $|\chi| = |-|^{-s}$. *If* τ^+ *is not generic with respect to* ψ_2, *then*

$$\Theta(\pi) = 0.$$

On the other hand, we assume τ^+ *is generic. Then:*

(a) *If* $\chi = 1_{F^\times}$ *and* $BC(\tau^+)$ *is an essentially discrete series representation, then*

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F} \cdot \omega_{\tau^+}))_{gen} & \text{if } \pi \text{ is generic,} \\ I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F} \cdot \omega_{\tau^+}))_{ng} & \text{otherwise.} \end{cases}$$

(b) *If* $\chi = \chi_{E/F}$, $\mathrm{Ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2} \tau^+$ *is reducible and* $BC(\tau^+)$ *is supercuspidal, then*

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{P_H}(1_{F^\times}, \pi(BC(\tau^+), \chi_{E/F}\omega_{\tau^+})) & \text{if } \pi \text{ is generic,} \\ 0 & \text{otherwise.} \end{cases}$$

(c) *If* $\chi = 1_{F^\times}$ *or* $\chi = \chi_{E/F}$, *and* $BC(\tau^+) = I_{B_2(E)}^{\mathrm{GL}_2(E)}(\xi \boxtimes \xi^\sigma)$, *then*

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_B(\chi\chi_{E/F}, \chi_{E/F}; \xi) & \text{if } \pi \text{ is generic,} \\ 0 & \text{otherwise.} \end{cases}$$

(d) *Otherwise, we have*

$$\theta(\pi) = J_{P_H}(\chi^{-1} \cdot \chi_{E/F}, \pi(BC(\chi \cdot \tau^+), \chi_{E/F} \cdot \chi^2 \omega_{\tau^+})).$$

- *Suppose* $\pi \hookrightarrow I_{P_G}(\tau, \chi_+)$ *with some irreducible representation* τ *of* GL_2 *and* χ *of* $F^\times$.

(a) *Suppose* $\tau \otimes \chi_{E/F} = \tau$ *with* $\omega_\tau = \chi_{E/F}$. *Then we have*

$$\theta(\pi) = \Theta(\pi) = \begin{cases} I_{Q_H}(\tau, \chi_E)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{Q_H}(\tau, \chi_E)_{ng} & \text{if } \pi \text{ is nongeneric.} \end{cases}$$

(b) *Otherwise,*

$$\theta(\pi) = J_{Q_H}((\tau \otimes \chi_{E/F})^\vee, (\chi\omega_\tau)_E).$$

- *Suppose* $\pi \hookrightarrow I_{B_G}(\chi_1, \chi_2; \chi_+)$.

(a) *In the case* (3a) *in Lemma* 4.2, *i.e., if* $\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^-)$, *then*

$$\theta(\pi) = 0.$$

On the other hand, if $\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^+)$,

$$\theta(\pi) = J_{B_H}(\chi_1^{-1}\chi_{E/F}[s_1], 1, \chi_{1,E}\chi_E[-s_1]).$$

(b) *In the case* (3b) *in Lemma* 4.2, *i.e., if* $\pi \hookrightarrow I_{Q_G}(\chi_0, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_0\chi_{E/F} \boxtimes \chi)_+)$, *then*

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{ng} & \text{if } \pi \text{ is nongeneric.} \end{cases}$$

(c) *In the case* (3c) *in Lemma* 4.2, *i.e., if* $\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, (\chi \cdot |-|^{-1/2}) \circ \det)$, *then*

$$\theta(\pi) = J_{B_H}(\chi^{-1}\chi_{E/F}, \chi_{E/F}[1]; (\chi_1\chi)_E[-1]).$$

(d) *In the case* (3d) *in Lemma* 4.2, *i.e., if* π *is a unique irreducible submodule of* $I_{Q_G}(\chi_1|-|^{-s_1}, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_2|-|^{-s_2}\boxtimes\chi)_+)$, *then*

$$\theta(\pi) = J_{B_H}(\chi_1^{-1}\chi_{E/F}[s_1], \chi_2^{-1}\chi_{E/F}[s_2]; \chi_{1,E}\chi_{2,E}\chi_E[-s_1-s_2]).$$

Before proceeding with a computation, we state two simple facts [13, p. 24]:

(1) if π is an irreducible representation of GSp_4^+, then

$$\Theta(\pi)^* \simeq \mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, \pi).$$

(2) if Π is an irreducible representation of $\mathrm{GSO}_{4,2}$ such that

$$\Pi^\vee \hookrightarrow \mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, \Sigma),$$

where Σ is not necessarily irreducible (typically, Σ is a principal representation), then there is a nonzero equivalent map

$$\Omega \to \Pi \boxtimes \Sigma.$$

In particular, $\Theta(\Pi) \neq 0$. The analogue result with roles of GSp_4^+ and $\mathrm{GSO}_{4,2}$ exchanged also holds.

6.4.1. *Supercuspidal.* We shall compute the theta correspondence for supercuspidal representation π of GSp_4^+. We note that $\Theta(\pi) = \theta(\pi)$ is irreducible or zero by Proposition 6.1.

Suppose that $n^+(\pi) > 6$. Then from the definition of $n^+(\pi)$, we have

$$\theta(\pi) = 0.$$

Suppose that $n^+(\pi) = 6$. Then $\theta(\pi)$ is nonzero, and it is known that $\theta(\pi)$ is supercuspidal.

Finally, we suppose that $n^+(\pi) < 6$. With the notation in Lemma 6.15, we have

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(A_0, \pi) = 0$$

because π is supercuspidal. From Proposition 6.1, we see that there exists a unique irreducible representation σ of $\mathrm{GO}_{3,1}$ such that

$$\mathrm{Hom}_{\mathrm{GSp}_4^+\times\mathrm{GO}_{3,1}}(\Omega_{4,2}, \pi\boxtimes\sigma) \neq 0$$

where $\Omega_{4,2}$ is the induced Weil representation of $\mathrm{GSp}_4^+ \times \mathrm{GO}_{3,1}$. Thus, $\theta(\pi)$ is an irreducible constituent of $I_{P_H}(1, \sigma|_{\mathrm{GSO}_{3,1}})$.

Let σ_0 be an irreducible constituent of $\sigma|_{\mathrm{GSO}_{3,1}}$. Suppose that σ_0 is invariant but not distinguished. Then $\sigma|_{\mathrm{GSO}_{3,1}} = \sigma_0$. Let us denote by $\sigma_0 = \pi(\tau, \mu)$ with irreducible representations τ of $\mathrm{GL}_2(E)$ and μ of $F^\times$ such that $\omega_\tau = \mu_E$. By Proposition 5.11, $I_{P_H}(1, \sigma_0)$ is reducible since $\pi(\tau, \mu)$ is not distinguished, that is, $\tau \notin \Phi_\mu$. Then we have (see Remark 5.13)

$$I_{P_H}(1, \sigma_0) = I_{P_H}(1, \sigma_0)_{gen} \oplus I_{P_H}(1, \sigma_0)_{ng},$$

and thus we obtain, by Corollary 6.7,

$$\theta(\pi) = \begin{cases} I_{P_H}(1, \sigma_0)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{P_H}(1, \sigma_0)_{ng} & \text{if } \pi \text{ is nongeneric.} \end{cases}$$

Suppose that σ_0 is not invariant or distinguished. In both cases, by Proposition 5.11, $I_{P_H}(1, \sigma_0)$ is irreducible. Thus, we see that

$$\theta(\pi) = I_{P_H}(1, \sigma_0).$$

Moreover, we give a remark on a choice of σ_0. Since $\theta(\pi)$ extends to a representation of $\mathrm{GO}_{4,2}$, $\theta(\pi)^g \simeq \theta(\pi)$ for any $g \in \mathrm{GO}_{4,2}$. Thus, $I_{P_H}(1, \sigma_0)$ does not depend on a choice of an irreducible constituent σ_0 of $\sigma|_{\mathrm{GSO}_{3,1}}$.

Finally, we note that π is generic with respect to ψ_N by [8, Theorem A.11 (ii)], and thus we have

$$I_{P_H}(1, \sigma_0) = I_{P_H}(1, \sigma_0)_{gen}.$$

6.4.2. *Discrete series representation.*

The generalized Steinberg representation $St(\chi, \tau^+)$. Let us consider the theta lift of $\pi = St(\chi, \tau^+)$. Since

$$St(\chi, \tau^+) \hookrightarrow I_{Q_G}(\chi|-|, |-|^{-1/2} \cdot \tau^+),$$

it is deduced from Proposition 6.17 and fact (1) that

$$I_{P_H}(\chi\chi_{E/F}[-1], \pi(BC(\tau^+)[1/2], \chi_{E/F}\omega_{\tau^+}[1])) \twoheadrightarrow \Theta(St(\chi, \tau^+))$$

if τ^+ is generic with respect to ψ_2. Otherwise, $\Theta(St(\chi, \tau^+)) = 0$.

Suppose that τ^+ is generic with respect to ψ_2. Then $St(\chi, \tau^+)$ is generic, and $\Theta(\pi)$ contains a generic constituent. In particular, we have $\Theta(\pi) \neq 0$.

Assume that τ^+ does not participate in the theta correspondence with GSO_2. Then we know that $BC(\tau^+)$ is supercuspidal. Moreover, $BC(\tau^+)$ is in $\Phi_{\chi\chi_{E/F}}$ if and only if $I_{P_H}(\chi\chi_{E/F}[-1], \pi(BC(\tau^+)[1/2], \chi_{E/F}\omega_{\tau^+}[1]))$ is reducible by Proposition 5.11. If this condition is satisfied, it has a unique irreducible quotient. Otherwise, the above induced representation is irreducible. Thus, we obtain

$$\theta(\pi) = \begin{cases} \delta_{P_H,2}(\chi\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F}\omega_{\tau^+}))^\vee & \text{if } BC(\tau^+) \in \Phi_{\chi\chi_{E/F}}, \\ I_{P_H}(\chi\chi_{E/F}[-1], \pi(BC(\tau^+)[1/2], \chi_{E/F}\omega_{\tau^+}[1])) & \text{if } BC(\tau^+) \notin \Phi_{\chi\chi_{E/F}}. \end{cases}$$

On the other hand, suppose that τ^+ participates in the theta correspondence with GSO_2. In this case, $BC(\tau^+)$ is not supercuspidal, and indeed it is the parabolic induction

$$I_{B_2(E)}^{\mathrm{GL}_2(E)}(\xi^\sigma \boxtimes \xi)$$

when $\tau^+ = \theta(\xi)$ with a character ξ of $\mathrm{GSO}_2 \simeq E^\times$. Then we have

$$\begin{aligned} &I_{P_H}(\chi\chi_{E/F}[-1], \pi(BC(\tau^+)[1/2], \chi_{E/F}\omega_{\tau^+}[1])) \\ =&I_{B_H}(\chi\chi_{E/F}[-1], \chi_{E/F}; \chi_E\xi[1/2]). \end{aligned}$$

If $\chi_E\xi^\sigma \neq \xi$, this parabolic induction is irreducible (see Section 5.2.2). Then we obtain

$$\theta(\pi) = I_{B_H}(\chi\chi_{E/F}[-1], \chi_{E/F}; \chi_E\xi[1/2]).$$

On the other hand, if we have $\chi_E\xi^\sigma = \xi$, then by Proposition 5.9, this induced representation has a unique irreducible quotient $I_{P_H}(\chi_{E/F}, \pi(\chi_E\xi^\sigma \cdot St, \chi\chi_{E/F}\xi_F))$. Thus, we obtain

$$\theta(\pi) = I_{P_H}(\chi_{E/F}, \pi(\chi_E\xi^\sigma \cdot St, \chi\chi_{E/F}\xi_F)).$$

The generalized Steinberg representation $St(\tau,\mu_+)$ and $St(st_{\chi_{E/F}},\mu_+)^{\pm}$. Let us consider the case $\pi = St(\tau,\mu_+)$ where τ is supercuspidal with trivial central character or $\tau = st_\chi$ with nontrivial quadratic character $\chi \neq \chi_{E/F}$. Then we have

$$\pi \hookrightarrow I_{P_G}(\tau|-|^{1/2}, (\mu|-|^{-1/2})_+)$$

as a unique irreducible submodule. Since π is generic, $\Theta(\pi)$ contains the generic constituent, in particular, $\Theta(\pi) \neq 0$.

Suppose that τ is supercuspidal. Then Proposition 6.18 shows that

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{P_G}(\tau|-|^{1/2}, (\mu|-|^{-1/2})_+)) = I_{Q_H}((\tau\otimes\chi_{E/F})^\vee[-1/2], (\omega_\tau\mu)_E[1/2])^*.$$

Then the above induced representation is reducible if and only if $\omega_\tau = 1_{F^\times}$ by Proposition 5.12. Further, if it is reducible, it has a unique irreducible quotient $\delta_{Q_H}(\tau\otimes\chi_{E/F},\mu_E)^\vee$ which is an essentially discrete series representation. Thus, we see that

$$\theta(\pi) = \delta_{Q_H,2}(\tau\otimes\chi_{E/F},\mu_E)^\vee$$

since we know $\theta(\pi) \neq 0$.

Suppose that $\tau = st_\chi$ with $\chi \neq \chi_{E/F}$. Then $I_{Q_H}((\tau\otimes\chi_{E/F})^\vee[-1/2], (\omega_\tau\mu)_E[1/2])$ is reducible if and only if $\chi = 1_{F^\times}$ by Proposition 5.7 and the result in Section 5.2.2. Hence, in this case, it is irreducible, and we obtain

$$\theta(\pi) = I_{Q_H}((\tau\otimes\chi_{E/F})^\vee[-1/2], (\omega_\tau\mu)_E[1/2])).$$

Let us consider the case of $\pi = St(st_{\chi_{E/F}},\mu_+)^{\pm}$. Recall that

$$n^-(St(st_{\chi_{E/F}},\mu_+)^-) \leq 4 \quad \text{and} \quad n^+(St(st_{\chi_{E/F}},\mu_+)^+) \leq 4$$

by [8, Theorem A.11 (iii)]. Thus, we see that

$$\Theta(St(st_{\chi_{E/F}},\mu_+)^-) = 0.$$

As in the above computation, we see that

$$I_{Q_H}(st[-1/2],\mu_E[1/2]) \twoheadrightarrow \Theta(St(st_{\chi_{E/F}},\mu_+)^+).$$

Since this induced representation has the unique irreducible quotient $\tau_+(1_{F^\times},\mu_E^{-1})^\vee$ by Proposition 5.7 and $\Theta(\pi) \neq 0$, we see that

$$\theta(\pi) = \tau_+(1_{F^\times},\mu_E^{-1})^\vee.$$

Twisted Steinberg representation. Let us compute the theta lift of $\chi\mathrm{St}_{\mathrm{GSp}_4^+}$. Since we know

$$\chi\mathrm{St}_{\mathrm{GSp}_4^+} \hookrightarrow I_{Q_G}(|-|^2, (\chi|-|^{-1})\cdot st_+)$$

and (see Proposition 5.6)

$$\delta(\chi_{E/F},\chi_E) \hookrightarrow I_{P_H}(\chi_{E/F}[2], \pi(\chi_E^{-1}St_{\mathrm{GL}_2(E)}[-1], \chi_{E/F}\chi^{-2}[-2])),$$

we can deduce from Proposition 6.17 and Proposition 6.19 that

$$\theta(\delta(\chi_{E/F},\chi_E)) = \chi\mathrm{St}_{\mathrm{GSp}_4^+} \text{ and } \theta(\chi\mathrm{St}_{\mathrm{GSp}_4^+}) = \delta(\chi_{E/F},\chi_E)$$

in a similar way as above (see also a proof of [13, Theorem 8.3] for generalized Steinberg representations).

6.4.3. *Nondiscrete series representation.* In this section, we shall consider theta lifts of representations given in Lemma 4.2.

Parabolic induction for Q_G. Let us consider the case where π is an irreducible submodule of

$$I_{Q_G}(\chi, \tau^+)$$

with an essentially discrete series representation τ^+ of GL_2^+ and a character χ of $F^\times$ such that $|\chi| = |-|^{-s}$. Then by Proposition 6.17, we have as a representation of $\mathrm{GSO}_{4,2}$,

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q_G}(\chi, \tau^+)) = I_{P_H}(\chi^{-1} \cdot \chi_{E/F}, \pi(BC(\chi \cdot \tau^+), \chi_{E/F} \cdot \chi^2 \omega_{\tau^+}))^*$$

if τ^+ is generic with respect to ψ_2. Otherwise we have

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{Q_G}(\chi, \tau^+)) = 0,$$

in particular, $\Theta(\pi) = 0$. Hereafter, we suppose that τ^+ is generic with respect to ψ_2. Then we have

$$I_{P_H}(\chi^{-1} \cdot \chi_{E/F}, \pi(BC(\chi \cdot \tau^+), \chi_{E/F} \cdot \chi^2 \omega_{\tau^+})) \twoheadrightarrow \Theta(\pi). \tag{6.1}$$

Suppose that

(1) $\chi = 1_{F^\times}$ or,
(2) $\chi = \chi_{E/F}$ and $\mathrm{Ind}_{\mathrm{GL}_2^+}^{\mathrm{GL}_2} \tau^+$ is reducible.

In both cases, $I_{Q_G}(\chi, \tau^+)$ decomposes into a direct sum $\pi \oplus \pi'$ with some irreducible representation π'. Thus, we obtain

$$I_{P_H}(\chi^{-1} \cdot \chi_{E/F}, \pi(BC(\chi \cdot \tau^+), \chi_{E/F} \cdot \chi^2 \omega_{\tau^+})) = \Theta(\pi) \oplus \Theta(\pi').$$

Suppose that $BC(\tau^+)$ is supercuspidal. In the case (1), since we know $BC(\tau^+) \notin \Phi_{\omega_{\tau^+}}$, the induced representation in (6.1) decomposes into a direct sum of two irreducible representations. Since we have $n^\pm(\pi) \geq 6$ and $n^\pm(\pi') \geq 6$ by [8, Theorem A.11 (iv)], $\Theta(\pi)$ and $\Theta(\pi')$ are nonzero. Thus, we obtain

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F} \cdot \omega_{\tau^+}))_{gen} & \text{if } \pi \text{ is generic,} \\ I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F} \cdot \omega_{\tau^+}))_{ng} & \text{otherwise.} \end{cases}$$

On the other hand, in case (2), since $BC(\tau^+) \in \Phi_{\chi_{E/F}\omega_{\tau^+}}$, $I_{P_H}(1_{F^\times}, \pi(BC(\tau^+), \chi_{E/F}\omega_{\tau^+}))$ is irreducible. Thus, $\Theta(\pi)$ should be irreducible or zero. Moreover, we note that either π or π' is generic, and the theta lift of generic one is not zero by Corollary 6.7. Hence, we obtain

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{P_H}(1_{F^\times}, \pi(BC(\tau^+), \chi_{E/F}\omega_{\tau^+})) & \text{if } \pi \text{ is generic,} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $BC(\tau^+) = I_{B_2(E)}^{\mathrm{GL}_2(E)}(\xi \boxtimes \xi^\sigma)$ with a character ξ of $E^\times$ such that $\xi \neq \xi^\sigma$. Then we have

$$I_{P_H}(\chi\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F} \cdot \omega_{\tau^+})) = I_{B_H}(\chi\chi_{E/F}, \chi_{E/F}; \xi).$$

By Lemma 5.2, this induced representation is irreducible. Thus, as in the above case, we obtain

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{B_H}(\chi\chi_{E/F}, \chi_{E/F}; \xi) & \text{if } \pi \text{ is generic,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let us consider the case where $\chi = 1$ and $\tau^+ = st_\chi$. Then π and π' do not participate in the theta correspondence with $\mathrm{GSO}_{3,1}$ [8, Theorem A.11]. Thus,

$\Theta(\pi)$ and $\Theta(\pi')$ are nonzero. Since $I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F}\cdot\omega_{\tau^+}))$ is reducible by Lemma 5.5, we obtain

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F}\cdot\omega_{\tau^+}))_{gen} & \text{if } \pi \text{ is generic,} \\ I_{P_H}(\chi_{E/F}, \pi(BC(\tau^+), \chi_{E/F}\cdot\omega_{\tau^+}))_{ng} & \text{otherwise.} \end{cases}$$

Assume that π is not in the above cases. Then π is a unique irreducible submodule of $I_{Q_G}(\chi, \tau^+)$. As in the case of twisted Steinberg representation, we can show that

$$\theta(\pi) = J_{P_H}(\chi^{-1}\cdot\chi_{E/F}, \pi(BC(\chi\cdot\tau^+), \chi_{E/F}\cdot\chi^2\omega_{\tau^+}))$$

since $I_{P_H}(\chi^{-1}\cdot\chi_{E/F}, \pi(BC(\chi\cdot\tau^+), \chi_{E/F}\cdot\chi^2\omega_{\tau^+}))$ is irreducible, or it has a unique irreducible quotient (cf. Remark 5.13).

Parabolic induction for P_G. Let us consider the case where π is an irreducible submodule of

$$I_{P_G}(\tau, \chi_+)$$

with an irreducible discrete series representation τ of GL_2 and a character χ of $F^\times$. By Proposition 6.18, as a representation of $\mathrm{GSO}_{4,2}$, we have

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{P_G}(\tau, \chi_+)) \hookrightarrow I_{Q_H}((\tau\otimes\chi_{E/F})^\vee, (\chi\omega_\tau)_E)^*.$$

Suppose that τ is unitary, $\tau\otimes\chi_{E/F} = \tau$ and $\omega_\tau = \chi_{E/F}$. Then τ should be supercuspidal, and thus we have

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega, I_{P_G}(\tau, \chi_+)) = I_{Q_H}((\tau\otimes\chi_{E/F})^\vee, (\chi\omega_\tau)_E)^*.$$

Further, we note that

$$I_{P_G}(\tau, \chi_+) = \pi \oplus \pi'$$

with some irreducible representation π' of GSp_4^+. Therefore, we obtain

$$\Theta(\pi) \oplus \Theta(\pi') = I_{Q_H}(\tau, \chi_E).$$

This induced representation is reducible by Proposition 5.12. Thus, taking Corollary 6.7 into account, we see that

$$\Theta(\pi) = \begin{cases} I_{Q_H}(\tau, \chi_E)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{Q_H}(\tau, \chi_E)_{ng} & \text{otherwise.} \end{cases}$$

We suppose that the above condition is not satisfied. Then π is a unique irreducible submodule of $I_{P_G}(\tau, \chi^+)$.

It is easy to see that by Proposition 5.12, $I_{Q_H}((\tau\otimes\chi_{E/F})^\vee, (\chi\omega_\tau)_E)$ has a unique irreducible quotient. Thus, we obtain

$$\theta(\pi) \subset J_{Q_H}((\tau\otimes\chi_{E/F})^\vee, (\chi\omega_\tau)_E).$$

From Theorem 6.4, we should have $n^+(\pi) \le 6$ or $n^-(\pi) \le 6$. If $n^-(\pi) \le 6$, $\Theta(\pi^g)$ should not be zero since

$$\mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega_-, \pi) \simeq \mathrm{Hom}_{\mathrm{GSp}_4^+}(\Omega_+, \pi^g)$$

for any $g \in \mathrm{GSp}_4 \setminus \mathrm{GSp}_4^+$. On the other hand, in this case, we have $\pi^g \simeq \pi$ for any $g \in \mathrm{GSp}_4 \setminus \mathrm{GSp}_4^+$. Thus, $\Theta(\pi) \neq 0$, and we have

$$\theta(\pi) = J_{Q_H}((\tau\otimes\chi_{E/F})^\vee, (\chi\omega_\tau)_E).$$

Parabolic induction for B_G. We consider the case where

$$\pi \hookrightarrow I_{B_G}(\chi_1, \chi_2; \chi_+).$$

We note that

$$I_{B_G}(\chi_1, \chi_2; \chi_+) = I_{Q_G}(\chi_1, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_2 \boxtimes 1)_+).$$

As in the previous cases, by Proposition 6.17, we obtain

$$I_{P_H}(\chi_1^{-1} \cdot \chi_{E/F}, \pi(BC(\chi_1 \cdot \tau^+), \chi_{E/F} \cdot \chi_1^2 \omega_{\tau^+})) \twoheadrightarrow \Theta(\pi)$$

with $\tau^+ = \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_2 \boxtimes 1)_+$. Moreover, we note that

$$\begin{aligned} &I_{P_H}(\chi_1^{-1} \cdot \chi_{E/F}, \pi(BC(\chi_1 \cdot \tau^+), \chi_{E/F} \cdot \chi_1^2 \omega_{\tau^+})) \\ =&I_{B_H}(\chi_1^{-1}\chi_{E/F}, \chi_2^{-1}\chi_{E/F}; \chi_{1,E}\chi_{2,E}\chi_E). \end{aligned}$$

Let us consider the case (3a) in Lemma 4.2, i.e.,

$$\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^{\pm})$$

with $s_1 > 0$. If

$$\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^{-}),$$

then by Proposition 6.17, $\Theta(\pi) = 0$. Suppose that

$$\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^{+}).$$

Then by Proposition 6.17, we obtain

$$I_{B_H}(\chi_1^{-1}\chi_{E/F}[s_1], 1, \chi_{1,E}\chi_E[-s_1]) \twoheadrightarrow \Theta(\pi).$$

This induced representation is a standard module, so that it has the unique irreducible quotient $J_{B_H}(\chi_1^{-1}[s_1]\chi_{E/F}, 1, \chi_{1,E}\chi_E[-s_1])$. Thus, we obtain

$$\theta(\pi) \subset J_{B_H}(\chi_1^{-1}[s_1]\chi_{E/F}, 1, \chi_{1,E}\chi_E[-s_1]).$$

On the other hand, by Proposition 6.19, we see that

$$\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega_+, I_{P_H}(\chi_1^{-1}[-s_1], \pi((\chi_1\chi)_E|-|_E^{-s_1} \cdot I_{B_2(E)}^{\mathrm{GL}_2(E)}(1_{E^\times} \boxtimes 1_{E^\times}), (\chi_1\chi[-s_1])^2))$$

is isomorphic to

$$I_{Q_G}(\chi_1|-|^{s_1}, \chi \cdot I_{B_2}^{\mathrm{GL}_2}(\chi_{E/F} \boxtimes 1)^{+}).$$

Thus, we find that $\theta(\pi) \neq 0$ and

$$\theta(\pi) = J_{B_H}(\chi_1^{-1}[s_1]\chi_{E/F}, 1, \chi_{1,E}\chi_E[-s_1]).$$

Let us consider the case (3b) in Lemma 4.2. Then by Lemma 5.2, we have

$$I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E) = I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{gen} \oplus I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{ng}.$$

Thus, we see that

$$\Theta(\pi) \oplus \Theta(\pi^g) = I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{gen} \oplus I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{ng}$$

for $g \in \mathrm{GSp}_4 \setminus \mathrm{GSp}_4^+$. From [8, Theorem A.11], π and π^g does not participate in the theta correspondence with $\mathrm{GSO}_{3,1}$. Thus, $\Theta(\pi)$ and $\Theta(\pi^g)$ are not zero, and by Corollary 6.7, we obtain

$$\Theta(\pi) = \theta(\pi) = \begin{cases} I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{gen} & \text{if } \pi \text{ is generic,} \\ I_{B_H}(\chi_0\chi_{E/F}, \chi_0; \chi_E)_{ng} & \text{if } \pi \text{ is nongeneric.} \end{cases}$$

Let us consider the case (3c) in Lemma 4.2, i.e.

$$\pi \hookrightarrow I_{Q_G}(\chi_1, (\chi \cdot |\cdot|^{-1/2}) \circ \det)$$

as the unique irreducible submodule such that $|\chi_1| = |\cdot|^{-s_1}$ with $s_1 \geq 1$. As in the above discussions, by Proposition 6.17, we see that $\Theta(\pi)$ is a quotient of $I_{P_H}(\chi_1^{-1}\chi_{E/F}, \pi((\chi_1\chi|\cdot|^{-1/2})_E, \chi_{E/F}\chi_1^2\chi^2|\cdot|^{-1}))$. But this induced representation is a quotient of

$$I_{B_H}(\chi^{-1}\chi_{E/F}, \chi_{E/F}[1]; (\chi_1\chi)_E[-1]).$$

This shows that

$$\theta(\pi) \subset \Pi := J_{B_H}(\chi^{-1}\chi_{E/F}, \chi_{E/F}[1]; (\chi_1\chi)_E[-1]).$$

On the other hand, we consider the theta lift of

$$\Pi \hookrightarrow I_{P_H}(\chi_1\chi_{E/F}, \pi((\chi_1^{-1}\chi^{-1})_E[1/2], \chi_{E/F}\chi_1^{-2}\chi^{-2}[1])).$$

Then by Proposition 6.19, we obtain

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GSO}_{4,2}}(\Omega, I_{P_H}(\chi_1\chi_{E/F}, \pi((\chi_1^{-1}\chi^{-1})_E[1/2], \chi_{E/F}\chi_1^{-2}\chi^{-2}[1])))\\ =&I_{Q_G}(\chi_1^{-1}, (\chi^{-1}|\cdot|^{1/2}) \circ \det).\end{aligned}$$

This shows that $\theta(\pi) \neq 0$, and we obtain

$$\theta(\pi) = J_{B_H}(\chi^{-1}\chi_{E/F}, \chi_{E/F}[1]; (\chi_1\chi)_E[-1]).$$

Finally, we consider the case (3d) in Lemma 4.2, i.e.,

$$\pi \hookrightarrow I_{Q_G}(\chi_1|-|^{-s_1}, I_{B_2}^{\mathrm{GL}_2}(\chi\chi_2|-|^{-s_2} \boxtimes \chi)_+).$$

Then in a similar calculation as in the case (3a), we can easily show that

$$\theta(\pi) = J_{B_H}(\chi_1^{-1}[s_1]\chi_{E/F}, \chi_2^{-1}\chi_{E/F}[s_2]; \chi_{1,E}\chi_{2,E}\chi_E[-s_1-s_2]).$$

This completes a proof of Theorem 6.21.

As we remarked in Section 6.1, the assumption in Lemma 6.2 holds by Theorem 6.21. Hence, we obtain the following corollary.

Corollary 6.22. *For the dual pair* $(\mathrm{GSp}_4^+, \mathrm{GSO}_{4,2})$*, the Howe duality holds over any nonarchimedean local field of characteristic zero, namely for any irreducible admissible representation of* π *of* GSp_4^+ *or* $\mathrm{GSO}_{4,2}$*, we have*

- $\theta(\pi)$ *is irreducible whenever* $\Theta(\pi)$ *is nonzero,*
- *the map* $\pi \mapsto \theta(\pi)$ *is injective on its domain.*

Further, from Theorem 6.21, we obtain a relation between Satake parameters of GSp_4 and $\mathrm{GSO}_{4,2}$ under the theta correspondence.

Corollary 6.23. *Let* σ *be an unramified irreducible representation of* $\mathrm{GSO}_{4,2}$ *with trivial central character. Suppose that* E/F *is an unramified quadratic extension. Then the theta lift* $\theta(\sigma)$ *to* GSp_4^+ *is an irreducible unramified representation, and we have*

$$L(s, \sigma, \wedge_t^2) = L(s, \pi, std \otimes \chi_{E/F})(1-q^{-s})^{-1}$$

where q *is the cardinality of the residual field of* F*, and* π *is an irreducible representation of* GSp_4 *such that* $\theta(\sigma) \subset \pi_+$*.*

6.5. **Shalika period and local theta lift.** Let us study irreducible representations of $\mathrm{GSO}_{4,2}$ which have Shalika period. Indeed, we characterize them in terms of a local theta correspondence.

Proposition 6.24. *Let σ be an irreducible representation of $\mathrm{GSO}_{4,2}$. Suppose that σ is essentially tempered. If σ has Shalika period, then $\theta(\sigma)$ is generic.*

Proof. Since $\theta(\sigma \otimes \chi) = \theta(\sigma) \otimes \chi$ for any character χ of $F^\times$, we may suppose that σ is tempered. Then we note that $\theta(\sigma)$ is tempered by Theorem 6.21. We recall that $\Theta(\sigma)$ contains a generic constituent (see Corollary 6.8). In particular, if $\Theta(\sigma)$ is irreducible, we obtain our required result.

If σ is supercuspidal, then we have $\Theta(\sigma) = \theta(\sigma)$ by Proposition 6.1, and our assertion follows from the above argument.

Suppose that σ is not supercuspidal representation, but it is discrete series representation. Then from Remark 5.13, σ is generic. Thus, $\theta(\sigma)$ is generic by Theorem 6.9.

We suppose that σ is a submodule of $I_{Q_H}(\tau, \xi)$ with unitary representations τ of GL_2 and ξ of $E^\times$. Then from Proposition 6.20, we obtain

$$I_{P_G}(\chi_{E/F} \cdot \tau^\vee, \omega_\tau \xi_F^2) \twoheadrightarrow \Theta(\sigma).$$

This induced representation is semisimple, and so is $\Theta(\sigma)$. Since we know the maximal semisimple quotient of $\Theta(\sigma)$ is irreducible, $\Theta(\sigma)$ should be irreducible.

Let us assume that σ is a submodule of parabolic induction $I_{P_H}(\chi, \pi)$ with a discrete series representation π of $\mathrm{GSO}_{3,1}$ and a unitary character χ of $F^\times$. If $\chi \neq \chi_{E/F}$ and $\pi = \theta(\tau)$ for an irreducible representation τ of GL_2, by Proposition 6.19, we obtain

$$I_{Q_G}(\chi^{-1}\chi_{E/F}, \tau \otimes \chi\chi_{E/F}) \twoheadrightarrow \Theta(\sigma).$$

Since τ should be discrete series, this parabolic induction is completely reducible, and so is $\Theta(\sigma)$. However, the maximal semisimple quotient of $\Theta(\sigma)$ is irreducible, so that $\Theta(\sigma)$ is irreducible.

If $\chi = \chi_{E/F}$ and π does not participate in the theta correspondence with GL_2^+,

$$\Theta_{W,V_2}(\pi) \twoheadrightarrow \Theta(\sigma)$$

where $\Theta_{W,V_2}(\pi)$ is the big theta lift of π to GSp_4^+. From [8, Theorem A.11], $\Theta(\pi)$ is irreducible. Thus, $\Theta(\pi) \simeq \Theta(\sigma)$ and $\Theta(\sigma)$ is irreducible.

Finally, suppose that $\chi = 1$ and $\pi = \theta(\tau)$. Then we have

$$\theta(\sigma) \supset \theta_{W,V_2}(\pi) = \Theta_{W,V_2}(\pi).$$

In this case, π is invariant and distinguished, and thus $\theta_{W,V_2}(\pi)$ is generic from [8, Theorem A.11]. Hence, $\theta(\sigma)$ is also generic. □

The following proposition follows from Theorem 6.21.

Proposition 6.25. *Let π be an irreducible representation of GSp_4^+. Suppose that π is essentially tempered. If π is generic, $\theta(\pi)$ is generic.*

Moreover, we can prove the following relationship between Shalika period and genericity.

Theorem 6.26. *Let σ be an irreducible essentially tempered representation of $\mathrm{GSO}_{4,2}$. If σ has Shalika period, then σ is generic.*

Proof. First, we note that if σ has Shalika period, $\theta(\sigma) \neq 0$ by Corollary 6.8. If σ is supercuspidal, from Theorem 6.21, $\theta(\sigma)$ should be supercuspidal. Thus, our assertion follows from Proposition 6.25. If σ is not supercuspidal but an essentially discrete series representation, σ is always generic by Remark 5.13. Otherwise, since $\theta(\theta(\sigma)) = \sigma$, the above two propositions show our assertion. □

Then we can characterize essentially tempered irreducible representations of $\mathrm{GSO}_{4,2}$ which have Shalika period, using a theta correspondence.

Theorem 6.27. *Let σ be an essentially tempered irreducible representation of $\mathrm{GSO}_{4,2}$. Then σ has Shalika period if and only if $\sigma = \theta(\pi)$ with some irreducible generic representation π of GSp_4^+.*

Proof. If σ has Shalika period, σ is generic by Theorem 6.26. Thus, our assertion follows from Theorem 6.9. On the other hand, suppose that $\sigma = \theta(\pi)$ with some irreducible generic representation π of GSp_4^+. Thus, $\Theta(\sigma)$ has a generic constituent, so that σ has Shalika period by Corollary 6.8. □

Finally, we remark that the assumption on the essentially temperedness was crucial for our proof of Theorem 6.27. Indeed, if we remove this assumption, Theorem 6.26 fails as the following example indicates.

Let τ be a supercuspidal irreducible representation of GL_2^+ such that it is generic with respect to ψ_2 and $\tau \otimes \chi_{E/F} \not\cong \tau$. Then by Proposition 5.11 and Lemma 6.12, the induced representation

$$I_{P_H}(|-|^{-1}, \theta(\tau)[1/4])$$

has a unique irreducible nontempered nongeneric submodule $Sp(\tau)$, where $\theta(\tau) \neq 0$ is the theta lift of τ to $\mathrm{GSO}_{3,1}$ (cf. Corollary 6.11). Then from Proposition 6.19, we see that

$$\Theta(Sp(\tau)) \hookrightarrow I_{Q_G}(\chi_{E/F}[1], (\tau \otimes \chi_{E/F})[-1/2]_+).$$

Since this induced representation is irreducible, we find that

$$\Theta(Sp(\tau)) = I_{Q_G}(\chi_{E/F}[1], (\tau \otimes \chi_{E/F})[-1/2]_+),$$

and $\Theta(Sp(\tau))$ is generic. Therefore, $Sp(\tau)$ has Shalika period by Corollary 6.8, but it is nongeneric. This implies that our proof of Theorem 6.27 does not work for $Sp(\tau)$. However, we note that $Sp(\tau)$ has Shalika period and $\theta(Sp(\tau))$ is generic.

7. Local theta correspondence: archimedean case

In this section, for the sake of completeness of a study of local theta correspondence for the dual pair $(\mathrm{GSp}_4, \mathrm{GSO}_{4,2})$ and also for the sake of convenience of the reader, we shall recall a result by Paul [28] on the local theta correspondence for discrete series representations over the real field $\mathbb{R}$.

Let n, p, and q be nonnegative integers such that $p+q$ is even, and let $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{O}_{p,q}(\mathbb{R})$ be the group of isometries of bilinear forms on $\mathbb{R}^{2n}$ or $\mathbb{R}^{p+q}$ given respectively by

$$\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}.$$

First, we note that as in the nonarchimedean case, we may replace $\mathrm{O}_{4,2}(\mathbb{R})$ by $\mathrm{SO}_{4,2}(\mathbb{R})$. Thus, we shall consider theta lifts from $\mathrm{Sp}_4(\mathbb{R})$ to $\mathrm{SO}_{4,2}(\mathbb{R})$.

Let us recall a classification of discrete series representations of $\mathrm{Sp}(4, \mathbb{R})$ and $\mathrm{SO}_{4,2}(\mathbb{R})$. Let us consider the case of $\mathrm{Sp}_4(\mathbb{R})$. We identify the root space with

$\mathbb{R}^2$ by putting $e_1 = (1,0)$ and $e_2 = (0,1)$. Then the set of positive compact roots is given by $\Sigma_c^+ = \{(1,-1)\}$ and the set of dominant weight is given by $\{(\lambda_1,\lambda_2) \in \mathbb{Z}^2 | \lambda_1 \geq \lambda_2\}$. To classify discrete series, we describe all the positive root systems compatible with Σ_c^+. Indeed, these root systems are given by

$$\begin{aligned}
\Sigma_i^+ &= \{(1,-1),(2,0),(1,1),(0,2)\},\\
\Sigma_{ii}^+ &= \{(1,-1),(2,0),(1,1),(0,-2)\},\\
\Sigma_{iii}^+ &= \{(1,-1),(2,0),(-1,-1),(0,-2)\},\\
\Sigma_{iv}^+ &= \{(1,-1),(-2,0),(-1,-1),(0,-2)\}.
\end{aligned}$$

Define a subset Ξ_J of dominant weights by

$$\Xi_J = \{\Lambda = (\Lambda_1,\Lambda_2) \mid \langle \Lambda, \beta\rangle > 0 \text{ for all } \beta \in \Sigma_J^+\}.$$

Then the set $\bigcup_{J=i}^{iv} \Xi_J$ gives the Harish-Chandra parametrization of the discrete series representations. The discrete series representations corresponding to the parameter in $\Xi_{ii}^+ \cup \Xi_{iii}^+$ are generic. On the other hand, the discrete series representations corresponding to the parameter in Ξ_i^+ (resp. Ξ_{iv}^+) are holomorphic (resp. anti-holomorphic).

Let us recall a classification of discrete series representations of $\mathrm{SO}_{4,2}(\mathbb{R})$. In this case, there are 6 positive root systems given by

$$\begin{aligned}
\Psi_k^+ &= \Psi_c^+ \cup \{e_i \pm e_3\,;\, 1 \leq i \leq k-1\} \cup \{e_3 \pm e_i\,;\, k \leq i \leq 2\} \quad (1 \leq k \leq 3),\\
\Psi_k^- &= \Psi_c^+ \cup \{e_i \pm e_3\,;\, 1 \leq i \leq k-1\} \cup \{-e_3 \pm e_i\,;\, k \leq i \leq 2\} \quad (1 \leq k \leq 2),\\
\Psi_3^- &= \Psi_c^+ \cup \{e_1 \pm e_3\} \cup \{-e_2 \pm e_3\},
\end{aligned}$$

where Ψ_c^+ is the set of compact positive roots. Then as in the case of $\mathrm{Sp}_4(\mathbb{R})$, the discrete series representations corresponding to Ψ_1^+ (resp. Ψ_1^-) are holomorphic (resp. anti-holomorphic), and the discrete series representations corresponding to $\Psi_2^\pm$ are generic. Finally, the discrete series representations corresponding to $\Psi_3^\pm$ are called middle discrete series.

With the above notations, let us consider the theta lift from $\mathrm{Sp}(4,\mathbb{R})$ to $\mathrm{SO}_{4,2}(\mathbb{R})$. We only need to consider Σ_i^+ and Σ_{ii}^+ because of the relation $\theta_{q,p}(\pi) = \theta_{p,q}(\pi^*)$ [28, Lemma 20].

For a discrete series representation corresponding to Σ_i^+, its Harish-Chandra parameter is given by (Λ_1,Λ_2) with $\Lambda_1 > \Lambda_2 > 0$. Then by [28, Theorem 15], the theta lifts of this representation are nonzero for the following four orthogonal groups

$$\mathrm{O}(4,0), \quad \mathrm{O}(5,1), \quad \mathrm{O}(4,2), \quad \mathrm{O}(6,0)$$

and, in particular, the Harish-Chandra parameter of the lift to $\mathrm{SO}(4,2)$ is given by $(\Lambda_1,\Lambda_2,0)$. Since this parameter corresponds to Ψ_3^+, we see that the theta lifts of holomorphic discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ to $\mathrm{SO}(4,2)$ are middle discrete series representations. We note that from [28, Proposition 22] and the above result, the theta lifts of holomorphic discrete series representations of $\mathrm{Sp}\,(4,\mathbb{R})$ to $\mathrm{O}(3,1)$ and $\mathrm{O}(1,3)$ are zero.

For a discrete series representation corresponding to Σ_{ii}^+, its Harish-Chandra parameter is given by (Λ_1,Λ_2) with $\Lambda_1 > -\Lambda_2 > 0$. Thus, by [28, Theorem 15], the theta lifts of this representation are nonzero for the following four orthogonal groups

$$\mathrm{O}(2,2), \quad \mathrm{O}(3,3), \quad \mathrm{O}(4,2), \quad \mathrm{O}(2,4)$$

and, in particular, the Harish-Chandra parameter of the theta lift to $\mathrm{SO}(4,2)$ is given by $(\Lambda_1, 0, -\Lambda_2)$. Since this parameter corresponds to Ψ_1^+, we see that the theta lifts of generic discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ to $\mathrm{SO}(4,2)$ are generic discrete series representations.

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