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DISTINGUISHED CONJUGACY CLASSES AND ELLIPTIC WEYL GROUP ELEMENTS

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ABSTRACT. We define and study a correspondence between the set of distinguished G^0-conjugacy classes in a fixed connected component of a reductive group G (with G^0 almost simple) and the set of (twisted) elliptic conjugacy classes in the Weyl group. We also prove a homogeneity property related to this correspondence.

INTRODUCTION

0.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic $\underline{p} \geq 0$ and let G be a (possibly disconnected) reductive algebraic group over $\mathbf{k}$. Let W be the Weyl group of G^0. (For an algebraic group H, H^0 denotes the identity component of H.) We view W as an indexing set for the orbits of G^0 acting diagonally on $\mathcal{B} \times \mathcal{B}$ where $\mathcal{B}$ is the variety of Borel subgroups of G^0; we denote by $\mathcal{O}_w$ the orbit corresponding to $w \in W$. Note that W is naturally a Coxeter group; its length function is denoted by $\underline{l} : W \to \mathbf{N}$. Let I be the set of simple reflections of W; for any $J \subset I$ let W_J be the subgroup of W generated by J.

Now any $\delta \in G/G^0$ defines a group automorphism $\epsilon_\delta : W \to W$ preserving length, by the requirement that

$$(B, B') \in \mathcal{O}_w, g \in \delta \implies (gBg^{-1}, gB'g^{-1}) \in \mathcal{O}_{\epsilon_\delta(w)}.$$

The orbits of the W-action $w_1 : w \mapsto w_1^{-1} w \epsilon_\delta(w_1)$ on W are said to be the ϵ_δ-conjugacy classes in W. Let $\underline{W}_\delta$ be the set of ϵ_D-conjugacy classes in W. We say that $C \in \underline{W}_\delta$ is elliptic if for any $J \subsetneqq I$ such that $\epsilon_D(J) = J$ we have $C \cap W_J = \emptyset$. For any $C \in \underline{W}_\delta$ let C_{min} be the set of elements of C where the length function $\underline{l} : C \to \mathbf{N}$ reaches its minimum value. Let $\mathbf{c}$ be a G^0-conjugacy class of G. Let δ be the connected component of G that contains $\mathbf{c}$ and let $C \in \underline{W}_\delta$ be elliptic. For any $w \in C_{min}$ we set

$$\mathfrak{B}_w^{\mathbf{c}} = \{(g, B) \in \mathbf{c} \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}.$$

Note that G^0 acts on $\mathfrak{B}_w^{\mathbf{c}}$ by $x : (g, B) \mapsto (xgx^{-1}, xBx^{-1})$. We write $C \clubsuit \mathbf{c}$ if the following condition is satisfied: for some/any $w \in C_{min}$, $\mathfrak{B}_w^{\mathbf{c}}$ is a single G^0-orbit for the action above (in particular it is nonempty). The equivalence of "some" and "any" follows from [L5, 1.15(a)] (which is based on results in [GP]).

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0.2. For an algebraic group H we denote by $\mathcal{Z}_H$ the center of H; for $h \in H$ we denote by $Z_H(h)$ the centralizer of h in H. An element $g \in G$ or its G^0-conjugacy class is said to be *distinguished* if $Z_G(g)^0/(\mathcal{Z}_{G^0} \cap Z_G(g))^0$ is a unipotent group. The notion of distinguished element appeared in [BC] in the case where g is unipotent and $G = G^0$.

The following is the main result of this paper.

Theorem 0.3. *Assume that G^0 is almost simple and that $|G/G^0| \le 2$. If G^0 is of exceptional type assume further that $G = G^0$ and that $\underline{p}$ is either 0 or a good prime for G. Then for any distinguished G^0-conjugacy class $\mathbf{c}$ in G contained in a connected component δ of G, there exists an elliptic $C \in \underline{W}_\delta$ such that $C \clubsuit \mathbf{c}$.*

In the case where $\mathbf{c}$ is unipotent the theorem is known from [L1, Theorem 0.2]. In particular, the theorem holds when $\underline{p} = 2$. Thus we may assume that $\underline{p} \ne 2$. We may also assume that $G/G^0 \to \mathrm{Aut}(\underline{W})$, $\delta \mapsto \epsilon_\delta$ is injective. It is enough to verify the theorem assuming that G^0 is simply connected (the theorem then automatically holds without that assumption). If G^0 is of type A and $G = G^0$, then $\mathbf{c}$ must be a regular unipotent class times a central element and we can take C to be the Coxeter class. The case where $G = G^0$ is of type B or C is treated in §1. The case where G^0 is of type D is treated also in §1. (In this case we may assume that $|G/G^0| = 2$.) The case where G^0 is of type A and $|G/G^0| = 2$ is treated in §2. (In this case we may assume that $\mathbf{c} \notin G^0$.) The case where G is of exceptional type is treated in §3.

We will show elsewhere that C in the theorem is unique (in the case where $\mathbf{c}$ is unipotent this is known from [L1]).

0.4. The results of this paper have applications to the study of character sheaves. We will show elsewhere how they can be used to prove that an irreducible cuspidal local system on $\mathbf{c}$ (a distinguished G^0-conjugacy class in a connected component δ of G), extended by 0 on $\delta - \mathbf{c}$, is (up to shift) a character sheaf on δ. In the case where $\delta = G^0$ this gives a new, constructive proof of a known result, but in the case where $\delta \ne G^0$, it is a new result.

0.5. For any integers x, y such that $y \ge 0$ we set $\binom{x}{y} = x(x-1)\dots(x-y+1)(y!)^{-1}$. Thus $\binom{x}{0} = 1$.

1. Isometries

1.0. In this section we assume that $\underline{p} \ne 2$. Let $\epsilon \in \{1, -1\}$. Let V be a $\mathbf{k}$-vector space of finite dimension $\mathbf{n}$ with a given nondegenerate bilinear form $(,) : V \times V \to \mathbf{k}$ such that $(x, y) = \epsilon(y, x)$ for all x, y; we then say that $(,)$ is ϵ-symmetric. Let $Is(V)$ be the group of isometries of $(,)$.

Assume that we are given $g \in Is(V)$. For any $z \in V$ and $i \in \mathbf{Z}$ we set $z_i = g^i z \in V$. Similarly, for any line L in V and $i \in \mathbf{Z}$ we set $L_i = g^i L \subset V$. For any z, z' in V and any $i, j, k \in \mathbf{Z}$ we have

(a) $$(z_{i+k}, z'_{j+k}) = (z_i, z'_j).$$

Let $a_1 \ge a_2 \ge \dots$, $b_1 \ge b_2 \ge \dots$ be two sequences in $\mathbf{N}$ such that

if $i \ge 1$, $a_i = a_{i+1}$, then $a_{i+1} = 0$,
if $i \ge 1$, $b_i = b_{i+1}$, then $b_{i+1} = 0$,
if $a_i > 0$, then $(-1)^{a_i} = -\epsilon$,
if $b_i > 0$, then $(-1)^{b_i} = -\epsilon$.

It follows that $a_i = 0$ for large i and $b_i = 0$ for large i. Thus, (a_i), (b_i) are strictly decreasing sequences of integers ≥ 0 of fixed parity as long as they are nonzero. We assume that

$$\mathbf{n} = (a_1 + a_2 + \dots) + (b_1 + b_2 + \dots).$$

Define $\kappa \in \{0,1\}$ by $\mathbf{n} - \kappa \in 2\mathbf{N}$. Note that if $\epsilon = -1$ we have $\kappa = 0$. Define $k \geq 0$ by $\{i \geq 1; a_i b_i > 0\} = [1,k]$. For $i \geq 1$ we set $c_i = a_i + b_i$. We have $c_1 \geq c_2 \geq \dots$. We define $p_i \in \mathbf{N}$ for $i \geq 1$ as follows.

If $\epsilon = -1$ we have $c_i \in 2\mathbf{N}$ and we set $p_i = c_i/2$ for $i \geq 1$.

If $\epsilon = 1$ and $i \in [1,k]$ we again have $c_i \in 2\mathbf{N}$ and we set $p_i = c_i/2$.

If $\epsilon = 1$ and $i > k$ we have $c_i \in 2\mathbf{N} + 1$ or $c_i = 0$ and we define p_i by requiring that for $s = 1, 3, 5, \dots$ we have:

$$\begin{aligned}
&(p_{k+s}, p_{k+s+1}) = ((c_{k+s} - 1)/2, (c_{k+s+1} + 1)/2) && \text{if } c_{k+s} \geq 1, c_{k+s+1} \geq 1,\\
&(p_{k+s}, p_{k+s+1}) = ((c_{k+s} - 1)/2, 0) && \text{if } c_{k+s} \geq 1, c_{k+s+1} = 0,\\
&(p_{k+s}, p_{k+s+1}) = (0,0) && \text{if } c_{k+s} = 0, c_{k+s+1} = 0.
\end{aligned}$$

We define σ as follows. We have $p_1 \geq p_2 \geq \dots \geq p_\sigma$ where $p_i \in \mathbf{N}_{>0}$ for $i \in [1,\sigma]$, $p_i = 0$ if $i > \sigma$. This defines σ. If $\mathbf{n} = 0$ or $\mathbf{n} = 1$ we have $\sigma = 0$. We set $p'_t = p_t$ if $t \in [1,\sigma]$, $p'_t = 1/2$ if $\kappa = 1, t = \sigma + 1$. We have

$$2(p_1 + p_2 + \dots + p_\sigma) + \kappa = 2(p'_1 + p'_2 + \dots + p'_{\sigma+\kappa}) = \mathbf{n}.$$

Let $\mathcal{C}^V_{a_*,b_*}$ be the set of all $g \in Is(V)$ such that $g^2 : V \to V$ is unipotent and such that on the generalized 1-eigenspace of g, g has Jordan blocks of sizes given by the nonzero numbers in $a_1, a_2, \dots$ and on the generalized (-1)-eigenspace of g, $-g$ has Jordan blocks of sizes given by the nonzero numbers in $b_1, b_2, \dots$. (Note that the union of the sets $\mathcal{C}^V_{a_*,b_*}$ where a_*, b_* as above vary is exactly the set of elements of $Is(V)$ which are distinguished in the sense of 0.2.)

For $g \in \mathcal{C}^V_{a_*,b_*}$ let $\tilde{\mathcal{C}}^V_{g;a_*,b_*}$ be the set consisting of all $L^1, L^2, \dots, L^{\sigma+\kappa}$ where $L^t (t \in [1,\sigma+\kappa])$ are lines in V (the upper scripts are not powers) such that for $i, j \in \mathbf{Z}$ we have:

$$\begin{aligned}
&(L^t_i, L^t_j) = 0 \quad \text{if } |i-j| < p_t, (L^t_i, L^t_j) \neq 0 && \text{if } j - i = p_t (t \in [1,\sigma+\kappa]),\\
&(L^t_i, L^r_j) = 0 \quad \text{if } i - j \in [-p_r, 2p_t - p_r - 1] && \text{and } 1 \leq t < r \leq \sigma+\kappa.
\end{aligned}$$

Here $L^t_i = g^i L^t$. We then have:

(b) $V = \bigoplus_{t\in[1,\sigma+\kappa], i\in[0,2p'_t-1]} L^t_i$.

(See [L3, 1.3].) Let $\tilde{\mathcal{C}}^V_{a_*,b_*}$ be the set of all $(g, L^1, L^2, \dots, L^{\sigma+\kappa})$ such that $g \in \mathcal{C}^V_{a_*,b_*}$ and $(L^1, L^2, \dots, L^{\sigma+\kappa}) \in \tilde{\mathcal{C}}^V_{g;a_*,b_*}$.

Now $Is(V)$ acts on $\mathcal{C}^V_{a_*,b_*}$ by $\gamma : g \mapsto (\gamma g \gamma^{-1})$ and on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ by

(c) $$\gamma : (g, L^1, L^2, \dots, L^{\sigma+\kappa}) \mapsto (\gamma g \gamma^{-1}, \gamma(L^1), \gamma(L^2), \dots, \gamma(L^{\sigma+\kappa})).$$

Let $\mathcal{I}' = \prod_{t\in[1,\sigma+\kappa]}\{1,-1\}$. If $\epsilon = -1$ let $\mathcal{I} = \mathcal{I}'$. If $\epsilon = 1$ let $\mathcal{I}$ be the subgroup of $\mathcal{I}'$ consisting of all $(\omega_t)_{t\in[1,\sigma+\kappa]}$ such that $\omega_t = \omega_{t+1}$ for any t such that $\{t, t+1\} \subset [k+1, \sigma+\kappa], t = k+1 \mod 2$. Thus $\mathcal{I}$ is a finite elementary abelian 2-group. The following is the main result of this section.

Theorem 1.1. (a) *$\tilde{\mathcal{C}}^V_{a_*,b_*}$ is nonempty;*

(b) *the action 1.0(c) of $Is(V)$ on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ is transitive;*

(c) *the isotropy group in $Is(V)$ at any point of $\tilde{\mathcal{C}}^V_{a_*,b_*}$ is canonically isomorphic to $\mathcal{I}$.*

The proof (by induction on $\mathbf{n}$) is given in 1.2–1.20.

1.2. We start with the case where a_*, b_* have a single nonzero term. Let $a \in \mathbf{N}, b \in \mathbf{N}, p \in \mathbf{N}_{>0}$ be such that $a + b = 2p$. We set $-\epsilon = (-1)^a = (-1)^b$. For $e \in \mathbf{N}$ we define $n_e \in \mathbf{Z}$ by $(1-T)^a(1+T)^b = \sum_{e\in\mathbf{N}} n_e T^e$. We have $n_0 = 1$, $n_{2p-i} = -\epsilon n_i$ for $i \in [0, 2p]$, $n_e = 0$ if $e > 2p$. We define $x_e \in \mathbf{Z}$ for $e \in \mathbf{N}$ by $x_0 = 1$ and $n_0x_e + n_1x_{e-1} + \dots + n_ex_0 = 0$ for $e \ge 1$.

1.3. In the setup of 1.2, let V be a $\mathbf{k}$-vector space with basis $\{w_i; i \in [0, 2p-1]\}$. Define $g \in GL(V)$ by

$$gw_i = w_{i+1} \text{ if } i \in [0, 2p-2], \quad gw_{2p-1} = \epsilon \sum_{i\in[0,2p-1]} n_iw_i.$$

We have the identity $(1-g)^a(1+g)^b = 0 : V \to V$, that is (setting $\tau = \sum_{i\in[0,2p]} n_ig^i : V \to V$), we have $\tau = 0$. Define a bilinear form $(,)$ on V by

$$(w_i, w_j) = 0 \text{ if } i, j \in [0, 2p-1], |i-j| < p,$$
$$(w_i, w_j) = x_s \text{ if } i, j \in [0, 2p-1], j-i = p+s, s \ge 0,$$
$$(w_i, w_j) = \epsilon x_s \text{ if } i, j \in [0, 2p-1], i-j = p+s, s \ge 0.$$

Clearly $(x, y) = \epsilon(y, x)$ for all x, y and $(,)$ is nondegenerate; the determinant of the matrix $((w_i, w_j))$ is ± 1. We show that g is an isometry of $(,)$. It is enough to show that

$$(gw_i, gw_j) = 0 \text{ if } |i-j| < p,$$
$$(gw_i, gw_j) = x_s \text{ if } j-i = p+s, s \ge 0,$$
$$(gw_i, gw_j) = \epsilon x_s \text{ if } i-j = p+s, s \ge 0.$$

This is obvious except if one or both i, j are $2p-1$. If $i = 2p-1, p-1 < j < 2p-1$, we must check that

$$(\epsilon \sum_{i'\in[0,2p-1]} n_{i'}w_{i'}, w_{j+1}) = 0,$$

that is,

$$\sum_{i'\in[0,j+1-p]} n_{i'}x_{j+1-i'-p} = 0,$$

which is true since $j+1-p > 0$. If $i = 2p-1, 0 \le j < p-1$, we must check that

$$(\epsilon \sum_{i'\in[0,2p-1]} n_{i'}w_{i'}, w_{j+1}) = \epsilon x_{2p-1-j-p},$$

that is,

$$\sum_{i'\in[j+1+p,2p-1]} n_{i'}x_{i'-j-1-p} = \epsilon x_{p-1-j},$$

that is,

$$-\epsilon \sum_{i'\in[j+1+p,2p-1]} n_{2p-i'}x_{i'-j-1-p} = \epsilon x_{p-1-j},$$

that is,

$$\sum_{i'\in[j+1+p,2p]} n_{2p-i'}x_{i'-j-1-p}=0,$$

which is true since $p-j-1>0$.

If $i=2p-1, j=p-1$, we must check that

$$(\epsilon\sum_{i'\in[0,2p-1]} n_{i'}w_{i'}, w_p)=\epsilon x_0,$$

that is, $n_0x_0=x_0$, which is obvious. The case where $j=2p-1, i<2p-1$ is entirely similar. It remains to show (in the case where $i=j=2p-1$) that

$$(\epsilon\sum_{i'\in[0,2p-1]} n_{i'}w_{i'},\epsilon\sum_{i'\in[0,2p-1]} n_{i'}w_{i'})=0.$$

If $\epsilon=-1$ this is obvious since $(x,x)=0$ for any x. Now assume that $\epsilon=1$. We must show

$$2\sum_{i'\in[0,p-1]}\sum_{u\in[0,p-1-i']} n_un_{u+p+i'}x_{i'}=0,$$

that is,

$$\sum_{u\in[0,p-1]} n_u\sum_{i'\in[0,p-1-u]} n_{p-u-i'}x_{i'}=0.$$

We have $\sum_{i'\in[0,p-u]} n_{p-u-i'}x_{i'}=0$ if $p>u$ hence it is enough to show that

$$\sum_{u\in[0,p-1]} n_un_0x_{p-u}=0,$$

that is,

$$\sum_{u\in[0,p-1]} n_ux_{p-u}=0.$$

We have

$$\sum_{u\in[0,p]} n_ux_{p-u}=0$$

since $p>0$. Hence it is enough to show that $n_p=0$. This follows from $n_p=-\epsilon n_p$. (We use that $\epsilon=1$.)

Now $g\in GL(V)$ is regular in the sense of Steinberg and satisfies $(g-1)^a(g+1)^b=0$ on V. Hence $V=V^+\oplus V^-$ where g acts on V^+ as a single unipotent Jordan block of size a and $-g$ acts on V^- as a single unipotent Jordan block of size b. Note that if $\epsilon=1$ we have $\det(g)=(-1)^b=-1$. It follows that, if L is the line spanned by w_0 and $a_*=(a,0,0,\dots), b_*=(b,0,0,\dots)$, then $(g,L)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$. In particular, $\tilde{\mathcal{C}}^V_{a_*,b_*}\neq\emptyset$.

1.4. In the setup of 1.2, let $V,(,)$ be as in 1.0. (Recall that $-\epsilon=(-1)^a=(-1)^b$.) Let $g\in Is(V)$. We assume that $\dim V=2p$ and that on the generalized 1-eigenspace of g, g is a single unipotent Jordan block of size a or is 1 (if $a=0$) and on the generalized (-1)-eigenspace of g, $-g$ is a single unipotent Jordan block of size b or is 1 (if $b=0$). Moreover, we assume that we are given $w\in V$ such that (with notation of 1.0) we have for $i,j\in\mathbf{Z}$:

$$(w_i,w_j)=0 \text{ if } |i-j|<p; (w_i,w_j)=1 \text{ if } j-i=p.$$

We show:

(a) *The following equalities hold for any* i, j *in* $\mathbf{Z}$:
(a1) $(w_i, w_j) = 0$ if $|i-j| < p$,
(a2) $(w_i, w_j) = x_s$ *if* $j - i = p + s$, $s \geq 0$,
(a3) $(w_i, w_j) = \epsilon x_s$ *if* $i - j = p + s$, $s \geq 0$.

Note that (a3) follows from (a2). In (a1) and (a2) we can assume that $i = 0$. (We use 1.0(a).) Since $(w_0, w_j) = \epsilon(w_0, w_{-j})$ for any j we can also assume in (a1) that $j \geq 0$ so that $j \in [0, p-1]$ and (a1) holds. We prove (a2) with $i = 0, j = p + s$ by induction on $s \geq 0$. If $s = 0$ the result is already known. Assume now that $s \geq 1$. Applying $(1-g)^a(g+1)^b = 0$ to w_{s-p} we obtain $\sum_{e \in [0,2p]} n_e w_{s-p+e} = 0$. Taking $(w_0,)$ we obtain $\sum_{e \in [0,2p]} n_e (w_0, w_{s-p+e}) = 0$. For e in the sum we have $s - p + e \geq -p + 1$; hence by (a1) we can assume that we have $s - p + e \geq p$. Thus $\sum_{e \in [0,2p]; s-p+e \geq p} n_e (w_0, w_{s-p+e}) = 0$. By the induction hypothesis this implies

$$\sum_{e \in [0,2p-1]; s-p+e \geq p} n_e x_{s-2p+e} + (w_0, w_{s+p}) = 0.$$

It is then enough to show that

$$\sum_{e \in [0,2p-1]; s-p+e \geq p} n_e x_{s-2p+e} + x_s = 0$$

or that

$$\sum_{e \in [0,2p]; s-p+e \geq p} n_{2p-e} x_{s-2p+e} = 0$$

or that

$$\sum_{h \geq 0, h' \geq 0; h+h'=s} n_h x_{h'} = 0.$$

But this holds by the definition of x_e since $s \geq 1$.

1.5. Let $p \geq 0$. For $e \geq 0$ we set

$$n_e = (-1)^e \binom{2p+1}{e}$$

so that $(1-T)^{2p+1} = \sum_{e \geq 0} n_e T^e$. For $e \geq 1$ we set $x_e = 2(p+e)(2p+1)(2p+2)\dots(2p+e-1)e!^{-1}$ (note that $x_1 = 2p+2$). We set $x_0 = 1$ if $p > 0$ and $x_0 = 2$ if $p = 0$. If $p > 0$, then for any $u \geq 2$ we have

(a) $$\sum_{j \in [0,u]} n_j x_{u-j} = 0.$$

(See [L3, line 4 of p. 134]) This shows by induction on e that $x_e \in \mathbf{N}$ for any $e \geq 0$.

For $u \in \mathbf{Z}$ we set $f_p(u) = 0$ if $|u| < p$ and $f_p(u) = x_e$ if $|u| = p + e$ with $e \geq 0$. For $u \in \mathbf{Z}$ we have

(b) $$f_p(u) = 2(2p)!^{-1} \prod_{k \in [0,p-1]} (u^2 - k^2).$$

For example, $f_0(u) = 2$. Also, $f_p(p) = 1$ if $p \geq 1$.

Setting $A_p = \sum_{e \geq 0} f_p(p+e)T^e = \sum_{e \geq 0} x_e T^e$ (where T is an indeterminate) we have, by (a), $(1-T)^{2p+1}A_p = 1 + T$ hence

(c) $$A_p = (1-T)^{-2p-1}(1+T).$$

1.6. In the setup of 1.5 let E be a $\mathbf{k}$-vector with basis $w_0, w_1, \dots, w_{2p}$. We define a symmetric bilinear form $(,) : E \times E \to \mathbf{k}$ by $(w_i, w_j) = (-1)^p f_p(i-j)$ for $i, j \in [0, 2p]$. We define $g \in GL(E)$ by $gw_i = w_{i+1}$ if $i \in [0, 2p-1]$, $gw_{2p} = \sum_{j\in[0,2p]} n_j w_j$.

We have $(g-1)^{2p+1} = 0$ hence $g : E \to E$ is unipotent (with a single Jordan block). We show that g is an isometry of $(,)$. We can assume that $p > 0$. It is enough to show that $(w_{i+1}, gw_{2p}) = (w_i, w_{2p})$ for $i \in [0, 2p-1]$ and $(gw_{2p}, gw_{2p}) = 0$. Thus we must show that

(a) $$\sum_{j\in[0,2p+1],e\ge 0,|i+1-j|=e+p} n_j x_e = 0 \text{ if } i \in [0, 2p-1],$$

(b) $$\sum_{j,j'\in[0,2p],e\ge 0,|j-j'|=e+p} n_j n_{j'} x_e = 0.$$

Now (a) for i is equivalent to (a) for $2p-1-i$ (we use the substitution $j \mapsto 2p+1-j$); hence it is enough to prove (a) for $i \in [p, 2p-1]$. Now (a) for $i = p$ reads $x_1 - (2p+1)x_0 - x_0 = 0$, that is, $x_1 = 2p+2$, which is true. For $i \in [p+1, 2p-1]$, (a) reads $\sum_{j\in[0,2p+1],i+1-j\ge p} n_j x_{i+1-j-p} = 0$, that is (setting $u = i+1-p$), $\sum_{j\in[0,u]} n_j x_{u-j} = 0$. This follows from 1.5(a) since $u \ge 2$. This proves (a).

We prove (b). The left hand side of (b) equals

$$\begin{aligned}
&\sum_{j'\in[0,2p]} n_{j'} \sum_{j\in[0,2p],e\ge 0,|j-j'|=e+p} n_j x_e \\
&= \sum_{j\in[0,2p],e\ge 0,|j|=e+p} n_j x_e + \sum_{j'\in[1,2p]} n_{j'} \sum_{j\in[0,2p],e\ge 0,|j-j'|=e+p} n_j x_e \\
&= \sum_{j\in[0,2p],e\ge 0,|j|=e+p} n_j x_e + \sum_{j'\in[1,2p]} n_{j'} \sum_{j\in[0,2p+1],e\ge 0,|j-j'|=e+p} n_j x_e \\
&- \sum_{j'\in[1,2p]} n_{j'} \sum_{e\ge 0,|2p+1-j'|=e+p} n_{2p+1} x_e.
\end{aligned}$$

In the last expression the second sum over j is zero by (a) and the second sum over j' becomes (setting $j = 2p+1-j'$)

$$\sum_{j\in[1,2p]} n_j \sum_{e\ge 0,|j|=e+p} x_e.$$

Hence the left hand side of (b) equals

$$\sum_{j\in[0,2p],e\ge 0,|j|=e+p} n_j x_e - \sum_{j\in[1,2p]} n_j \sum_{e\ge 0,|j|=e+p} x_e = \sum_{e\ge 0,|0|=e+p} x_e$$

and this is zero since $e + p > 0$. Thus (b) holds.

For any $i \in \mathbf{Z}$ we set $w_i = g^i w_0$. This agrees with the earlier notation when $i \in [0, 2p]$. We show:

(c) $$(w_i, w_j) = (-1)^p f_p(i-j) \text{ if } i, j \in \mathbf{Z}.$$

If $p = 0$ there is nothing to prove since $g = 1$; thus we can assume that $p \ge 1$. We will prove (c) assuming only the identities

(d1) $(w_{p-1}, w_j) = 0$ if $j \in [0, 2p-2]$,

(d2) $(w_{p-1}, w_{2p-1}) = (-1)^p$.

If $|i-j|<p$, then (c) follows from (d1); if $|i-j|=p$, then (c) follows from (d2). Thus we can assume that $|i-j|\ge p+1$. We can also assume that $i=0$ and $j\ge 0$ (hence $j\ge p+1$). We must only prove that

$$(w_0,w_j)=(-1)^p x_{j-p} \text{ if } j\ge p.$$

We argue by induction on j. For $j=p$ the result is known. Assume that $j\ge p+1$. From $(g-1)^{2p+1}w_{j-2p-1}=0$ we deduce $\sum_{h\in[0,2p+1]} n_h w_{j-2p-1+h}=0$. Hence $\sum_{h'\in[0,2p+1]} n_{h'} w_{j-h'}=0$ and $\sum_{h\in[0,2p+1]} n_h(w_0,w_{j-h})=0$. If $j=p+1$ we can assume that $h=0, h=1$ or $h=2p+1$ (the other terms are zero); thus,

$$n_0(w_0,w_{p+1})+n_1(w_0,w_p)+n_{2p+1}(w_0,w_{-p})=0.$$

We see that $(w_0,w_{p+1})-(2p+1)(-1)^p-(-1)^p=0$ so that $(w_0,w_{p+1})=(-1)^p(2p+2)$ as required. Now assume that $j\ge p+2$. We have

$$\sum_{h\in[0,2p+1];j-h\ge p} n_h(w_0,w_{j-h})=0.$$

Using the induction hypothesis this implies

$$\sum_{h\in[1,2p+1];j-h\ge p} n_h(-1)^p x_{j-h-p}+(w_0,w_j)=0$$

hence it is enough to show that

$$\sum_{h\in[0,2p+1];j-h\ge p} n_h x_{j-h-p}=0,$$

that is,

$$\sum_{h\in[0,j-p]} n_h x_{j-h-p}=0.$$

This follows from 1.5(a) with $u=j-p$ since $j-p\ge 2$.

1.7. We preserve the setup of 1.6. The subspace E' of E spanned by $\{w_i; i\in[0,2p-1]\}$ is clearly nondegenerate for $(,)$ hence there exists $\tilde w\in E$ such that $(w_i,\tilde w)=0$ for $i\in[0,2p-1]$ and $(\tilde w,\tilde w)=2$. Moreover, $\tilde w$ is unique up to multiplication by ±1. We have $\tilde w\notin E'$. We can write $\tilde w=\sum_{i\in[0,2p]} c_i w_i$ where $c_i\in\mathbf{k}$ are uniquely defined and $c_*:=c_{2p}\ne 0$. Taking $(w_h,)$ and setting $\bar c_i=c_i/c_*$ we obtain

$$(*)\qquad \sum_{i\in[0,2p]} \bar c_i f_p(i-h)=0 \text{ for } h\in[0,2p-1].$$

We show (setting $l_j=\binom{2p+1}{j}$):

$$\bar c_i=(-1)^{i-1}(l_0+l_1+\dots+l_i) \text{ if } i\in[0,p-1],$$
$$\bar c_i=(-1)^{i}(l_0+l_1+\dots+l_{2p-i}) \text{ if } i\in[p,2p].$$

We can assume that $p\ge 1$. Clearly $(*)$ has a unique solution $\bar c_i (i\in[0,2p-1])$. Note that $\bar c_{2p}=1$. If $h=p$, then $(*)$ is $\bar c_0+1=0$. If $h\in[p+1,2p-1]$, then $(*)$ is $\sum_{i\in[0,h-p]}\bar c_i f_p(i-h)=0$. If $h\in[0,p-1]$, then $(*)$ is $\sum_{i\in[h+p,2p]}\bar c_i f_p(i-h)=0$. It is enough to show:

$$\text{(a)}\qquad \sum_{i\in[0,h-p]} (-1)^{i-1}(l_0+\dots+l_i)x(h-i-p)=0 \text{ if } h\in[p+1,2p-1],$$

(b) $$\sum_{i\in[h+p,2p]}(-1)^i(l_0+\dots+l_{2p-i})x(i-h-p)=0 \text{ if } h\in[0,p-1].$$

We rewrite equation (b) (using $i\mapsto 2p-i$ and $h\mapsto 2p-h$) as

(c) $$\sum_{i\in[0,h-p]}(-1)^i(l_0+\dots+l_i)x(h-i-p)=0.$$

Here $h\in[p+1,2p]$. Note that (c) contains (a) as a special case. Thus it is enough to prove (c). We prove (c) by induction on h. If $h=p+1$, then equation (c) is $l_0x_1-(l_0+l_1)x_0=0$, that is, $2p+2-(2p+2)=0$, which is correct. If $h\ge p+2$ we have $\sum_{i\in[0,h-p]}(-1)^il_ix(h-i-p)=0$. Hence in this case (c) is equivalent to $\sum_{i\in[1,h-p]}(-1)^i(l_0+\dots+l_{i-1})x(h-i-p)=0$ which is the same as equation (c) with h replaced by $h-1$ (this holds by the induction hypothesis). This proves (c) hence (a),(b).

We show:

(d) $$(w_{2p},\tilde w)c_*=2.$$

Indeed, we have

$$2=(\tilde w,\tilde w)=(\sum_{i\in[0,2p]}c_iw_i,\tilde w)=c_{2p}(w_{2p},\tilde w),$$

as desired. We show:

(e) $$c_*^2=2^{-2p}.$$

We have

$$2=(w_{2p},\tilde w)c_*=(w_{2p},\sum_{i\in[0,2p]}c_iw_i)c_*=\sum_{i\in[0,2p]}c_i(-1)^pf_p(2p-i)c_*.$$

Thus

$$2c_*^{-2}=\sum_{i\in[0,p]}\bar c_i(-1)^pf_p(2p-i).$$

If $p=0$, this reads $2c_*^{-2}=\bar c_0f_0(0)=2$ hence (e) follows. If $p\ge 1$, we have $(w_0,\tilde w)=0$ hence $0=\sum_{i\in[0,2p]}\bar c_i(-1)^pf_p(i)$ hence $0=\sum_{i\in[p,2p]}\bar c_i(-1)^pf_p(i)$, that is,

$$0=\sum_{i\in[0,p]}\bar c_{2p-i}(-1)^pf_p(2p-i).$$

Adding to

$$2c_*^{-2}=\sum_{i\in[0,p]}\bar c_i(-1)^pf_p(2p-i)$$

we get

$$2c_*^{-2}=\sum_{i\in[0,p]}(\bar c_i+\bar c_{2p-i})(-1)^pf_p(2p-i).$$

Now $\bar c_i+\bar c_{2p-i}=0$ if $i\in[0,p-1]$ hence

$$2c_*^{-2}=2(-1)^p\bar c_p=2(l_0+l_1+\dots+l_p)=2^{2p+1}$$

and (e) follows.

From (e) we see that, by replacing if necessary, $\tilde w$ by $-\tilde w$ we can assume that

(f) $$c_*=2^{-p}.$$

This condition determines $\tilde w$ uniquely.

We show that for $h\in\mathbf{Z}$:

(g) $$(w_h,\tilde w)=2^{p+1}\binom{h}{2p}.$$

We must show that for $h\in\mathbf{Z}$:

$$\sum_{i\in[0,2p]}c_i(-1)^p f_p(i-h)=2^{p+1}\binom{h}{2p}$$

or that

$$\sum_{i\in[0,2p]}\bar c_i(-1)^p f_p(i-h)=2^{2p+1}\binom{h}{2p}.$$

It is enough to prove this equality in $\mathbf{Z}$. The left hand side is a polynomial in h with rational coefficients of degree $\le 2p$ which vanishes for $h\in[0,2p-1]$ in which the coefficient of h^{2p} is

$$\sum_{i\in[0,2p]}\bar c_i(-1)^p2(2p)!^{-1}=(-1)^p\bar c_p2(2p)!$$
$$=(l_0+l_1+\dots+l_p)2(2p)!^{-1}=(-1)^p2^{2p}2(2p)!^{-1}.$$

Hence it is equal to the right hand side.

For any $h\in\mathbf{Z}$, $\tilde w_h$ is defined as in 1.0. We show:

(h) $(\tilde w_0,\tilde w_h)=2(-1)^h$ if $h\in[0,p]$; $(\tilde w_0,\tilde w_{p+1})=2(-1)^{p+1}+(-1)^p2^{2p+2}$.

We can assume that $h\ge 1$. We have

$$(\tilde w_0,\tilde w_h)=(\sum_{i\in[0,2p]}c_iw_i,\tilde w_h)=\sum_{i\in[0,2p]}c_i(w_{i-h},\tilde w_0)=\sum_{i\in[0,2p]}2\bar c_i\binom{i-h}{2p}$$
$$=\sum_{i\in[0,h-1];i\ne p}2(-1)^{i-1}(l_0+\dots+l_i)\binom{i-h}{2p}+\delta_{h,p+1}2(-1)^p(l_0+\dots+l_p)\binom{p-h}{2p}$$
$$=\sum_{i\in[0,h-1]}2(-1)^{i-1}(l_0+\dots+l_i)\binom{i-p}{2p}+2\delta_{h,p+1}2(-1)^p(l_0+\dots+l_p).$$

Now $4(-1)^p(l_0+\dots+l_p)=(-1)^p2^{2p+2}$. It remains to show that

$$\sum_{i\in[0,h-1]}(-1)^{i-1}(l_0+\dots+l_i)\binom{h-i+2p-1}{2p}=(-1)^h$$

for $h\in[1,p+1]$, or setting $h'=h-1,u=h'-i$:

$$\sum_{i\ge0,u\ge0,i+u=h'}(-1)^i(l_0+\dots+l_i)\binom{u+2p}{2p}=(-1)^{h'}$$

for $h'\in[0,p]$. We shall actually show that this holds for any $h'\ge 0$. It is enough to show that for an indeterminate T we have

$$\sum_{i\ge0,u\ge0}(-1)^i(l_0+\dots+l_i)T^i\binom{u+2p}{2p}T^u=\sum_{h'\ge0}(-1)^{h'}T^{h'}$$

or that

$$\sum_{i\ge0}(-1)^i(l_0+\dots+l_i)T^i(1-T)^{-2p-1}=(1+T)^{-1}$$

or that

$$l_0(1-T+T^2-\dots)+l_1(-T+T^2-T^3)+\dots)(1-T)^{-2p-1}=(1+T)^{-1}$$

or that

$$(1+T)^{-1}(l_0-l_1T+l_2T^2-\dots)(1-T)^{-2p-1}=(1+T)^{-1}.$$

This is obvious.

1.8. We preserve the setup of 1.7. For $h\in\mathbf{Z}$ we show

(a) $(\tilde w_0,\tilde w_h)=\sum_{r\in[0,p]}(-1)^r2^{2r}f_r(h)$. *In particular,* $(\tilde w_0,\tilde w_h)\in 2\mathbf{Z}$.

We must prove the equality

$$\text{(a}'\text{)}\qquad \sum_{i\in[0,2p]}2\bar c_i\binom{i-h}{2p}=\sum_{r\in[0,p]}(-1)^r2^{2r}f_r(h)$$

in $\mathbf{k}$. It is enough to prove that (a′) holds in $\mathbf{Z}$. Let $F_p(h)$ be the left hand side of (a′). It can be viewed as a polynomial with rational coefficients in h of degree $\le 2p$ in which the coefficient of h^{2p} is

$$\sum_{i\in[0,2p]}2\bar c_i(2p)!^{-1}=2\bar c_p(2p)!^{-1}=2(-1)^p(l_0+\dots+l_p)(2p)!^{-1}=2(-1)^p2^{2p}(2p)!^{-1}.$$

(We have used that $\bar c_i+\bar c_{2p-i}=0$ if $i\ne p$.) Thus

$$F_p(h)=(-1)^p2^{2p+1}(2p)!^{-1}h^{2p}+\text{ lower powers of }h.$$

In the case where $p=0$ this implies that $F_p(h)=2$ so that (a′) holds. We now assume that $p\ge 1$. Note that $F_p(-h)=F_p(h)$ for $h\in\mathbf{Z}$; an equivalent statement is that $(\tilde w_0,\tilde w_h)=(\tilde w_0,\tilde w_{-h})$, which follows from the definitions. We see that $F_p(-h)=F_p(h)$ as polynomials in h. Now F_p-F_{p-1} is a polynomial of degree $2p$ in h whose value at $h\in[0,p-1]$ is $2(-1)^h-2(-1)^h=0$. Using this and $F_p(-h)=F_p(h)$ we see that

$$F_p(h)-F_{p-1}(h)=(-1)^p2^{2p+1}(2p)!^{-1}h^2(h^2-1)\dots(h^2-(p-1)^2).$$

From this we see by induction on p that (a′) holds.

It follows that, if L is the line spanned by w_0, L' is the line spanned by $\tilde w_0$ and $a_*=(2p+1,0,0,\dots),b_*=(0,0,0,\dots)$, then $(g,L,L')\in\tilde{\mathcal{C}}^E_{a_*,b_*}$. In particular, $\tilde{\mathcal{C}}^E_{a_*,b_*}\ne\emptyset$.

1.9. In the setup of 1.5, we consider a $\mathbf{k}$-vector space E of dimension $2p+1$ with a given nondegenerate symmetric bilinear form $(,):E\times E\to\mathbf{k}$ and a unipotent isometry $g:E\to E$ of $(,)$ such that g is a single unipotent Jordan block (of size $2p+1$). Moreover, we assume that we are given $\tilde w\in E$ and (if $p\ge 1$) $w\in E$ such that (with notation of 1.0) for $i,j\in\mathbf{Z}$ we have:

$(w_i,w_j)=0$ if $|i-j|<p$; $(w_i,w_j)=(-1)^p$ if $|i-j|=p$ (with $p\ge 1$),

$(w_i,\tilde w_j)=0$ if $i-j\in[0,2p-1]$,

$(\tilde w_i,\tilde w_j)=2$ if $i=j$.

We show:

(a) *After possibly replacing $\tilde w$ by $-\tilde w$, the following equalities hold for any i,h in* $\mathbf{Z}$:

(a1) $(w_i, w_h) = (-1)^p f_p(i-h)$ *if* $p \ge 1$,
(a2) $(w_h, \tilde{w}_0) = 2^{p+1} h(h-1)(h-2)\dots(h-2p+1)(2p)!^{-1}$ *if* $p \ge 1$,
(a3) $(\tilde{w}_0, \tilde{w}_h) = \sum_{r\in[0,p]}(-1)^r 2^{2r} f_r(h)$.

Now the proof of (a1) is exactly as in 1.6. We show:

(b) if $p \ge 1$, then $\{w_i; i \in [0, 2p]\}$ is linearly independent.

Assume that this is not true. Then w_{2p} belongs to E', the span of $\{w_i; i \in [0, 2p-1]\}$; hence E' is a g-stable hyperplane. Note that g acts on E' as a unipotent linear map with a single Jordan block (of size $2p$). By (a1), $(,)_{E'}$ is nondegenerate. Hence $g : E \to E$ has a Jordan block of size $2p$ and one of size 1; this contradicts our assumption that g has a single Jordan block of size $2p + 1$. This contradiction proves (b).

By (b) we can write uniquely (assuming $p \ge 1$) $\tilde{w}_0 = \sum_{i\in[0,2p]} c_i w_i$ where $c_i \in \mathbf{k}$. Note that $c_{2p} \ne 0$. (Otherwise, $\tilde{w}_0$ would be contained in E'; on the other hand, $\tilde{w}_0$ is perpendicular to E' contradicting the nondegeneracy of $(,)|_{E'}$.) We set $c_* = c_{2p}$, $\bar{c}_i = c_i c_*^{-1}$ $(i \in [0, 2p])$. By repeating the arguments in 1.7 we see that $c_* = \pm 2^{-p}$. Replacing if necessary $\tilde{w}$ by $-\tilde{w}$ we can assume that $c_* = 2^{-p}$. Now (a2) and (a3) are proved exactly as in 1.7 and 1.8. If $p = 0$, then $\tilde{w}_h = \tilde{w}_0$ for any $h \in \mathbf{Z}$ hence $(\tilde{w}_0, \tilde{w}_h) = (\tilde{w}_0, \tilde{w}_0) = f_0(0) = 2$. Thus (a3) holds again.

1.10. We fix two integers p_1, p_2 such that $p_1 \ge p_2 \ge 1$. Let V', V'' be two $\mathbf{k}$-vector spaces of dimension $2p_1 + 1, 2p_2 - 1$, respectively. Let $V = V' \oplus V''$. Assume that V' has a given basis $z_0, z_1, \dots, z_{2p_1}$ and that V'' has a given basis $v_0, v_1, \dots, v_{2p_2-2}$. We define a symmetric bilinear form $(,)$ on V by

$$(z_i, z_j) = (-1)^{p_1} f_{p_1}(i-j) \text{ for } i, j \in [0, 2p_1],$$
$$(v_i, v_j) = (-1)^{p_2-1} f_{p_2-1}(i-j) \text{ for } i, j \in [0, 2p_2-2],$$
$$(z_i, v_j) = (v_j, z_i) = 0 \text{ for } i \in [0, 2p_1], j \in [0, 2p_2-2].$$

(Notation of 1.5.) We define $g \in GL(V)$ by

$$gz_i = z_{i+1} \text{ if } i \in [0, 2p_1 - 1],$$
$$gz_{2p_1} = \sum_{j\in[0,2p_1]} (-1)^j \binom{2p_1+1}{j} z_j,$$
$$gv_i = v_{i+1} \text{ if } i \in [0, 2p_2 - 3],$$
$$gv_{2p_2-2} = \sum_{j\in[0,2p_2-2]} (-1)^j \binom{2p_2-1}{j} v_j.$$

Note that $g : V \to V$ is unipotent and that V', V'' are g-stable (g has a single Jordan block on V' and a single Jordan block on V''). By 1.6, $g : V \to V$ is an isometry. For $i \in \mathbf{Z}$ we set $z_i = g^i z_0 \in V', v_i = g^z v_0 \in V''$. This agrees with our earlier notation. By 1.6 we have for $i, j \in \mathbf{Z}$:

$$(z_i, z_j) = (-1)^{p_1} f_{p_1}(i-j), \quad (v_i, v_j) = (-1)^{p_2-1} f_{p_2-1}(i-j).$$

As in 1.7, 1.8, there is a unique vector $\tilde{z}_0 \in V'$ and a unique vector $\tilde{v}_0 \in V''$ such that for any $h \in \mathbf{Z}$ we have

$$(z_h, \tilde{z}_0) = 2^{p_1+1} h(h-1)(h-2)\dots(h-2p_1+1)(2p_1)!^{-1},$$
$$(\tilde{z}_0, \tilde{z}_h) = \sum_{r\in[0,p_1]} (-1)^r 2^{2r} f_r(h),$$

$$(v_h,\tilde v_0)=2^{p_2}h(h-1)(h-2)\dots(h-2p_2+3)(2p_2-2)!^{-1},$$
$$(\tilde v_0,\tilde v_h)=\sum_{r\in[0,p_2-1]}(-1)^r2^{2r}f_r(h).$$

For $i\in\mathbf{Z}$ we set $\tilde z_i=g^i\tilde z_0\in V',\tilde v_i=g^i\tilde v_0\in V''$. By 1.7 we have

$$(\tilde z_0,\tilde z_h)=2(-1)^h \text{ if } h\in[0,p_1],$$
$$(\tilde z_0,\tilde z_{p_1+1})=2(-1)^{p_1+1}+(-1)^{p_1}2^{2p_1+2},$$
$$(\tilde v_0,\tilde v_h)=2(-1)^h \text{ if } h\in[0,p_2-1],$$
$$(\tilde v_0,\tilde v_{p_2})=2(-1)^{p_2}+(-1)^{p_2-1}2^{2p_2}.$$

We fix $\zeta\in\mathbf{k}$ such that $\zeta^2=-1$. We set

$$\xi=2^{-p_2}(\tilde z_{-p_2}+\zeta\tilde v_0)\in V.$$

Let $h\in\mathbf{Z}$. We have

$$(z_h,\xi)=2^{-p_2}(z_h,\tilde z_{-p_2})=2^{-p_2}(z_{h+p_2},\tilde z_0)=2^{p_1-p_2+1}\binom{h+p_2}{2p_1}.$$

In particular, we have $(z_h,\xi)\in2\mathbf{Z}$; moreover,

$$(\zeta_h,\xi)=0 \text{ if } h\in[-p_2,2p_1-p_2-1].$$

Let $h\in\mathbf{Z}$. We set $\xi_h=g^h\xi$. Using the definitions we see that

$$(\xi_0,\xi_h)=2^{-2p_2}((\tilde z_0,\tilde z_h)-(\tilde v_0,\tilde v_h)).$$

From this we deduce using the formulas above that

$$(\xi_0,\xi_h)=0 \text{ if } h\in[-p_2+1,p_2-1],$$
$$(\xi_0,\xi_h)=(-1)^{p_2} \text{ if } h=p_2,$$
$$(\xi_0,\xi_h)=\sum_{r\in[p_2,p_1]}(-1)^r2^{2r-2p_2}f_r(h) \text{ for } h\in\mathbf{Z}.$$

It follows that, if L is the line in V spanned by z_0, L' is the line in V spanned by ξ and $a_*=(2p_1+1,2p_2-1,0,0,\dots),b_*=(0,0,\dots)$, then $(g,L,L')\in\tilde{\mathcal{C}}^V_{a_*,b_*}$. In particular, $\tilde{\mathcal{C}}^V_{a_*,b_*}\ne\emptyset$.

1.11. Let p_1,p_2 be as in 1.10; let $V,\epsilon,(,)$ be as in 1.0. Let $g\in Is(V)$. We assume that $\epsilon=1,\dim V=2p_1+2p_2$ and that g is unipotent with exactly two Jordan blocks: one of size $2p_1+1$ and one of size $2p_2-1$. Moreover, we assume that we are given $z\in V,\xi\in V$ such that (with notation of 1.0) we have for $i,j\in\mathbf{Z}$:

$$(z_i,z_j)=0 \text{ if } |i-j|<p_1,(z_i,z_j)=(-1)^{p_1} \text{ if } |i-j|=p_1,$$
$$(\xi_i,\xi_j)=0 \text{ if } |i-j|<p_2,(\xi_i,\xi_j)=(-1)^{p_2} \text{ if } |i-j|=p_2,$$
$$(z_i,\xi_j)=0 \text{ if } i-j\in[-p_2,2p_1-p_2-1].$$

We show:

(a) *After possibly replacing ξ by $-\xi$, the following equalities hold for any $u\in\mathbf{Z}$ and any $i,j\in\mathbf{Z}$ such that $i-j=u$:*

(a1) $(z_i,z_j)=(-1)^{p_1}f_{p_1}(u)$,
(a2) $(z_i,x_j)=2^{p_1-p_2+1}\binom{u+p_2}{2p_1}$,
(a3) $(\xi_i,\xi_j)=\sum_{r\in[p_2,p_1]}(-1)^r2^{2r-2p_2}f_r(u)$.

(Notation of 1.5.) Let $\alpha_u, \gamma_u, \beta_u$ be the left hand side of (a1), (a2), (a3), respectively. (These are well defined by 1.0(a).) Note that $\alpha_u = a_{-u}, \beta_u = \beta_{-u}$. When z_i, ξ_i are replaced by the vectors with the same name in 1.10, the quantities $\alpha_u, \beta_u, \gamma_u$ become $\alpha_u^0, \beta_u^0, \gamma_u^0$ (which were computed in 1.10). Then (a1)–(a3) are equivalent to the equalities $\alpha_u = \alpha_u^0, \beta_u = \beta_u^0, \gamma_u = \gamma_u^0$.

We prove (a1). (See also the proof of 1.6(c).) If $|u| \le p_1$, then (a1) is clear. Thus we can assume that $|u| \ge p_1 + 1$. We can also assume that $u \ge 0$ (hence $u \ge p_1 + 1$). We must only prove that $(z_0, z_u) = (-1)^{p_1} x_{u-p_1}$ if $u \ge p_1$ where x_h is as in 1.5 (with $p = p_1$). As in the proof of 1.6(c) we argue by induction on u. For $u = p_1$ the result is known. Assume that $u \ge p_1 + 1$. We have $(g-1)^{2p_1+1} = 0$ on V hence $(g-1)^{2p_1+1} z_{u-2p_1-1} = 0$, that is,

$$\sum_{h \in [0, 2p_1+1]} n_h z_{u-2p_1-1+h} = 0.$$

Hence

$$\sum_{h' \in [0, 2p_1+1]} n_{h'} z_{u-h'} = 0$$

and

$$\sum_{h \in [0, 2p_1+1]} n_h (z_0, z_{u-h}) = 0.$$

If $u = p_1 + 1$ we can assume that $h = 0, h = 1$ or $h = 2p_1 + 1$ (the other terms are zero); thus,

$$n_0(z_0, z_{p_1+1}) + n_1(z_0, z_{p_1}) + n_{2p_1+1}(z_0, z_{-p_1}) = 0.$$

We see that $(z_0, z_{p_1+1}) - (-1)^{p_1}(2p_1 + 1) - (-1)^{p_1} = 0$ so that

$$(z_0, z_{p_1+1}) = (-1)^{p_1}(2p_1 + 2),$$

as required. Now assume that $u \ge p_1 + 2$. We have

$$\sum_{h \in [0, 2p_1+1]; j-h \ge p_1} n_h (z_0, z_{u-h}) = 0.$$

Using the induction hypothesis this implies

$$\sum_{h \in [1, 2p_1+1]; u-h \ge p_1} n_h (-1)^{p_1} x_{u-h-p_1} + (z_0, z_u) = 0,$$

hence it is enough to show that

$$\sum_{h \in [0, 2p_1+1]; u-h \ge p_1} n_h x_{u-h-p_1} = 0,$$

that is,

$$\sum_{h \in [0, u-p_1]} n_h x_{u-h-p_1} = 0.$$

This follows from 1.5(a) with u replaced by $u - p_1$ since $u - p_1 \ge 2$.

The proof of (a2) and (a3) will be given in 1.12–1.16 where the setup of this subsection is preserved.

1.12. We show:

(a) the set $\{z_i; i \in [0, 2p_1]\}$ is linearly independent.

Assume that this is not true. Then $z_{2p_1} \in E$, the span of $\{z_i; i \in [0, 2p_1 - 1]\}$. Hence E is g-stable and its perpendicular $E^\perp$ is g-stable. By assumption we have $\xi_{p_2} \in E^\perp$. Since $E^\perp$ is g-stable we see that $\xi_i \in E^\perp$ for all $i \in \mathbf{Z}$. Thus E', the span of $\{\xi_i; i \in [0, 2p_2 - 1]\}$, is contained in $E^\perp$. By assumption, E' has dimension $2p_2$ which is the same as $\dim E^\perp$. Hence $E' = E^\perp$. Since $V = E \oplus E'$, we see that $V = E \oplus E^\perp$ with both summands being g-stable. Now g acts on E as a single unipotent Jordan block of size $2p_1$. Thus $g : V \to V$ has a Jordan block of size $2p_1$. This contradicts the assumption that the Jordan blocks of $g : V \to V$ have sizes $2p_1 + 1, 2p_2 - 1$. This proves (a).

We set $N = g - 1, e = p_1 - p_2$. Let $\mathcal{L}$ be the span of $\{N^i z_0; i \in [2p_2, 2p_1]\}$ or equivalently the span of $\{N^{2p_2} z_i; i \in [0, 2e]\}$. We show:

(b) $$\dim \mathcal{L} = 2e + 1.$$

Let $\mathcal{L}'$ be the span of $\{N^i z_0; i \in [2p_2, 2p_1 - 1]\}$. We have $\dim \mathcal{L}' = 2e$ since $\{N^i z_0; i \in [0, 2p_1 - 1]\}$ is a linearly independent set. If (b) is false we would have $N^{2p_2} z_0 \in \mathcal{L}'$. Then the span of $\{N^i z_0; i \in [0, 2p_1 - 1]\}$ is N-stable. Hence the span of $\{g^i z_0; i \in [0, 2p_1 - 1]\}$ is g-stable. This contradicts the proof of (a).

We show:

(c) $$N^{2p_2} \xi_0 \in \mathcal{L}.$$

From the structure of Jordan blocks of $N : V \to V$ we see that $\dim N^{2p_2} V = 2e+1$. Clearly, $\mathcal{L} \subset N^{2p_2} V$. Hence using (b) it follows that $\mathcal{L} = N^{2p_2} V$ so that (c) holds.

Using (c) we deduce

(d) $$N^{2p_2} \xi_0 = \sum_{i \in [0,2e]} c_i N^{2p_2} z_i$$

where $c_i \in \mathbf{k}$ ($i \in [0, 2e]$) are uniquely determined.

1.13. For $j \in \mathbf{N}$ we set $m_j = (-1)^j \binom{2p_2}{j}$ so that $N^{2p_2} = \sum_{j \in [0,2p_2]} m_j g^j$. From 1.12(d) we deduce

(a) $$\sum_{j \in [0,2p_2]} m_j \xi_j = \sum_{i \in [0,2e], j \in [0,2p_2]} c_i m_j z_{i+j}.$$

Taking $(, z_u)$ with $u \in \mathbf{Z}$, we deduce

(b) $$\sum_{j \in [0,2p_2]} m_j \gamma_{u-j} = \sum_{i \in [0,2e], j \in [0,2p_2]} c_i m_j \alpha_{u-i-j}.$$

We show:

(c1) If $u \in [p_2, 2p_1 - p_2 - 1]$, then the left hand side of (b) is 0.

(c2) If $u = 2p_1 - p_2$, then the left hand side of (b) is $\gamma_{2p_1 - p_2}$.

For (c1) it is enough to show: if u is as in (c1) and $j \in [0, 2p_2]$, then $u - j + p_2 \in [0, 2p_1 - 1]$. (Indeed, we have $u - j + p_2 \le 2p_1 - p_2 - 1 + p_2 = 2p_1 - 1$ and $u - j + p_2 \ge p_2 - 2p_2 + p_2 = 0$.) For (c2) it is enough to show: if $j \in [1, 2p_2]$, then $2p_1 - p_2 - j + p_2 \in [0, 2p_1 - 1]$. (Indeed, we have $2p_1 - j \le 2p_1 - 1$ and $2p_1 - j \ge 2e \ge 0$.)

If $u \in [p_2, p_1-1]$, then in the right hand side of (b) we have $u-i-j<p_1$; we can assume then that $u-i-j \le -p_1$ hence $i \ge u-j+p_1 \ge p_2-2p_2+p_1=e$. Thus in this case (b) becomes (using (c1) and setting $u=p_1-t$):

$$\sum_{i\in[e,2e],j\in[0,2p_2]} c_i m_j \alpha_{p_1-t-i-j}=0 \text{ for } t\in[1,e].$$

Setting $c'_h=c_{2e-h}$ for $h\in[0,e]$ and with the change of variable $j\mapsto 2p_2-j$, $i\mapsto 2e-i$ we obtain

(d) $$\sum_{i\in[0,e],j\in[0,2p_2]} c'_i m_j \alpha_{-p_1-t+i+j}=0 \text{ for } t\in[1,e].$$

In the last sum we have $-p_1-t+i+j<p_1$. Indeed, we have

$$-p_1-t+i+j\le -p_1-1+p_1-p_2+2p_2=p_2-1<p_1.$$

Hence we can restrict the sum to indices such that $-p_1-t+i+j\le -p_1$, that is, $-t+i+j=-s$ where $s\ge 0$. Thus we have

$$\sum_{i\in[0,e],j\ge 0,s\ge 0;i+s+j=t} c'_i m_j \alpha_{-p_1-s}=0 \text{ for } t\in[1,e].$$

Hence

$$(\sum_{i\in[0,e]} c'_i T^i)(\sum_{j\ge 0} m_j T^j)(\sum_{s\ge 0} f_{p_1}(p_1+s)T^s)=c'_0+\text{terms of degree} > e \text{ in } T.$$

Thus

$$(\sum_{i\in[0,e]} c'_i T^i)(1-T)^{2p_2}A_{p_1}=c'_0+\text{terms of degree} > e \text{ in } T,$$

where A_{p_1} is as in 1.5. Using 1.5(c) we obtain

$$(\sum_{i\in[0,e]} c'_i T^i)(1-T)^{2p_2}(1+T)(1-T)^{-2p_1-1}=c'_0+\text{terms of degree} > e \text{ in } T$$

hence

$$\sum_{i\in[0,e]} c'_i T^i=(1+T)^{-1}(1-T)^{2e+1}(c'_0+\text{terms of degree} > e \text{ in } T).$$

We have $(1-T)^{2e+1}=\sum_{j\in[0,2e+1]}(-1)^j l_j T^j$ where $l_j=\binom{2e+1}{j}$. Hence

$$(1+T)^{-1}(1-T)^{2e+1}=\sum_{j\in[0,e]}(-1)^j(l_0+l_1+\dots+l_j)T^j+\text{terms of degree} > e \text{ in } T.$$

We see that

(e) $$c'_i=(-1)^i c'_0(l_0+l_1+\dots+l_i) \text{ for } i\in[0,e].$$

In the remainder of this subsection we assume that $e>0$. If $u=p_1$, then in the right hand side of (b) we have $u-i-j\in[-p_1,p_1]$; we can then assume that $u-i-j$ is $-p_1$ or p_1. Hence $i+j$ is $2p_1$ or 0 and (i,j) is $(2e,2p_2)$ or $(0,0)$. Thus in this case (b) becomes (using (c1)) $c_0+c_{2e}=0$, that is, $c_0=-c'_0$ (to apply (c1) we use that $e>0$).

If $u\in[p_1+1,2p_1-p_2-1]$, then in the right hand side of (b) we have $u-i-j>-p_1$; we can assume then that $u-i-j\ge p_1$ hence

$$i\le u-j-p_1\le 2p_1-p_2-1-p_1=e-1.$$

Using this and (c1) we see that (b) becomes (setting $u = p_1 + t$):

$$\sum_{i\in[0,e-1],j\in[0,2p_2]} c_i m_j \alpha_{p_1+t-i-j} = 0 \text{ for } t \in [1, e-1].$$

Note that in the sum we have $p_1 + t - i - j > -p_1$. Indeed, we have

$$p_1 + t - i - j \geq p_1 + 1 - p_1 + p_2 + 1 - 2p_2 = -p_2 + 2 > -p_1.$$

Hence we can restrict the sum to indices such that $p_1 + t - i - j \geq p_1$, that is, $p_1 + t - i - j = p_1 + s$ where $s \geq 0$. Thus we have

$$\sum_{i\in[0,e-1],j\geq 0,s\geq 0;i+s+j=t} c_i m_j \alpha_{p_1+s} = 0 \text{ for } t \in [1, e-1].$$

For such t we have also

$$\sum_{i\in[0,e-1],j\geq 0,s\geq 0;i+s+j=t} c'_i m_j \alpha_{-p_1-s} = 0$$

as we have seen earlier; the index i cannot take the value e since $i \leq t$. Adding the last two equations and using $\alpha_{p_1+s} = \alpha_{-p_1-s}$ we obtain

$$\sum_{i\in[0,e-1],j\geq 0,s\geq 0;i+s+j=t} (c_i + c'_i) m_j \alpha_{-p_1-s} = 0 \text{ for } t \in [1, e-1].$$

Thus,

$$(\sum_{i\in[0,e-1]} (c_i + c'_i) T^i)(\sum_{j\geq 0} m_j T^j)(\sum_{s\geq 0} f_{p_1}(p_1 + s) T^s) = c + \text{terms of degree} > e \text{ in } T,$$

where $c \in \mathbf{k}$. We see that

$$(\sum_{i\in[0,e-1]} (c_i + c'_i) T^i)(1-T)^{2p_2} A_{p_1} = c + \text{terms of degree} > e \text{ in } T.$$

Using again 1.5(c), we obtain

$$(\sum_{i\in[0,e-1]} (c_i + c'_i) T^i)(1-T)^{2p_2}(1+T)(1-T)^{-2p_1-1} = c + \text{terms of degree} > e \text{ in } T$$

hence

$$\sum_{i\in[0,e-1]} (c_i + c'_i) T^i = (1+T)^{-1}(1-T)^{2e+1}(c + \text{terms of degree} > e \text{ in } T),$$

that is,

$$\sum_{i\in[0,e-1]} (c_i + c'_i) T^i = c + \text{terms of degree} > e \text{ in } T.$$

We see that $c_i + c'_i = 0$ for $i \in [1, e-1]$. Using also (e) we see that

(f) $$c_i = (-1)^{i+1} c'_0 (l_0 + l_1 + \cdots + l_i) \text{ for } i \in [0, e-1].$$

(In the case where $i = 0$ this is just $c_0 = -c'_0$ which is already known.)

1.14. If $u = 2p_1 - p_2$ then, using 1.13(b) and 1.13(c2), we have

(a) $$\gamma_{2p_1-p_2} = \sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \alpha_{2p_1-p_2-i-j}.$$

Taking $(,\xi_{p_2})$ with 1.13(a) we obtain

$$\sum_{j\in[0,2p_2]} m_j \beta_{p_2-j} = \sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \gamma_{i+j-p_2}.$$

In the left hand side only the contribution of $j = 0$ and $j = 2p_2$ is $\neq 0$; it is $(-1)^{p_2}$; in the right hand side we can assume that $i + j - p_2 \geq 2p_1 - p_2$ (since $i+j-p_2 \geq -p_2$); hence we have $i+j \geq 2p_1$ and $i = 2e, j = 2p_2$ and the right hand side is $c_{2e}\gamma_{2p_1-p_2} = c'_0\gamma_{2p_1-p_2}$. Thus

(b) $$2(-1)^{p_2} = c'_0\gamma_{2p_1-p_2}.$$

We see that $c'_0 \neq 0$ and using (a),(b) we have

$$2(-1)^{p_2}{c'_0}^{-1} = \sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \alpha_{2p_1-p_2-i-j}.$$

In the right hand side we have $2p_1 - p_2 - i - j \geq -p_1$; we can assume then that either $2p_1 - p_2 - i - j = -p_1$ (hence $i = 2e, j = 2p_2$) or $2p_1 - p_2 - i - j \geq p_1$ (hence $i \leq e$). The first case can arise only if $e = 0$, hence it is included in the second case. Thus

(c) $$2(-1)^{p_2}{c'_0}^{-1} = \sum_{i\in[0,e],j\in[0,2p_2]} c_i m_j \alpha_{2p_1-p_2-i-j}.$$

Assume now that $e > 0$. From 1.13(d) with $t = e$, we have

(d) $$0 = \sum_{i\in[0,e],j\in[0,2p_2]} c'_i m_j \alpha_{-2p_1+p_2+i+j}.$$

We now add (c) and (d) and use that $c_i + c'_i = 0$ if $i \in [0, e-1]$ and $c_e = c'_e$. We get

$$2(-1)^{p_2}{c'_0}^{-1} = 2c'_e \sum_{j\in[0,2p_2]} m_j \alpha_{p_1-j}.$$

If $j \in [1, 2p_2]$ we have $p_1 - j \in [-p_1 + 1, p_1 - 1]$ hence $\alpha_{p_1-j} = 0$. Thus

$$2(-1)^{p_2}{c'_0}^{-1} = 2c'_e\alpha_{p_1} = 2(-1)^{p_1}c'_e.$$

By 1.13(e) we have $c'_e = (-1)^e c'_0(l_0 + l_1 + \cdots + l_e) = (-1)^e c'_0 2^{2e}$ hence

$$2(-1)^{p_2}{c'_0}^{-1} = 2(-1)^{p_1}(-1)^e c'_0 2^{2e}$$

so that ${c'_0}^2 = 2^{-2e}$ and $c'_0 = \pm 2^{-e}$. Changing if necessary ξ by $-\xi$ we can therefore assume that

(e) $$c'_0 = 2^{-e}.$$

Assume now that $e = 0$. We have $c'_0 = c_0$ and (c) becomes

$$2(-1)^{p_2}{c_0}^{-1} = \sum_{j\in[0,2p_2]} c_0 m_j \alpha_{p_1-j},$$

that is, $2(-1)^{p_2}{c_0}^{-1} = 2c_0(-1)^{p_1}$ hence $c_0^2 = 1$ and $c_0 = \pm 1$. Changing if necessary ξ by $-\xi$ we can therefore assume that $c_0 = 1$. Thus (e) holds without the assumption $e > 0$.

Using (e) we rewrite 1.13(e), 1.13(f) as follows:

(f) $$c_{2e-i} = (-1)^i 2^{-e}(l_0 + l_1 + \dots + l_i) \text{ for } i \in [0, e],$$

(g) $$c_i = (-1)^{i+1} 2^{-e}(l_0 + l_1 + \dots + l_i) \text{ for } i \in [0, e-1].$$

When z_i, ξ_i are replaced by the vectors with the same name in 1.10, the quantities c_i become the quantities c_i^0. (Here $i \in [0, 2e]$.) We show that

(h) $$c_i = c_i^0 \text{ for } i \in [0, 2e].$$

By the analogue of (b) we have $2(-1)^{p_2} = c_{2e}^0 \gamma^0_{2p_1-p_2}$. By results in 1.10 we have $\gamma^0_{2p_1-p_2} = 2^{e+1}$. Hence $c_{2e}^0 = (-1)^{p_2} 2^{-e}$. Using this and the analogues of 1.13(e), 1.13(f), we see that c_i^0 are given by the same formulas as c_i in (e) and (f). This proves (h).

1.15. Let $C = \sum_{s \ge 0} \gamma_{2p_1-p_2+s} T^s$, $C^0 = \sum_{s \ge 0} \gamma^0_{2p_1-p_2+s} T^s$. If $u = 2p_1 - p_2 + t, t \ge 0$, then for any j that contributes to the left hand side of 1.13(b) we have $u - j \ge -p_2$ (indeed, $u - j \ge 2p_1 - p_2 - 2p_2 \ge -p_2$) hence we can assume that in the left hand side of 1.13(b) we have $u - j \ge 2p_1 - p_2$. Muliplying both sides of 1.13(b) by T^t and summing over all $t \ge 0$ we thus obtain

$$\sum_{t \ge 0} \sum_{j \in [0, 2p_2]; t-j \ge 0} m_j \gamma_{2p_1-p_2+t-j} T^t = \sum_{t \ge 0} \sum_{i \in [0,2e], j \in [0, 2p_2]} c_i m_j \alpha_{2p_1-p_2+t-i-j} T^t.$$

The left hand side equals

$$(\sum_{j \in [0, 2p_2]} m_j T^j)(\sum_{t' \ge 0} \gamma_{2p_1-p_2+t'} T^{t'}) = (1-T)^{2p_2} C.$$

Thus

$$C = (1-T)^{-2p_2} (\sum_{t \ge 0} \sum_{i \in [0,2e], j \in [0, 2p_2]} c_i m_j \alpha_{2p_1-p_2+t-i-j} T^t).$$

Similarly we have

$$C^0 = (1-T)^{-2p_2} (\sum_{t \ge 0} \sum_{i \in [0,2e], j \in [0, 2p_2]} c_i^0 m_j \alpha^0_{2p_1-p_2+t-i-j} T^t).$$

By 1.14(h) we have $c_i = c_i^0$. By 1.11(a1) we have $\alpha_{2p_1-p_2+t-i-j} = \alpha^0_{2p_1-p_2+t-i-j}$ for any i, j, t. It follows that $C = C^0$. Hence

(a) $$\gamma_{2p_1-p_2+s} = \gamma^0_{2p_1-p_2+s}$$

for any $s \ge 0$. We set $C' = \sum_{t \ge 0} \gamma_{-p_2-1-t} T^t$, $C'^0 = \sum_{t \ge 0} \gamma^0_{-p_2-1-t} T^t$. If $u = p_2 - 1 - t, t \ge 0$, then for any j that contributes to the left hand side of 1.13(b) we have $u - j \le 2p_1 - p_2 - 1$ (indeed $u - j \le p_2 - 1 - j \le p_2 - 1 \le 2p_1 - p_2 - 1$) hence we can assume that in the left hand side of 1.13(b) we have $u - j \le -p_2 - 1$. With the substitution $j \mapsto 2p_2 - j$ the previous inequality becomes $j - t \le 0$ and the left hand side of 1.13(b) becomes

$$\sum_{j \in [0, 2p_2]} m_j \gamma_{u-2p_2+j} = \sum_{j \in [0, 2p_2]} m_j \gamma_{-p_2-1+j-t}.$$

Muliplying both sides of 1.13(b) by T^t and summing over all $t \ge 0$ we thus obtain

$$\sum_{t \ge 0, j \ge 0; t-j \ge 0} m_j \gamma_{-p_2-1+j-t} T^t = \sum_{t \ge 0} \sum_{i \in [0,2e], j \in [0, 2p_2]} c_i m_j \alpha_{p_2-1-t-i-j} T^t.$$

The left hand side equals

$$(\sum_{j\in[0,2p_2]} m_j T^j)(\sum_{t'\geq 0}\gamma_{-p_2-1-t'}T^{t'}) = (1-T)^{2p_2}C'.$$

Thus

$$C' = (1-T)^{-2p_2}(\sum_{t\geq 0}\sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \alpha_{p_2-1-t-i-j}T^t).$$

Similarly we have

$$C'^0 = (1-T)^{-2p_2}(\sum_{t\geq 0}\sum_{i\in[0,2e],j\in[0,2p_2]} c_i^0 m_j \alpha^0_{p_2-1-t-i-j}T^t).$$

By 1.14(h) we have $c_i = c_i^0$. By 1.11(a1) we have $\alpha_{p_2-1-t-i-j} = \alpha^0_{p_2-1-t-i-j}$ for any i,j,t. It follows that $C' = C'^0$. Hence

(b) $$\gamma_{-p_2-1-t} = \gamma^0_{-p_2-1-t}$$

for any $t \geq 0$. Clearly (a) and (b) imply 1.11(a2).

1.16. We set $B = \sum_{s\geq 0}\beta_{p_2+s}T^s$, $B^0 = \sum_{s\geq 0}\beta^0_{p_2+s}T^s$. Let $t \geq 1$. Taking $(,\xi_{p_2+t})$ with 1.13(a) we obtain

(a) $$\sum_{j\in[0,2p_2]} m_j\beta_{p_2+t-j} = \sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \gamma_{i+j-p_2-t}.$$

For any j that contributes to the left hand side of (a) we have $p_2+t-j \geq -p_2+1$ (indeed, $p_2+t-j \geq p_2+1-2p_2 = -p_2+1$) hence we can assume that in the left hand side of (a) we have $p_2+t-j \geq p_2$, that is, $t \geq j$. Multiplying both sides of (a) by T^t and summing over all $t \geq 1$ we thus obtain

$$\sum_{t\geq 1}\sum_{j\in[0,2p_2];t\geq j} m_j\beta_{p_2+t-j}T^t = \sum_{t\geq 1}\sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \gamma_{i+j-p_2-t}T^t.$$

The left hand side equals

$$-(-1)^{p_2} + (\sum_{j\in[0,2p_2]} m_j T^j)(\sum_{t'\geq 0}\beta_{p_2+t'}T^{t'}) = -(-1)^{p_2} + (1-T)^{2p_2}B.$$

Thus

$$B = (1-T)^{-2p_2}((-1)^{p_2} + \sum_{t\geq 1}\sum_{i\in[0,2e],j\in[0,2p_2]} c_i m_j \gamma_{i+j-p_2-t}T^t).$$

Similarly we have

$$B^0 = (1-T)^{-2p_2}((-1)^{p_2} + \sum_{t\geq 1}\sum_{i\in[0,2e],j\in[0,2p_2]} c_i^0 m_j \gamma^0_{i+j-p_2-t}T^t).$$

By 1.14(h) we have $c_i = c_i^0$. By 1.11(a2) we have $\gamma_{i+j-p_2-t} = \gamma^0_{i+j-p_2-t}$ for any i,j,t. It follows that $B = B^0$. Hence $\beta_{p_2+s} = \beta^0_{p_2+s}$ for any $s \geq 0$. This clearly implies 1.11(a3).

1.17. In the setup of 1.1 we show that 1.1(a) holds by induction on $\mathbf{n}$. If $\mathbf{n}=0$ we have $V=0$ and $a_i=b_i=c_i=p_i=0$ for all i. We take $g=0$ and (L^t) to be the empty set of lines. We obtain an element of $\tilde{\mathcal{C}}^V_{a_*,b_*}$. Now assume that $\mathbf{n}>0$.

Assume first that either $a_1\ge 1, b_1\ge 1$ or that $\epsilon=-1$. We can find a direct sum decomposition $V=V'\oplus V''$ such that $(V',V'')=0$ and $\dim V'=a_1+b_1=2p_1$. Let a'_* be the sequence $a_1,0,0,\dots$; let b'_* be the sequence $b_1,0,0,\dots$; let a''_* be the sequence $a_2,a_3,\dots$; let b''_* be the sequence $b_2,b_3,\dots$. By the induction hypothesis we have $\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}\ne 0$. By 1.3 we have $\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}\ne\emptyset$. Let $(g',L^1)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ and let $(g'',L^2,L^3,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$. Clearly, $(g'\oplus g'',L^1,L^2,\dots)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$ hence 1.1(a) holds in this case. Thus we can assume that $\epsilon=1$ and either

(i) $a_1>0$ and $b_1=0$ or
(ii) $a_1=0$ and $b_1>0$.

Assume that we are in case (i). We have $b_1=b_2=\dots=0$ and g is unipotent. If $a_2=0$, then 1.1(a) holds by 1.6 with $p=(a_1-1)/2$. If $a_2>0$ we can find a direct sum decomposition $V=V'\oplus V''$ such that $(V',V'')=0$ and $\dim V'=a_1+a_2$. Let a'_* be the sequence $a_1,a_2,0,\dots$; let a''_* be the sequence $a_3,a_4,\dots$; let $b'_*=b''_*$ be the sequence $0,0,\dots$. By the induction hypothesis we have $\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}\ne\emptyset$. By 1.10 we have $\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}\ne\emptyset$. Let $(g',L^1,L^2)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ and let $(g'',L^3,L^4,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$. Clearly $(g'\oplus g'',L^1,L^2,\dots)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$ hence 1.1(a) holds in this case. This completes the proof in case (i).

Assume now that we are in case (ii) so that $-g$ is unipotent. It is easy to check that $\tilde{\mathcal{C}}^V_{g;a_*,b_*}=\tilde{\mathcal{C}}^V_{-g;b_*,a_*}$ and the last set is nonempty by the earlier part of the argument. Hence $\tilde{\mathcal{C}}^V_{g;a_*,b_*}\ne\emptyset$. This completes the inductive proof of 1.1(a).

In the following result (which is needed in the proof of 1.1(b),(c)) we preserve the setup of 1.1.

Proposition 1.18. *Let $(g,L^1,L^2,\dots,L^{\sigma+\kappa})\in\tilde{\mathcal{C}}^V_{a_*,b_*}$. Let f_r be as in 1.5. There exist vectors $z^t\in L^t-\{0\}$ for $t\in[1,\sigma+\kappa]$ such that (i), (ii), (iii) below hold for any $i,j\in\mathbf{Z}$.*

(i) *Assume that either $t\in[1,\sigma],\epsilon=-1$ or $t\in[1,k]$. Then $(z^t_i,z^t_j)=0$ if $|i-j|<p_t$, $(z^t_i,z^t_j)=x_s$ if $j-i=p_t+s, s\ge 0$ (x_s as in 1.5 with $p=p_t$); $(z^t_i,z^{t'}_j)=0$ if $t'\in[1,\sigma+\kappa], t'\ne t$.*

(ii) *Assume that $\{t,t+1\}\subset[k+1,\sigma+\kappa], t=k+1 \mod 2$ and $\epsilon=1$. We set $\delta=1$ if $a_t>0$, $\delta=-1$ if $b_t>0$. Then*

$$(z^t_i,z^t_j)=(-1)^{p_t}\delta^{i-j}f_{p_t}(i-j),$$
$$(z^{t+1}_i,z^{t+1}_j)=\delta^{i-j}\sum_{r\in[p_{t+1},p_t]}(-1)^r 2^{2r-2p_{t+1}}f_r(i-j),$$
$$(z^t_i,z^{t+1}_j)=\delta^{i-j}2^{p_t-p_{t+1}+1}\binom{i-j+p_{t+1}}{2p_t},$$
$$(z^t_i,z^{t'}_j)=0 \text{ if } t'\in[1,\sigma+\kappa], t'\notin\{t,t+1\}.$$

(iii) *Assume that $\epsilon=1,\kappa=1,t=\sigma+1$. We set $\delta=1$ if $a_t>0$, $\delta=-1$ if $b_t>0$. (We have $p_t=0$.) Then*

$$(z^t_i,z^t_j)=2\delta^{i-j},$$
$$(z^t_i,z^{t'}_j)=0 \text{ if } t'\in[1,\sigma].$$

We argue by induction on $\mathbf{n}$. When $\mathbf{n}=0$ the result is obvious. Now assume that $\mathbf{n}\ge 1$.

Case 1. Assume first that either $a_1\ge 1, b_1\ge 1$ or that $\epsilon=-1$. We have $a_1+b_1=2p_1$. Let $V'=\bigoplus_{i\in[0,2p_1-1]}L^1_i\subset V$. We show that

(a) $$gV'=V'.$$

It is enough to show that $gL^1_{2p_1-1}\subset V'$, that is, $g^{2p_1}L^1_0\subset V'$. Since $g^iL^1_0\subset V'$ for $i\in[0,2p_1-1]$ and $a_1+b_1=2p_1$, it is enough to show that $(g-1)^{a_1}(g+1)^{b_1}L^1_0=0$. It is also enough to show that $(g-1)^{a_1}(g+1)^{b_1}=0$ on V. But this follows from the fact that $g\in\mathcal{C}^V_{a_*,b_*}$.

Now let $V''=\bigoplus_{t\in[2,\sigma+\kappa],i\in[0,2p_t-1]}L^t_i\subset V$. We show that

(b) $$V''=V'^{\perp}\text{ (the perpendicular to }V')\text{ and }V=V'\oplus V'^{\perp}.$$

For $t\in[2,\sigma]$, $i\in[0,2p_1-1]$ we have $(L^1_i,L^t_{p_t})=0$; thus $L^t_{p_t}\in V'^{\perp}$. Since $V'^{\perp}$ is g-stable it follows that $L^t_i\subset V'^{\perp}$ for $t\in[2,\sigma], i\in\mathbf{Z}$. If $\kappa=1$ we have $(L^1_i,L^{\sigma+1}_0)=0$ for $i\in[0,2p_1-1]$; thus $L^{\sigma+1}_0\subset V'^{\perp}$. Hence $V''\subset V'^{\perp}$. But these two vector spaces have the same dimension so that $V''=V'^{\perp}$. Since $V=V'\oplus V''$ it follows that $V=V'\oplus V'^{\perp}$. This proves (b).

Let $g'=g|_{V'}$, $g''=g_{V''}$. We show:

(c) *g' restricted to the generalized 1-eigenspace of g' is unipotent with a single Jordan block of size a_1; $-g'$ restricted to the generalized (-1)-eigenspace of g' is unipotent with a single Jordan block of size b_1; g'' restricted to the generalized 1-eigenspace of g'' is unipotent with Jordan blocks of sizes given by the nonzero numbers in $a_2,a_3,\dots$; $-g''$ restricted to the generalized (-1)-eigenspace of g'' is unipotent with Jordan blocks of sizes given by the nonzero numbers in $b_2,b_3,\dots$.*

As we have seen earlier we have $(g-1)^{a_1}(g+1)^{b_1}=0$ on V' (even on V). Also $g'\in GL(V')$ is regular in the sense of Steinberg and $\dim V'=a_1+b_1$. This implies (c).

Let a'_* be the sequence $a_1,0,0,\dots$; let b'_* be the sequence $b_1,0,0,\dots$; let a''_* be the sequence $a_2,a_3,\dots$; let b''_* be the sequence $b_2,b_3,\dots$.

Now the proposition holds when $(g,L^1,L^2,\dots)$ is replaced by $(g'',L^2,L^3,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$ (by the induction hypothesis) or by $(g',L^1)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ (we choose any $z^1\in L^1-\{0\}$ such that $(z^1_i,z^1_j)=1$ for $|i-j|=p_1$ and we apply 1.4). Hence the proposition holds for $(g,L^1,L^2,\dots)$ (since $(V',V'')=0$).

Case 2. Next we assume that $k=0$, $\epsilon=1$, $a_1>0,a_2>0$. Then $b_1=b_2=\dots=0$. We have $a_1=2p_1+1,a_2=2p_2-1$. Let $V'=\bigoplus_{t\in[1,2],i\in[0,2p_t-1]}L^t_i\subset V$. We show that

(d) $$gV'=V'.$$

Let $N=g-1$. Then $V=\bigoplus_{t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]}N^iL^t_0$ is a direct sum decomposition into lines and $p_i=p'_i$ if $i\in[1,2]$. Now $N^{2p_2-1}(V)$ contains the lines:

$(*)$ $$N^{2p_2-1+i}L^1_0(i=0,1,\dots,2p_1-2p_2)\quad\text{and}\quad N^{2p_2-1}L^2_0$$

(whose number is $2p_1-2p_2+2$); moreover, since N has Jordan blocks of sizes $a_1=2p_1+1,a_2=2p_2-1$ and others of size $<a_2$, we see that $\dim N^{2p_2-1}(V)=2p_1-2p_2+2$ so that $N^{2p_2-1}(V)$ is equal to the subspace spanned by $(*)$ and $N^{2p_2-1}(V)\subset V'$. Now V' is the subspace of V spanned by the lines $N^iL^t_0$ with

$t \in [1,2], i \in [0, 2p_t - 1]$. It is enough to show that $NV' \subset V'$ or that $N^{2p_t} L_0^t \subset V'$ for $t = 1, 2$. But for $t = 1, 2$ we have $N^{2p_t} L_0^t \subset N^{2p_2 - 1} V \subset V'$ since $2p_t - 2p_2 + 1 \geq 0$. This proves (d).

Let $V'' = \bigoplus_{t \in [3, \sigma+\kappa], i \in [0, 2p'_t - 1]} L_i^t \subset V$. We show that

(e) $V'' = V'^{\perp}$ (the perpendicular to V') and $V = V' \oplus V'^{\perp}$.

For $t \in [1,2], r \in [3, \sigma]$, $i \in [0, 2p_t - 1]$ we have $(L_i^t, L_{p_r}^r) = 0$. Thus $L_{p_r}^r \subset V'^{\perp}$ for $r \in [3, \sigma]$. Since $V'^{\perp}$ is g-stable it follows that $L_i^r \subset V'^{\perp}$ for $r \in [3, \sigma], i \in \mathbf{Z}$. If $\kappa = 1$ we have $(L_i^t, L_0^{\sigma+1}) = 0$ for $i \in [0, 2p_t - 1], t \in [1, 2]$. Thus $L_0^{\sigma+1} \in V'^{\perp}$. Hence $V'' \subset V'^{\perp}$. But these two vector spaces have the same dimension so that $V'' = V'^{\perp}$. Since $V = V' \oplus V''$ it follows that $V = V' \oplus V'^{\perp}$. This proves (e).

Let $g' = g|_{V'}$, $g'' = g_{V''}$. We show:

(f) *g' is unipotent with exactly two Jordan blocks of size a_1, a_2. Moreover, g'' is unipotent with Jordan blocks of sizes given by the nonzero numbers in $a_3, a_4, \dots$.*

Since V' is the direct sum of the lines $N^i L_0^t$, $t \in [1, 2], i \in [0, 2p_t - 1]$ and V' is N-stable, we see that the kernel of $N : V' \to V'$ has dimension ≤ 2. Hence $N : V' \to V'$ has either a single Jordan block of size $2p_1 + 2p_2 = a_1 + a_2$ or two Jordan blocks of sizes $a'_1 \geq a'_2$ where $a'_1 + a'_2 = a_1 + a_2$. The first alternative does not occur since the Jordan blocks of $N : V' \to V'$ have sizes $\leq a_1$ (by (e)). Thus the second alternative holds. Since a'_1, a'_2 must form a subsequence of $a_1 > a_2 > a_3 > \dots$ and $a'_1 + a'_2 = a_1 + a_2$ it follows that $a'_1 = a_1$, $a'_2 = a_2$. This implies (f).

Let a'_* be the sequence $a_1, a_2, 0, \dots$; let a''_* be the sequence $a_3, a_4, \dots$; let $b'_* = b''_*$ be the sequence $0, 0, \dots$. Now the proposition holds when $(g, L^1, L^2, \dots)$ is replaced by $(g'', L^3, L^4, \dots) \in \tilde{\mathcal{C}}^{V''}_{a''_*, b''_*}$ (by the induction hypothesis) or by $(g', L^1, L^2) \in \tilde{\mathcal{C}}^{V'}_{a'_*, b'_*}$ (we choose any $z^1 \in L^1 - \{0\}$ such that $(z_i^1, z_j^1) = (-1)^{p_1}$ for $|i - j| = p_1$ and any $z^2 \in L^2 - \{0\}$ such that $(z_i^2, z_j^2) = (-1)^{p_2}$ for $|i - j| = p_2$ and we apply 1.11 by possibly changing z^2 to $-z^2$). Hence the proposition holds for $(g, L^1, L^2, \dots)$ (since $(V', V'') = 0$).

Case 3. Next we assume that $k = 0$, $\epsilon = 1$, $a_1 > 0, a_2 = 0$. Then $b_1 = b_2 = \dots = 0$ and $\sigma = 1, \kappa = 1$. We have $a_1 = 2p_1 + 1, p_2 = 0, p'_2 = 1/2$. We choose any $z^1 \in L^1 - \{0\}$ such that $(z_i^1, z_j^1) = (-1)^{p_1}$ for $|i - j| = p_1$ and any $z^2 \in L^2 - \{0\}$ such that $(z_i^2, z_j^2) = 2$ for $|i - j| = p_2$ and we apply 1.9 by possibly changing z^2 to $-z^2$. We see that the proposition holds for $(g, L^1, L^2, \dots)$.

Case 4. Finally assume that $k = 0$, $\epsilon = 1$, $b_1 > 0$. Then $(-g, L^1, L^2, \dots) \in \tilde{\mathcal{C}}^V_{b_*, a_*}$ is as in Case 2 or 3. Let (z^t) be the corresponding sequence of vectors in V. This sequence is the desired sequence for $(g, L^1, L^2, \dots)$. This completes the proof.

1.19. In the setup of 1.1, we show that 1.1(b) holds. We must show that

(a) any two elements $(g, L^1, L^2, \dots, L^{\sigma+\kappa})$, $(g', L'^1, L'^2, \dots, L'^{\sigma+\kappa})$ of $\tilde{\mathcal{C}}^V_{a_*, b_*}$ are in the same $Is(V)$-orbit.

Since $Is(V)$ acts transitively on $\mathcal{C}^V_{a_*, b_*}$ we can assume that $g = g'$. Let $z^t \in L^t$ ($t \in [1, \sigma + \kappa]$) be as in 1.18. Let $z'^t \in L'^t$ ($t \in [1, \sigma + \kappa]$) be the analogous vectors for $(g, L'^1, L'^2, \dots)$ instead of $(g, L^1, L^2, \dots)$. By 1.18 we have

(b) $$(z_i^t, z_j^{t'}) = ({z'}_i^t, {z'}_j^{t'})$$

for any $i,j\in\mathbf{Z}$ and any $t,t'\in[1,\sigma+\kappa]$. Since $\{z_i^t;t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]\}$ and $\{z'^t_i;t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]\}$ are bases of V (see 1.0(b)) we see that there is a unique $\gamma\in GL(V)$ such that $\gamma(z_i^t)=z'^t_i$ for any $t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]$. From (b) we see that $\gamma\in Is(V)$. We show that

(c) $$\gamma(z_{i+1}^t)=z'^t_{i+1}\text{ for any }t\in[1,\sigma+\kappa],i\in[0,2p'_t-1].$$

When $i+1\in[0,2p'_t-1]$ this follows from the definition of γ. Thus we can assume that $i=2p'_t-1$ and we must show that $\gamma(z_{2p'_t}^t)=z'^t_{2p'_t}$ for any $t\in[1,\sigma+\kappa]$. It is enough to show that $(\gamma(z_{2p'_t}^t),z'^{t'}_j)=(z'^t_{2p'_t},z'^{t'}_j)$ for any $t'\in[1,\sigma+\kappa],j\in[0,2p'_t-1]$ (we use again that $\{z'^t_i;t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]\}$ is a basis of V). We have $(\gamma(z_{2p'_t}^t),z'^{t'}_j)=(\gamma(z_{2p'_t}^t),\gamma(z_j^{t'}))=(z_{2p'_t}^t,z_j^{t'})$ and this is equal to $(z'^t_{2p'_t},z'^{t'}_j)$ by (b). Thus (c) holds. From (c) we see that $\gamma(g(z_i^t))=g(\gamma(z_i^t))$ for any $t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]$. It follows that $\gamma g=g\gamma$. From the definition it is clear that $\gamma(L^t)=L'^t$ for $t\in[1,\sigma+\kappa]$. Thus (a) holds (with $g'=g$). This proves 1.1(b).

1.20. In the setup of 1.1, we show that 1.1(c) holds. Let $(g,L^1,L^2,\dots,L^{\sigma+\kappa})\in\tilde{\mathcal{C}}^V_{a_*,b_*}$ and let I be the set of all $\gamma\in Is(V)$ such that $\gamma g\gamma^{-1}=g$, $\gamma(L^t)=L^t$ for $t\in[1,\sigma+\kappa]$. Let $z^t\in L^t(t\in[1,\sigma+\kappa])$ be as in 1.18. Let $\gamma\in I$. If $t\in[1,\sigma+\kappa]$, we have $\gamma(z^t)=\omega_t^\gamma z^t$ where $\omega_t^\gamma=\pm1$. If $\{t,t+1\}\subset[k+1,\sigma+\kappa],t=k+1\mod 2$ and $\epsilon=1$, we have $\omega_t^\gamma=\omega_{t+1}^\gamma$. Indeed, for some $\iota\in\{1,-1\}$ we have

$$\begin{aligned}\iota 2^{p_t-p_{t+1}-1}&=(z_{-1}^t,z_{p_{t+1}}^{t+1})=(\gamma(z_{-1}^t),\gamma(z_{p_{t+1}}^{t+1}))\\&=\omega_t^\gamma\omega_{t+1}^\gamma(z_{-1}^t,z_{p_{t+1}}^{t+1})=\omega_t^\gamma\omega_{t+1}^\gamma\iota 2^{p_t-p_{t+1}-1}\end{aligned}$$

hence $\omega_t^\gamma\omega_{t+1}^\gamma=1$ and our claim follows. Thus, $\gamma\mapsto(\omega_t^\gamma)$ is a homomorphism $\psi:I\to\mathcal{I}$ (notation of 1.0). Assume that γ is in the kernel of ψ. Then γ restricts to the identity map $L^t\to L^t$ for $t\in[1,\sigma+\kappa]$. Since γ commutes with g it follows that γ restricts to the identity map on each of the lines g^iL^t ($t\in[1,\sigma+\kappa]$, $i\in\mathbf{Z}$). Since these lines generate V (see 1.0(b)) we see that $\gamma=1$. Thus ψ is injective. Now let $(\omega_t)\in\mathcal{I}$. We define $\gamma\in GL(V)$ by $\gamma(z_i^t)=\omega_tz_i^t$ for $t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]$. From the definitions we see that

(a) $$(\omega_tz_i^t,\omega_{t'}z_j^{t'})=(z_i^t,z_j^{t'})$$

for any $i,j\in\mathbf{Z}$ and any $t,t'\in[1,\sigma+\kappa]$.

From (a) we see that $\gamma\in Is(V)$. We show that

(b) $$\gamma(z_{i+1}^t)=\omega_tz_{i+1}^t\text{ for any }t\in[1,\sigma+\kappa],i\in[0,2p'_t-1].$$

(This is similar to 1.19(c).) When $i+1\in[0,2p'_t-1]$ this follows from the definition of γ. Thus we can assume that $i=2p'_t-1$ and we must show that $\gamma(z_{2p'_t}^t)=\omega_tz_{2p'_t}^t$ for any $t\in[1,\sigma+\kappa]$. It is enough to show that $(\gamma(z_{2p'_t}^t),\omega_{t'}z_j^{t'})=(\omega_tz_{2p'_t}^t,\omega_{t'}z_j^{t'})$ for any $t'\in[1,\sigma+\kappa],j\in[0,2p'_t-1]$ (we use again that $\{z_i^t;t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]\}$ is a basis of V). We have

$$(\gamma(z_{2p'_t}^t),\omega_{t'}z_j^{t'})=(\gamma(z_{2p'_t}^t),\gamma(z_j^{t'}))=(z_{2p'_t}^t,z_j^{t'})$$

and this is equal to $(z_{2p'_t}^t,z_j^{t'})$ by (a). Thus (b) holds.

From (b) we see that $\gamma(g(z_i^t))=g(\gamma(z_i^t))$ for any $t\in[1,\sigma+\kappa],i\in[0,2p'_t-1]$. It follows that $\gamma g=g\gamma$. From the definition it is clear that $\gamma(L^t)=L^t$ for $t\in[1,\sigma+\kappa]$. Thus $\gamma\in I$. We see that ψ is surjective hence an isomorphism. This proves 1.1(c).

1.21. In the setup of 1.1, assume that $\mathbf{n}$ is even ≥ 2 and $\epsilon = 1$. Let Ω be the set of $Is(V)^0$-orbits on the set of $(\mathbf{n}/2)$-dimensional subspaces of V which are isotropic for $(,)$; note that $|\Omega| = 2$. If $(g, L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$, then the $(\mathbf{n}/2)$-dimensional subspace $\bigoplus_{t\in[1,\sigma], i\in[p_t,2p_t-1]} L^t_i$ of V is isotropic for $(,)$. Hence we have a partition

$$\tilde{\mathcal{C}}^V_{a_*,b_*} = \bigsqcup_{\mathcal{O}\in\Omega} \tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$$

where for $\mathcal{O} \in \Omega$, $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$ is the set of all $(g, L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$ such that $\bigoplus_{t\in[1,\sigma], i\in[p_t,2p_t-1]} L^t_i \in \mathcal{O}$. Now

(a) *the action 1.0(c) of $Is(V)$ restricts for any $\mathcal{O} \in \Omega$ to an action of $Is(V)^0$ on $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$;*

(b) *if $\gamma \in Is(V) - Is(V)^0$ then the action of γ on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ maps $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$ onto $\tilde{\mathcal{C}}^V_{a_*,b_*;\Omega-\mathcal{O}}$.*

For any $\mathcal{O} \in \Omega$ we have the following variant of Theorem 1.1:

(c) $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}} \neq \emptyset$;

(d) *the action (a) of $Is(V)^0$ on $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$ is transitive;*

(e) *the isotropy group in $Is(V)^0$ at any point of $\tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$ is canonically isomorphic to $\mathcal{I}$.*

Now (c) follows immediately from (b) and 1.1(a). We prove (d). Let

$$(g, L^1, L^2, \dots) \in \tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}, \quad (g', L'^1, L'^2, \dots) \in \tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}.$$

By 1.1(b) we can find $\gamma \in Is(V)$ which carries $(g, L^1, L^2, \dots)$ to $(g', L'^1, L'^2, \dots)$. By (b) we have automatically $\gamma \in Is(V)^0$. Hence (d) holds.

To prove (e) it is enough to show that if γ is in the isotropy group in $Is(V)$ at $(g, L^1, L^2, \dots)$, then $\det(\gamma) = 1$. Let $(\omega_t) = \psi(\gamma)$ be as in 1.20. From the proof in 1.20 we see that $\det(\gamma) = \prod_{t\in[1,\sigma]} \omega_t^{2p_t}$. Since $\omega_t = \pm 1$ we see that $\det(\gamma) = 1$, as required.

We now show:

(f) *If $a_1 > 0$, $b_1 > 0$ and $(g, L^1, L^2, \dots) \in \tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$, then there exists $\gamma \in I'$ (the isotropy group in $Is(V)^0$ at $(g, L^1, L^2, \dots)$) such that for $\delta \in \{1, -1\}$, the restriction of γ to the generalized δ-eigenspace of g has determinant -1.*

Define (ω_t) by $\omega_1 = -1, \omega_t = 1$ for $t \in [2, \sigma]$. In our case we have $k \geq 1$ hence $(\omega_t) \in \mathcal{I}$. Let $V' = \sum_{i\in\mathbf{Z}} L^1_i$, $V'' = \sum_{t\in[2,\sigma], i\in\mathbf{Z}} L^t_i$. By 1.18, $V = V' \oplus V''$ (orthogonal direct sum). Define $\gamma \in I'$ by $\psi(\gamma) = (\omega_t)$ (notation of 1.20). Then γ acts as identity on V'' and as -1 times the identity on V'. It is enough to prove that the restriction of γ to the generalized δ-eigenspace of $g_{V'}$ has determinant -1 or that this generalized δ-eigenspace has odd dimension. But this dimension is a_1 (if $\delta = 1$) and b_1 (if $\delta = -1$) and a_1, b_1 are odd.

1.22. In the setup of 1.1, assume that $\mathbf{n}$ is odd (hence $\epsilon = 1$) and that $\mathcal{C}^V_{a_*,b_*} \subset Is(V)^0$. We have the following variant of Theorem 1.1:

(a) *the restriction of the action 1.0(c) to $Is(V)^0$ is transitive on $\tilde{\mathcal{C}}^V_{a_*,b_*}$;*

(b) *the isotropy group in $Is(V)^0$ at any point of $\tilde{\mathcal{C}}^V_{a_*,b_*}$ is canonically isomorphic to a subgroup of $\mathcal{I}$ of index 2.*

Note that if $\gamma \in Is(V) - Is(V)^0$, then $-\gamma \in Is(V)^0$. Moreover, $-1 \in Is(V)$ acts trivially on $\tilde{\mathcal{C}}^V_{a_*,b_*}$; hence (a) follows from 1.1(b). Now let γ be in the isotropy group in $Is(V)$ at $(g, L^1, L^2, \dots)$ and let $(\omega_t) = \psi(\gamma)$ be as in 1.20. We have

$$\det(\gamma) = \omega_{\sigma+1} \prod_{t \in [1,\sigma]} \omega_t^{2p_t} = \omega_{\sigma+1}.$$

Thus the condition that $\gamma \in Is(V)^0$ is equivalent to the condition that $\omega_{\sigma+1} = 1$. This proves (b).

We now show:

(c) *If $a_1 > 0$, $b_1 > 0$ and $(g, L^1, L^2, \dots) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$ with $g \in Is(V)^0$ then there exists $\gamma \in I'$ (the isotropy group in $Is(V)^0$ at $(g, L^1, L^2, \dots)$) such that for $\delta \in \{1, -1\}$, the restriction of γ to the generalized δ-eigenspace of g has determinant -1.*

Define (ω_t) by $\omega_1 = -1, \omega_t = 1$ for $t \in [2, \sigma + 1]$. In our case we have $k \geq 1$ hence $(\omega_t) \in \mathcal{I}$. Let $V' = \sum_{i \in \mathbf{Z}} L^1_i$, $V'' = \sum_{t \in [2,\sigma+1], i \in \mathbf{Z}} L^t_i$. By 1.18, we have $V = V' \oplus V''$ (orthogonal direct sum). Define $\gamma \in I'$ by $\psi(\gamma) = (\omega_t)$ (notation of 1.20). Then γ acts as identity on V'' and as -1 times the identity on V'. It is enough to prove that the restriction of γ to the generalized δ-eigenspace of $g_{V'}$ has determinant -1 or that this generalized δ-eigenspace has odd dimension. But this dimension is a_1 (if $\delta = 1$) and b_1 (if $\delta = -1$) and a_1, b_1 are odd.

1.23. In the setup of 1.1, assume that $\mathbf{n} \geq 3$ and $\epsilon = 1$. When $\mathbf{n}$ is odd we assume that $\mathcal{C}^V_{a_*,b_*} \subset Is(V)^0$ and let $\pi : \Gamma \to Is(V)^0$ be a surjective morphism of algebraic groups with kernel of order 2 such that Γ is connected and simply connected. When $\mathbf{n}$ is even let $\pi : \Gamma \to Is(V)$ be a surjective morphism of algebraic groups with kernel of order 2 such that $\pi^{-1}(Is(V)^0)$ is connected and simply connected.

Let $\mathbf{c}$ be a γ^0-conjugacy class contained in $\pi^{-1}(\mathcal{C}^V_{a_*,b_*})$. (If $a_1 b_1 > 0$ we have $\mathbf{c} = \pi^{-1}(\mathcal{C}^V_{a_*,b_*})$; if $a_1 b_1 = 0$ there are two choices for $\mathbf{c}$.) For $\mathbf{n}$ odd let X be the set of all $(\tilde{g}, L^1, L^2, \dots, L^{\sigma+1})$ where $\tilde{g} \in \mathbf{c}$ and $(\pi(\tilde{g}), L^1, L^2, \dots, L^{\sigma+1}) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$. For $\mathbf{n}$ even let X be the set of all $(\tilde{g}, L^1, L^2, \dots, L^\sigma)$ where $\tilde{g} \in \mathbf{c}$ and $(\pi(\tilde{g}), L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{a_*,b_*;\mathcal{O}}$. Note that $X \neq \emptyset$. Now γ^0 acts on X by

$$\gamma : (\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa}) \mapsto (\gamma\tilde{g}\gamma^{-1}, \pi(\gamma)L^1, \pi(\gamma)L^2, \dots, \pi(\gamma)L^{\sigma+\kappa}).$$

We show:

(a) *This action is transitive.*

If $a_1 b_1 = 0$, then (a) follows trivially from 1.21(d), 1.22(a). Assume now that $a_1 b_1 > 0$. Let $(\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa}) \in X$ and let c be the nontrivial element in $\ker \pi$. Let $g = \pi(\tilde{g})$. We define γ in terms of $(g, L^1, L^2, \dots, L^{\sigma+\kappa})$ as in 1.21(f) or 1.22(c). Let $\tilde{\gamma} \in \pi^{-1}(\gamma)$. Since $\gamma g \gamma^{-1} = g$ we see that either $\tilde{\gamma}\tilde{g}\tilde{\gamma}^{-1} = \tilde{g}$ or $\tilde{\gamma}\tilde{g}\tilde{\gamma}^{-1} = c\tilde{g}$. In the first case $\tilde{\gamma}$ is in the centralizer in γ^0 of $\tilde{g}_s$ (the semisimple part of $\tilde{g}$). This centralizer is a connected algebraic group (by a result of Steinberg). Thus its image under π is connected hence it is contained in the connected centralizer of g_s (the semisimple part of g) in $Is(V)^0$. Thus $\gamma = \pi(\tilde{\gamma})$ is contained in the connected centralizer of g_s in $Is(V)^0$. But then the restriction of γ to the 1-eigenspace of g_s would have determinant 1, contradicting the choice of γ. We see that we must have

(b) $\tilde{\gamma}\tilde{g}\tilde{\gamma}^{-1} = c\tilde{g}$.

Using 1.21(d), 1.22(a), we see that any γ^0-orbit on X contains either $(\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa})$ or $(c\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa})$. From (b) and the definition of $\tilde{\gamma}$ we see that the action of $\tilde{g}$ takes $(\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa})$ to $(c\tilde{g}, L^1, L^2, \dots, L^{\sigma+\kappa})$. This shows that (a) holds.

1.24. As in [L1, §3], [L5, §3] we see that 1.23 (resp. 1.1) implies that Theorem 0.3 holds when G is Γ in 1.23 (resp. $G = Is(V)$ with $\mathbf{n} \geq 2$, $\epsilon = -1$).

2. Bilinear forms

2.0. For any subset S of $\mathbf{Z}$ we write $S'' = S \cap (2\mathbf{Z})$, $S' = S \cap (2\mathbf{Z}+1)$.

Let V be a $\mathbf{k}$-vector space of finite dimension n. Let $(,) : V \times V^* \to \mathbf{k}$ be the obvious pairing. Let $G_V = GL(V)$ and let G_V^1 be the set of all vector space isomorphisms $V \xrightarrow{\sim} V^*$. Note that an element of G_V^1 can be viewed as a bilinear form $V \times V \to \mathbf{k}$. For $\gamma \in G_V$ we define $\check{\gamma} \in G_{V^*}$ by $(\gamma(x), \check{\gamma}(\xi)) = (x, \xi)$ for all $x \in V, \xi \in V^*$. For $g \in G_V^1$ we define $\breve{g} \in G_{V^*}^1$ by $(\breve{g}z', gz) = (z, z')$ for any $z \in V, z' \in V^*$. There is a well-defined group structure on $G := G_V \sqcup G_V^1$ denoted by $*$ such that for γ, γ' in G_V and g, g' in G_V^1 we have

$$\gamma * \gamma' = \gamma\gamma' \in G_V; \quad \gamma * g' = \check{\gamma} g' \in G_V^1; \quad g * g' = \breve{g} g' \in G_V; \quad g * \gamma' = g\gamma' \in G_V^1.$$

Now let $g \in G_V^1$. For $i \in \mathbf{Z}$ let g^{*i} be the i-th power of g for the multiplication $*$. In particular, we have $g^{*2} = g * g = \breve{g} g$. For $i \in \mathbf{Z}''$ we have $g^{*i} \in G_V$. For $i \in \mathbf{Z}'$ we have $g^{*i} \in G_V^1$. For any $z \in V$ and $i \in \mathbf{Z}$ we set $z_i = g^{*i} z$; we have $z_i \in V$ if $i \in \mathbf{Z}''$ and $z_i \in V^*$ if $i \in \mathbf{Z}'$. Similarly, for any line L in V and $i \in \mathbf{Z}$ we set $L_i = g^{*i} L$; this is a line in V if $i \in \mathbf{Z}''$ and a line in V^* if $i \in \mathbf{Z}'$.

For any z, z' in V and any $i \in \mathbf{Z}'', j \in \mathbf{Z}', k \in \mathbf{Z}''$, we show:

(a) $$(z_{i+k}, z'_{j+k}) = (z_i, z'_j),$$

(b) $$(z_i, z'_j) = (z'_{-i}, z_{-j}).$$

Indeed, we have

(c) $$\begin{aligned}(z_i, z'_j) &= (z_i, g z'_{j-1}) = (z'_{j-1}, (\breve{g})^{-1} z_i) = (z'_{j-1}, g(\breve{g} g)^{-1} z_i) \\ &= (z'_{j-1}, g z_{i-2}) = (z'_{j-1}, z_{i-1}).\end{aligned}$$

Repeating this we get $(z'_{j-1}, z_{i-1}) = (z_{i-2}, z'_{j-2})$. Combining with (c) we get $(z_i, z'_j) = (z_{i-2}, z'_{j-2})$; hence $(z_i, z'_j) = \phi(i-j)$ where $\phi : \mathbf{Z}' \to \mathbf{k}$; by (c) we have $(z'_i, z_j) = \phi(j-i)$ for $i \in \mathbf{Z}'', j \in \mathbf{Z}'$. In particular, (a) and (b) hold.

Let $a_1 \geq a_2 \geq \dots$, $b_1 \geq b_2 \geq \dots$ be two sequences of integers ≥ 0 in $\mathbf{N}$ such that

if $i \geq 1$, $a_i = a_{i+1}$, then $a_{i+1} = 0$;
if $i \geq 1$, $b_i = b_{i+1}$, then $b_{i+1} = 0$;
if $a_i > 0$, then $a_i \in \mathbf{Z}'$;
if $b_i > 0$, then $b_i \in \mathbf{Z}''$;

$(a_1 + a_2 + \dots) + (b_1 + b_2 + \dots) = n$.

It follows that $a_i = 0$ for large i and $b_i = 0$ for large i. Define $k \geq 0$ by $\{i \geq 1; a_i b_i > 0\} = [1, k]$. We define $p_i \in \mathbf{N}$ for $i \geq 1$ as follows. If $i \in [1, k]$, we have $p_i = (a_i + b_i + 1)/2$. If $i > k$ we define p_i by requiring that for $s = 1, 3, 5, \dots$ we have:

$$(p_{k+s}, p_{k+s+1}) = (b_{k+s}/2, (b_{k+s+1} + 2)/2) \text{ if } b_{k+s} > 0,$$

$$(p_{k+s}, p_{k+s+1}) = ((a_{k+s} + 1)/2, (a_{k+s+1} + 1))/2) \text{ if } a_{k+s} > 0, a_{k+s+1} > 0,$$

$$(p_{k+s}, p_{k+s+1}) = ((a_{k+s} + 1)/2, 0) \text{ if } a_{k+s} > 0, a_{k+s+1} = 0,$$

$$(p_{k+s}, p_{k+s+1}) = (0, 0) \text{ if } a_{k+s} = a_{k+s+1} = 0.$$

We define σ as follows. If $n = 0$ we set $\sigma = 0$. If $n \ge 1$ let σ be the largest i such that $p_i > 0$. We have $p_1 \ge p_2 \ge p_\sigma$ and

$$(2p_1 - 1) + (2p_2 - 1) + \cdots + (2p_\sigma - 1) = n.$$

Let $\mathcal{C}^V_{a_*,b_*}$ be the set of all $g \in G^1_V$ such that $g^{*4} \in G_V$ is unipotent and such that on the generalized 1-eigenspace of g^{*2}, g^{*2} has Jordan blocks of sizes given by the nonzero numbers in $a_1, a_2, \dots$ and on the generalized (-1)-eigenspace of g^{*2}, $-g^{*2}$ has Jordan blocks of sizes given by the nonzero numbers in $b_1, b_2, \dots$. (Note that the union of the sets $\mathcal{C}^V_{a_*,b_*}$ where a_*, b_* as above vary is exactly the set of elements of G^1_V which are distinguished in G in the sense of 0.2.)

For $g \in \mathcal{C}^V_{a_*,b_*}$ let $\tilde{\mathcal{C}}^V_{g;a_*,b_*}$ be the set consisting of all $L^1, L^2, \dots, L^\sigma$ where $L^t (t \in [1, \sigma])$ are lines in V (the upper scripts are not powers) such that for $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ we have:

$$(L^t_i, L^t_j) = 0 \text{ if } i - j \in [-2p_t + 3, 2p_t - 3]',$$
$$(L^t_i, L^t_j) \ne 0 \text{ if } |i - j| = 2p_t - 1 \ (t \in [1, \sigma]),$$
$$(L^r_i, L^t_j) = 0 \text{ if } j - i \in [1 - 2p_r, 4p_t - 2p_r - 3]', 1 \le t < r \le \sigma.$$

Here $L^t_i = g^{*i} L^t$. We then have:

(d) $V = \bigoplus_{t \in [1,\sigma], i \in [0, 2p_t - 2]} L^t_i$.

(See [L5, 4.8(a)].) Let $\tilde{\mathcal{C}}^V_{a_*,b_*}$ be the set of all $(g, L^1, L^2, \dots, L^\sigma)$ such that $g \in \mathcal{C}^V_{a_*,b_*}$ and $(L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{g;a_*,b_*}$.

Note that G_V acts on G^1_V by "twisted conjugation" that is by $\gamma : g \mapsto \check{\gamma} g \gamma^{-1}$. Also G_V acts on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ by

(e) $$\gamma : (g, L^1, L^2, \dots, L^\sigma) \mapsto (\check{\gamma} g \gamma^{-1}, \gamma(L^1), \gamma(L^2), \dots, \gamma(L^\sigma)).$$

Now let $\mathcal{I}$ be the subgroup of $\prod_{t \in [1,\sigma]} \{1, -1\}$ consisting of all $(\omega_t)_{t \in [1,\sigma]}$ such that $\omega_t = \omega_{t+1}$ for any t such that $\{t, t+1\} \subset [k+1, \sigma], t = k+1 \mod 2, b_t > 0$. Thus $\mathcal{I}$ is a finite elementary abelian 2-group.

The following is the main result of this section.

Theorem 2.1. (a) *$\tilde{\mathcal{C}}^V_{a_*,b_*}$ is nonempty;*

(b) *the action 2.0(e) of G_V on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ is transitive;*

(c) *the isotropy group in G_V at any point of $\tilde{\mathcal{C}}^V_{a_*,b_*}$ is canonically isomorphic to $\mathcal{I}$.*

The proof (by induction on n) follows the same lines as that of Theorem 1.1; it is given in 2.2–2.20. The numbering of the subsections is such that the material in 2.2, 2.3, …, 2.20 is analogous to the material in 1.2, 1.3, …, 1.20, respectively.

2.2. Let $a \in \mathbf{N}', b \in \mathbf{N}'', p \in \mathbf{N}_{>0}$ be such that $a+b=2p-1$. For $e \in \mathbf{N}''$ we define $n_e \in \mathbf{Z}$ by

$$(1-T^2)^a(1+T^2)^b = \sum_{e\in\mathbf{N}''} n_e T^e.$$

We have $n_0 = 1, n_{4p-2-e} = -n_e$, $n_e = 0$ if $e > 4p-2$. We define $x_e \in \mathbf{Z}$ for $e \in \mathbf{N}''$ by $x_0 = 1$ and

(a) $$n_0 x_e + n_2 x_{e-2} + \cdots + n_e x_0 = 0 \text{ for } e \geq 2.$$

For $h \in \mathbf{Z}'$ we set $x'_h = 0$ if $|h| < 2p-1$, $x'_h = x_{|h|-2p+1}$ if $|h| \geq 2p-1$. We show:

(b) $$\sum_{e\in\mathbf{N}''} n_e x'_{e-j-1} = 0 \text{ for any } j \in [0, 4p-4]''.$$

Assume first that $j \in [2p, 4p-4]''$. We have

$$e-j-1 \leq e-2p-1 \leq 4p-2-2p-1 \leq 2p-3.$$

Hence we can assume that $e-j-1 \leq -2p+1$ so that $x'_{e-j-1} = x_{j+1-e-2p+1}$ and we must show that

$$\sum_{e;e\leq j+1-2p+1} n_e x_{j+1-e-2p+1} = 0.$$

This holds since $j+1-2p+1 \geq 2$. Assume next that $j \in [0, 2p-4]''$. We have $e-j-1 \geq e-2p+4-1 \geq -2p+3$. Hence we can assume that $e-j-1 \geq 2p-1$ so that $x'_{e-j-1} = x_{e-j-1-2p+1}$ and we must show that

$$\sum_{e;e\geq j+1+2p-1} n_e x_{e-j-1-2p+1} = 0,$$

that is,

$$\sum_{e;e\geq j+1+2p-1} n_{4p-2-e} x_{e-j-1-2p+1} = 0,$$

that is,

$$\sum_{e';4p-2-e'\geq j+1+2p-1} n_{e'} x_{4p-2-e'-j-1-2p+1} = 0,$$

that is,

$$\sum_{e';e'\leq 2p-2-j} n_{e'} x_{2p-2-e'-j} = 0,$$

and this holds since $2p-2-j \geq 2$. Assume next that $j = 2p-2$. In the sum over e we can assume that $e-j-1 \geq 2p-1$ or $e-j-1 \leq -2p+1$, that is, $e \geq 4p-2$ or $e \leq 0$. Thus $e = 0$ or $e = 4p-2$. Thus the sum is $n_0 x'_{-2p+1} + n_{4p-2} x'_{2p-1} = n_0 + n_{4p-2} = 0$.

2.3. In the setup of 2.2 let V be a $\mathbf{k}$-vector space of dimension $2p-1$. Assume that we are given a basis $\{w_i; i \in [0, 4p-4]''\}$ of V. Let $\{w_i; i \in [1, 4p-3]'\}$ be the basis of V^* such that

$$(w_i, w_j) = x'_{i-j} = x'_{j-i} \text{ if } i \in [0, 4p-4]'', j \in [1, 4p-3]'.$$

Thus $(w_i, w_j) = 0$ if $|i-j| < 2p-1$. We define $g \in G^1_V$ by $gw_i = w_{i+1}$ for $i \in [0, 4p-4]''$. Let $\check{g} \in G^1_{V^*}$ be as in 2.0. We have

$$\check{g} w_i = w_{i+1} \text{ if } i \in [1, 4p-5]';$$

we must check that $(w_{i+1}, w_{j+1}) = (w_j, w_i)$ for $i \in [1, 4p-3]', j \in [0, 4p-4]''$; we use that $|i+1-(j+1)| = |j-i|$.

We show:

$$\check{g}w_{4p-3} = \sum_{i\in[0,4p-4]''} n_i w_i,$$

that is,

$$\sum_{i\in[0,4p-4]''} n_i(w_i, w_{j+1}) = (w_j, w_{4p-3}) \text{ for any } j \in [0, 4p-4]'',$$

that is,

$$\sum_{i\in[0,4p-4]''} n_i x'_{i-j-1} = x'_{4p-3-j} \text{ for any } j \in [0, 4p-4]'',$$

that is,

$$\sum_{i\in[0,4p-2]''} n_i x'_{i-j-1} = 0 \text{ for any } j \in [0, 4p-4]''.$$

This has been seen in 2.2(b).

We have $g^{*2}(w_i) = w_{i+2}$ for $i \in [0, 4p-6]''$, $g^{*2}(w_{4p-4}) = \sum_{i\in[0,4p-4]''} n_i w_i$. Hence $(g^{*2}-1)^a(g^{*2}+1)^b = 0$ on V. Indeed this holds on w_0 and then it holds automatically on $w_i, i \in [0, 4p-4]''$. Now $g^{*2} \in G_V$ is regular in the sense of Steinberg and satisfies $(g^{*2}-1)^a(g^{*2}+1)^b = 0$ on V. Hence $V = V^+ \oplus V^-$ where g^{*2} acts on V^+ as a single unipotent Jordan block of size a and $-g^{*2}$ acts on V^- as a single unipotent Jordan block of size b.

It follows that, if L is the line in V spanned by w_0 and $a_* = (a, 0, 0, \dots)$, $b_* = (b, 0, 0, \dots)$, then $(g, L) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$; in particular, $\tilde{\mathcal{C}}^V_{a_*,b_*} \neq \emptyset$.

We now consider a variant of the situation above. Let V' be a $\mathbf{k}$-vector space of dimension $2p-1$ with a given element $g \in G^1_{V'}$ such that $g^{*4} = 1$, on the generalized 1-eigenspace of g^{*2}, g^{*2} is a single unipotent Jordan block of size a and on the generalized (-1)-eigenspace of g^{*2}, $-g^{*2}$ is a single unipotent Jordan block of size b. Moreover, we assume that we are given $w \in V'$ such (with notation of 2.0) we have

$$(w_i, w_j) = 0 \text{ if } i \in \mathbf{Z}'', j \in \mathbf{Z}', |i-j| < 2p-1 \text{ and}$$
$$(w_i, w_j) = 1 \text{ if } i \in \mathbf{Z}'', j \in \mathbf{Z}', |i-j| = 2p-1.$$

We show:

(a) *for any $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ we have $(w_i, w_j) = x'_{i-j}$.*

We can assume that $i = 0$ and $j \geq 1$. The equality in (a) is already known if $j \leq 2p-1$. It is enough to show that $(w_0, w_{2p-1+2t}) = x_{2t}$ for $t \in \mathbf{N}$. We argue by induction on t; for $t = 0$ the result is already known. Now assume that $t \geq 1$. Applying $(g^{*2}-1)^a(g^{*2}+1)^b = 0$ to $w_{2t-2p+2}$ we obtain $\sum_{e\in[0,4p-2]''} n_e w_{2t-2p+2+e} = 0$. Taking $(, w_1)$ we obtain

$$\sum_{e\in[0,4p-2]''} n_e(w_{2t-2p+2+e}, w_1) = 0,$$

that is,

$$\sum_{e\in[0,4p-2]''} n_e(w_0, w_{2t-2p+1+e}) = 0.$$

For e in the sum we have $2t-2p+1+e\ge -2p+3$; hence we can assume that we have $2t-2p+1+e\ge 2p-1$. Thus

$$\sum_{e\in[0,4p-2]'';2t-2p+1+e\ge 2p-1} n_e(w_0,w_{2t-2p+1+e})=0.$$

By the induction hypothesis this implies

$$\sum_{e\in[0,4p-4]'';2t-2p+1+e\ge 2p-1} n_e x_{2t-4p+2+e}-(w_0,w_{2t+2p-1})=0.$$

It is then enough to show that

$$\sum_{e\in[0,4p-4]'';2t-2p+1+e\ge 2p-1} n_e x_{2t-4p+2+e}-x_{2t}=0,$$

or that

$$\sum_{e\in[0,4p-2]'';2t-2p+1+e\ge 2p-1} n_{4p-2-e} x_{2t-4p+2+e}=0,$$

or that

$$\sum_{h,h'\in\mathbf{N}'';h+h'=2t} n_h x_{h'}=0.$$

But this holds by the definition of x_e since $2t\ge 2$.

2.4. Let $p\in\mathbf{N}_{>0}$. We define n_e for $e\in\mathbf{N}''$ by $n_e=\binom{2p}{e/2}$. We define x_e for $e\in\mathbf{N}''$ by $x_0=1, x_2=-(2p+1)$, and $n_0x_e+n_2x_{e-2}+\dots+n_ex_0=0$ for $e\ge 4$. For $e=2$ we have

$$n_0x_e+n_2x_{e-2}+\dots+n_ex_0=n_0x_2+n_2x_0=-(2p+1)+2p=-1.$$

For $d\in\mathbf{Z}'$ we set $\phi_p(d)=0$ if $|d|<2p-1$, $\phi_p(d)=x_{|d|-2p+1}$ if $|d|\ge 2p-1$. We show for any $h\in\mathbf{Z}'$:

(a) $$\sum_{e\in[0,4p]''} n_e\phi_p(e+h)=0$$

Assume that $h\le -1$. We set $h=-j-1$ so that $j\in\mathbf{N}''$. Assume first that $j\ge 2p+2$. We have $e-j-1\le e-2p-2-1\le 4p-2p-2-1\le 2p-3$. Hence we can assume that $e-j-1\le -2p+1$ so that $\phi_p(e-j-1)=x_{j+1-e-2p+1}$ and we must show

$$\sum_{e\in\mathbf{N}'';e\le j+1-2p+1} n_e x_{j+1-e-2p+1}=0.$$

This holds since $j+1-2p+1\ge 4$.

Assume next that $j\le 2p-4$. We have $e-j-1\ge e-2p+4-1\ge -2p+3$. Hence we can assume that $e-j-1\ge 2p-1$ so that $\phi_p(e-j-1)=x_{e-j-1-2p+1}$ and we must show:

$$\sum_{e\in\mathbf{N}'';e\ge j+1+2p-1} n_e x_{e-j-1-2p+1}=0,$$

that is,

$$\sum_{e\in\mathbf{N}'';e\ge j+1+2p-1} n_{4p-e} x_{e-j-1-2p+1}=0,$$

that is,

$$\sum_{e'\in\mathbf{N}'';4p-e'\ge j+1+2p-1} n_{e'} x_{4p-e'-j-1-2p+1}=0,$$

that is,

$$\sum_{e'\in\mathbf{N};e'\le 2p-j} n_{e'}x_{2p-e'-j}=0$$

and this holds since $2p-j\ge 4$.

Assume next that $j=2p-2$. In the sum we can assume that $e-j-1\ge 2p-1$ or $e-j-1\le -2p+1$ that is $e\ge 4p-2$ or $e\le 0$. Thus $e=0$ or $e=4p-2$ or $e=4p$. Thus the sum is

$$n_0\phi_p(-2p+1)+n_{4p-2}\phi_p(2p-1)+n_{4p}\phi_p(2p+1)=x_0+2px_0+x_2=x_2+2p+1=0.$$

Assume next that $j=2p$. In the sum we can assume that $e-j-1\ge 2p-1$ or $e-j-1\le -2p+1$, that is, $e\ge 4p$ or $e\le 2$. Thus $e=0,2$ or $4p$. Thus the sum is

$$n_0\phi_p(-2p-1)+n_2\phi_p(-2p+1)+n_{4p}\phi_p(2p-1)=x_2+2p+1=0.$$

Thus the desired formula holds when $h\le -1$. Now assume that $h\ge 1$. We have

$$\sum_{e\in\mathbf{N}''} n_e\phi_p(e+h)=\sum_{e\in[0,4p]''} n_{4p-e}\phi_p(e+h)$$
$$=\sum_{e\in[0,4p]''} n_e\phi_p(4p-e+h)=\sum_{e\in[0,4p]''} n_e\phi_p(-4p+e-h)$$

and this is 0 by the first part of the proof since $-4p-h\le -1$.

We have

$$\sum_{e\in\mathbf{N}'',j\in\mathbf{N}''} n_eT^ex_jT^j=1-T^2,$$

hence

$$(1+T^2)^{2p}\sum_{j\in\mathbf{N}''} x_jT^j=1-T^2$$

and

$$\sum_{j\in\mathbf{N}''} x_jT^j=(1-T^2)(1+T^2)^{-2p}=(1-T^2)(\sum_{k\ge 0}(-1)^k\binom{2p-1+k}{2p-1}T^{2k}).$$

Thus,

$$x_{2k}=(-1)^k\binom{2p-1+k}{2p-1}-(-1)^{k-1}\binom{2p-1+k-1}{2p-1}$$
$$=(-1)^k\frac{(2p-2+k)!(2p-1+k+k)}{k!(2p-1)!}$$
$$=(-1)^k(2p+2k-1)(2p-2+k)(2p-2+k-1)\dots(k+1)(2p-1)!^{-1}.$$

We show for any $h\in\mathbf{Z}'$:

(b) $\phi_p(h)=(-1)^{(h+2p+1)/2}2h(h+2p-3)(h+2p-5)\dots(h-2p+3)(4p-2)!!^{-1}$

where

$$(4p-2)!!:=2\times 4\times\dots(4p-2)=2^{2p-1}(2p-1)!.$$

Assume first that $h = 2d + 1 \geq 2p - 1$. We have

$$\begin{aligned}&\phi_p(h) = x_{2d+1-2p+1} = x_{2d-2p+2}\\ &= (-1)^{d-p+1}(2p + 2d - 2p + 2 - 1)(2p - 2 + d - p + 1)\\ &\times (2p - 2 + d - p + 1 - 1) \dots (d - p + 2)(2p - 1)!^{-1}\\ &= (-1)^{d-p+1}(2d + 1)(p + d - 1)(p + d - 2) \dots (d - p + 2)(2p - 1)!^{-1}\end{aligned}$$

so that the result holds in this case. Now both sides of (b) are invariant under $h \mapsto -h$. Hence (b) also holds if $h \leq -2p + 1$. If $h \in [-2p + 3, 2p - 3]'$, both sides of (b) are zero. Hence (b) holds for any $h \in \mathbf{Z}'$.

In particular we have $\phi_p(2p + 1) = -(2p + 1)$.

2.5. In the setup of 2.4, let E be a $\mathbf{k}$-vector space of dimension $2p$. Assume that we are given a basis $\{w_i; i \in [0, 4p - 2]''\}$ of E. We define a basis $\{w_i; i \in [1, 4p - 1]'\}$ of E^* by

$$(w_i, w_j) = \phi_p(i - j) = \phi_p(j - i) \text{ for } i \in [0, 4p - 2]'', j \in [1, 4p - 1]'.$$

Thus $(w_i, w_j) = 0$ if $i \in [0, 4p - 2]'', j \in [1, 4p - 1]'$, $|i - j| < 2p - 1$. We define $g \in G^1_E$ by $gw_i = w_{i+1}$ for $i \in [0, 4p - 2]''$. We have

$$\check{g}w_i = w_{i+1} \text{ if } i \in [1, 4p - 3]';$$

we must check that $(w_{i+1}, w_{j+1}) = (w_j, w_i)$ for $i \in [1, 4p - 1]', j \in [0, 4p - 2]''$; we use that $|i + 1 - (j + 1)| = |j - i|$.

We show:

$$\check{g}w_{4p-1} = -\sum_{i\in[0,4p-4]''} n_i w_i.$$

We must show for any $j \in [0, 4p - 2]''$ that

$$-\sum_{i\in[0,4p-2]''} n_i(w_i, w_{j+1}) = (w_j, w_{4p-1}),$$

that is,

$$-\sum_{i\in[0,4p-2]''} n_i\phi_p(i - j - 1) = \phi_p(4p - 1 - j),$$

that is,

$$\sum_{i\in[0,4p]''} n_i\phi_p(i - j - 1) = 0;$$

note that $n_{4p} = -1$. This has been seen in 2.4(a).

We have

$$\begin{aligned}g^{*2}(w_i) &= w_{i+2} \text{ for } i \in [0, 4p - 4],\\ g^{*2}(w_{4p-2}) &= -\sum_{i\in[0,4p-2]''} n_i w_i.\end{aligned}$$

Hence

(a) $$(g^{*2} + 1)^{2p} = 0 \text{ on } E.$$

Indeed this holds on w_0 and then it holds automatically on $w_i, i \in [0, 4p - 2]''$. Now $g^{*2} \in GL(E)$ is regular in the sense of Steinberg and satisfies (a). Hence $-g^{*2}$ acts on E as a single unipotent Jordan block of size $2p$.

2.6. For $i \in \mathbf{Z}$ we write w_i instead of $(w_0)_i$. This agrees with our earlier notation for w_i when $i \in [0, 4p-1]$. We show:

(a) $$(w_i, w_j) = \phi_p(i-j) = \phi_p(j-i) \text{ for any } i \in \mathbf{Z}'', j \in \mathbf{Z}'.$$

By 2.0(a) there exists a function $f : \mathbf{Z}' \to \mathbf{k}$ such that $(w_i, w_j) = f(i-j)$ for any $i \in \mathbf{Z}'', j \in \mathbf{Z}'$. We must show that $f(h) = \phi_p(h)$ for $h \in \mathbf{Z}'$. We set $f'(h) = f(h) - \phi_p(h)$. We must show that $f'(h) = 0$ for all $h \in \mathbf{Z}'$. This is clearly true when $h \in [-2p+1, 2p-1]'$. Applying $\sum_{e \in [0,4p]''} n_e g^{*e} = 0$ to $w_i, i \in \mathbf{Z}''$, we deduce

$$\sum_{e \in [0,4p]''} n_e w_{i+e} = 0;$$

hence

$$\sum_{e \in [0,4p]''} n_e (w_{i+e}, w_j) = 0 \text{ for } i \in \mathbf{Z}'', j \in \mathbf{Z}'.$$

Thus, $\sum_{e \in [0,4p]''} n_e f(i-j+e) = 0$ for $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ and $\sum_{e \in [0,4p]''} n_e f(h+e) = 0$ for $h \in \mathbf{Z}''$. Combining this with $\sum_{e \in [0,4p]''} n_e \phi_p(h+e) = 0$ for $h \in \mathbf{Z}''$ (see 2.4(a)), we deduce $\sum_{e \in [0,4p]''} n_e f'(h+e) = 0$ for $h \in \mathbf{Z}''$. We show that $f'(h) = 0$ for $h \geq 2p-1$ by induction on h. For $h = 2p-1$ this is already known. Now assume that $h \geq 2p+1$. We have $\sum_{e \in [0,4p]''} n_e f'(h+e-4p) = 0$. If $e \in [0, 4p-2]''$ we have $h+e-4p \in [-2p+1, h-2]$ hence $f'(h+e-4p) = 0$ and the sum over e becomes $n_{4p} f'(h) = 0$ so that $f'(h) = 0$. This completes the induction. We now show that $f'(h) = 0$ for $h \leq -2p+1$ by descending induction on h. For $h = -2p+1$ this is known. Now assume that $h \leq -2p-1$. If $e \in [2, 4p]''$ we have $h+e \in [h+2, 2p-1]$ hence $f'(e+h) = 0$ and the equation $\sum_{e \in [0,4p]''} n_e f'(h+e) = 0$ becomes $n_0 f'(h) = 0$ so that $f'(h) = 0$. This completes the descending induction and completes the proof of (a).

2.7. We preserve the setup of 2.5. Let $\tilde{w}$ be a nonzero vector in E such that

(a) $(\tilde{w}, w_i) = 0$ for $i \in [1, 4p-3]'$.

Note that $\tilde{w}$ is uniquely determined up to a nonzero scalar. Then $\tilde{w}_i$ is defined for any $i \in \mathbf{Z}$ as in 2.0; in particular, $\tilde{w}_0 = \tilde{w}, \tilde{w}_1 = g\tilde{w}$. We have

(b) $(w_i, \tilde{w}_1) = 0$ for $i \in [2, 4p-2]''$.

Indeed, using 2.0(a),(b) we have $(w_i, \tilde{w}_1) = (\tilde{w}_{-i}, w_{-1}) = (\tilde{w}_0, w_{i-1})$ and this is zero since $i-1 \in [1, 4p-3]'$.

We show that $(\tilde{w}_0, \tilde{w}_1) \neq 0$. Let E_1 be the span of $\{w_i; i \in [2, 4p-2]''\}$ and let E'_1 be the span of $\{w_i; i \in [1, 4p-3]'\}$. The canonical pairing $(,) : E \times E^* \to \mathbf{k}$ restricts to a nondegenerate pairing $E_1 \times E'_1 \to \mathbf{k}$ (by the formulas for (w_i, w_j) in 2.5). Since $\tilde{w}_0$ is in the annihilator of E'_1 in E, it follows that $\tilde{w}_0 \notin E_1$. Since $\tilde{w}_1$ is in the annihilator of E_1 in E^*, it follows that $\tilde{w}_0$ is not in the annihilator of $\tilde{w}_1$ in E. The claim follows.

If $\tilde{w}$ is replaced by $a\tilde{w}$ with $a \in \mathbf{k}^*$, then $(\tilde{w}_0, \tilde{w}_1)$ is replaced by $a^2(\tilde{w}_0, \tilde{w}_1)$ which, for a suitable a, is equal to 1. Thus we can assume that

(c) $$(\tilde{w}_0, \tilde{w}_1) = 1.$$

Then $\tilde{w}_0$ is uniquely determined up to multiplication by ± 1. We have

$$\tilde{w}_0 = \sum_{i \in [0, 4p-2]''} c_i w_i$$

where $c_i \in \mathbf{k}$ are uniquely determined. Since $\tilde{w}_0 \notin E_1$ we see that $c_* := c_{4p-2} \neq 0$. We set $\bar{c}_i = c_i c_*^{-1} \in \mathbf{k}$. Note that $\bar{c}_{4p-2} = 1$. We have the following result (with n_i as in 2.4):

(d) $$\bar{c}_i = -(n_0 + n_2 + \cdots + n_i) \text{ if } i \in [0, 2p-2]'',$$

(e) $$\bar{c}_i = (n_0 + n_2 + \cdots + n_{4p-2-i}) \text{ if } i \in [2p, 4p-2]'',$$

(f) $$c_* = \pm 2^{-p}.$$

We can rewrite (a) as follows.

(∗) $$\sum_{i\in[0,4p-2]''} \bar{c}_i \phi_p(i-h) = 0 \text{ for } h \in [1, 4p-3]'.$$

If $h = 2p-1$, then (∗) is $\bar{c}_0 + 1 = 0$. If $h \in [2p+1, 4p-3]'$, then (∗) is

$$\sum_{i\in[0,h-2p+1]''} \bar{c}_i \phi_p(i-h) = 0.$$

If $h \in [1, 2p-3]'$, then (∗) is

$$\sum_{i\in[h+2p-1,4p-2]''} \bar{c}_i \phi_p(i-h) = 0.$$

To prove (d) and (e) it is enough to show:

(d′) $$-\sum_{i\in[0,h-2p+1]''} (n_0 + n_2 + \cdots + n_i)\phi_p(i-h) = 0 \text{ if } h \in [2p+1, 4p-3]',$$

(e′) $$\sum_{i\in[h+2p-1,4p-2]''} (n_0 + n_2 + \cdots + n_{4p-2-i})\phi_p(i-h) = 0 \text{ if } h \in [1, 2p-3]'.$$

We rewrite equation (e′) using $i \mapsto 4p-2-i$ and $h \mapsto 4p-2-h$ as

$$\sum_{i\in[0,h-2p+1]''} (n_0 + n_2 + \cdots + n_i)\phi_p(h-i) = 0 \text{ if } h \in [2p+1, 4p-3]',$$

which is the same as (d′). Thus it is enough to prove (d′). We argue by induction on h. If $h = 2p+1$, equation (d′) is

$$n_0\phi_p(-2p-1) + (n_0 + n_2)\phi_p(-2p+1) = 0,$$

that is, $-(2p+1) + (1+2p) = 0$, which is correct. If $h \geq 2p+3$ we have

$$\sum_{i\in[0,h-2p+1]''} n_i x_{h-i-2p+1} = 0$$

since $h - 2p + 1 \geq 4$. Hence in this case (d′) is equivalent to

$$\sum_{i\in[2,h-2p+1]''} (n_0 + n_2 + \cdots + n_{i-2})\phi_p(i-h) = 0,$$

which is the same as equation (d′) with h replaced by $h-2$ (this holds by the induction hypothesis). This proves (d) and (e).

The equation $(\tilde{w}_0, \tilde{w}_1) = 1$ can be written as

$$1 = (\tilde{w}_0, \sum_{i\in[0,4p-2]''} c_i w_{i+1}) = (\tilde{w}_0, c_{4p-2} w_{4p-1}),$$

that is,

(g) $$1 = c_*(\tilde{w}_0, w_{4p-1}).$$

We deduce that

$$1 = c_* \sum_{i\in[0,4p-2]''} c_i(w_i, w_{4p-1}),$$

that is,

$$c_*^{-2} = \sum_{i\in[0,4p-2]''} \bar{c}_i\phi_p(4p-1-i).$$

We have $4p-i-1 \ge -2p+3$ hence we can assume $4p-i-1 \ge 2p-1$. Thus

$$\begin{aligned} c_*^{-2} &= \sum_{i\in[0,2p]''} \bar{c}_i\phi_p(4p-1-i) \\ &= -\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)\phi_p(4p-1-i) + n_0+n_2+\dots+n_{2p-2} \\ &= -\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)x_{2p-i} + n_0+n_2+\dots+n_{2p-2}. \end{aligned}$$

Thus,

$$\begin{aligned} c_*^{-2} &= -\sum_{i\in\mathbf{N}'',j\in\mathbf{N}'';i+j\le 2p,j\ge 2} n_ix_j + n_0+n_2+\dots+n_{2p-2} \\ &= -\sum_{i\in\mathbf{N}'',j\in\mathbf{N}'';i+j\le 2p} n_ix_j + \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= -\sum_{k\in[0,2p]''}\sum_{i\in\mathbf{N}'',j\in\mathbf{N}'';i+j=k} n_ix_j + \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= -\sum_{k\in[0,2p]'';k=0,2}\sum_{i\in\mathbf{N}'',j\in\mathbf{N}'';i+j=k} n_ix_j + \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= -1 + n_0x_2 + n_2x_0 + \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= -1-(n_2+1)+n_2 + \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= \sum_{i\in\mathbf{N}'';i\le 2p} n_i + (n_0+n_2+\dots+n_{2p-2}) \\ &= n_0+n_2+\dots+n_{2p}+n_{2p+2}+\dots+n_{4p} = 2^{2p}. \end{aligned}$$

Thus $c_*^{-2} = 2^{2p}$ and (f) follows.

If $\tilde{w}$ is replaced by $-\tilde{w}$, then c_* is changed to $-c_*$. Hence $\tilde{w}$ can be chosen uniquely so that

(f′) $$c_* = 2^{-p}.$$

2.8. We preserve the setup of 2.5. For $h \in \mathbf{Z}'$ we show

(a) $$(\tilde{w}_0, w_h) = (-1)^{(h+1)/2}2^p(h-1)(h-3)\dots(h-4p+3)(4p-2)!!^{-1} \in 2\mathbf{Z}.$$

We have $(\tilde{w}_0, w_h) = \sum_{i\in[0,4p-2]''} c_i\phi_p(i-h)$. Since $c_i = 2^{-p}\bar{c}_i$ it is enough to prove

(b) $$\sum_{i\in[0,4p-2]''} \bar{c}_i(-1)^{(h+1)/2}\phi_p(i-h) = 2^{2p}(h-1)(h-3)\dots(h-4p+3)(4p-2)!!^{-1}.$$

It is also enough to prove this equality in $\mathbf{Z}$. For fixed i, $(-1)^{(h+1)/2}\phi_p(i-h)$ is a polynomial in h with rational coefficients of degree $\le 2p-1$. Hence the left hand side of (b) is a polynomial in h with rational coefficients of degree $\le 2p-1$. Since $(\tilde{w}_0, w_h) = 0$ for $h \in [1, 4p-3]'$, this polynomial is zero for $h \in [1, 4p-3]'$ (that is for $2p-1$ values of h). It follows that

$$(-1)^{(h+1)/2} \sum_{i\in[0,4p-2]''} \bar{c}_i \phi_p(i-h) = a(h-1)(h-3)\dots(h-4p+3)$$

for some rational number a. (The left hand side is $(-1)^{(h+1)/2} 2^p (\tilde{w}_0, w_h)$.) For $h = 4p-1$ we have $(\tilde{w}_0, w_h) = c_*^{-1} = 2^p$ (see 2.7(g)), hence

$$2^{2p} = a(4p-2)(4p-4)\dots 2 = a(4-2)!!,$$

that is, $a = 2^{2p}(4p-2)!!^{-1}$. It remains to show that

$$(-1)^{(h-1)/2} 2^p (h-1)(h-3)\dots(h-4p+3)(4p-2)!!^{-1} \in 2\mathbf{Z}.$$

Setting $h = 2s+1$ it is enough to show that

$$2^p(2s+1-1)(2s+1-3)\dots(2s+1-4p+3)(4p-2)!!^{-1} \in 2\mathbf{Z}$$

or that

$$2^p s(s+1)\dots(s-2p+2)(2p-1)!^{-1} \in 2\mathbf{Z}.$$

This is obvious since $p \ge 1$.

2.9. We preserve the setup of 2.5. We will show:

(a) $$(\tilde{w}_0, \tilde{w}_h) = \sum_{k\in[1,p]} 2^{2k-2}\phi_k(h) \in \mathbf{k} \text{ for } h \in \mathbf{Z}';$$

(b) $$(\tilde{w}_0, \tilde{w}_h) = 1 \text{ if } h \in [-2p+1, 2p-1]';$$

(c) $$(\tilde{w}_0, \tilde{w}_{2p+1}) = 1 - 2^{2p}.$$

We prove (a). We have

$$(\tilde{w}_0, \tilde{w}_h) = \sum_{i\in[0,4p-2]''} c_i(\tilde{w}_0, w_{i+h}) = \sum_{i\in[0,4p-2]''} (-1)^{(i+h+1)/2} c_i 2^p (i+h-1)$$

(d) $$\times (i+h-3)\dots(i+h-4p+3) \times (4p-2)!!^{-1}.$$

Thus, (a) would follow from the equality

$$\sum_{i\in[0,4p-2]''} (-1)^{i/2}\bar{c}_i(i+h-1)(i+h-3)\dots(i+h-4p+3)(4p-2)!!^{-1}$$

(e) $$= \sum_{k\in[1,p]} (-1)^{(h+1)/2} 2^{2k-2}\phi_k(h)$$

in $\mathbf{k}$. It is enough to prove that (e) holds in $\mathbf{Z}$. We will do that assuming that (b) holds. Let $F_p(h)$ be the left hand side of (e). It can be viewed as a polynomial with rational coefficients in h of degree $\le 2p-1$ in which the coefficient of h^{2p-1} is

$$
\begin{aligned}
&(4p-2)!!^{-1}\sum_{i\in[0,4p-2]''}\bar{c}_i(-1)^{-i/2}\\
&=-(4p-2)!!^{-1}\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)(-1)^{i/2}\\
&+(4p-2)!!^{-1}\sum_{i\in[2p,4p-2]''}(n_0+n_2+\dots+n_{4p-2-i})(-1)^{i/2}\\
&=-(4p-2)!!^{-1}\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)(-1)^{i/2}\\
&+(4p-2)!!^{-1}\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)(-1)^{(4p-2-i)/2}\\
&=-2(4p-2)!!^{-1}\sum_{i\in[0,2p-2]''}(n_0+n_2+\dots+n_i)(-1)^{i/2}\\
&=-2(4p-2)!!^{-1}(-1)^{p-1}(n_{2p-2}+n_{2p-6}+n_{2p-10}+\dots)\\
&=-2(4p-2)!!^{-1}(-1)^{p-1}2^{2p-2}\\
&=(-1)^p2^{2p-1}(4p-2)!!^{-1}=(-1)^p(2p-1)!^{-1}.
\end{aligned}
$$

Thus,

$$F_p(h)=(-1)^p(2p-1)!^{-1}h^{2p-1}+\text{ lower powers of }h.$$

Note that $F_p(-h)=-F_p(h)$ for $h\in\mathbf{Z}'$. An equivalent statement is that

$$(-1)^{(h+1)/2}(\tilde{w}_0,\tilde{w}_h)=-(-1)^{(-h+1)/2}(\tilde{w}_0,\tilde{w}_{-h})$$

which follows from $(\tilde{w}_0,\tilde{w}_h)=(\tilde{w}_0,\tilde{w}_{-h})$; see 2.0. It follows that $F_p(-h)=-F_p(h)$ as polynomials in h. Specializing this for $h=0$ we see that

(g) $$F_p(0)=0.$$

In the case where $p=1$, from (f) and (g) we see that $F_1(h)=-h$ so that (e) holds in this case (we have $(-1)^{(h+1)/2}\phi_1(h)=-h$). We now assume that $p\ge 2$. Now F_p-F_{p-1} is a polynomial of degree $2p-1$ in h whose value at $h\in[-2p+3,2p-3]'$ is $(-1)^{(h+1)/2}-(-1)^{(h+1)/2}=0$ (we use (b) for p and $p-1$) and whose value at 0 is 0 (see (e)); moreover, the coefficient of h^{2p-1} in F_p-F_{p-1} is $(-1)^p(2p-1)!^{-1}$ (see (f)). It follows that $F_p-F_{p-1}=(-1)^{(h+1)/2}2^{2p-2}\phi_p(h)$. From this we see by induction on p that (e) holds.

It remains to prove (b) and (c) (without assuming (a)). To prove (b) we can assume that $h\ge 1$ (we use that $(\tilde{w}_0,\tilde{w}_h)=(\tilde{w}_0,\tilde{w}_{-h})$, see 2.0). Thus it is enough to prove (b) for $h\in[1,2p-1]'$ and (c). If $h=1$, (b) holds by the definition of $\tilde{w}_0$. Assume now that $h\in[3,2p+1]'$. In the right hand side of (e) the sum over i can be restricted to those i such that $i+h\notin\{1,3,\dots,4p-3\}$ hence such that $i+h\ge 4p-1$; for such i we have $i\ge 4p-1-h\ge(4p-1)-(2p+1)$ hence

$i \geq 2p-2$. Moreover, if $i = 2p-2$, then we must have $h = 2p+1$. Thus we have

$$
\begin{aligned}
&(\tilde{w}_0, \tilde{w}_h) = (-1)^{(h+1)/2} \sum_{i \in [4p-1-h, 4p-2]''} (-1)^{i/2} \bar{c}_i (i+h-1) \\
&\times (i+h-3)\dots(i+h-4p+3)(4p-2)!!^{-1} \\
&= (-1)^{(h+1)/2} \sum_{i \in [4p-1-h, 4p-2]''; i \geq 2p} (-1)^{i/2} (n_0 + n_2 + \dots + n_{4p-2-i}) \\
&\times (i+h-1)(i+h-3)\dots(i+h-4p+3)(4p-2)!!^{-1} \\
&- (-1)^{(h+1)/2} \delta_{h,2p+1} (-1)^{(2p-2)/2} (n_0 + n_2 + \dots + n_{2p-2}) \\
&= (-1)^{(h+1)/2} \sum_{i \in [4p-1-h, 4p-2]''} (-1)^{i/2} (n_0 + n_2 + \dots + n_{4p-2-i}) \\
&\times (i+h-1)(i+h-3)\dots(i+h-4p+3)(4p-2)!!^{-1} \\
&- (-1)^{p+1} \delta_{h,2p+1} (-1)^{p-1} (n_0 + n_2 + \dots + n_{2p}) \\
&- (-1)^{p+1} \delta_{h,2p+1} (-1)^{p-1} (n_0 + n_2 + \dots + n_{2p-2}) \\
&= x - \delta_{h,2p+1} (n_0 + n_2 + \dots + n_{2p} + n_0 + n_2 + \dots + n_{2p-2}) \\
&= x - \delta_{h,2p+1} (n_0 + n_2 + \dots + n_{2p} + n_{2p+2} + \dots + n_{4p}) \\
&= x - \delta_{h,2p+1} 2^{2p}
\end{aligned}
$$

where

$$
\begin{aligned}
x = (-1)^{(h+1)/2} &\sum_{i \in [4p-1-h, 4p-2]''} (-1)^{i/2} (n_0 + n_2 + \dots + n_{4p-2-i}) \\
&\times (i+h-1)(i+h-3)\dots(i+h-4p+3)(4p-2)!!^{-1}.
\end{aligned}
$$

It remains to show that $x = 1$. Setting $h = 2h'+1, i = 4p-2-2i'$ we have

$$
\begin{aligned}
x &= \sum_{i' \in [0,h']} (-1)^{i'+h'} (n_0 + n_2 + \dots + n_{2i'}) \binom{2p-i+h'-1}{2p-1} \\
&= \sum_{i' \geq 0, u \geq 0; i'+u=h'} (-1)^u (n_0 + n_2 + ... + n_{2i'}) r_u
\end{aligned}
$$

where $r_u = \binom{u+2p-1}{2p-1}$. Note that

$$
\sum_{i \geq 0, u \geq 0; i+u=e} (-1)^i n_{2i} r_e = \delta_{e,0}
$$

for any $e \in \mathbf{N}$. Hence

$$
\begin{aligned}
x &= \sum_{i' \geq 0, j \geq 0, r \geq 0, u \geq 0; i'=j+r, i'+u=h'} (-1)^{h'+j+r} n_{2j} r_u \\
&= \sum_{r \in [0,h']} (-1)^{h'+r} \sum_{j,u \geq 0; j+u=h'-r} (-1)^j n_{2j} r_u \\
&= \sum_{r \in [0,h']} (-1)^{h'+r} \delta_{h'-r} = (-1)^{h'+h'} = 1.
\end{aligned}
$$

This completes the proof of (a), (b), (c).

2.10. We fix two integers p_1, p_2 such that $p_1 \ge p_2 \ge 1$. Let V', V'' be two $\mathbf{k}$-vector spaces of dimension $2p_1, 2p_2 - 2$, respectively, and let $V = V' \oplus V''$. We identify $V^* = V'^* \oplus V''^*$ in the obvious way. Let $(,) : V \times V^* \to \mathbf{k}$ be the obvious pairing. Assume that V' has a given basis $\{z_i; i \in [0, 4p_1 - 2]''\}$ and that V'' has a given basis $\{v_i; i \in [0, 4p_2 - 6]''\}$. There is a unique basis $\{z_i; i \in [1, 4p_1 - 1]'\}$ of V'^* and a unique basis $\{v_i; i \in [1, 4p_2 - 5]'\}$ of V''^* such that

$$(z_i, z_j) = \phi_{p_1}(i - j) \text{ for } i \in [0, 4p_1 - 2]'', j \in [1, 4p_1 - 1]',$$

$$(v_i, v_j) = \phi_{p_2 - 1}(i - j) \text{ for } i \in [0, 4p_2 - 6]'', j \in [1, 4p_2 - 5]'.$$

(Notation of 2.4; the basis of V'' and V''^* is empty when $p_2 = 1$.) We define $g \in G_V^1$ by $gz_i = z_{i+1}$ for $i \in [0, 4p_1 - 2]''$, $gv_i = v_{i+1}$ for $i \in [0, 4p_2 - 6]''$. We have

$$g^{*2}(z_i) = z_{i+2} \text{ for } i \in [0, 4p_1 - 4], \quad g^{*2}(v_i) = v_{i+2} \text{ for } i \in [0, 4p_2 - 8],$$

$$(g^{*2} + 1)^{2p_1} = 0 \text{ on } V', \quad (g^{*2} + 1)^{2p_2 - 2} = 0 \text{ on } V''.$$

(See 2.5.) Hence $-g^{*2}$ acts on V' as a single unipotent Jordan block of size $2p_1$ and on V'' as a single unipotent Jordan block of size $2p_2 - 2$. (When $p_2 = 1$, $-g^{*2} = 0$ on $V'' = 0$.)

For $i \in \mathbf{Z}$ we write z_i instead of $(z_0)_i$ (as in 2.0); when $p_2 \ge 2$ we write v_i instead of $(v_0)_i$. This agrees with our earlier notation for z_i when $i \in [0, 4p_1 - 1]$ and v_i for $i \in [0, 4p_2 - 5]$. We have

$$(z_i, z_j) = \phi_{p_1}(i - j) \text{ for } i \in \mathbf{Z}'', j \in \mathbf{Z}';$$

$$(v_i, v_j) = \phi_{p_2 - 1}(i - j) \text{ for } i \in \mathbf{Z}'', j \in \mathbf{Z}' \text{ (assuming } p_2 \ge 2).$$

(See 2.6(a).) If $p_2 \ge 2$ we clearly we have

$$(z_i, v_j) = 0, (v_i, z_j) = 0 \text{ for } i \in \mathbf{Z}'', j \in \mathbf{Z}'.$$

As in 2.7 and 2.8, there is a unique vector $\tilde{z} \in V'$ such that for any $h \in \mathbf{Z}'$ we have

$$(\tilde{z}_0, z_h) = 2^{p_1}(-1)^{(h+1)/2}(h - 1)(h - 3) \dots (h - 4p_1 + 3)(4p_1 - 2)!!^{-1},$$

Similarly, if $p_2 \ge 2$, there is a unique vector $\tilde{v} \in V''$ such that for any $h \in \mathbf{Z}'$ we have

$$(\tilde{v}_0, v_h) = 2^{p_2 - 1}(-1)^{(h+1)/2}(h - 1)(h - 3)...(h - 4p_2 + 7)(4p_2 - 6)!!^{-1}.$$

(Notation of 2.0.) If $p_2 = 1$ we set $\tilde{v}_i = 0$ for all $i \in \mathbf{Z}$. As in 2.9, we have

(a) $$(\tilde{z}_0, \tilde{z}_h) = \sum_{k \in [1, p_1]} 2^{2k-2}\phi_k(h),$$

(b) $$(\tilde{v}_0, \tilde{v}_h) = \sum_{k \in [1, p_2 - 1]} 2^{2k-2}\phi_k(h), \text{ (if } p_2 \ge 2),$$

(c) $$(\tilde{z}_0, \tilde{z}_h) = 1 \text{ if } h \in [-2p_1 + 1, 2p_1 - 1]'; \quad (\tilde{z}_0, \tilde{z}_{2p_1 + 1}) = 1 - 2^{2p_1},$$

(d) $$(\tilde{v}_0, \tilde{v}_h) = 1 \text{ if } h \in [-2p_2 + 3, 2p_2 - 3]'; \quad (\tilde{v}_0, \tilde{v}_{2p_2 - 1}) = 1 - 2^{2p_2 - 2} \text{ (if } p_2 \ge 2).$$

Let $\zeta \in \mathbf{k}$ be such that $\zeta^2 = -1$. We set

$$\xi = 2^{-p_2 + 1}\tilde{z}_{-2p_2} + 2^{-p_2 + 1}\zeta\tilde{v}_0 \in V.$$

Let $h \in \mathbf{Z}'$. We show:

$$(\xi_0, z_h) = 2^{p_1-p_2+1}(-1)^{(h+2p_2+1)/2}(h+2p_2-1)(h+2p_2-3)\dots \times (h+2p_2-4p_1+3)(4p_1-2)!!^{-1} \in 2\mathbf{Z}.$$

Indeed,

$$(\xi_0, z_h) = 2^{-p_2+1}(\tilde{z}_{-2p_2}, z_h) = 2^{-p_2+1}(\tilde{z}_0, z_{2p_2+h}) = 2^{-p_2+1}2^{p_1}(-1)^{(2p_2+h+1)/2} \times (2p_2+h-1)(2p_2+h-3)\dots(2p_2+h-4p_1+3)(4p_1-2)!!^{-1},$$

as desired. In particular, we have

$$(\xi_0, z_h) = 0 \text{ if } h \in [1-2p_2, 4p_1-2p_2-3]'.$$

Let $h \in \mathbf{Z}'$. From the definitions we have $(\xi_0, \xi_h) = 2^{-2p_2+2}((\tilde{z}_0, \tilde{z}_h) - (\tilde{v}_0, \tilde{v}_h))$. From this we deduce using (a)–(d) that

$$(\xi_0, \xi_h) = \sum_{k \in [p_2, p_1]} 2^{2k-2p_2}\phi_k(h) \in \mathbf{Z} \text{ for } h \in \mathbf{Z}',$$

$$(\xi_0, \xi_h) = 0 \text{ if } h \in [-2p_2+3, 2p_2-3]'; (\xi_0, \xi_{2p_2-1}) = 1.$$

It follows that, if L is the line in V spanned by z_0, L' is the line in V spanned by ξ_0 and $a_* = (0,0,0,\dots)$, $b_* = (2p_1, 2p_2-2, 0, \dots)$, then $(g, L, L') \in \tilde{\mathcal{C}}^V_{a_*,b_*}$; in particular, $\tilde{\mathcal{C}}^V_{a_*,b_*} \neq \emptyset$.

2.11. Let p_1, p_2 be integers such that $p_1 \geq p_2 \geq 1$. We consider a $\mathbf{k}$-vector space V of dimension $2p_1 + 2p_2 - 2$ with a given bilinear form $g \in G^1_V$ such that (with notation of 2.0) $-g^{*2} \in G_V$ is unipotent with a single Jordan block of size $2p_1$ (if $p_2 = 1$) or with two Jordan blocks, one of size $2p_1$ and one of size $2p_2-2$ (if $p_2 \geq 2$). We assume given two vectors z, ξ in V such that (with notation of 2.0), setting for $h \in \mathbf{Z}'$:

$$\alpha_h = (z_i, z_j), \beta_h = (\xi_i, \xi_j), \gamma_h = (\xi_i, z_j) \text{ where } i \in \mathbf{Z}'', j \in \mathbf{Z}', h = j - i,$$

we have

$$\alpha_h = 0 \text{ if } h \in [-2p_1+3, 2p_1-3]', \alpha_{2p_1-1} = 1,$$
$$\beta_h = 0 \text{ if } h \in [-2p_2+3, 2p_2-3]', \beta_{2p_2-1} = 1,$$
$$\gamma_h = 0 \text{ if } h \in [1-2p_2, 4p_1-2p_2-3]'.$$

We show:

(a) *After possibly replacing ξ by $-\xi$, the following equalities hold for any $h \in \mathbf{Z}'$:*
(a1) $\alpha_h = \phi_{p_1}(h) \in \mathbf{Z}$,
(a2) $\beta_h = \sum_{k \in [p_2, p_1]} 2^{2k-2p_2}\phi_k(h) \in \mathbf{Z}$,
(a3)

$$\gamma_h = 2^{p_1-p_2+1}(-1)^{(h+2p_2+1)/2} \times \frac{(h+2p_2-1)(h+2p_2-3)\dots(h+2p_2-4p_1+3)}{(4p_1-2)!!} \in 2\mathbf{Z}.$$

(ϕ_p as in 2.4.) We prove (a1) (see also 2.6). If $|h| \leq 2p_1 - 1$, then (a1) is clear. Thus we can assume that $|h| \geq 2p_1 + 1$. Since $\alpha_h = \alpha_{-h}$ we can also assume that $h \geq 1$ (hence $h \geq 2p_1 + 1$). We must only prove that

(b) $\alpha_h = x_{h-2p_1+1}$ if $h \geq 2p_1 - 1$ is odd, where x_e is as in 2.4 (with $p = p_1$). We have $(g^{*2} + 1)^{2p_1} = 0$ on V hence applying to z_0, we have$\sum_{j\in[0,2p_1]} r_j z_{2j} = 0$ where $r_j = \binom{2p_1}{j}$. Taking $(, z_{2p_1+2s-1})$ we get $\sum_{j\ge 0} r_j \alpha_{2p_1+2s-1-2j} = 0$. The coefficient of T^s ($s \in \mathbf{N}$) in

$$(\sum_{j\in\mathbf{N}} r_j T^j)(\sum_{u\in\mathbf{N}} \alpha_{2p_1-1+2u} T^u)$$

is

$$k_s = \sum_{j\in[0,s]} r_j \alpha_{2p_1-1+2s-2j}.$$

If $s \ge 2, j > s, j \le 2p_1$, we have $\alpha_{2p_1-1+2s-2j} = 0$ since $2p_1-3 \ge 2p_1-1+2s-2j \ge -2p_1+3$; hence $k_s = \sum_{j\ge 0} r_j \alpha_{2p_1-1+2s-2j}$ for $s \ge 2$. We have

$$r_0\alpha_{2p_1+1} + r_1\alpha_{2p_1-1} + r_{2p_1}\alpha_{-2p_1+1} = 0$$

hence $\alpha_{2p_1+1} = -(2p_1 + 1)$ and

$$k_1 = r_0\alpha_{2p_1+1} + r_1\alpha_{2p_1-1} = -1.$$

Also, $k_0 = 1$. Thus $\sum_{s\ge 0} k_s T^s = 1 - T$. The left hand side is

$$(\sum_{j\ge 0} r_j T^j)(\sum_{u\ge 0} \alpha_{2p_1-1+2u} T^u).$$

Thus $\sum_{u\ge 0} \alpha_{2p_1-1+2u} T^u = (1 - T)(1 + T)^{-2p_1}$. On the other hand, from the definition of x_{2u} we have $\sum_{u\ge 0} x_{2u} T^u = (1-T)(1+T)^{-2p_1}$. This proves (b) hence (a1).

Note that

(c) $\{z_i; i \in [0, 4p_1 - 4]''\})$ together with $\{\xi_i; i \in [0, 4p_2 - 4]''\}$ form a basis of V.

2.12. We show:

(a) $\{z_{2i}; i \in [0, 2p_1 - 1]\}$ are linearly independent.

Assume that this is not true. Then $z_{4p_1-2} \in E$, the span of $\{z_i; i \in [0, 4p_1 - 4]''\}$ hence E is g^{*2}-stable and the annihilator $(gE)^\perp$ of gE in V is g^{*2}-stable. For $i \in [0, 2p_1 - 2]$ we have $(\xi_{2p_2}, z_{2i+1}) = 0$ hence $\xi_{2p_2} \in (gE)^\perp$. Since $(gE)^\perp$ is g^{*2}-stable we see that $\xi_i \in (gE)^\perp$ for all $i \in \mathbf{Z}''$. Thus E', the span of $\{\xi_i; i \in [0, 4p_2 - 4]''\}$, is contained in $(gE)^\perp$. Now E' has dimension $2p_2 - 1$ which is the same as $\dim(gE)^\perp$. Hence $E' = (gE)^\perp$. Since $V = E\oplus E'$ (see 2.11(c)) we see that $V = E\oplus(gE)^\perp$ with both summands g^{*2}-stable. Now $-g^{*2}$ acts on E as a single Jordan block of size $2p_1 - 1$. Thus $-g^{*2} : V \to V$ has a Jordan block of size $2p_1 - 1$. This contradicts the assumption that the Jordan blocks of $-g^{*2} : V \to V$ have even sizes. This proves (a).

We set $N = g^{*2}+1, e = p_1-p_2$. Let $\mathcal{L}$ be the span of $\{N^i z_0; i \in [2p_2-1, 2p_1-1]\}$ or equivalently the span of $\{N^{2p_2-1} z_i; i \in [0, 4e]''\}$. We show that

(b) $\dim \mathcal{L} = 2e + 1$.

Let $\mathcal{L}'$ be the span of $\{N^i z_0; i \in [2p_2 - 1, 2p_1 - 2]\}$. We have $\dim \mathcal{L}' = 2e$ since $\{N^i z_0; i \in [0, 2p_1 - 2]\}$ is a linearly independent set. If (b) is false we would have $N^{2p_1-1} z_0 \in \mathcal{L}'$. Then the span of $\{N^i z_0; i \in [0, 2p_1 - 2]\}$ is N-stable. Hence the span of $\{g^{*(2i)} z_0; i \in [0, 2p_1 - 2]\}$ is g^{*2}-stable. This contradicts the proof of (a).

We show:

(c) $N^{2p_2-1}\xi_0 \in \mathcal{L}$.

From the structure of Jordan blocks of $N : V \to V$ we see that $\dim N^{2p_2-1}V = 2e+1$. Clearly, $\mathcal{L} \subset N^{2p_2-1}V$. Hence using (b) it follows that $\mathcal{L} = N^{2p_2-1}V$ so that (c) holds.

Using (c) we deduce

(d) $$N^{2p_2-1}\xi_0 = \sum_{i\in[0,2e]} c_{2i}N^{2p_2-1}z_{2i}$$

where $c_{2i} \in \mathbf{k}$ ($i \in [0, 2e]$) are uniquely determined.

2.13. For $j \in \mathbf{N}$ we set $m_j = \binom{2p_2-1}{j}$ so that $N^{2p_2-1} = \sum_{j\in[0,2p_2-1]} m_j g^{*(2j)}$. From 2.13(d) we deduce

(a) $$\sum_{j\in[0,2p_2-1]} m_j\xi_{2j} = \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j z_{2i+2j}.$$

Taking $(, z_u)$ with $u \in \mathbf{Z}'$ we deduce

(b) $$\sum_{j\in[0,2p_2-1]} m_j\gamma_{u-2j} = \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{u-2i-2j}.$$

We show:

(c1) If $u \in [2p_2-1, 4p_1-2p_2-3]'$, then the left hand side of (b) is 0.
(c2) If $u = 4p_1-2p_2-1$, then the left hand side of (b) is $\gamma_{4p_1-2p_2-1}$.

For (c1) it is enough to show: if u is as in (c1) and $j \in [0, 2p_2-1]$ then $u-2j+2p_2 \in [1, 4p_1-3]$. Indeed we have

$$u-2j+2p_2 \le 4p_1-2p_2-3+2p_2 = 4p_1-3$$

and

$$u-2j+2p_2 \ge 2p_2-1-4p_2+2+2p_2 = 1.$$

For (c2) it is enough to show: if $j \in [1, 2p_2-1]$, then $4p_1-2p_2-1-2j+2p_2 \in [1, 4p_1-3]$ or that $4p_1-1-2j \in [1, 4p_1-3]$. This is clear.

If $u \in [2p_2-1, 2p_1-3]'$, then in the right hand side of (b) we have $u-2i-2j < 2p_1-1$; we can assume then that $u-2i-2j \le -2p_1+1$ hence

$$2i \ge u-2j+2p_1-1 \ge 2p_2-1-(4p_2-2)+2p_1-1 = 2e$$

and $i \ge e$. Thus in this case (b) becomes (using (c1) and setting $u = 2p_1-1-2t$):

$$\sum_{i\in[e,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{2p_1-1-2t-2i-2j}$$

for $t \in [1, e]$. Setting $c'_h = c_{4e-h}$ for $h \in [0, 2e]''$ and with the change of variable $j \mapsto 2p_2-1-j$, $i \mapsto 2e-i$ we obtain

(d) $$\sum_{i\in[0,e],j\in[0,2p_2-1]} c'_{2i}m_j\alpha_{-2p_1+1-2t+2i+2j} = 0 \text{ for } t \in [1, e].$$

In the last sum we have $-2p_1+1-t+2i+2j < 2p_1-1$. Indeed, we have

$$-2p_1+1-2t+2i+2j \le -2p_1-1+2p_1-2p_2+4p_2-2 = 2p_2-3 < 2p_1-1.$$

Hence we can restrict the sum to indices such that $-2p_1+1-2t+2i+2j \le -2p_1+1$, that is, $-t+i+j=-2s$ where $s \ge 0$. Thus we have

$$\sum_{i\in[0,e],j\ge 0,s\ge 0,i+j+s=t} c'_{2i}m_j\alpha_{-2p_1+1-2s}=0 \text{ for } t\in[1,e].$$

Hence

$$(\sum_{i\in[0,e]} c'_{2i}T^i)(\sum_{j\ge 0} m_jT^j)(\sum_{s\ge 0}\alpha_{-2p_1+1-s}T^s)=c'_0+ \text{ terms of degree } > e \text{ in } T.$$

Using results in 2.11 this can be written as

$$(\sum_{i\in[0,e]} c'_{2i}T^i)(1+T)^{2p_2-1}(1-T)(1+T)^{-2p_1}=c'_0+ \text{ terms of degree } > e \text{ in } T,$$

that is,

$$(\sum_{i\in[0,e]} c'_{2i}T^i)(1+T)^{-2e-1}(1-T)=c'_0+ \text{ terms of degree } > e \text{ in } T,$$

hence

$$\sum_{i\in[0,e]} c'_{2i}T^i=(1-T)^{-1}(1+T)^{2e+1}(c'_0+ \text{ terms of degree } > e \text{ in } T).$$

We have $(1+T)^{2e+1}=\sum_{j\in[0,2e+1]} l_jT^j$ where $l_j=\binom{2e+1}{j}$. Hence

$$(1-T)^{-1}(1+T)^{2e+1}=\sum_{j\in[0,e]}(l_0+l_1+\dots+l_j)T^j+ \text{ terms of degree } > e \text{ in } T.$$

We see that

(e) $c'_{2i}=c'_0(l_0+l_1+\dots+l_i)$ for $i\in[0,e]$.

In the remainder of this subsection we assume that $e>0$. If $u=2p_1-1$, then in the right hand side of (b) we have $u-2i-2j\in[-2p_1+1,2p_1-1]$; we can then assume that $u-2i-2j$ is $-2p_1+1$ or $2p_1-1$. Hence $i+j$ is $2p_1-1$ or 0 and (i,j) is $(2e,2p_2-1)$ or $(0,0)$. Thus in this case (b) becomes (using (c1)) $c_0+c_{4e}=0$, that is, $c_0=-c'_0$. (The left hand side of (b) is 0 by (c1); here we use that $e>0$.)

If $u\in[2p_1+1,4p_1-2p_2-3]'$, then in the right hand side of (b) we have $u-2i-2j>-2p_1+1$; we can then assume that $u-2i-2j\ge 2p_1-1$ hence

$$2i\le u-2j-2p_1+1\le 4p_1-2p_2-3-2p_1+1=2e-2$$

and $i\le e-1$. Using this and (c1) we see that (b) becomes (setting $u=2p_1-1+2t$):

$$\sum_{i\in[0,e-1],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{2p_1-1+2t-2i-2j}=0 \text{ for } t\in[1,e-1].$$

Note that in the sum we have $2p_1-1+2t-2i-2j>-2p_1+1$. (Indeed we have $2p_1-1+2t-2i-2j\ge 2p_1+1-2p_1+2p_2+2-4p_2+2=-2p_2+5>-2p_1+1$.) Hence we can restrict the sum to indices such that $2p_1-1+2t-2i-2j\ge 2p_1-1$, that is, $2p_1-1+2t-2i-2j=2p_1-1+2s$ where $s\ge 0$. Thus we have

$$\sum_{i\in[0,e-1],j\ge 0,s\ge 0;i+s+j=t} c_{2i}m_j\alpha_{2p_1-1+2s}=0 \text{ for } t\in[1,e-1].$$

For such t we have also

$$\sum_{i\in[0,e-1],j\ge 0,s\ge 0;i+s+j=t} c'_{2i}m_j\alpha_{-2p_1+1-2s}=0$$

as we have seen earlier; the index i cannot take the value e since $i \le t$. Adding the last two equations and using $\alpha_{2p_1-1+2s} = \alpha_{-2p_1+1-2s}$ we obtain

$$(*) \qquad \sum_{i\in[0,e-1],j\ge 0,s\ge 0;i+s+j=t} (c_{2i}+c'_{2i})m_j\alpha_{-2p_1+1-2s} = 0 \text{ for } t\in[1,e-1].$$

We show that $c_{2i}+c'_{2i}=0$ for $i\in[0,e-1]$. For $i=0$ this is already known; the general case follows from $(*)$ by induction on i. Using also (e), we see that

$$\text{(f)} \qquad c_{2i} = -c'_0(l_0+l_1+\cdots+l_i) \text{ for } i\in[0,e-1].$$

(In the case where $i=0$, this is just $c_0=-c'_0$ which is already known.)

2.14. If $u=4p_1-2p_2-1$, then using 2.13(b) and 2.13(c2) we have

$$\text{(a)} \qquad \gamma_{4p_1-2p_2-1} = \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{4p_1-2p_2-1-2i-2j}.$$

Taking $(,\xi_{2p_2-1})$ with 2.13(a) we obtain

$$\sum_{j\in[0,2p_2-1]} m_j\beta_{2p_2-1-2j} = \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\gamma_{2i+2j-2p_2+1}.$$

In the left hand side only the contribution of $j=0$ and $j=2p_2-1$ is $\ne 0$; it is 1; in the right hand side we have $2i+2j-2p_2+1\ge -2p_2+1$ hence we can assume that $2i+2j-2p_2+1 > 4p_1-2p_2-3$, that is, $2i+2j\ge 4p_1-2$; hence we have $i=2e, j=2p_2-1$ and the right hand side is $c_{4e}\gamma_{4p_1-2p_2-1}$. Thus

$$\text{(b)} \qquad 2 = c'_0\gamma_{4p_1-2p_2-1}.$$

We see that $c'_0\ne 0$ and using (a) and (b) we have

$$2{c'_0}^{-1} = \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{4p_1-2p_2-1-2i-2j}.$$

In the right hand side we have $4p_1-2p_2-1-2i-2j\ge -2p_1+1$; we can assume then that either $4p_1-2p_2-1-2i-2j=-2p_1+1$ (hence $i=2e, j=2p_2-1$) or $4p_1-2p_2-1-2i-2j\ge 2p_1-1$ (hence $i\le e$). The first case can arise only if $e=0$ hence it is included in the second case. Thus

$$\text{(c)} \qquad 2{c'_0}^{-1} = \sum_{i\in[0,e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{4p_1-2p_2-1-2i-2j}.$$

Assume now that $e>0$. From 2.13(d) with $t=e$ we have

$$\text{(d)} \qquad 0 = \sum_{i\in[0,e],j\in[0,2p_2-1]} c'_{2i}m_j\alpha_{-4p_1+2p_2+1+2i+2j}.$$

We now add (c) and (d) and use that $c_{2i}+c'_{2i}=0$ if $i\in[0,e-1]$ and $c_e=c'_e$. We get

$$2{c'_0}^{-1} = 2c'_{2e}\sum_{j\in[0,2p_2-1]} m_j\alpha_{2p_1-1-2j}.$$

If $j\in[1,2p_2-1]$ we have $2p_1-1-2j\in[-2p_1+3,2p_1-3]$ hence $\alpha_{2p_1-1-j}=0$. Thus $2{c'_0}^{-1}=2c'_{2e}=2c'_0 2^{2e}$ and ${c'_0}^2=2^{-2e}$. Changing if necessary ξ by $-\xi$ we can therefore assume that

$$\text{(e)} \qquad c'_0 = 2^{-e}.$$

Assume now that $e = 0$. We have $c'_0 = c_0$ and (c) becomes

$$2c_0^{-1} = \sum_{j\in[0,2p_2-1]} c_0 m_j \alpha_{2p_1-1-2j},$$

that is, $2c_0^{-1} = 2c_0$ hence $c_0^2 = 1$. Changing if necessary ξ by $-\xi$ we can therefore assume that $c_0 = 1$. Thus (e) holds without the assumption $e > 0$.

Using (e) we rewrite 2.13(e) and 2.13(f) as follows:

(f) $$c_{2e-i} = 2^{-e}(l_0 + l_1 + \cdots + l_i) \text{ for } i \in [0, e],$$

(g) $$c_i = -2^{-e}(l_0 + l_1 + \cdots + l_i) \text{ for } i \in [0, e-1].$$

When z_i, ξ_i are replaced by the vectors with the same name in 2.10, the quantities c_{2i} become the quantities c^0_{2i}. (Here $i \in [0, 2e]$.) We show that

(h) $$c_{2i} = c^0_{2i} \text{ for } i \in [0, 2e].$$

By the analogue of (b) we have $2 = c^0_{4e}\gamma^0_{4p_1-2p_2-1}$. By results in 2.10 we have $\gamma^0_{4p_1-2p_2-1} = 2^{e+1}$. Hence $c^0_{4e} = 2^{-e}$. Using this and the analogues of 2.13(e), 2.13(f) we see that c^0_{2i} are given by the same formulas as c_{2i} in (e) and (f). This proves (h).

2.15. Let $C = \sum_{t\geq 0} \gamma_{4p_1-2p_2-1+2t}T^t$, $C^0 = \sum_{t\geq 0} \gamma^0_{4p_1-2p_2-1+2t}T^t$. If $u = 4p_1 - 2p_2 - 1 + 2t, t \geq 0$, then for any j that contributes to the left hand side of 2.13(b) we have $u - 2j \geq -2p_2 + 1$. Indeed,

$$u - 2j \geq 4p_1 - 2p_2 - 1 - 2j \geq 4p_1 - 2p_2 - 1 - 4p_2 + 2 \geq -2p_2 + 1$$

hence we can assume that in the left hand side of 2.13(b) we have $u - 2j \geq 4p_1 - 2p_2 - 1$. Muliplying both sides of 2.13(b) with T^t and summing over all $t \geq 0$ we thus obtain

$$\sum_{t\geq 0} \sum_{j\in[0,2p_2-1];t-j\geq 0} m_j \gamma_{4p_1-2p_2-1+2t-2j} T^t$$
$$= \sum_{t\geq 0} \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i} m_j \alpha_{4p_1-2p_2-1+2t-2i-2j} T^t.$$

The left hand side equals

$$(\sum_{j\in[0,2p_2-1]} m_j T^j)(\sum_{t'\geq 0} \gamma_{4p_1-2p_2-1+2t'} T^{t'}) = (1+T)^{2p_2-1} C.$$

Thus,

$$C = (1+T)^{-2p_2+1} \sum_{t\geq 0} \sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i} m_j \alpha_{4p_1-2p_2-1+2t-2i-2j} T^t.$$

Similarly we have

$$C^0 = (1+T)^{-2p_2+1} \sum_{t\geq 0} \sum_{i\in[0,2e],j\in[0,2p_2-1]} c^0_{2i} m_j \alpha^0_{4p_1-2p_2-1+2t-2i-2j} T^t.$$

By 2.14(h) we have $c_{2i} = c^0_{2i}$. By 2.11(a1) we have

$$\alpha_{4p_1-2p_2-1+2t-2i-2j} = \alpha^0_{4p_1-2p_2-1+2t-2i-2j}$$

for all i, j, t. It follows that $C = C^0$ hence

(a) $$\gamma_{4p_1-2p_2-1+2t} = \gamma^0_{4p_1-2p_2-1+2t} \text{ for any } t \geq 0.$$

We set $C'=\sum_{t\ge0}\gamma_{2p_2-3-2t}T^t$, $C'^0=\sum_{t\ge0}\gamma^0_{2p_2-3-2t}T^t$. If $u=2p_2-3-2t, t\ge0$, then for any j that contributes to the left hand side of 2.13(b) we have $u-2j\le 4p_1-2p_2-3$ (indeed, $u-2j\le 2p_2-3-2j\le 2p_2-3\le 4p_1-2p_2-3$) hence we can assume that in the left hand side of 2.13(b) we have $u-2j\le -2p_2-1$. With the substitution $j\mapsto 2p_2-1-j$ the previous inequality becomes $j-t\le0$ and the left hand side of 2.13(b) becomes

$$\sum_{j\in[0,2p_2-1]} m_j\gamma_{u-4p_2+2+2j}=\sum_{j\in[0,2p_2-1]} m_j\gamma_{-2p_2-1+2(j-t)}.$$

Muliplying both sides of 2.13(b) with T^t and summing over all $t\ge0$ we thus obtain

$$\sum_{t\ge0,j\ge0;t-j\ge0} m_j\gamma_{-2p_2-1+2(j-t)}T^t=\sum_{t\ge0}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{2p_2-3-2t-2i-2j}T^t.$$

The left hand side equals

$$(\sum_{j\in[0,2p_2-1]} m_jT^j)(\sum_{t'\ge0}\gamma_{-2p_2-1-2t'}T^{t'})=(1+T)^{2p_2-1}C'.$$

Thus,

$$C'=(1+T)^{-2p_2+1}\sum_{t\ge0}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\alpha_{2p_2-3-2t-2i-2j}T^t.$$

Similarly we have

$$C'^0=(1+T)^{-2p_2+1}\sum_{t\ge0}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c^0_{2i}m_j\alpha^0_{2p_2-3-2t-2i-2j}T^t.$$

By 2.14(h) we have $c_{2i}=c^0_{2i}$. By 2.11(a1) we have

$$\alpha_{2p_2-3-2t-2i-2j}=\alpha^0_{2p_2-3-2t-2i-2j}$$

for all i,j,t. It follows that $C'=C'^0$ hence

(b) $$\gamma_{2p_2-3-2t}=\gamma^0_{2p_2-3-2t} \text{ for any } t\ge0.$$

Clearly, (a) and (b) imply 2.11(a3).

2.16. We set $B=\sum_{s\ge0}\beta_{2p_2-1+2s}T^s$, $B^0=\sum_{s\ge0}\beta^0_{2p_2-1+2s}T^s$. Let $t\ge1$. Taking $(,\xi_{2p_2-1+2t})$ with 2.13(a) we obtain

(a) $$\sum_{j\in[0,2p_2-1]} m_j\beta_{2p_2-1+2t-2j}=\sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\gamma_{2i+2j-2p_2+1-2t}.$$

For any j that contributes to the left hand side of (a) we have $2p_2-1+2t-2j\ge -2p_2+3$ (indeed, $2p_2-1+2t-2j\ge 2p_2+1-4p_2+2=-2p_2+3$) hence we can assume that in the left hand side of (a) we have $2p_2-1+2t-2j\ge 2p_2-1$, that is, $t\ge j$. Multiplying both sides of (a) by T^t and summing over all $t\ge1$, we thus obtain

$$\sum_{t\ge1}\sum_{j\in[0,2p_2-1];t\ge j} m_j\beta_{2p_2-1+2t-2j}T^t$$
$$=\sum_{t\ge1}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\gamma_{2i+2j-2p_2+1-2t}T^t.$$

The left hand side equals

$$-1+(\sum_{j\in[0,2p_2-1]} m_jT^j)(\sum_{t'\ge 0}\beta_{2p_2-1+t'}T^{t'})=1+(T+1)^{2p_2-1}B.$$

Thus,

$$B=(T+1)^{2p_2-1}(1+\sum_{t\ge 1}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c_{2i}m_j\gamma_{2i+2j-2p_2+1-2t}T^t).$$

Similarly we have

$$B^0=(T+1)^{2p_2-1}(1+\sum_{t\ge 1}\sum_{i\in[0,2e],j\in[0,2p_2-1]} c^0_{2i}m_j\gamma^0_{2i+2j-2p_2+1-2t}T^t).$$

By 2.14(h) we have $c_{2i}=c^0_{2i}$. By 2.11(a3) we have

$$\gamma_{2i+2j-2p_2+1-2t}=\gamma^0_{2i+2j-2p_2+1-2t}$$

for any i,j,t. It follows that $B=B^0$. Hence

$$\beta_{2p_2-1+2s}=\beta^0_{2p_2-1+2s}$$

for any $s\ge 0$. This clearly implies 2.11(a2).

2.17. We preserve the setup of 2.1. We prove 2.1(a) by induction on n. If $n=0$ we have $V=0$ and $a_i=b_i=p_i=0$ for all i. We take $g=0$ and (L^t) to be the empty set of lines. We obtain an element of $\tilde{\mathcal{C}}^V_{a_*,b_*}$. Now assume that $n>0$.

Assume first that $a_1\ge 1$. We can find a direct sum decomposition $V=V'\oplus V''$ such that $\dim V'=a_1+b_1=2p_1-1$. We identify $V^*=V'^*\oplus V''^*$ in the obvious way. Let a'_* be the sequence $a_1,0,0,\dots$; let b'_* be the sequence $b_1,0,0,\dots$; let a''_* be the sequence $a_2,a_3,\dots$; let b''_* be the sequence $b_2,b_3,\dots$. By the induction hypothesis we have $\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}\ne\emptyset$. By 2.3 we have $\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}\ne\emptyset$. Let $(g',L^1)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ and let $(g'',L^2,L^3,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$. Here $g'\in G^1_{V'}$, $g''\in G^1_{V''}$. Let $g=g'\oplus g''\in G^1_V$. Clearly, $(g,L^1,L^2,\dots)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$ hence 2.1(a) holds in this case. Thus we may assume that $a_1=a_2=\dots=0$ and $b_1>0$. We see that $-g^{*2}$ is unipotent. We can find a direct sum decomposition $V=V'\oplus V''$ such that $\dim V'=b_1+b_2$. We identify $V^*=V'^*\oplus V''^*$ in the obvious way. Let b'_* be the sequence $b_1,b_2,0,\dots$; let b''_* be the sequence $b_3,b_4,\dots$; let $a'_*=a''_*$ be the sequence $0,0,\dots$. By the induction hypothesis we have $\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}\ne\emptyset$. By 2.11 we have $\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}\ne\emptyset$. Let $(g',L^1,L^2)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ and let $(g'',L^3,L^4,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$. Here $g'\in G^1_{V'}$, $g''\in G^1_{V''}$. Clearly, $(g'\oplus g'',L^1,L^2,\dots)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$ hence 2.1(a) holds in this case. This completes the proof of 2.1(a).

In the following result (which is needed in the proof of 2.1(b),(c)) we preserve the setup of 2.1.

Proposition 2.18. *Let $(g,L^1,L^2,\dots,L^\sigma)\in\tilde{\mathcal{C}}^V_{a_*,b_*}$. Let ϕ_r be as in 2.4. There exist vectors $z^t\in L^t-\{0\}$ for $t\in[1,\sigma]$ such that (i) and (ii) below hold for $i\in\mathbf{Z}'',j\in\mathbf{Z}'$.*

(i) *Assume that $t\in[1,\sigma],a_t>0$. Then $(z^t_i,z^t_j)=x'_{i-j}$ (x'_h as in 2.2 with $p=p_t$); $(z^t_i,z^{t'}_j)=0$ if $t'\in[1,\sigma],t'\ne t$.*

(ii) *Assume that* $\{t,t+1\}\subset[k+1,\sigma], t=k+1 \mod 2$ *and* $a_t=0$. *Then*

$$(z_i^t,z_j^t)=\phi_{p_t}(i-j),$$
$$(z_i^{t+1},z_j^{t+1})=\sum_{r\in[p_{t+1},p_t]}2^{2r-2p_{t+1}}\phi_r(i-j),$$
$$(z_i^t,z_j^{t+1})=2^{p_t-p_{t+1}+1}(-1)^{(i-j+2p_2+1)/2}(i-j+2p_{t+1}-1)(i-j+2p_{t+1}-3)\dots$$
$$\times(i-j+2p_{t+1}-4p_t+3)(4p_t-2)!!^{-1},$$
$$(z_i^t,z_j^{t'})=0 \text{ if } t'\in[1,\sigma], t'\notin\{t,t+1\}.$$

We argue by induction on n. When $n=0$ the result is obvious. Now assume that $n\ge 1$.

Case 1. Assume first that $a_1\ge 1$. We have $a_1+b_1=2p_1-1$. Let $V'=\bigoplus_{i\in[0,4p_1-4]''}L_i^1\subset V$. We show that

(a) $$g^{*2}V'=V'.$$

It is enough to show that $g^{*2}L^1_{4p_1-4}\subset V'$. Since $g^{*i}L_0^1\in V'$ for $i\in[0,4p_1-4]''$ and $a_1+b_1=2p_1-1$ it is enough to show that $(g^{*2}-1)^{a_1}(g^{*2}+1)^{b_1}L_0^1=0$. It is also enough to show that $(g^{*2}-1)^{a_1}(g^{*2}+1)^{b_1}=0$ on V. But this follows from the fact that $g\in\mathcal{C}^V_{a_*,b_*}$.

Now let $V''=\bigoplus_{t\in[2,\sigma],i\in[0,2p_t-2]}L_i^t\subset V$. We show that

(b) $V''=(gV')^\perp$, *the annihilator of* gV' *in* V. *Hence* V'' *is* g^{*2}*-stable and* $V=V'\oplus(gV')^\perp$.

We have $(L^r_{2p_r},L^1_{i+1})=0$ for $r\in[2,\sigma]$, $i\in[0,4p_1-4]''$. Thus $L^r_{2p_r}\subset(gV')^\perp$. Since $(gV')^\perp$ is g^{*2}-stable (we use (a) and 2.0(a)) it follows that $L_i^r\subset(gV')^\perp$ for any $i\in\mathbf{Z}'', r\in[2,\sigma]$. Thus $V''\subset(gV')^\perp$. But these two vector spaces have the same dimension so that $V''=(gV')^\perp$ and (b) follows.

We identify $V^*=V'^*\oplus V''^*$ in the obvious way. From (a),(b) we see that $g\in G_V^1$ restricts to an isomorphism $g':V'\to V'^*$ and to an isomorphism $g'':V''\to V''^*$. We show:

(c) g'^{*2} *restricted to the generalized* 1*-eigenspace of* g'^{*2} *is unipotent with a single Jordan block of size* a_1; $-g'^{*2}$ *restricted to the generalized* (-1)*-eigenspace of* g'^{*2} *is unipotent with a single Jordan block of size* b_1 *(if that eigenspace is* $\neq 0$*). Moreover,* g''^{*2} *restricted to the generalized* 1*-eigenspace of* g''^{*2} *is unipotent with Jordan blocks of sizes given by the nonzero numbers in* $a_2,a_3,\dots$; $-g''^{*2}$ *restricted to the generalized* (-1)*-eigenspace of* g''^{*2} *is unipotent with Jordan blocks of sizes given by the nonzero numbers in* $b_2,b_3,\dots$.

As we have seen earlier we have $(g^{*2}-1)^{a_1}(g^{*2}+1)^{b_1}=0$ on V' (even on V). Also $g'^{*2}\in GL(V')$ is regular in the sense of Steinberg and $\dim V'=a_1+b_1$. This implies (c).

Let a'_* be the sequence $a_1,0,0,\dots$; let b'_* be the sequence $b_1,0,0,\dots$; let a''_* be the sequence $a_2,a_3,\dots$; let b''_* be the sequence $b_2,b_3,\dots$. Now the proposition holds when $(g,L^1,L^2,\dots)$ is replaced by $(g'',L^2,L^3,\dots)\in\tilde{\mathcal{C}}^{V''}_{a''_*,b''_*}$ (by the induction hypothesis) or by $(g',L^1)\in\tilde{\mathcal{C}}^{V'}_{a'_*,b'_*}$ (we choose any $z^1\in L^1-\{0\}$ such that $(z_i^1,z_j^1)=1$ for $i\in\mathbf{Z}'',j\in\mathbf{Z}'$, $|i-j|=2p_1-1$ and we apply 2.3). Hence the proposition holds for $(g,L^1,L^2,\dots)$ (we use (b)).

Case 2. Next we assume that $k = 0$, $b_1 > 0$. Then $a_1 = a_2 = \cdots = 0$. We have $b_1 = 2p_1, b_2 = 2p_2 - 2$. Let $V' = \bigoplus_{t\in[1,2],i\in[0,4p_t-4]''} L_i^t \subset V$. We show that

$$g^{*2}V' = V'. \tag{d}$$

Let $N = g^{*2} + 1$. Then $V = \bigoplus_{t\in[1,\sigma],i\in[0,4p_t-4]''} N^{i/2}L_0^t$ is a direct sum decomposition into lines. Now $N^{2p_2-2}(V)$ contains the lines

$$N^{2p_2-2+(i/2)}L_0^1(i \in [0, 4p_1 - 4p_2]'') \quad \text{and} \quad N^{2p_2-2}L_0^2 \tag{$*$}$$

(whose number is $2p_1 - 2p_2 + 2$); moreover, since N has Jordan blocks of sizes $b_1 = 2p_1, b_2 = 2p_2 - 2$ and others of size $< b_2$ we see that $\dim N^{2p_2-2}(V) = 2p_1 - 2p_2 + 2$ so that $N^{2p_2-2}(V)$ is equal to the subspace spanned by $(*)$ and $N^{2p_2-2}(V) \subset V'$. Now V' is the subspace of V spanned by the lines $N^iL_0^t$ with $t \in [1, 2], i \in [0, 2p_t-2]$. It is enough to show that $NV' \subset V'$ or that $N^{2p_t-1}L_0^t \subset V'$ for $t = 1, 2$. But for $t = 1, 2$ we have $N^{2p_t-1}L_0^t \subset N^{2p_2-2}V \subset V'$ since $2p_t - 2p_2 + 1 \geq 0$. This proves (d).

Let $V'' = \bigoplus_{t\in[3,\sigma],i\in[0,4p_t-4]''} L_i^t \subset V$. We show that

(e) $V'' = (gV')^\perp$, *the annihilator of* gV' *in* V. *Hence* V'' *is* g^{*2}*-stable and* $V = V' \oplus (gV')^\perp$.

We have $(L_{2p_r}^r, L_{i+1}^t) = 0$ for $t \in [1, 2], r \in [3, \sigma]$, $i \in [0, 4p_t - 4]''$. Thus $L_{2p_r}^r \subset (gV')^\perp$. Since $(gV')^\perp$ is g^{*2}-stable (we use (d) and 2.0(a)) it follows that $L_i^r \subset (gV')^\perp$ for any $i \in \mathbf{Z}'', r \in [3, \sigma]$. Thus $V'' \subset (gV')^\perp$. But these two vector spaces have the same dimension so that $V'' = (gV')^\perp$ and (e) follows.

We identify $V^* = V'^* \oplus V''^*$ in the obvious way. From (d) and (e) we see that $g : V \to V^*$ restricts to an isomorphism $g' : V' \to V'^*$ and to an isomorphism $g'' : V'' \to V''^*$. We show:

(f) $-g'^{*2}$ *is unipotent with a single Jordan block of size* b_1 *(if* $b_2 = 0$*) or with two Jordan blocks of size* b_1, b_2 *(if* $b_2 > 0$*). Moreover,* $-g''^{*2}$ *is unipotent with Jordan blocks of sizes given by the nonzero numbers in* $b_3, b_4, \ldots$*.*

Since V' is the direct sum of the lines $N^iL_0^t$, $t \in [1, 2], i \in [0, 2p_t - 2]$, and V' is N-stable, we see that the kernel of $N : V' \to V'$ has dimension ≤ 2. Hence $N : V' \to V'$ has either a single Jordan block of size $2p_1 + 2p_2 - 2 = b_1 + b_2$ or two Jordan blocks of sizes $b_1' \geq b_2'$ where $b_1' + b_2' = b_1 + b_2$. In the first case we must have $b_2 = 0$ (since the Jordan blocks of $N : V' \to V'$ have sizes $\leq b_1$ (by (e)). In the second case, since b_1', b_2' must form a subsequence of $b_1 > b_2 > b_3 > \ldots$ and $b_1' + b_2' = b_1 + b_2$ it follows that $b_1' = b_1$, $b_2' = b_2$. This implies (f). This completes the proof.

2.19. In the setup of 2.1, we show that 2.1(b) holds. We must show that

(a) any two elements $(g, L^1, L^2, \ldots, L^\sigma)$, $(g', L'^1, L'^2, \ldots, L'^\sigma)$ of $\tilde{\mathcal{C}}_{a_*,b_*}^V$ are in the same G_V-orbit.

Since G_V acts transitively on $\mathcal{C}_{a_*,b_*}^V$ we can assume that $g = g'$. Let $z^t \in L^t$ $(t \in [1, \sigma])$ be as in 2.18. Let $z'^t \in L'^t$ $(t \in [1, \sigma])$ be the analogous vectors for $(g, L'^1, L'^2, \ldots)$ instead of $(g, L^1, L^2, \ldots)$. By 2.18 we have

$$(z_i^t, z_j^{t'}) = (z'^t_i, z'^{t'}_j) \tag{b}$$

for any $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ and any $t, t' \in [1, \sigma]$. Since $\{z_i^t; t \in [1, \sigma], i \in [0, 4p_t - 4]''\}$ and $\{z'^t_i; t \in [1, \sigma], i \in [0, 4p_t - 4]''\}$ are bases of V (see 2.0(d)) we see that there is a unique $\gamma \in GL(V)$ such that $\gamma(z_i^t) = z'^t_i$ for any $t \in [1,\sigma], i \in [0, 4p_t - 4]$. We show that

(c) $$\check{\gamma}(z_{j+1}^t) = z'^t_{j+1} \text{ for any } t \in [1,\sigma], j \in [0, 4p_t - 4]''.$$

It is enough to show that $(z'^{t'}_i, z'^t_{j+1}) = (z'^{t'}_i, \check{\gamma}(z_{j+1}^t))$, that is, $(z'^{t'}_i, z'^t_{j+1}) = (z_i^{t'}, z_{j+1}^t)$ for any $t, t' \in [1,\sigma]$ and any $i, j \in [0, 4p_t - 4]''$. This follows from (b). From (c) we see that $\check{\gamma}(g(z_j^t)) = g(\gamma(z_j^t))$ for any $t \in [1,\sigma], j \in [0, 4p_t - 4]''$. It follows that $\check{\gamma} g = g\gamma$. From the definition it is clear that $\gamma(L^t) = L'^t$ for $t \in [1,\sigma]$. Thus (a) holds (with $g' = g$). This proves 2.1(b).

2.20. In the setup of 2.1, we show that 2.1(c) holds. Let $(g, L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$ and let I be the set of all $\gamma \in G_V$ such that $\check{\gamma} g \gamma^{-1} = g$, $\gamma(L^t) = L^t$ for $t \in [1,\sigma]$. Let $z^t \in L^t (t \in [1,\sigma])$ be as in 2.18. Let $\gamma \in I$. If $t \in [1,\sigma]$ we have $\gamma(z^t) = \omega_t^\gamma z^t$ where $\omega_t^\gamma \in \mathbf{k} - \{0\}$. Since γ commutes with g^{*2}, it follows that $\gamma(z_i^t) = \omega_t^\gamma z_i^t$ for $i \in \mathbf{Z}''$. For $t \in [1,\sigma], j \in \mathbf{Z}'$ we have

$$\check{\gamma}(z_j^t) = \check{\gamma}(g(z_{j-1}^t)) = g(\gamma(z_{j-1}^t)) = g(\omega_t^\gamma z_{j-1}^t) = \omega_t^\gamma z_j^t;$$

thus, $\check{\gamma}(z_j^t) = \omega_t^\gamma z_j^t$. For any $t, t' \in [1,\sigma]$, $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ we have

$$(z_i^{t'}, \omega_t^\gamma z_j^t) = (z_i^{t'}, \check{\gamma}(z_j^t)) = (\gamma^{-1}(z_i^{t'}), z_j^t) = (\omega_{t'}^\gamma)^{-1}(z_i^{t'}, z_j^t).$$

Thus, $(\omega_t^\gamma - (\omega_{t'}^\gamma)^{-1})(z_i^{t'}, z_j^t) = 0$. Taking $t' = t, i - j = 2p_t - 1$ we deduce that $\omega_t^\gamma - (\omega_t^\gamma)^{-1} = 0$ hence $\omega_t^\gamma = \pm 1$. Taking $t' = t+1$ (where $\{t, t+1\} \subset [k+1,\sigma], t = k+1 \mod 2, a_t = 0$) and using that

$$(z_i^{t+1}, z_j^t) = (z_{-i}^t, z_{-j}^{t+1}) = \pm 2^{p_t - p_{t+1} + 1} \text{ if } j - i + 2p_{t+1} = -1$$

we see that $(\omega_t^\gamma - (\omega_{t+1}^\gamma)^{-1}) 2^{p_t - p_{t+1} + 1} = 0$ hence $\omega_t^\gamma - (\omega_{t+1}^\gamma)^{-1} = 0$ and $\omega_t^\gamma = \omega_{t+1}^\gamma$. We see that $\gamma \mapsto (\omega_t^\gamma)$ is a homomorphism $\psi : I \to \mathcal{I}$ (notation of 2.0). Assume that γ is in the kernel of ψ. Then γ restricts to the identity map $L^t \to L^t$ for $t \in [1,\sigma]$. Since γ commutes with g^{*2} it follows that γ restricts to the identity map on each of the lines $g^{*i} L^t$ ($t \in [1,\sigma]$, $i \in \mathbf{Z}''$). Since these lines generate V (see 2.0) we see that $\gamma = 1$. Thus, ψ is injective. Now let $(\omega_t) \in \mathcal{I}$. We define $\gamma \in GL(V)$ by $\gamma(z_i^t) = \omega_t z_i^t$ for $t \in [1,\sigma], i \in [0, 4p_t - 4]''$. From the definitions we see that

(a) $$(\omega_t z_i^t, \omega_{t'} z_j^{t'}) = (z_i^t, z_j^{t'})$$

for any $i \in \mathbf{Z}'', j \in \mathbf{Z}'$ and any $t, t' \in [1,\sigma]$. We show that

(b) $$\check{\gamma}(z_{i+1}^t) = \omega_t z_{i+1}^t \text{ for any } t \in [1,\sigma], i \in [0, 4p_t - 4]''.$$

It is enough to show that $(\gamma(z_j^{t'}), \omega_t z_{i+1}^t) = (z_j^{t'}, z_{i+1}^t)$ for any $t' \in [1,\sigma], j \in [0, 4p_{t'} - 4]''$ or that $(\omega_{t'} z_j^{t'}, \omega_t z_{i+1}^t) = (z_j^{t'}, z_{i+1}^t)$ or that

$$(\omega_{t'}\omega_t - 1)(z_j^{t'}, z_{i+1}^t) = 0.$$

The second factor is zero unless either $t = t'$ or $t' = t+1$ (where $\{t, t+1\} \subset [k+1,\sigma], t = k+1 \mod 2, a_t = 0$) in which case the first factor is zero. This proves (b).

From (b) we see that $\check{\gamma}(g(z_i^t)) = g(\gamma(z_i^t))$ for any $t \in [1,\sigma], i \in [0, 4p_t - 4]''$. It follows that $\check{\gamma} g = g\gamma$. From the definition it is clear that $\gamma(L^t) = L^t$ for $t \in [1,\sigma]$. Thus $\gamma \in I$. We see that ψ is surjective hence an isomorphism. This proves 2.1(c).

2.21. We now assume that $n \geq 1$. We denote by $\overset{n}{V}$ (resp. $\overset{n}{V^*}$) the n-th exterior power of V (resp. V^*); we have naturally $\overset{n}{V^*} = (\overset{n}{V})^*$. Any $\gamma \in G_V$ induces an element $\overset{n}{\gamma} : \overset{n}{V} \xrightarrow{\sim} \overset{n}{V}$; any $g \in G_V^1$ induces an element $\overset{n}{g} : \overset{n}{V} \xrightarrow{\sim} \overset{n}{V^*}$. For any $\theta \in \overset{n}{V} - \{0\}$ we denote by θ^* the unique element in $\overset{n}{V^*} - \{0\}$ such that $(\theta, \theta^*) = 1$.

We show:

(a) For any $g \in G_V^1$ we have $\check{g}g \in SL(V)$.

Let (e_i) be a basis of V; let (e_i^*) be the dual basis of V^*. We have $ge_i = \sum_j x_{ij} e_j^*$, $\check{g}e_k^* = \sum_h y_{kh} e_h$ where $X = (x_{ij}), Y = (y_{ij})$ are square matrices. Now

$$\delta_{ki} = (\check{g}e_k^*, ge_i) = (\sum_h y_{kh} e_h, \sum_j x_{ij} e_j^*) = \sum_h y_{kh} x_{ih}.$$

Thus $YX^t = I$ where X^t is the transpose of X. We have $\check{g}ge_i = \sum_{j,h} x_{ij} y_{jh} e_h$. Thus the matrix of $\check{g}g$ is XY. We have

$$\det(XY) = \det(X)\det(Y) = \det(X^t)\det(Y) = \det(YX^t) = 1,$$

as required.

We now fix $\theta \in \overset{n}{V} - \{0\}$ and we set

$$\Gamma^1 = \{g \in G_V^1; \overset{n}{g} \text{ takes } \theta \text{ to } \theta^*\}.$$

If $g \in \Gamma^1$ then, using (a), we see that $\overset{n}{\check{g}}$ takes θ^* to θ. We see that $\Gamma := SL(V) \sqcup \Gamma^1$ is a subgroup of $G_V \sqcup G_V^1$. Let $SL(V)' = \{\Gamma \in G_V; \det(\Gamma) = \pm 1\}$.

We show:

(b) *Let $g, g' \in G_V^1$, $\gamma \in G_V$ be such that $\check{\gamma} g \gamma^{-1} = g'$. If $g, g' \in \gamma^1$, then $\gamma \in SL(V)'$. Conversely, if $g \in \Gamma^1$ and $\Gamma \in SL(V)'$, then $g' \in \Gamma^1$.*

Replacing V by $\overset{n}{V}$ we can assume that $n = 1$. We have $g\theta = \theta^*$, $g'\theta = \theta^*$, $\gamma\theta = a\theta$ where $a \in \mathbf{k} - \{0\}$. We have $\theta^* = \check{\gamma} g \gamma^{-1}(\theta) = \check{\gamma} g a^{-1} \theta = \check{\gamma} a^{-1} \theta^* = a^{-2} \theta^*$ hence $a^2 = 1$ and $a = \pm 1$ proving the first assertion of (b). The second assertion is proved similarly.

2.22. Assuming that $a_1 > 0$ we show:

(a) $\mathcal{C}^V_{a_*, b_*} \cap \Gamma^1$ is a single $SL(V)$-conjugacy class in Γ.

Let $g, g' \in \mathcal{C}^V_{a_*, b_*} \cap \Gamma^1$. From Theorem 2.1(b) we see that $\check{\gamma} g \gamma^{-1} = g'$ for some $\gamma \in G_V$. Using 2.21(b) we see that $\det(\gamma) = \pm 1$. If $\det(\gamma) = 1$, then g, g' are in the same $SL(V)$-conjugacy class, as required. Assume now that $\det(\gamma) = -1$. We complete g to an element $(g, L^1, L^2, \dots) \in \tilde{\mathcal{C}}^V_{a_*, b_*}$ and we write $V = V' \oplus V''$, $V^* = V'^* \oplus V''^*$ as in the proof of 2.18 (Case 1). Let $\gamma_0 \in GL(V)$ be such that $\gamma_0|_{V'} = -1$, $\gamma_0|_{V''} = 1$. Since $\dim V'$ is odd we have $\det(\gamma_0) = -1$. We have $\check{\gamma}_0 g \gamma_0^{-1} = g$ hence $\check{\gamma}\check{\gamma}_0 g \gamma_0^{-1} \gamma^{-1} = g'$. We have $\gamma\gamma_0 \in SL(V)$ so that g, g' are in the same $SL(V)$-conjugacy class, as required.

2.23. Assuming that $a_1 = 0$ (hence $b_1 > 0$) we show:

(a) $\mathcal{C}^V_{a_*, b_*} \cap \Gamma^1$ is a union of two $SL(V)$-conjugacy classes in Γ.

Let $g \in \mathcal{C}^V_{a_*, b_*} \cap \Gamma^1$. Let $C(g)$ (resp. $C'(g)$) be the set of elements of the form $\check{\gamma} g \gamma^{-1} = g'$ for some $\gamma \in G_V$ such that $\det(\gamma) = 1$ (resp. $\det(\gamma) = -1$). It is clear that $C(g)$ and $C'(g)$ are $SL(V)$-conjugacy classes. As in the proof of 2.22 we see, using 2.1(b) and 2.21(b), that $\mathcal{C}^V_{a_*, b_*} \cap \Gamma^1 = C(g) \cup C'(g)$. It remains to prove that $C(g) \cap C'(g) = \emptyset$. Assume that $C(g) \cap C'(g) \neq \emptyset$. It follows that there exists $\gamma_0 \in G_V$ such that $\check{\gamma}_0 g \gamma_0^{-1} = g$ and satisfies $\det(\gamma_0) = -1$. Let g_s be the semisimple part of g. Then γ_0 is in the centralizer of g_s in G_V which is a symplectic group all of whose elements have necessarily determinant 1. This contradicts $\det(\gamma_0) = -1$.

2.24. Let $\mathbf{c}$ be an $SL(V)$-conjugacy class contained in $\mathcal{C}^V_{a_*,b_*} \cap \Gamma^1$. (See 2.22(a), 2.23(a).) Let X be the set of all $(g, L^1, L^2, \dots, L^\sigma) \in \tilde{\mathcal{C}}^V_{a_*,b_*}$ where $g \in \mathbf{c}$. Note that $X \neq \emptyset$. Now $SL(V)'$ acts on X by the restriction of the G_V-action on $\tilde{\mathcal{C}}^V_{a_*,b_*}$ (see 2.21(b)). Using 2.1(b) and 2.21(b), we see that this $SL(V)'$-action is transitive. We now restrict this action to $SL(V)$.

We show:

(a) *This $SL(V)$-action is transitive.*

Let $(g, L^1, L^2, \dots, L^\sigma) \in X$, $(g', L'^1, L'^2, \dots, L'^\sigma) \in X$. We must show that these two sequences are in the same $SL(V)$-orbit. As we have seen, we can find $\gamma \in SL(V)'$ which conjugates $(g, L^1, L^2, \dots, L^\sigma)$ to $(g', L'^1, L'^2, \dots, L'^\sigma)$. If $a_1 = 0$ this implies by the argument in 2.3 that $\det(\gamma) = 1$ so that in this case (a) holds. We can thus assume that $a_1 > 0$. If $\det(\gamma) = 1$, then the proof is finished. We now assume that $\det(\gamma) = -1$. Let $\gamma_0 \in G_V$ be as in 2.22. We have $\det(\gamma_0) = -1$ and γ_0 conjugates $(g, L^1, L^2, \dots, L^\sigma)$ to itself. Hence $\gamma\gamma_0$ conjugates $(g, L^1, L^2, \dots, L^\sigma)$ to $(g', L'^1, L'^2, \dots, L'^\sigma)$. We have $\gamma\gamma_0 \in SL(V)$. This proves (a).

2.25. Assume that $n \geq 3$. As in [L5, §4] we see that 2.24(a) implies that Theorem 0.3 holds for Γ instead of G.

3. Exceptional groups

3.1. In this section we assume that $G = G^0$ (as in 0.2) is simple of exceptional type. In the case where $\mathbf{c}$ is a distinguished unipotent class this follows from [L3] where it was proved by a reduction to a computer calculation. In the nonunipotent case the same method works but it uses instead of [L1, 1.2(c)], the more general formula [L6, 5.3(a)]. The needed computer calculation was actually done at the time of preparing [L6]. (I thank Frank Lübeck for providing to me tables of Green functions for groups of rank ≤ 8 in GAP format. I also thank Gongqin Li for her help with programming in GAP to perform the computer calculation.)

We will describe below the result in the form of a list of rows in each case; each row corresponds to an ϵ_D-elliptic ϵ_D-conjugacy class in W. For example, the row

$$12; \Phi_{20}; (E_8(a_2))_{E_8}, (E_7(a_2)A_1)_{E_7A_1}, (J_{11}J_5)_{D_8}$$

in type E_8 corresponds to the elliptic conjugacy class C in W such that the characteristic polynomial in the reflection representation of any $w \in C$ is the cyclotomic polynomial Φ_{20} and the length of any element in C_{min} is $d_C = 12$. The row also includes the names of the three distinguished conjugacy classes $\mathbf{c}$ such that $C♣\mathbf{c}$ (see 0.1); for example, $(E_7(a_2)J_2)_{E_7A_1}$ is the conjugacy class of $su = us$ where s is a semisimple element with $Z_G(s)^0$ of type E_7A_1 (in the subscript) and u is a unipotent element of $Z_G(s)^0$ whose E_7 component is of type $E_7(a_2)$ (notation as in [L1, 4.3]) and whose A_1-component has a single Jordan block of size 2 in the standard representation of A_1. On the other hand, $(J_{11}J_5)_{D_8}$ is the conjugacy class of $su = us$ where s is a semisimple element with $Z_G(s)^0$ of type D_8 and u is a unipotent element of $Z_G(s)^0$ with Jordan blocks of sizes $11, 5$ in the standard representation of D_8.

Type E_8.

$$8;\Phi_{30};(E_8)_{E_8},(E_7J_2)_{E_7A_1},(E_6J_3)_{E_6A_2},(J_9J_1J_4)_{D_5A_3},(J_5J_5)_{A_4A_4},$$
$$(J_6J_3J_2)_{A_5A_2A_1},(J_9)_{A_8},(J_8J_2)_{A_7A_1},(J_{15}J_1)_{D_8},$$

$$10;\Phi_{24};(E_8(a_1))_{E_8},(E_7(a_1)J_2)_{E_7A_1},(E_6(a_1)J_3)_{E_6A_2},(J_7J_3J_4)_{D_5A_3},(J_{13}J_3)_{D_8}.$$

$$12;\Phi_{20};(E_8(a_2))_{E_8},(E_7(a_2)J_2)_{E_7A_1},(J_{11}J_5)_{D_8},$$

$$14;\Phi_6\Phi_{18};(E_7A_1)_{E_8},(E_7(a_3)J_2)_{E_7A_1},(J_9J_7)_{D_8},$$

$$16;\Phi_{15};(D_8)_{E_8},(E_7(a_4)J_2)_{E_7A_1},$$

$$18;\Phi_2^2\Phi_{14};(E_7(a_1)A_1)_{E_8},$$

$$20;\Phi_{12}^2;(D_8(a_1))_{E_8},(J_7J_5J_3J_1)_{D_8},$$

$$22;\Phi_6^2\Phi_{12};(E_7(a_2)A_1)_{E_8},(E_7(a_5)J_2)_{E_7A_1},$$

$$24;\Phi_{10}^2;(A_8)_{E_8},$$

$$28;\Phi_3\Phi_9;(D_8(a_3))_{E_8},$$

$$40;\Phi_6^4;(2A_4)_{E_8}.$$

Type E_7.

$$7;\Phi_2\Phi_{18};(E_7)_{E_7},(J_{11}J_1J_2)_{D_6A_1},(J_6J_3)_{A_5A_2},(J_4J_4J_2)_{A_3A_3A_1},(J_8)_{A_7},$$

$$9;\Phi_2\Phi_{14};(E_7(a_1))_{E_7},((J_9J_3)A_1)_{D_6A_1},$$

$$11;\Phi_2\Phi_6\Phi_{12};(E_7(a_2))_{E_7},(J_7J_5J_2)_{D_6A_1},$$

$$13;\Phi_2\Phi_6\Phi_{10};(D_6A_1)_{E_7},$$

$$17;\Phi_2\Phi_4\Phi_8;(D_6(a_1)A_1)_{E_7},$$

$$21;\Phi_2\Phi_6^3;(D_6(a_2)A_1)_{E_7}.$$

Type E_6.

$$6;\Phi_3\Phi_{12};(E_6)_{E_6},(J_6J_2)_{A_5A_1},(J_3J_3J_3)_{A_2A_2A_2},$$

$$8;\Phi_9;(E_6(a_1))_{E_6},$$

$$12;\Phi_3\Phi_6^2;(A_5A_1)_{E_6}.$$

Type F_4.

$$4;\Phi_{12};(F_4)_{F_4},(J_6J_2)_{C_3A_1},(J_3J_3)_{A_2A_2},(J_4J_2)_{A_3A_1},(J_9)_{B_4},$$

$$6;\Phi_8;(F_4(a_1))_{F_4},(J_4J_2J_2)_{C_3A_1},$$

$$8;\Phi_6^2;(F_4(a_2))_{F_4},(J_5J_3J_1)_{B_4},$$

$$12;\Phi_4^2;(F_4(a_3))_{F_4}.$$

Type G_2.

$$2;\Phi_6;(G_2)_{G_2},(J_3)_{A_2},(J_2J_2)_{A_1A_1},$$

$$4;\Phi_3;(G_2(a_1))_{G_2}.$$

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