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CORRECTIONS TO: "ON THE EQUIVARIANT K-THEORY OF THE NILPOTENT CONE IN THE GENERAL LINEAR GROUP"

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ABSTRACT. In the paper [P. Achar, *On the equivariant K-theory of the nilpotent cone in the general linear group*, Represent. Theory **8** (2004), 180–211], the author gave a combinatorial algorithm for computing the Lusztig–Vogan bijection for $GL(n,\mathbb{C})$. However, that paper failed to mention one easy case that may sometimes arise, making the description of the algorithm incomplete. This note fills in that gap.

1. INTRODUCTION

The main result of [1, 2] is the existence of a natural bijection between the set Λ^+ of dominant weights for $GL(n,\mathbb{C})$ and the set Ω of pairs $(C,\mathcal{E})$, where C is a nilpotent orbit and $\mathcal{E}$ is an irreducible equivariant vector bundle on C (up to isomorphism). Combinatorial objects called *weight diagrams* play a key role, serving as an intermediary between Λ^+ and Ω. Specifically, there are maps

$$\kappa : D_n \to \Omega \qquad \text{and} \qquad \tau : D_n \to \Lambda^+$$

that both become bijections when restricted to the set $D_n^\circ \subset D_n$ of *distinguished weight diagrams*. The proofs of both assertions consist of giving algorithms for computing the inverse map.

The algorithm for the map $\Omega \to D_n^\circ$ involves choosing some (not necessarily distinguished) weight diagram $X \in \kappa^{-1}(C,\mathcal{E})$, and then applying a sequence of "moves" to make X distinguished. At each step, one chooses a move to make by examining various properties, denoted by $\mathbf{P_1}(r)$, $\mathbf{P_2}(r)$, $\mathbf{P_3}(r)$, and $\mathbf{P_4}(r)$.

Unfortunately, the proofs in [2] do not cover the case where $\mathbf{P_4}(1)$ is false for the initial weight diagram. Moreover, this omission is somewhat "hidden", because the statements of various propositions purport to cover this case. Happily, these mistakes are quite easy to correct. In this note, we indicate the appropriate corrections to statements in [2], and we prove one new assertion, Proposition 7.9 below, to cover the missing case.

These mistakes came to light in November 2015, when David Vogan informed me of unexpected behavior[1] in the software implementation [3] in a specific example. I initially thought that this would be a matter of tracking down a software bug. It turned out instead to be a reasoning bug. Indeed, the software faithfully implemented the algorithm of [2], including the failure to handle the case where $\mathbf{P_4}(1)$ is false. The software has now been updated to incorporate the corrections described below. Vogan's example is reproduced in Section 3.

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The author was partially supported by NSF Grant No. DMS-1500890.
[1]This is a euphemism for "The program crashed."

2. Corrections to [2]

2.1. On page 190, line 5, change

> Every Y is said to have properties $\mathbf{p_1}(1)$ and $\mathbf{p_3}(1)$, for convenience.

to

> Every Y is said to have properties $\mathbf{p_1}(1)$, $\mathbf{p_2}(1)$, and $\mathbf{p_3}(1)$, for convenience.

2.2. On page 190, lines 8–9, change

> We suppose that the properties $\mathbf{P_1}(0)$, $\mathbf{P_3}(0)$, and $\mathbf{P_4}(0)$ always hold, as do $\mathbf{P_1}(1)$ and $\mathbf{P_3}(1)$.

to

> We suppose that the properties $\mathbf{P_1}(0)$, $\mathbf{P_2}(0)$, $\mathbf{P_3}(0)$, and $\mathbf{P_4}(0)$ always hold, as do $\mathbf{P_1}(1)$, $\mathbf{P_2}(1)$, and $\mathbf{P_3}(1)$.

2.3. On page 197, in Table 2, the definitions of q_5 and q_6 define the parameter r incorrectly. They should both be amended to say

$$r = -q_4(X) + 1.$$

Also, the definition of $\tilde{q}_{5;ir}$ should be amended to consider the case $r = 1$ separately. The corrected definition is

$$\tilde{q}_{5;ir}(X) = \begin{cases} 0 & \text{if } r = 1, \\ \max\{EX_{ir} - (EX_{i,r-1} + 1), EX_{i,r-1} - EX_{ir}\} & \text{if } r \text{ is odd and } r \neq 1, \\ \max\{EX_{ir} - EX_{i,r-1}, (EX_{i,r-1} - 1) - EX_{ir}\} & \text{if } r \text{ is even.} \end{cases}$$

2.4. On page 200, the statement of Proposition 7.8 should include "$r > 1$" as a hypothesis. The corrected statement is:

Proposition 7.8. *Suppose $r > 1$, and suppose rows i and i' of X are such that* $\mathbf{B}$ *or* $\mathbf{C}$ *might be performed on them: in particular, they agree in their first $r - 1$ entries, and intervening rows have length less than $r - 1$. Suppose furthermore that $q_4(X) = -(r-1)$, and that row i has length at least r.*

(a) *Suppose X has no entry at position $i'r$. If $\tilde{q}_{5;ir} > 0$ and $EX_{ir} < EX_{i,r-1}$,* $\mathbf{B}X$ *is well-behaved of order ≥ 4.*

(b) *Suppose that X does have an entry at position $i'r$. If $X_{ir} < X_{i'r}$, then* $\mathbf{C}X$ *is well-behaved of order ≥ 4.*

2.5. On page 202, before the last sentence of §7, insert the following new proposition:

Proposition 7.9. *Suppose $q_4(X) = 0$, and let $r = 1$. Let i and $i' = i + 1$ be two consecutive rows of X such that $X_{ir} < X_{i'r}$. Then* $\mathbf{C}X$ *is well-behaved of order ≥ 4.*

Proof. As in the proof of Proposition 7.8, this move preserves q_1, q_2, and q_3. The assumption implies that X does not have $\mathbf{P_4}(1)$ and that $q_6(X) > 0$. It is easy to see from the definition that

$$\tilde{q}_{6;jr}(\mathbf{C}X) = \begin{cases} \tilde{q}_{6;jr}(X) & \text{if } j \neq i, i', \\ \tilde{q}_{6,jr} - (X_{i'r} - X_{ir}) & \text{if } j = i \text{ or } i'. \end{cases}$$

TABLE 3. Well-behavedness of moves under various hypotheses

Conditions	*Move*	*Well-Behavedness*
$q_4(X) \leq -r$.		Order ≥ 1.
$X_{i_m s}$ is lowerable, $X_{i_1 r}$ is raisable, and $EX_{i_k s} - EX_{i_k r} \geq 1$ for $k = 1, \dots, m$.	**A**	
$X_{i_1 s}$ is raisable, $X_{i_m r}$ is lowerable, and $EX_{i_k s} - EX_{i_k r} \leq -1$ for $k = 1, \dots, m$.	$\mathbf{A}^{-1}$	
$m = 1$; $q_4(X) = -(r-1)$. $X_{i_1 s}$ is lowerable. $\tilde{q}_{3;i_1 r}(X) \neq 0$. If $X_{jr} = X_{ir}$, then $j \geq i$ (for **A**) or $j \leq i$ (for $\mathbf{A}^{-1}$).		Order $= 1$.
$EX_{i_1 s} - EX_{i_1 r} \geq 2$.	**A**	
$EX_{i_1 s} - EX_{i_1 r} \leq -2$.	$\mathbf{A}^{-1}$	
r odd, s even, and $EX_{i_1 s} - EX_{i_1 r} = 1$.	**A**	
r even, s odd, and $EX_{i_1 s} - EX_{i_1 r} = -1$.	$\mathbf{A}^{-1}$	
$q_4(X) = -(r-1)$; $\tilde{q}_{5;i_1 r}(X) > 0$		Order ≥ 4.
$EX_{i_1, r-1} > EX_{i_1 r}$.	**B**	
$EX_{i_1, r-1} < EX_{i_1 r}$.	$\mathbf{B}^{-1}$	
$X_{ir} < X_{i'r}$.	**C**	
$q_4(X) = 0$; $i' = i+1$; $r = 1$		Order ≥ 4.
$X_{ir} < X_{i'r}$.	**C**	

Therefore, $q_6(\mathbf{C}X) < q_6(X)$. If $q_6(\mathbf{C}X) = 0$, then $\mathbf{C}X$ has $\mathbf{P_4}(1)$, so $q_4(\mathbf{C}X) \leq -1 < q_4(X)$, and hence $\mathbf{C}X$ is well-behaved of order 4. On the other hand, if $q_6(\mathbf{C}X) > 0$, then we still have $q_4(\mathbf{C}X) = 0$. In that case, we also have $q_5(\mathbf{C}X) = q_5(X) = 0$, so $\mathbf{C}X$ is well-behaved of order 6. □

2.6. On page 202, the last sentence of §7 should be amended to mention the new Proposition 7.9. It should say:

> The facts in Propositions 7.5, 7.7, 7.8, and 7.9 are collected and summarized in Table 3.

In addition, Table 3 should be amended to include the case covered by the new Proposition 7.9. The corrected table is shown above.

2.7. On page 203, the statements of Proposition 8.3 and Corollary 8.4 should include "$r > 1$" as a hypothesis. The corrected statements are:

Proposition 8.3. *Suppose $r > 1$. If EX has $\mathbf{p_3}(r)$ and $\mathbf{p_4}(r-1)$, then it also has $\mathbf{p_4}(r)$.*

Corollary 8.4. *Suppose $r > 1$, and that EX has $\mathbf{P_1}(r-1)$, $\mathbf{P_2}(r-1)$, $\mathbf{P_3}(r-1)$, and $\mathbf{P_4}(r-1)$. If it does not have $\mathbf{P_4}(r)$, then it also does not have $\mathbf{P_3}(r)$.* □

2.8. On page 204, the statement of Proposition 8.5 should be amended to say

Proposition 8.5. *Suppose EX has $\mathbf{P_1}(r-1)$, $\mathbf{P_2}(r-1)$, $\mathbf{P_3}(r-1)$, and $\mathbf{P_4}(r-1)$. If it does not have $\mathbf{P_3}(r)$ or $\mathbf{P_4}(r)$, then X satisfies some hypothesis in the left-hand column of Table* 3.

Its proof should have the following new paragraph inserted at the beginning:

Proof. Suppose first that $r = 1$. In this case, $\mathbf{P_3}(1)$ holds automatically, so it must fail to have $\mathbf{p_4}(1)$. This means that there is some entry X_{i1} that is smaller than its column-successor $X_{i'1}$, where necessarily $i' = i + 1$. In other words, we are in the setting of Proposition 7.9, and we can do move **C**.

For the remainder of the proof, assume that $r > 1$. Corollary 8.4 tells us that $\mathbf{P_3}(r)$ must fail, so we know that $q_5(X) > 0$. [The rest of the proof is as in [2].] □

3. Vogan's example

We will compute the distinguished weight diagram in $\kappa^{-1}(C, \mathcal{E})$ for the pair $(C, \mathcal{E}) = ([3^3, 2^2]; ((5, 0, -3), (4, -6)))$. In the calculation below, which follows the pattern of [2, §9.3], Steps 2 and 3 rely on the new Proposition 7.9. Note that for the initial weight diagram X, EX fails to satisfy $\mathbf{P_4}(1)$. Moreover, it only satisfies the hypothesis of the newly added row in Table 3. Thus, this example demonstrates the incompleteness of the description of the algorithm given in [2].

(1) Choose a weight diagram X such that $\kappa(X) = ([3^3, 2^2]; ((5, 0, -3), (4, -6)))$.

$X =$

5	0	0
0	0	0
−3	0	0
4	0	
−6	0	

$EX =$

9	4	2
0	2	0
−5	0	−2
6	−2	
−10	−4	

(2) Perform **C** with $r = 1$, $i = 3$, and $i' = 4$.

$X =$

5	0	0
0	0	0
4	0	
−3	0	0
−6	0	

$EX =$

9	4	2
0	2	0
6	0	
−5	−2	−2
−10	−4	

(3) Perform **C** with $r = 1$, $i = 2$, and $i' = 3$.

$X =$

5	0	0
4	0	
0	0	0
−3	0	0
−6	0	

$EX =$

9	4	2
6	2	
0	0	0
−5	−2	−2
−10	−4	

(4) Perform **A** four times with $s = 1$, $r = 2$, on rows 1 and 2.

$X =$

3	2	0
2	2	
0	0	0
−3	0	0
−6	0	

$EX =$

7	6	2
4	4	
0	0	0
−5	−2	−2
−10	−4	

(5) Perform $\mathbf{A}^{-1}$ five times with $s = 1$, $r = 2$, on rows 4 and 5.

$X =$

3	2	0
2	2	
0	0	0
−1	−2	0
−3	−3	

$EX =$

7	6	2
4	4	
0	0	0
−3	−4	−2
−7	−7	

(6) Perform **A** with $s = 1$, $r = 3$ on row 1.

$X =$

2	2	1
2	2	
0	0	0
−1	−2	0
−3	−3	

$EX =$

6	6	3
4	4	
0	0	0
−3	−4	−2
−7	−7	

(7) Perform $\mathbf{A}^{-1}$ with $s = 2$, $r = 3$ on row 4.

$$X = \begin{array}{ccc} 2 & 2 & 1 \\ 2 & 2 & \\ 0 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & -3 & \end{array} \qquad EX = \begin{array}{ccc} 6 & 6 & 3 \\ 4 & 4 & \\ 0 & 0 & 0 \\ -3 & -3 & -3 \\ -7 & -7 & \end{array}$$

(8) Perform $\mathbf{B}$ on rows 1 and 2.

$$X = \begin{array}{ccc} 2 & 2 & \\ 2 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & -3 & \end{array} \qquad EX = \begin{array}{ccc} 6 & 6 & \\ 4 & 4 & 3 \\ 0 & 0 & 0 \\ -3 & -3 & -3 \\ -7 & -7 & \end{array}$$

(9) Perform $\mathbf{A}$ with $s = 2$, $r = 3$ on row 2.

$$X = \begin{array}{ccc} 2 & 2 & \\ 2 & 1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \\ -3 & -3 & \end{array} \qquad EX = \begin{array}{ccc} 6 & 6 & \\ 4 & 3 & 4 \\ 0 & 0 & 0 \\ -3 & -3 & -3 \\ -7 & -7 & \end{array}$$

From the last step, we obtain

$$\gamma([3^3, 2^2]; ((5, 0, -3), (4, -6))) = \tau(X) = (6, 6, 4, 4, 3, 0, 0, 0, -3, -3, -7, -7).$$

References

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