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UNIPOTENT REPRESENTATIONS AS A CATEGORICAL CENTRE

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ABSTRACT. Let $G(F_q)$ be the group of rational points of a split connected reductive group G over the finite field F_q. In this paper we show that the category of representations of $G(F_q)$ which are finite direct sums of unipotent representations in a fixed two-sided cell is equivalent to the centre of a certain monoidal category of sheaves on the flag manifold of $G \times G$. We also consider a version of this for nonsplit groups.

INTRODUCTION

0.1. Let $\mathbf{k}$ be an algebraic closure of the finite field $\mathbf{F}_p$ with p elements. For any power q of p let $\mathbf{F}_q$ be the subfield of $\mathbf{k}$ with q elements. Let G be a reductive connected group over $\mathbf{k}$, assumed to be adjoint. Let $\mathcal{B}$ be the variety of Borel subgroups of G.

Let W be the Weyl group of G and let $\mathbf{c}$ be a two-sided cell of W. Let $s \in \mathbf{Z}_{>0}$ and let $F : G \to G$ be the Frobenius map for an $\mathbf{F}_{p^s}$-rational structure on G. Let $G(\mathbf{F}_{p^s}) = G^F = \{g \in G; F(g) = g\}$ be a finite group. Let $\mathrm{Rep}^{\spadesuit}(G^F)$ (resp. $\mathrm{Rep}^{\mathbf{c}}(G^F)$) be the category of representations of G^F over $\bar{\mathbf{Q}}_l$ which are finite direct sums of unipotent representations in the sense of [DL] (resp., of unipotent representations whose associated two-sided cell (see 1.3) is $\mathbf{c}$); here l is a fixed prime number invertible in $\mathbf{k}$.

In the rest of this subsection we assume for simplicity that the $\mathbf{F}_{p^s}$-rational structure on G is split. The simple objects of $\mathrm{Rep}^{\mathbf{c}}(G^F)$ were classified in [L1]. The classification turns out to be the same as that [L4] of unipotent character sheaves on G whose associated two-sided cell is $\mathbf{c}$. The fact that

(a) *these two classification problems have the same solution*

has not until now been adequately explained.

One of the guiding ideas of this paper (already formulated in [L12]) is that the theory of unipotent character sheaves on G should be regarded as the limit for $s \mapsto 0$ of the theory of unipotent representations of the finite group $G(\mathbf{F}_{p^s})$; equivalently, the theory of unipotent representations of $G(\mathbf{F}_{p^s})$ should be regarded as a q-analogue or quantum version of the theory of character sheaves on G.

In [L13] we have shown that the category of perverse sheaves on G which are direct sums of unipotent character sheaves with associated two-sided cell $\mathbf{c}$ is naturally equivalent to the centre of a certain monoidal category $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ of sheaves on $\mathcal{B}^2$ introduced in [L9] for which the induced ring structure on the Grothendieck group is the J-ring attached to $\mathbf{c}$; see [L10, 18.3]. (The analogous statement for

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D-modules on a reductive group over $\mathbf{C}$ was proved earlier in a quite different way in [BFO].) In this paper we show that $\mathrm{Rep}^{\mathbf{c}}(G^F)$ is also naturally equivalent to the centre of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ (see Theorem 6.3).

This implies in particular that the set of simple objects of $\mathrm{Rep}^{\mathbf{c}}(G^F)$ (up to isomorphism) is naturally in bijection with the set of unipotent character sheaves (up to isomorphism) with associated two-sided cell $\mathbf{c}$, which explains (a). Note that the bijection between these two sets is achieved by showing that both sets are in bijection with the set of simple objects of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ (up to isomorphism). (A less conceptual bijection between these two sets is given in [L11]; we expect that it is the same as the bijection of this paper.) We deduce that the set of simple objects $\mathrm{Rep}_{\mathbf{c}}(G^F)$ is "independent" of the choice of s; in fact, as we show in 7.1, it is also independent of the characteristic of $\mathbf{k}$. It follows that to classify the unipotent representations of G^F it is enough to classify the unipotent character sheaves on G in sufficiently large characteristic; for the latter classification one can use the scheme of [L11] which uses the unipotent support of a character sheaf.

The methods of this paper are extensions of those in [L13]. We replace $\mathrm{Rep}^{\mathbf{c}}G^F$ by an equivalent category consisting of certain G-equivariant perverse sheaves on G_s, the set of all Frobenius maps $G \to G$ corresponding to split F_{p^s}-rational structures on G; we view G_s as an algebraic variety in a natural way.

We construct functors $\underline{\chi}_s, \underline{\zeta}_s$ between this category and the category $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ which are analogues of the truncated induction and truncated restriction $\underline{\chi}, \underline{\zeta}$ of [L13] and we show that most properties of $\underline{\chi}, \underline{\zeta}$ are preserved. We also define a truncated convolution product from our sheaves on G_s and on $G_{s'}$ to our sheaves on $G_{s+s'}$ which is analogous to the truncated convolution of character sheaves in [L13]; we also give a meaning for this even when s, s' are arbitrary integers. The main application of this truncated convolution product is in the case where $s' = -s$, the result of the product being a direct sum of character sheaves on G; this is used in the proof of a weak form of an adjunction formula between $\underline{\chi}_s, \underline{\zeta}_s$ which is then used to prove the main result (Theorem 6.3).

0.2. In this paper we also prove extensions of the results in 0.1 to the case where $F : G \to G$ is the Frobenius map of a nonsplit F_{p^s}-rational structure. In this case the role of unipotent character sheaves on G is taken by the unipotent character sheaves on a connected component of the group of automorphisms of G. Moreover, in this case the centre of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ is replaced by a slight generalization of the centre (the ϵ-centre) which depends on the connected component above.

Many arguments in this paper are very similar to arguments in [L13] and are often replaced by references to the corresponding arguments in [L13].

The analogues of our results in the case of nonunipotent character sheaves are considered in [L14].

0.3. *Notation.* We assume that we are given a split $\mathbf{F}_p$-rational structure on G with Frobenius map $F_0 : G \to G$. Let $\nu = \dim \mathcal{B}$, $\Delta = \dim(G)$, $\rho = \mathrm{rk}(G)$. We shall view W as an indexing set for the orbits of G acting on $\mathcal{B}^2 := \mathcal{B} \times \mathcal{B}$ by simultaneous conjugation; let $\mathcal{O}_w$ be the orbit corresponding to $w \in W$ and let $\bar{\mathcal{O}}_w$ be the closure of $\mathcal{O}_w$ in $\mathcal{B}^2$. For $w \in W$ we set $|w| = \dim \mathcal{O}_w - \nu$ (the length of w). Let w_{max} be the unique element of W such that $|w_{max}| = \nu$.

As in [L1, 3.1], we say that an automorphism $\epsilon : W \to W$ is *ordinary* if it leaves stable the set $\{s \in W; |s| = 1\}$ and for any two elements $s \neq s'$ in that set which are in the same orbit of ϵ, the product ss' has order ≤ 3. For example, if W is of type B_2, G_2 or F_4 and $\epsilon : W \to W$ has order 2 and preserves $\{s \in W; |s| = 1\}$ (as in the case of Ree and Suzuki groups), then ϵ is not ordinary. Let $\mathfrak{A}$ be the group of ordinary automorphisms of W.

For $B \in \mathcal{B}$, let U_B be the unipotent radical of B. Then B/U_B is independent of B; it is "the" maximal torus T of G. Let $\mathcal{X}$ be the group of characters of T.

Let Rep$W$ be the category of finite dimensional representations of W over $\mathbf{Q}$; let Irr$W$ be a set of representatives for the isomorphism classes of irreducible objects of Rep$W$.

For an algebraic variety X over $\mathbf{k}$ we denote by $\mathcal{D}(X)$ the bounded derived category of constructible $\bar{\mathbf{Q}}_l$-sheaves on X (l is a fixed prime number invertible in $\mathbf{k}$); let $\mathcal{M}(X)$ be the subcategory of $\mathcal{D}(X)$ consisting of perverse sheaves on X. If X has a fixed $\mathbf{F}_q$-structure X_0, we denote by $\mathcal{D}_m(X)$ what in [BBD, 5.1.5] is denoted by $\mathcal{D}^b_m(X_0, \bar{\mathbf{Q}}_l)$. (When X is $G, \mathcal{B}, \mathcal{O}_w$ or $\bar{\mathcal{O}}_w$, the subscript $_m$ refers to the $\mathbf{F}_{p^{s_0}}$-structure defined by $F_0^{s_0}$ for a sufficiently large $s_0 > 0$.) Note that any object $K \in \mathcal{D}_m(X)$ can be viewed as an object of $\mathcal{D}(X)$ which will be denoted again by K. For $K \in \mathcal{D}(X)$ and $i \in \mathbf{Z}$ let $\mathcal{H}^i K$ be the i-th cohomology sheaf of K, $\mathcal{H}^i_x K$ its stalk at $x \in X$, and let K^i be the i-th perverse cohomology sheaf of K. For $K \in \mathcal{D}(X)$ (or $K \in \mathcal{D}_m(X)$) and $n \in \mathbf{Z}$ we write $K[[n]] = K[n](n/2)$ where $[n]$ is a shift and $(n/2)$ is a Tate twist; we write $\mathfrak{D}(K)$ for the Verdier dual of K. Let $\mathcal{M}_m(X)$ be the subcategory of $\mathcal{D}_m(X)$ whose objects are in $\mathcal{M}(X)$. If $K \in \mathcal{M}_m(X)$ and $j \in \mathbf{Z}$ we denote by $\mathcal{W}^j K$ the subobject of K which has weight $\le j$ and is such that $K/\mathcal{W}^j K$ has weight $> j$; see [BBD, 5.3.5]. Let $gr_j K = \mathcal{W}^j K/\mathcal{W}^{j-1} K$ be the associated pure perverse sheaf of weight j. For $K \in \mathcal{D}_m(X)$ we shall often write $K^{\{i\}}$ instead of $gr_i(K^i)(i/2)$; recall that $K^{\{i\}}$ is a semisimple perverse sheaf (being pure of weight zero).

If $K \in \mathcal{M}(X)$ and A is a simple object of $\mathcal{M}(X)$ we denote by $(A : K)$ the multiplicity of A in a Jordan-Hölder series of K.

Assume that $C \in \mathcal{D}_m(X)$ and that $\{C_i; i \in I\}$ is a family of objects of $\mathcal{D}_m(X)$. We shall write $C \approx \{C_i; i \in I\}$ if the following condition is satisfied: there exist distinct elements $i_1, i_2, \dots, i_s$ in I, objects $C'_j \in \mathcal{D}_m(X)$ ($j = 0, 1, \dots, s$) and distinguished triangles $(C'_{j-1}, C'_j, C_{i_j})$ for $j = 1, 2, \dots, s$ such that $C'_0 = 0$, $C'_s = C$; moreover, $C_i = 0$ unless $i = i_j$ for some $j \in [1, s]$.

If X, X' are algebraic varieties over $\mathbf{k}$, we say that a map of sets $f : X \to X'$ is a quasi-morphism if for some $\mathbf{F}_q$-rational structure on X and X' with Frobenius maps F and F' and some integer $t \ge 0$, $fF^t : X \to X'$ is a morphism equal to $F'^t f$. If, in addition, $fF = F'f$, then we have well-defined functors $f_! : \mathcal{D}_m(X) \to \mathcal{D}_m(X')$, $f^* : \mathcal{D}_m(X') \to \mathcal{D}_m(X)$ such that $f_!$ is the composition of usual functors $(fF^t)_!(F^t)^* = (F'^t)^*(F'^t f)_!$ and f^* is the composition of usual functors $(F^t)_!(fF^t)^* = (F'^t f)^*(F'^t)_!$. The usual properties of $f_!, f^*$ for morphisms continue to hold for quasi-morphisms.

We will denote by $\mathbf{p}$ the variety consisting of one point. For any variety X let $\mathfrak{L}_X = \alpha_! \bar{\mathbf{Q}}_l \in \mathcal{D}_m X$ where $\alpha : X \times T \to X$ is the obvious projection. We sometimes write $\mathfrak{L}$ instead of $\mathfrak{L}_X$. For example, we have $\mathfrak{L}_{\mathbf{p}} = \alpha_! \bar{\mathbf{Q}}_l$ where $\alpha : T \to \mathbf{p}$ is the obvious map.

Let v be an indeterminate. For any $\phi \in \mathbf{Q}[v, v^{-1}]$ and any $k \in \mathbf{Z}$ we write $(k; \phi)$ for the coefficient of v^k in ϕ. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$.

Contents

1. Truncated induction

1.1. For $y \in W$ let $L_y \in \mathcal{D}_m(\mathcal{B}^2)$ be the constructible sheaf which is $\bar{\mathbf{Q}}_l$ (with the standard mixed structure of pure weight 0) on $\mathcal{O}_y$ and is 0 on $\mathcal{B}^2 - \mathcal{O}_y$; let $L_y^\sharp \in \mathcal{D}_m(\mathcal{B}^2)$ be its extension to an intersection cohomology complex of $\bar{\mathcal{O}}_y$ (equal to 0 on $\mathcal{B}^2 - \bar{\mathcal{O}}_y$). Let $\mathbf{L}_y = L_y^\sharp[[|y| + \nu]] \in \mathcal{D}_m(\mathcal{B}^2)$. This is a simple perverse sheaf which is pure of weight 0.

Let $r \geq 1$. For $\mathbf{w} = (w_1, w_2, \dots, w_r) \in W^r$ we set $|\mathbf{w}| = |w_1| + \dots + |w_r|$. For any $i < i'$ in $[1, r]$ let $p_{i,i'} : \mathcal{B}^{r+1} \to \mathcal{B}^2$ be the projection to the i, i' factors. From the definitions we see that

$$L_{\mathbf{w}}^{[1,r]} := p_{01}^* L_{w_1}^\sharp \otimes p_{12}^* L_{w_2}^\sharp \otimes p_{r-1,r}^* L_{w_r}^\sharp \in \mathcal{D}_m(\mathcal{B}^{r+1})$$

is the intersection cohomology complex of the projective variety

$$\mathcal{O}_{\mathbf{w}}^{[1,r]} = \{(B_0, B_1, \dots, B_r) \in \mathcal{B}^{r+1}; (B_{i-1}, B_i) \in \bar{\mathcal{O}}_{w_i} \forall i \in [1, r]\}$$

extended by 0 on $\mathcal{B}^{r+1} - \mathcal{O}_{\mathbf{w}}^{[1,r]}$ (it has the standard mixed structure of pure weight 0). For any $J \subset [1, r]$ we set

$$\mathcal{O}_{\mathbf{w}}^J = \{(B_0, B_1, \dots, B_r) \in \mathcal{B}^{r+1}; (B_{i-1}, B_i) \in \bar{\mathcal{O}}_{w_i} \forall i \in J, (B_{i-1}, B_i) \in \mathcal{O}_{w_i} \\ \forall i \in [1, r] - J\}.$$

Let $i_J : \mathcal{O}_{\mathbf{w}}^J \to \mathcal{O}_{\mathbf{w}}^{[1,r]}$ (resp. $i'_J : \mathcal{O}_{\mathbf{w}}^{[1,r]} - \mathcal{O}_{\mathbf{w}}^J \to \mathcal{O}_{\mathbf{w}}^{[1,r]}$) be the obvious open (resp. closed) imbedding and let $L_{\mathbf{w}}^J \in \mathcal{D}_m(\mathcal{B}^{r+1})$ (resp. $\dot{L}_{\mathbf{w}}^J \in \mathcal{D}_m(\mathcal{B}^{r+1})$) be $i_J^* L_{\mathbf{w}}^{[1,r]}$ (resp. $i'^*_J L_{\mathbf{w}}^{[1,r]}$) extended by 0 on $\mathcal{B}^{r+1} - \mathcal{O}_{\mathbf{w}}^J$ (resp. $\mathcal{B}^{r+1} - (\mathcal{O}_{\mathbf{w}}^{[1,r]} - \mathcal{O}_{\mathbf{w}}^J)$); we have a distinguished triangle

(a) $$(L_{\mathbf{w}}^J, L_{\mathbf{w}}^{[1,r]}, \dot{L}_{\mathbf{w}}^J)$$

in $\mathcal{D}_m(\mathcal{B}^{r+1})$.

For ${}^1L, {}^2L, \dots, {}^rL$ in $\mathcal{D}_m(\mathcal{B}^2)$ we set

$${}^1L \bullet {}^2L \bullet {}^rL = p_{0r!}(p_{01}^* {}^1L \otimes p_{12}^* {}^2L \otimes p_{r-1,r}^* {}^rL) \in \mathcal{D}_m(\mathcal{B}^2).$$

1.2. Let $\mathbf{H}$ be the free $\mathcal{A}$-module with basis $\{T_w; w \in W\}$. It is well known that $\mathbf{H}$ has a unique structure of associative $\mathcal{A}$-algebra with $1 = T_1$ (Hecke algebra) such that $T_w T_{w'} = T_{ww'}$ if $w, w' \in W$, $|ww'| = |w| + |w'|$ and $T_s^2 = 1 + (v - v^{-1})T_s$ if $s \in W$, $|s| = 1$. Let $\{c_w; w \in W\}$ be the "new" basis of $\mathbf{H}$ defined as in [L10, 5.2] with $L(w) = |w|$. For example, if $w = s \in W$, $|s| = 1$. we have $c_s = T_s + v^{-1}$.

For $x, y \in W$, the relations $x \preceq y$, $x \sim y$, $x \sim_L y$ on W are defined as in [L13, 1.3]. If $\mathbf{c}$ is a two-sided cell of W and $w \in W$, the relations $w \preceq \mathbf{c}$, $\mathbf{c} \preceq w$, $w \prec \mathbf{c}$, $\mathbf{c} \prec w$ are defined as in [L13, 1.3]. If $\mathbf{c}, \mathbf{c}'$ are two-sided cells of W, the relations $\mathbf{c} \preceq \mathbf{c}'$, $\mathbf{c} \prec \mathbf{c}'$ are defined as in [L13, 1.3]. Let $\mathbf{a} : W \to \mathbf{N}$ be the $\mathbf{a}$-function in [L10, 13.6]. If $\mathbf{c}$ is a two-sided cell of W, then for all $w \in \mathbf{c}$ we have $\mathbf{a}(w) = \mathbf{a}(\mathbf{c})$ where $\mathbf{a}(\mathbf{c})$ is a constant.

Let $\mathbf{J}$ be the free $\mathbf{Z}$-module with basis $\{t_z; z \in W\}$ with the structure of associative ring (with 1) as in [L13, 1.3]. For a two-sided cell $\mathbf{c}$ of W let $\mathbf{J}^{\mathbf{c}}$ be the subgroup of $\mathbf{J}$ generated by $\{t_z; z \in \mathbf{c}\}$; it is a subring of $\mathbf{J}$ with unit element $\sum_{d \in \mathbf{D}_{\mathbf{c}}} t_d$ where $\mathbf{D}_{\mathbf{c}}$ is the set of distinguished involutions of $\mathbf{c}$. We have $\mathbf{J} = \bigoplus_{\mathbf{c}} \mathbf{J}^{\mathbf{c}}$ as rings.

For $E \in \mathrm{Irr}W$ we define a simple $\mathbf{Q} \otimes \mathbf{J}$-module E_∞ and a simple $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathbf{H}$-module $E(v)$ as in [L13, 1.3]; there is a unique two-sided cell $\mathbf{c}_E$ of W such that $\mathbf{J}^{\mathbf{c}_E} E_\infty \neq 0$.

Let $\epsilon \in \mathfrak{A}$. Let $E \in \mathrm{Irr}W$. We say that $E \in \mathrm{Irr}_\epsilon W$ if $\mathrm{tr}(\epsilon(w), E) = \mathrm{tr}(w, E)$ for any $w \in W$. In this case there exists a linear transformation of finite order $e_E : E \to E$ such that $e_E w e_E^{-1} = \epsilon(w) : E \to E$ for any $w \in W$; moreover e_E is unique up to multiplication by -1. See ([L1, 3.2]). For each $E \in \mathrm{Irr}_\epsilon W$ we choose e_E as above. As a $\mathbf{Q}$-vector space we have $E_\infty = E$, $E(v) = \mathbf{Q}(v) \otimes_{\mathbf{Q}} E$; hence, if $E \in \mathrm{Irr}_\epsilon W$, $e_E : E \to E$ can be viewed as a $\mathbf{Q}$-linear map (of finite order) $e_E : E_\infty \to E_\infty$ and as a $\mathbf{Q}(v)$-linear map (of finite order) $e_E : E(v) \to E(v)$. From the definitions we see that $e_E t_w e_E^{-1} = t_{\epsilon(w)} : E_\infty \to E_\infty$ and $e_E T_w e_E^{-1} = T_{\epsilon(w)} : E(v) \to E(v)$ for any $w \in W$.

If $E \in \mathrm{Irr}_\epsilon W$, then $\epsilon(\mathbf{c}_E) = \mathbf{c}_E$. Let $\mathrm{Irr}_{\epsilon,\mathbf{c}} W = \{E \in \mathrm{Irr}_\epsilon W; \mathbf{c}_E = \mathbf{c}\}$.

1.3. For any $\epsilon \in \mathfrak{A}, s \in \mathbf{Z}$ let $G_{\epsilon,s}$ be the set of bijections $F : G \to G$ such that

(i) if $s > 0$, then F is the Frobenius map for an F_{p^s}-rational structure on G;

(ii) if $s < 0$, then F^{-1} is the Frobenius map for an $F_{p^{-s}}$-rational structure on G;

(iii) if $s = 0$, then F is an automorphism of G;

moreover in each case (i)–(iii) we require that the following holds: for any $w \in W$ and any $(B, B') \in \mathcal{O}_w$ we have $(F(B), F(B')) \in \mathcal{O}_{\epsilon(w)}$.

(If $\epsilon = 1, s = 0$ we can identify G and $G_{\epsilon,s}$ by $g \mapsto \mathrm{Ad}(g)$; recall that G is assumed to be adjoint.) Now G acts on $G_{\epsilon,s}$ by $g : F \mapsto \mathrm{Ad}(g) F \mathrm{Ad}(g^{-1})$. If $s \neq 0$, this action is transitive and the stabilizer of a point $F \in G_{\epsilon,s}$ is the finite group $G^F = \{g \in G; F(g) = g\}$. (The transitivity follows from the classification of $\mathbf{F}_q$-rational structures on G where $q = p^s$ if $s > 0$ and $q = p^{-s}$ if $s < 0$.)

For any $s \in \mathbf{Z}$ and any $\tilde{F} \in G_{\epsilon,s}$, the maps $\ell : G \to G_{\epsilon,s}$, $g \mapsto \mathrm{Ad}(g)\tilde{F}$ and $\ell' : G \to G_{\epsilon,s}$, $g \mapsto \tilde{F}\mathrm{Ad}(g)$ are bijections. (The fact that ℓ, ℓ' are injective is obvious. We show that ℓ, ℓ' are surjective. This is obvious when $s = 0$. Assume now that $s \neq 0$. If $F \in G_{\epsilon,s}$, we can write, by the transitivity property above, $F = \mathrm{Ad}(h)\tilde{F}\mathrm{Ad}(h^{-1})$ for some $h \in G$. Thus, $F = \mathrm{Ad}(g)\tilde{F}$ where $g = -\tilde{F}(h^{-1}) \in G$ and $F = \tilde{F}\mathrm{Ad}(g')$ where $g' = \tilde{F}^{-1}(h)h^{-1} \in G$, proving the surjectivity of ℓ and ℓ'.)

We use ℓ (resp. ℓ') to view $G_{\epsilon,s}$ with $s \geq 0$ (resp. $s \leq 0$) as an affine algebraic variety isomorphic to G; this algebraic variety structure on $G_{\epsilon,s}$ is independent of the choice of $\tilde{F}$. We have $\dim G_{\epsilon,s} = \Delta$. When $X = G_{\epsilon,s}$ then the subscript $_m$ in $\mathcal{D}_m(X), \mathcal{M}_m(X)$ refers to the $\mathbf{F}_{p^{s_0}}$-structure with Frobenius map $F \mapsto F_0^{s_0} F F_0^{-s_0}$ (with F_0, s_0 as in 0.3).

Note that $\bigsqcup_{\epsilon \in \mathfrak{A}, s \in \mathbf{Z}} G_{\epsilon,s}$ is a group under composition of maps: if $F \in G_{\epsilon,s}, F' \in G_{\epsilon',s'}$, then $FF' \in G_{\epsilon\epsilon',s+s'}$. (It is enough to show that for some $F \in G_{\epsilon,s}, F' \in G_{\epsilon',s'}$ we have $FF' \in G_{\epsilon\epsilon',s+s'}$. We take $F = \mathrm{Ad}(\gamma)F_0^s$, $F' = \mathrm{Ad}(\gamma')F_0^{s'}$ where $\gamma \in G_{\epsilon,1}$ and $\gamma' \in G_{\epsilon',1}$ commute with F_0; then $FF' = \mathrm{Ad}(\gamma\gamma')F_0^{s+s'}$ and $\gamma\gamma' \in G_{\epsilon\epsilon',1}$ commutes with F_0 hence $FF' \in G_{\epsilon\epsilon',s+s'}$.) Note that the composition $G_{\epsilon,s} \times$

$G_{\epsilon',s'} \to G_{\epsilon\epsilon',s+s'}$ is not in general a morphism of algebraic varieties but only a quasi-morphism (see 0.3), which is good enough for our purposes.

Until the end of Section 2 we fix $\epsilon \in \mathfrak{A}$.

Let $s \in \mathbf{Z}$. We consider the maps $\mathcal{B}^2 \xleftarrow{f} X_{\epsilon,s} \xrightarrow{\pi} G_{\epsilon,s}$ where

$$X_{\epsilon,s} = \{(B, B', F) \in \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F(B) = B'\},$$
$$f(B, B', F) = (B, B'), \pi(B, B', F) = F.$$

Note that f, π are G-equivariant where G acts on $G_{\epsilon,\sigma}$ by $g : F \mapsto \mathrm{Ad}(g)F\mathrm{Ad}(g^{-1})$ and on $X_{\epsilon,s}$ by $g : (B, B', F) \mapsto (gBg^{-1}, gB'g^{-1}, \mathrm{Ad}(g)F\mathrm{Ad}(g^{-1})$.

Now $L \mapsto \chi_{\epsilon,s}(L) = \pi_! f^* L$ defines a functor $\mathcal{D}_m(\mathcal{B}^2) \to \mathcal{D}_m(G_{\epsilon,s})$. (When $\epsilon = 1, s = 0$, $c_{\epsilon,s}$ coincides with the functor χ defined in [L13, 1.5]). For $i \in \mathbf{Z}, L \in \mathcal{D}_m(\mathcal{B}^2)$ we write $\chi^i_{\epsilon,s}(L)$ instead of $(\chi_{\epsilon,s}(L))^i$, the i-th perverse cohomology sheaf of $\chi_{\epsilon,s}(L)$. For any $z \in W$ we set $R_{\epsilon,s,z} = \chi_{\epsilon,s}(L^\sharp_z) \in \mathcal{D}_m(G_{\epsilon,s})$. (When $\epsilon = 1, s = 0$ this is the same as R_z in [L13, 1.5].)

Let $b : G_{\epsilon^{-1},-s} \xrightarrow{\sim} G_{\epsilon,s}$ be the isomorphism $F \mapsto F^{-1}$ and let $b' : \mathcal{B}^2 \xrightarrow{\sim} \mathcal{B}^2$ be the isomorphism $(B, B') \mapsto (B', B)$. Using the definitions and base change, we see that for $L \in \mathcal{D}_m(\mathcal{B}^2)$ we have $\chi_{\epsilon,s}(b'_! L) = b_! \chi_{\epsilon^{-1},-s}(L)$.

Let $CS(G_{\epsilon,s})$ be a set of representatives for the isomorphism classes of simple perverse sheaves $A \in \mathcal{M}(G_{\epsilon,s})$ such that $(A : R^j_{\epsilon,s,z}) \neq 0$ for some $z \in W, j \in \mathbf{Z}$. (When $\epsilon = 1, s = 0$ this agrees with the definition of $CS(G)$ in [L13, 1.5].) Now let $A \in CS(G_{\epsilon,s})$. We associate to A a two-sided cell $\mathbf{c}_A$ as follows.

Assume first that $s \neq 0$. Since A is G-equivariant and the conjugation action of G on $G_{\epsilon,s}$ is transitive, for any $F \in G_{\epsilon,s}$ we have $A|_{\{F\}} = r_{A,F}[\Delta]$ where $r_{A,F}$ is an irreducible G^F-module. From the definitions, for any $z \in W$ and any $F \in G_{\epsilon,s}$ we have

$$(A : R^j_{s,z}) = (r_{A,F} : IH^{j-\Delta}\{(B; (B, FB) \in \bar{\mathcal{O}}_z\})_{G^F}$$

where the right-hand side is the multiplicity of $r_{A,F}$ in the G^F-module

$$IH^{j-\Delta}\{(B; (B, FB) \in \bar{\mathcal{O}}_z\};$$

here IH denotes intersection cohomology with coefficients in $\bar{\mathbf{Q}}_l$. In particular, $r_{A,F}$ is a unipotent representation of G^F. By [L1, 3.8], for any $A \in CS(G_{\epsilon,s})$, any $F \in G_{\epsilon,s}$, any $z \in W$ and any $j \in \mathbf{Z}$ we have

$$(r_{A,F} : IH^{j-\Delta}\{(B; (B, FB) \in \bar{\mathcal{O}}_z\})_{G^F}$$
$$= (j - \Delta - |z|; (-1)^{j-\Delta} \sum_{E \in \mathrm{Irr}_\epsilon W} c_{A,E,e_E} \mathrm{tr}(e_E c_z, E(v)))$$

or, equivalently,

(a) $$(A : R^j_{\epsilon,s,z}) = (j - \Delta - |z|; (-1)^{j-\Delta} \sum_{E \in \mathrm{Irr}_\epsilon W} c_{A,E,e_E} \mathrm{tr}(e_E c_z, E(v)))$$

where c_{A,E,e_E} are uniquely defined rational numbers; now (a) also holds when $s = 0$, see [L13, 1.5(a)] when $\epsilon = 1$ and [L6, 34.19, 35.22], [L8, 44.7(e)] for general ϵ. Moreover, if $s \neq 0$ then, by [L1, 6.17], given A as above, there is a unique two-sided cell $\mathbf{c}_A$ of W such that $\epsilon(\mathbf{c}_A) = \mathbf{c}_A$ and $c_{A,E,e_E} = 0$ whenever $E \in \mathrm{Irr}_\epsilon W$ satisfies $\mathbf{c}_E \neq \mathbf{c}_A$. The same holds when $s = 0$; see [L13, 1.5] when $\epsilon = 1$ and [L7, §41] for general ϵ.

When $s \neq 0$, $\mathbf{c}_A$ differs from the two-sided cell associated to $r_{A,F}$ in [L1, 4.23] by multiplication on the left or right by w_{max}. (For example, if $G = PGL_2(\mathbf{k})$, then there are two objects $A_0 \ne A_1$ in $CS(G_{\epsilon,s})$ where A_0 corresponds to the unit representation and A_1 to the Steinberg representation. We have $\mathbf{c}_{A_0} = \{w_{max}\}$, $\mathbf{c}_{A_1} = \{1\}$.) Similarly, when $s = 0$, $\mathbf{c}_A$ differs from the two-sided cell associated to A in [L7, §41] by multiplication on the left or right by w_{max}.

As in [L13, 1.5(b)], for $s \in \mathbf{Z}$ we have

(b) $(A : R^j_{\epsilon,s,z}) \ne 0$ *for some* $z \in \mathbf{c}_A, j \in \mathbf{Z}$ *and conversely, if* $(A : R^j_{\epsilon,s,z}) \ne 0$ *for* $z \in W, j \in \mathbf{Z}$*, then* $\mathbf{c}_A \preceq z$.

For $s \in \mathbf{Z}$, $A \in CS(G_{\epsilon,s})$ let a_A be the value of the $\mathbf{a}$-function on $\mathbf{c}_A$. If $z \in W, E \in \mathrm{Irr}_\epsilon W$ satisfy $\mathrm{tr}(e_E c_z, E(v)) \ne 0$, then $\mathbf{c}_E \preceq z$; if in addition we have $z \in \mathbf{c}_E$, then

$$\mathrm{tr}(e_E c_z, E(v)) = \gamma_{z,E,e_E} v^{a_E} + \text{lower powers of } v$$

where $\gamma_{z,E,e_E} \in \mathbf{Z}$ and a_E is the value of the $\mathbf{a}$-function on $\mathbf{c}_E$. Hence from (a) we see that

(c) $(A : R^j_{\epsilon,s,z}) = 0$ *unless* $\mathbf{c}_A \preceq z$ *and, if* $z \in \mathbf{c}_A$*, then*

$$(A : R^j_{\epsilon,s,z})$$
$$= (-1)^{j+\Delta}(j - \Delta - |z|; (\sum_{E \in \mathrm{Irr}_\epsilon W; \mathbf{c}_E = \mathbf{c}_A} c_{A,E,e_E}\gamma_{z,E,e_E})v^{a_A} + \text{ lower powers of } v)$$

which is 0 *unless* $j - \Delta - |z| \le a_A$.

In the remainder of this section we fix a two-sided cell $\mathbf{c}$ *of* W *such that* $\epsilon(\mathbf{c}) = \mathbf{c}$*; we set* $a = \mathbf{a}(\mathbf{c})$.

For $s \in \mathbf{Z}$ and $Y = G_{\epsilon,s}$ or $Y = \mathcal{B}^2$ let $\mathcal{M}^\spadesuit Y$ be the category of perverse sheaves on Y whose composition factors are all of the form $A \in CS(G_{\epsilon,s})$, when $Y = G_{\epsilon,s}$, or of the form $\mathbf{L}_z$ with $z \in W$, when $Y = \mathcal{B}^2$. Let $\mathcal{M}^\preceq Y$ (resp. $\mathcal{M}^\prec Y$) be the category of perverse sheaves on Y whose composition factors are all of the form $A \in CS(G_{\epsilon,s})$ with $\mathbf{c}_A \preceq \mathbf{c}$ (resp. $\mathbf{c}_A \prec \mathbf{c}$), when $Y = G_{\epsilon,s}$, or of the form $\mathbf{L}_z$ with $z \preceq \mathbf{c}$ (resp. $z \prec \mathbf{c}$) when $Y = \mathcal{B}^2$. Let $\mathcal{D}^\spadesuit Y$ (resp. $\mathcal{D}^\preceq Y$ or $\mathcal{D}^\prec Y$) be the category of all $K \in \mathcal{D}(Y)$ such that $K^i \in \mathcal{M}^\spadesuit Y$ (resp. $K^i \in \mathcal{M}^\prec Y$ or $K^i \in \mathcal{M}^\prec Y$) for all $i \in \mathbf{Z}$. Let $\mathcal{M}^\spadesuit_m Y$ (or $\mathcal{M}^\preceq_m Y$, or $\mathcal{M}^\prec_m Y$) be the category of all $K \in \mathcal{M}_m Y$ which are also in $\mathcal{M}^\spadesuit Y$ (or $\mathcal{M}^\preceq Y$ or $\mathcal{M}^\prec Y$). Let $\mathcal{D}^\spadesuit_m Y$ (or $\mathcal{D}^\preceq_m Y$, or $\mathcal{D}^\prec_m Y$) be the category of all $K \in \mathcal{D}_m Y$ which are also in $\mathcal{D}^\spadesuit Y$ (or $\mathcal{D}^\preceq Y$ or $\mathcal{D}^\prec Y$). From (c) we deduce:

(d) *If* $z \preceq \mathbf{c}$*, then* $R^j_{\epsilon,s,z} \in \mathcal{M}^\preceq G_{\epsilon,s}$ *for all* $j \in \mathbf{Z}$*. If* $z \in \mathbf{c}$ *and* $j > a + \Delta + |z|$*, then* $R^j_{\epsilon,s,z} \in \mathcal{M}^\prec G_{\epsilon,s}$*. If* $z \prec \mathbf{c}$*, then* $R^j_{\epsilon,s,z} \in \mathcal{M}^\prec G_{\epsilon,s}$ *for all* $j \in \mathbf{Z}$.

Lemma 1.4. *Let* $s \in \mathbf{Z}$*. Let* $r \ge 1$*,* $J \subset [1, r]$*,* $J \ne \emptyset$ *and* $\mathbf{w} = (w_1, w_2, \dots, w_r) \in W^r$.

(a) Assume that $w_i \in \mathbf{c}$ *for some* $i \in [1, r]$*. If* $j \in \mathbf{Z}$ *(resp.* $j > \Delta + ra$*), then* $\chi^j_{\epsilon,s}(p_{0r!}L^{[1,r]}_{\mathbf{w}}[|\mathbf{w}|])$ *is in* $\mathcal{M}^\preceq G_{\epsilon,s}$ *(resp.* $\mathcal{M}^\prec G_{\epsilon,s}$*).*

(b) Assume that $w_i \in \mathbf{c}$ *for some* $i \in J$*. If* $j \in \mathbf{Z}$ *(resp.* $j \ge \Delta + ra$*), then* $\chi^j_{\epsilon,s}(p_{0r!}\dot{L}^J_{\mathbf{w}}[|\mathbf{w}|])$ *is in* $\mathcal{M}^\preceq G_{\epsilon,s}$ *(resp.* $\mathcal{M}^\prec G_{\epsilon,s}$*).*

(c) Assume that $w_i \in \mathbf{c}$ *for some* $i \in J$*. If* $j \ge \Delta + ra$*, then the cokernel of the map*

$$\chi^j_{\epsilon,s}(p_{0r!}L^J_{\mathbf{w}}[|\mathbf{w}|]) \to \chi^j_{\epsilon,s}(p_{0r!}L^{[1,r]}_{\mathbf{w}}[|\mathbf{w}|])$$

associated to 1.1(a) is in $\mathcal{M}^\prec G_{\epsilon,s}$.

(d) Assume that $w_i \in \mathbf{c}$ *for some* $i \in J$*. If* $j \in \mathbf{Z}$ *(resp.* $j > \Delta + ra$*), then* $\chi^j_{\epsilon,s}(p_{0r!}L^J_{\mathbf{w}}[|\mathbf{w}|])$ *is in* $\mathcal{M}^\preceq G_{\epsilon,s}$ *(resp.* $\mathcal{M}^\prec G_{\epsilon,s}$*).*

(e) Assume that $w_i \prec \mathbf{c}$ *for some* $i \in J$*. If* $j \in \mathbf{Z}$*, then* $\chi^j_{\epsilon,s}(p_{0r!}L^{[1,r]}_{\mathbf{w}}[|\mathbf{w}|]) \in \mathcal{M}^\prec G_{\epsilon,s}$ *and* $\chi^j_{\epsilon,s}(p_{0r!}L^J_{\mathbf{w}}[|\mathbf{w}|]) \in \mathcal{M}^\prec G_{\epsilon,s}$.

When $\epsilon = 1, s = 0$ this is just [L13, 1.6]; the proof in the general case is entirely similar (it uses 1.3(b), 1.3(c)).

1.5. Let $s \in \mathbf{Z}$. Let $CS_{\epsilon,s,\mathbf{c}} = \{A \in CS(G_{\epsilon,s}); \mathbf{c}_A = \mathbf{c}\}$. For any $z \in \mathbf{c}$ we set $n_z = a + \Delta + |z|$. Let $A \in CS_{\epsilon,s,\mathbf{c}}$ and let $z \in \mathbf{c}$. We have:

(a) $$(A : R^{n_z}_{\epsilon,s,z}) = (-1)^{a+|z|} \sum_{E \in \mathrm{Irr}_{\epsilon,\mathbf{c}} W} c_{A,E,e_E} \mathrm{tr}(e_E t_z, E_\infty).$$

When $\epsilon = 1, s = 0$ this is just [L13, 1.7(a)]. In the general case, from 1.3(a) we have
$$(A : R^{n_z}_{\epsilon,s,z}) = (-1)^{a+|z|} \sum_{E \in \mathrm{Irr}_\epsilon W} c_{A,E,e_E}(a; \mathrm{tr}(e_E c_z, E(v)))$$
and it remains to use that $(a; \mathrm{tr}(e_E c_z, E(v)))$ is equal to $\mathrm{tr}(e_E t_z, E_\infty)$ if $E \in \mathrm{Irr}_{\epsilon,\mathbf{c}} W$ and to 0, otherwise. We have:

(b) *For any $A \in CS_{\epsilon,s,\mathbf{c}}$ there exists $z \in \mathbf{c}$ such that $(A : R^{n_z}_{\epsilon,s,z}) \neq 0$.*

The proof, based on (a), is the same as that in the case $\epsilon = 1, s = 0$ given in [L13, 1.7(b)].

Let $\mathbf{c}^0 = \{z \in \mathbf{c}; z \sim_L \epsilon(z\imath)\}$. If $z \in \mathbf{c} - \mathbf{c}^0$ and $E \in \mathrm{Irr}_{\epsilon,\mathbf{c}} W$, then $\mathrm{tr}(e_E t_z, E_\infty) = 0$. (We can write $E_\infty = \bigoplus_{d \in \mathbf{D_c}} t_d E_\infty$ and $e_E t_z : E_\infty \to E_\infty$ maps the summand $t_d E_\infty$ (where $z \sim_L d$) into $t_{\epsilon(d')} E_\infty$ (where $d' \in \mathbf{D_c}$, $d' \sim_L z^{-1}$) and all other summands to 0. If $\mathrm{tr}(e_E t_z, E_\infty) \neq 0$, we must have $t_d E_\infty = t_{\epsilon(d')} E_\infty \neq 0$ and $d = \epsilon(d')$ and $z \sim_L \epsilon(z^{-1})$.) From this and (a) we deduce:

(c) *If $z \in \mathbf{c} - \mathbf{c}^0$, then $R^{n_z}_{\epsilon,s,z} = 0$.*

1.6. Let $s \in \mathbf{Z}$. For $Y = G_{\epsilon,s}$ or $\mathcal{B}^2$ let $\mathcal{C}^\spadesuit Y$ be the subcategory of $\mathcal{M}^\spadesuit Y$ consisting of semisimple objects; let $\mathcal{C}^\spadesuit_0 Y$ be the subcategory of $\mathcal{M}_m Y$ consisting of those $K \in \mathcal{M}_m Y$ such that K is pure of weight 0 (hence $K \in \mathcal{M}(Y)$ is semisimple) and such that as an object of $\mathcal{M}(Y)$, K belongs to $\mathcal{C}^\spadesuit Y$. Let $\mathcal{C}^{\mathbf{c}} Y$ be the subcategory of $\mathcal{M}^\spadesuit Y$ consisting of objects which are direct sums of objects in $CS_{\epsilon,s,\mathbf{c}}$ (if $Y = G_{\epsilon,s}$) or of the form $\mathbf{L}_z$ with $z \in \mathbf{c}$ (if $Y = \mathcal{B}^2$). Let $\mathcal{C}^{\mathbf{c}}_0 Y$ be the subcategory of $\mathcal{C}^\spadesuit_0 Y$ consisting of those $K \in \mathcal{C}^\spadesuit_0 Y$ such that as an object of $\mathcal{C}^\spadesuit Y$, K belongs to $\mathcal{C}^{\mathbf{c}} Y$. For $K \in \mathcal{C}^\spadesuit_0 Y$, let $\underline{K}$ be the largest subobject of K such that, as an object of $\mathcal{C}^\spadesuit Y$, we have $\underline{K} \in \mathcal{C}^{\mathbf{c}} Y$.

For $L \in \mathcal{C}^\spadesuit_0 \mathcal{B}^2$ we define ${}^\epsilon L \in \mathcal{C}^\spadesuit_0 \mathcal{B}^2$ as follows. We have canonically $L = \bigoplus_{y \in W} V_y \otimes \mathbf{L}_y$ where V_y are finite dimensional $\bar{\mathbf{Q}}_l$-vector spaces; we set ${}^\epsilon L = \bigoplus_{y \in W} V_y \otimes \mathbf{L}_{\epsilon^{-1}(y)}$. We show:

(a) *Let $s \in \mathbf{N}$. Define $u : G_{\epsilon,s} \times \mathcal{B}^2 \to G_{\epsilon,s} \times \mathcal{B}^2$ by*
$$(F, (B_1, B_2)) \mapsto (F, F(B_1), F(B_2))$$
and let $L \in \mathcal{C}^\spadesuit_0 \mathcal{B}^2$. We have canonically $u^*(\bar{\mathbf{Q}}_l \boxtimes L) = \bar{\mathbf{Q}}_l \boxtimes {}^\epsilon L$. (Note that for $F \in G_{\epsilon,s}$, the perverse sheaf $F^* L$ is isomorphic to ${}^\epsilon L$.)

We can assume that $L = \mathbf{L}_y$ where $y \in W$; we must show that $u^*(\bar{\mathbf{Q}}_l \boxtimes \mathbf{L}_y) = \bar{\mathbf{Q}}_l \boxtimes \mathbf{L}_{\epsilon^{-1}(y)}$ or that $u^*(\bar{\mathbf{Q}}_l \boxtimes L^\sharp_y) = \bar{\mathbf{Q}}_l \boxtimes L^\sharp_{\epsilon^{-1}(y)}$. Now $\bar{\mathbf{Q}}_l \boxtimes L^\sharp_y$ is the intersection cohomology complex of $G_{\epsilon,s} \times \bar{\mathcal{O}}_y$ with coefficients in $\bar{\mathbf{Q}}_l$ (extended by 0 on $G_{\epsilon,s} \times (\mathcal{B}^2 - \bar{\mathcal{O}}_y)$). Hence $u^*(\bar{\mathbf{Q}}_l \boxtimes L^\sharp_y)$ is the intersection cohomology complex of $u^{-1}(G_{\epsilon,s} \times \bar{\mathcal{O}}_y)$ with coefficients in $\bar{\mathbf{Q}}_l$ (extended by 0 on $G_{\epsilon,s} \times u^{-1}(\mathcal{B}^2 - \bar{\mathcal{O}}_y)$); that is, the intersection cohomology complex of $G_{\epsilon,s} \times \bar{\mathcal{O}}_{\epsilon^{-1}(y)}$ with coefficients in $\bar{\mathbf{Q}}_l$ (extended by 0 on $G_{\epsilon,s} \times (\mathcal{B}^2 - \bar{\mathcal{O}}_{\epsilon^{-1}(y)})$). This is $\bar{\mathbf{Q}}_l \boxtimes L^\sharp_{\epsilon^{-1}(y)}$, as required.

Assume that $s \in \mathbf{Z}_{>0}$ and let $F \in G_{\epsilon,s}$. For any $A \in \mathcal{C}^{\spadesuit}G_{\epsilon,s}$ we have $A|_{\{F\}} = r_{A,F}[\Delta]$ where $r_{A,F} \in \mathrm{Rep}^{\spadesuit}(G^F)$ (see 0.1). Moreover, from the definitions we see that

(b) $A \mapsto r_{A,F}$ *is an equivalence of categories* $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \xrightarrow{\sim} \mathrm{Rep}^{\mathbf{c}}(G^F)$ *(see 0.1).*

Proposition 1.7. *Let* $s \in \mathbf{Z}$.

(a) If $L \in \mathcal{D}^{\preceq}\mathcal{B}^2$*, then* $\chi_{\epsilon,s}(L) \in \mathcal{D}^{\preceq}G_{\epsilon,s}$*. If* $L \in \mathcal{D}^{\prec}\mathcal{B}^2$*, then* $\chi_{\epsilon,s}(L) \in \mathcal{D}^{\prec}G_{\epsilon,s}$*.*

(b) If $L \in \mathcal{M}^{\preceq}\mathcal{B}^2$ *and* $j > a+\nu+\rho$*, then* $\chi^j_{\epsilon,s}(L) \in \mathcal{M}^{\prec}G_{\epsilon,s}$*.*

When $\epsilon = 1, s = 0$ this is just [L13, 1.9]; the proof in the general case is entirely similar (it uses 1.4(a),(e)).

1.8. Let $s \in \mathbf{Z}$. For $L \in \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2$ we set

$$\underline{\chi}_{\epsilon,s}(L) = \underline{\chi^{a+\nu+\rho}_{\epsilon,s}(L)}((a+\nu+\rho)/2) = \underline{(\chi_{\epsilon,s}(L))^{\{a+\nu+\rho\}}} \in \mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s}.$$

This is the projection onto $\mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s}$ of the pure (of weight 0) semisimple perverse sheaf $(\chi_{\epsilon,s}(L))^{\{a+\nu+\rho\}}$.

The functor $\underline{\chi}_{\epsilon,s} : \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s}$ is called *truncated induction*. For $z \in \mathbf{c}$ we have

(a) $$\underline{\chi}_{\epsilon,s}(\mathbf{L}_z) = \underline{R^{n_z}_{\epsilon,s,z}}(n_z/2).$$

When $\epsilon = 1, s = 0$ this is just [L13, 1.10(a)]; the proof in the general case is entirely similar.

We shall denote by $\tau : \mathbf{J}^{\mathbf{c}} \to \mathbf{Z}$ the group homomorphism such that $\tau(t_z) = 1$ if $z \in \mathbf{D_c}$ and $\tau(t_z) = 0$, otherwise. For $z, u \in \mathbf{c}$ we have:

(b) $$\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) = \sum_{y \in \mathbf{c}} \tau(t_{y^{-1}}t_z t_{\epsilon(y)} t_{u^{-1}}).$$

When $\epsilon = 1, s = 0$ this is just [L13, 1.10(b)]. We now consider the general case.

Using (a) and the definitions we see that the left-hand side of (b) equals

$$\sum_{A \in CS_{\epsilon,s,\mathbf{c}}} (A : R^{n_z}_{\epsilon,s,z})(A : R^{n_u}_{\epsilon,s,u}),$$

hence, using 1.5(a) it equals

$$\sum_{E,E' \in \mathrm{Irr}_{\epsilon,\mathbf{c}}W} (-1)^{|z|+|u|} \sum_{A \in CS_{\epsilon,s,\mathbf{c}}} c_{A,E,e_E} c_{A,E',e_{E'}} \mathrm{tr}(e_E t_z, E_\infty)\mathrm{tr}(e_{E'} t_u, E'_\infty).$$

Replacing in the last sum $\sum_{A \in CS_{\epsilon,s,\mathbf{c}}} c_{A,E,e_E} c_{A,E',e_{E'}}$ by 1 if $E = E'$ and by 0 if $E \neq E'$ (see [L1, 3.9] in the case $s \neq 0$ and [L3, 13.12], [L6, 35.18(g)] in the case $s = 0$) we obtain

$$\sum_{E \in \mathrm{Irr}_{\epsilon,\mathbf{c}}W} (-1)^{|z|+|u|}\mathrm{tr}(e_E t_z, E_\infty)\mathrm{tr}(e_E t_u, E_\infty).$$

This is equal to $(-1)^{|z|+|u|}$ times the trace of the operator $\xi \mapsto t_z\epsilon(\xi)t_{u^{-1}}$ on $\mathbf{Q} \otimes \mathbf{J}^{\mathbf{c}}$ (see [L6, 34.14(a), 34.17]). The last trace is equal to the sum over $y \in \mathbf{c}$ of the coefficient of t_y in $t_z t_{\epsilon(y)} t_{u^{-1}}$; this coefficient is equal to $\tau(t_{y^{-1}} t_z t_{\epsilon(y)} t_{u^{-1}})$ since for $y, y' \in \mathbf{c}$, $\tau(t_{y'}t_y)$ is 1 if $y' = y^{-1}$ and is 0 if $y' \neq y^{-1}$ (see [L10, 20.1(b)]). Thus we have

$$\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) = (-1)^{|u|+|z|} \sum_{y \in \mathbf{c}} \tau(t_{y^{-1}}t_z t_{\epsilon(y)} t_{u^{-1}}).$$

Since $\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_z), \underline{\chi}_{\epsilon,s}(\mathbf{L}_u)) \in \mathbf{N}$ and $\sum_{y\in\mathbf{c}} \tau(t_{y^{-1}}t_z t_{\epsilon(y)}t_{u^{-1}}) \in \mathbf{N}$, it follows that (b) holds.

The following lemma will be used several times to transfer properties of usual functors and operations to their truncated analogue.

Lemma 1.9. *Let $s \in \mathbf{N}$. Let Y_1, Y_2 be among $G_{\epsilon,s}, \mathcal{B}^2$ and let $\mathbf{X} \in \mathcal{D}_m^{\preceq} Y_1$. Let c, c' be integers and let $\Phi : \mathcal{D}_m^{\preceq} Y_1 \to \mathcal{D}_m^{\preceq} Y_2$ be a functor which takes distinguished triangles to distinguished triangles, commutes with shifts, maps $\mathcal{D}_m^{\prec} Y_1$ into $\mathcal{D}_m^{\prec} Y_2$ and maps complexes of weight $\le i$ to complexes of weight $\le i$ (for any i). Assume that (a) and (b) below hold:*

(a) $$(\Phi(\mathbf{X}_0))^h \in \mathcal{M}_m^{\preceq} Y_2 \text{ for any } \mathbf{X}_0 \in \mathcal{M}_m^{\preceq} Y_1 \text{ and any } h > c;$$

(b) $$\mathbf{X} \text{ has weight} \le 0 \text{ and } \mathbf{X}^i \in \mathcal{M}^{\prec} Y_1 \text{ for any } i > c'.$$

Then

(c) $$(\Phi(\mathbf{X}))^j \in \mathcal{M}^{\prec} Y_2 \text{ for any } j > c + c',$$

and we have canonically

(d) $$\underline{(\Phi(\underline{\mathbf{X}^{\{c'\}}}))^{\{c\}}} = \underline{(\Phi(\mathbf{X}))^{\{c+c'\}}}.$$

When $\epsilon = 1, s = 0$ this is just [L13, 1.12]; the proof in the general case is entirely similar.

1.10. Let $s \in \mathbf{Z}$. Let $L \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$. We have $\mathfrak{D}(L) \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$. Moreover, we have canonically:

(a) $$\underline{\chi}_{\epsilon,s}(\mathfrak{D}(L)) = \mathfrak{D}(\underline{\chi}_{\epsilon,s}(L)).$$

When $\epsilon = 1, s = 0$ this is just [L13, 1.13]; the proof in the general case is entirely similar.

1.11. Let $L \in \mathcal{C}_0^{\spadesuit} \mathcal{B}^2$, $L' \in \mathcal{D}_m(\mathcal{B}^2)$. We have canonically

(a) $$\chi_{\epsilon,s}(L \bullet L') = \chi_{\epsilon,s}(L' \bullet {}^{\epsilon}L).$$

In the case where $\epsilon = 1, s = 0$ this appears in [L13, 1.11]; the proof in the general case is entirely similar.

2. Truncated restriction

2.1. Recall that $\epsilon \in \mathfrak{A}$ is fixed. In this section we fix $s \in \mathbf{Z}$. Let π, f be as in 1.3. Now $K \mapsto \zeta_{\epsilon,s}(K) = f_! \pi^* K$ defines a functor $\mathcal{D}_m(G_{\epsilon,s}) \to \mathcal{D}_m(\mathcal{B}^2)$. (When $\epsilon = 1, s = 0$, $\zeta_{\epsilon,s}$ is the same as ζ of [L13, 2.5].) For $i \in \mathbf{Z}, K \in \mathcal{D}_m(G_{\epsilon,s})$ we write $\zeta^i_{\epsilon,s}(K)$ instead of $(\zeta_{\epsilon,s}(K))^i$ (the i-th perverse cohomology sheaf of $\zeta_{\epsilon,s}(K)$).

Let $b : G_{\epsilon^{-1},-s} \xrightarrow{\sim} G_{\epsilon,s}$, $b' : \mathcal{B}^2 \xrightarrow{\sim} \mathcal{B}^2$ be as in 1.3. From the definitions we see that for $K \in \mathcal{D}_m(G_{\epsilon^{-1},-s})$ we have

(a) $$\zeta_{\epsilon,s}(b_! K) = b'_! \zeta_{\epsilon^{-1},-s}(K).$$

Proposition 2.2. *For any $L \in \mathcal{D}_m(\mathcal{B}^2)$ we have*

(a) $$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \approx \{ \bigoplus_{y\in W; |y|=k} L_y \bullet L \bullet L_{\epsilon(y)^{-1}} \otimes \mathfrak{L}[[2k-2\nu]]; k \in \mathbf{N}\},$$

(b)
$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \Bumpeq$$
$$\{\bigoplus_{y\in W;|y|=k} L_y \bullet L \bullet L_{\epsilon(y)^{-1}}[[2k-2\nu-2\rho]]$$
$$\otimes \Lambda^d \mathcal{X}[[d]](d/2); k\in \mathbf{N}, d\in[0,\rho]\},$$

where $\mathfrak{L}, \mathcal{X}$ *are as in 0.3.*

When $\epsilon = 1, s = 0$ this is proved in [L13, 2.6]. The proof in the general case will be quite similar to that in the case $\epsilon = 1, s = 0$. Let

$$Y = \{(B_1,B_2,B_3,B_4,F)\in \mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B}\times G_s; F(B_1)=B_4, F(B_2)=B_3\}.$$

For $ij = 14$ or 23 we define $h'_{ij}: Y \to X_{\epsilon,s}$ by $(B_1,B_2,B_3,B_4,F)\mapsto (B_i,B_j,F)$ and $h_{ij}: Y\to \mathcal{B}^2$ by $(B_1,B_2,B_3,B_4,F)\mapsto (B_i,B_j)$. We have $\pi^*\pi_! = h'_{14!}h'^{*}_{23}$; hence

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) = f_!\pi^*\pi_! f^*(L) = f_! h'_{14!}h'^{*}_{23} f^*(L) = h_{14!}h^*_{23}L.$$

For $k\in\mathbf{N}$ let $Y^k = \bigcup_{y\in W;|y|=k} Y_y$ where

$$Y_y = \{(B_1,B_2,B_3,B_4,F)\in Y; (B_1,B_2)\in\mathcal{O}_y, (B_3,B_4)\in \mathcal{O}_{\epsilon(y)^{-1}}\}$$

and let $Y^{\ge k} := \bigcup_{k';k'\ge k} Y^{k'}$, an open subset of Y; let $h^k_{ij}: Y^k\to\mathcal{B}^2$, $h^{\ge k}_{ij}: Y^{\le k}\to \mathcal{B}^2$ be the restrictions of h_{ij}. For any $k\in\mathbf{N}$ we have a distinguished triangle

$$(h^{\ge k+1}_{14!}h^{\ge k+1*}_{23}L), h^{\ge k}_{14!}h^{\ge k*}_{23}L, h^k_{14!}h^{k*}_{23}L).$$

It follows that we have

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \Bumpeq \{h^k_{14!}h^{k*}_{23}L; k\in\mathbf{N}\}.$$

For $k\in\mathbf{N}$ let $Z^k = \bigcup_{y\in W;|y|=k} Z_y$ where

$$Z_y = \{(B_1,B_2,B_3,B_4)\in\mathcal{B}^4; (B_1,B_2)\in\mathcal{O}_y, (B_3,B_4)\in\mathcal{O}_{\epsilon(y)^{-1}}\};$$

for $i,j\in[1,4]$ we define $\tilde h^k_{ij}: Z^k\to\mathcal{B}^2$ and $\tilde h^y_{ij}: Z_y\to\mathcal{B}^2$ by $(B_1,B_2,B_3,B_4)\mapsto (B_i,B_j)$. We have an obvious morphism $u: Y^k\to Z^k$. The fibre of u at $(B_1,B_2,B_3,B_4)\in Z^k$ can be identified with the set of all $F\in G_{\epsilon,s}$ such that $F(B_1) = B_4, F(B_2) = B_3$. Since $(B_1,B_2)\in\mathcal{O}_y, (B_3,B_4)\in\mathcal{O}_{\epsilon(y)^{-1}}$ for some $y\in W$, we can find $\tilde F\in G_{\epsilon,s}$ such that $\tilde F(B_1) = B_4, \tilde F(B_2) = B_3$; hence the fibre above can be identified with

$$\{g\in G; \mathrm{Ad}(g)\tilde F(B_1) = B_4, \mathrm{Ad}(g)\tilde F(B_2) = B_3\}$$
$$= \{g\in G; \mathrm{Ad}(g)(B_4) = B_4, \mathrm{Ad}(g)(B_3) = B_3\} = B_3\cap B_4$$

if $s\ge 0$ and with

$$\{g\in G; \tilde F\mathrm{Ad}(g)(B_1) = B_4, \tilde F\mathrm{Ad}(g)(B_2) = B_3\}$$
$$= \{g\in G; \mathrm{Ad}(g)(B_1) = B_1, \mathrm{Ad}(g)(B_2) = B_2\} = B_1\cap B_2$$

if $s<0$, which is a $\mathbf{k}^{\nu-k}$-bundle over T via the obvious map $B_3\cap B_4\to B_4/U_{B_4} = T$ (if $s\ge 0$) or $B_1\cap B_2\to B_1/U_{B_1} = T$ (if $s<0$). We have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{B}^2 & \xleftarrow{h^k_{23}} & Y^k & \xrightarrow{h^k_{14}} & \mathcal{B}^2 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 1} \\
\mathcal{B}^2 & \xleftarrow{\tilde h^k_{23}} & Z^k & \xrightarrow{\tilde h^k_{14}} & \mathcal{B}^2
\end{array}$$

We have

$$h^k_{14!}h^{k*}_{23}L = \tilde{h}^k_{14!}u_!u^*\tilde{h}^{k*}_{23}L = \tilde{h}^k_{14!}(\tilde{h}^{k*}_{23}L \otimes u_!\bar{\mathbf{Q}}_l) = (\tilde{h}^k_{14!}\tilde{h}^{k*}_{23}L) \otimes \mathfrak{L}[[-2\nu + 2k]].$$

(The the last equality uses the description of the fibres of u given above.) We deduce that

$$\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)) \approx \{(\tilde{h}^k_{14!}\tilde{h}^{k*}_{23}L) \otimes \mathfrak{L}[[-2\nu + 2k]]; k \in \mathbf{N}\}.$$

Since Z^k is the union of open and closed subvarieties $Z_y, |y| = k$, we have

$$\tilde{h}^k_{14!}\tilde{h}^{k*}_{23}L = \bigoplus_{y \in W; |y|=k} \tilde{h}^y_{14!}\tilde{h}^{y*}_{23}L.$$

From the definitions we have

$$\tilde{h}^y_{14!}\tilde{h}^{y*}_{23}L = L_y \bullet L \bullet L_{\epsilon(y)^{-1}}.$$

This completes the proof of (a). Now (b) follows from (a) using

$$\mathfrak{L}[[2\rho]] \approx \{\bar{\mathbf{Q}}_l \otimes \Lambda^d\mathcal{X}[[d]](d/2); d \in [0, \rho]\}$$

which follows from the definitions.

In the remainder of this section we fix a two-sided cell $\mathbf{c}$ *of* W *such that* $\epsilon(\mathbf{c}) = \mathbf{c}$*; we set* $a = \mathbf{a}(\mathbf{c})$.

Proposition 2.3. *Let* $w \in W$ *and let* $j \in \mathbf{Z}$*. We set* $S = \zeta_{\epsilon,s}(R_{\epsilon,s,w})[[2\rho + 2\nu + |w|]] \in \mathcal{D}_m(\mathcal{B}^2)$.

(a) If $w \preceq \mathbf{c}$*, then* $S^j \in \mathcal{M}^{\preceq}\mathcal{B}^2$.

(b) If $w \in \mathbf{c}$ *and* $j > \nu + 2a$*, then* $S^j \in \mathcal{M}^{\prec}\mathcal{B}^2$.

(c) If $w \prec \mathbf{c}$*, then* $S^j \in \mathcal{M}^{\prec}\mathcal{B}^2$.

(d) S^j *is mixed of weight* $\le j$.

(e) If $j \neq \nu + 2a$ *and* $w \in \mathbf{c}$*, then* $gr_{\nu+2a}S^j \in \mathcal{M}^{\prec}\mathcal{B}^2$.

(f) If $k > \nu + 2a$ *and* $w \in \mathbf{c}$*, then* $gr_k S^j \in \mathcal{M}^{\prec}\mathcal{B}^2$.

When $\epsilon = 1, s = 0$ this is just [L13, 2.7]. The proof in the general case is entirely similar; it uses Proposition 2.2.

Proposition 2.4. *(a) If* $K \in \mathcal{D}^{\preceq}G_{\epsilon,s}$*, then* $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\preceq}\mathcal{B}^2$*. If* $K \in \mathcal{D}^{\prec}G_{\epsilon,s}$*, then* $\zeta_{\epsilon,s}(K) \in \mathcal{D}^{\prec}\mathcal{B}^2$.

(b) If $K \in \mathcal{M}^{\preceq}G_{\epsilon,s}$ *and* $j > \rho + \nu + a$*, then* $\zeta^j_{\epsilon,s}(K) \in \mathcal{M}^{\prec}\mathcal{B}^2$.

When $\epsilon = 1, s = 0$ this is just [L13, 2.8]. The proof in the general case is similar. It is enough to prove (a) assuming in addition that $K = A \in CS_{\epsilon,s}(G)$. By 1.5(b) we can find $w \in \mathbf{c}$ such that $(A : R^{n_w}_{\epsilon,s,w}) \neq 0$. Then $A[-n_w]$ is a direct summand of $R_{\epsilon,s,w}$. Hence $\zeta_{\epsilon,s}(A)$ is a direct summand of $\zeta_{\epsilon,s}(R_{\epsilon,s,w})[\Delta + a + |w|]$ and $\zeta^j_{\epsilon,s}(A)$ is a direct summand of

$$\zeta^{j+\Delta+a+|w|}_{\epsilon,s}(R_{\epsilon,s,w}) = \zeta^{j-\rho+a}_{\epsilon,s}(R_{\epsilon,s,w}[2\rho + 2\nu + |w|]).$$

Using Proposition 2.3 we deduce that (a) holds. A similar argument, based on 2.3, proves (b).

2.5. For $K \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}$ we set

$$\underline{\zeta}_{\epsilon,s}(K) = \underline{(\zeta_{\epsilon,s}(K))^{\{\rho+\nu+a\}}} \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2.$$

We say that $\underline{\zeta}_{\epsilon,s}(K)$ is the *truncated restriction* of K.

Proposition 2.6. *Let $K \in \mathcal{D}_m(G_{\epsilon,s})$ and let $L \in \mathcal{C}_0^{\spadesuit} \mathcal{B}^2$. Then there is a canonical isomorphism ${}^{\epsilon}L \bullet \zeta_{\epsilon,s}(K) \xrightarrow{\sim} \zeta_{\epsilon,s}(K) \bullet L$.*

When $\epsilon = 1, s = 0$ this follows from [L13, 2.10(a)]. We now consider the general case. Let $u : G_{\epsilon,s} \times \mathcal{B}^2 \to G_{\epsilon,s} \times \mathcal{B}^2$ be as in 1.6(a). Using the base change theorem we have $\zeta_{\epsilon,s}(K) \bullet L = c_! d^*(K \boxtimes L)$ where $Z = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times \mathcal{B}^3; F(B) = B''\}$, $d : Z \to G_{\epsilon,s} \times \mathcal{B}^2$ is $(F, (B, B'', B')) \mapsto (F, (B'', B'))$, $c : Z \to \mathcal{B}^2$ is $(F, (B, B'', B')) \mapsto (B, B')$. We have ${}^{\epsilon}L \bullet \zeta_{\epsilon,s}(K) = c'_! d'^*(K \boxtimes {}^{\epsilon}L)$ where $Z' = \{(F, (B, B'', B')) \in G_{\epsilon,s} \times \mathcal{B}^3; F(B'') = B'\}$, $d' : Z' \to G_{\epsilon,s} \times \mathcal{B}^2$ is $(F, (B, B'', B')) \mapsto (F, (B, B''))$, $c' : Z' \to \mathcal{B}^2$ is $(F, (B, B'', B')) \mapsto (B, B')$. Using 1.6(a) we have $K \boxtimes {}^{\epsilon}L = u^*(K \boxtimes L)$ hence it is enough to show that $c_! d^*(K \boxtimes L) = c'_! d'^* u^*(K \boxtimes L)$. We have $c_! d^*(K \boxtimes L) = c_{1!} d_1^*(K \boxtimes L) = c'_! d'^* u^*(K \boxtimes L)$ where $d_1 : G_{\epsilon,s} \times \mathcal{B}^2 \to G_{\epsilon,s} \times \mathcal{B}^2$ is $(F, (B, B')) \mapsto (F, (F(B), B'))$, $c_1 : G_{\epsilon,s} \times \mathcal{B}^2 \to \mathcal{B}^2$ is $(F, (B, B')) \mapsto (B, B')$. The proposition follows.

Proposition 2.7. *(a) If $L \in \mathcal{M}^{\preceq} \mathcal{B}^2$ and $j > 2a + 2\nu + 2\rho$, then $(\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))^j \in \mathcal{M}^{\prec} \mathcal{B}^2$.*

(b) If $L \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$, we have canonically

$$\underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L)) = \underline{(\zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))^{\{2a+2\nu+2\rho\}}} \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2.$$

We apply Lemma 1.9 with $\Phi = \zeta_{\epsilon,s} : \mathcal{D}_m(G_{\epsilon,s}) \to \mathcal{D}_m(\mathcal{B}^2)$ and with $\mathbf{X} = \chi_{\epsilon,s}(L)$, $(c, c') = (a + \nu + \rho, a + \nu + \rho)$; see Propositions 2.4 and 1.7. The result follows.

2.8. For $L, L' \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$, we set (as in [L13, 3.2])

(a) $$L \underline{\bullet} L' = \underline{(L \bullet L')^{\{a-\nu\}}} \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2.$$

This defines an associative tensor product structure on $\mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$.

Proposition 2.9. *Let $K \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}, L \in \mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$. There is a canonical isomorphism*

(a) $${}^{\epsilon}L \underline{\bullet} \underline{\zeta}_{\epsilon,s}(K) \xrightarrow{\sim} \underline{\zeta}_{\epsilon,s}(K) \underline{\bullet} L.$$

Applying Lemma 1.9 with $\Phi : \mathcal{D}_m^{\preceq} \mathcal{B}^2 \to \mathcal{D}_m^{\preceq} \mathcal{B}^2$, $L' \mapsto L' \bullet L$, $\mathbf{X} = \zeta_{\epsilon,s}(K)$, $(c, c') = (a-\nu, a+\rho+\nu)$ (see [L13, 3.1] and 2.4), we deduce that we have canonically

(b) $$\underline{((\zeta_{\epsilon,s}(K))^{\{a+\rho+\nu\}} \bullet L)^{\{a-\nu\}}} = \underline{(\zeta_{\epsilon,s}(K) \bullet L)^{\{2a+\rho\}}}.$$

Using Lemma 1.9 with $\Phi : \mathcal{D}_m^{\preceq} \mathcal{B}^2 \to \mathcal{D}_m^{\preceq} \mathcal{B}^2$, $L' \mapsto {}^{\epsilon}L \bullet L'$, $\mathbf{X} = \zeta_{\epsilon,s}(K)$, $(c, c') = (a - \nu, a + \rho + \nu)$ (see [L13, 3.1] and 2.8), we deduce that we have canonically

(c) $$\underline{({}^{\epsilon}L \bullet (\zeta_{\epsilon,s}(K))^{\{a+\rho+\nu\}})^{\{a-\nu\}}} = \underline{({}^{\epsilon}L \bullet \zeta_{\epsilon,s}(K))^{\{2a+\rho\}}}.$$

We now combine (b) and (c) with Proposition 2.6; we obtain the isomorphism (a).

2.10. Define $c : G_{\epsilon,s} \times \mathcal{B}^2 \to \mathcal{B}^2$ by $(F, B, B') \mapsto (F(B), F(B'))$. We show that for $K \in \mathcal{C}^{\spadesuit} G_{\epsilon,s}$ we have canonically

(a) $$c^* \zeta_{\epsilon,s} K = \bar{\mathbf{Q}}_l \boxtimes \zeta_{\epsilon,s} K.$$

We have a commutative diagram with cartesian left squares:

$$\begin{array}{ccccccc}
G_{\epsilon,s} \times \mathcal{B}^2 & \xleftarrow{f''} & X''_{\epsilon,s} & \xrightarrow{\pi''} & G_{\epsilon,s} \times G_{\epsilon,s} & \xrightarrow{e} & G_{\epsilon,s} \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d''} & & \\
G_{\epsilon,s} \times \mathcal{B}^2 & \xleftarrow{f'} & X'_{\epsilon,s} & \xrightarrow{\pi'} & G_{\epsilon,s} & & \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle e''} & & \\
\mathcal{B}^2 & \xleftarrow{f} & X_{\epsilon,s} & \xrightarrow{\pi} & G_{\epsilon,s} & &
\end{array}$$

where f, g are as in 1.3,
$X'_{\epsilon,s} = \{(\tilde{F}, B, B', F) \in G_{\epsilon,s} \times \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F\tilde{F}(B) = \tilde{F}(B')\}$,
$X''_{\epsilon,s} = \{(\tilde{F}, B, B', F) \in G_{\epsilon,s} \times \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s}; F(B) = B'\}$,
$f'(\tilde{F}, B, B', F) = (\tilde{F}, B, B')$, $f''(\tilde{F}, B, B', F) = (\tilde{F}, B, B')$,
$\pi'(\tilde{F}, B, B', F) = F$, $\pi''(\tilde{F}, B, B', F) = (\tilde{F}, F)$,
$c'(\tilde{F}, B, B', F) = (F, \tilde{F}(B), \tilde{F}(B'))$, $c''(F) = F$, $d(\tilde{F}, B, B') = (\tilde{F}, B, B')$,
$d'(\tilde{F}, B, B', F) = (\tilde{F}, \tilde{F}^{-1} F \tilde{F}, B, B')$, $d''(\tilde{F}, F) = \tilde{F}^{-1} F \tilde{F}$, $e(\tilde{F}, F) = F$.
It is enough to show that $d^* f'_! c'^* \pi^* K = f''_! \pi''^* e^* K$, or that $f''_! d'^* \pi'^* K = f''_! \pi''^* e^* K$. It is enough to show that $d'^* \pi'^* K = \pi''^* e^* K$, or that $\pi''^* d''^* K = \pi''^* e^* K$. Hence it is enough to show that $d''^* K = e^* K$. We identify $G \times G_{\epsilon,s} \leftrightarrow G_{\epsilon,s} \times G_{\epsilon,s}$ by $(g, F) \leftrightarrow (F\mathrm{Ad}(g), F)$. Then $d'', e : G_{\epsilon,s} \times G_{\epsilon,s} \to G_{\epsilon,s}$ become the maps $d_1, e_1 : G \times G_{\epsilon,s} \to G_{\epsilon,s}$ given by $(g, F) \mapsto \mathrm{Ad}(g)^{-1} F \mathrm{Ad}(g)$, $(g, F) = F$, respectively, and we have $d_1^* K = e_1^* K$ by the G-equivariance of K. Hence $d''^* K = e^* K$ as required.

Using (a) and the definitions we see that for any $K \in \mathcal{C}_0^{\mathfrak{c}} G_{\epsilon,s}$ we have canonically

(b) $$c^* \underline{\zeta}_{\epsilon,s} K = \bar{\mathbf{Q}}_l \boxtimes \underline{\zeta}_{\epsilon,s} K.$$

From the definitions (see 1.6) for any $L \in \mathcal{C}_0^{\spadesuit} \mathcal{B}^2$ we have $c^* L = \bar{\mathbf{Q}}_l \boxtimes {}^{\epsilon} L$. Comparing with (b) we deduce that we have canonically

(c) $$\epsilon(\underline{\zeta}_{\epsilon,s} K) = \underline{\zeta}_{\epsilon,s} K$$

for any $K \in \mathcal{C}_0^{\mathfrak{c}} G_{\epsilon,s}$.

3. Truncated convolution from $G_{\epsilon,s} \times G_{\epsilon',s'}$ to $G_{\epsilon\epsilon',s+s'}$

3.1. Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. We define $\mu : G_{\epsilon,s} \times G_{\epsilon',s'} \to G_{\epsilon\epsilon',s+s'}$ by $(F, F') = FF'$ (composition of maps $G \to G$); this is a quasi-morphism; see 1.3. For $K \in \mathcal{D}_m(G_{\epsilon,s}), K' \in \mathcal{D}_m(G_{\epsilon',s'})$ we define the convolution $K * K' \in \mathcal{D}_m(G_{\epsilon\epsilon',s+s'})$ by $K * K' = \mu_!(K \boxtimes K')$. If $\epsilon'' \in \mathfrak{A}$, $s'' \in \mathbf{Z}$, then for K, K' as above and $K'' \in \mathcal{D}_m(G_{\epsilon'',s''})$, we have canonically $(K * K') * K'' = K * (K' * K'') \in \mathcal{D}_m(G_{\epsilon\epsilon'\epsilon'',s+s'+s''})$ (and we denote this by $K * K' * K''$).

Lemma 3.2. *Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. Let $K \in \mathcal{D}_m(G_{\epsilon,s})$, $L \in \mathcal{D}_m(\mathcal{B}^2)$. We have canonically $K * \chi_{\epsilon',s'}(L) = \chi_{\epsilon\epsilon',s+s'}(L \bullet \zeta_{\epsilon,s}(K))$.*

Let

$$Z = \{(F_1, F_2, B, B') \in G_{\epsilon,s} \times G_{\epsilon',s'} \times \mathcal{B} \times \mathcal{B}; F_2(B) = B'\}.$$

Define $c : Z \to G_{\epsilon,s} \times \mathcal{B}^2$ by $(F_1, F_2, B, B') \mapsto (F_1, (B, B'))$ and $d : Z \to G_{\epsilon\epsilon',s+s'}$ by $(F_1, F_2, B, B') \mapsto F_1F_2$. From the definitions we see that both

$$K * \chi_{\epsilon',s'}(L), \chi_{\epsilon\epsilon',s+s'}(L \bullet \zeta_{\epsilon,s}(K))$$

can be identified with $d_! c^*(K \boxtimes L)$. The lemma follows. (In the case where $\epsilon = \epsilon' = 1$ and $s = s' = 0$ this reduces to [L13, 4.2].)

Proposition 3.3. *Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. For any $L, L' \in \mathcal{D}_m(\mathcal{B}^2)$ we have*

$$\chi_{\epsilon,s}(L) * \chi_{\epsilon',s'}(L')[[2\rho + 2\nu]]$$
$$\approx \{\chi_{\epsilon\epsilon',s+s'}(L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}})[[2|y|]] \otimes \Lambda^d\mathcal{X}[[d]](d/2); d \in [0, \rho], y \in W\}.$$

From 2.2(b) we deduce

$$L' \bullet \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L))[[2\nu + 2\rho]]$$
$$\approx \{L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}}[[2|y|]] \otimes \Lambda^d\mathcal{X}[[d]](d/2); y \in W, d \in [0, \rho]\}$$

and

$$\chi_{\epsilon\epsilon',s+s'}(L' \bullet \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L)))[[2\nu + 2\rho]]$$
$$\approx \{\chi_{\epsilon\epsilon',s+s'}(L' \bullet L_y \bullet L \bullet L_{\epsilon(y)^{-1}})[[2|y|]] \otimes \Lambda^d\mathcal{X}[[d]](d/2); y \in W, d \in [0, \rho]\}.$$

It remains to show that $\chi_{\epsilon\epsilon',s+s'}(L' \bullet \zeta_{\epsilon,s}(\chi_{\epsilon,s}(L))) = \chi_{\epsilon,s}(L) * \chi_{\epsilon',s'}(L')$. This follows from Lemma 3.2 with K, L replaced by $\chi_{\epsilon,s}(L), L'$.

In the remainder of this section we fix a two-sided cell $\mathbf{c}$ of W; we set $a = \mathbf{a}(\mathbf{c})$.

Proposition 3.4. *Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. Assume that $\epsilon(\mathbf{c}) = \mathbf{c}$, $\epsilon'(\mathbf{c}) = \mathbf{c}$. Let $w, w' \in W$ and let $j \in \mathbf{Z}$. We set $C = R_{\epsilon,s,w} * R_{\epsilon',s',w'}[[2\rho + 2\nu + |w| + |w'|]] \in \mathcal{D}_m(G_{\epsilon\epsilon',s+s'})$.*

(a) If $w \preceq \mathbf{c}$ or $w' \preceq \mathbf{c}$, then $C^j \in \mathcal{M}^{\preceq}G_{\epsilon\epsilon',s+s'}$.

(b) If $j > \Delta + 4a$ and either $w \in \mathbf{c}$ or $w' \in \mathbf{c}$, then $C^j \in \mathcal{M}^{\prec}G_{\epsilon\epsilon',s+s'}$.

(c) If $w \prec \mathbf{c}$ or $w' \prec \mathbf{c}$, then $C^j \in \mathcal{M}^{\prec}G_{\epsilon\epsilon',s+s'}$.

(d) C^j is mixed of weight $\le j$.

(e) If $j \neq \Delta + 4a$ and either $w \in \mathbf{c}$ or $w' \in \mathbf{c}$, then $gr_{\Delta+4a}C^j \in \mathcal{M}^{\prec}G_{\epsilon\epsilon',s+s'}$.

(f) If $k > \Delta + 4a$ and $w \in \mathbf{c}$ or $w' \in \mathbf{c}$, then $gr_kC^j \in \mathcal{M}^{\prec}G_{\epsilon\epsilon',s+s'}$.

When $\epsilon = \epsilon' = 1$, $s = s' = 0$, this is just [L13, 4.4]. The proof in the general case is entirely similar; it uses Proposition 3.3 and Lemma 1.4(d),(e).

Proposition 3.5. *Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. Assume that $\epsilon(\mathbf{c}) = \mathbf{c}$, $\epsilon'(\mathbf{c}) = \mathbf{c}$. Let $K \in \mathcal{D}^{\spadesuit}_m(G_{\epsilon,s})$, $K' \in \mathcal{D}^{\spadesuit}_m(G_{\epsilon',s'})$.*

(a) If $K \in \mathcal{D}^{\preceq}G_{\epsilon,s}$ or $K' \in \mathcal{D}^{\preceq}G_{\epsilon',s'}$, then $K * K' \in \mathcal{D}^{\preceq}G_{\epsilon\epsilon',s+s'}$; if $K \in \mathcal{D}^{\prec}G_{\epsilon,s}$ or $K' \in \mathcal{D}^{\prec}G_{\epsilon',s'}$, then $K * K' \in \mathcal{D}^{\prec}G_{\epsilon\epsilon',s+s'}$.

(b) If $K \in \mathcal{M}^{\preceq}G_{\epsilon,s}$, $K' \in \mathcal{M}^{\preceq}G_{\epsilon',s'}$ and $j > \rho + 2a$, then $(K * K')^j \in \mathcal{M}^{\prec}G_{\epsilon\epsilon',s+s'}$.

When $\epsilon = \epsilon' = 1$, $s = s' = 0$, this is just [L13, 4.5]. The proof in the general case is entirely similar. It is enough to prove the proposition assuming in addition that $K = A \in CS(G_{\epsilon,s})$, $K' = A' \in CS(G_{\epsilon',s'})$. By 1.5(b), we can find $w \in \mathbf{c}_A$, $w' \in \mathbf{c}_{A'}$ such that $(A : R^{n_w}_{\epsilon,s,w}) \neq 0$, $(A' : R^{n_{w'}}_{\epsilon',s',w'}) \neq 0$. Then $A[-n_w]$ is a direct summand of $R_{\epsilon,s,w}$ and $A'[-n_{w'}]$ is a direct summand of $R_{\epsilon',s',w'}$. Hence $A * A'$ is a direct summand of $R_{\epsilon,s,w} * R_{\epsilon',s',w'}[2\Delta + \mathbf{a}(w) + \mathbf{a}(w') + |w| + |w'|]$ and $(A * A')^j$ is a direct summand of

$$(R_{\epsilon,s,w} * R_{\epsilon',s',w'}[2\rho + 2\nu + |w| + |w'|])^{j+\mathbf{a}(w)+\mathbf{a}(w')+2\nu}.$$

Using Proposition 3.4 we deduce that (a) holds and that $(A*A')^j \in \mathcal{M}^{\prec}G$ provided that $j+\mathbf{a}(w)+\mathbf{a}(w')+2\nu > \Delta+4a$. Hence (b) holds. (To prove (b) we can assume, by (a), that $w \in \mathbf{c}, w' \in \mathbf{c}$ hence $\mathbf{a}(w) = \mathbf{a}(w') = a$.)

3.6. Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. Assume that $\epsilon(\mathbf{c}) = \mathbf{c}$, $\epsilon'(\mathbf{c}) = \mathbf{c}$. For $K \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}$, $K' \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon',s'}$ we set

$$K \underline{*} K' = \underline{(K * K')^{\{2a+\rho\}}} \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon\epsilon',s+s'}.$$

We say that $K\underline{*}K'$ is the *truncated convolution* of K, K'.

Proposition 3.7. *Let $\epsilon, \epsilon', \epsilon'' \in \mathfrak{A}$, $s, s', s'' \in \mathbf{Z}$. Assume that $\epsilon(\mathbf{c}) = \mathbf{c}$, $\epsilon'(\mathbf{c}) = \mathbf{c}$, $\epsilon''(\mathbf{c}) = \mathbf{c}$. Let K, K', K'' be in $\mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}, \mathcal{C}_0^{\mathbf{c}} G_{\epsilon',s'}, \mathcal{C}_0^{\mathbf{c}} G_{\epsilon'',s''}$, respectively. There is a canonical isomorphism*

(a) $$(K\underline{*}K')\underline{*}K'' \xrightarrow{\sim} K\underline{*}(K'\underline{*}K'').$$

When $\epsilon = \epsilon' = \epsilon'' = 1$, $s = s' = s'' = 0$, this is just [L13, 4.7]. The proof in the general case is entirely similar; it uses Lemma 1.9 and Proposition 3.5.

Proposition 3.8. *Let $\epsilon, \epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. Assume that $\epsilon(\mathbf{c}) = \mathbf{c}$, $\epsilon'(\mathbf{c}) = \mathbf{c}$. Let $K \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon,s}$, $K' \in \mathcal{C}_0^{\mathbf{c}} G_{\epsilon',s'}$. There is a canonical isomorphism (in $\mathcal{C}_0^{\mathbf{c}} \mathcal{B}^2$):*

$$\underline{\zeta}_{\epsilon',s'}(K') \underline{\bullet} \underline{\zeta}_{\epsilon,s}(K) \xrightarrow{\sim} \underline{\zeta}_{\epsilon\epsilon',s+s'}(K\underline{*}K').$$

When $\epsilon = \epsilon' = 1$, $s = s' = 0$ this is just [L13, 5.2]. In the general case, we first show that for any $\mathfrak{K} \in \mathcal{D}_m(G_{\epsilon,s}), \mathfrak{K}' \in \mathcal{D}_m(G_{\epsilon',s'})$ we have canonically

(a) $$\zeta_{\epsilon',s'}(\mathfrak{K}') \bullet \zeta_{\epsilon,s}(\mathfrak{K}) = \zeta_{\epsilon\epsilon',s+s'}(\mathfrak{K} * \mathfrak{K}').$$

(When $\epsilon = \epsilon' = 1$, $s = s' = 0$ this is just [L13, 5.5], which goes back to the work of Ginzburg.) Let

$$Y = \{(B_1, B_2, F_1, F_2) \in \mathcal{B} \times \mathcal{B} \times G_{\epsilon,s} \times G_{\epsilon',s'}; F_1F_2(B_1) = B_2\}.$$

Define $f' : Y \to \mathcal{B}\times\mathcal{B}$ by $f'(B_1, B_2, F_1, F_2) = (B_1, B_2)$. Define $\pi' : Y \to G_{\epsilon,s}\times G_{\epsilon',s'}$ by $\pi'(B_1, B_2, F_1, F_2) = (F_1, F_2)$. From the definitions we see that $f'_! \pi'^*(\mathfrak{K} \boxtimes \mathfrak{K}')$ is canonically isomorphic to $\zeta_{\epsilon',s'}(\mathfrak{K}') \bullet \zeta_{\epsilon,s}(\mathfrak{K})$ and to $\zeta_{\epsilon\epsilon',s+s'}(\mathfrak{K} * \mathfrak{K}')$. Thus (a) holds. Now the lemma follows from (a), using Lemma 1.9, by an argument similar to that in [L13, 5.6].

4. Analysis of the composition $\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}$

4.1. In the remainder of this paper we fix a two-sided cell $\mathbf{c}$ of W; we set $a = \mathbf{a}(\mathbf{c})$. We also fix $\epsilon \in \mathfrak{A}$ such that $\epsilon(\mathbf{c}) = \mathbf{c}$. In this section we fix $s \in \mathbf{Z}$. Let e, f, e' be integers such that $e \le f \le e' - 3$ and let $\mathbf{e} = e' - e + 1$; we have $\mathbf{e} \ge 4$. We set

$$\mathcal{Y} = \{((B_e, B_{e+1}, \dots, B_{e'}), F) \in \mathcal{B}^{\mathbf{e}} \times G_{\epsilon,s}; F(B_f) = B_{f+3}, F(B_{f+1}) = B_{f+2}\}.$$

Define $\vartheta : \mathcal{Y} \to \mathcal{B}^{\mathbf{e}}$ by $((B_e, B_{e+1}, \dots, B_{e'}), F) \mapsto (B_e, B_{e+1}, \dots, B_{e'})$. For i, j in $\{e, e + 1, \dots, e'\}$ let $p_{ij} : \mathcal{B}^e \to \mathcal{B}^2$ be the projection to the i, j coordinate; define

$h_{ij}:\mathcal{Y}\to\mathcal{B}^2$ by $h_{ij}=p_{ij}\vartheta$. Now $G^{\mathbf{e}-2}$ acts on $\mathcal{Y}$ by

$$(g_e,\dots,g_f,g_{f+3},\dots,g_{e'}):((B_e,B_{e+1},\dots,B_{e'}),F)\mapsto$$
$$(\mathrm{Ad}(g_e)(B_e),\mathrm{Ad}(g_{e+1}((B_{e+1}),\dots,\mathrm{Ad}(g_{f-1})(B_{f-1}),\mathrm{Ad}(g_f)(B_f),\mathrm{Ad}(g_f)(B_{f+1}),$$
$$\mathrm{Ad}(g_{f+3})(B_{f+2}),\mathrm{Ad}(g_{f+3})(B_{f+3}),\mathrm{Ad}(g_{f+4})(B_{f+4}),\dots,\mathrm{Ad}(g_{e'})(B_{e'})),$$
$$\mathrm{Ad}(g_{f+3})F\mathrm{Ad}(g_f^{-1}));$$

this induces a $G^{\mathbf{e}-2}$-action on $\mathcal{B}^{\mathbf{e}}$ so that ϑ is $G^{\mathbf{e}-2}$-equivariant.

Let $E=\{e,e+1,\dots,e'-1\}-\{f,f+2\}$. Assume that $x_n\in\mathbf{c}$ are given for $n\in E$. Let $P=\bigotimes_{n\in E}p^*_{n,n+1}\mathbf{L}_{x_n}\in\mathcal{D}_m\mathcal{B}^{\mathbf{e}}$, $\tilde P=\bigotimes_{n\in E}h^*_{n,n+1}\mathbf{L}_{x_n}=\vartheta^*P\in\mathcal{D}_m\mathcal{Y}$. In 4.1–4.7 we will study

$$h_{ee'!}\tilde P\in\mathcal{D}_m\mathcal{B}^2.$$

Setting $\Xi=\vartheta_!\bar{\mathbf{Q}}_l\in\mathcal{D}_m\mathcal{B}^{\mathbf{e}}$, we have

$$h_{ee'!}\tilde P=p_{ee!}(\Xi\otimes P).$$

Clearly, Ξ^j is $G^{\mathbf{e}-2}$-equivariant for any j. For any y,y' in W we set

$$Z_{y,y'}:=\{(B_e,B_{e+1},\dots,B_{e'})\in\mathcal{B}^{\mathbf{e}};(B_f,B_{f+1})\in\mathcal{O}_y,(B_{f+2},B_{f+3})\in\mathcal{O}_{y'}\}.$$

These are the orbits of the $G^{\mathbf{e}-2}$-action on $\mathcal{B}^{\mathbf{e}}$. Note that the fibre of ϑ over a point of $Z_{y,y'}$ is isomorphic to $T\times\mathbf{k}^{\nu-|y|}$ if $y'=\epsilon(y)^{-1}$ and is empty if $y'\ne\epsilon(y)^{-1}$. Thus,

(a) $\Xi|_{Z_{y,y'}}$ is 0 if $y'\ne\epsilon(y)^{-1}$

and for any $y\in W$ we have

(b)

$$\mathcal{H}^h\Xi|_{Z_{y,\epsilon(y)^{-1}}}=0 \text{ if } h>2\nu-2|y|+2\rho,\quad \mathcal{H}^{2\nu-2|y|+2\rho}\Xi|_{Z_{y,\epsilon(y)^{-1}}}=\bar{\mathbf{Q}}_l(-\nu+|y|-\rho).$$

The closure of $Z_{y,y'}$ in $\mathcal{B}^{\mathbf{e}}$ is denoted by $\bar Z_{y,y'}$. We set $k_{\mathbf{e}}=\mathbf{e}\nu+2\rho$. We have the following result.

Lemma 4.2. *(a) We have $\Xi^j=0$ for any $j>k_{\mathbf{e}}$. Hence, setting $\Xi'=\tau_{\le k_{\mathbf{e}}-1}\Xi$, we have a canonical distinguished triangle $(\Xi',\Xi,\Xi^{k_{\mathbf{e}}}[-k_{\mathbf{e}}])$.*

(b) If $\xi\in Z_{y,y'}$ and $i=2\nu-|y|-|y'|+2\rho$, the induced homomorphism $\mathcal{H}^i_\xi\Xi\to\mathcal{H}^{i-k_{\mathbf{e}}}_\xi(\Xi^{k_{\mathbf{e}}})$ is an isomorphism.

When $\epsilon=1,s=0$ this is just [L13, 6.2]. The proof in the general case is entirely similar; it uses 4.1(a),(b).

4.3. For any y,y' in W let $\mathfrak{T}_{y,y'}$ be the intersection cohomology complex of $\bar Z_{y,y'}$ extended by 0 on $\mathcal{B}^{\mathbf{e}}-\bar Z_{y,y'}$, to which $[[(\mathbf{e}-2)\nu+|y|+|y'|]]$ is applied. It is a pure perverse sheaf of weight 0. Note that

(a) $$\mathfrak{T}_{y,y'}=p^*_{f,f+1}\mathbf{L}_y\otimes p^*_{f+2,f+3}\mathbf{L}_{y'}[[(\mathbf{e}-4)\nu]].$$

We have the following result.

Lemma 4.4. *We have canonically $gr_0(\Xi^{k_{\mathbf{e}}}(k_{\mathbf{e}}/2))=\bigoplus_{y\in W}\mathfrak{T}_{y,\epsilon(y)^{-1}}$.*

When $\epsilon=1,s=0$ this is just [L13, 6.4]. The proof in the general case is entirely similar; it uses Lemma 4.2(b) and 4.1.

4.5. Let $y, \tilde{y} \in W$. Using the definitions and 1.2(a) we have

$$p_{ee'!}(\mathfrak{T}_{y,\tilde{y}} \otimes P[[(6-2\mathbf{e})\nu]])$$

(a) $$= L^\sharp_{x_1} \bullet L^\sharp_{x_{f-1}} L^\sharp_y L^\sharp_{x_{f+1}} L^\sharp_{\tilde{y}} L^\sharp_{x_{f+3}} L^\sharp_{x_{e'}}[[\nu + |y| + |\tilde{y}| + \sum_{n \in E} |x_n|]].$$

Lemma 4.6. *The map $\Xi \to \Xi^{k_\mathbf{e}}[-k_\mathbf{e}]$ (coming from $(\Xi', \Xi, \Xi^{k_\mathbf{e}}[-k_\mathbf{e}])$ in Lemma 4.2(a)) induces a morphism*

$$(p_{ee'!}(\Xi \otimes P))^{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho} \to (p_{ee'!}(\Xi^{k_\mathbf{e}} \otimes P))^{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho-k_\mathbf{e}}$$

whose kernel and cokernel are in $\mathcal{M}^{\preceq}_m \mathcal{B}^2$.

When $\epsilon = 1, s = 0$ this is just [L13, 6.6]. The proof in the general case is entirely similar; it uses Lemma 4.5(a) and [L13, 2.2(a)].

Lemma 4.7. *We have canonically*

$$\underline{(h_{ee'!}\tilde{P})^{\{(\mathbf{e}-2)a+(6-\mathbf{e})\nu+2\rho\}}} = \bigoplus_{y \in \mathbf{c}} Q_y$$

where

$$Q_y = \underline{(p_{ee'!}(\mathfrak{T}_{y,\epsilon(y)^{-1}} \otimes P))^{\{(\mathbf{e}-2)a+(6-2\mathbf{e})\nu\}}}$$
$$= \mathbf{L}_{x_1} \underline{\bullet} \mathbf{L}_{x_{f-1}} \mathbf{L}_y \mathbf{L}_{x_{f+1}} \mathbf{L}_{\epsilon(y)^{-1}} \mathbf{L}_{x_{f+3}} \mathbf{L}_{x_{e'}}.$$

When $\epsilon = 1, s = 0$ this is just [L13, 6.7]. The proof in the general case is entirely similar; it uses 4.6, 4.5(a) and [L13, 2.2(a), 2.3, 3.2].

Theorem 4.8. *Let $x \in \mathbf{c}$. We have canonically*

(a) $$\underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)) = \bigoplus_{y \in \mathbf{c}} \mathbf{L}_y \underline{\bullet} \mathbf{L}_x \underline{\bullet} \mathbf{L}_{\epsilon(y)^{-1}}.$$

When $\epsilon = 1, s = 0$ this is just [L13, 6.8]. The proof in the general case is entirely similar; it uses Lemma 4.7, the proofs of Propositions 2.2 and 2.7(b).

4.9. Using [L13, 2.4] we see that Theorem 4.8(a) implies

(a) $$\underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} \mathbf{L}_x \cong \bigoplus_{z \in \mathbf{c}} (\mathbf{L}_z)^{\oplus \psi_x(z)}$$

in $\mathcal{C}^\mathbf{c}\mathcal{B}^2$ where $\psi_x(z) \in \mathbf{N}$ are given by the following equation in $\mathbf{J}^\mathbf{c}$:

$$\sum_{y \in \mathbf{c}} t_y t_x t_{\epsilon(y)^{-1}} = \sum_{z \in \mathbf{c}} \psi_x(z) t_z.$$

5. Adjunction formula (weak form)

Proposition 5.1. *Let $\epsilon' \in \mathfrak{A}$, $s, s' \in \mathbf{Z}$. We assume that $\epsilon'(\mathbf{c}) = \mathbf{c}$. Let $K \in \mathcal{C}^\mathbf{c}_0(G_{\epsilon,s})$, $L \in \mathcal{C}^\mathbf{c}_0(\mathcal{B}^2)$. We have canonically*

(a) $$K \underline{*} \underline{\chi}_{\epsilon',s'}(L) = \underline{\chi}_{\epsilon\epsilon',s+s'}(L \underline{\bullet} \underline{\zeta}_{\epsilon,s}(K)).$$

When $\epsilon = \epsilon' = 1, s = s' = 0$ this is just [L13, 8.1]. The proof in the general case is entirely similar; it uses Lemmas 3.2 and 1.9.

For future use we recall the following result in [L13, 8.7]. (Here $\mathbf{1}'$ is as in [L13, 8.6] and $b' : \mathcal{B}^2 \to \mathcal{B}^2$ is given by $(B, B') \mapsto (B', B)$, see 1.3.)

(b) *Let $L, L' \in \mathcal{C}^\mathbf{c}\mathcal{B}^2$. We have canonically*

$$\mathrm{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathbf{1}', L \underline{\bullet} L') = \mathrm{Hom}_{\mathcal{C}^\mathbf{c}\mathcal{B}^2}(\mathfrak{D}(b'_! L'), L).$$

5.2. Let $s \in \mathbf{Z}$. When $\epsilon = 1, s = 0$, the arguments in this subsection reduce to arguments in [L13, 8.8]. Let $u' : G_{\epsilon^{-1},-s} \to \mathbf{p}$ be the obvious map. From [L2, (7.4.1),(7.4.2)] and its proof we see that if $K, K' \in \mathcal{M}_m^{\preceq} G_{\epsilon^{-1},-s}$ are semisimple, then we have canonically

$$(u'_!(K \otimes K'))^0 = \mathrm{Hom}_{\mathcal{M}G_{\epsilon^{-1},-s}}(\mathfrak{D}(K), K'), \quad (u'_!(K \otimes K'))^j = 0 \text{ if } j > 0.$$

We deduce that if K, K' are also pure of weight 0, then $(u'_!(K \otimes K'))^0$ is pure of weight zero, that is, $(u'_!(K \otimes K'))^0 = gr_0(u'_!(K \otimes K'))^0$. Let $\iota : \mathbf{p} \to G = G_{1,0}$ be the map with image 1. From the definitions we see that we have $u'_!(K \otimes K') = \iota^*(b_!(K) * K')$ where $b : G_{\epsilon^{-1},-s} \to G_{\epsilon,s}$ is given by $F \mapsto F^{-1}$. Hence for $K, K' \in \mathcal{C}_0^{\mathfrak{c}} G_{\epsilon^{-1},-s}$ we have

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon^{-1},-s}}(\mathfrak{D}(K), K') = (\iota^*(b_!(K) * K'))^0 = (\iota^*(b_!(K) * K'))^{\{0\}}.$$

Applying [L13, 8.2] with $\Phi : \mathcal{D}_m^{\preceq} G_{1,0} \to \mathcal{D}_m \mathbf{p}$, $K_1 \mapsto \iota^* K_1$, $c = -2a - \rho$ (see [L13, 8.3(a)], K replaced by $b_!(K) * K' \in \mathcal{D}_m(G_{1,0})$ and $c' = 2a + \rho$ we see that we have canonically

$$(\iota^*(b_!(K) \underline{*} K'))^{\{-2a-\rho\}} \subset (\iota^*(b_!(K) * K'))^{\{0\}}.$$

In particular, if $L, L' \in \mathcal{C}_0^{\mathfrak{c}} \mathcal{B}^2$, then we have canonically

$$(\iota^*(\underline{\chi}_{\epsilon,s}(L') \underline{*} \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{-2a-\rho\}} \subset (\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{0\}}.$$

Using the equality

$$(\iota^*(\underline{\chi}_{\epsilon,s}(L') \underline{*} \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{-2a-\rho\}} = (\iota^*(\underline{\chi}_{1,0}(L \underline{\bullet} \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}}$$

which comes from Proposition 5.1, we deduce that we have canonically

$$(\iota^*(\underline{\chi}_{1,0}(L \underline{\bullet} \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}} \subset (\iota^*(\underline{\chi}_{\epsilon,s}(L') * \underline{\chi}_{\epsilon^{-1},-s}(L)))^{\{0\}}$$

or equivalently, using (a) with K, K' replaced by $b^* \underline{\chi}_{\epsilon,s}(L')$, $\underline{\chi}_{\epsilon^{-1},-s}(L)$:

$$\begin{aligned}(\iota^*(\underline{\chi}_{1,0}(L \underline{\bullet} \underline{\zeta}_{\epsilon,s}(\underline{\chi}_{\epsilon,s}(L')))))^{\{-2a-\rho\}} &\subset \mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon^{-1},-s}}(\mathfrak{D}(b^* \underline{\chi}_{\epsilon,s}(L')), \underline{\chi}_{\epsilon^{-1},-s}(L)) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon,s}}(\mathfrak{D}(b_! \underline{\chi}_{\epsilon^{-1},-s}(L)), \underline{\chi}_{\epsilon,s}(L')).\end{aligned}$$

Now using [L13, 8.6(c)], we deduce that we have canonically

$$\mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}\mathcal{B}^2}(\mathbf{1}', L \underline{\bullet} \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon,s}}(\mathfrak{D}(b_! \underline{\chi}_{\epsilon^{-1},-s}(L), \underline{\chi}_{\epsilon,s}(L'))$$

or equivalently (see Proposition 5.1(b)):

$$\mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}\mathcal{B}^2}(\mathfrak{D}(b'_! L), \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon,s}}(\mathfrak{D}(b_! \underline{\chi}_{\epsilon^{-1},-s}(L), \underline{\chi}_{\epsilon,s}(L')).$$

We now set $'L = \mathfrak{D}(b'_! L)$ and note that

$$\mathfrak{D}(b_! \underline{\chi}_{\epsilon^{-1},-s}(L)) = \mathfrak{D}(\underline{\chi}_{\epsilon,s}(b'_! L)) = \underline{\chi}_{\epsilon,s}(\mathfrak{D}(b'_! L)) = \underline{\chi}_{\epsilon,s}('L);$$

see 1.3, 1.10(a). We obtain

(b) $$\mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}\mathcal{B}^2}('L, \underline{\zeta}_{\epsilon,s} \underline{\chi}_{\epsilon,s} L') \subset \mathrm{Hom}_{\mathcal{C}^{\mathfrak{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}('L), \underline{\chi}_{\epsilon,s}(L'))$$

for any $'L, L' \in \mathcal{C}_0^{\mathfrak{c}} \mathcal{B}^2$.

We have the following result which is a weak form of an adjunction formula, of which the full form will be proved in Theorem 6.6.

Proposition 5.3. *Let $s \in \mathbf{Z}$. For any ${}'L, L' \in \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2$ we have canonically*

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}({}'L, \underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(L')) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}({}'L), \underline{\chi}_{\epsilon,s}(L'))$$

We can assume that ${}'L = \mathbf{L}_z, L' = \mathbf{L}_u$ where $z, u \in \mathbf{c}$. By 4.9(a) and 1.8(b), both sides of the inclusion Proposition 5.2(b) have dimension $\sum_{y\in\mathbf{c}} \tau(t_{y^{-1}}t_z t_{\epsilon(y)} t_{u^{-1}})$. Hence that inclusion is an equality. The proposition is proved. (The case where $\epsilon = 1, s = 0$ is treated in [L13, 8.9].)

6. Equivalence of $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}$ with the ϵ-centre of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$

6.1. For $\epsilon' \in \mathfrak{A}$ such that $\epsilon'(\mathbf{c}) = \mathbf{c}$ and $s, s' \in \mathbf{Z}$, the bifunctor $\mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s} \times \mathcal{C}_0^{\mathbf{c}}G_{\epsilon',s'} \to \mathcal{C}_0^{\mathbf{c}}G_{\epsilon\epsilon',s+s'}$, $K, K' \mapsto K\underline{*}K'$ in 3.6 defines a bifunctor $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \times \mathcal{C}^{\mathbf{c}}G_{\epsilon',s'} \to \mathcal{C}^{\mathbf{c}}G_{\epsilon\epsilon',s+s'}$ denoted again by $K, K' \mapsto K\underline{*}K'$ as follows. Let $K \in \mathcal{C}^{\mathbf{c}}G_{\epsilon,s}$, $K' \in \mathcal{C}^{\mathbf{c}}G_{\epsilon',s'}$; we choose mixed structures of pure weight 0 on K, K' (this is possible if s_0 in 0.3 is large enough), we define $K\underline{*}K' \in \mathcal{C}_0^{\mathbf{c}}G_{\epsilon\epsilon',s+s'}$ as in 3.6 in terms of these mixed structures and we then disregard the mixed structure on $K\underline{*}K'$. The resulting object of $\mathcal{C}^{\mathbf{c}}G_{\epsilon\epsilon',s+s'}$ is denoted again by $K\underline{*}K'$; it is independent of the choices made.

In the same way, the bifunctor $\mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2 \times \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2$, $L, L' \mapsto L\underline{\bullet}L'$ gives rise to a bifunctor $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2 \times \mathcal{C}^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ denoted again by $L, L' \mapsto L\underline{\bullet}L'$; the functor $\underline{\chi}_{\epsilon,s} : \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s}$ gives rise to a functor $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}^{\mathbf{c}}G_s$ denoted again by $\underline{\chi}_{\epsilon,s}$ (it is again called *truncated induction*); the functor $\underline{\zeta}_{\epsilon,s} : \mathcal{C}_0^{\mathbf{c}}G_{\epsilon,s} \to \mathcal{C}_0^{\mathbf{c}}\mathcal{B}^2$ gives rise to a functor $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \to \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ denoted again by $\underline{\zeta}_{\epsilon,s}$ (it is again called *truncated restriction*).

The operation $K\underline{*}K'$ is again called *truncated convolution*. It has a canonical associativity isomorphism (deduced from that in Proposition 3.7) which satisfies the pentagon property.

The operation $L\underline{\bullet}L'$ makes $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ into a monoidal abelian category (see also [L9]) which has a unit object (see [L13, 9.2]) and is rigid (see [L13, 9.3]).

Note that $L \mapsto {}^{\epsilon}L$ (see 1.6) can be regarded as a functor $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2 \to \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$.

6.2. Extending slightly a definition in [Mu, 3.1] we define a *ϵ-half braiding* for an object $\mathcal{L} \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ as a collection $e_{\mathcal{L}} = \{e_{\mathcal{L}}(L); L \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2\}$ where $e_{\mathcal{L}}(L)$ are isomorphisms ${}^{\epsilon}L\underline{\bullet}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\underline{\bullet}L$ such that (i) and (ii) below hold:

(i) If $L \xrightarrow{t} L'$ is any morphism in $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$, then the diagram

$$\begin{array}{ccc} {}^{\epsilon}L\underline{\bullet}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L}\underline{\bullet}L \\ {\scriptstyle {}^{\epsilon}t\underline{\bullet}1}\downarrow & & \downarrow{\scriptstyle 1\underline{\bullet}t} \\ {}^{\epsilon}L'\underline{\bullet}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L}\underline{\bullet}L' \end{array}$$

is commutative.

(ii) If $L, L' \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$, then $e_{\mathcal{L}}(L\underline{\bullet}L') : {}^{\epsilon}(L\underline{\bullet}L')\underline{\bullet}\mathcal{L} \to \mathcal{L}\underline{\bullet}(L\underline{\bullet}L')$ is equal to the composition

$${}^{\epsilon}L\underline{\bullet}{}^{\epsilon}L'\underline{\bullet}\mathcal{L} \xrightarrow{1\underline{\bullet}e_{\mathcal{L}}(L')} {}^{\epsilon}L\underline{\bullet}\mathcal{L}\underline{\bullet}L' \xrightarrow{e_{\mathcal{L}}(L)\underline{\bullet}1} \mathcal{L}\underline{\bullet}L\underline{\bullet}L'.$$

When $\epsilon = 1$, this reduces to the definition of a half-braiding for $\mathcal{L}$ given in [Mu, 3.1]. Let $\mathcal{Z}_{\epsilon}^{\mathbf{c}}$ the category whose objects are the pairs consisting of an object $\mathcal{L}$ of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ and an ϵ-half braiding for $\mathcal{L}$. For $(\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})$ in $\mathcal{Z}_{\epsilon}^{\mathbf{c}}$ we define

$\mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}((\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'}))$ to be the vector space consisting of all $t \in \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathcal{L}, \mathcal{L}')$ such that for any $L \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ the diagram

$$\begin{array}{ccc} {}^{\epsilon}L\underline{\bullet}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L}\underline{\bullet}L \\ {\scriptstyle 1\underline{\bullet}t}\downarrow & & \downarrow{\scriptstyle t\underline{\bullet}1} \\ {}^{\epsilon}L\underline{\bullet}\mathcal{L}' & \xrightarrow{e_{\mathcal{L}'}(L)} & \mathcal{L}'\underline{\bullet}L \end{array}$$

is commutative. We say that $\mathcal{Z}_\epsilon^{\mathbf{c}}$ is the ϵ-centre of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$. (When $\epsilon = 1$, it reduces to the centre of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$; see [Mu, 3.2].)

If $s \in \mathbf{Z}$ and $K \in \mathcal{C}^{\mathbf{c}}G_s$, then the isomorphisms Proposition 2.9(a) provide an ϵ-half braiding for $\underline{\zeta}_{\epsilon,s}(K) \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ so that $\underline{\zeta}_{\epsilon,s}(K)$ can be naturally viewed as an object of $\mathcal{Z}_\epsilon^{\mathbf{c}}$ denoted by $\overline{\underline{\zeta}_{\epsilon,s}(K)}$. (Note that 2.9 is stated in the mixed category but, as above, it implies the corresponding result in the unmixed category.) Then $K \mapsto \overline{\underline{\zeta}_{\epsilon,s}(K)}$ is a functor $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \to \mathcal{Z}_\epsilon^{\mathbf{c}}$. We have the following result.

Theorem 6.3. *Let $s \in \mathbf{Z}$. The functor $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \to \mathcal{Z}_\epsilon^{\mathbf{c}}$, $K \mapsto \overline{\underline{\zeta}_{\epsilon,s}(K)}$ is an equivalence of abelian categories.*

When $\epsilon = 1, s = 0$ this reduces to [L13, 9.5]. The general case will be proved in 6.5.

Note that, when combined with 1.6(b), the theorem yields for any $F \in G_{\epsilon,s}$ (with $s > 0$) an equivalence of abelian categories

(a) $$\mathrm{Rep}^{\mathbf{c}}(G^F) \xrightarrow{\sim} \mathcal{Z}_\epsilon^{\mathbf{c}}.$$

6.4. By a variation of a general result on semisimple rigid monoidal categories in [ENO, Proposition 5.4], for any $L \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ one can define directly an ϵ-half braiding on the object $I_\epsilon(L) := \bigoplus_{y \in \mathbf{c}} \mathbf{L}_y\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(y)^{-1}}$ of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ such that, denoting by $\overline{I_\epsilon(L)}$ the corresponding object of $\mathcal{Z}_\epsilon^{\mathbf{c}}$, we have canonically

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(L, \mathcal{L}) = \mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{I_\epsilon(L)}, (\mathcal{L}, e_{\mathcal{L}}))$$

for any $(\mathcal{L}, e_{\mathcal{L}}) \in \mathcal{Z}_\epsilon^{\mathbf{c}}$.

The ϵ-half braiding on $I_\epsilon(L)$ can be described as follows: for any $L' \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ we have canonically

$$\begin{aligned} {}^{\epsilon}L'\underline{\bullet}I_\epsilon(L) &= \bigoplus_{y \in \mathbf{c}} ({}^{\epsilon}L'\underline{\bullet}\mathbf{L}_y\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(y)^{-1}}) \\ &= \bigoplus_{y,z \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathbf{L}_z, {}^{\epsilon}L'\underline{\bullet}\mathbf{L}_y) \otimes (\mathbf{L}_z\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(y)^{-1}}) \\ &= \bigoplus_{y,z \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathbf{L}_{y^{-1}}, \mathbf{L}_{z^{-1}}\underline{\bullet}{}^{\epsilon}L') \otimes (\mathbf{L}_z\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(y)^{-1}}) \\ &= \bigoplus_{y,z \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathbf{L}_{\epsilon(y)^{-1}}, \mathbf{L}_{\epsilon(z)^{-1}}\underline{\bullet}L') \otimes (\mathbf{L}_z\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(y)^{-1}}) \\ &= \bigoplus_{z \in \mathbf{c}} (\mathbf{L}_z\underline{\bullet}L\underline{\bullet}\mathbf{L}_{\epsilon(z)^{-1}}\underline{\bullet}L') = I_\epsilon(L)\underline{\bullet}L'. \end{aligned}$$

(We have used [L13, 7.7].) By a variation of results in [Mu, 3.3], [ENO, 2.15], we see that $\mathcal{Z}_\epsilon^{\mathbf{c}}$ is a semisimple $\bar{\mathbf{Q}}_l$-linear category with finitely many simple objects up to isomorphism. Note that

(b) *if* $\sigma = (\mathcal{L}, e_{\mathcal{L}})$ *is a simple object of* $\mathcal{Z}_\epsilon^{\mathbf{c}}$*, then* σ *is a summand of* $\overline{I_\epsilon(\mathbf{L}_z)}$ *for some* $z \in \mathbf{c}$*.*

Indeed, let $z \in \mathbf{c}$ be such that $\mathbf{L}_z$ is a summand of $\mathcal{L}$ in $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$; then by (a), σ is a summand of $\overline{I_\epsilon(\mathbf{L}_z)}$.

6.5. Let $s \in \mathbf{Z}$. For $x \in \mathbf{c}$ we have canonically $\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x = I_\epsilon(\mathbf{L}_x)$ as objects of $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$; see Theorem 4.8. This identification is compatible with the ϵ-half braidings (see 6.2, 6.4). (When $\epsilon = 1, s = 0$ this follows from the last commutative diagram in [L13, 7.9]; in the general case we have an analogous commutative diagram, which is established using the results in Section 4.) It follows that

(a) $$\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x} = \overline{I_\epsilon(\mathbf{L}_x)}.$$

Using this and 6.4(a) with $\mathcal{L} = \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}}$, $\tilde{L} \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$, we see that

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathbf{L}_x, \underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}) = \mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x}, \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\tilde{L}}).$$

Combining this with Proposition 5.3 we obtain for $\tilde{L} = \mathbf{L}_{x'}$ (with $x' \in \mathbf{c}$):

(b) $$\mathring{A}_{x,x'} = \mathring{A}'_{x,x'}$$

where

$$\mathring{A}_{x,x'} = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(\mathbf{L}_x), \underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})), \mathring{A}'_{x,x'} = \mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_x}, \overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}\mathbf{L}_{x'}}).$$

Note that the identification (b) is induced by the functor $K \mapsto \overline{\underline{\zeta}_{\epsilon,s}(K)}$. Let $\mathring{A} = \bigoplus_{x,x' \in \mathbf{c}} \mathring{A}_{x,x'}, \mathring{A}' = \bigoplus_{x,x' \in \mathbf{c}} \mathring{A}'_{x,x'}$. Then from (b) we have $\mathring{A} = \mathring{A}'$. Note that this identification is compatible with the obvious algebra structures of $\mathring{A}, \mathring{A}'$.

For any $A \in CS_{\epsilon,s,\mathbf{c}}$ we denote by $\mathring{A}_A$ the set of all $f \in \mathring{A}$ such that for any x, x', the (x, x')-component of f maps the A-isotypic component of $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$ to the A-isotypic component of $\underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})$ and any other isotypic component of $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$ to 0. Then $\mathring{A} = \bigoplus_{A \in CS_{\epsilon,s,\mathbf{c}}} \mathring{A}_A$ is the decomposition of $\mathring{A}$ into a sum of simple algebras (each $\mathring{A}_A$ is $\neq 0$ since, by 1.5(b) and 1.8(a), any A is a summand of some $\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)$).

Let $\mathfrak{S}$ be a set of representatives for the isomorphism classes of simple objects of $\mathcal{Z}_\epsilon^{\mathbf{c}}$. For any $\sigma \in \mathfrak{S}$ we denote by $\mathring{A}'_\sigma$ the set of all $f' \in \mathring{A}'$ such that for any x, x', the (x, x')-component of f' maps the σ-isotypic component of $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)}$ to the σ-isotypic component of $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_{x'})}$ and any other isotypic component of $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_x)}$ to 0. Then $\mathring{A}' = \bigoplus_{\sigma \in \mathfrak{S}} \mathring{A}'_\sigma$ is the decomposition of $\mathring{A}'$ into a sum of simple algebras (each $\mathring{A}'_\sigma$ is $\neq 0$ since any σ is a summand of some $\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(\mathbf{L}_z)}$ (we use 6.4(b), 6.5(a)).

Since $\mathring{A} = \mathring{A}'$, from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection $CS_{\epsilon,s,\mathbf{c}} \leftrightarrow \mathfrak{S}$, $A \leftrightarrow \sigma_A$ such that the identification $\mathring{A} = \mathring{A}'$ restricts to an identification $\mathring{A}_A = \mathring{A}'_{\sigma_A}$ for any $A \in CS_{\epsilon,s,\mathbf{c}}$. From the definitions we now see that for any $A \in CS_{\epsilon,s,\mathbf{c}}$ we have $\overline{\underline{\zeta}_{\epsilon,s}A} \cong \sigma_A$. Therefore Theorem 6.3 holds.

Theorem 6.6. *Let* $s \in \mathbf{Z}$*. Let* $L \in \mathcal{C}^{\mathbf{c}}\mathcal{B}^2$*,* $K \in \mathcal{C}^{\mathbf{c}}G_{\epsilon,s}$*. We have canonically*

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(L, \underline{\zeta}_{\epsilon,s}(K)) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(L), K).$$

Moreover, in $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ *we have* $\underline{\zeta}_{e,s}(K) \cong \bigoplus_{z\in\mathbf{c}^0} \mathbf{L}_z^{\bigoplus m_z}$ *where* $\mathbf{c}^0$ *is as in 1.5 and* $m_z \in \mathbf{N}$*.*

From Theorems 6.3 and 6.5, we see that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}G_{\epsilon,s}}(\underline{\chi}_{\epsilon,s}(L), K) &= \mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{\underline{\zeta}_{\epsilon,s}\underline{\chi}_{\epsilon,s}(L)}, \overline{\underline{\zeta}_{\epsilon,s}K})\\ &= \mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{I_\epsilon(L)}, \overline{\underline{\zeta}_{\epsilon,s}K}).\end{aligned}$$

Using 6.4(a) we see that

$$\mathrm{Hom}_{\mathcal{Z}_\epsilon^{\mathbf{c}}}(\overline{I_\epsilon(L)}, \overline{\underline{\zeta}_{\epsilon,s}K}) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(L, \underline{\zeta}_{\epsilon,s}(K))$$

and (a) follows. To prove the second assertion of the theorem it is enough to show that for any $z \in \mathbf{c} - \mathbf{c}^0$ we have $\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\mathcal{B}^2}(\mathbf{L}_z, \underline{\zeta}_{\epsilon,s}(K)) = 0$; by (a), it is enough to show that $\underline{\chi}_{\epsilon,s}(\mathbf{L}_z) = 0$ and this follows from 1.5(c). (The case where $\epsilon = 1, s = 0$ is just [L13, 9.8].)

6.7. Let $s \in \mathbf{Z}$. For $K \in \mathcal{C}^{\mathbf{c}}G_{\epsilon,s}$ we have canonically

(a) $$\mathfrak{D}(\underline{\zeta}_{\epsilon,s}(\mathfrak{D}(K))) = \underline{\zeta}_{\epsilon,s}(K).$$

When $\epsilon = 1, s = 0$ this is proved in [L13, 9.9]. The proof in the general case is entirely similar; it uses Theorem 6.6(a) and 1.10(a).

6.8. In this subsection we assume that $\epsilon = 1$. The monoidal structure on $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ induces a monoidal structure on $\mathcal{Z}_1^{\mathbf{c}}$. Moreover, the category

(a) $$\bigsqcup_{s\in\mathbf{Z}} \mathcal{C}^{\mathbf{c}}G_{1,s} = \mathcal{C}^{\mathbf{c}}G_{1,0} \bigsqcup_{s\in\mathbf{Z}; s\neq 0} \mathrm{Rep}^{\mathbf{c}}(G^{F_0^s})$$

(see 1.6(b)) has a monoidal structure given by truncated convolution; see 6.1. Moreover, Theorem 6.3 provides a functor from (a) to $\mathcal{Z}_1^{\mathbf{c}}$ which is an equivalence when restricted to any $\mathcal{C}^{\mathbf{c}}G_s$. This functor is compatible with the monoidal structures (this can be deduced from Proposition 3.8 and from the fact that the monoidal structure of $\mathcal{Z}_1^{\mathbf{c}}$ is equivalent to its opposite). Note that $\mathcal{C}^{\mathbf{c}}G_{1,0}$ is a monoidal subcategory of (a), whose unit object, described in [L13, 9.10], is also a unit object for the monoidal category (a).

6.9. The functor $L \mapsto {}^\epsilon L$ from $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ into itself induces a functor $\mathcal{Z}_\epsilon^{\mathbf{c}} \to \mathcal{Z}_\epsilon^{\mathbf{c}}$ which carries any simple object (L, e_L) of $\mathcal{Z}_\epsilon^{\mathbf{c}}$ into an object isomorphic to (L, e_L); this follows from 2.10(c), using Theorem 6.3.

6.10. Let $s \in \mathbf{Z}$. For any $A \in CS_{\epsilon,s,\mathbf{c}}$ and any $x \in \mathbf{c}$ we denote by $n_{A,x}$ the multiplicity of A in $\chi_{\epsilon,s}\mathbf{L}_x \in \mathcal{C}^{\mathbf{c}}G_{\epsilon,s}$. From Theorem 6.3 and its proof we see that if σ is the simple object of $\mathcal{Z}_\epsilon^{\mathbf{c}}$ corresponding to A, then $n_{A,x}$ is equal to the multiplicity of σ in $\overline{I_\epsilon(\mathbf{L}_x)} \in \mathcal{Z}_\epsilon^{\mathbf{c}}$.

7. Relation with Soergel bimodules

7.1. Let R be the algebra of polynomials functions on a fixed reflection representation of W (over $\bar{\mathbf{Q}}_l$). Then for each $x \in W$, the indecomposable Soergel graded R-bimodule B_x is defined as in [So, 6.16]. Let $C_{\mathbf{c}}$ be the category of graded R-bimodules which are isomorphic to finite direct sums of graded R-bimodules of the form B_x ($x \in \mathbf{c}$) without shift. There is a well defined functor $M \mapsto {}^\epsilon M$ from $C_{\mathbf{c}}$ to $C_{\mathbf{c}}$ which is linear and satisfies ${}^\epsilon B_x = B_{\epsilon^{-1}(x)}$ for $x \in \mathbf{c}$. Now $C_{\mathbf{c}}$ has a natural monoidal structure (see [L13, 10.1]) defined purely in terms of $R, W, \mathbf{c}$. (Its definition makes use of the results in [EW].) From the definition we see that $C_{\mathbf{c}}$ is equivalent to $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ as monoidal categories so that $M \mapsto {}^{\epsilon}M$ corresponds to $L \mapsto {}^{\epsilon}L$ from $\mathcal{C}^{\mathbf{c}}\mathcal{B}^2$ to itself. Then the ϵ-centre of $C_{\mathbf{c}}$ is defined as in 6.2. It is naturally equivalent to $\mathcal{Z}_{\epsilon}^{\mathbf{c}}$. Thus we can restate Theorem 6.3 as follows.

(a) *For any $s \in \mathbf{Z}$, the category $\mathcal{C}^{\mathbf{c}}G_{\epsilon,s} \to \mathcal{Z}_{\epsilon}^{\mathbf{c}}$ is naturally equivalent to the ϵ-centre of the monoidal category $C_{\mathbf{c}}$.*

This, combined with 1.6(b), shows that for $F \in G_{\epsilon,s}$ (with $s > 0$), the category $\text{Rep}^{\mathbf{c}}(G^F)$ is equivalent to the ϵ-centre of the monoidal category $C_{\mathbf{c}}$; thus, the set of simple objects of $\text{Rep}^{\mathbf{c}}(G^F)$ is not only independent of s but also independent of the characteristic of $\mathbf{k}$, since the ϵ-centre of $C_{\mathbf{c}}$ is so. (Here we identify $\bar{\mathbf{Q}}_l$ with the complex numbers.)

7.2. As mentioned in [L13, 10.1], the definition of the monoidal category $C_{\mathbf{c}}$ makes sense even when W is replaced by any (say finite, irreducible) Coxeter group and $\mathbf{c}$ is a two-sided cell in W. Assume now that $\epsilon : W \to W$ is an automorphism of W which leaves stable the set of simple reflections and leaves stable $\mathbf{c}$. Then the definition of the ϵ-centre of $C_{\mathbf{c}}$ makes sense even if W is noncrystallographic. We expect that the indecomposable objects of the ϵ-centre of $C_{\mathbf{c}}$ are in bijection with the "unipotent characters" associated to $W, \epsilon, \mathbf{c}$ in [L5]. (For $\epsilon = 1$ this expectation has already been stated in [L13, 10.1].)

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