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ON REDUCIBILITY OF p-ADIC PRINCIPAL SERIES REPRESENTATIONS OF p-ADIC GROUPS

DUBRAVKA BAN AND JOSEPH HUNDLEY

Abstract. We study the continuous principal series representations of split connected reductive p-adic groups over p-adic fields. We show that such representations are irreducible when the inducing character lies in a certain cone. This is consistent with a conjecture of Schneider regarding reducibility in the semisimple case.

1. Introduction

The theory of p-adic Banach space representations of p-adic groups was developed by Schneider and Teitelbaum in [ScT02]. These representations play a fundamental role in the p-adic Langlands program [BBr10], [Col10]. Important examples of p-adic Banach space representations are continuous principal series. In [Sch06], Schneider formulated a conjecture about the irreducibility of principal series representations. In this paper, we confirm the validity of the conjecture for certain characters. These characters form a "cone" in the group of characters, as we will explain below.

Let $\mathbb{Q}_p \subseteq L \subseteq K$ be a sequence of finite extensions. Let $\mathbf{G}$ be a split and connected reductive algebraic $\mathbb{Z}$-group, and $G = \mathbf{G}(L)$. We fix a maximal split torus $\mathbf{T}$ in $\mathbf{G}$ and a minimal parabolic subgroup $\mathbf{P}$ containing $\mathbf{T}$. Set $T = \mathbf{T}(L)$ and $P = \mathbf{P}(L)$.

Let $\chi : T \to K^\times$ be a continuous character. Let

$$\mathrm{Ind}_P^G(\chi^{-1}) = \{f : G \to K \text{ continuous } \mid f(gp) = \chi(p)f(g)\, \forall p \in P, g \in G\},$$

where G acts on the left by $g \cdot f(h) = f(g^{-1}h)$.

Let $X(\mathbf{T})$ be the lattice of rational characters of $\mathbf{T}$. We select a basis $\lambda_1, \ldots, \lambda_r$ for $X(\mathbf{T})$ consisting of dominant elements. If $\eta : L^\times \to K^\times$ is a continuous character, then there exists an integer $e(\eta)$ such that $ord_K \circ \eta = e(\eta) \cdot ord_L$ (see Definition 7.2). Our main result is the following theorem.

Theorem 1.1. *Let $\chi_1, \ldots, \chi_r : L^\times \to K^\times$ be continuous characters such that $e(\chi_i) < 0$ for $1 \leq i \leq r$. Define $\chi : T \to K^\times$ by $\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i})$. Then $\mathrm{Ind}_P^G \chi^{-1}$ is topologically irreducible (that is, it has no proper nontrivial closed invariant subspaces).*

To explain how this theorem relates to Conjecture 2.5 of [Sch06], assume that G is semisimple and simply connected. Then we can take $\lambda_1, \ldots, \lambda_r$ to be the fundamental weights. Let $\delta = \sum_{i=1}^r \lambda_i$. The character $\chi : T \to K^\times$ is called

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anti-dominant if $\chi\delta \circ \alpha^\vee \neq (\,)^m$ for any integer $m \geq 1$ and any positive root α. In [Sch06], Schneider conjectures that the G-representation $\mathrm{Ind}_P^G(\chi^{-1})$ is topologically irreducible if χ is anti-dominant. The conjecture is known to be true for $G = GL_2(\mathbb{Q}_p)$.

Some instances of irreducibility follow from the understanding of locally analytic principal series. Let $\mathrm{Ind}_P^G(\chi^{-1})^{an}$ denote the set of locally analytic vectors in $\mathrm{Ind}_P^G(\chi^{-1})$. The reducibility of $\mathrm{Ind}_P^G(\chi^{-1})^{an}$ is known from [Fr03] and [OStr10]. If $L = \mathbb{Q}_p$, then $\mathrm{Ind}_P^G(\chi^{-1})^{an}$ is dense in $\mathrm{Ind}_P^G(\chi^{-1})$, and irreducibility of locally analytic principal series implies irreducibility of continuous principal series.

Our Theorem 1.1 proves irreducibility for the characters χ belonging to the cone $\{\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i}) \mid e(\chi_i) < 0 \text{ for } 1 \leq i \leq r\}$. These characters are anti-dominant. However, the set of anti-dominant elements is much larger than the cone. Still, the value of Theorem 1.1 is in its generality: L is any finite extension of $\mathbb{Q}_p$ and G is a split connected reductive L-group.

To relate our result with the locally analytic case, we mention that the condition for locally analytic representations is given in terms of the derivative of χ, and hence concerns the behavior of χ_i in a small neighborhood of 1. Our condition, on the other hand, is a condition on $\chi_i(\varpi_L)$, where ϖ_L is a uniformizer of L.

The proof of the main theorem relies on the duality theory developed by Schneider and Teitelbaum in [ScT02]. Let o_L denote the ring of integers of L. Set $G_0 = \mathbf{G}(o_L)$, $P_0 = \mathbf{P}(o_L)$ and $T_0 = \mathbf{T}(o_L)$. Denote by χ_0 the restriction of χ to T_0. Let $K[[G_0]]$ be the completed group algebra defined in section 6.2. The dual of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ is $K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$. Then the isomorphism $\mathrm{Ind}_P^G(\chi^{-1}) \cong \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ induces a G-module structure on $M^{(\chi)} = K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$. For χ as in Theorem 1.1, we prove that $M^{(\chi)}$ has no G-invariant $K[[G_0]]$-submodules (Theorem 7.5).

We briefly describe the content of the paper. In section 2, we introduce notation. In section 3 we recall some results that we need from the theory of algebraic representations. These results are used in section 4 to construct a convenient, explicit, model for the space G/P. The main technical result of the paper concerns the action of T on this model, and is proved in section 5. In section 6 we recall some facts about principal series representations and their duals, and deduce information about the action of T on these vector spaces from the main technical result concerning its action on G/P. Finally, in section 7, we prove the main theorem.

Our group G will be a split connected reductive L-group. The group G is determined (up to an L-isomorphism) by its root datum ([Spr98], Theorem 16.3.2). Since we also need the corresponding group of o_L-points, we use the existence of the split reductive $\mathbb{Z}$-group with the same root datum ([SGA3], XXV.1.2).

2. Notation

Let L be a finite extension of $\mathbb{Q}_p$, o_L its ring of integers, and $\mathfrak{p}_L$ the unique maximal ideal of o_L. We denote the discrete valuation of L by ord_L. Let K be a finite extension of L and define $o_K, \mathfrak{p}_K$ and ord_K analogously.

We work with algebraic $\mathbb{Z}$-groups as defined in [Jan03]. We denote such an algebraic group by a boldface letter, such as $\mathbf{H}$. Then $H = \mathbf{H}(L)$ is the L-points, while $H_0 := \mathbf{H}(o_L)$ is the o_L-points. For each integer n, there is a canonical projection $H_0 \to \mathbf{H}(o_L/\mathfrak{p}_L^n)$. We denote the kernel by H_n.

Let $\mathbf{G}$ be a split and connected reductive algebraic $\mathbb{Z}$-group. We fix a split maximal torus $\mathbf{T}$ and a $\mathbf{T}$-stable maximal unipotent subgroup $\mathbf{U}$. We let $\mathbf{P} = \mathbf{TU}$ be the corresponding minimal parabolic subgroup. Also, let $\mathbf{U}^-$ denote the opposite minimal parabolic subgroup.

Let Φ denote the set of roots of $\mathbf{T}$ in $\mathbf{G}$. For each root $\alpha \in \Phi$ we write $\mathbf{U}_\alpha$ for the corresponding root subgroup. We let W denote the Weyl group of $\mathbf{G}$ which we realize as a quotient of the normalizer $N_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. We fix a set $\dot{W}$ of representatives for W in G_0, but do not assume that they form a subgroup. We write $X(\mathbf{T})$ for the lattice of rational characters of T. We define a partial order on it by declaring that $\lambda > \mu$ if $\lambda - \mu$ is a sum of positive roots. The group action of $N_{\mathbf{G}}(\mathbf{T})$ on $\mathbf{T}$ by conjugation induces actions of W on $\mathbf{T}$ and $X(\mathbf{T})$. An element of $X(\mathbf{T})$ is said to be dominant if it is maximal in its W-orbit, with respect to the above partial ordering. An equivalent condition is that λ is dominant if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for each simple root α. This extends the definition of "dominant" to the real vector space $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Lemma 2.1. *The lattice $X(\mathbf{T})$ has a $\mathbb{Z}$-basis which consists of dominant elements.*

Proof. Fix a $\mathbb{Z}$-basis $\underline{\omega} = \{\omega_1, \dots, \omega_r\}$ of $X(\mathbf{T})$. If $\underline{\lambda} = \{\lambda_1, \dots, \lambda_r\}$ is a set of linearly independent elements in $X(\mathbf{T})$, let $\mathcal{P}(\underline{\lambda})$ denote the fundamental parallelepiped $\{t_1\lambda_1 + \cdots + t_r\lambda_r : t_1, \dots, t_r \in [0,1)\}$. Let $M(\underline{\lambda})$ be the change of basis matrix from $\underline{\omega}$ to $\underline{\lambda}$ and define $d(\underline{\lambda}) = |\det M(\underline{\lambda})|$ (the volume of $\mathcal{P}(\underline{\lambda})$). Then $d(\underline{\lambda}) \in \mathbb{N}$ and $\underline{\lambda}$ is a $\mathbb{Z}$-basis of $X(\mathbf{T})$ if and only if $d(\underline{\lambda}) = 1$.

Write $\mathcal{D}$ for the set of dominant elements in $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then:

(1) $\mathcal{D}$ has nonempty interior,
(2) $a\mathcal{D} = \mathcal{D}$ for any positive real number a, and
(3) $\mathcal{D}$ is convex.

Any subset of $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ with the first two properties has the additional property that $\mathcal{D} \cap X(\mathbf{T})$ contains a basis for $X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$. Fix some such basis $\underline{\lambda} = \{\lambda_1, \dots, \lambda_r\}$. If $\mathcal{P}(\underline{\lambda}) \cap X(\mathbf{T}) = \{0\}$, then $\underline{\lambda}$ is a $\mathbb{Z}$-basis for $X(\mathbf{T})$. Otherwise, take a nontrivial element $\lambda' = t_1\lambda_1 + \cdots + t_r\lambda_r$ of $\mathcal{P}(\underline{\lambda}) \cap X(\mathbf{T})$. Select a nonzero coefficient t_i. Let $\underline{\lambda}'$ be the basis obtained from $\underline{\lambda}$ by replacing λ_i by λ'. As $\mathcal{D}$ is convex, the new basis is still contained in $\mathcal{D} \cap X(\mathbf{T})$. Moreover, $d(\underline{\lambda}') = |t_i| d(\underline{\lambda}) < d(\underline{\lambda})$. After a finite number of steps, we obtain a basis $\underline{\lambda}$ such that $d(\underline{\lambda}) = 1$. □

3. Some results from the theory of algebraic representations

As an algebraic $\mathbb{Z}$-group, $\mathbf{G}$ is a $\mathbb{Z}$-group functor. If k is a $\mathbb{Z}$-algebra, we denote by $\mathbf{G}_k$ the associated k-functor ([Jan03], I.1.10). Let M be a k-module. A representation of $\mathbf{G}_k$ on M is an action of $\mathbf{G}_k$ on the k-functor M_a such that for any k-algebra A, the group $\mathbf{G}_k(A)$ acts on $M_a(A) = M \otimes A$ through A-linear maps.

Theorem 3.1. *Let $\lambda \in X(\mathbf{T})$ be dominant. Then:*

(1) *There is an irreducible finite-dimensional algebraic representation $V_{\mathbb{Q}}$ of $\mathbf{G}_{\mathbb{Q}}$ such that $V_{\mathbb{Q}}^{\mathbf{U}}$ is a one-dimensional space on which $\mathbf{T}$ acts by λ. It is unique up to isomorphism.*
(2) *If α is a positive root, then $\mathbf{U}_{-\alpha}$ acts trivially on $V_{\mathbb{Q}}^{\mathbf{U}}$ if and only if $\langle \lambda, \alpha^\vee \rangle = 0$.*

(3) *Let $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} L$. Then V has a basis $\mathcal{B}$ consisting of weight vectors such that:*
   (a) *The o_L-span of $\mathcal{B}$ is preserved by the action of G_0.*
   (b) *If v is a weight vector on which T acts by λ and $u \in \mathbf{U}_\alpha(o_L)$, then $u.v - v$ is in the span of*
   $$\{b \in \mathcal{B} : T \text{ acts on } b \text{ by } \lambda + n\alpha, \text{ some positive integer } n\}.$$
   (c) *If v is in the o_L-span of $\mathcal{B}$ and $u \in \mathbf{U}_\alpha(\mathfrak{p}_L)$, then $u.v - v$ is in the $\mathfrak{p}_L$-span of $\mathcal{B}$.*

Proof. (1) follows from [Jan03], Proposition II.2.4.

(2) follows from the representation theory of $\mathfrak{sl}_2$. Indeed, if $v \in V^{\mathbf{U}}$, then v generates an irreducible representation of the copy of $\mathfrak{sl}_2$ generated by the root subgroups attached to $\pm\alpha$. It is also a highest weight vector in this $\mathfrak{sl}_2$-module, and the highest weight is $\langle \lambda, \alpha^\vee \rangle$. The module is one dimensional if and only if its highest weight is 0. Thus, v is annihilated by the Lie algebra of $\mathbf{U}_{-\alpha}$ if and only if $\langle \lambda, \alpha^\vee \rangle = 0$. And this is equivalent to being fixed by $\mathbf{U}_{-\alpha}$ itself.

(3) Let $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$. Let $\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_1, \dots, H_r\}$ be the basis of $\mathfrak{g}$ defined in [Jan03], Part II, 1.11. Denote by $\mathcal{U}_{\mathbb{Q}}$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{Q}}$. Let $\mathcal{U}_{\mathbb{Z}}$ be the $\mathbb{Z}$-algebra of $\mathcal{U}_{\mathbb{Q}}$ generated by all $X_\alpha^n/(n!)$ with $\alpha \in \Phi$ and $n \in \mathbb{N}$, and by all $\begin{pmatrix} H \\ m \end{pmatrix}$ with $H \in \mathrm{Lie}(\mathbf{T})$ and $m \in \mathbb{N}$. Let v^+ be a highest weight vector in V. Define
$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} v^+, \quad V_0 = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} o_L.$$
Then we know from [Jan03], Part II, 8.3, that $V_{\mathbb{Z}}$ is a $\mathbf{G}$-stable lattice in $V_{\mathbb{Q}}$ and $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} L = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} L = V$ (since L has characteristic zero). In addition, V_0 is a G_0-invariant o_L-lattice in V.

Let $X_{\alpha,n} = X_\alpha^n/(n!) \otimes 1 \in \mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} o_L$. The T_0-module V_0 decomposes into a direct sum of weight spaces
$$V_0 = \bigoplus_{\mu \in X(\mathbf{T})} (V_0)_\mu.$$
We have
$$V_\mu = (V_0)_\mu \otimes_{o_L} L.$$
For each weight space $(V_0)_\mu$, select an o_L-basis $\mathcal{B}_\mu$. Let $\mathcal{B} = \bigcup_\mu \mathcal{B}_\mu$. Then (a) is satisfied, and (b) and (c) follow from the expression
$$x_\alpha(a)(v \otimes 1) = \sum_{n \geq 0} (X_{\alpha,n} v) \otimes a^n \tag{3.1}$$
(see [Jan03], Part II, 1.19, eq. (6)). Here, $x_\alpha : G_a \to \mathbf{G}$ is a root homomorphism as in [Jan03], Part II, 1.2. □

4. A convenient model for G/P

Let $\lambda_1, \dots, \lambda_r$ be a basis for $X(\mathbf{T})$ consisting of dominant elements. For each i, let $V_i = V(\lambda_i)$ be the unique irreducible representation of G with highest weight λ_i (as in Theorem 3.1, part (3)), fix a basis $\mathcal{B}_i$ of weight vectors whose o_L-span is preserved by G_0, and let v_i be the highest weight vector in this basis. Then let $V := \bigoplus_{i=1}^r V_i$. It is equipped with the obvious action of G. Let $\mathcal{B} = \mathcal{B}_1 \cup \dots \cup \mathcal{B}_r$. Let
$$x_0 := (v_i)_{i=1}^r \in V.$$

Lemma 4.1. *For every $n \in \mathbb{N}$ there exists $k \geq n$ satisfying: if $u \in \mathbf{U}_\alpha(L)$, α is negative, and $u.x_0 - x_0$ is in the $\mathfrak{p}_L^k$-span of $\mathcal{B}$, then $u \in \mathbf{U}_\alpha(\mathfrak{p}_L^n)$.*

Proof. For each positive root α, we select $\lambda_\alpha \in \{\lambda_1, \dots, \lambda_r\}$ such that

$$m_\alpha = \langle \lambda_\alpha, \alpha^\vee \rangle > 0.$$

Define $m = \max\{m_\alpha \mid \alpha \in \Phi^+\}$. Let $V^\alpha = V(\lambda_\alpha)$. We denote the corresponding highest weight vector by v_α.

We have $s_\alpha(\lambda_\alpha) = \lambda_\alpha - m_\alpha\alpha$, so $s_\alpha(V^\alpha_{\lambda_\alpha}) = V^\alpha_{\lambda_\alpha - m_\alpha\alpha}$. It follows that

$$\dim V^\alpha_{\lambda_\alpha - m_\alpha\alpha} = \dim V^\alpha_{\lambda_\alpha} = 1.$$

Let $\mathcal{B}^\alpha_{\lambda_\alpha - m_\alpha\alpha} = \{v_{\lambda_\alpha - m_\alpha\alpha}\}$ be the selected o_L-basis of $(V_0^\alpha)_{\lambda_\alpha - m_\alpha\alpha}$. From Theorem 3.1 (2), $X_{-\alpha, m_\alpha} v_\alpha \neq 0$. Since $X_{-\alpha, m_\alpha} v_\alpha \in (V_0^\alpha)_{\lambda_\alpha - m_\alpha\alpha}$, we have

$$X_{-\alpha, m_\alpha} v_\alpha = a_\alpha v_{\lambda_\alpha - m_\alpha\alpha}, \quad a_\alpha \in o_L, a_\alpha \neq 0.$$

Define

$$c = \max\{ord_L(a_\alpha) \mid \alpha \in \Phi^+\}.$$

Given $n \in \mathbb{N}$, let $k = mn + c$. Assume $u \in U_{-\alpha}(L)$, $\alpha \in \Phi^+$, and $u.x_0 - x_0$ is in the $\mathfrak{p}_L^k$-span of $\mathcal{B}$. Write $u = x_{-\alpha}(a)$, $a \in L$. Since $u.v_\alpha - v_\alpha$ is in the $\mathfrak{p}_L^k$-span of $\mathcal{B}$, the same holds for its $(V_0^\alpha)_{\lambda_\alpha - m_\alpha\alpha}$-component

$$a^{m_\alpha} X_{-\alpha, m_\alpha} v_\alpha = a^{m_\alpha} a_\alpha v_{\lambda_\alpha - m_\alpha\alpha}.$$

It follows that $a^{m_\alpha} a_\alpha \in \mathfrak{p}_L^k$. Then

$$ord_L(a^{m_\alpha}) \geq k - ord_L(a_\alpha) \geq mn,$$

so $ord_L(a) \geq mn/m_\alpha \geq n$. This proves $u \in U_{-\alpha}(\mathfrak{p}_L^n)$. □

Definition 4.2. Given $n \in \mathbb{N}$, we define $k(n)$ to be the least integer $\geq n$ satisfying the condition of Lemma 4.1.

Lemma 4.3. *Let $u \in U_0^-$. If $u.x_0 - x_0$ is in the $\mathfrak{p}_L^{k(n)}$-span of $\mathcal{B}$, then $u \in U_n^-$.*

Proof. Take $u \in U_0^-$, number the negative roots $\gamma_1, \dots, \gamma_N$, and write $u = u_1 \dots u_N$, where $u_i \in (U_{\gamma_i})_0$ for $1 \leq i \leq N$. Let V_0 denote the o_L-span of $\mathcal{B}$, and let $\bar{x}_0$ denote the image of x_0 in $V_0/\mathfrak{p}_L^{k(n)} V_0$. Likewise, for $1 \leq j \leq r$ let $V_{j,0}$ denote the o_L-span of the basis $\mathcal{B}_j$ for V_j, and let $\bar{v}_j$ denote the image of v_j in $V_{j,0}/\mathfrak{p}_L^{k(n)} V_{j,0}$. Then G_0 fixes $V_{j,0}$ and $\mathfrak{p}_L^{k(n)} V_{j,0}$, and hence acts on the quotient, and it is given that u fixes the point $\bar{v}_j$ for $1 \leq j \leq r$. We claim that u_i fixes $\bar{v}_j$ for each $1 \leq i \leq N$ and $1 \leq j \leq r$. Suppose not. For $1 \leq i \leq N$ let $u_{\geq i} = u_i \dots u_N$, and for completeness let $u_{\geq N+1}$ denote the identity. For each weight μ of the representation V_j, write $(V_j)_\mu$ for the μ weight space and p_μ for the projection $V_{j,0} \to (V_{j,0})_\mu / \mathfrak{p}_L^{k(n)} (V_{j,0})_\mu$. For $1 \leq i \leq N$ let $S_i = \{\mu : p_\mu(u_{\geq i}.v_j) \neq 0\}$. By assumption $S' := \bigcup_{i=1}^N S_i \setminus \{\lambda_j\}$ is nonempty. Hence it has some element μ which is maximal in the sense that $\eta - \mu$ is not a sum of positive roots for any $\eta \in S'$. But it follows from the expression (3.1) that if $p_\mu(u_{\geq i}.v_j) \neq p_\mu(u_{\geq i+1}.v_j)$, then there exists η such that $p_\eta(u_{\geq i}.v_j) \neq 0$ and $\mu - \eta$ is an integral multiple of γ_i with positive factor. By maximality of μ, the weight η can only be the highest weight λ_j.

By hypothesis there exists an index i such that $p_\mu(u_{\geq i}.v_j) \neq 0$. On the other hand $p_\mu(u_{\geq 1}.v_j) = p_\mu(v_j) = 0$. This forces the weight $\mu - \lambda_j$ to be an integral multiple with positive factor of two distinct elements of $\{\gamma_1, \dots, \gamma_N\}$. But this is not possible in a reduced root system. □

We identify $X(\mathbf{P})$ with $X(\mathbf{T})$ via the projection $\mathbf{P} \to \mathbf{T}$. We use exponential notation for rational characters. Then $p \cdot x_0 = (p^{\lambda_i} \cdot v_i)_{i=1}^r$ for $p \in P$. Let

$$X = Gx_0 \subset V \quad \text{and} \quad X_0 = G_0 x_0.$$

We have the obvious action of G on X. We also have an obvious action of $(L^\times)^r$ on V by scaling in each factor. As $(L^\times)^r$ is abelian, this can be viewed as a left or right action and it is convenient to view it as a right action. Explicitly,

$$(x_1, \dots, x_r) \cdot (a_1, \dots, a_r) = (a_1 x_1, \dots, a_r x_r), \qquad x_i \in V_i, a_i \in L^\times, i = 1, \dots, r.$$

We have a homomorphism $P \to (L^\times)^r$ given by $p \mapsto (p^{\lambda_i})_{i=1}^r$. Hence we may pull our right action of $(L^\times)^r$ back to a right action of P.

Write $[V_i]$ for $(V_i \setminus \{0\})/GL_1$ and $[V]$ for $\prod_{i=1}^r [V_i]$. For $x \in V_i$ (resp. V) write $[x]$ for the image in $[V_i]$ (resp. $[V]$). If $\lambda \in X(\mathbf{T})$ is dominant, let $\mathbf{P}_\lambda$ be the standard parabolic subgroup such that a simple root α of $\mathbf{G}$ is a simple root of the Levi factor if $\langle \lambda, \alpha^\vee \rangle = 0$ and a root of the unipotent radical if $\langle \lambda, \alpha^\vee \rangle > 0$.

Proposition 4.4. *We have:*

1. *The stabilizer of* $[v_i] \in [V_i]$ *is* P_{λ_i}.
2. *The stabilizer of* $[x_0] \in [V]$ *is* P.
3. *The stabilizer of* $v_i \in V_i$ *is the kernel of a rational character* $\widetilde{\lambda}_i$ *whose restriction to* P_{λ_i} *is* λ_i.
4. *The stabilizer of* $x_0 \in V$ *is* U.

Proof. It is clear from the definitions that P stabilizes $[v_i]$ for all i and hence also $[x_0]$. It then follows from Corollary 21.3.B from [Hum75] that the stabilizer of $[v_i]$ is a standard parabolic subgroup for each i. It then follows from Theorem 3.1, part (2), that the stabilizer of $[v_i]$ is P_{λ_i} for each i. Since P_{λ_i} stabilizes $[v_i]$, it must act on v_i by a rational character. We denote this rational character by $\widetilde{\lambda}_i$ and it is evident that the restriction to $\mathbf{T}$ is λ_i.

The stabilizer of $[x_0]$ is $\bigcap_{i=1}^r P_{\lambda_i}$, which contains P. On the other hand, since $\lambda_1, \dots, \lambda_r$ span $X(\mathbf{T})$, it follows that for any $0 \neq \varphi \in X^\vee(\mathbf{T})$, there exists i with $\langle \lambda_i, \varphi \rangle \neq 0$. Applying this to the coroots, we deduce that $\bigcap_{i=1}^r P_{\lambda_i}$ does not contain U_α for any negative root α. Thus $\bigcap_{i=1}^r P_{\lambda_i} = P$. Similarly, if $t \in T$ stabilizes x_0, then it is in the kernel of λ_i for all i, and hence is trivial. $\square$

4.1. **Coset representatives for G/P.** We know that $G_0 P = G$. Write l for $o_L/\mathfrak{p}_L$ and $\bar{H}$ for $\mathbf{H}(l)$ (identified with the image of H_0 in $\mathbf{G}(l)$) for any algebraic subgroup of $\mathbf{G}$. Write B for the preimage of $\bar{P}$ in G_0. So $B = G_1 P_0$. Also, for each $w \in W$, let $\mathbf{U}_w = \mathbf{U} \cap w\mathbf{U}^- w^{-1}$. We have

$$\bar{G} = \coprod_w \bar{U}_w w \bar{P}.$$

Pulling back, we have

$$G_0 = \coprod_w (U_0 \cap wU^- w^{-1}) w G_1 P_0 = \coprod_w U_{w,0} w U_1^- P_0 = \coprod_w w U^-_{w,\frac{1}{2}} P_0, \tag{4.1}$$

where $U^-_{w,\frac{1}{2}} = w^{-1} U_{w,0} w U_1^-$. Note that $U^-_{w,\frac{1}{2}}$ is a subgroup of $U^-_{w,0}$. Indeed $w^{-1} U_w w = U^- \cap w^{-1} U w$ is a subgroup of U^-, and $U^-_{w,\frac{1}{2}}$ is the preimage of $w^{-1} \bar{U}_w w$ in U_0^-. It may be understood explicitly as follows. Fix an order on the negative roots. Then multiplication gives a bijection $\prod_{\alpha<0} U_\alpha \to U^-$. (This is not, in general, a group homomorphism.) Then $U^-_{w,\frac{1}{2}}$ corresponds to $\prod_{\alpha<0,w\alpha>0} U_{\alpha,0} \times \prod_{\alpha<0,w\alpha<0} U_{\alpha,1}$.

This can also be written as

$$\coprod_w (U_1^- \cap wU^-w^{-1})(U_0 \cap wU^-w^{-1})wP_0.$$

Let $\dot{w}$ be our fixed representative of $w \in W$. Then we have that $\coprod_w \dot{w}U^-_{w,\frac{1}{2}}$ or $\coprod_w (U_1^- \cap wU^-w^{-1})(U_0 \cap wU^-w^{-1})\dot{w}$ maps injectively into X and onto $[X]$.

For later use, we record one key property of these coset representatives for G/P.

Lemma 4.5. *Fix $w_0 \in W$ and $u_0 \in U^-_{w_0,\frac{1}{2}}$. Let $n \geq 1$. Then*

$$u_0^{-1}\dot{w}_0^{-1} \cdot (\coprod_w \dot{w}U^-_{w,\frac{1}{2}}) \cap G_nP_0 = U_n^-,$$

that is, if $w \in W, u \in U^-_{w,\frac{1}{2}}$ and $u_0^{-1}\dot{w}_0^{-1}\dot{w}u \in G_nP_0$, then $w = w_0$ and $u_0^{-1}u \in U_n^-$.

Proof. Consider the projection from G_0 to $\bar{G}$ (the points of $\mathbf{G}$ over the finite field $l = o_L/\mathfrak{p}_L$). The sets $\dot{w}U^-_{w,\frac{1}{2}}$ for $w \in W$ all project to distinct Bruhat cells. Hence

$$w_0u_0G_1P_0 \cap \dot{w}U^-_{w,\frac{1}{2}} \neq \emptyset \implies w = w_0.$$

Assume $w = w_0$. Then $u_0^{-1}\dot{w}_0^{-1}\dot{w}U^-_{w,\frac{1}{2}} = u_0^{-1}U^-_{w,\frac{1}{2}} \subset U_0^-$. Hence, it is enough to prove $U_0^- \cap G_nP_0 \subset G_n$. To show this, we consider the projection to G_0/G_n. Since $U_0^- \cap P_0 = \{1\}$, the only element of G_0/G_n which is in the image of both P_0 and U_0^- is the identity. □

Let

$$X_0^1 = \coprod_w \dot{w}U^-_{w,\frac{1}{2}}x_0.$$

Since $\coprod_w \dot{w}U^-_{w,\frac{1}{2}}$ is a set of coset representatives for G/P, and since P acts on x_0 through the map to GL_1^r, it follows that

$$\forall x \in X \exists x_0^1 \in X_0^1, a \in GL_1^r \text{ such that } x = x_0^1 \cdot a.$$

Since the stabilizer of x_0 is U, one can say that x_0^1 and a are unique. Since the map $T \to GL_1^r$ is a bijection, we can also say

$$\forall x \in X \exists! x_0^1 \in X_0^1, t \in T \text{ such that } x = x_0^1 \cdot t.$$

The next lemma permits us to compute x_0^1 and t explicitly, given x. Moreover, the set X_0^1 is given as a disjoint union of components indexed by the elements of W, and the next lemma also provides a means of determining which component x_0^1 is in.

Lemma 4.6. *Let $x = (x_1, \ldots, x_r)$ be an element of X_0^1. For $1 \leq i \leq r$ let $\mathcal{B}_i = (b_{i,1}, \ldots, b_{i,\dim V_i})$ be a basis for V_i as in part (3) of Theorem 3.1. By hypothesis, then, t acts from the left on $b_{i,j}$ by some weight $\lambda_{i,j} \in X(\mathbf{T})$. Write*

$$x_i = \sum_{j=1}^{\dim V_i} c_{ij}b_{i,j}.$$

Then:

1. $c_{ij} \in o_L \forall i, j$.
2. *For each i, there exists j with $c_{i,j} \in o_L^\times$.*
3. *For each i, $\{\lambda : \exists j$ with $\lambda_{i,j} = \lambda$ and $c_{i,j} \in o_L^\times\}$ has a unique minimal element. It is $w \cdot \lambda_i$, where $x \in \dot{w} U^-_{w,\frac{1}{2}} \cdot x_0$.*

Proof. We know that $x = \dot{w}u.x_0$ for some unique w in the Weyl group and $u \in U^-_{w,\frac{1}{2}}$. Part (1) follows from the fact that both u and $\dot{w}$ are in G_0 and the fact that the o_L-span of $\mathcal{B}$ is G_0-stable.

To prove parts (2) and (3), we use the fact that

$$\dot{w} U^-_{w,\frac{1}{2}} = (U_1^- \cap wU^- w^{-1})(U_0 \cap wU^- w^{-1})\dot{w}$$

to write $x = u_1 u_0 \dot{w}.x_0$, where $u_0 \in U_0 \cap wU^- w^{-1}$ and $u_1 \in U_1^- \cap wU^- w^{-1}$.

Let $v_i' = \dot{w} v_i$. Then t acts on v_i' by the $w \cdot \lambda_i$. The space of highest weight vectors is one dimensional, so the space of weight vectors in V_i attached to the weight $w \cdot \lambda_i$ is one dimensional as well. Thus, there exists j_0 such that $v_i' = c b_{i,j_0}$. Further, c is a unit, because $(\dot{w^{-1}})\dot{w} \in T_0$.

Next, write $v_i'' = u_0 \cdot v_i' = \sum_{j=0}^{\dim V_i} d_{i,j} b_{i,j}$, where $d_{i,j} \in o_L$. Then it follows from Theorem 3.1, part (3)(b), that $d_{i,j} = 0$ unless $\lambda_{i,j} - w \cdot \lambda_i$ is zero or a sum of positive roots, and that d_{i,j_0} is equal to the constant $c \in o_L^\times$ from the previous paragraph.

Since $x_i = u_1 \cdot v_i''$, we know from Theorem 3.1, part (3)(c), that $x_i - v_i''$ is in the $\mathfrak{p}_L$-span of $\mathcal{B}$. It follows that $c_{i,j} \equiv d_{i,j} \pmod{\mathfrak{p}_L}$ for all i, j. In particular, c_{i,j_0} is a unit, and $c_{i,j} \equiv 0 \pmod{\mathfrak{p}_L}$ unless $\lambda_{i,j} - w \cdot \lambda_i$ is zero or a sum of positive roots. □

5. Key technical result

Lemma 5.1.

1. *Take $x \in X_0^1$ and $t \in T$. Assume that $ord_L(t^\alpha) \geq 1$ for each simple root α. Let $t' \in T$ and $x' \in X_0^1$ be the unique elements satisfying $t \cdot x = x' \cdot t'$. Then*
$$ord_L((t')^{\lambda_i}) \leq ord_L(t^{\lambda_i}) \qquad (1 \leq i \leq r)$$
and equality holds in all places if and only if $x \in U_t^- x_0$, where U_t^- is a neighborhood of the identity in U_0^- which depends on t.
2. *Fix a positive integer n. If $ord_L(t^\alpha) \geq k(n)$ for each simple root α, then the neighborhood U_t^- from the first part is contained in U_n^-.*

Proof. (1) Let $x = (x_1, \dots, x_r)$ and $x' = (x_1', \dots, x_r')$. For $1 \leq i \leq r$, let $\mathcal{B}_i = (b_{i,1}, \dots, b_{i,\dim V_i})$ be a basis for V_i as in part (3) of Theorem 3.1. Write $x_i = \sum_{j=1}^{\dim V_i} x_{i,j} b_{i,j}$ and $x_i' = \sum_{j=1}^{\dim V_i} x_{i,j}' b_{i,j}$, where $x_{i,j}, x_{i,j}' \in o_L$. Set $t \cdot x =: y =: (y_1, \dots, y_r)$ with

$$y_i = \sum_{j=1}^{\dim V_i} y_{i,j} b_{i,j} = \sum_{j=1}^{\dim V_i} t^{\lambda_{i,j}} x_{i,j} b_{i,j}.$$

Since the right action of t' on x_i' is multiplication by $(t')^{\lambda_i}$, we have $y_i = (t')^{\lambda_i} x_i'$. We know from Lemma 4.6 that one of the coefficients $x_{i,j}'$ is a unit. It follows that

$$ord_L((t')^{\lambda_i}) = \min_j ord_L(y_{i,j}) = \min_j (ord_L(t^{\lambda_{i,j}} x_{i,j})).$$

Now, if λ is a weight of T in V_i and $\lambda \neq \lambda_i$, then $\lambda_i - \lambda$ is a sum of positive roots. Let us assume that the numbering is such that $b_{i,1}$ is the highest weight vector. Then $ord_L(t^{\lambda_i - \lambda_{i,j}}) \geq 1$ for all $j > 1$ and all i.

Now, we know that for each i there exists j_0 such that $x_{ij_0} \in o_L^\times$. Hence

$$\min_{1 \leq j \leq \dim V_i} ord_L(y_{i,j}) \leq ord_L(t^{\lambda_{i,j_0}}) \leq ord_L(t^{\lambda_i}),$$

proving $ord_L((t')^{\lambda_i}) \leq ord_L(t^{\lambda_i})$ for all i.

Equality holds in all places if and only if $ord_L(t^{\lambda_{i,j}} x_{i,j}) \geq ord_L(t^{\lambda_i})$ for all j, that is, $ord_L(x_{i,j}) \geq ord_L(t^{\lambda_i - \lambda_{i,j}}) \geq 1$ for all $j > 1$. Assume this is the case. Then for each i, the set $\{\lambda : \exists j \text{ with } \lambda_{i,j} = \lambda \text{ and } x_{i,j} \in o_L^\times\}$ contains only element λ_i. Now we apply Lemma 4.6 to find the unique element w in the Weyl group such that x belongs to $\dot{w}U^-_{w,\frac{1}{2}}x_0$. Lemma 4.6 (3) implies that $w \cdot \lambda_i = \lambda_i$ for all i. Thus, $w = e$ (the identity in the Weyl group) and $x = ux_0$ for some $u \in U^-_{e,\frac{1}{2}} = U^-_1$. Thus, the set of points x which satisfy $ord_L(x_{i,j}) \geq ord_L(t^{\lambda_i - \lambda_{i,j}}) \geq 1$ for all $j > 1$ and all i corresponds to an open neighborhood of the identity in U^-_1. This is our neighborhood U^-_t.

(2) Assume $ord_L(t^\alpha) \geq k(n)$ for each simple root α. If $x = u.x_0$ for some $u \in U^-_t$, then $ord_L(x_{i,j}) \geq ord_L(t^{\lambda_i - \lambda_{i,j}}) \geq k(n)$ for all i and all $j > 1$. It follows that $u.x_0 - x_0$ is in the $\mathfrak{p}_L^{k(n)}$-span of $\mathcal{B}$, and Lemma 4.3 implies that $u \in U^-_n$. □

Corollary 5.2. *Fix a positive integer n. Take $g \in \coprod_w \dot{w}U^-_{w,\frac{1}{2}}$ and $t \in T$. Assume that $ord_L(t^\alpha) \geq 1$ for each simple root α. Let $t' \in T$ and $g' \in \coprod_w \dot{w}U^-_{w,\frac{1}{2}}$ be the unique elements satisfying $t \cdot g \in g' \cdot t'U$. Then*

$$ord_L((t')^{\lambda_i}) \leq ord_L(t^{\lambda_i}) \qquad (1 \leq i \leq r)$$

and equality holds in all places if and only if $g \in U^-_t$.

Proof. Apply the previous statement to $x = g.x_0$. Then g' is the unique element of $\coprod_w \dot{w}U^-_{w,\frac{1}{2}}$ such that $x' = g'.x_0$. □

Corollary 5.3. *Fix a positive integer n. Take $g \in G_0$ and $t \in T$. Assume that $ord_L(t^\alpha) \geq k(n)$ for each simple root α. If $tg \in G_0t'U$, then*

$$ord_L((t')^{\lambda_i}) \leq ord_L(t^{\lambda_i}) \qquad (1 \leq i \leq r)$$

and equality holds in all places if and only if $g \in U^-_t P_0$.

Proof. Write $g = g_1p_0$ with $p_0 \in P_0$ and $g_1 \in \coprod_w \dot{w}U^-_{w,\frac{1}{2}}$. Use the previous corollary to write $tg_1 = g't''u$, where $g' \in \coprod_w \dot{w}U^-_{w,\frac{1}{2}}$, $t'' \in T$, and $u \in U$, and get $tg = g't''up_0$. If $tg = g_0t'U$, then $g_0^{-1}g' \in t'Up_0^{-1}u^{-1}t''^{-1}$. It follows that $t't''^{-1} \in T_0$. □

6. Induced representations and their duals

With the required technical result in hand, we are ready to proceed to the main theorem, which applies these technical results to the problem of reducibility of principal series representations of p-adic groups over p-adic fields. Recall that we have fixed a finite extension K of L.

6.1. **Principal series representations.** Let $\chi : T \to K^\times$ be a continuous homomorphism, and let χ_0 denote the restriction of χ to T_0. Also, let

$$I = \mathrm{Ind}_P^G(\chi^{-1}) = \{f \in C(G,K) \mid f(gp) = \chi(p)f(g)\, \forall p \in P, g \in G\}$$

be the corresponding principal series representation. Restriction gives an isomorphism

$$I \cong \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1}).$$

If S is an open and closed subset of G_0, then we may identify $C(S,K)$ with $\{f \in C(G_0,K) : \mathrm{supp}(f) \subset S\}$. Then $C(G_0,K) = C(S,K) \oplus C(S^c,K)$, where S^c denotes the complement of S in G_0. If S is preserved by P_0 acting on the right, then this decomposition induces a corresponding decomposition of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$. In particular, the decomposition (4.1) of G_0 gives rise to a decomposition of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ as

$$\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1}) = \bigoplus_{w \in W} \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w, \tag{6.1}$$

where

$$\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w := \{f \in \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1}) : \mathrm{supp}(f) \subset wU^-_{w,\frac{1}{2}}P_0\}.$$

Recall that we fixed a set $\dot{W}$ of representatives for the elements of W. Let $\dot{w}$ be the representative for w. Notice that an element of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w$ is determined by its restriction to $\dot{w}U^-_{w,\frac{1}{2}}$, which is continuous. Moreover, given a continuous function $h : U^-_{w,\frac{1}{2}} \to K$, we may define

$$f_h(g) = \begin{cases} h(u)\chi_0(p), & g = \dot{w}up,\ u \in U^-_{w,\frac{1}{2}},\ p \in P_0, \\ 0, & g \notin wU^-_{w,\frac{1}{2}}P_0. \end{cases} \tag{6.2}$$

Thus, we have a vector space isomorphism $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w \cong C(U^-_{w,\frac{1}{2}}, K)$.

6.2. **Completed group algebras.** If $\bar{H}$ is a finite group, we have the usual group algebra $o_K[\bar{H}]$, and the augmentation homomorphism $\mathrm{aug} : o_K[\bar{H}] \to o_K$ given by

$$\mathrm{aug} \sum_{\bar{h} \in \bar{H}} a_{\bar{h}}\bar{h} := \sum_{\bar{h} \in \bar{H}} a_{\bar{h}}.$$

If H is a compact p-adic Lie group (such as G_i, where $i \geq 0$) we have the projective limit $o_K[[H]] := \mathrm{proj\,lim}_{H'}\, o_K[H/H']$ taken over compact open normal subgroups H' of H. The augmentation homomorphism extends canonically to $o_K[[H]]$. If $H = G_i$ for some i, the projective limit may be taken over the groups $G_j, j \geq i$.

The ring $o_K[H/H']$ is canonically identified with the space of o_K-linear maps $C(H/H', o_K) \cong C(H, o_K)^{H'} \to o_K$. This induces canonical isomorphisms of $o_K[[H]]$ with the space of o_K-linear maps $C(H, o_K) \to o_K$, and of $K[[H]] := K \otimes_{o_K} o_K[[H]]$ with the space of all distributions on H (i.e., continuous K-linear maps $C(H,K) \to K$). A distribution $\nu \in K[[H]]$ may be written as $\nu_0 \otimes 1$ with $\nu_0 \in o_K[[H]]$ if and only if it maps $C(H, o_K)$ into o_K.

Lemma 6.1.

$$1 + \varpi_K o_K[[G_0]] \subset o_K[[G_0]]^\times.$$

Proof. Let $Jac(o_K[[G_0]])$ denote the Jacobson radical of $o_K[[G_0]]$. We know from [Sch11], Proposition 19.7, that $\varpi_K \in Jac(o_K[[G_1]])$. Let N be a simple $o_K[[G_0]]$-module. Then $\varpi_K \cdot N = 0$ or N. Since N is a finitely generated $o_K[[G_1]]$-module, Nakayama's lemma ([Lam01], Chapter 2, Lemma 4.22) implies that $\varpi_K \cdot N \neq N$. It follows that $\varpi_K \cdot N = 0$. Then Lemmas 4.1 and 4.3 of [Lam01], Chapter 2, imply that $\varpi_K \in Jac(o_K[[G_0]])$ and $1 + \varpi_K \mu$ is a unit for any $\mu \in o_K[[G_0]]$. □

6.3. **The dual of a principal series representation.** The dual of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ is $K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$. The isomorphism $I \cong \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ induces a G-module structure on $K[[G_0]] \otimes_{K[[P_0]]} K^{(\chi_0)}$ which depends on χ. We denote this G-module $M^{(\chi)}$. Notice that $M^{(\chi)}$ is a quotient of $K[[G_0]]$. If $\mu \in K[[G_0]]$, we denote its image in $M^{(\chi)}$ by $[\mu]$. We denote the canonical pairing $M^{(\chi)} \times \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1}) \to K$ by $\langle\ ,\ \rangle$. We use the same notation for the canonical pairing $K[[G_0]] \times C(G_0, K) \to K$. This is justified, for if $f \in \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ and $\mu \in K[[G_0]]$, then $\langle \mu, f\rangle$ depends only on $[\mu]$ and is equal to $\langle [\mu], f\rangle$. The decomposition (6.1) of the induced representation gives rise to a corresponding decomposition of $M^{(\chi)}$:

$$M^{(\chi)} = \bigoplus_{w\in W} M_w^{(\chi)}, \tag{6.3}$$

where

$$M_w^{(\chi)} := \{[\mu] \in M^{(\chi)} : \langle [\mu], f\rangle = 0,\ f \in \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_{w'},\ w' \neq w\}.$$

Now, notice that the inclusion $U^-_{w,\frac{1}{2}} \hookrightarrow G_0$ induces an inclusion $o_K[[U^-_{w,\frac{1}{2}}]] \hookrightarrow o_K[[G_0]]$ and hence $K[[U^-_{w,\frac{1}{2}}]] \hookrightarrow K[[G_0]]$.

Lemma 6.2. *Fix $w \in W, h \in C(U^-_{w,\frac{1}{2}}, K)$ and $\eta \in K[[U^-_{w,\frac{1}{2}}]]$. Let $\dot{w}$ be the representative for w in $\dot{W}$. Define $f_h \in \mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})_w$ by (6.2). Then $\langle f_h, [\dot{w}\eta]\rangle = \langle h, \eta\rangle$.*

Proof. Indeed, for general $f \in C(G_0, K)$, the value of $\langle f, \dot{w}\eta\rangle$ is obtained by applying η to the function $u \mapsto f(\dot{w}u)$ $(u \in U^-_{w,\frac{1}{2}})$. And, if $f = f_h$ this is precisely h. □

Combining Lemma 6.2 with (6.3) yields the following.

Corollary 6.3. *For each w, the subspace $\dot{w}K[[U^-_{w,\frac{1}{2}}]] \subset K[[G_0]]$ maps isomorphically onto $M_w^{(\chi)}$. Consequently, $\bigoplus_{w\in W} \dot{w}K[[U^-_{w,\frac{1}{2}}]]$ maps isomorphically onto $M^{(\chi)}$.*

Now, recall that $\mu \in K[[G_0]]$ lies in $o_K[[G_0]]$ if and only if it maps elements of $C(G_0, o_K) \subset C(G_0, K)$ into o_K. It follows that, for any such μ, the image $[\mu] \in M^{(\chi)}$ maps o_K-valued elements of $\mathrm{Ind}_{P_0}^{G_0}(\chi_0^{-1})$ into o_K. Using Corollary 6.3 we show that this characterizes the image of $o_K[[G_0]]$ in $M^{(\chi)}$.

Lemma 6.4.

(1) *The image of $o_K[[G_0]]$ in $M^{(\chi)}$ is the set of elements which map o_K-valued elements of $\mathrm{Ind}_{P_0}^{G_0}\chi_0^{-1}$ into o_K.*
(2) *Likewise, for each integer r, the image of $\varpi_K^r \cdot o_K[[G_0]]$ is the set of elements which map o_K-valued elements of $\mathrm{Ind}_{P_0}^{G_0}\chi_0^{-1}$ into $\mathfrak{p}_K^r$.*

Proof. The second part of the lemma follows easily from the first.

For the first part, the issue is to show that any class $[\mu]$ in $M^{(\chi)}$ which maps o_K-valued elements to o_K has a representative in $o_K[[G_0]]$. By Corollary 6.3, we can take a representative of the form $\eta = \sum_{w\in W} \dot{w}\eta_w$ with $\eta_w \in K[[U^-_{w,\frac{1}{2}}]]$. Now, fix w and take $h \in C(U^-_{w,\frac{1}{2}}, K)$. Clearly, the function f_h defined by (6.2) is o_K valued whenever h is. It follows that η_w maps $C(U^-_{w,\frac{1}{2}}, o_K)$ into o_K and hence lies in $o_K[[U^-_{w,\frac{1}{2}}]]$. As this holds for all w and the representatives $\dot{w}$ were taken from G_0, it follows that $\eta \in o_K[[G_0]]$. □

7. Main theorem

We begin with an important result from [Sch11].

Proposition 7.1. *Assume $n \geq 1$.*

1. *$o_K[[G_n]]$ is a local ring with residue field $o_K/\mathfrak{p}_K$.*
2. *The maximal ideal in $o_K[[G_n]]$ is*
$$\{\mu \in o_K[[G_n]] : \operatorname{aug}(\mu) \in \mathfrak{p}_K\}.$$

Next we need to define an important invariant.

Definition 7.2. Let $\eta : L^\times \to K^\times$ be a character (continuous homomorphism). As η must map $o_L^\times$ into $o_K^\times$, it induces a map $L^\times/o_L^\times \to K^\times/o_K^\times$. That is, for $a \in L^\times$, $ord_K(\eta(a))$ depends only on $ord_L(a)$. Let $e(\eta)$ denote the integer such that $ord_K \circ \eta = e(\eta) \cdot ord_L$.

Proposition 7.3. *Fix a positive integer n. Take $t \in T$. Assume that $ord_L(t^\alpha) \geq k(n)$ for each simple root α. Take $\chi_1, \ldots, \chi_r : L^\times \to K^\times$ and assume that $e(\chi_i) < 0$ for $1 \leq i \leq r$. Define $\chi : T \to K^\times$ by $\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i})$, where $\lambda_1, \ldots, \lambda_r$ are dominant and form a basis for $X(\mathbf{T})$. Take $f \in \operatorname{Ind}_P^G \chi^{-1}$ which maps G_0 into o_K. Write L^ι for left-inverse translation. Then $L^\iota(t^{-1}).f$ maps $G_0 \setminus G_nP_0$ into $\chi(t)\mathfrak{p}_K$.*

Proof. Take $g \in G_0\setminus G_nP_0$. We have $[L^\iota(t^{-1}).f](g) = f(tg)$. Write $tg = g't'u$, where $g' \in G_0, t' \in T$, and $u \in U$. Then $f(tg) = f(g')\chi(t')$. As $f(g') \in o_K$ we just need to show that $\chi(t') \in \chi(t)\mathfrak{p}_K$. Equivalently, we need to show that $ord_K(\chi(t't^{-1})) > 0$. But
$$ord_K(\chi(t't^{-1})) = \sum_{i=1}^r e(\chi_i)\left(ord_L((t')^{\lambda_i}) - ord_L((t)^{\lambda_i})\right).$$
Let U_t^- be defined as in Lemma 5.1. Then, by that lemma, $U_t^- \subset U_n^-$ and so $g \notin U_t^- P_0$. Corollary 5.3 implies that each of the integers $ord_L((t')^{\lambda_i}) - ord_L((t)^{\lambda_i})$ is nonpositive and at least one is nonzero. Hence if $e(\chi_i)$ is strictly negative for each i, it follows that the sum will be strictly positive. □

Corollary 7.4. *Fix a positive integer n. Take $t \in T$ such that $ord_L(t^\alpha) \geq k(n)$ for each simple root α, and take $\chi_1, \ldots, \chi_r : L^\times \to K^\times$ such that $e(\chi_i) < 0$ for $1 \leq i \leq r$. Define $\chi : T \to K^\times$ by $\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i})$, where $\lambda_1, \ldots, \lambda_r$ are dominant and form a basis for $X(\mathbf{T})$. Let $\nu \in M^{(\chi)}$ lie in the image of $o_K[[G_0]] < K[[G_0]]$ and vanish on G_nP. Then $t \cdot \nu$ lies in the image of $\chi(t)\varpi_K \cdot o_K[[G_0]]$.*

Proof. Take $f \in \operatorname{Ind}_P^G \chi^{-1}$ which maps G_0 into o_K. Then
$$\langle t.\nu, f\rangle = \langle \nu, L^\iota(t^{-1}).f\rangle = \langle \nu, \pi_2(L^\iota(t^{-1}).f)\rangle,$$
where π_2 is projection onto the second factor in the decomposition $C(G, K) = C(G_nP, K) \oplus C(G \setminus G_nP, K)$. By the previous proposition, $\pi_2(L^\iota(t^{-1}).f)$ maps G_0 into $\chi(t)\mathfrak{p}_K$. It then follows that $t.\nu$ maps f into $\chi(t)\mathfrak{p}_K$. □

Theorem 7.5. *Define* $\chi : T \to K^\times$ *by* $\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i})$, *where* $\lambda_1, \dots, \lambda_r$ *are dominant and form a basis for* $X(\mathbf{T})$. *Assume that* $e(\chi_i) < 0$ *for* $1 \le i \le r$. *Then* $M^{(\chi)}$ *has no proper nontrivial* G*-invariant* $K[[G_0]]$*-submodules.*

Proof. Choose a nontrivial element of $M^{(\chi)}$, and construct a representative $\eta = \sum_{w\in W} \dot{w}\eta_w$ for it as in Corollary 6.3. Here $\eta_w \in K[[U^-_{w,\frac{1}{2}}]]$ for each $w \in W$. We want to show that the submodule generated by $[\eta]$ is all of $M^{(\chi)}$.

By scaling, we may assume that $\eta_w = (\eta_{w,\ell})_{\ell=0}^\infty \in o_K[[U^-_{w,\frac{1}{2}}]]$ for each w, and that there exist $n \ge 1, w_0 \in W$ and $\bar{u}_0 \in U^-_{w,\frac{1}{2}}/U^-_n$ such that the coefficient c_0 of $\bar{u}_0$ in $\eta_{w,n}$ is a unit. Choose $u_0 \in U^-_{w,\frac{1}{2}}$ which projects to $\bar{u}_0$, and let $\mu = u_0^{-1}\dot{w}_0^{-1}\eta$. If we write μ as an element of the projective limit $\mu = (\mu_\ell)_{\ell=0}^\infty$, then

$$\mu_n = c_0 + \sum_{\substack{\bar{g}\in G_0/G_n \\ \bar{g}\neq 1}} c_{\bar{g}}\bar{g}, \quad c_0 \in o_K^\times, c_{\bar{g}} \in o_K. \tag{7.1}$$

Now, observe that the partition of G_0 as $G_n \cup (G_0 \setminus G_n)$ gives rise to direct sum decompositions of $C(G_0, K), o_K[[G_0]]$ and $K[[G_0]]$. Moreover $\{\lambda \in K[[G_0]] : \mathrm{supp}(\lambda) \subset G_n\}$ is canonically identified with $K[[G_n]]$. Let

$$\mu = \mu' + \mu'', \qquad \mu' \in o_K[[G_n]],\ \mathrm{supp}(\mu'') \subset G_0 \setminus G_n,$$

and note that, by Lemma 4.5, the support of μ'' is actually disjoint from G_nP_0. The image of μ' under the augmentation map is precisely c_0, the coefficient of the identity coset of μ_n in equation (7.1). Since c_0 is a unit, we know from Proposition 7.1 that μ' is an invertible element of $o_K[[G_n]]$. Multiplying by its inverse, we obtain an element of the form $1 + \nu$ where the support of ν is disjoint from G_nP_0.

Note that the elements $[\eta]$, $[\mu]$ and $[1+\nu]$ generate the same submodule of $M^{(\chi)}$.

Now choose t so that $ord_L(t^\alpha) > k(n)$ for each simple root α. Then it follows directly from the definitions that $[t \cdot 1] = [\chi(t) \cdot 1]$. Further, by Corollary 7.4,

$$[t \cdot \nu] \in [\chi(t) \cdot \varpi_K \cdot o_K[[G_0]]].$$

If we act by $\chi(t)^{-1}t$, we see that the submodule of $M^{(\chi)}$ generated by $[\eta]$ contains

$$\chi(t)^{-1}t \cdot [1 + \nu] = [1 + \chi(t)^{-1}t \cdot \nu] \in [1 + \varpi_K \cdot o_K[[G_0]]].$$

But Lemma 6.1 implies that the elements of $1 + \varpi_K \cdot o_K[[G_0]]$ are units. Hence the submodule generated by $[\eta]$ is all of $M^{(\chi)}$. □

Theorem 7.6. *Define* $\chi : T \to K^\times$ *by* $\chi(t) = \prod_{i=1}^r \chi_i(t^{\lambda_i})$, *where* $\lambda_1, \dots, \lambda_r$ *are dominant and form a basis for* $X(\mathbf{T})$. *Assume that* $e(\chi_i) < 0$ *for* $1 \le i \le r$. *Then* $\mathrm{Ind}_P^G \chi^{-1}$ *is topologically irreducible (that is, it has no proper nontrivial closed invariant subspaces).*

Proof. The theorem follows from the duality between $\mathrm{Ind}_P^G \chi^{-1}$ and $M^{(\chi)}$. □

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Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901
E-mail address: dban@siu.edu

244 Mathematics Building, University at Buffalo, Buffalo, New York 14260-2900
E-mail address: jahundle@buffalo.edu