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CUSPIDAL REPRESENTATIONS OF REDUCTIVE P-ADIC GROUPS ARE RELATIVELY INJECTIVE AND PROJECTIVE

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ABSTRACT. Cuspidal representations of a reductive p-adic group G over a field of characteristic different from p are relatively injective and projective with respect to extensions that split by a U-equivariant linear map for any subgroup U that is compact modulo the centre. The category of smooth representations over a field whose characteristic does not divide the pro-order of G is the product of the subcategories of cuspidal representations and of subrepresentations of direct sums of parabolically induced representations.

1. INTRODUCTION

Let G be a reductive linear algebraic group over a non-Archimedean local field with residue field of characteristic p; we briefly call G a *reductive p-adic group*. Let R be a commutative ring in which p is invertible. Let $\mathfrak{Rep}_R(G)$ be the category of smooth representations of G on R-modules.

A smooth representation is *cuspidal* if it is killed by the parabolic restriction functors for all proper parabolic subgroups. We call an extension $V' \rightarrowtail V \twoheadrightarrow V''$ in $\mathfrak{Rep}_R(G)$ *cmc-split exact* if it splits in $\mathfrak{Rep}_R(K)$ for any subgroup K of G that is compact modulo the centre $Z(G)$ of G (cmc). A smooth representation is *cmc-projective* or *cmc-injective*, respectively, if it is projective or injective with respect to cmc-split extensions.

Theorem 1.1. *Cuspidal representations are cmc-projective and cmc-injective.*

We call R a *good field* if it is a field whose characteristic ℓ does not divide the pro-order of G; that is, ℓ does not divide $[U_1 : U_2]$ for any compact open subgroups $U_2 \subseteq U_1 \subseteq G$. Our theorem implies:

Theorem 1.2. *Any cuspidal representation over a good field is a quotient of a cuspidal projective representation and contained in a cuspidal injective representation.*

This implies that the category of smooth representations of G over a good field is the product of the subcategory of cuspidal representations and the subcategory of representations that are contained in a sum of parabolically induced representations. This splitting is a crucial part of the Bernstein decomposition.

Our main theorem follows quickly from the theory of support projections in [1]. We recall the relevant notation and results in Section 2. Section 3 proves the assertions involving relative projectivity and applies it to show that parabolically induced representations have no cuspidal subquotients if R is a good field. Section 4

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proves the assertions about relative injectivity. This implies that any smooth representation over a good field is a direct sum of a cuspidal representation and a subrepresentation of a parabolically induced representation.

2. Support projections

Let $\mathcal{BT}$ be the affine Bruhat–Tits building of G. We treat $\mathcal{BT}$ as a partially ordered set of polysimplices, with the relation $\sigma \prec \tau$ if σ is a facet of τ. The group G acts on $\mathcal{BT}$. We denote the stabiliser of $\sigma \in \mathcal{BT}$ by

$$G_\sigma := \{g \in G \mid g\sigma = \sigma\};$$

its elements may permute the vertices of σ non-trivially.

The group G_σ is open and compact modulo the centre $Z(G)$ of G. Any subgroup that is compact modulo the centre is contained in G_σ for some $\sigma \in \mathcal{BT}$ because it fixes some point in the geometric realisation of $\mathcal{BT}$. Hence an extension is cmc-split exact (see Definition 3.3) if and only if it splits G_σ-equivariantly for each $\sigma \in \mathcal{BT}$.

The normalised Haar measure on a compact, open, pro-p subgroup $U \subseteq G$ gives an idempotent element $\langle U\rangle$ in the Hecke algebra $\mathcal{H} = \mathcal{H}(G, R)$ of G with coefficients in R because $p^{-1} \in R$. Schneider and Stuhler [3] and Vignéras [5] use certain compact, open, pro-p subgroups U_σ^n for $\sigma \in \mathcal{BT}$, $n \in \mathbb{N}$ to construct resolutions for smooth representations of G. Let $e_\sigma^n := \langle U_\sigma^n\rangle$. For fixed $\sigma \in \mathcal{BT}$, the groups $(U_\sigma^n)_{n\in\mathbb{N}}$ form a neighbourhood basis of the unit in G. Hence $(e_\sigma^n)_{n\in\mathbb{N}}$ is an approximate unit in $\mathcal{H}$.

Theorem 2.1. *Let $\Sigma \subseteq \mathcal{BT}$ be a finite, convex subcomplex. Define*

$$u_\Sigma^n := \sum_{\sigma\in\Sigma} (-1)^{\dim(\sigma)} e_\sigma^n \in \mathcal{H}.$$

This element is idempotent and acts on any smooth representation such that

$$\operatorname{im}(u_\Sigma^n) = \sum_{x\in\Sigma^0} \operatorname{im}(e_x^n), \qquad \ker(u_\Sigma^n) = \bigcap_{x\in\Sigma^0} \ker(e_x^n). \tag{1}$$

Proof. The idempotents $(e_x^n)_{x\in\mathcal{BT}^0}$ satisfy the conditions in [1, Definition 2.1], and $e_\sigma^n = \prod_{x\prec\sigma} e_x^n$ for $\sigma \in \mathcal{BT}$. Hence everything follows from [1, Theorem 2.12]. □

The idempotent u_Σ^n is called the *support projection* of Σ in [1].

3. Relative projectivity of cuspidal representations

Definition 3.1. A representation $V \in \mathfrak{Rep}_R(G)$ is *cuspidal* if it is killed by the parabolic restriction functor for any proper parabolic subgroup.

Proposition 3.2 ([4, II.2.7])**.** *A representation V is cuspidal if and only if for each compact open pro-p subgroup $U \subseteq G$ and each $v \in V$, the set of $g \in G$ with $\langle U\rangle gv \neq 0$ is compact modulo $Z(G)$.*

Since $e_\sigma^n = U_\sigma^n$ and the subgroups U_σ^n form a neighbourhood basis in G, V is cuspidal if and only if the set of $g \in G$ with $e_\sigma^n gv \neq 0$ is compact modulo the centre of G for all $n \in \mathbb{N}$, $\sigma \in \mathcal{BT}$, $v \in V$. The set of these $g \in G$ is closed under multiplication by elements of the centre $Z(G)$ because $e_\sigma^n zgv = ze_\sigma^n gv = 0$ if and only if $e_\sigma^n gv = 0$; and it is closed under left multiplication by elements of U_σ^n. Thus being cuspidal means that for each n, σ, v, the set of $g \in G$ with $e_\sigma^n gv \neq 0$ consists of only finitely many cosets $U_\sigma^n Z(G)g$.

Definition 3.3. An *extension* $W' \rightarrowtail W \twoheadrightarrow W''$ in $\mathfrak{Rep}_R(G)$ consists of smooth representations W', W and W'' of G on R-modules and G-equivariant R-module maps $i\colon W' \to W$ and $p\colon W \to W''$ such that i is injective, p is surjective, and $i(W')$ is the kernel of p. This extension is *c-split* if there is a U-equivariant R-module map $s\colon W'' \to W$ with $p \circ s = \mathrm{Id}_{W''}$ for each compact subgroup $U \subseteq G$, and *cmc-split* if the same happens for each subgroup $U \subseteq G$ for which $U/(Z(G) \cap U)$ is compact.

Definition 3.4. An object V of $\mathfrak{Rep}_R(G)$ is *projective*, *c-projective* or *cmc-projective* if the functor $\mathrm{Hom}_{R,G}(V, \llcorner\!\lrcorner)$ is exact on all extensions, all c-split extensions or all cmc-split extensions in $\mathfrak{Rep}_R(G)$, respectively. It is *injective*, *c-injective* or *cmc-injective* if $\mathrm{Hom}_{R,G}(\llcorner\!\lrcorner, V)$ is exact on the appropriate extensions.

Theorem 3.5. *Cuspidal representations in $\mathfrak{Rep}_R(G)$ are cmc-projective.*

Theorem 3.6. *For any cuspidal representation W'' in $\mathfrak{Rep}_R(G)$ there is a c-split extension $W' \rightarrowtail W \twoheadrightarrow W''$ with cuspidal and c-projective W.*

We are going to prove Theorems 3.5 and 3.6 in Sections 3.1 and 3.2, respectively. In Section 3.3, we specialise to the case where R is a good field and deduce an orthogonality result for cuspidal and parabolically induced representations.

3.1. **Cuspidal representations are cmc-projective.** We are going to prove Theorem 3.5 about cuspidal representations being cmc-projective. Let V be a cuspidal representation, so it verifies the condition in Proposition 3.2. Fix $n \in \mathbb{N}$. Let $e_\sigma^n V \subseteq V$ for $\sigma \in \mathcal{BT}$ be the image of e_σ^n on V. Let

$$F^n := \bigoplus_{\sigma \in \mathcal{BT}} e_\sigma^n V.$$

Elements of F^n are functions $\psi\colon \mathcal{BT} \to V$ with finite support and $e_\sigma^n \psi(\sigma) = \psi(\sigma)$ for all $\sigma \in \mathcal{BT}$. We let G act on F^n by $(g\psi)(\sigma) := g \cdot (\psi(g^{-1}\sigma))$ for $g \in G$, $\sigma \in \mathcal{BT}$, $\psi \in F^n$. This belongs to F^n again because $e_{g\sigma}^n = g e_\sigma^n g^{-1}$, and it defines a smooth representation, so $F^n \in \mathfrak{Rep}_R(G)$.

Lemma 3.7. *The smooth representation F^n is cmc-projective.*

Proof. We may decompose F^n as a sum over the finite set of G-orbits $G\backslash\mathcal{BT}$, where each summand is the subspace F_σ^n of F^n of functions supported on $G\sigma \cong G/G_\sigma$. When we apply compact induction from G_σ to G to the representation of G_σ on the G_σ-invariant subspace $e_\sigma^n V \subseteq V$, we get the representation F_σ^n. Since compact induction for the open subgroup $G_\sigma \subseteq G$ is left adjoint to restriction, we get

$$\mathrm{Hom}_G(F_\sigma, \llcorner\!\lrcorner) \cong \mathrm{Hom}_{G_\sigma}(e_\sigma^n V, \llcorner\!\lrcorner).$$

Since the groups $G_\sigma/Z(G)$ are compact, this functor is exact on cmc-split extensions. $\square$

The inclusion maps $e_\sigma^n V \hookrightarrow V$ define a G-equivariant R-module homomorphism $\pi^n\colon F^n \to V$. Since V is cuspidal, we also get a map in the opposite direction:

Lemma 3.8. *Let V be a cuspidal representation. There is a G-equivariant R-module homomorphism $\alpha^n\colon V \to F^n$ defined by*

$$\alpha^n(v)(\sigma) := (-1)^{\dim(\sigma)} e_\sigma^n(v) \qquad \text{for } v \in V,\ \sigma \in \mathcal{BT}.$$

The map $\pi^n \circ \alpha^n\colon V \to V$ is idempotent with image $\sum_{x \in \mathcal{BT}^0} e_x^n V$.

Proof. Let $v \in V$ and let $\sigma \in \mathcal{BT}$. Proposition 3.2 shows that the set of $g \in G$ with $e^n_{g\sigma} v = g\langle U^n_\sigma\rangle g^{-1} v \neq 0$ is compact modulo $Z(G)$. Hence $\alpha^n(v)(\tau)$ vanishes on all but finitely many $\tau \in G\sigma$. Since $G\backslash\mathcal{BT}$ is finite, $\alpha^n(v)$ has finite support. We have $\alpha^n(v)(\sigma) \in e^n_\sigma V$ by construction, so $\alpha^n(v) \in F^n$. The map $\alpha^n \colon V \to F^n$ is R-linear and G-equivariant because $e^n_{g\sigma} = g e^n_\sigma g^{-1}$. Theorem 2.1 implies that $\pi^n \circ \alpha^n \colon V \to V$ is idempotent with the asserted image: write $\mathcal{BT}$ as a union of finite convex subcomplexes and use that all sums that occur are finite. □

If V is finitely generated, then $V = \sum_{x\in\mathcal{BT}^0} e^n_x V$ for some $n \geq 0$, so that $\pi^n \circ \alpha^n = \mathrm{Id}_V$ and V is a direct summand of F^n. For general V, we let $F := \bigoplus_{n=0}^\infty F^n$. This is cmc-projective by Lemma 3.7. Let $\pi \colon F \to V$ be induced by the maps $\pi^n \colon F^n \to V$. Define $\alpha \colon V \to F$ by $\alpha(v) = (\alpha(v)_n)_{n\in\mathbb{N}}$ with

$$\alpha(v)_n := \alpha^n \circ (\mathrm{Id}_V - \pi^{n-1} \circ \alpha^{n-1})(v)$$

for $n \in \mathbb{N}$, with the convention $\pi^{-1}\circ\alpha^{-1} = 0$ for $n = 0$. For any $v \in V$ and $\sigma \in \mathcal{BT}^0$, there is $n_0 \geq 0$ with $e^n_\sigma v = v$ for all $n \geq n_0$. This implies $(\mathrm{Id}_V - \pi^n \circ \alpha^n)(v) = 0$ and hence $\alpha(v)_n = 0$ for $n \geq n_0$. Thus $\alpha(v) \in F$. Since the maps α^n and π^n are G-equivariant R-module homomorphisms, so is α. The sequence of idempotent maps $\pi^n \circ \alpha^n$ on V is increasing, that is, $\pi^n \circ \alpha^n \circ \pi^{n-1} \circ \alpha^{n-1} = \pi^{n-1} \circ \alpha^{n-1}$. Hence

$$\pi \circ \alpha(v) = \sum_{n=0}^{\infty} (\pi^n \circ \alpha^n - \pi^{n-1} \circ \alpha^{n-1})(v) = v.$$

This finishes the proof that cuspidal representations are cmc-projective.

3.2. **Enough c-projective cuspidal representations.** We are going to prove Theorem 3.6 about cuspidal representations being c-split quotients of c-projective cuspidal representations. An important tool that we will also use for other purposes is the subgroup ${}^0G \subseteq G$, that is, the intersection of the kernels of all unramified characters. The following is proved in [2, Section V.2.3–6]:

Proposition 3.9. *The subgroup ${}^0G$ is open and normal in G and contains any compact subgroup of G. The quotient group $G/{}^0G$ is free Abelian of finite rank. The image of $Z(G)$ in $G/{}^0G$ has finite index, and ${}^0Z := Z(G)\cap{}^0G$ is compact.* □

The subgroup ${}^0G$ is equal to the subgroup generated by the compact elements of G used in [4]. We modify the criterion for cuspidal representations in Proposition 3.2:

Proposition 3.10. *A smooth representation $V \in \mathfrak{Rep}_R(G)$ is cuspidal if and only if for each $v \in V$ and $x \in \mathcal{H}$, the set of $g \in {}^0G$ with $xgv \neq 0$ is compact.*

Proof. It makes no difference whether we use all $x \in \mathcal{H}$ or only $\langle U\rangle$ for compact open pro-p subgroups $U \subseteq G$ because for any $x \in \mathcal{H}$ there is U with $x = x\langle U\rangle$. Assume first that for each $v \in V$, the set of $g \in {}^0G$ with $xgv \neq 0$ is compact. Let $h_1, \dots, h_n \in G$ be representatives for the finite quotient group $G/{}^0GZ(G)$. So every element of G is of the form gzh_i with $g \in {}^0G$, $z \in Z(G)$, $i \in \{1, \dots, n\}$. For each i, the set K_i of $g \in {}^0G$ with $xgh_iv \neq 0$ is compact. The set of $g \in G$ with $xgv \neq 0$ is $\bigsqcup_{i=1}^n K_i h_i Z(G)$, which is compact modulo the centre. Hence V verifies the criterion for being cuspidal in Proposition 3.2. Conversely, if V verifies that criterion, then the set of $g \in {}^0G$ with $xgv \neq 0$ has to be compact because ${}^0Z$ is compact. □

We use the criterion in Proposition 3.10 to define which smooth representations of ${}^0G$ are cuspidal. Then Proposition 3.10 says that a representation of G is cuspidal if and only if its restriction to ${}^0G$ is cuspidal.

Theorem 3.11. *Cuspidal representations of ${}^0G$ are c-projective in $\mathfrak{Rep}_R({}^0G)$.*

Proof. Since ${}^0Z$ is compact, cmc- and c-projectives in $\mathfrak{Rep}_R({}^0G)$ are the same. Hence the assertion follows as in the proof of Theorem 3.5. ☐

The functor that restricts a smooth G-representation to ${}^0G$ and then applies compact induction to G, maps $W'' \in \mathfrak{Rep}_R(G)$ to $W = \bigoplus_{[g]\in G/{}^0G} W''$, the space of finitely supported functions $G/{}^0G \to W''$, with the (smooth) representation of G by $(g\psi)[h] = g \cdot (\psi[g^{-1}h])$ for $g, h \in G$ and $\psi\colon G/{}^0G \to W''$; here we use that G already acts on W'' to simplify the result. Since this action of G on W restricts to the given representation of ${}^0G$ on W'' on each summand, Proposition 3.10 shows that W is cuspidal if and only if W'' is.

Define $\pi\colon W \to W''$, $\pi(\psi) = \sum_{[g]\in G/{}^0G} \psi[g]$. This map is G-equivariant and R-linear. It splits by the ${}^0G$-equivariant R-linear map $s\colon W'' \to W$ defined by $s(v)[1] := v$ and $s(v)[g] := 0$ for $[g] \neq [1]$ in $G/{}^0G$. Since all compact subgroups are contained in ${}^0G$, π is a c-split surjection. The adjointness between compact induction and restriction for the open subgroup ${}^0G \subseteq G$ gives

$$\mathrm{Hom}_G(W, \text{␣}) \cong \mathrm{Hom}_{{}^0G}(W'', \text{␣}).$$

This functor is exact on c-split extensions by Theorem 3.11 because W'' is cuspidal. Thus W'' is c-projective and cuspidal. Take $W' := \ker(\pi)$ to finish the proof of Theorem 3.6.

3.3. **Representations over fields of good characteristic.** Now let R be a *good field*, that is, its characteristic does not divide the pro-order of G.

Corollary 3.12. *Let R be a good field. Cuspidal representations of G are projective as representations of ${}^0G$, and quotients of projective, cuspidal representations of G.*

Proof. Any extension of vector spaces over the field R splits, and we can make the section U-invariant for a given compact subgroup $U \subseteq G$ by averaging over the normalised Haar measure of U; this measure has values in R because $[U : U']$ is invertible in R for any open subgroup $U' \subseteq U$ by our assumption on the characteristic of R. Thus all extensions are c-split exact, and c-projective objects are projective. Now everything follows from Theorems 3.11 and 3.6. ☐

Definition 3.13. A representation is *subinduced* if it is contained in a direct sum of parabolically induced representations from proper parabolic subgroups.

Theorem 3.14. *Let R be a good field. Let $V \in \mathfrak{Rep}_R(G)$ be cuspidal and let $W \in \mathfrak{Rep}_R(G)$ be subinduced. Any map $W \to V$ or $V \to W$ vanishes. Subinduced representations have no non-zero cuspidal subquotients.*

Proof. The vanishing of maps $V \to W$ is well known and follows from the (easy) left adjointness of parabolic restriction to parabolic induction (see [4, II.2.3] or [2, Section VI.7.2]). Assume there is a non-zero map $f\colon W \to V$. Since subrepresentations of cuspidal representations remain cuspidal, $\mathrm{im}(f)$ is cuspidal. By Corollary 3.12, $\mathrm{im}(f)$ is a quotient of a projective, cuspidal representation, $p\colon V' \twoheadrightarrow \mathrm{im}(f)$. Since V'

is projective, there is a map $h\colon V' \to W$ with $f \circ h = p$. Since $\mathrm{im}(f) \neq 0$, we have $p \neq 0$ and thus $h \neq 0$. But since V' is cuspidal and W is subinduced, there is no non-zero map $V' \to W$. Thus there cannot be a non-zero map $W \to V$.

If subinduced representations may have non-zero cuspidal subquotients, they also may have non-zero cuspidal quotients. This would give a non-zero map from a subinduced representation to a cuspidal representation, which is impossible. □

An example mentioned in [4, II.2.5] shows that parabolically induced representations of $\mathrm{Gl}(2, \mathbb{Q}_5)$ over the field with 3 elements may have cuspidal subquotients. Hence the assertions of Theorem 3.14 and Corollary 3.12 become false in this case. Since 3 divides the pro-order of $\mathrm{Gl}(2, \mathbb{Q}_5)$, there is no contradiction.

Remark 3.15. Vignéras [5] proves that *irreducible* cuspidal representations are projective in the category of representations with a fixed central character; this is essentially the same as proving that they are projective as ${}^0G$-representations. The proof depends on Schur's Lemma, which implies that any non-zero map on an irreducible representation is invertible. There are only finitely many isomorphism classes of cuspidal representations of ${}^0G$ that contain a K-fixed vector for any fixed compact subgroup $K \subseteq {}^0G$. This deep fact and the projectivity of irreducible cuspidal representations imply a product decomposition of the category of representations of ${}^0G$ into one factor for each irreducible cuspidal representation and one factor for those representations that have no cuspidal subquotients (compare [2, Section VI.3.4]). This product decomposition shows that *all* cuspidal representations are projective as representations of ${}^0G$ because this holds for irreducible cuspidal representations. In contrast, we directly prove that arbitrary cuspidal representations are projective and also injective as representations of ${}^0G$. Much of the direct product decomposition in [2, Section VI.3.4] follows from such projectivity and injectivity results, compare Theorem 4.7 below.

4. Relative injectivity of cuspidal representations

Definition 4.1. A representation V is *uniformly cuspidal* if for each $x, y \in \mathcal{H}$ the set of $g \in G$ for which $xgy \in \mathcal{H}$ acts non-trivially on V is compact modulo $Z(G)$.

It is shown in [4, II.2.16] that cuspidal representations are uniformly cuspidal. This allows us to prove relative injectivity results for cuspidal representations.

4.1. **Cmc-injectivity.**

Theorem 4.2. *Cuspidal representations in $\mathfrak{Rep}_R(G)$ are cmc-injective.*

For $n \geq 0$, let

$$\tilde{I}^n := \prod_{\sigma \in \mathcal{BT}} e_\sigma^n V.$$

Elements of $\tilde{I}^n$ are functions $\psi\colon \mathcal{BT} \to V$ with $e_\sigma^n \psi(\sigma) = \psi(\sigma)$ for all $\sigma \in \mathcal{BT}$, and G acts on such functions as above, by $(g\psi)(\sigma) := g \cdot (\psi(g^{-1}\sigma))$ for $g \in G$, $\sigma \in \mathcal{BT}$, $\psi \in \tilde{I}^n$. This representation is not smooth. Let $I^n \subseteq \tilde{I}^n$ be the subspace of smooth elements, that is, elements fixed by some compact open subgroup of G.

Lemma 4.3. *The smooth representation I^n is cmc-injective.*

Proof. This is proved in the same way as Lemma 3.7. Let I^n_σ be the restriction of I^n to the orbit of σ. Then $I^n \cong \prod_{\sigma\in G\backslash\mathcal{BT}} I^n_\sigma$ because $G\backslash\mathcal{BT}$ is finite. Thus I^n is cmc-injective if and only if I^n_σ is cmc-injective for all $\sigma \in G\backslash\mathcal{BT}$. The induction functor from G_σ to G maps V to I^n_σ and is right adjoint to the restriction functor. Thus the same argument as in the proof of Lemma 3.7 shows that I^n_σ is cmc-injective. $\square$

Since $F^n \subseteq \tilde{I}^n$ is a smooth representation, we have $F^n \subseteq I^n$. Thus the map α^n constructed in Section 3.1 is also a G-equivariant R-module map $\alpha^n\colon V \to I^n$.

Lemma 4.4. *If V is cuspidal, then the map $\pi^n\colon F^n \to V$ extends to a G-equivariant R-module map $\bar{\pi}^n\colon I^n \to V$.*

Proof. Let $\psi \in I^n \subseteq \tilde{I}^n$. Since ψ is a smooth element, it is fixed by some compact open pro-p subgroup U of G. Let $\sigma \in \mathcal{BT}$. The set of $g \in G$ with $\langle U\rangle g e^n_\sigma \neq 0$ is compact modulo $Z(G)$ because V is uniformly cuspidal (see [4, II.2.16]). Since $\psi(g\sigma) \in e^n_{g\sigma}V = g e^n_\sigma g^{-1}V = g e^n_\sigma V$, the set of $g \in G$ with $\langle U\rangle\psi(g\sigma) \neq 0$ is compact modulo $Z(G)$. Thus $\langle U\rangle\psi(g\sigma) = 0$ for all but finitely many elements in the orbit $G\sigma$ because the stabiliser G_σ of σ contains $Z(G)$. Since $G\backslash\mathcal{BT}$ is finite, the sum

$$\bar{\pi}^n_U(\psi) := \sum_{\sigma\in\mathcal{BT}} \langle U\rangle\psi(\sigma)$$

is finite. If ψ has finite support, then we may pull $\langle U\rangle$ out of the sum and get $\bar{\pi}^n_U(\psi) = \langle U\rangle\pi^n(\psi) = \pi^n(\langle U\rangle\psi) = \pi^n(\psi)$ because π^n is G-equivariant.

We claim that $\bar{\pi}^n_U$ does not depend on U, so we get a well-defined map $\bar{\pi}^n\colon I^n \to V$. Taking this for granted, it is routine to check that $\bar{\pi}^n$ is a G-equivariant R-module homomorphism. We checked already that it extends π^n, so this finishes the proof.

It remains to prove that $\bar{\pi}^n$ does not depend on U. If U_1, U_2 are compact open pro-p subgroups fixing ψ, then there is an open subgroup $U_3 \subseteq U_1 \cap U_2$ which is normal in U_1, and U_3 contains an open subgroup U_4 that is normal in U_2 and hence in U_3. Thus we are done if we show that $\bar{\pi}^n_U = \bar{\pi}^n_{U'}$ if U is a normal open subgroup of U'. Then $\langle U'\rangle = [U' : U]^{-1}\sum_{g\in U'/U}\langle U\rangle g$. We compute

$$\begin{aligned}
\bar{\pi}^n_{U'}(\psi) &= \frac{1}{[U':U]}\sum_{\substack{g\in U'/U\\ \sigma\in\mathcal{BT}}} \langle U\rangle g\cdot(\psi(\sigma)) = \frac{1}{[U':U]}\sum_{\substack{g\in U'/U\\ \sigma\in\mathcal{BT}}} \langle U\rangle g\cdot(\psi(g^{-1}\sigma))\\
&= \frac{1}{[U':U]}\sum_{\substack{g\in U'/U\\ \sigma\in\mathcal{BT}}} \langle U\rangle(g\psi)(\sigma) = \bar{\pi}^n_U(\psi);
\end{aligned}$$

the second step reindexes the sum over σ; the third step is the definition of the G-action on I^n, and the last step uses $g\psi = \psi$ for $g \in U'$. $\square$

Lemma 3.8 shows that $\bar{\pi}^n\circ\alpha^n = \pi^n\circ\alpha^n$ is idempotent with image $\sum_{x\in\mathcal{BT}^0} e^n_x V$. If $V = \sum_{x\in\mathcal{BT}^0} e^n_x V$, then Lemmas 3.8 and 4.4 show that V is cmc-injective because it is a direct summand in the cmc-injective smooth representation I^n.

In general, let $V^n := (\pi^n\alpha^n - \pi^{n-1}\alpha^{n-1})V$ for $n \in \mathbb{N}$. These are direct summands in V with $V = \bigoplus_{n\in\mathbb{N}} V^n$. We claim that any element of $\prod_{n\in\mathbb{N}} V^n$ that is smooth for the G-action already belongs to the direct sum $\bigoplus_{n\in\mathbb{N}} V^n$. Indeed, a smooth vector is in the image of e_x^m for some $m \in \mathbb{N}$ because these form an approximate unit, and then it is killed by $\pi^n\alpha^n - \pi^{n-1}\alpha^{n-1}$ for $n > m$. Hence V is also the *product* of the V^n in the category of smooth representations. The proof above shows that the factors V^n are cmc-injective; so V is also because products of cmc-injective representations remain cmc-injective. This finishes the proof of Theorem 4.2.

4.2. **Enough c-injective cuspidal representations.**

Theorem 4.5. *For any cuspidal representation W' in $\mathfrak{Rep}_R(G)$ there is a c-split extension $W' \rightarrowtail W \twoheadrightarrow W''$ with cuspidal and c-injective W. Cuspidal representations of ${}^0G$ are c-injective in $\mathfrak{Rep}_R({}^0G)$.*

Proof. This is analogous to the proofs of Theorem 3.6 and 3.11. For the first statement, we restrict the representation on W' to ${}^0G$ and now apply ordinary induction to define W; this produces a c-injective representation of G because induction is right adjoint to restriction. There is a canonical G-equivariant map $W' \to W$, which splits ${}^0G$-equivariantly by restriction to ${}^0G \subseteq G$. Letting $W'' := W/W'$, we get the desired c-split extension. □

Corollary 4.6. *Let R be a good field. Any cuspidal representation in $\mathfrak{Rep}_R(G)$ is contained in a cuspidal, injective representation.*

Proof. See the proof of Corollary 3.12. □

Theorem 4.7. *Let R be a good field. The category $\mathfrak{Rep}_R(G)$ is the product of the two subcategories of cuspidal and subinduced representations.*

Proof. Theorem 3.14 shows that there are no arrows in either direction between the subcategories of cuspidal and subinduced representations. It remains to show that every representation is a product of a cuspidal and a subinduced representation.

Let S be the set of proper standard parabolic subgroups of G. Let i_P and r_P for $P \in S$ be the parabolic induction and restriction functors. The right adjointness of i_P to r_P gives natural maps $\beta_P\colon V \to i_P r_P(V)$ for $V \in \mathfrak{Rep}_R(G)$, which we put together into a natural map $\beta\colon V \to \bigoplus_{P\in S} i_P r_P(V)$. Let $V' := \ker(\beta)$, $V'' := \operatorname{im}(\beta)$. These form an extension $V' \rightarrowtail V \twoheadrightarrow V''$ in $\mathfrak{Rep}_R(G)$ with subinduced V''. Inspection shows that $\beta(v) = 0$ for $v \in V$ if and only if for each unipotent subgroup $N \subseteq G$ there is a compact open subgroup U_N with $\langle U_N\rangle v = 0$; thus V' is cuspidal.

We may embed V' in an injective, cuspidal representation W by Corollary 4.6. The embedding $i\colon V' \to W$ extends to a map $\varphi\colon V \to W$ because W is injective. Since $\varphi(V') = i(V')$, φ induces a map $\dot\varphi\colon V'' \to W/V'$. The quotient W/V' is again cuspidal, so Theorem 3.14 gives $\dot\varphi = 0$. Thus $\varphi(V) \subseteq i(V')$, so $\varphi\colon V \to V'$ is a section for our extension. Hence $V \cong V' \oplus V''$. □

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