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COCENTERS AND REPRESENTATIONS OF PRO-p HECKE ALGEBRAS

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Abstract. In this paper, we study the relation between the cocenter $\bar{\tilde{\mathcal{H}}}$ and the representations of an affine pro-p Hecke algebra $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(0, -)$. As a consequence, we obtain a new criterion on supersingular representations: a (virtual) representation of $\tilde{\mathcal{H}}$ is supersingular if and only if its character vanishes on the non-supersingular part of the cocenter $\bar{\tilde{\mathcal{H}}}$.

Introduction

0.1. Let G be a connected reductive p-adic group and $\tilde{W}$ be its Iwahori-Weyl group. The Iwahori-Hecke algebra $\tilde{\mathcal{H}}_q$ is a deformation of the group algebra of $\tilde{W}$. It plays an important role in the study of the ordinary representations of G.

For representations of G in characteristic p (the defining characteristic), one expects that there is a close relation between the mod-p representations of G and of the pro-p Hecke algebra $\tilde{\mathcal{H}}$ of G. The pro-p Hecke algebra is a deformation of the group algebra $\tilde{W}(1)$, with parameter $q = 0$. Here $\tilde{W}(1)$ is the pro-p Iwahori-Weyl group, an extension of $\tilde{W}$ by a finite torus.

0.2. For a pro-p Hecke algebra of a connected reductive p-adic group (i.e. the associated group $\tilde{W}(1)$ is the pro-p Iwahori-Weyl group of a connected reductive p-adic group), the representations are studied by Abe [1] and Vignéras [22], based on the Bernstein presentation and Satake-type isomorphism.

In this paper, we study the representations via a different approach, the "cocenter program".

Let us first provide some background on the "cocenter program".

For a group algebra of a finite group, the cocenter is very simple. The elements in the same conjugacy class of the group have the same image in the cocenter of the group algebra and the cocenter has a standard basis given by the conjugacy classes. There is a perfect pairing between the cocenter of the group algebra and the Grothendieck group of finite dimensional (complex) representations, via the trace map. This is a "toy model" for the "cocenter program".

For finite or affine Hecke algebras (with non-zero parameters), the cocenter is more complicated. The elements in the same conjugacy class may not have the same image in the cocenter. However, based on some remarkable properties on the

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minimal length elements in the Weyl group, one may show that the elements of minimal length in a conjugacy class of the Weyl group still have the same image in the cocenter of the Hecke algebra and the cocenter is still indexed by the conjugacy classes of the group.

For finite Hecke algebras (with generic parameters), the relation between the cocenter and the representations is fairly simple. The dimension of the cocenter equals the number of irreducible representations, and the trace map gives a perfect pairing between the cocenter and representations. For affine Hecke algebras, both the dimension of the cocenter and the number of irreducible representations are infinite and the counting-number method does not simply apply. In [4], we introduced the rigid cocenter and rigid quotient of the Grothendieck group of representations, and showed that they form a perfect pairing under the trace map, and the whole cocenter and all the finite dimensional representations can be understood via the rigid part of the parabolic subalgebras.

0.3. For finite and affine pro-p Hecke algebra (with parameter $q = 0$), the trace map from the cocenter to the "good" linear functions on the Grothendieck group of finite dimensional representations, is surjective, but not injective. However, the knowledge of the structure of the cocenter is a big help in the understanding of representations.

We first show in Theorem 2.4 that

Theorem 0.1. *For finite and affine pro-p Hecke algebra $\tilde{\mathcal{H}}$, the cocenter is spanned by the image of T_w, where w runs over the elements of $\tilde{W}(1)$ which are of minimal lengths in their conjugacy classes.*

For finite pro-p Hecke algebras, the irreducible representations are just characters. For affine pro-p Hecke algebras, it is easy to construct a family $\{\pi_{J,\Gamma,\Xi,V}\}$ of representations by taking the parabolic induction from characters of the parabolic algebras of $\tilde{\mathcal{H}}$. See §6.2 for the precise definition.

One of the main results in this paper is the following (see Theorem 7.1).

Theorem 0.2. *Let $\tilde{\mathcal{H}}$ be an affine pro-p Hecke algebra. The set $\{\pi_{J,\Gamma,\Xi,V}\}$ is a basis of the Grothendieck group $R(\tilde{\mathcal{H}})$.*

For pro-p Hecke algebras of connected reductive p-adic groups, the above result is obtained by Abe [1] and Vignéras [22]. Our strategy is different from Abe and Vignéras. We use the structure of the cocenter (see Theorem 2.4) and the character formula established in §6.

0.4. Among all the representations of the affine pro-p Hecke algebra $\tilde{\mathcal{H}}$, the supersingular representations are the most important ones. Abe [1] showed that for pro-p Hecke algebras of connected reductive p-adic groups, any irreducible representation can be obtained from supersingular ones by the parabolic inductions. The classification of supersingular representations for affine pro-p Hecke algebras is obtained by Vignéras [22]. In Theorem 7.8, we give a new proof of the classification (in the more general setting) and we also give a new criterion of supersingular representations.

Theorem 0.3. *A virtual representation π of $\tilde{\mathcal{H}}$ is supersingular if and only if $Tr(\overline{\tilde{\mathcal{H}}}^{nss}, \pi) = 0$, where $\overline{\tilde{\mathcal{H}}}^{nss}$ is the non-supersingular part of the cocenter defined in §7.3.*

0.5. The paper is organized as follows.

In §1, we recall the definition of pro-p Hecke algebras and trace maps. In §2, we study the cocenters of finite and affine pro-p Hecke algebras. In §3, we discuss the cocenter and representations of finite pro-p Hecke algebras. In §4, we define the parabolic subalgebras of affine pro-p Hecke algebras. In §5, we discuss the standard representatives associated to minimal length elements and study their powers. Such results are used in §6, in which we study the character formula for affine pro-p Hecke algebras. Finally, in §7, we give a basis of the Grothendieck group of finite dimensional modules of affine pro-p Hecke algebras and give a new criterion of supersingular representations.

1. Preliminary

1.1. We start with a sextuple $(W, S, \Omega, \tilde{W}, Z, \tilde{W}(1))$, where (W, S) is a Coxeter system, Ω is a group acting on W and stabilizing S, $\tilde{W} = W \rtimes \Omega$, Z is a finite commutative group, and we have a short exact sequence

$$1 \to Z \to \tilde{W}(1) \xrightarrow{\pi} \tilde{W} \to 1.$$

The Bruhat order $\leqslant$ on W extends in a natural way to a partial order on $\tilde{W}$, which we still denote by $\leqslant$. Let ℓ be the length function on W. It extends to a length function on $\tilde{W}$ by requiring that $\ell(\tau) = 0$ for $\tau \in \Omega$, and inflates to a length function ℓ on $\tilde{W}(1)$.

Since Z is commutative, the conjugation action of $\tilde{W}(1)$ on Z induces an action of $\tilde{W}$ on Z. We denote both actions on $\bullet$.

For any subset D of $\tilde{W}$, we denote by $D(1)$ the inverse image of D in $\tilde{W}(1)$.

Let $w \in W$. The support of w is defined to be the set of simple reflections that appear in some (or equivalently, any) reduced expression of w and is denoted by $\mathrm{supp}(w)$. For $w \in W$ and $\tau \in \Omega$, we define $\mathrm{supp}(w\tau)$ to be $\bigcup_{i \in \mathbb{N}} \tau^i \mathrm{supp}(w) \tau^{-i}$. For $\tilde{w} \in \tilde{W}(1)$, we define $\mathrm{supp}(\tilde{w}) = \mathrm{supp}(\pi(\tilde{w}))$.

1.2. Now we recall the definition of generic pro-p Hecke algebra introduced by Vignéras in [21].

Let $T = \bigcup_{w \in W} wSw^{-1} \subseteq W$ be the set of reflections in W. Let $\mathbf{k}$ be an algebraically closed field. The actions of $\tilde{W}$ and $\tilde{W}(1)$ on Z extends linearly to actions on $\mathbf{k}[Z]$. We still denote these actions by $\bullet$. We choose $c_t \in \mathbf{k}[Z]$ and indeterminates q_t for $t \in T(1)$ such that

- $q_{wtw^{-1}} = q_t$ for $w \in \tilde{W}(1)$ and $q_{tz} = q_t$ for $z \in Z$.
- $c_{wtw^{-1}} = w \bullet c_t$ for $w \in \tilde{W}(1)$ and $c_{tz} = c_t z$ for $z \in Z$.

Note that $q_{\_}$ and $c_{\_}$ are determined by their values on $S(1)$. We may extend q to a multiplicative function $w \mapsto q(w)$ on $\tilde{W}(1)$ by requiring that $q(\omega) = 1$ if $\omega \in \Omega(1)$ and $q(s) = q_s$ if $s \in S(1)$.

We simply write $\mathbf{k}[\mathbf{q}]$ for $\mathbf{k}[\{q_t\}_{t \in T(1)}]$ and $\mathbf{k}[\mathbf{q}^{\pm 1}]$ for $\mathbf{k}[\{q_t^{\pm 1}\}_{t \in T(1)}]$. Let $\tilde{\mathcal{H}}(\mathbf{q}, c)$ be the associative $\mathbf{k}[\mathbf{q}]$-algebra with basis $(T_w)_{w \in \tilde{W}(1)}$ subject to the following relations:

$$T_w T_{w'} = T_{ww'}, \quad \text{for } w, w' \in \tilde{W}(1) \text{ with } \ell(ww') = \ell(w) + \ell(w');$$
$$T_s^2 = q_s T_{s^2} + c_s T_s, \quad \text{for } s \in S(1).$$

Note that $\mathbf{k}[Z]$ injects into $\tilde{\mathcal{H}}(\mathbf{q}, c)$ via $z \mapsto T_z$. We denote by $\mathcal{H}(\mathbf{q}, c)$ the subalgebra of $\tilde{\mathcal{H}}(\mathbf{q}, c)$ spanned by T_w for $w \in W(1)$.

We denote by $\tilde{\mathcal{H}}(0, c)$ the specialization of $\tilde{\mathcal{H}}(\mathbf{q}, c)$ at $q_s \equiv 0$. In this case, the second relation becomes $T_s^2 = c_s T_s$ for $s \in S(1)$. We may simply write $\tilde{\mathcal{H}}$ for $\tilde{\mathcal{H}}(0, c)$. This plays an important role in the study of mod-p representations of reductive groups over finite fields of characteristic p and over p-adic fields.

In the case where W is a finite Coxeter group, we call $\tilde{\mathcal{H}}$ a *finite pro-p Hecke algebra*. In the case where W is an affine Weyl group, we call $\tilde{\mathcal{H}}$ an *affine pro-p Hecke algebra*.

Note that for any $w \in \tilde{W}(1)$, T_w^{-1} is defined in $\tilde{\mathcal{H}}(\mathbf{q}, c) \otimes_{\mathbf{k}[\mathbf{q}]} \mathbf{k}[\mathbf{q}^{\pm 1}]$, but is not contained in $\tilde{\mathcal{H}}(\mathbf{q}, c)$ in general. However, it is proved in [20, Corollary 2] that $\mathbf{q}(w) T_w^{-1} \in \tilde{\mathcal{H}}(\mathbf{q}, c)$ for any $w \in \tilde{W}(1)$ and the map

$$T_w \mapsto {}^{\iota}T_w := (-1)^{\ell(w)} q(w) T_{w^{-1}}^{-1}$$

gives an involution ι of $\tilde{\mathcal{H}}(\mathbf{q}, c)$. We still denoted by ι the induced involution of $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(0, c)$.

1.3. Let $[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$ be the commutator of $\tilde{\mathcal{H}}$, the subspace of $\tilde{\mathcal{H}}$ spanned by $[T_w, T_{w'}] := T_w T_{w'} - T_{w'} T_w$ for $w, w' \in \tilde{W}(1)$. Let $\overline{\tilde{\mathcal{H}}} = \tilde{\mathcal{H}}/[\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$ be the cocenter of $\tilde{\mathcal{H}}$. Denote by $R(\tilde{\mathcal{H}})_{\mathbf{k}}$ the ($\mathbf{k}$-span of the) Grothendieck group of finite dimensional representations of $\tilde{\mathcal{H}}$ over $\mathbf{k}$, i.e., the $\mathbf{k}$-vector space with basis given by the isomorphism classes of irreducible representations of $\tilde{\mathcal{H}}$. Note that in this paper, we deal with left modules of $\tilde{\mathcal{H}}$, while some references (e.g. [22]) deal with right modules.

We consider the trace map

$$Tr : \overline{\tilde{\mathcal{H}}} \to R(\tilde{\mathcal{H}})_{\mathbf{k}}^*, \qquad h \mapsto (V \mapsto Tr(h, V)).$$

Similar maps for affine Hecke algebras with generic non-zero parameters are studied in the joint work of Ciubotaru and the first-named author [4]. It is proved in [4] that the trace map is injective and there is a "perfect pairing" between the rigid-cocenter and rigid-representations of $\tilde{\mathcal{H}}(\mathbf{q}, c)$.

2. Cocenter of $\tilde{\mathcal{H}}$

2.1. Let $w, w' \in \tilde{W}(1)$ and $s \in S(1)$. We write $w \xrightarrow{s} w'$ if $\pi(w) \neq \pi(w')$, $w' = sws^{-1}$ and $\ell(w') \leqslant \ell(w)$. We write $w \to w'$ if $w = w'$ or there exists a sequence $w = w_1, w_2, \cdots, w_n = w'$ of elements in $\tilde{W}(1)$ such that for any k, $w_{k-1} \xrightarrow{s_k} w_k$ for some $s_k \in S(1)$. We write $w \approx w'$ if $w \to \tau w' \tau^{-1}$ and $\tau w' \tau^{-1} \to w$ for some $\tau \in \Omega(1)$. In this case, we say that w and w' are in the same cyclic-shift class. It is easy to see that if $w \to w'$ and $\ell(w) = \ell(w')$, then $w \approx w'$.

Lemma 2.1. *Let $w \in \tilde{W}(1)$ and $s \in S(1)$.*
(1) If $\ell(sws^{-1}) = \ell(w)$ and $\pi(sws^{-1}) \neq \pi(w)$, then $T_w \equiv T_{sws^{-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$.
(2) If $\ell(sws^{-1}) < \ell(w)$, then $T_w \equiv c_{s^{-1}} T_{ws} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$.

Proof. (1) By [5, Lemma 1.6.4], we have $\ell(sw) = \ell(w) - 1$ or $\ell(ws^{-1}) = \ell(w) - 1$.
If $\ell(sw) = \ell(w) - 1$, then

$$T_w = T_{s^{-1}} T_{sw} \equiv T_{sw} T_{s^{-1}} = T_{sws^{-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

Here the last equality follows from the fact that $\ell(sw) = \ell(sws^{-1}) - 1$.

If $\ell(ws^{-1}) = \ell(w) - 1$, then

$$T_w = T_{ws^{-1}} T_s \equiv T_s T_{ws^{-1}} = T_{sws^{-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

Here the last equality follows from the fact that $\ell(ws) = \ell(sws^{-1}) - 1$.

(2) We have $\ell(sws) = \ell(w) - 2$. So

$$T_w = T_{s^{-1}} T_{sws} T_{s^{-1}} \equiv T_{s^{-1}}^2 T_{sws} = c_{s^{-1}} T_{s^{-1}} T_{sws} = c_{s^{-1}} T_{ws} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

Here the last equality follows from the fact that $\ell(sws) = \ell(sw) - 1$. □

The following consequence follows easily from Lemma 2.1 (1).

Corollary 2.2. *Let $w, w' \in \tilde{W}(1)$ with $w \approx w'$. Then*

$$T_w \equiv T_{w'} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

Proof. By definition, there exists a sequence $w = w_0, w_1, \cdots, w_n = w'$ such that for any $1 \leqslant k \leqslant n$, $\ell(w_k) = \ell(w_{k-1})$, $w_k = s_k w_{k-1} s_k^{-1}$ and $\pi(w_k) \neq \pi(w_{k-1})$ for some $s_k \in S(1)$ and $w' = \tau w_{n-1} \tau^{-1}$ for some $\tau \in \Omega(1)$. By Lemma 2.1(1), $T_w \equiv T_{w_{n-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$. By definition,

$$T_{w'} = T_\tau T_{w_{n-1}} T_{\tau^{-1}} \equiv T_{w_{n-1}} T_{\tau^{-1}} T_\tau = T_{w_{n-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

The corollary is proved. □

Let $\tilde{W}(1)_{\min}$ be the set of elements in $\tilde{W}(1)$ that are of minimal length in their conjugacy classes. We have the following result.

Theorem 2.3. *Assume that W is a finite Coxeter group or an affine Weyl group. Then for any $w \in \tilde{W}(1)$, there exists $w' \in \tilde{W}(1)_{\min}$ such that $w \to w'$.*

Proof. Since the length function on $\tilde{W}(1)$ is induced from the length function on $\tilde{W}$ through π, the statement follows directly from [7, Theorem 1.1] and [6, Theorem 2.6] (see also [11]) if W is a finite Coxeter group, and from [12, Theorem 2.9] if W is an affine Weyl group. □

Now we prove the main result of this section.

Theorem 2.4. *Let $\tilde{\mathcal{H}}$ be a finite or an affine pro-p Hecke algebra. Then the cocenter $\overline{\tilde{\mathcal{H}}}$ is spanned by the image of T_w for $w \in \tilde{W}(1)_{\min}$.*

Proof. Let $x \in \tilde{W}(1)$. We prove by induction that the image of T_x in $\overline{\tilde{\mathcal{H}}}$ is spanned by T_w for $w \in \tilde{W}(1)_{\min}$.

If $x \in \tilde{W}(1)_{\min}$, then the statement is obvious. If $x \notin \tilde{W}(1)_{\min}$, then by Theorem 2.3 there exists $x' \in \tilde{W}(1)$ and $s \in S(1)$ with $x \approx x'$ and $\ell(sx's^{-1}) < \ell(x') = \ell(x)$. By Corollary 2.2 and Lemma 2.1 (2), we have

$$T_x \equiv T_{x'} \equiv c_s T_{x's^{-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

Note that $\ell(x's^{-1}) < \ell(x') = \ell(x)$. By inductive hypothesis, the image of $T_{x's^{-1}}$ in $\overline{\tilde{\mathcal{H}}}$ is spanned by T_w for $w \in \tilde{W}(1)_{\min}$. Hence the image of T_x in $\overline{\tilde{\mathcal{H}}}$ is spanned by T_w for $w \in \tilde{W}(1)_{\min}$. □

2.2. Let $\mathrm{Cyc}(\tilde{W}(1)_{\min})$ be the set of cyclic-shift classes in $\tilde{W}(1)_{\min}$. For $\Sigma \in \mathrm{Cyc}(\tilde{W}(1)_{\min})$, we denote by T_Σ the image of T_w in $\overline{\tilde{\mathcal{H}}}$ for any $w \in \Sigma$. By Corollary 2.2, T_Σ is well-defined. By Theorem 2.4, for a finite or an affine pro-p Hecke algebra, its cocenter is spanned by $(T_\Sigma)_{\Sigma \in \mathrm{Cyc}(\tilde{W}(1)_{\min})}$. It is interesting to see if the spanning set is in fact a basis. It is known to be true for finite 0-Hecke algebras [10, Theorem 5.4] and for affine 0-Hecke algebras [13, Theorem 0.1].

3. Finite pro-p Hecke algebras

In this section, we assume that $\tilde{\mathcal{H}}$ is a finite pro-p Hecke algebra and we discuss the relation between the cocenter and representations of $\tilde{\mathcal{H}}$.

3.1. Recall that $\mathcal{H}$ is the subalgebra of $\tilde{\mathcal{H}}$ spanned by T_w for $w \in W(1)$. It is proved by Vignéras [22, Proposition 2.1 & Proposition 2.2], every irreducible representation of $\mathcal{H}$ is a character and is of the form $\Xi_{\chi,\Gamma}$, where χ is a character of Z and

$$\Gamma \subseteq \{s \in S; \chi(c_{\tilde{s}}) \neq 0 \text{ for some } \tilde{s} \in \pi^{-1}(s)\}.$$

Here the character $\Xi_{\chi,\Gamma}$ is defined to be

$$\Xi_{\chi,\Gamma}(T_s) = \begin{cases} \chi(c_s), & \text{if } s \in \Gamma(1); \\ 0, & \text{otherwise.} \end{cases}$$

We set $\Gamma_\Xi = \Gamma$ for $\Xi = \Xi_{\chi,\Gamma}$.

As $\Omega(1)$ consists of length-zero elements, the conjugation action of $\Omega(1)$ on $\tilde{W}(1)$ preserves the subset $S(1)$ and the conjugation action of $\Omega(1)$ on $\mathcal{H}$ preserves the subset $\{T_s; s \in S(1)\}$. Let $\Omega(\Xi)(1)$ be the stabilizer of Ξ in $\Omega(1)$. Let V be an irreducible representation of $\Omega(\Xi)(1)$. We say the pair (Ξ, V) is *permissible* with respect to $(W(1), \Omega(1))$ if $Z \subseteq \Omega(\Xi)(1)$ acts on V via Ξ. Set

$$I(\Xi, V) = \tilde{\mathcal{H}} \otimes_{\mathcal{H} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega(\Xi)(1)]} (\Xi \otimes V).$$

This is a left module of $\tilde{\mathcal{H}}$.

We say that two permissible pairs (Ξ, V) and (Ξ', V') are equivalent if there exists $\gamma \in \Omega(1)$ such that $({}^\gamma\Xi, {}^\gamma V) \cong (\Xi', V')$. Here ${}^\gamma V$ denotes the twisted module of $\mathcal{H}$ by γ and $\Omega({}^\gamma\Xi)(1) = \gamma\Omega(\Xi)(1)\gamma^{-1}$. In this case, we write $(\Xi, V) \sim (\Xi', V')$. It is proved by Vignéras [22, Proposition 6.17] that every irreducible representation of $\tilde{\mathcal{H}}$ is of the form $I(\Xi, V)$ and $I(\Xi, V) \cong I(\Xi', V')$ if and only if $(\Xi, V) \sim (\Xi', V')$.

The following formula follows easily from the definition of induced modules.

Lemma 3.1. *Let (Ξ, V) be a permissible pair. For $w \in W(1)$ and $\tau \in \Omega(1)$ we have*

$$Tr(T_{w\tau}, I(\Xi, V)) = \sum_{\gamma \in \Omega(1)/\Omega(\Xi)(1)} {}^\gamma\Xi(T_w) Tr(\tau, {}^\gamma V).$$

Here we set $Tr(\tau, {}^\gamma V) = 0$ if $\tau \notin \Omega({}^\gamma\Xi)(1)$.

3.2. We denote by $\tilde{\mathcal{H}}^{=S} \subseteq \tilde{\mathcal{H}}$ the $\mathbf{k}$-linear space generated by $T_{w\tau}$, where $w \in W(1)$ with $\mathrm{supp}(w) = S$ and $\tau \in \Omega(1)$. Denote by $R(\tilde{\mathcal{H}})_{=S}$ the $\mathbf{k}$-linear space spanned by the simple $\tilde{\mathcal{H}}$-modules $I(\Xi, V)$, where (Ξ, V) is a permissible pair such that $\Gamma_\Xi = S$.

By Dedekind's Theorem, the trace map $Tr : A \to R(A)^*_{\mathbf{k}}$ is surjective for any $\mathbf{k}$-algebra A. For finite pro-p Hecke algebra $\tilde{\mathcal{H}}$, we have the following refinement.

Proposition 3.2. *The trace map $Tr : \tilde{\mathcal{H}}^{=S} \to R(\tilde{\mathcal{H}})_{=S}{}^*$ is surjective.*

Proof. Let $M \in R(\tilde{\mathcal{H}})_{=S}$ such that $Tr(\tilde{\mathcal{H}}^{=S}, M) = 0$. We show that $M = 0$.

Assume $M = \sum_{[(\Xi,V)]} a_{[(\Xi,V)]} I(\Xi, V)$, where $a_{[(\Xi,V)]} \in \mathbf{k}$ and $[(\Xi, V)]$ ranges over the $\sim$-equivalence classes of permissible pairs with $\Gamma_\Xi = S$.

Now we show that each coefficient $a_{[(\Xi,V)]}$ vanishes. Fix $w_0 \in W(1)$ with $\text{supp}(w_0) = S$. For any $w \in W(1)$, $T_{w_0} T_w$ is a linear combination of $T_{w'}$ with $w' \in W(1)$ such that $\text{supp}(w') = S$. So for any $\tau \in \Omega(1)$ we have $Tr(T_{w_0} T_w T_\tau, M) = 0$ by assumption. Using Lemma 3.1, we have

$$\begin{aligned} 0 = Tr(T_{w_0} T_w T_\tau, M) &= \sum_{[(\Xi,V)]} \sum_{\gamma \in \Omega(1)/\Omega(\Xi)(1)} a_{[(\Xi,V)]} {}^\gamma\Xi(T_{w_0}) {}^\gamma\Xi(T_w) Tr(\tau, {}^\gamma V) \\ &= \sum_{(\Xi',V')} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V') \Xi'(T_w), \end{aligned}$$

where in the last expression, (Ξ', V') ranges over permissible pairs such that $\Gamma_{\Xi'} = S$, and $a_{(\Xi',V')} = a_{[(\Xi,V)]}$ for $(\Xi', V') \in [(\Xi, V)]$.

Now we regard $\sum_{(\Xi',V')} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V') \Xi'(T_w)$ as the virtual character $\sum_{(\Xi',V')} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V') \Xi'$ evaluated at T_w. Since w runs over all the elements in $W(1)$, we have

$$\sum_{(\Xi',V')} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V') \Xi' = 0.$$

Therefore, for each Ξ', we have

$$\sum_{V';\ (\Xi',V') \text{ is permissible}} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V') = 0.$$

We regard $\sum_{V';\ (\Xi',V') \text{ is permissible}} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(\tau, V')$ as the virtual characters $\sum_{V';\ (\Xi',V') \text{ is permissible}} a_{(\Xi',V')} \Xi'(T_{w_0}) Tr(-, V')$ evaluated at τ. Since τ runs over all the elements in $\Omega(\Xi')(1)$, we have $a_{(\Xi',V')} \Xi'(T_{w_0}) = 0$ for any V'. In particular, $a_{(\Xi,V)} \Xi(T_{w_0}) = 0$. Since $\Xi(T_{w_0}) \neq 0$, we have $a_{[(\Xi,V)]} = a_{(\Xi,V)} = 0$ as desried. $\square$

4. Affine pro-p Hecke algebras and parabolic algebras

We start with a general result on the generic pro-p Hecke algebra.

Lemma 4.1. *Let D be a submonoid of $\tilde{W}$ that is closed under the Bruhat order $\leqslant$, i.e., if $u \leqslant u'$ and $u' \in D$, then $u \in D$. Then $\bigoplus_{x \in D(1)} \mathbf{k}[\mathbf{q}] T_x$ is a subalgebra of $\tilde{\mathcal{H}}(\mathbf{q}, c)$.*

Proof. Let $x, x' \in D(1)$. We argue by induction on $\ell(x)$ that

$$T_x T_{x'} \in \bigoplus_{x'' \in D(1)} \mathbf{k}[\mathbf{q}] T_{x''}.$$

By definition, if $\ell(x) + \ell(x') = \ell(xx')$, then $T_x T_{x'} = T_{xx'}$ and $xx' \in D$ since D is a monoid. Now we assume that $\ell(xx') < \ell(x) + \ell(x')$. Let $x' = s_1 s_2 \cdots s_n \omega$, where $n = \ell(x')$, $\omega \in \Omega(1)$ and $s_1, \cdots, s_n \in S(1)$. Then there exists $1 \leqslant m \leqslant n$ such that $\ell(x s_1 \cdots s_{m-1}) = \ell(x) + m - 1$ and $\ell(x s_1 \cdots s_m) < \ell(x) + m$. By the exchange condition on the Coxeter group W, we have $x s_1 \cdots s_m = y s_1 \cdots s_{m-1}$ for some $y \in \tilde{W}(1)$ with $\pi(y) < \pi(x)$ and $\ell(xs_1 \cdots s_m) = \ell(y) + m - 1$. Since D is closed under $\leqslant$, $y \in D(1)$. Then we have

$$\begin{aligned} T_x T_{x'} &= T_x T_{s_1 \cdots s_{m-1}} T_{s_m} T_{s_{m+1}} \cdots T_{s_n} T_\omega \\ &= T_y T_{s_1 \cdots s_{m-1}} T_{s_m}^2 T_{s_{m+1}} \cdots T_{s_n} T_\omega \\ &= T_y T_{s_1 \cdots s_{m-1}} (c_{s_m} T_{s_m} + q_{s_m} T_{s_m^2}) T_{s_{m+1}} \cdots T_{s_n} T_\omega \\ &= c_s T_y T_{x'} + q_s T_{s^2} T_y T_{s_1 \cdots s_{m-1}} T_{s_{m+1}} \cdots T_{s_n} T_\omega, \end{aligned}$$

where $s = (ys_1 \cdots s_{m-1}) s_m (ys_1 \cdots s_{m-1})^{-1}$. Note that $T_{s_1 \cdots s_{m-1}} T_{s_{m+1}} \cdots T_{s_n} T_\omega$ is a linear combination of $T_{x''}$ with $x'' < x'$. Now the lemma follows from inductive hypothesis. □

4.1. Let $\mathfrak{R} = (X, R, Y, R^\vee, F_0)$ be a based root datum, where X and Y are free abelian groups of finite rank together with a perfect pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$, $R \subseteq X$ is the set of roots, $R^\vee \subseteq Y$ is the set of coroots and $F_0 \subseteq R$ is the set of simple roots. Let $\alpha \mapsto \alpha^\vee$ be the natural bijection from R to $R^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. For $\alpha \in R$, we denote by $s_\alpha : X \to X$ the corresponding reflections stabilizing R. Let $S_0 = \{s_\alpha; \alpha \in F_0\}$ be the set of simple reflections of the associated finite Weyl group W_0. Let $R^+ \subseteq R$ be the set of positive roots determined by F_0. Let $X^+ = \{\lambda \in X; \langle \lambda, \alpha^\vee \rangle \geqslant 0, \forall \alpha \in R^+\}$. For any $v \in X_{\mathbb{Q}}$, we set $J_v = \{s_\alpha \in S_0; \langle v, \alpha^\vee \rangle = 0\}$. For any $J \subseteq S_0$, we set $X^+(J) = \{\lambda \in X^+; J_\lambda = J\}$.

4.2. Let $W_{\mathrm{a}ff} = \mathbb{Z}R \rtimes W_0$ be the affine Weyl group and $S_{\mathrm{a}ff} \supset S_0$ be the set of simple reflections in $W_{\mathrm{a}ff}$. Then $(W_{\mathrm{a}ff}, S_{\mathrm{a}ff})$ is a Coxeter group. Let $\tilde{W} = X \rtimes W_0$ be the extended affine Weyl group. Then $W_{\mathrm{a}ff}$ is a subgroup of $\tilde{W}$. For $\lambda \in X$, we denote by $t^\lambda \in \tilde{W}$ the corresponding translation element.

Let $V = X \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in R$ and $k \in \mathbb{Z}$, set

$$H_{\alpha,k} = \{v \in V; \langle v, \alpha^\vee \rangle = k\}.$$

Let $\mathfrak{H} = \{H_{\alpha,k}; \alpha \in R, k \in \mathbb{Z}\}$. Connected components of $V - \cup_{H \in \mathfrak{H}} H$ are called alcoves. Let

$$C_0 = \{v \in V; 0 < \langle v, \alpha^\vee \rangle < 1, \forall \alpha \in R^+\}$$

be the fundamental alcove. We may regard $W_{\mathrm{a}ff}$ and $\tilde{W}$ as subgroups of affine transformations of V, where t^λ acts by translation $v \mapsto v + \lambda$ on V. The actions of $W_{\mathrm{a}ff}$ and $\tilde{W}$ on V preserve the set of alcoves.

For any $\tilde{w} \in \tilde{W}$, we denote by $\ell(\tilde{w})$ the number of hyperplanes in $\mathfrak{H}$ separating C_0 from $\tilde{w}C_0$. Then $\tilde{W} = W_{\mathrm{a}ff} \rtimes \Omega$, where $\Omega = \{\tilde{w} \in \tilde{W}; \ell(\tilde{w}) = 0\}$ is the subgroup of $\tilde{W}$ stabilizing fundamental alcove C_0. The conjugation action of Ω on $\tilde{W}$ preserves the set $S_{\mathrm{a}ff}$ of simple reflections in $W_{\mathrm{a}ff}$.

4.3. For any $J \subseteq S_0$, we denote by R_J the set of roots spanned by J and set $R_J^\vee = \{\alpha^\vee; \alpha \in R_J\}$. Let $\mathfrak{R}_J = (X, R_J, Y, R_J^\vee, J)$ be the based root datum corresponding to J. Let $W_J \subseteq W_0$ and $\tilde{W}_J = X \rtimes W_J$ be the Weyl group and the extended affine Weyl group of $\mathfrak{R}_J$ respectively. We say $\tilde{w} \in \tilde{W}_J$ is J-positive if $\tilde{w} \in t^\lambda W_J$ for some $\lambda \in X$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for $\alpha \in R^+ \smallsetminus R_J$. Denote by $\tilde{W}_J^+$ the set of J-positive elements. Then

(1) $\tilde{W}_J^+$ is a submonoid of $\tilde{W}_J$. See [2, Section 6] and [19, II.4].
(2) $\tilde{W}_J^+$ is closed under the Bruhat order $\leqslant_J$. See [1, Lemma 4.1].

We set $\mathfrak{H}_J = \{H_{\alpha,k} \in \mathfrak{H}; \alpha \in R_J, k \in \mathbb{Z}\}$ and $C_J = \{v \in V; 0 < \langle v, \alpha^\vee \rangle < 1, \alpha \in R_J^+\}$. For any $\tilde{w} \in \tilde{W}_J$, we denote by $\ell_J(\tilde{w})$ the number of hyperplanes in $\mathfrak{H}_J$ separating C_J from $\tilde{w}C_J$.

Let $(W_J)_{\mathrm{a}ff} = \mathbb{Z}R_J \rtimes W_J$ and let $J_{\mathrm{a}ff} \supseteq J$ be the set of simple reflections of $(W_J)_{\mathrm{a}ff}$. Then $\tilde{W}_J = (W_J)_{\mathrm{a}ff} \rtimes \Omega_J$, where $\Omega_J = \{\tilde{w} \in \tilde{W}_J; \ell_J(\tilde{w}) = 0\}$. We denote by $\leqslant_J$ the Bruhat order on $\tilde{W}_J$, the partial order induced from the Bruhat order of the Coxeter group $(W_J)_{\mathrm{a}ff}$. Note that $\leqslant_J$ differs from the restriction to $\tilde{W}_J$ of the Bruhat order on $\tilde{W}$.

We denote by $\tilde{W}^J$ (resp. ${}^J\tilde{W}$) the set of minimal coset representatives in $\tilde{W}/W_J$ (resp. $W_J \setminus \tilde{W}$). For $J, K \subseteq S_0$, we simply write $\tilde{W}^J \cap {}^K\tilde{W}$ as ${}^K\tilde{W}^J$. We define ${}^JW_0, W_0^J$ and ${}^JW_0^K$ in a similar way.

4.4. We denote by $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$ the generic affine pro-p Hecke algebra of $\tilde{W}_J(1)$ with parameters defined by $q_s^J = q_s$ and $c_s^J = c_s$ for $s \in J_{\mathrm{a}ff}(1)$ and we denote by $\{T_{J,u}\}_{u \in \tilde{W}_J(1)}$ the standard basis of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$. Let $\tilde{\mathcal{H}}_J^+(\mathbf{q}^J, c^J)$ be the $\mathbf{k}[\mathbf{q}^J]$-submodule of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$ spanned by $T_{J,u}$ for $u \in \tilde{W}_J^+(1)$. By Lemma 4.1 and §4.3 (1) & (2), $\tilde{\mathcal{H}}_J^+(\mathbf{q}^J, c^J)$ is a subalgebra of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$.

4.5. Let $\lambda \in X(1)$. Then $\lambda = \lambda_1 \lambda_2^{-1}$ for some $\lambda_1, \lambda_2 \in X^+(1)$. We set

$$\theta_\lambda = T_{\lambda_1} T_{\lambda_2}^{-1} \in \tilde{\mathcal{H}}(\mathbf{q}, c) \otimes_{\mathbf{k}[\mathbf{q}]} \mathbf{k}[\mathbf{q}^{\pm 1}].$$

It is easy to see that θ_λ does not depend on the choices of λ_1 and λ_2. For $\lambda, \lambda' \in X(1)$ we have $\theta_\lambda \theta_{\lambda'} = \theta_{\lambda\lambda'}$. For $u \in \tilde{W}_J(1)$, there exist $u' \in \tilde{W}_J^+(1)$ and $\lambda \in X^+(J)(1)$ such that $u = u'\lambda^{-1}$. We set $T_u^J = T_{u'} T_\lambda^{-1}$. It is easy to see that T_u^J does not depend on the choices of u' and λ.

Let

$$\tilde{\mathcal{H}}(J; \mathbf{q}, c) = \bigoplus_{u \in \tilde{W}_J(1)} \mathbf{k}[\mathbf{q}^{\pm 1}] T_u^J$$

be the submodule of $\tilde{\mathcal{H}}(\mathbf{q}, c) \otimes_{\mathbf{k}[\mathbf{q}]} \mathbf{k}[\mathbf{q}^{\pm 1}]$.

Note that $T_u^J \in \tilde{\mathcal{H}}(\mathbf{q}, c)$ for $u \in \tilde{W}_J^+(1)$. Let

$$\tilde{\mathcal{H}}^+(J; \mathbf{q}, c) = \bigoplus_{u \in \tilde{W}_J^+(1)} \mathbf{k}[\mathbf{q}] T_u^J$$

be the submodule of $\tilde{\mathcal{H}}(\mathbf{q}, c)$. We denote by $\tilde{\mathcal{H}}^+(J; 0, c)$ the specialization of $\tilde{\mathcal{H}}^+(J; \mathbf{q}, c)$ at $\mathbf{q}_s \equiv 0$.

Theorem 4.2. *Let $J \subseteq S_0$. Then the multiplication map on $\tilde{\mathcal{H}}(\mathbf{q}, c)$ gives $\tilde{\mathcal{H}}(J; \mathbf{q}, c)$ an algebra structure and we have an isomorphism of algebras*

$$\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J) \otimes_{\mathbf{k}[\mathbf{q}^J]} \mathbf{k}[\mathbf{q}^{\pm 1}] \cong \tilde{\mathcal{H}}(J; \mathbf{q}, c), \quad T_{J,u} \mapsto T_u^J.$$

In particular $\tilde{\mathcal{H}}^+(J; \mathbf{q}, c)$ is an algebra and we have an isomorphism of algebras

$$\tilde{\mathcal{H}}_J^+(\mathbf{q}^J, c^J) \cong \tilde{\mathcal{H}}^+(J; \mathbf{q}, c), \quad T_{J,u} \mapsto T_u^J.$$

Remark. We have a natural embedding

$$\tilde{\mathcal{H}}_J^+(\mathbf{q}^J, c^J) \hookrightarrow \tilde{\mathcal{H}}(\mathbf{q}, c), \qquad T_{J,u} \mapsto T_u^J.$$

Notice that this embedding does not extend to an algebra homomorphism $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J) \to \tilde{\mathcal{H}}(\mathbf{q}, c)$ since $T_{J,\lambda}$ for $\lambda \in X^+(J)(1)$ is invertible in $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$, but not invertible in $\tilde{\mathcal{H}}(\mathbf{q}, c)$.

Proof. By definition, $\{T_{J,u}\}_{u\in\tilde{W}_J(1)}$ is a $\mathbf{k}[\mathbf{q}^{\pm 1}]$-basis of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)\otimes_{\mathbf{k}[\mathbf{q}^J]}\mathbf{k}[\mathbf{q}^{\pm 1}]$ and $\{T^J_u\}_{u\in\tilde{W}_J(1)}$ is a $\mathbf{k}[\mathbf{q}^{\pm 1}]$-basis of $\tilde{\mathcal{H}}(J;\mathbf{q}, c)$. Thus the map $T_{J,u}\mapsto T^J_u$ gives a bijection between the $\mathbf{k}[\mathbf{q}^{\pm 1}]$-modules $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)\otimes_{\mathbf{k}[\mathbf{q}^J]}\mathbf{k}[\mathbf{q}^{\pm 1}]$ and $\tilde{\mathcal{H}}(J;\mathbf{q}, c)$. It remains to check that the relations in the definition of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$ in §1.2 are preserved when mapped to $\tilde{\mathcal{H}}(J;\mathbf{q}, c)$. We check those relations in Lemma 4.4 and Corollary 4.6 below.

The "in particular" part follows from the fact that the bijection $T^J_u \mapsto T_{J,u}$ gives a bijection between $\tilde{\mathcal{H}}^+(J;\mathbf{q}, c)$ and $\tilde{\mathcal{H}}^+_J(\mathbf{q}^J, c^J)$ and that $\tilde{\mathcal{H}}^+_J(\mathbf{q}^J, c^J)$ is a subalgebra of $\tilde{\mathcal{H}}_J(\mathbf{q}^J, c^J)$ (see §4.4). □

We now discuss some relations on θ, which are essentially due to Lusztig [15, Lemmas 2.5 & 2.7].

Lemma 4.3. *Let $s\in S_0(1)$ and $\chi\in X(1)$. Denote by α_s the simple root corresponding to s. Then we have the following relations in $\tilde{\mathcal{H}}(\mathbf{q}, c)\otimes_{\mathbf{k}[\mathbf{q}]}\mathbf{k}[\mathbf{q}^{\pm 1}]$:*

(1) If $\langle\chi, \alpha_s^\vee\rangle = 0$, then $T_s\theta_\chi = \theta_{s\chi s^{-1}}T_s$.

(2) If $\langle\chi, \alpha_s^\vee\rangle = 1$, then $T^{-1}_{s^{-1}}\theta_\chi T^{-1}_{s^{-1}} = \theta_{s\chi s}$.

(3) If $\langle\chi, \alpha_s^\vee\rangle = 2$, $\chi\in X^+(1)$ and $\alpha_s^\vee\in 2Y$, then

(i) $T_{w'}T_{w''}\theta_{\chi^{-1}} = T_s\theta_{s^{-1}\chi s}$;

(ii) $T_{w'}T_{\tilde{s}}T_{w''}\theta_{\chi^{-1}} = \theta_{w'\tilde{s}w''\chi^{-1}}$;

(iii) $T_{w'}T_{\tilde{s}^{-1}}T_{\tilde{s}}T_{w''}\theta_{\chi^{-1}} = \theta_\chi T^{-1}_{s^{-1}}$.

Here $\tilde{s}\in S_{aff}(1)\smallsetminus S_0(1)$ such that $\pi(\tilde{s}), \pi(s)$ are conjugate in $\tilde{W}$, and $w', w''\in \tilde{W}(1)$ such that $w'w'' = \chi s\chi$ and $\pi(w'\tilde{s}w''\chi^{-1}s^{-1}\chi^{-1}) = t^{\alpha_s}s_{\alpha_s}$.

Lemma 4.4. *For $t\in J_{aff}(1)$, we have the quadratic relation*

$$(T^J_t)^2 = c_tT^J_t + q_tT_{t^2}.$$

Proof. If $t\in J(1)$, then $T^J_t = T_t$ and the statement is trivial. Now we assume $t\in J_{aff}(1)\smallsetminus J(1)$. Since Z is finite and $\tilde{W}_J(1)$ is finitely generated, there exists an element $\mu\in X^+(J)(1)$ which is central in $\tilde{W}_J(1)$ such that $t\mu\in\tilde{W}^+_J(1)$. Then $T^J_t = T_{t\mu}T^{-1}_\mu = T^{-1}_\mu T_{\mu t}$. It remains to show $T^2_{\mu t} = c_tT_{\mu^2 t} + q_tt^2T_{\mu^2}$.

Assume $\pi(t) = t^\alpha s_\alpha\in (W_J)_{\rm aff}$ for some maximal short root $\alpha\in R^+_J$. Let $w\in W_J(1)$ and $s\in J(1)$ such that $s_\alpha = \pi(wsw^{-1})$ and $\ell(s_\alpha) = 2\ell(w)+1$. Choose $\delta\in\tilde{W}(1)$ such that $\pi(\delta) = \underline{\lambda}_s\pi(s)\underline{\lambda}_s^{-1}$. Here $\underline{\lambda}_s$ is the fundamental weight corresponding to s.

Since the quadratic relation of T_t is equivalent to that of T_{tz} for any $z\in Z$, we can assume $t = w\delta w^{-1}$. Set $\lambda = \delta s^{-1}\in X(1)$. We have $T^J_{\mu t} = \theta_{w\mu\lambda w^{-1}}T^{-1}_{ws^{-1}w^{-1}} = \theta_{w\mu\lambda w^{-1}}T^{-1}_{w^{-1}}T^{-1}_{s^{-1}}T^{-1}_w$. Let $w = s_1\cdots s_n$ be a reduced expression of w with each $s_i\in J(1)$. Then $\langle s_{k-1}\cdots s_1(\alpha), \alpha_k^\vee\rangle = 1$ for $k = 1, \ldots, n$, where α_i is the simple root corresponding to s_i. Applying Lemma 4.3 (2), we have

(a) $$T^{-1}_w\theta_{w\mu\lambda w^{-1}}T^{-1}_{w^{-1}} = T^{-1}_{s_2\cdots s_n}\theta_{s_1^{-1}w\mu\lambda w^{-1}s_1}T^{-1}_{(s_2\cdots s_n)^{-1}} = \cdots = \theta_{\mu\lambda}.$$

Similarly, we have

(b) $$T^{-1}_{w^{-1}}\theta_{s\mu\lambda s}T^{-1}_w = \theta_{w\mu\lambda w^{-1}}.$$

Case 1 ($\alpha^\vee \notin 2Y$). Then $q_t = q_s$ and there exists $\lambda_s \in X(1)$ such that $\pi(\lambda_s) = \underline{\lambda}_s$. Hence we can and do assume $\lambda = \lambda_s s \lambda_s^{-1} s^{-1}$. Then

$$\begin{aligned}
(T^J_{\mu t})^2 &= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_{\mu\lambda} T^{-1}_{s^{-1}} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_{\mu\lambda_s} T^{-1}_s T_s \theta_{s\lambda_s^{-1}s^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} \theta_{s\mu\lambda_s s^{-1}} (c_s + q_s s^2 T^{-1}_s) \theta_{s\lambda_s^{-1}s^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \\
&= ((w\lambda_s) \bullet c_s) \theta_{w\mu^2\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \\
&\quad + q_s \theta_{w\mu^2\lambda w^{-1}} T^{-1}_{w^{-1}} \theta_{s\mu\lambda_s s^{-1}} s^2 T^{-1}_s \theta_{s\lambda_s^{-1}s^{-1}} T^{-1}_{s^{-1}} T^{-1}_w \\
&= c_t \theta_{w\mu^2\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w + q_s \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} \theta_{\mu\lambda s\mu\lambda_s s\lambda_s^{-1}} T^{-1}_w \\
&= c_t \theta_{w\mu^2\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w + q_s \theta_{w\mu\lambda s\mu\lambda_s s\lambda_s^{-1} w^{-1}} \\
&= c_t T_{\mu^2 t} + q_t t^2 T_{\mu^2},
\end{aligned}$$

where the second equality follows from (a); the sixth one follows from Lemma 4.3 (2); the seventh follows from Lemma 4.3 (1).

Case 2 ($\alpha^\vee \in 2Y$). Then $\underline{\lambda}_s \notin X$. Let $\chi, w', w'', \tilde{s}$ be as in Lemma 4.3 (3). Since $w'w''\chi^{-1}s^{-1}\chi^{-1}\mu\lambda s = \mu\lambda s = \mu\delta$ and

$$\pi(w'\tilde{s}^{-1}w''\chi^{-1}s^{-1}\chi^{-1}\mu\lambda s) = \pi(\mu),$$

there exists $r \in Z$ such that

(c) $$w'\tilde{s}^{-1}w'^{-1} = r\delta^{-1}, \quad T_{w'}c_{\tilde{s}} = c_\delta T_{r^{-1}} T_{w'}.$$

Then one computes that

$$\begin{aligned}
(T^J_{\mu t})^2 &= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_{\mu\lambda} T^{-1}_{s^{-1}} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_\chi \theta_{\chi^{-1}\mu\lambda} T^{-1}_{s^{-1}} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_\chi T^{-1}_{s^{-1}} \theta_{s^{-1}\chi^{-1}\mu\lambda s} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T_{w'} T_{\tilde{s}^{-1}} T_{\tilde{s}} T_{w''} \theta_{\chi^{-1}s^{-1}\chi^{-1}\mu\lambda s} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T_{w'} (c_{\tilde{s}} T_{\tilde{s}^{-1}} + q_{\tilde{s}}) T_{w''} \theta_{\chi^{-1}s^{-1}\chi^{-1}\mu\lambda s} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} ((ws) \bullet c_\delta) T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T_{r^{-1}} \theta_{w'\tilde{s}^{-1}w''\chi^{-1}s^{-1}\chi^{-1}\mu\lambda s} T^{-1}_w \\
&\quad + q_{\tilde{s}} \theta_{w\mu\lambda w^{-1}} T^{-1}_{w^{-1}} \theta_{s\mu\lambda s} T^{-1}_w \\
&= \theta_{w\mu\lambda w^{-1}} ((ws) \bullet c_\delta) T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} \theta_\mu T^{-1}_w + q_{\tilde{s}} \theta_{w\mu\lambda s\mu\lambda s w^{-1}} \\
&= \theta_{w\mu^2\lambda w^{-1}} ((ws) \bullet c_\delta) T^{-1}_{w^{-1}} T^{-1}_{s^{-1}} T^{-1}_w + q_{\tilde{s}} t^2 T_{\mu^2},
\end{aligned}$$

where the first equality follows from (a); the third one follows from Lemma 4.3 (1); the fourth follows from Lemma 4.3 (3) (iii); the sixth follows from (c) and Lemma 4.3 (3) (ii); the seventh follows from (b). Note that $q_{\tilde{s}} = q_t$ since $\pi(\tilde{s})$ and $\pi(t)$ are conjugate under $\tilde{W}$. It remains to check

$$w\mu^2\lambda s c_\delta s^{-1} w^{-1} = c_t w\mu^2\lambda w^{-1},$$

which follows from the equalities: $w^{-1}c_t w = c_\delta$ and $\delta = \lambda s$ and $\delta c_\delta = c_\delta \delta$. The proof is finished. □

Lemma 4.5. *Let $x, x' \in \tilde{W}^+_J(1)$ such that $\ell_J(xx') = \ell_J(x) + \ell_J(x')$. Then $\ell(xx') = \ell(x) + \ell(x')$.*

Proof. Write $x = \lambda u$ and $x' = \lambda' u'$, where $\lambda, \lambda' \in X(1)$ and $u, u' \in W_J(1)$. Since $x, x', xx' \in \tilde{W}_J^+(1)$, one computes that

$$\begin{aligned}
\ell(xx') &= \ell_J(xx') + \sum_{\alpha \in R^+ \smallsetminus R_J} |\langle \lambda + \pi(u)(\lambda'), \alpha^\vee \rangle| \\
&= \ell_J(x) + \ell_J(x') + \sum_{\alpha \in R^+ \smallsetminus R_J} \langle \lambda, \alpha^\vee \rangle + \sum_{\alpha \in R^+ \smallsetminus R_J} \langle \pi(u)(\lambda'), \alpha^\vee \rangle \\
&= \ell_J(x) + \sum_{\alpha \in R^+ \smallsetminus R_J} \langle \lambda, \alpha^\vee \rangle + \ell_J(x') + \sum_{\alpha \in R^+ \smallsetminus R_J} \langle \lambda', \alpha^\vee \rangle \\
&= \ell(x) + \ell(x')
\end{aligned}$$

as desired. □

As a consequence, we have

Corollary 4.6. *Let $y, y' \in \tilde{W}_J(1)$ such that $\ell_J(yy') = \ell_J(y) + \ell_J(y')$. Then $T_y^J T_{y'}^J = T_{yy'}^J$.*

Proof. Let $\mu \in X^+(J)(1)$ such that $\mu y, y'\mu \in \tilde{W}_J^+$. Since $\ell_J(\mu) = 0$, we have $\ell_J(\mu y) + \ell_J(y'\mu) = \ell_J(\mu y y' \mu)$. Applying Lemma 4.5, we deduce that

$$T_\mu^J T_y^J T_{y'}^J T_\mu^J = T_{\mu y}^J T_{y'\mu}^J = T_{\mu y} T_{y'\mu} = T_{\mu y y' \mu} = T_{\mu y y' \mu}^J = T_\mu^J T_{yy'}^J T_\mu^J,$$

which implies $T_y^J T_{y'}^J = T_{yy'}^J$ as desired. □

5. Standard representatives

Recall that $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(0, c)$. We simply write $\tilde{\mathcal{H}}_J$ for $\tilde{\mathcal{H}}_J(0, c^J)$ and write $\tilde{\mathcal{H}}_J^+$ for $\tilde{\mathcal{H}}_J^+(0, c^J)$. We have shown in Theorem 4.2 that $\tilde{\mathcal{H}}_J^+$ is a subalgebra of $\tilde{\mathcal{H}}$.

By Theorem 2.4, the cocenter $\overline{\tilde{\mathcal{H}}}$ of an affine pro-p Hecke algebra $\tilde{\mathcal{H}}$ is spanned by the image of $T_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}(1)_{\min}$. In this section, we will compute the trace of $T_{\tilde{w}}$ for $\tilde{w} \in \tilde{W}(1)_{\min}$ using certain elements in the parabolic subalgebra $\tilde{\mathcal{H}}_J^+$ of $\tilde{\mathcal{H}}$.

5.1. Let $n_0 = \sharp W_0$. For any $\tilde{w} \in \tilde{W}(1)$, $\tilde{w}^{n_0} = \lambda$ for some $\lambda \in X(1)$. Let $\nu_{\tilde{w}} = \pi(\lambda)/n_0 \in X_{\mathbb{Q}}$ and $\bar{\nu}_{\tilde{w}} \in X_{\mathbb{Q}}^+$ be the unique dominant element in the W_0-orbit of ν_w. It is easy to see that the map $\tilde{W} \to V, \tilde{w} \mapsto \bar{\nu}_{\tilde{w}}$ is constant on each conjugacy class of $\tilde{W}$.

We say that an element $\tilde{w} \in \tilde{W}(1)$ is straight if $\ell(\tilde{w}^n) = n\ell(\tilde{w})$ for any $n \in \mathbb{N}$. By [9, Lemma 8.1], $\tilde{w}$ is straight if and only if $\ell(\tilde{w}) = \langle \bar{\nu}_{\tilde{w}}, 2\rho^\vee \rangle$, where $\rho^\vee$ is the half sum of positive coroots. A conjugacy class that contains a straight element is called a straight conjugacy class.

It is proved in [12, Proposition 2.7] that for each cyclic-shift class in $\tilde{W}(1)_{\min}$, we have some nice representatives.

Proposition 5.1. *For any $\tilde{w} \in \tilde{W}(1)_{\min}$, there exist a subset $K' \subseteq S_{aff}$ with $W_{K'}$ finite, a straight element $y \in {}^{K'}\tilde{W}^{K'}(1)$ with $yK'(1)y^{-1} = K'(1)$, and an element $w \in W_{K'}(1)$ such that $\tilde{w} \approx wy$. Here $W_{K'} \subseteq W_{aff}$ denotes the subgroup generated by reflections of K'.*

5.2. In the situation of Proposition 5.1, we call wy a standard representative of the cyclic-shift class of $\tilde{w}$. Since the finite group $W_{K'}$ is normalized by $\pi(y)$, there exists some integer $m > 0$ such that $(\pi(wy))^m = \pi(y)^m$, which means $\bar{\nu}_{\tilde{w}} = \bar{\nu}_{wy} = \bar{\nu}_y$. The expression of standard representative relates each conjugacy class of $\tilde{W}$ with a straight conjugacy class. It plays an important role in the study of combinatorial properties of conjugacy classes of affine Weyl groups [12], σ-conjugacy classes of connected reductive p-adic groups [8] and representations of affine Hecke algebras with non-zero parameters [4].

However, for a given cyclic-shift class in $\tilde{W}_{\min}(1)$, the standard representatives are in general, not unique. This leads to some difficulty in understanding the cyclic-shift classes in $\tilde{W}_{\min}(1)$ and their relations to the representations of $\tilde{\mathcal{H}}$. To overcome this difficulty, we introduce the notion of *standard pairs*.

Let wy and K' be as in Proposition 5.1. Let $J = J_{\bar{\nu}_y} \subseteq S_0$ and

$$K = \cup_i \pi(y)^i \mathrm{supp}(w) \pi(y)^{-i} \subseteq K'.$$

In particular, W_K is finite (since so is $W_{K'}$). Let $h \in {}^J W_0(1)$ such that $h(\nu_y) = \bar{\nu}_y$. Set $\Gamma = \pi(h) K \pi(h)^{-1} \subseteq J_{aff}$. We call (J, Γ) the *associated standard pair*. Notice that $hwyh^{-1}$ might not lie in $\tilde{W}_J^+(1)$. However, we have

$$hwyh^{-1} \in xW_\Gamma(1) = W_\Gamma(1)x,$$

where $x = hyh^{-1} \in \Omega_J(1)$ and there exists $m > 0$ such that $x^m \in X^+(J)(1)$. Thus for sufficiently large $n > 0$, we have $ux^{k+mn} \in \tilde{W}_J^+(1)$ for each $u \in W_J(1)$ and each $0 \leqslant k \leqslant m-1$. Notice that $(T^J_{hwyh^{-1}})^n$ is a linear combination of T_{ux^n} for $u \in W_J(1)$. Therefore, $(T^J_{hwyh^{-1}})^n \in \tilde{\mathcal{H}}_J^+$ for $n \gg 0$.

Now we state the main result of this section.

Proposition 5.2. *Let w, y, h, J, K be as in §5.2. Then for $n \gg 0$,*

$$T_{wy}^n \equiv (T^J_{hwyh^{-1}})^n \in \tilde{\mathcal{H}}_J^+ \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}].$$

The following is a variation of the length formula in [14].

Lemma 5.3. *For $w \in W_0(1)$ and $\alpha \in R$, set*

$$\delta_w(\alpha) = \begin{cases} 0, & \text{if } w\alpha \in R^+; \\ 1, & \text{if } w\alpha \in R^-. \end{cases}$$

Then for any $x, y \in W_0(1)$ and $\mu \in X(1)$, we have that

$$\ell(x\mu y) = \sum_{\alpha \in R^+} |\langle \mu, \alpha^\vee \rangle + \delta_x(\alpha) - \delta_{y^{-1}}(\alpha)|.$$

Proposition 5.4. *Let $J \subseteq S_0$ and let $x \in \Omega_J(1)$ such that $\nu_x \in X_{\mathbb{Q}}^+(J)$. Then for any $u \in \tilde{W}_J(1)$, we have*

(1) for $n \gg 0$ and $h \in {}^J W_0(1)$, $\ell(h^{-1} u x^n h) = \ell(ux^n)$;

(2) for $n \gg 0$, $\ell(ux^{n+n_0}) = \ell(ux^n) + \ell(x^{n_0})$, where $n_0 = \sharp W_0$.

Proof. We have $ux^n = \lambda w$ for some $\lambda \in X(1)$ and $w \in W_J(1)$. Since $\langle \nu_x, \alpha^\vee \rangle > 0$ for any $\alpha \in R^+ \smallsetminus R_J$, we have $\langle \lambda, \alpha^\vee \rangle > 0$ for any $\alpha \in R^+ \smallsetminus R_J^+$ as $n \gg 0$.

Notice that for $\alpha \in R_J$, $\delta_{h^{-1}w^{-1}(\alpha)} = \delta_{w^{-1}(\alpha)}$. Now

$$
\begin{aligned}
\ell(h^{-1}ux^n h) &= \sum_{\alpha \in R^+} |\langle \lambda, \alpha^\vee\rangle + \delta_{h^{-1}}(\alpha) - \delta_{h^{-1}w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} |\langle \lambda, \alpha^\vee\rangle + \delta_{h^{-1}}(\alpha) - \delta_{h^{-1}w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \big(\langle \lambda, \alpha^\vee\rangle + \delta_{h^{-1}}(\alpha) - \delta_{h^{-1}w^{-1}}(\alpha)\big) \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle \lambda, \alpha^\vee\rangle + \sharp\{\alpha \in R^+ \smallsetminus R_J^+, h^{-1}(\alpha) \in R^-\} \\
&\quad - \sharp\{\alpha \in R^+ \smallsetminus R_J^+, h^{-1}w^{-1}(\alpha) \in R^-\} \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle \lambda, \alpha^\vee\rangle + \ell(h) - \ell(h) \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle \lambda, \alpha^\vee\rangle.
\end{aligned}
$$

This proves part (1).

For part (2),

$$
\begin{aligned}
\ell(ux^{n+n_0}) &= \sum_{\alpha \in R^+} |\langle \lambda + n_0\nu_x, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda + n_0\nu_x, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle \lambda + n_0\nu_x, \alpha^\vee\rangle \\
&= \sum_{\alpha \in R_J^+} |\langle \lambda, \alpha^\vee\rangle - \delta_{w^{-1}}(\alpha)| + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle \lambda, \alpha^\vee\rangle + \sum_{\alpha \in R^+ \smallsetminus R_J^+} \langle n_0\nu_x, \alpha^\vee\rangle \\
&= \ell(ux^n) + \ell(x^{n_0}).
\end{aligned}
$$

□

As a consequence, we have

Corollary 5.5. *Let w, y, h, J be as in §5.2. Then for $n \gg 0$ we have $wy^n \approx hwy^n h^{-1}$.*

Proof. Let $x = hyh^{-1}$ and $u = hwh^{-1}$. Suppose that $h = s_1 \cdots s_k$ for $s_1, \cdots, s_k \in S_0(1)$. Set $h_i = s_1 \cdots s_i$ for $1 \leqslant i \leqslant k$. Then $h_i \in {}^J W_0(1)$. By Proposition 5.4 (1), $\ell(h_i^{-1}ux^nh_i) = \ell(h_{i+1}^{-1}ux^nh_{i+1})$ for $0 \leqslant i \leqslant k-1$. Moreover, since $J = J_{\bar\nu_y} = J_{\nu_{ux}}$, $h_i^{-1}(\bar\nu_y) \neq h_{i+1}^{-1}(\bar\nu_y)$. In particular, $\pi(h_i^{-1}ux^nh_i) \neq \pi(h_{i+1}^{-1}ux^nh_{i+1})$. Thus $h_i^{-1}ux^nh_i \approx h_{i+1}^{-1}ux^nh_{i+1}$ for $0 \leqslant i \leqslant k-1$. Therefore $ux^n \approx h^{-1}ux^nh = wy^n$. □

5.3. **Proof of Proposition 5.2.** Assume

$$T_w T_{ywy^{-1}} \cdots T_{y^{n-1}wy^{1-n}} = \sum_{w' \in W_K(1)} a_{w'} T_{w'}$$

with $a_{w'} \in \mathbf{k}$. Let $\mathcal{H}_K \subseteq \tilde{\mathcal{H}}$ (resp. $\mathcal{H}_{J,\Gamma} \subseteq \tilde{\mathcal{H}}_J$) be the subalgebra generated by T_w for $w \in W_K(1)$ (resp. by T_w^J for $w \in W_\Gamma(1)$). By Lemma 4.4, the map $T_{w'} \mapsto T^J_{hw'h^{-1}}$ gives an algebra isomorphism between $\mathcal{H}_K$ and $\mathcal{H}_{J,\Gamma}$. Thus

$$T^J_{hwh^{-1}} T^J_{hywy^{-1}h^{-1}} \cdots T^J_{hy^{n-1}wy^{1-n}h^{-1}} = \sum_{w' \in W_K(1)} a_{w'} T^J_{hw'h^{-1}}.$$

Now one computes that

$$\begin{aligned} T_{wy}^n &= T_w T_{ywy^{-1}} \cdots T_{y^{n-1}wy^{1-n}} T_{y^n} = (\sum_{w' \in W_K(1)} a_{w'} T_{w'}) T_{y^n} \\ &= \sum_{w' \in W_K(1)} a_{w'} T_{w' y^n} \equiv \sum_{w' \in W_K(1)} a_{w'} T_{h w' y^n h^{-1}} \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]. \end{aligned}$$

Moreover

$$\begin{aligned} \sum_{w' \in W_K(1)} a_{w'} T_{h w' y^n h^{-1}} &= \sum_{w' \in W_K(1)} a_{w'} T^J_{h w' y^n h^{-1}} \\ &= (\sum_{w' \in W_K(1)} a_{w'} T^J_{h w' h^{-1}}) T^J_{h y^n h^{-1}} \\ &= T^J_{hwh^{-1}} T^J_{hywy^{-1}h^{-1}} \cdots T^J_{hy^{n-1}wy^{1-n}h^{-1}} T^J_{hy^n h^{-1}} \\ &= (T^J_{hwyh^{-1}})^n. \end{aligned}$$

6. Some character formulas

6.1. Let M be a representation of $\tilde{\mathcal{H}}$ over $\mathbf{k}$. For any $J \subseteq S_0$, we set $M_J = \bigcap_{\lambda \in X^+(J)(1)} T_\lambda M$. Since M is finite dimensional, there exists $\mu \in X^+(J)(1)$ such that $M_J = T_\mu M$. Moreover, since the action of T_λ on M_J is invertible for any $\lambda \in X^+(J)(1)$, we may regard M_J as an $\tilde{\mathcal{H}}_J$-module. For $\Gamma \subseteq J_{aff}$ with $\sharp W_\Gamma < \infty$, let

$$\Omega_J(\Gamma) = \{\tau \in \Omega_J(1); \pi(\tau)\Gamma\pi(\tau)^{-1} = \Gamma\}$$

and $M_{J,\Gamma} = T^J_{w_\Gamma} M_J$, where $w_\Gamma \in \tilde{W}_J(1)$ such that $\pi(w_\Gamma)$ is the longest element of W_Γ. Then $M_{J,\Gamma}$ is an $\Omega_J(\Gamma)(1)$-module. We extend the map $M \mapsto M_{J,\Gamma}$ linearly to the whole Grothendieck group $R(\tilde{\mathcal{H}})_{\mathbf{k}}$.

Let $\mathcal{H}_{J,\Gamma} \subseteq \tilde{\mathcal{H}}_J$ be the subalgebra spanned by T^J_u with $u \in W_\Gamma(1)$. By §3.1, each irreducible $\mathcal{H}_{J,\Gamma} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)]$-module is of the form $I(\Xi, V)$ for some permissible pair (Ξ, V) with respect to $(W_\Gamma(1), \Omega_J(\Gamma)(1))$. Let $u \in \tilde{W}_J(1)$ such that $\mathrm{supp}^J(u) = \mathrm{supp}^J(w_\Gamma) = \Gamma$. One checks directly that $T^J_u I(\Xi, V) = T^J_{w_\Gamma} I(\Xi, V)$. View M_J as a virtual $\mathcal{H}_{J,\Gamma} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)]$-module. Then it is a sum of the modules $I(\Xi, V)$. So we have $M_{J,\Gamma} = T^J_{w_\Gamma} M_J = T^J_u M_J \in R(\mathcal{H}_{J,\Gamma} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)])_{\mathbf{k}}$.

6.2. Let $\aleph = \{(J, \Gamma); J \subseteq S_0, \Gamma \subseteq J_{aff}\}$ and let $\aleph^* = \{(J, \Gamma) \in \aleph; \sharp W_\Gamma < +\infty\}$ be the set of standard pairs. We define an equivalence relation $\sim$ and a partial order $<$ on $\aleph$ as follows. Let $(J, \Gamma), (J', \Gamma') \in \aleph$. We say $(J, \Gamma) \sim (J', \Gamma')$ if $J = J'$ and $\Gamma' = \pi(\tau)\Gamma\pi(\tau)^{-1}$ for some $\tau \in \Omega_J(1)$. We say that $(J, \Gamma) < (J', \Gamma')$ if either $J \subsetneq J'$ or $J = J'$ and $\Gamma \supsetneq \pi(\tau)\Gamma'\pi(\tau)^{-1}$ for some $\tau \in \Omega_J(1)$.

For $(J, \Gamma) \in \aleph$, denote by $\mathcal{P}(J, \Gamma)$ the set of permissible pairs (Ξ, V) with respect to $(W_\Gamma(1), \Omega_J(\Gamma)(1))$ such that $\Xi(T^J_w) \neq 0$ for each $w \in W_\Gamma(1)$. Let $(\Xi, V) \in \mathcal{P}(J, \Gamma)$ and let $I(\Xi, V)$ be the $\mathcal{H}_{J,\Gamma} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)]$-module constructed as in §3. Set $\tilde{\mathcal{H}}_J(\Gamma) = \mathcal{H}_J \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)]$. We denote by $I(\Xi, V)_0$ the extension of $I(\Xi, V)$ by zero as a module of $\tilde{\mathcal{H}}_J(\Gamma)$ by requiring that $T_w I(\Xi, V)_0 = 0$ if $\pi(w) \notin W_\Gamma \rtimes \Omega_J(\Gamma)$. Define

$$\pi_{J,\Gamma,\Xi,V} = \tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{H}}_J^+} (\tilde{\mathcal{H}}_J \otimes_{\tilde{\mathcal{H}}_J(\Gamma)} I(\Xi, V)_0).$$

It is easy to see that $\pi_{J,\Gamma,\Xi,V}$ and $\pi_{J,{}^\gamma\Gamma,{}^\gamma\Xi,{}^\gamma V}$ are isomorphic as $\tilde{\mathcal{H}}$-modules for $\gamma \in \Omega_J(1)$.

If one considers the right module instead of left module, then our construction becomes $-\otimes_{\tilde{\mathcal{H}}_J^-}\tilde{\mathcal{H}}$, while the construction in [24] uses $-\otimes_{\tilde{\mathcal{H}}_J^+}\tilde{\mathcal{H}}$. We do not know the explicit relationship between the two constructions. It is an interesting question to consider.

Theorem 6.1. *Let* $(J,\Gamma),(J',\Gamma')\in\aleph^*$ *and* $(\Xi,V)\in\mathcal{P}(J,\Gamma)$. *Then*

$$(\pi_{J,\Gamma,\Xi,V})_{J'}=\begin{cases}\oplus_{\gamma\in\Omega_J(1)/\Omega_J(\Gamma)(1)}{}^{\gamma}I(\Xi,V)_0, & \text{if } J=J';\\ 0, & \text{if } J\nsubseteq J'.\end{cases}$$

Moreover,

$$(\pi_{J,\Gamma,\Xi,V})_{J',\Gamma'}=\begin{cases}I(\Xi,V), & \text{if } (J,\Gamma)=(J',\Gamma');\\ 0, & \text{if } (J,\Gamma)\nleq(J',\Gamma').\end{cases}$$

Proof. Let $\lambda\in X^+(J')(1)$ such that $T_\lambda\pi_{J,\Gamma,\Xi,V}=(\pi_{J,\Gamma,\Xi,V})_{J'}$. We may replace λ with some appropriate power of itself so that $T^J_{\gamma^{-1}\lambda\gamma}\in\tilde{\mathcal{H}}_J(\Gamma)$ for any $\gamma\in\Omega_J(1)$. Let

$$M=\bigoplus_{\gamma\in\Omega_J(1)/\Omega_J(\Gamma)(1)}T^J_\gamma\otimes I(\Xi,V)_0.$$

By [16, Proposition 5.2], we have

$$\pi_{J,\Gamma,\Xi,V}=\bigoplus_{d\in W_0^J(1)/Z}{}^{\iota}T_d\otimes M.$$

Let $s\in S_0(1)$. If $s\notin J'(1)$, then $\lambda s<\lambda$ (since $\lambda\in X^+(J')(1)$), which means $T_\lambda{}^{\iota}T_s=T_{\lambda s^{-1}}T_s{}^{\iota}T_s=0$. Otherwise, $\pi(\lambda),\pi(s)$ commute with each other, which means $T_\lambda{}^{\iota}T_s={}^{\iota}T_sT_{s^{-1}\lambda s}$. In summary, we have

$$T_\lambda{}^{\iota}T_s=\begin{cases}{}^{\iota}T_sT_{s^{-1}\lambda s}, & \text{if } s\in J'(1);\\ 0, & \text{otherwise.}\end{cases}$$

Thus, by induction on the length of $d\in W_0^J(1)$, we deduce that

(a) $$T_\lambda{}^{\iota}T_d=\begin{cases}{}^{\iota}T_dT_{d^{-1}\lambda d}, & \text{if } d\in W_{J'}(1);\\ 0, & \text{otherwise,}\end{cases}$$

where $d^{-1}\lambda d\in\lambda Z$ since $\lambda\in X(J')(1)$.

Assume $J\nsubseteq J'$. We show that $(\pi_{J,\Gamma,\Xi,V})_{J'}=T_\lambda\pi_{J,\Gamma,\Xi,V}=0$. By (a), it suffices to show $T_\lambda M=0$. One checks that

$$T_\lambda(T^J_\gamma\otimes I(\Xi,V)_0)=T^J_\lambda(T^J_\gamma\otimes I(\Xi,V)_0)=T^J_\gamma\otimes(T^J_{\gamma^{-1}\lambda\gamma}I(\Xi,V)_0).$$

Since $J\nsubseteq J'$, there exists $\beta\in R_J$ such that $\langle\nu_\lambda,\beta^\vee\rangle\neq0$. We have $\gamma^{-1}\lambda\gamma=w\lambda w^{-1}$ for some $w\in W_J(1)$. Thus $\langle\nu_{\gamma^{-1}\lambda\gamma},\pi(w)^{-1}(\beta^\vee)\rangle\neq0$. Hence $\gamma^{-1}\lambda\gamma\notin W_\Gamma\rtimes\Omega_J(\Gamma)(1)$ and $T^J_{\gamma^{-1}\lambda\gamma}I(\Xi,V)_0=0$.

Assume $J=J'$. By (a), we have

$$(\pi_{J,\Gamma,\Xi,V})_{J'}=T_\lambda\pi_{J,\Gamma,\Xi,V}=M=\bigoplus_{\gamma\in\Omega_J(1)/\Omega_J(\Gamma)(1)}I({}^{\gamma}\Xi,{}^{\gamma}V)_0.$$

Let $u\in W_{\Gamma'}$ with $\mathrm{supp}^J(u)=\Gamma'$. If $(J,\Gamma)\nleq(J,\Gamma')$, that is, $\Gamma'\nsubseteq{}^{\gamma}\Gamma$ for each $\gamma\in\Omega_J(1)$. So ${}^{\gamma}\Xi(T_u)=0$ and hence

$$(\pi_{J,\Gamma,\Xi,V})_{J,\Gamma}=T_uM=\oplus_{\gamma\in\Omega_J(1)/\Omega_J(\Gamma)(1)}T_uI({}^{\gamma}\Xi,{}^{\gamma}V)=0.$$

If $\Gamma = \Gamma'$, once checks similarly $T_u I({}^\gamma\Xi, {}^\gamma V) \neq 0$ if and only if $\gamma \in \Omega_J(\Gamma)(1)$. So $(\pi_{J,\Gamma,\Xi,V})_{J,\Gamma} = I(\Xi, V)$ and the proof is finished. □

Now we state the main result of this section.

Theorem 6.2. *Let $(J,\Gamma) \in \aleph^*$ and $(\Xi, V) \in \mathcal{P}(J,\Gamma)$. Let wy be a standard representative, and let (J',Γ') be its associated standard pair. Let $h \in {}^J W_0(1)$ with $h(\nu_y) = \bar{\nu}_y$. Then*

$$Tr(T_{wy}, \pi_{J,\Gamma,\Xi,V}) = \begin{cases} \sum_{\gamma \in \Omega_J(\Gamma)(1)/\Omega_J(\Gamma,\Xi)(1)} {}^\gamma\Xi(T_{hwh^{-1}}) Tr(hyh^{-1}, {}^\gamma V), & \text{if } (J,\Gamma) = (J',\Gamma'); \\ 0, & \text{if } (J,\Gamma) \nleq (J',\Gamma'). \end{cases}$$

Here $\Omega_J(\Gamma,\Xi)(1)$ is the stabilizer of Ξ in $\Omega_J(\Gamma)(1)$.

Lemma 6.3. *Let (J,Γ) be a standard pair. Let $x, x' \in \Omega_J(\Gamma)(1)$ such that $\nu_x \in X^+(J)_\mathbb{Q}$, and let $u \in W_\Gamma(1)$ with $\mathrm{supp}^J(u) = \Gamma$. Let $M \in R(\tilde{\mathcal{H}})_\mathbf{k}$. Then for $n \gg 0$, we have*

$$Tr(T_{ux'x^n}, M) = Tr(T^J_{ux'x^n}, M_{J,\Gamma}).$$

Proof. Let $\mu \in X^+(J)(1)$ such that $M_J = T_\mu M$. Notice that $n_0\nu_x \in X^+(J)$, where $n_0 = \sharp W_0$. There exists $m \in \mathbb{N}$ such that $x^{mn_0}\mu^{-1} \in X^+(J)(1)$. By Proposition 5.4 (2), for $n \gg 0$,

$$\ell(ux'x^{n+mn_0}) = \ell(ux'x^n) + \ell(x^{mn_0}) = \ell(ux'x^n) + \ell(x^{mn_0}\mu^{-1}) + \ell(\mu).$$

Here the last equality follows from the fact that $\mu, x^{mn_0}\mu^{-1} \in X^+(J)(1)$. We have

$$T_{ux'x^{n+mn_0}} = T_{ux'x^n} T_{x^{mn_0}\mu^{-1}} T_\mu.$$

Moreover, for $n \gg 0$, $ux'x^{n+mn_0} \in \tilde{W}^+_J(1)$ and $T_{ux'x^{n+mn_0}} = T^J_{ux'x^{n+mn_0}}$. Since $0 \to \ker(T_\mu : M \to M) \to M \to M_J \to 0$, we have

$$\begin{aligned} Tr(T_{ux'x^{n+mn_0}}, M) &= Tr(T_{ux'x^n} T_{x^{mn_0}\mu^{-1}} T_\mu, M) = Tr(T_{ux'x^{n+mn_0}}, M_J) \\ &= Tr(T^J_{ux'x^{n+mn_0}}, M_J). \end{aligned}$$

By Corollary4.6, $T^J_{ux'x^{n+mn_0}} = T^J_{ux'}(T^J_x)^{n+mn_0} = T^J_{x'}(T^J_x)^{n+mn_0} T^J_{u'}$ with $u' = (x'x^{n+mn_0})^{-1} u (x'x^{n+mn_0})$. Noticing that $x', x \in \Omega_J(\Gamma)(1)$, we have $\mathrm{supp}^J(u') = \mathrm{supp}^J(u) = \Gamma$. Since $0 \to \ker(T^J_{u'} : M_J \to M_J) \to M_J \to M_{J,\Gamma} \to 0$, we have

$$Tr(T^J_{ux^{n+mn_0}}, M_J) = Tr(T^J_{x'}(T^J_x)^{n+mn_0} T^J_{u'}, M_J) = Tr(T^J_{ux'x^{n+mn_0}}, M_{J,\Gamma})$$

as desired. □

6.3. **Proof of Theorem 6.2.** Let $x = hyh^{-1} \in \Omega_{J'}(1)$ and $u = hwh^{-1} \in W_{\Gamma'}(1)$ with $\mathrm{supp}^{J'}(u) = \Gamma'$. Let $n \in \mathbb{N}$. Then $(T^{J'}_{hwyh^{-1}})^n = (T^{J'}_{ux})^n$ is a linear combination of $T_{u'x^n}$ with $u' \in W_{\Gamma'}(1)$. Thus, for any representation M and $n \gg 0$ we have that

$$Tr(T^n_{wy}, M) = Tr((T^{J'}_{hwyh^{-1}})^n, M) = Tr((T^{J'}_{ux})^n, M_{J',\Gamma'}),$$

where the first equality follows from Proposition 5.2, and the second one follows from Lemma 6.3. By [3, Corollary 4.2], we have

$$Tr(T_{wy}, \pi_{J,\Gamma,\Xi,V}) = Tr(T^{J'}_{ux}, (\pi_{J,\Gamma,\Xi,V})_{J',\Gamma'}).$$

By Theorem 6.1,

$$Tr(T_{wy}, \pi_{J,\Gamma,\Xi,V}) = \begin{cases} Tr(T^{J'}_{ux}, I(\Xi, V)), & \text{if } (J,\Gamma) = (J',\Gamma'); \\ 0, & \text{if } (J,\Gamma) \nleq (J',\Gamma'). \end{cases}$$

Now the statement follows from Lemma 3.1.

7. Representations of $\tilde{\mathcal{H}}$

We first give a basis of $R(\tilde{\mathcal{H}})_{\mathbf{k}}$.

Theorem 7.1. *The set*

$$\{\pi_{J,\Gamma,\Xi,V}; (J,\Gamma) \in \aleph^*/\sim, (\Xi, V) \in \mathcal{P}(J,\Gamma)/\sim\}$$

is a $\mathbf{k}$*-basis of* $R(\tilde{\mathcal{H}})_{\mathbf{k}}$.

Lemma 7.2. *Let A be a* $\mathbf{k}$*-algebra. Let $\tau \in A$ and $\zeta \in R(A)_{\mathbf{k}}$. Assume there exists an invertible central element $\mu \in A$ such that $Tr(\tau\mu^n, \zeta) = 0$ for $n \gg 0$. Then $Tr(\tau, \zeta) = 0$.*

Proof. Assume $\zeta = \sum_V a_V V$, where $a_V \in \mathbf{k}$ and V ranges over finite dimensional simple modules. Since μ is central, μ acts on V by a scalar $\chi_{V,\mu} \in \mathbf{k}^\times$. By assumption, for $n \gg 0$ we have

$$0 = Tr(\tau\mu^n, \zeta) = \sum_V a_V Tr(\tau\mu^n, V) = \sum_V \chi^n_{V,\mu} a_V Tr(\tau, V).$$

Due to the non-vanishing of Vandermonde determinant, for each $f \in \mathbf{k}^\times$, we have

$$\sum_{V, \chi_{V,\mu}=f} a_V Tr(\tau, V) = 0.$$

So $Tr(\tau, \zeta) = \sum_V a_V Tr(\tau, V) = 0$. □

7.1. First we show that

$$\{\pi_{J,\Gamma,\Xi,V}; (J,\Gamma) \in \aleph^*/\sim, (\Xi, V) \in \mathcal{P}(J,\Gamma)/\sim\}$$

is linearly independent in $R(\tilde{\mathcal{H}})_{\mathbf{k}}$.

Suppose $\sum_{J,\Gamma,\Xi,V} a_{J,\Gamma,\Xi,V} \pi_{J,\Gamma,\Xi,V} = 0$ with $a_{J,\Gamma,\Xi,V} \in \mathbf{k}$.

Let $(J_1, \Gamma_1) \in \aleph^*/\sim$ be a minimal element such that $a_{J_1,\Gamma_1,\Xi_1,V_1} \neq 0$ for some (Ξ_1, V_1). Since Z is finite and Ω is finitely generated, there exists a central element μ of $\Omega_{J_1}(\Gamma_1)(1)$ with $\mu \in X^+(J_1)(1)$. Let $u \in W_{\Gamma_1}(1)$ with $\operatorname{supp}^{J_1}(u) = \Gamma_1$ and $x \in \Omega_{J_1}(\Gamma_1)(1)$. Combining Lemma 6.3 with Theorem 6.1, we deduce that for $n \gg 0$

$$\begin{aligned} 0 &= \sum_{J,\Gamma,\Xi,V} a_{J,\Gamma,\Xi,V} Tr(T_{ux\mu^n}, \pi_{J,\Gamma,\Xi,V}) \\ &= \sum_{(\Xi,V) \in \mathcal{P}(J_1,\Gamma_1)/\sim} a_{J_1,\Gamma_1,\Xi,V} Tr(T_{ux\mu^n}, \pi_{J_1,\Gamma_1,\Xi,V}) \\ &= \sum_{(\Xi,V) \in \mathcal{P}(J_1,\Gamma_1)/\sim} a_{J_1,\Gamma_1,\Xi,V} Tr(T^J_{ux\mu^n}, (\pi_{J_1,\Gamma_1,\Xi,V})_{J_1,\Gamma_1}) \\ &= \sum_{(\Xi,V) \in \mathcal{P}(J_1,\Gamma_1)/\sim} a_{J_1,\Gamma_1,\Xi,V} Tr(T^J_{ux\mu^n}, I(\Xi, V)). \end{aligned}$$

By Lemma 7.2, we have that

$$\sum_{(\Xi,V)\in\mathcal{P}(J_1,\Gamma_1)/\sim} a_{J_1,\Gamma_1,\Xi,V} Tr(T^J_{ux}, I(\Xi,V)) = 0.$$

Thanks to Proposition 3.2 (where we take $S = \Gamma_1$ and $\Omega = \Omega_{J_1}(\Gamma_1)$), $a_{J_1,\Gamma_1,\Xi,V} = 0$ for every (Ξ, V). That is a contradiction.

7.2. Next we show that $(\pi_{J,\Gamma,\Xi,V})_{(J,\Gamma)\in\aleph^*,(\Xi,V)\in\mathcal{P}(J,\Gamma)}$ spans $R(\tilde{\mathcal{H}})_{\mathbf{k}}$.

For any $M \in R(\tilde{\mathcal{H}})_{\mathbf{k}}$, let $\aleph^*(M)$ be the set of pairs (J,Γ) in $\aleph^*/\sim$ which is associated to some standard representative $\tilde{w} \in \tilde{W}(1)_{\min}$ such that $Tr(T_{\tilde{w}}, M) \neq 0$.

Fix a total order on $\aleph^*$ that is compatible with the partial order given in §6.2. We argue by induction on the minimal element in $\aleph^*(M)$.

If $\aleph^*(M) = \emptyset$, then $Tr(T_w, M) = 0$ for all $w \in \tilde{W}(1)_{\min}$. By Theorem 2.4, $Tr(h, M) = 0$ for all $h \in \tilde{\mathcal{H}}$. Hence $M = 0$.

Now suppose that $\aleph^*(M) \neq \emptyset$. Let (J,Γ) be the minimal element in $\aleph^*(M)$. We regard $M_{J,\Gamma}$ as a virtual $\mathcal{H}_{J,\Gamma} \otimes_{\mathbf{k}[Z]} \mathbf{k}[\Omega_J(\Gamma)(1)]$-module. Then $M_{J,\Gamma}$ is a linear combination of $I(\Xi,V)$, where (Ξ,V) ranges over a permissible pair with respect to $(W_\Gamma(1), \Omega_J(\Gamma)(1))$. Therefore,

$$M_{J,\Gamma} = \sum_{(\Xi,V)\in\mathcal{P}(J,\Gamma)/\sim} a_{(\Xi,V)} I(\Xi,V),$$

where $a_{(\Xi,V)} \in \mathbf{k}$. We set

$$U(J,\Gamma) = \sum_{(\Xi,V)\in\mathcal{P}(J,\Gamma)/\sim} a_{(\Xi,V)} \pi_{J,\Gamma,\Xi,V}.$$

By Theorem 6.1, $M_{J,\Gamma} = U(J,\Gamma)_{J,\Gamma}$. By Theorem 6.2, we deduce that $Tr(T_{\tilde{w}}, M) = Tr(T_{\tilde{w}}, U(J,\Gamma))$ for each standard representative $\tilde{w} \in \tilde{W}(1)_{\min}$ whose associated standard pair is equivalent to (J,Γ).

Set $M' = M - U(J,\Gamma)$. By Theorem 6.1, for any standard representative $\tilde{w}' \in \tilde{W}(1)_{\min}$ with $Tr(T_{\tilde{w}'}, M') \neq 0$, its associated standard pair (J',Γ') is larger than (J,Γ) in the fixed linear order. By inductive hypothesis, M' is a linear combination of $(\pi_{J,\Gamma,\Xi,V})_{(J,\Gamma)\in\aleph^*,(\Xi,V)\in\mathcal{P}(J,\Gamma)}$. So does M, as desired.

7.3. Motivated by [4], we introduce a rigid part of the cocenter. Here we recall that $\{T_w\}_{w\in\tilde{W}(1)_{\min}}$ spans $\overline{\tilde{\mathcal{H}}}$.

Let $\overline{\tilde{\mathcal{H}}}^{rig}$ be the subspace of $\overline{\tilde{\mathcal{H}}}$ spanned by T_w for $w \in \tilde{W}(1)_{\min}$ with ν_w central and let $\overline{\tilde{\mathcal{H}}}^{nrig}$ be the subspace of $\overline{\tilde{\mathcal{H}}}$ spanned by T_w for $w \in \tilde{W}(1)_{\min}$ with ν_w non-central. We call $\overline{\tilde{\mathcal{H}}}^{rig}$ the *rigid part* of the cocenter and $\overline{\tilde{\mathcal{H}}}^{nrig}$ the *non-rigid part* of the cocenter.

Let $\overline{\tilde{\mathcal{H}}}^{nss} = \overline{\tilde{\mathcal{H}}}^{nrig} + \iota(\overline{\tilde{\mathcal{H}}}^{nrig})$. We call $\overline{\tilde{\mathcal{H}}}^{nss}$ the *non-supersingular part* of the cocenter.

Lemma 7.3. *Let $w \in \tilde{W}(1)$ such that $\sharp W_{\mathrm{supp}(w)} = +\infty$; then the image of T_w in $\overline{\tilde{\mathcal{H}}}$ lies in $\overline{\tilde{\mathcal{H}}}^{nrig}$.*

Proof. If $w \in \tilde{W}(1)_{\min}$, by the same proof of [12, Corollary 2.8], we see that ν_w is non-central and the statement follows. Assume the statement holds for any $w'' \in \tilde{W}(1)$ with $\ell(w'') < \ell(w)$. Assume $w \notin \tilde{W}(1)_{\min}$. By Theorem 2.3, there exist $w' \in \tilde{W}(1)$ and $s \in S_{aff}(1)$ such that $w \approx w'$ and $sw's < w'$. We have $T_w = T_{w'} = c_s T_{w's^{-1}} \in \bar{\bar{\tilde{\mathcal{H}}}}$ and $\mathrm{supp}(w) = \mathrm{supp}(w') = \mathrm{supp}(w's^{-1})$. Note that $\ell(w's) < \ell(w') = \ell(w)$; the statement follows by induction hypothesis. $\square$

Let $M \in R(\tilde{\mathcal{H}})_{\mathbf{k}}$. We say M is *rigid* if $Tr(\overline{\tilde{\mathcal{H}}}^{nrig}, M) = 0$.

Proposition 7.4. *Let $M \in R(\tilde{\mathcal{H}})_{\mathbf{k}}$. Then M is rigid if and only if M is spanned by $(\pi_{S_0,\Gamma,\Xi,V})_{(S_0,\Gamma)\in\aleph^*,(\Xi,V)\in\mathcal{P}(S_0,\Gamma)}$.*

Proof. By Theorem 6.2, $\pi_{S_0,\Gamma,\Xi,V}$ is rigid.

On the other hand, assume that

$$M = \sum_{(J,\Gamma)\in\aleph^*/\sim \text{ with } J\subsetneq S_0,(\Xi,V)\in\mathcal{P}(J,\Gamma)/\sim} a_{J,\Gamma,\Xi,V}\pi_{J,\Gamma,\Xi,V}$$

with $a_{J,\Gamma,\Xi,V} \in \mathbf{k}$. Let (J_1, Γ_1) be a minimal standard pair such that $a_{J_1,\Gamma_1,\Xi_1,V_1} \neq 0$ for some (Ξ_1, V_1). Using Lemma 7.3 and the same argument as in §7.1, one deduces that $a_{J_1,\Gamma_1,\Xi_1,V_1} = 0$ for all (Ξ_1, V_1). That is a contradiction. $\square$

Now we describe the image of rigid representations of $\tilde{\mathcal{H}}$ under the involution ι.

Proposition 7.5. *Let $(S_0, \Gamma) \in \aleph$ and let $(\Xi, V) \in \mathcal{P}(S_0, \Gamma)$. Let*

$$\Gamma' = \{s \in S_{aff} \smallsetminus \Gamma; \Xi(c_{\tilde{s}}) \neq 0 \text{ for } \tilde{s} \in \pi^{-1}(s)\}$$

and Ξ' be the character of $\mathcal{H}_{\Gamma'}$ defined by $\Xi'|_Z = \Xi|_Z$ and $\Xi'(T_s) = \Xi(c_s)$ for $s \in \Gamma'$. Then

$$\iota(\pi_{S_0,\Gamma,\Xi,V}) \cong \pi_{S_0,\Gamma',\Xi',V}.$$

Lemma 7.6. *Let Υ be a character on $\mathcal{H}$. Then*

$$\Omega(\Upsilon)(1) = \Omega(\Upsilon|_Z)(1) \cap \Omega(\Gamma_\Upsilon)(1),$$

where $\Gamma_\Upsilon = \{s \in S_{aff}; \Upsilon(T_{\tilde{s}}) \neq 0 \text{ for } \tilde{s} \in \pi^{-1}(s)\}$.

Proof. By definition, $\Omega(\Upsilon)(1) \subseteq \Omega(\Upsilon|_Z)(1) \cap \Omega(\Gamma_\Upsilon)(1)$. On the other hand, let $\gamma \in \Omega(\Upsilon|_Z)(1) \cap \Omega(\Gamma_\Upsilon)(1)$. If $s \in S_{aff}(1) \smallsetminus \Gamma_\Upsilon(1)$, then $\gamma s\gamma^{-1} \in S_{aff}(1) \smallsetminus \Gamma_\Upsilon(1)$ and $\Upsilon(T_{\gamma s\gamma^{-1}}) = \Upsilon(T_s) = 0$. If $s \in \Gamma_\Upsilon(1)$, then $\gamma s\gamma^{-1} \in \Gamma_\Upsilon(1)$ and $\Upsilon(T_{\gamma s\gamma^{-1}}) = \Upsilon(c_{\gamma s\gamma^{-1}}) = \Upsilon(\gamma c_s\gamma^{-1}) = \Upsilon(c_s) = \Upsilon(T_s)$. Therefore, $\gamma \in \Omega(\Upsilon)(1)$. $\square$

7.4. **Proof of Proposition 7.5.** We denote by Ξ_0 the extension of Ξ on $\mathcal{H}$ defined by $\Xi_0(T_w) = 0$ if $w \notin W_\Gamma(1)$. Note that for any $s \in S_{aff}(1)$, $\iota(T_s) = -T_s + c_s$. By Lemma 7.6,

$$\Omega(\Xi_0)(1) = \Omega(\Xi|_Z)(1) \cap \Omega(\Gamma)(1),$$
$$\Omega(\iota(\Xi_0))(1) = \Omega(\Xi|_Z)(1) \cap \Omega(\Gamma')(1).$$

Let $\gamma \in \Omega(\Xi|_Z)(1)$. Then γ preserves $\{s \in S_{aff}(1); \Xi(c_s) \neq 0\}$. Hence $\gamma \in \Omega(\Gamma)(1)$ if and only if $\gamma \in \Omega(\Gamma')(1)$. Thus $\Omega(\Xi_0)(1) = \Omega(\iota(\Xi_0))(1)$.

We have

$$\begin{aligned}\pi_{S_0,\Gamma,\Xi,V} &= \tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{H}}(\Gamma)} I(\Xi, V)_0 \cong \tilde{\mathcal{H}} \otimes_{\tilde{\mathcal{H}}(\Gamma)} (\tilde{\mathcal{H}}(\Gamma) \otimes_{\mathcal{H}_\Gamma\otimes_{\mathbf{k}[Z]}\Omega(\Xi_0)(1)} (\Xi_0 \otimes V)) \\ &\cong \tilde{\mathcal{H}} \otimes_{\mathcal{H}_\Gamma\otimes_{\mathbf{k}[Z]}\Omega(\Xi_0)(1)} (\Xi_0 \otimes V).\end{aligned}$$

Note that $\iota(\Xi_0) = \Xi_0'$. Then

$$\begin{aligned}\iota(\pi_{S_0,\Gamma,\Xi,V}) &\cong \iota(\tilde{\mathcal{H}} \otimes_{\mathcal{H}_\Gamma \otimes_{\mathbf{k}[Z]} \Omega(\Xi_0)(1)} (\Xi_0 \otimes V)) \\ &\cong \tilde{\mathcal{H}} \otimes_{\mathcal{H}_\Gamma \otimes_{\mathbf{k}[Z]} \Omega(\iota(\Xi_0))(1)} (\iota(\Xi_0) \otimes V) \\ &\cong \tilde{\mathcal{H}} \otimes_{\mathcal{H}_\Gamma \otimes_{\mathbf{k}[Z]} \Omega(\Xi_0')(1)} (\Xi_0' \otimes V) \\ &\cong \pi_{S_0,\Gamma',\Xi',V}.\end{aligned}$$

7.5. Let $L : \tilde{W} \to \mathbb{Z}_{\geqslant 0}$ be a weight function, i.e.

(1) $L(x) \geqslant 1$ if $\ell(x) \geqslant 1$;

(2) $L(xx') = L(x) + L(x')$ if $\ell(xx') = \ell(x) + \ell(x')$.

For $t \in T$ we set $L_t = L(xtx^{-1})$, where $x \in \tilde{W}$ such that $xtx^{-1} \in S_{aff}$. We set $L(w) = L(\pi(w))$ for $w \in \tilde{W}(1)$ and $L_s = L_{\pi(s)}$ for $s \in T(1)$.

Lemma 7.7. *Let $x \in \tilde{W}$ and $t \in T$ such that $xt < x$. Then $L(x) \geqslant L(xt) + L_t$.*

Proof. Let $x = \omega s_1 \cdots s_r$ be a reduced expression of x, where $\omega \in \Omega$ and each s_i is a simple reflection in S_{aff}. In particular, $L(x) = L(s_1) + \cdots + L(s_r)$. Since $xt < x$, there exists $1 \leqslant i \leqslant r$ such that $s_i s_{i+1} \cdots s_r = s_{i+1} \cdots s_r t$. Thus we have $L(xt) \leqslant L(x) - L(s_i) = L(x) - L_t$ as desired. □

7.6. Let q be an indeterminant. We denote by $\tilde{\mathcal{H}}(L, c)$ the pro-p Hecke algebra $\tilde{\mathcal{H}}(\mathbf{q}, c)$ over $\mathbf{k}[q^{\pm 1}]$ such that $q_t = q^{L_t}$ for all $t \in T(1)$ (see §1.2).

In application to p-adic groups G, the indeterminant $q(w)$ is specialized to $q(w) \mapsto [IwI : I]$, where I is an Iwahori subgroup of G. If G is split, then $[IwI : I] = (q')^{\ell(w)}$ for any $w \in \tilde{W}$, where q' is the cardinality of the residue field. In general, there exists a weight function L such that $[IwI : I] = (q')^{L(w)}$ for any $w \in \tilde{W}$ (see [18, Proposition 1.11]).

7.7. Let $\tilde{\mathcal{H}}(L, c)' \subseteq \tilde{\mathcal{H}}(L, c)$ be the $\mathbf{k}[q]$-submodule generated by T_x for $x \in \tilde{W}(1)$. Then $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(L, c)'/q\tilde{\mathcal{H}}(L, c)'$. Let $\tilde{w} \in \tilde{W}(1)$. There exists $w \in W_0(1)$, $\lambda_1, \lambda_2 \in X^+(1)$ such that $\tilde{w} = w\lambda_1\lambda_2^{-1}$. Following Vignéras, we define

$$E_{\tilde{w}} = q^{\frac{1}{2}(L(\lambda_2) - L(\lambda_1) - L(w) + L(\tilde{w}))} T_{w\lambda_1} T_{\lambda_2}^{-1}.$$

It is known that $E_{\tilde{w}}$ is independent of the choices of w, λ_1 and λ_2. By [21], $E_{\tilde{w}} \in \tilde{\mathcal{H}}(L, c)'$ and the set $\{E_{\tilde{w}}; \tilde{w} \in \tilde{W}(1)\}$ form a basis of $\tilde{\mathcal{H}}(L, c)'$ over $\mathbf{k}[q]$.

We say that $M \in R(\tilde{\mathcal{H}})_{\mathbf{k}}$ is *supersingular* if there exists $m > 0$ such that $E_w M = 0$ for $w \in \tilde{W}(1)$ with $\ell(w) \geqslant m$ (or equivalently, $L(w) \geqslant m$). We have the following criterions on the supersingular representations.

Theorem 7.8. *Let $M \in R(\tilde{\mathcal{H}})_{\mathbf{k}}$. The following conditions are equivalent:*

(1) M is supersingular.

(2) $Tr(\overline{\tilde{\mathcal{H}}}^{nss}, M) = 0$.

(3) M is spanned by $\pi_{S_0,\Gamma,\Xi,V}$ for $(S_0, \Gamma) \in \aleph^*$ and $(\Xi, V) \in \mathcal{P}(S_0, \Gamma)$ with $\sharp W_\Gamma, \sharp W_{\Gamma'} < \infty$, where $\Gamma' = \{s \in S_{aff} \smallsetminus \Gamma; \Xi(c_{\tilde{s}}) \neq 0 \text{ for } \tilde{s} \in \pi^{-1}(s)\}$.

Remark. The equivalence between (1) and (3) is first obtained by Ollivier in [17, Theorem 5.14] and Vignéras in [22, Proposition 6.9 & Theorem 6.18] if $\tilde{\mathcal{H}}$ is the pro-p Iwahori-Hecke algebra of a connected reductive p-adic group.

Lemma 7.9. *Let $x, y \in \tilde{W}(1)$. Then*

$$q^{\frac{1}{2}(L(x)-L(y)+L(yx))}T_yT_{x^{-1}}^{-1} \in \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(x)+L(yx))}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)',$$

$$q^{\frac{1}{2}(L(y)-L(x)+L(xy))}T_xT_{y^{-1}}^{-1} \in \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(x)+L(xy))}\mathbb{Z}{}^{\iota}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)'.$$

Proof. We prove the first statement. The second one can be proved in the same way.

We argue by induction on $\ell(x)$. If $\ell(x) = 0$, then the statement is obvious. Assume $\ell(x) \geqslant 1$ and the statement holds for any x' with $\ell(x') < \ell(x)$. Let $s \in S_{aff}(1)$ such that $sx < x$. Then $L(x) = L(sx) + L(s)$.

If $ys < y$, that is, $L(y) = L(ys) + L(s)$, then

$$q^{\frac{1}{2}(L(x)-L(y)+L(yx))}T_yT_{x^{-1}}^{-1} = q^{\frac{1}{2}(L(sx)-L(ys^{-1})+L(ys^{-1}sx))}T_{ys^{-1}}T_{(sx)^{-1}}^{-1}$$

and $L(y)-L(x)+L(yx) = L(ys)-L(sx)+L(ys^{-1}sx)$. The statement follows from induction hypothesis.

If $ys > y$, that is, $L(y) = L(ys) - L(s)$, then

$$q^{\frac{1}{2}(L(x)-L(y)+L(yx))}T_yT_{x^{-1}}^{-1} = q^{\frac{1}{2}(L(sx)-L(ys)+L(yx))}T_{ys^{-1}}T_{(sx)^{-1}}^{-1} - q^{\frac{1}{2}(L(sx)-L(ys)+L(yx))}T_yc_{s^{-1}}T_{(sx)^{-1}}^{-1}.$$

By inductive hypothesis,

$$\begin{aligned} q^{\frac{1}{2}(L(sx)-L(ys)+L(yx))}&T_{ys^{-1}}T_{(sx)^{-1}}^{-1} \\ &\in \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(ys)-L(sx)+L(yx))}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)' \\ &= \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(x)+L(yx))+L(s)}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)' \\ &\subseteq \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(x)+L(yx))}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)'. \end{aligned}$$

Let α be the simple root associated to s and $\beta = \pi(x)^{-1}(\alpha)$. Then $\beta < 0$ since $sx < x$ and $\pi(yx)(\beta) = \pi(y)(\alpha) > 0$ since $ys > y$. Therefore, $ysx < yx$. By Lemma 7.7, we have $L(yx) \geqslant L(ysx) + L_{x^{-1}sx} = L(ysx) + L(s)$.

If $L(ysx) < L(yx)-L(s)$, then $L(sx)-L(ys)+L(yx) > L(sx)-L(y)+L(ysx)$ and by inductive hypothesis, $q^{\frac{1}{2}(L(sx)-L(ys)+L(yx))}T_yc_{s^{-1}}T_{(sx)^{-1}}^{-1} \in q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)'$ and the statement holds in this case.

If $L(ysx) = L(yx) - L(s)$, then $L(y) - L(x) + L(yx) = L(y) - L(sx) + L(ysx)$ and by inductive hypothesis,

$$\begin{aligned} q^{\frac{1}{2}(L(sx)-L(ys)+L(yx))}&T_yc_{s^{-1}}T_{(sx)^{-1}}^{-1} \\ &\in \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(sx)+L(ysx))}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)' \\ &= \left(\oplus_{z\in\tilde{W}(1),L(z)\geqslant\frac{1}{2}(L(y)-L(x)+L(yx))}\mathbb{Z}T_z\right) + q\mathbb{Z}[q]\tilde{\mathcal{H}}(L,c)'. \end{aligned}$$

The statement also holds in this case. □

Corollary 7.10. *Let $(S_0, \Gamma) \in \aleph^*$. Let $w \in \tilde{W}(1)$ such that $L(w) > 2\max L(W_\Gamma)$. Then, in $\bar{\mathcal{H}} = \tilde{\mathcal{H}}(L,c)'/q\tilde{\mathcal{H}}(L,c)'$, we have either $E_w \in \bigoplus_{z\in\tilde{W}(1),\mathrm{supp}(z)\not\subseteq\Gamma}\mathbf{k}\,T_z$ or $E_w \in \bigoplus_{z\in\tilde{W}(1),\mathrm{supp}(z)\not\subseteq\Gamma}\mathbf{k}\,{}^{\iota}T_z$.*

Proof. By definition, $E_w = q^{\frac{1}{2}(L(x)-L(y)+L(yx))} T_y T_{x^{-1}}^{-1}$ for some $x, y \in \tilde{W}(1)$ such that $yx = w$. Applying Lemma 7.9, we see that $E_w \in \bigoplus_{z \in \tilde{W}(1), L(z) > \max L(W_\Gamma)} \mathbf{k} T_z$ if $L(x) \leqslant L(y)$ and $E_w \in \bigoplus_{z \in \tilde{W}(1), L(z) > \max L(W_\Gamma)} \mathbf{k} {}^\iota T_z$ if $L(y) \leqslant L(x)$. The statement follows by noticing that $\mathrm{supp}(z) \nsubseteq \Gamma$ if $L(z) > \max L(W_\Gamma)$. □

7.8. **Proof of Theorem 7.8.** (1) $\Rightarrow$ (2). Let $w \in \tilde{W}(1)_{\min}$ with associated standard pair (J, Γ) such that $J = J_{\bar{\nu}_y} \subsetneq S_0$. It remains to show $Tr(T_w, M) = Tr({}^\iota T_{w^{-1}}, M) = 0$. By Proposition 5.2, there exists $x \in \Omega_J(1)$ such that $\nu_x = \bar{\nu}_y$ and $u \in W_\Gamma(1)$ such that $T_w^n \equiv (T_{(ux)}^J)^n \mod [\tilde{\mathcal{H}}, \tilde{\mathcal{H}}]$ for $n \gg 0$. Note that $(T_{(ux)}^J)^n$ is a linear combination of $T_{u'x^n}^J$, where $u' \in W_\Gamma(1)$. By definition, there exists a sufficiently large integer m_0 such that $x^{m_0} \in X^+(1)$ and $T_{x^{m_0}} M = E_{x^{m_0}} M = 0$. By Proposition 5.4, $\ell(u'x^n) = \ell(u'x^{n-m_0}) + \ell(x^{m_0})$ for $u' \in W_\Gamma(1)$ and $n \gg 0$. Thus $T_{u'x^n}^J M = T_{u'x^n} M = T_{u'x^{n-m_0}} T_{x^{m_0}} M = 0$. Therefore, $Tr(T_w^n, M) = Tr((T_{ux}^J)^n, M) = 0$ for $n \gg 0$. Hence $Tr(T_w, M) = 0$ as desired. The equality $Tr({}^\iota T_{w^{-1}}, M) = 0$ follows in a similar way by noticing that ${}^\iota T_{x^{-m_0}} = E_{x^{-m_0}}$.

(2) $\Rightarrow$ (3). By Proposition 7.4, M and $\iota(M)$ lie in the $\mathbf{k}$-span of

$$(\pi_{S_0,\Gamma,\Xi,V})_{(S_0,\Gamma)\in\aleph^*,(\Xi,V)\in\mathcal{P}(S_0,\Gamma)}.$$

Write

$$\iota(M) = \sum_{(S_0,\Gamma)\in\aleph^*/\sim,\ (\Xi,V)\in\mathcal{P}(S_0,\Gamma)} a_{\Gamma,\Xi,V} \pi_{S_0,\Gamma,\Xi,V}$$

for some $a_{\Gamma,\Xi,V} \in \mathbf{k}$. By Proposition 7.5,

$$M = \sum_{(S_0,\Gamma)\in\aleph^*/\sim,\ (\Xi,V)\in\mathcal{P}(S_0,\Gamma)} a_{\Gamma,\Xi,V} \pi_{S_0,\Gamma',\Xi',V},$$

where for any given (Ξ, Γ), $\Gamma' = \{s \in S_{aff} \smallsetminus \Gamma; \Xi(c_{\tilde{s}}) \neq 0 \text{ for } \tilde{s} \in \pi^{-1}(s)\}$ and Ξ' is the character of $\mathcal{H}_{\Gamma'}$ defined by $\Xi'|_Z = \Xi|_Z$ and $\Xi'(T_s) = \Xi(c_s)$ for $s \in \Gamma'(1)$. By Proposition 7.4, $a_{\Gamma,\Xi,V} = 0$ unless $\sharp W_{\Gamma'} < \infty$. Part (2) is proved.

(3) $\Rightarrow$ (1). By definition, $T_x \pi_{S_0,\Gamma,\Xi,V} = {}^\iota T_x \pi_{S_0,\Gamma',\Xi',V'} = 0$ for any $x \in \tilde{W}(1)$ such that $\mathrm{supp}(x) \nsubseteq \Gamma$ and $\mathrm{supp}(x) \nsubseteq \Gamma'$. Applying Corollary 7.10, $E_w \pi_{S_0,\Gamma,\Xi,V} = 0$ for $w \in \tilde{W}(1)$ with $L(w) > 2 \max L(W_\Gamma \cup W_{\Gamma'})$.

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