

## $\mathbf{Z}/m$ -GRADED LIE ALGEBRAS AND PERVERSE SHEAVES, II

GEORGE LUSZTIG AND ZHIWEI YUN

ABSTRACT. We consider a fixed block for the equivariant perverse sheaves with nilpotent support on the 1-graded component of a semisimple cyclically graded Lie algebra. We give a combinatorial parametrization of the simple objects in that block.

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### INTRODUCTION

As in [LY] we fix  $G$ , a semisimple simply connected algebraic group over  $\mathbf{k}$  (an algebraically closed field of characteristic  $p \geq 0$ ) and a  $\mathbf{Z}/m$ -grading  $\mathfrak{g} = \bigoplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$  for the Lie algebra  $\mathfrak{g}$  of  $G$ ; here  $m$  is an integer  $> 0$  and  $p$  is 0 or a large prime number. We fix  $\eta \in \mathbf{Z} - \{0\}$  and let  $\delta$  be the image of  $\eta$  in  $\mathbf{Z}/m$ . Let  $G_{\underline{0}}$  be the closed connected subgroup of  $G$  with Lie algebra  $\mathfrak{g}_{\underline{0}}$ ; this group acts naturally on  $\mathfrak{g}_{\delta}$  and on  $\mathfrak{g}_{\delta}^{nil} = \mathfrak{g}_{\delta} \cap \mathfrak{g}^{nil}$  where  $\mathfrak{g}^{nil}$  is the variety of nilpotent elements in  $\mathfrak{g}$ . Recall that  $\mathcal{I}(\mathfrak{g}_{\delta})$  denotes the set of pairs  $(\mathcal{O}, \mathcal{L})$  where  $\mathcal{O}$  is a  $G_{\underline{0}}$ -orbit on  $\mathfrak{g}_{\delta}^{nil}$  and  $\mathcal{L}$  is an irreducible  $G_{\underline{0}}$ -equivariant local system on  $\mathcal{O}$  up to isomorphism. Let  $\mathfrak{B}$  be the set of isomorphism classes of simple  $G_{\underline{0}}$ -equivariant perverse sheaves on  $\mathfrak{g}_{\delta}^{nil}$ . Then there is a natural bijection  $\mathcal{I}(\mathfrak{g}_{\delta}) \rightarrow \mathfrak{B}$  given by  $(\mathcal{O}, \mathcal{L}) \mapsto \mathcal{L}^{\sharp}[\dim \mathcal{O}]$  (for the notation  $\mathcal{L}^{\sharp}$  see [LY, 0.11]). In [LY, Theorem 0.6] we have shown that  $\mathfrak{B}$  can be naturally decomposed as a disjoint union of blocks  ${}^{\xi}\mathfrak{B}$  indexed by the  $G_{\underline{0}}$ -conjugacy classes of admissible systems  $\xi = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$  such that if  $B, B'$  are in different blocks then the  $G_{\underline{0}}$ -equivariant Ext-groups of  $B$  with  $B'$  are zero. This generalizes a result of [L4] in the  $\mathbf{Z}$ -graded case; its analogue in the ungraded case is the known partition of the set  $G$ -equivariant simple perverse sheaves on  $\mathfrak{g}^{nil}$  into blocks given by the generalized Springer correspondence.

In this paper we give an (essentially) combinatorial way to parametrize the objects in a fixed block  ${}^{\xi}\mathfrak{B}$  of  $\mathfrak{B}$ . It is known that in the ungraded case, the objects in

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Received by the editors October 12, 2016, and, in revised form, June 23, 2017.

2010 *Mathematics Subject Classification*. Primary 20G99.

The first author was supported by NSF grant DMS-1566618.

The second author was supported by NSF grant DMS-1302071 and the Packard Foundation.

a fixed block are indexed by the irreducible representations of a certain (relative) Weyl group attached to the block. In the  $\mathbf{Z}/m$ -graded case we will associate to the block  ${}^\xi\mathfrak{B}$  a  $\mathbf{Q}$ -vector space  $\mathbf{E}$  with a certain finite collection of hyperplanes. The complement of the union of these hyperplanes is naturally a union of finitely many (not necessarily conical) chambers which can be taken to form a basis of a  $\mathbf{Q}(v)$ -vector space  $\mathbf{V}'$  (here  $v$  is an indeterminate). The chambers represent various spiral induction associated to the block. The vector space  $\mathbf{V}'$  carries a natural, explicit, sesquilinear form  $(\cdot) : \mathbf{V}' \times \mathbf{V}' \rightarrow \mathbf{Q}(v)$  defined in terms of dimensions of Ext-groups between the spiral inductions (see [LY, 6.4]) which correspond to the chambers. We show that the left radical of this form is the same as its right radical. Taking the quotient of  $\mathbf{V}'$  by the left or right radical we obtain a  $\mathbf{Q}(v)$ -vector space  $\mathbf{V}$  with an induced sesquilinear form  $(\cdot)$ . It turns out that  $\mathbf{V}$  is naturally isomorphic to the Grothendieck group based on  ${}^\xi\mathfrak{B}$  (tensored with  $\mathbf{Q}(v)$ ) and, in particular, the number of elements in  ${}^\xi\mathfrak{B}$  is equal to  $\dim_{\mathbf{Q}(v)} \mathbf{V}$ . The vector space  $\mathbf{V}$  has a natural bar involution  $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$  and a natural  $\mathcal{A}$ -lattice  $\mathbf{V}_{\mathcal{A}}$  in  $\mathbf{V}$ , both defined combinatorially. (Here  $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ .) We then define a subset  $\mathbf{B}'$  of  $\mathbf{V}$  as the set of all  $b \in \mathbf{V}_{\mathcal{A}}$  such that  $\bar{b} = b$  and  $(b : b) \in 1 + v\mathbf{Z}[[v]]$ . We show that  $\mathbf{B}'$  is a signed basis of  $\mathbf{V}$ . (Although the definition of  $\mathbf{B}'$  is combinatorial, the proof of the fact that it is a well-defined signed basis is based on geometry; it is not combinatorial. It would be desirable to find a proof without using geometry.) Let  $\mathbf{B}'/\pm$  be the set of orbits of the  $\mathbf{Z}/2$ -action  $b \mapsto -b$  on  $\mathbf{B}'$ . We show that  $\mathbf{B}'/\pm$  is in natural bijection with the given block  ${}^\xi\mathfrak{B}$ . Thus  $\mathbf{B}'/\pm$  could be regarded as a combinatorial index set for  ${}^\xi\mathfrak{B}$ . (A similar result in the  $\mathbf{Z}$ -graded case appears in [L4].)

We now discuss the contents of the various sections. In Section 10 we define the  $\mathbf{Q}$ -vector vector  $\mathbf{E}$  with its hyperplane arrangement associated to a block. We also define the  $\mathbf{Q}(v)$ -vector space  $\mathbf{V}'$  with its sesquilinear form  $(\cdot)$ , its quotient space  $\mathbf{V}$  and the bar operator. In Section 11 we define the  $\mathcal{A}$ -lattice  $\mathbf{V}_{\mathcal{A}}$  in  $\mathbf{V}$  and the signed basis  $\mathbf{B}'$ . In Section 12 we prove some purity properties of the cohomology sheaves of the simple  $G_0$ -equivariant perverse sheaves on  $\mathfrak{g}_\delta^{nil}$ , which generalize those in the  $\mathbf{Z}$ -graded case given in [L4]. In Section 13 we generalize an argument in [L4] to express the matrix whose entries are the values of the  $(\cdot)$ -pairing at two elements of  $\mathfrak{B}$  as a product of three matrices. This is used in Section 14 to prove the vanishing of the odd cohomology sheaves of the intersection cohomology of the closure of any  $G_0$ -orbit in  $\mathfrak{g}_\delta^{nil}$  with coefficients in an irreducible  $G_0$ -equivariant local system on that orbit. This generalizes a result in [L4] in the  $\mathbf{Z}$ -graded case whose proof was quite different (it was based on geometric arguments which are not available in the present case).

We will adhere to the notation and assumptions of [LY].

## 10. THE VECTOR SPACE $\mathbf{V}$ AND THE SESQUILINEAR FORM $(\cdot)$

In this section, we introduce a combinatorial way of calculating the number of irreducible perverse sheaves in a block  ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta)$ . This is achieved by introducing a finite-dimensional vector space  $\mathbf{V}'$  over  $\mathbf{Q}(v)$  together with a sesquilinear form on it coming from the pairing between spiral inductions in the fixed block  $\xi$ . Then the number of irreducible perverse sheaves in  ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta)$  turns out to be the rank of this sesquilinear form on  $\mathbf{V}'$  (Proposition 10.19).

10.1. We fix  $\xi \in \mathfrak{T}_\eta$  and a representative  $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$  for  $\xi$ . We also fix  $\phi = (e, h, f) \in J^M$  such that  $e \in \overset{\circ}{\mathfrak{m}}_\eta$ ,  $h \in \mathfrak{m}_0$ ,  $f \in \mathfrak{m}_{-\eta}$ . We set  $\iota = \iota_\phi \in Y_M$ . Let

$$Z = \mathcal{Z}_M^0.$$

Since  $Z$  is a torus,  $Y_Z$  is naturally a free abelian group (with operation written as addition) and  $\mathbf{E} := Y_{Z, \mathbf{Q}}$  may be naturally identified with the  $\mathbf{Q}$ -vector space  $\mathbf{Q} \otimes Y_Z$ . Let  $X_Z = \text{Hom}(Y_Z, \mathbf{Z})$  and let  $\langle \cdot \rangle$  be the obvious perfect bilinear pairing  $Y_Z \times X_Z \rightarrow \mathbf{Z}$ . This extends to a bilinear pairing  $\mathbf{E} \times (\mathbf{Q} \otimes X_Z) \rightarrow \mathbf{Q}$  denoted again by  $\langle \cdot \rangle$ . For any  $\alpha \in X_Z$  let

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g}; \text{Ad}(z)x = \alpha(z)x \quad \forall z \in Z\}.$$

For any  $(\alpha, i) \in X_Z \times \mathbf{Z}/m$  let

$$\mathfrak{g}_i^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_i.$$

For any  $(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m$  let

$$\mathfrak{g}_i^{\alpha, n} = \mathfrak{g}_i^\alpha \cap ({}^t_n \mathfrak{g}_i).$$

For any  $i \in \mathbf{Z}/m$ , let  $\mathcal{R}_i = \{(\alpha, n) \in X_Z \times \mathbf{Z}; \mathfrak{g}_i^{\alpha, n} \neq 0\}$ ,  $\mathcal{R}_i^* = \{(\alpha, n) \in \mathcal{R}_i; \alpha \neq 0\}$ . Note that

(a)  $\dim \mathfrak{g}_i^{\alpha, n} = \dim \mathfrak{g}_{-i}^{-\alpha, -n}$  for any  $\alpha, n, i$ ; hence  $(\alpha, n) \mapsto (-\alpha, -n)$  is a bijection  $\mathcal{R}_i \xrightarrow{\sim} \mathcal{R}_{-i}$  and a bijection  $\mathcal{R}_i^* \xrightarrow{\sim} \mathcal{R}_{-i}^*$ .

We have

$$\mathfrak{g} = \bigoplus_{(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m} (\mathfrak{g}_i^{\alpha, n}) = \bigoplus_{i \in \mathbf{Z}/m, (\alpha, n) \in \mathcal{R}_i} (\mathfrak{g}_i^{\alpha, n}).$$

For any  $N \in \mathbf{Z}$  and any  $(\alpha, n) \in \mathcal{R}_N^*$  we set

$$\mathfrak{H}_{\alpha, n, N} = \{\varpi \in \mathbf{E}; \langle \varpi : \alpha \rangle = 2N/\eta - n\}.$$

This is an affine hyperplane in  $\mathbf{E}$ . We set

$$\mathbf{E}' = \mathbf{E} - \bigcup_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_N^*} \mathfrak{H}_{\alpha, n, N}.$$

10.2. Recall the notation  $Y_{H, \mathbf{Q}}$  for a connected algebraic group  $H$  with Lie algebra  $\mathfrak{h}$  from [LY, 0.11]. For  $\mu, \mu' \in Y_{H, \mathbf{Q}}$ , we say  $\mu$  commutes with  $\mu'$  if for some  $r, r'$  in  $\mathbf{Z}_{>0}$  such that  $r\mu, r'\mu' \in Y_H$ , the images of the homomorphisms  $r\mu, r'\mu' : \mathbf{k}^* \rightarrow H$  commute with each other. This property is independent of the choice of  $r, r'$ . If  $\mu$  commutes with  $\mu'$ , and  $r, r' \in \mathbf{Z}_{>0}$  are as above, then letting  $\lambda = r\mu, \lambda' = r'\mu'$ , we have the homomorphism  $\nu : \mathbf{k}^* \rightarrow H$  given by  $\nu(t) = \lambda(t^{r'})\lambda'(t^r)$ . We then define  $\mu + \mu' := \nu/(rr') \in Y_{H, \mathbf{Q}}$ . Then  $\mu + \mu'$  is independent of the choice of  $r, r'$ . Moreover, for any  $\kappa \in \mathbf{Q}$ , we have

(a) 
$$\overset{\mu+\mu'}{\kappa} \mathfrak{h} = \bigoplus_{s, t \in \mathbf{Q}; s+t=\kappa} (\overset{\mu}{s} \mathfrak{h} \cap \overset{\mu'}{t} \mathfrak{h}).$$

Let  $\varpi \in \mathbf{E}$ . Since  $\varpi \in \mathbf{E} = Y_{Z, \mathbf{Q}}$  commutes with  $\iota \in Y_{M_0}$ , by the above discussion, we may define

$$\underline{\varpi} = \frac{|\eta|}{2}(\varpi + \iota) \in Y_{M_0, \mathbf{Q}} \subset Y_{G_0, \mathbf{Q}}.$$

From the definitions,  $\underline{\varpi}$  may be computed as follows. We choose  $f \in \mathbf{Z}_{>0}$  such that  $\lambda' := f\varpi \in Y_Z$  and  $f/|\eta| \in \mathbf{Z}$ ; we define  $\lambda \in Y_{G_0}$  by  $\lambda(t) = \iota(t^f)\lambda'(t) = \lambda'(t)\iota(t^f)$  for all  $t \in \mathbf{k}^*$ ; we then have  $\underline{\varpi} = \frac{|\eta|}{2f}\lambda \in Y_{G_0, \mathbf{Q}}$ .

We shall set  $\epsilon = \dot{\eta} \in \{1, -1\}$  (the sign of  $\eta$ , see 0.12).

Now  $\overset{\mu}{\mathfrak{p}}_*$ ,  $\overset{\epsilon \underline{\varpi}}{\mathfrak{f}}_*$  are well-defined; see [LY, 2.5, 2.6].

We set

$$\epsilon^{\tilde{\varpi}} = \bigoplus_{N \in \mathbf{Z}} \epsilon^{\tilde{\varpi}}_N.$$

We show:

(b) We have  $\epsilon^{\tilde{\varpi}}_* = \mathfrak{m}_*$  if and only if  $\varpi \in \mathbf{E}'$ .

If  $x \in \mathfrak{g}_i^{\alpha,n}$  and  $t \in \mathbf{k}^*$ , then

$$\text{Ad}(\lambda(t))x = \text{Ad}(\iota(t^f)) \text{Ad}(\lambda'(t))x = t^{nf + \langle \lambda': \alpha \rangle} x$$

and  $\text{Ad}(\iota(t))x = t^n x$ . Thus,  $\mathfrak{g}_i^{\alpha,n} \subset \lambda_{nf + \langle \lambda': \alpha \rangle} \mathfrak{g}_i$  and  $\mathfrak{g}_i^{\alpha,n} \subset \iota_n \mathfrak{g}_i$ . It follows that for any  $k \in \mathbf{Z}$  we have

(c) 
$$\lambda_k \mathfrak{g}_i = \bigoplus_{(\alpha,n) \in \mathcal{R}_i; nf + \langle \lambda': \alpha \rangle = k} (\mathfrak{g}_i^{\alpha,n}),$$

(d) 
$$\iota_k \mathfrak{g}_i = \bigoplus_{(\alpha,n) \in \mathcal{R}_i; n = k} (\mathfrak{g}_i^{\alpha,n}).$$

For any  $N \in \mathbf{Z}$  we have  $\epsilon^{\tilde{\varpi}}_N = \lambda_{2fN/\eta} \mathfrak{g}_N$ . We see that

(e) 
$$\epsilon^{\tilde{\varpi}}_N = \bigoplus_{(\alpha,n) \in \mathcal{R}_N; nf + \langle \lambda': \alpha \rangle = 2fN/\eta} (\mathfrak{g}_N^{\alpha,n}).$$

Recall from 1.2(e) that if  $N \in \mathbf{Z}, \mathfrak{m}_N \neq 0$ , then  $N \in \eta\mathbf{Z}$ . Let  $N \in \eta\mathbf{Z}$ . From the arguments in 3.3 (or by [LY, 10.3(a)]) we see that, for some  $\varpi' \in \mathbf{E}$ , we have  $\epsilon^{\tilde{\varpi}'_*} = \mathfrak{m}_*$ . (Here  $\underline{\varpi}'$  is attached to  $\varpi'$  in the same way that  $\underline{\varpi}$  was attached to  $\varpi$ .) Hence, using (e) with  $\varpi$  replaced by  $\varpi'$ , we see that there exists a subset  $R_N$  of  $\mathcal{R}_N$  such that  $\mathfrak{m}_N = \bigoplus_{(\alpha,n) \in R_N} (\mathfrak{g}_N^{\alpha,n})$ . If  $(\alpha, n) \in R_N$ , then  $0 \neq \mathfrak{g}_N^{\alpha,n} \subset \mathfrak{m}_N$ . Hence for  $x \in \mathfrak{g}_N^{\alpha,n} - \{0\}$  and  $z \in Z, t \in \mathbf{k}^*$ , we have  $\text{Ad}(z)x = \alpha(z)x, \text{Ad}(\iota(t))x = t^n x$ ; but  $\text{Ad}(z)x = x$  since  $Z$  is in the center of  $M$  and  $\text{Ad}(\iota(t))x = t^{2N/\eta} x$  since  $x \in \mathfrak{m}_N$ . Thus  $\alpha(z) = 1$  (so that  $\alpha = 0$ ) and  $n = 2N/\eta$ . We see that  $R_N \subset \{(0, 2N/\eta)\}$ . Conversely, we show that  $\mathfrak{g}_N^{0,2N/\eta} \subset \mathfrak{m}_N$ . If  $x \in \mathfrak{g}_N^{0,2N/\eta}$ , then  $\text{Ad}(z)x = x$  for all  $z \in Z$  and  $\text{Ad}(\iota(t))x = t^{2N/\eta} x$  for all  $t \in \mathbf{k}^*$ . Since  $\xi$  in 9.1 is admissible, there exists  $t_0 \in \mathbf{k}^*, z \in Z$ , such that  $\mathfrak{m}$  is the fixed point set of  $\text{Ad}(\iota(t_0)z)\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ . Since  $\mathfrak{m}_\eta$  is contained in this fixed point set and  $\text{Ad}(z)$  acts trivially on it, we see that  $t_0^2 \zeta^\eta y = y$  for all  $y \in \mathfrak{m}_\eta$ .

If  $\mathfrak{m}_\eta \neq 0$ , it follows that  $t_0^2 \zeta^\eta = 1$ . Thus we have  $\text{Ad}(\iota(t)z)\theta(x) = t_0^{2N/\eta} \zeta^N x = (t_0^2 \zeta^\eta)^{N/\eta} x = x$ . (Here we use that  $N \in \eta\mathbf{Z}$ .) Thus  $x$  is contained in the fixed point set of  $\text{Ad}(\iota(t_0)z)\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ , so that  $x \in \mathfrak{m}_N$ . We see that for any  $N \in \eta\mathbf{Z}$  we have

(f) 
$$\mathfrak{m}_N = \mathfrak{g}_N^{0,2N/\eta}.$$

Now (f) also holds when  $\mathfrak{m}_\eta = 0$ . (In that case we must have  $\iota = 1$  hence  $\mathfrak{m} = \mathfrak{m}_0$ , so that  $\mathfrak{m}_N = 0$  for  $N \neq 0$ . Moreover, by [LY, 3.6(d)],  $\mathfrak{m}$  is a Cartan subalgebra of  $\mathfrak{g}_0$  and  $Z = M$ . We also have  $\mathfrak{g}_N^{0,2N/\eta} = 0$  for  $N \neq 0$ . Thus, for  $N \neq 0$ , (f) states that  $0 = 0$ . If  $N = 0$ , (f) states that  $\mathfrak{m}$  is its own centralizer in  $\mathfrak{g}_0$ , which is clear.)

From (e),(f) we see that  $\mathfrak{m}_N \subset \epsilon^{\tilde{\varpi}}_N$  for all  $N \in \eta\mathbf{Z}$  and that

(g) for  $N \in \eta\mathbf{Z}$  we have  $\mathfrak{m}_N = \epsilon^{\tilde{\varpi}}_N$  if and only if the following holds:  $\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta\}$  is equal to  $\{(0, 2N/\eta)\}$  if  $(0, 2N/\eta) \in \mathcal{R}_N$  and is empty if  $(0, 2N/\eta) \notin \mathcal{R}_N$ ;

(g') for  $N \in \mathbf{Z} - \eta\mathbf{Z}$  we have  $\mathfrak{m}_N = \epsilon^{\tilde{\varpi}}_N$  (or equivalently  $\epsilon^{\tilde{\varpi}}_N = 0$ ) if and only if  $\{(\alpha, n) \in \mathcal{R}_N; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset$ . The condition in (g) can be also expressed

as follows:

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}^*; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset \text{ for any } N \in \eta\mathbf{Z}.$$

The condition in  $(\mathfrak{g}')$  can be also expressed as follows:

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta\} = \emptyset \text{ for any } N \in \mathbf{Z} - \eta\mathbf{Z}.$$

Indeed, it is enough to show that if  $N \in \mathbf{Z} - \eta\mathbf{Z}$  and  $\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta\}$ , then we have automatically  $\alpha \neq 0$ . (Assume that  $\alpha = 0$ . Then  $n = 2N/\eta$  so that  $n$  is an odd integer. Since  $\mathfrak{g}^0 = \mathfrak{m}$  and  $\mathfrak{g}_{\underline{N}}^{0,n} \neq 0$  we see that  ${}^t_n \mathfrak{m} \neq 0$ . Using 1.2(d) we deduce that  $n$  is even, a contradiction.)

We see that (b) holds.

From (c) we deduce that

$${}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi}} = \bigoplus_{k \in \mathbf{Z}; k \geq 2fN/\eta} (\lambda_k \mathfrak{g}_{\underline{N}}) = \bigoplus_{k \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_{\underline{N}}; k \geq 2fN/\eta, nf + \langle \lambda' : \alpha \rangle = k} (\mathfrak{g}_{\underline{N}}^{\alpha, n}),$$

hence

$$(h) \quad {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi}} = \bigoplus_{(\alpha, n) \in \mathcal{R}_{\underline{N}}; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_{\underline{N}}^{\alpha, n}).$$

The nilradical  ${}^\epsilon \mathfrak{u}_{\overline{\varpi}}^*$  of  ${}^\epsilon \mathfrak{p}_{\overline{\varpi}}^*$  is given by

$$(i) \quad {}^\epsilon \mathfrak{u}_{\underline{N}}^{\overline{\varpi}} = \bigoplus_{(\alpha, n) \in \mathcal{R}_{\underline{N}}^*; \langle \varpi : \alpha \rangle > 2N/\eta - n} (\mathfrak{g}_{\underline{N}}^{\alpha, n}).$$

We see that for  $\varpi, \varpi'$  in  $\mathbf{E}$  and  $N \in \mathbf{Z}$ , the following two conditions are equivalent:

$$(I) \quad {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi}} = {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi'}}.$$

$$(II) \text{ For any } (\alpha, n) \in \mathcal{R}_{\underline{N}}^* \text{ we have } \langle \varpi : \alpha \rangle + n \geq 2N/\eta \Leftrightarrow \langle \varpi' : \alpha \rangle + n \geq 2N/\eta.$$

For  $\varpi, \varpi'$  in  $\mathbf{E}'$  we say that  $\varpi \equiv \varpi'$  if for any  $N \in \mathbf{Z}$  and any  $(\alpha, n) \in \mathcal{R}_{\underline{N}}^*$  we have

$$(\langle \varpi : \alpha \rangle + n - 2N/\eta)(\langle \varpi' : \alpha \rangle + n - 2N/\eta) > 0.$$

This is clearly an equivalence relation on  $\mathbf{E}'$ . From the equivalence of (I),(II) above, we see that

$$(j) \quad \varpi \equiv \varpi' \Leftrightarrow {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi}} = {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi'}} \quad \forall N \in \mathbf{Z}.$$

For any  $\varpi \in \mathbf{E}'$  we set

$$(k) \quad \begin{aligned} I_{\varpi} &= {}^\epsilon \text{Ind}_{{}^\epsilon \mathfrak{p}_{\underline{\eta}}^{\overline{\varpi}}}^{\mathfrak{g}_{\delta}}(\tilde{C}[-\dim \mathfrak{m}_{\eta}]) \in \mathcal{Q}(\mathfrak{g}_{\delta}), \\ \tilde{I}_{\varpi} &= {}^\epsilon \widetilde{\text{Ind}}_{{}^\epsilon \mathfrak{p}_{\underline{\eta}}^{\overline{\varpi}}}^{\mathfrak{g}_{\delta}}(\tilde{C}) \in \mathcal{Q}(\mathfrak{g}_{\delta}). \end{aligned}$$

Here we regard  $\tilde{C}$  as an object of  $\mathcal{Q}(\tilde{l}_{\underline{\eta}}^{\overline{\varpi}}) = \mathcal{Q}(\mathfrak{m}_{\eta})$ , see (b). Note that in  $\mathcal{K}(\mathfrak{g}_{\delta})$  we have

$$\tilde{I}_{\varpi} = v^{h(\varpi)} I_{\varpi},$$

where

$$h(\varpi) = \dim {}^\epsilon \mathfrak{u}_{\underline{0}}^{\overline{\varpi}} + \dim {}^\epsilon \mathfrak{u}_{\underline{\eta}}^{\overline{\varpi}} + \dim \mathfrak{m}_{\eta} = \dim {}^\epsilon \mathfrak{u}_{\underline{0}}^{\overline{\varpi}} + \dim {}^\epsilon \mathfrak{p}_{\underline{\eta}}^{\overline{\varpi}}.$$

We show:

$$(1) \text{ If } \varpi, \varpi' \in \mathbf{E}', \varpi \equiv \varpi', \text{ then } I_{\varpi} = I_{\varpi'}, h(\varpi) = h(\varpi').$$

Indeed, in this case we have  ${}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi}} = {}^\epsilon \mathfrak{p}_{\underline{N}}^{\overline{\varpi'}}$  for all  $N \in \mathbf{Z}$  (see (j)) and the result follows from the definitions.

10.3. We keep the setup of 10.1, 10.2. As in [LY, 2.9], for any  $N \in \mathbf{Z}$  we set

$$\tilde{\mathfrak{l}}_N^\phi = \iota_{2N/\eta} \mathfrak{g}_N \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}_N^\phi = 0 \text{ if } 2N/\eta \notin \mathbf{Z}.$$

Hence

$$(a) \quad \tilde{\mathfrak{l}}_N^\phi = \bigoplus_{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n=2N/\eta} \mathfrak{g}_N^{\alpha, n}.$$

We set  $\tilde{\mathfrak{l}}^\phi = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{l}}_N^\phi$ .

Let

$$\mathbf{E}'' = \mathbf{E} - \bigcup_{N \in \mathbf{Z}, (\alpha, n) \in \mathcal{R}_{\underline{N}}^*; n \neq 2N/\eta} \mathfrak{S}_{\alpha, n, N}.$$

For  $\varpi \in \mathbf{E}$  we show:

(b) *We have  ${}^\epsilon \tilde{\mathfrak{I}}^\varpi \subset \tilde{\mathfrak{l}}^\phi$  if and only if  $\varpi \in \mathbf{E}''$ .*

Using (a) and 10.2(e) we see that we have  $\bigoplus_N {}^\epsilon \tilde{\mathfrak{I}}_N^\varpi \subset \tilde{\mathfrak{l}}^\phi$  if and only if for any  $N \in \mathbf{Z}$  we have

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta\} \subset \{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n = 2N/\eta\},$$

or equivalently,

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}; n + \langle \varpi : \alpha \rangle = 2N/\eta; n \neq 2N/\eta\} = \emptyset,$$

or equivalently,

$$\{(\alpha, n) \in \mathcal{R}_{\underline{N}}^*; \langle \varpi : \alpha \rangle = 2N/\eta - n \neq 0\} = \emptyset.$$

This is the same as the condition that  $\varpi \in \mathbf{E}''$ . This proves (b).

10.4. We show:

(a) *Let  $\mathfrak{p}_*$  be an  $\epsilon$ -spiral with a splitting  $\mathfrak{m}'_*$  such that  $\mathfrak{m}$  is a Levi subalgebra of a parabolic subalgebra of  $\mathfrak{m}' = \bigoplus_N \mathfrak{m}'_N$  compatible with the  $\mathbf{Z}$ -gradings. Then for some  $\varpi \in \mathbf{E}$  we have  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$ ,  $\mathfrak{m}'_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$ .*

We can find  $\mu \in Y_{G_{\underline{Q}}}$  such that  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\mu$  and  $\mathfrak{m}'_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\mu$ . Let  $M'_0 = e^{\mathfrak{m}'_0}$ . We choose  $f \in \mathbf{Z}_{>0}$  such that  $\lambda_1 := f\mu \in Y_{G_{\underline{Q}}}$ . We have  $\mathfrak{m}'_N = \lambda_{fN} \mathfrak{g}_N$  for all  $N \in \mathbf{Z}$ . Let  $N \in \eta\mathbf{Z}$ . Since  $\mathfrak{m}_N \subset \mathfrak{m}'_N$ , for any  $x \in \mathfrak{m}_N$  and any  $t \in \mathbf{k}^*$  we have  $\text{Ad}(\lambda_1(t))x = t^{fN}x$ ; we have also  $\text{Ad}(\iota(t))x = t^{2N/\eta}x$ .

Hence  $\text{Ad}(\lambda_1(t^2)\iota(t^{-f|\eta}))x = x$  for any  $x \in \mathfrak{m}_N$ . Since this holds for any  $N \in \eta\mathbf{Z}$ , it follows that the image of  $\lambda_1^2 \iota^{-f|\eta} : \mathbf{k}^* \rightarrow M'_0$  commutes with  $M$ . Since  $M$  is a Levi subgroup of a parabolic subgroup of  $M'$ , the image of  $\lambda_1^2 \iota^{-f|\eta}$  is contained in  $\mathcal{Z}_M$ , hence in  $\mathcal{Z}_M^0 = Z$ . In particular, the images of  $\iota$  and  $\lambda_1^2 \iota^{-f|\eta}$  commute with each other, hence the images of  $\lambda_1$  and  $\iota$  commute with each other. It therefore makes sense to write  $\lambda' := 2\lambda_1 - f|\eta\iota \in Y_M$  and we actually have  $\lambda' \in Y_Z$ . Let  $\varpi = |\eta|^{-1}f^{-1}\lambda' \in Y_{Z, \mathbf{Q}} = \mathbf{E}$ . We have  $\varpi = |\eta|^{-1}(2\mu - |\eta|\iota)$  hence  $\varpi + \iota = 2|\eta|^{-1}\mu$ , that is,  $\mu = \varpi$  and (a) is proved.

Let  $\mathcal{C}'$  be the collection of  $\epsilon$ -spirals  $\mathfrak{p}_*$  such that  $\mathfrak{m}_*$  is a splitting of  $\mathfrak{p}_*$ . We show:

(b)  *$\mathcal{C}'$  coincides with the collection of  $\epsilon$ -spirals of the form  ${}^\epsilon \mathfrak{p}_*^\varpi$  with  $\varpi \in \mathbf{E}'$ .*

Assume first that  $\mathfrak{p}_* \in \mathcal{C}'$ . Using (a) with  $\mathfrak{m}'_* = \mathfrak{m}_*$  we see that for some  $\varpi \in \mathbf{E}$  we have  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$ ,  $\mathfrak{m}_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$ . Using 10.2(b), we see that  $\varpi \in \mathbf{E}'$ .

Conversely, assume that  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^\varpi$  for some  $\varpi \in \mathbf{E}'$ . From 10.2(b) we have  $\mathfrak{m}_* = {}^\epsilon \tilde{\mathfrak{l}}_*^\varpi$ . Thus  $\mathfrak{p}_* \in \mathcal{C}'$ . This proves (b).

Let  $\mathcal{C}''$  be the collection of  $\epsilon$ -spirals  $\mathfrak{p}_*$  with the following property: there exists a splitting  $\mathfrak{m}'_*$  of  $\mathfrak{p}_*$  such that  $\mathfrak{m}_N \subset \mathfrak{m}'_N \subset \tilde{\mathfrak{l}}_N^\phi$  for all  $N$ . We show:

(c)  $\mathcal{C}''$  coincides with the collection of  $\epsilon$ -spirals of the form  ${}^\epsilon\mathfrak{p}_*^{\tilde{\varpi}}$  with  $\tilde{\varpi} \in \mathbf{E}''$ .

Assume first that  $\mathfrak{p}_* \in \mathcal{C}''$  and let  $\mathfrak{m}'_*$  be a splitting of  $\mathfrak{p}_*$  as in the definition of  $\mathcal{C}''$ . Now  $\mathfrak{m}$  is a Levi subalgebra of a parabolic subalgebra of  $\mathfrak{m}' = \bigoplus_N \mathfrak{m}'_N$  compatible with the  $\mathbf{Z}$ -gradings of  $\mathfrak{m}$  and  $\mathfrak{m}'$ . (Indeed, from the proof of 3.7(c) we see that there exists  $\lambda \in Y_Z$  such that  $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}^\phi; \text{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$ . Since  $\mathfrak{m}' \subset \tilde{\mathfrak{l}}^\phi$  we see that  $\mathfrak{m} = \{y \in \mathfrak{m}'; \text{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$ , as required.) Using (a), we see that for some  $\tilde{\varpi} \in \mathbf{E}$  we have  $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^{\tilde{\varpi}}$ ,  $\mathfrak{m}'_* = {}^\epsilon\tilde{\mathfrak{l}}_*^{\tilde{\varpi}}$ . Since  ${}^\epsilon\tilde{\mathfrak{l}}_N^{\tilde{\varpi}} \subset \tilde{\mathfrak{l}}_N^\phi$  for all  $N$ , we see from 10.3(b) that  $\tilde{\varpi} \in \mathbf{E}''$ .

Conversely, assume that  $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^{\tilde{\varpi}}$  for some  $\tilde{\varpi} \in \mathbf{E}''$ . Let  $\mathfrak{m}'_* = {}^\epsilon\tilde{\mathfrak{l}}_*^{\tilde{\varpi}}$ . From 10.3(b) we see that  ${}^\epsilon\tilde{\mathfrak{l}}_N^{\tilde{\varpi}} \subset \tilde{\mathfrak{l}}_N^\phi$  for all  $N$ . We also have  $\mathfrak{m}_N \subset {}^\epsilon\tilde{\mathfrak{l}}_N^{\tilde{\varpi}}$  for all  $N$ . Thus  $\mathfrak{p}_* \in \mathcal{C}''$ . This proves (c).

Note that:

(d) If  $\mathfrak{p}_* \in \mathcal{C}''$ , then the splitting  $\mathfrak{m}'_*$  in the definition of  $\mathcal{C}''$  is in fact unique.

Indeed, assume that  $\mathfrak{m}'_*, \tilde{\mathfrak{m}}'_*$  are splittings of  $\mathfrak{p}_*$  such that  $\mathfrak{m}_N \subset \mathfrak{m}'_N, \mathfrak{m}_N \subset \tilde{\mathfrak{m}}'_N$  for all  $N$ . By [LY, 2.7] we can find  $u \in U_0$  ( $U_0$  as in [LY, 2.5]) such that  $\text{Ad}(u)\mathfrak{m}'_* = \tilde{\mathfrak{m}}'_*$ . In particular, we have  $\text{Ad}(u)\mathfrak{m}'_0 = \tilde{\mathfrak{m}}'_0$ . This implies  $u = 1$  since  $\mathfrak{m}'_0, \tilde{\mathfrak{m}}'_0$  are both Levi subalgebras of  $\mathfrak{p}_0$  containing  $\mathfrak{m}_0$ . Hence  $\mathfrak{m}'_* = \tilde{\mathfrak{m}}'_*$ .

10.5. Let  $\mathfrak{p}_*, \mathfrak{p}'_*$  be two  $\epsilon$ -spirals such that  $\mathfrak{p}_N \subset \mathfrak{p}'_N$  for all  $N$ . Let  $\mathfrak{u}_*, \mathfrak{u}'_*$  be their nilradicals. Then we have  $\mathfrak{u}'_N \subset \mathfrak{u}_N$  for all  $N$ . In particular,  $\mathfrak{u}'_N \subset \mathfrak{p}_N$  for all  $N$ . We show:

(a)  $\bigoplus_N \mathfrak{p}_N / \mathfrak{u}'_N$  is a parabolic subalgebra of  $\mathfrak{g}' = \bigoplus_N \mathfrak{p}'_N / \mathfrak{u}'_N$ .

Let  $P_0 = e^{\mathfrak{p}_0}, P'_0 = e^{\mathfrak{p}'_0}$  and let  $U_0 = U_{P_0}, U'_0 = U_{P'_0}$ . We have  $P_0 \subset P'_0, U'_0 \subset U_0$ . Now  $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}_*^\mu$  and  $\mathfrak{p}'_* = {}^\epsilon\mathfrak{p}_*^{\mu'}$  for some  $\mu = \lambda/r, \mu' = \lambda'/r'$ , with  $\lambda, \lambda' \in Y_{G_0}$  and  $r, r' \in \mathbf{Z}_{>0}$ . We have  $\lambda(\mathbf{k}^*) \subset P_0, \lambda'(\mathbf{k}^*) \subset P'_0$ . We can find Levi subgroups  $\tilde{L}_0, \tilde{L}'_0$  of  $P_0, P'_0$  such that  $\tilde{L}_0 \subset \tilde{L}'_0$ . By conjugating  $\lambda$  (resp.  $\lambda'$ ) by an element of  $U_0$  (resp.  $U'_0$ ) we can assume that  $\lambda \in \mathcal{Z}_{\tilde{L}_0}, \lambda' \in \mathcal{Z}_{\tilde{L}'_0}$ . Since  $\mathcal{Z}_{\tilde{L}'_0} \subset \mathcal{Z}_{\tilde{L}_0}$ , we have  $\lambda(t)\lambda'(t') = \lambda'(t')\lambda(t)$  for any  $t, t'$  in  $\mathbf{k}^*$ . Hence we have  $\mathfrak{g} = \bigoplus_{k,k' \in \mathbf{Q}; i \in \mathbf{Z}/m} (\begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_i)$ , where  $\begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_i = \begin{smallmatrix} \mu \\ k \end{smallmatrix} \mathfrak{g}_i \cap \begin{smallmatrix} \mu' \\ k' \end{smallmatrix} \mathfrak{g}_i$ . We have

$$\begin{aligned} \mathfrak{p}_N &= \bigoplus_{k,k' \in \mathbf{Q}; k \geq N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N, \\ \mathfrak{p}'_N &= \bigoplus_{k,k' \in \mathbf{Q}; k' \geq N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N, \\ \mathfrak{u}'_N &= \bigoplus_{k,k' \in \mathbf{Q}; k' > N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N. \end{aligned}$$

Since  $\mathfrak{u}'_N \subset \mathfrak{p}_N \subset \mathfrak{p}'_N$  we see that

$$\begin{aligned} \mathfrak{p}_N &= \bigoplus_{k,k' \in \mathbf{Q}; k \geq N, k' \geq N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N, \\ \mathfrak{u}'_N &= \bigoplus_{k,k' \in \mathbf{Q}; k \geq N, k' > N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N, \end{aligned}$$

hence

$$\mathfrak{p}_N / \mathfrak{u}'_N \cong \bigoplus_{k,k' \in \mathbf{Q}; k \geq N, k' = N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N.$$

This is a subspace of

$$\mathfrak{p}'_N / \mathfrak{u}'_N \cong \tilde{\mathfrak{l}}'_N := \bigoplus_{k,k' \in \mathbf{Q}; k' = N} \epsilon \begin{smallmatrix} \mu, \mu' \\ k, k' \end{smallmatrix} \mathfrak{g}_N.$$

Since  $\mu$  and  $\mu'$  commute with each other, it makes sense to define  $\nu = \mu - \mu' \in Y_{\tilde{L}'_0, \mathbf{Q}}$ . Let  $\tilde{\mathcal{V}} = \bigoplus_N \tilde{\mathcal{V}}'_N \subset \mathfrak{g}$ . Then  $\tilde{L}'_0$  acts on  $\mathcal{V}$  by the Ad-action and  $\nu$  induces a  $\mathbf{Q}$ -grading  $\mathcal{V} = \bigoplus_{k_1 \in \mathbf{Q}} \mathcal{V}'_{k_1}$ . From the definitions we see that  $\bigoplus_N \mathfrak{p}'_N / \mathfrak{u}_N = \bigoplus_{k_1 \in \mathbf{Q}; k_1 \geq 0} \mathcal{V}'_{k_1}$ . Thus (a) holds.

A similar argument shows:

(b) *If  $\mathfrak{m}_*$  is a splitting of  $\mathfrak{p}_*$ , the obvious map  $\mathfrak{m} = \bigoplus_N \mathfrak{m}_N \rightarrow \bigoplus_N \mathfrak{p}_N / \mathfrak{u}'_N$  defines an isomorphism of  $\mathfrak{m}$  onto a Levi subalgebra of  $\bigoplus_N \mathfrak{p}_N / \mathfrak{u}'_N$ .*

10.6. Let  $\varpi, \varpi'$  in  $\mathbf{E}'$ . For any  $t \in \mathbf{Q}$  such that  $0 \leq t \leq 1$  we set

$$\varpi_t := t\varpi + (1-t)\varpi'.$$

We assume that there is a unique hyperplane  $\mathfrak{H}$  of the form  $\mathfrak{H} = \mathfrak{H}_{\alpha_0, n_0, N_0}$  for some  $(\alpha_0, n_0, N_0)$  with  $N_0 \in \mathbf{Z}$ ,  $(\alpha_0, n_0) \in \mathcal{R}_{N_0}^*$  such that  $\varpi_t \in \mathfrak{H}$  for some  $t = s$ ; this  $s$  is necessarily unique since  $\varpi_0 \notin \mathfrak{H}$ . Note, however, that the triple  $(\alpha_0, n_0, N_0)$  is not uniquely determined by  $\mathfrak{H}$ . We set  $\varpi'' = \varpi_s$ ,

$$\begin{aligned} \mathfrak{p}_* &= {}^\epsilon \mathfrak{p}_*^{\varpi}, \mathfrak{p}'_* = {}^\epsilon \mathfrak{p}_*^{\varpi'}, \mathfrak{u}_* = {}^\epsilon \mathfrak{u}_*^{\varpi}, \mathfrak{u}'_* = {}^\epsilon \mathfrak{u}_*^{\varpi'}, \\ \mathfrak{p}''_* &= {}^\epsilon \mathfrak{p}_*^{\varpi''}, \tilde{\mathfrak{l}}''_* = {}^\epsilon \tilde{\mathfrak{l}}_*^{\varpi''}, \tilde{\mathfrak{l}}'' = \bigoplus_{N \in \mathbf{Z}} \tilde{\mathfrak{l}}''_N, \\ P_0 &= e^{\mathfrak{p}_0}, P'_0 = e^{\mathfrak{p}'_0}, P''_0 = e^{\mathfrak{p}''_0}. \end{aligned}$$

We show:

(a) *For any  $N \in \mathbf{Z}$  we have  $\mathfrak{p}_N \subset \mathfrak{p}''_N$ ; hence  $P_0 \subset P''_0$ . For any  $N \in \mathbf{Z}$  we have  $\mathfrak{p}'_N \subset \mathfrak{p}''_N$ ; hence  $P'_0 \subset P''_0$ .*

Using 10.2(h), we see that to prove the first sentence in (a) it is enough to show that for any  $(\alpha, n) \in \mathcal{R}_N^*$  such that  $\langle \varpi : \alpha \rangle \geq 2N/\eta - n$  we have also  $\langle \varpi'' : \alpha \rangle \geq 2N/\eta - n$ . (Since  $\varpi \in \mathbf{E}'$ , we must have  $\langle \varpi : \alpha \rangle > 2N/\eta - n$ . Since for  $t \in \mathbf{Q}$ ,  $s < t \leq 1$  we have  $\varpi_t \in \mathbf{E}'$ , it follows that for all such  $t$  we have  $\langle \varpi_t : \alpha \rangle > 2N/\eta - n$ . Taking the limit as  $t \mapsto s$  we obtain  $\langle \varpi'' : \alpha \rangle \geq 2N/\eta - n$ , as required). The second sentence in (a) is proved in a similar way.

We show:

(b) *Fix  $N \in \mathbf{Z}$ . If  $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, N}$  for any  $(\alpha, n) \in \mathcal{R}_N^*$ , then we have  $\mathfrak{p}_N = \mathfrak{p}''_N = \mathfrak{p}'_N$ .*

We first show that  $\mathfrak{p}_N = \mathfrak{p}''_N$ . By the equivalence of (I),(II) in 10.2, it is enough to show that for any  $(\alpha, n) \in \mathcal{R}_N^*$  we have

$$\langle \varpi : \alpha \rangle + n - 2N/\eta > 0 \Leftrightarrow \langle \varpi'' : \alpha \rangle + n - 2N/\eta > 0,$$

or equivalently, that  $c_1 c_s > 0$ , where  $c_t = \langle \varpi_t : \alpha \rangle + n - 2N/\eta$  for  $t \in \mathbf{Q}$  such that  $0 \leq t \leq 1$ . From our assumptions we see that  $c_t \neq 0$  for all  $t$ . It follows that either  $c_t > 0$  for all  $t$  or  $c_t < 0$  for all  $t$ . In particular,  $c_1 c_s > 0$ , as required. The proof of the equality  $\mathfrak{p}'_N = \mathfrak{p}''_N$  is entirely similar.

We show:

(c) *If  $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, 0}$  for any  $(\alpha, n) \in \mathcal{R}_0^*$  and  $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, \eta}$  for any  $(\alpha, n) \in \mathcal{R}_\delta^*$ , then  $I_\varpi = I_{\varpi'}$ ,  $h(\varpi) = h(\varpi')$ .*

Indeed, in this case, by (b) we have  $\mathfrak{p}_N = \mathfrak{p}'_N$  for  $N \in \{0, \eta\}$  and the equality  $I_\varpi = I_{\varpi'}$  follows from the definitions. From  $\mathfrak{p}_0 = \mathfrak{p}'_0$  we deduce that  $\mathfrak{u}_0 = \mathfrak{u}'_0$ . Using this and  $\mathfrak{p}_\eta = \mathfrak{p}'_\eta$  we deduce that  $h(\varpi) = h(\varpi')$ . This proves (c).

We show:

(d) *If  $\mathfrak{H} = \mathfrak{H}_{\alpha_0, n_0, 0}$  for some  $(\alpha_0, n_0) \in \mathcal{R}_0^*$  but  $\mathfrak{H} \neq \mathfrak{H}_{\alpha, n, \eta}$  for any  $(\alpha, n) \in \mathcal{R}_\delta^*$ , then  $I_\varpi \cong I_{\varpi'}$ ,  $h(\varpi) = h(\varpi')$ .*



By (a) we have  $\mathfrak{p}_N \subset \mathfrak{p}''_N, \mathfrak{p}'_N \subset \mathfrak{p}''_N$  for all  $N \in \mathbf{Z}$  and by (b) we have  $\mathfrak{p}_\eta = \mathfrak{p}''_\eta = \mathfrak{p}'_\eta$ . We can now apply [LY, 4.5(b)] twice to conclude that

$$I_\varpi \cong \bigoplus_{j \in J} I_{\varpi''}[-2a_j], I_{\varpi'} \cong \bigoplus_{j' \in J'} I_{\varpi''}[-2a'_{j'}],$$

where  $a_j$  (resp.  $a'_{j'}$ ) are integers such that

$$\rho_{P_0''/P_0} \bar{\mathbf{Q}}_l = \bigoplus_{j \in J} \bar{\mathbf{Q}}_l[-2a_j], \rho_{P_0''/P_0'} \bar{\mathbf{Q}}_l = \bigoplus_{j' \in J'} \bar{\mathbf{Q}}_l[-2a'_{j'}].$$

To show that  $I_\varpi \cong I_{\varpi'}$  it is then enough to show that

$$\bigoplus_{j \in J} \bar{\mathbf{Q}}_l[-2a_j] \cong \bigoplus_{j' \in J'} \bar{\mathbf{Q}}_l[-2a'_{j'}]$$

or that  $\rho_{P_0''/P_0} \bar{\mathbf{Q}}_l \cong \rho_{P_0''/P_0'} \bar{\mathbf{Q}}_l$ . This is clear since  $P_0''/P_0, P_0''/P_0'$  are partial flag manifolds of the reductive quotient  $L_0''$  of  $P_0''$  with respect to two associate parabolic subgroups. The above argument shows also that  $\dim U_{P_0} = \dim U_{P_0'}$  hence  $\dim \mathfrak{u}_0 = \dim \mathfrak{u}'_0$ . Moreover, from (b) we have  $\mathfrak{p}_\eta = \mathfrak{p}'_\eta$ ; we see that  $h(\varpi) = h(\varpi')$ . This proves (d).

For  $N \in \mathbf{Z}$  let  $\mathfrak{q}_N$  (resp.  $\mathfrak{q}'_N$ ) be the image of  $\mathfrak{p}_N$  (resp.  $\mathfrak{p}'_N$ ) under the obvious projection  $\mathfrak{p}''_N \rightarrow \tilde{l}''_N$ . From 10.5(a),(b), we see that  $\mathfrak{q} = \bigoplus_N \mathfrak{q}_N, \mathfrak{q}' = \bigoplus_N \mathfrak{q}'_N$  are parabolic subalgebras of  $\tilde{l}''$  and  $\mathfrak{m}$  is a Levi subalgebra of both  $\mathfrak{q}$  and  $\mathfrak{q}'$ . We show:

(e) *If  $\mathfrak{H} = \mathfrak{H}_{\alpha_0, 2, \eta}$  for some  $(\alpha_0, 2) \in \mathcal{R}_\delta^*$ , then  $I_\varpi \cong I_{\varpi'}$  and  $h(\varpi) = h(\varpi')$ .*

By (a) we have  $\mathfrak{p}_N \subset \mathfrak{p}''_N, \mathfrak{p}'_N \subset \mathfrak{p}''_N$  for all  $N \in \mathbf{Z}$ . Since  $\mathfrak{H}_{\alpha_0, 2, \eta}$  is the unique hyperplane (as in 10.1) on which  $\varpi''$  lies, we have  $\varpi'' \in \mathbf{E}''$ . Hence by 10.3(b) we have  $\tilde{l}''_N \subset \tilde{l}^\phi_N$  for all  $N \in \mathbf{Z}$ . In particular, the  $\mathbf{Z}$ -grading of  $\tilde{l}''$  is  $\eta$ -rigid and  $e \in \tilde{l}''_\eta$ , so that  $\mathfrak{m}_\eta \subset \tilde{l}''_\eta$ . Let  $A \in \mathcal{Q}(\tilde{l}''_\eta)$  be the simple perverse sheaf on  $\tilde{l}''_\eta$  such that the support of  $A$  is  $\tilde{l}''_\eta$  and  $A|_{\mathfrak{m}_\eta}$  is equal up to shift to  $\tilde{C}|_{\mathfrak{m}_\eta}$ . Applying (twice) [LY, 1.8(b)] and the transitivity formula 4.2(a) we deduce

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}) = {}^\epsilon \text{Ind}_{\mathfrak{p}''_\eta}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}_\eta}^{\tilde{l}''_\eta}(\tilde{C})) \cong \bigoplus_j {}^\epsilon \text{Ind}_{\mathfrak{p}''_\eta}^{\mathfrak{g}_\delta}(A)[-2s_j][\dim \mathfrak{m}_\eta - \dim \tilde{l}''_\eta],$$

$${}^\epsilon \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\tilde{C}) = {}^\epsilon \text{Ind}_{\mathfrak{p}''_\eta}^{\mathfrak{g}_\delta}(\text{ind}_{\mathfrak{q}'_\eta}^{\tilde{l}''_\eta}(\tilde{C})) \cong \bigoplus_j {}^\epsilon \text{Ind}_{\mathfrak{p}''_\eta}^{\mathfrak{g}_\delta}(A)[-2s_j][\dim \mathfrak{m}_\eta - \dim \tilde{l}''_\eta],$$

where  $(s_j)$  is a certain finite collection of integers. It follows that

$${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}) \cong {}^\epsilon \text{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(\tilde{C}).$$

This shows that  $I_\varpi, I_{\varpi'}$  are isomorphic. It remains to show that  $h(\varpi) = h(\varpi')$ , for which it suffices to show the following two equalities:

(f)  $\dim \mathfrak{u}_0 = \dim \mathfrak{u}'_0,$

(g)  $\dim \mathfrak{u}_\eta = \dim \mathfrak{u}'_\eta.$

Since  $\mathfrak{p}_0$  and  $\mathfrak{p}'_0$  are associate parabolic subalgebras of  $\mathfrak{p}''_0$  sharing the same Levi subalgebra  $\mathfrak{m}_0$ , their unipotent radicals  $\mathfrak{u}_0$  and  $\mathfrak{u}'_0$  have the same dimension. This proves (f).

Now we show (g). By 10.2(i), we have

$$\dim \mathfrak{u}_\eta - \dim \mathfrak{u}'_\eta = \sum_{\alpha \in X_Z; \langle \varpi: \alpha \rangle > 0, \langle \varpi': \alpha \rangle < 0} \dim \mathfrak{g}_\delta^{\alpha, 2} - \sum_{\alpha \in X_Z; \langle \varpi: \alpha \rangle < 0, \langle \varpi': \alpha \rangle > 0} \dim \mathfrak{g}_\delta^{\alpha, 2}.$$

We want to show the above difference is zero. For this, it suffices to show that  $\dim \mathfrak{g}_\delta^{\alpha, 2} = \dim \mathfrak{g}_\delta^{-\alpha, 2}$  for any  $\alpha \in X_Z$ . Note that  $\mathfrak{g}_\delta^{\alpha, 2} = \tilde{l}^\phi_\eta \cap \mathfrak{g}^\alpha$ , which is the  $\alpha$ -weight space of  $Z$  on  $\tilde{l}^\phi_\eta$ . By the property of the  $\mathfrak{sl}_2$ -action on  $\tilde{l}^\phi$  given by the

triple  $\phi = (e, h, f)$ , we have that  $\text{ad}(e)$  induces an isomorphism  $\tilde{\mathfrak{l}}_{-\eta}^\phi \cong \tilde{\mathfrak{l}}_\eta^\phi$ . Since  $Z$  commutes with  $e$ , this isomorphism restricts to an isomorphism of  $\alpha$ -weight spaces  $\tilde{\mathfrak{l}}_{-\eta}^\phi \cap \mathfrak{g}^\alpha \cong \tilde{\mathfrak{l}}_\eta^\phi \cap \mathfrak{g}^\alpha$ , that is,  $\mathfrak{g}_{-\delta}^{\alpha, -2} \cong \mathfrak{g}_\delta^{\alpha, 2}$ . Therefore  $\dim \mathfrak{g}_\delta^{\alpha, 2} = \dim \mathfrak{g}_{-\delta}^{\alpha, -2} = \dim \mathfrak{g}_\delta^{-\alpha, 2}$ , as desired. This proves (g) and finishes the proof of (e).

10.7. Let

$$\begin{aligned} \mathring{\mathbf{E}} &= \mathbf{E} - \cup_{(\alpha, n) \in \mathcal{R}_\delta^*; n \neq 2} \mathfrak{H}_{\alpha, n, \eta} \\ &= \{\varpi \in \mathbf{E}; \langle \varpi : \alpha \rangle + n - 2 \neq 0 \quad \forall (\alpha, n) \in \mathcal{R}_\delta^* \text{ such that } n \neq 2\}. \end{aligned}$$

Note that  $\mathbf{E}' \subset \mathbf{E}'' \subset \mathring{\mathbf{E}}$ . For  $'\varpi, ''\varpi$  in  $\mathring{\mathbf{E}}$  we say that  $'\varpi \sim ''\varpi$  if for any  $(\alpha, n) \in \mathcal{R}_\delta^*$  such that  $n \neq 2$  we have

$$(\langle '\varpi : \alpha \rangle + n - 2)(\langle ''\varpi : \alpha \rangle + n - 2) > 0.$$

This is an equivalence relation on  $\mathring{\mathbf{E}}$ . We show:

(a) Assume that  $'\varpi, ''\varpi$  in  $\mathbf{E}'$  satisfy  $'\varpi \sim ''\varpi$ . Then  $I_{'\varpi} \cong I_{''\varpi}$  and  $h('\varpi) = h(''\varpi)$ ; hence  $\tilde{I}_{'\varpi} \cong \tilde{I}_{''\varpi}$ .

By a known property of hyperplane arrangements, we can find a sequence  $\varpi_0, \varpi_1, \dots, \varpi_k$  in  $\mathbf{E}'$  such that  $\varpi_0 = '\varpi, \varpi_k = ''\varpi$  and such that for any  $j \in \{0, 1, \dots, k-1\}$  one of the following holds:

- $\varpi_j \equiv \varpi_{j+1}$ ;
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$  are as in 10.6(c);
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$  are as in 10.6(d);
- $\varpi_j = \varpi, \varpi_{j+1} = \varpi'$  are as in 10.6(e).

Using 10.2(1) or 10.6(c) or 10.6(d) or 10.6(e) we see that for  $j \in \{0, 1, \dots, k-1\}$  we have  $I_{\varpi_j} \cong I_{\varpi_{j+1}}$  and  $h(\varpi_j) = h(\varpi_{j+1})$ . This proves (a).

The set of equivalence classes on  $\mathring{\mathbf{E}}$  for  $\sim$  is denoted by  $\mathring{\mathbf{E}}$ ; it is a finite set.

10.8. For  $\varpi, \varpi'$  in  $\mathbf{E}$  we set

$$\begin{aligned} \tau(\varpi, \varpi') &= \sum_{(\alpha, n) \in \mathcal{R}_\delta^*; (\langle \varpi : \alpha \rangle + n - 2)(\langle \varpi' : \alpha \rangle + n - 2) < 0} \dim \mathfrak{g}_\delta^{\alpha, n}, \\ \text{(a)} \quad &- \sum_{(\alpha, n) \in \mathcal{R}_0^*; (\langle \varpi : \alpha \rangle + n)(\langle \varpi' : \alpha \rangle + n) < 0} \dim \mathfrak{g}_0^{\alpha, n}. \end{aligned}$$

Using 10.2(i) we see that when  $\varpi, \varpi'$  are in  $\mathbf{E}'$ , then

$$\text{(b)} \quad \tau(\varpi, \varpi') = -\dim \frac{\epsilon \mathfrak{u}_0^{\varpi} + \epsilon \mathfrak{u}_0^{\varpi'}}{\epsilon \mathfrak{u}_0^{\varpi} \cap \epsilon \mathfrak{u}_0^{\varpi'}} + \dim \frac{\epsilon \mathfrak{u}_\eta^{\varpi} + \epsilon \mathfrak{u}_\eta^{\varpi'}}{\epsilon \mathfrak{u}_\eta^{\varpi} \cap \epsilon \mathfrak{u}_\eta^{\varpi'}}.$$

10.9. We define  $G_\phi, M_\phi$  as in [LY, 3.6]. We show that:

(a) The obvious map  $M_\phi/M_\phi^0 \rightarrow (G_0 \cap G_\phi)/(G_0 \cap G_\phi)^0$  is an isomorphism.

Recall that  $\phi = (e, h, f)$  with  $e \in \mathring{\mathfrak{m}}_\eta$ . Let  $U$  (resp.  $U'$ ) be the unipotent radical of  $G(e)$  (resp.  $(M \cap G(e))^0$ ). We have  $G(e) = G_\phi U$  (semidirect product). Taking fixed point sets of  $\vartheta$  we obtain  $G_0 \cap G(e) = (G_0 \cap G_\phi)(G_0 \cap U)$  (semidirect product). We have  $M \cap G(e) = M_\phi U'$  (semidirect product). Taking fixed point set of  $\iota(t)$  for all  $t \in \mathbf{k}^*$  we obtain  $M_0 \cap G(e) = (M_0 \cap M_\phi)(M_0 \cap U') = M_\phi(M_0 \cap U')$  (semidirect product). (We have used that  $M_\phi \subset M_0$ .) It follows that we have canonically

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = M_\phi/M_\phi^0,$$

$$(G_{\underline{0}} \cap G(e))/(G_{\underline{0}} \cap G(e))^0 = (G_{\underline{0}} \cap G_{\phi})/(G_{\underline{0}} \cap G_{\phi})^0.$$

It remains to use that

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = (G_{\underline{0}} \cap G(e))/(G_{\underline{0}} \cap G(e))^0;$$

see [LY, 3.8(a)].

Recall from [LY, 3.6] that  $Z$  is a maximal torus of  $(G_{\phi} \cap G_{\underline{0}})^0$ . Let  $H$  be the normalizer of  $Z$  in  $(G_{\phi} \cap G_{\underline{0}})^0$ . Let  $H' = \{g \in G_{\underline{0}}; \text{Ad}(g)M = M, \text{Ad}(g)\mathfrak{m}_k = \mathfrak{m}_k \ \forall k \in \mathbf{Z}\}$ . Note that  $M_0$  is a normal subgroup of  $H'$ . Hence the groups  $H/Z$ ,  $H'/M_0$  are defined. We show:

(b)  $H \subset H'$ .

Let  $g \in (G_{\phi} \cap G_{\underline{0}})^0$  be such that  $gZg^{-1} = Z$ . Let

$$(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*) = (gMg^{-1}, gM_0g^{-1}, \text{Ad}(g)\mathfrak{m}, \text{Ad}(g)\mathfrak{m}_*).$$

Note that  $\mathcal{Z}_{M'}^0 = \mathcal{Z}_M^0 = Z$ . Repeating the argument in the last paragraph in the proof of [LY, 3.6(a)], we see that

$$(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*).$$

(The argument is applicable since  $g \in G_{\phi} \cap G_{\underline{0}}$ .) We see that  $g \in H'$ ; this proves (b).

We show:

(c)  $H \cap M_0 = Z$ .

From the injectivity of the map in (a) we see that  $M_{\phi} \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0$ . Hence we have

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset (M_0 \cap G_{\phi}) \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset M_{\phi} \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0,$$

so that

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 \subset M_{\phi}^0.$$

The opposite inclusion is also true since  $M_{\phi} \subset M_0$  and  $M_{\phi}^0 \subset G_{\underline{0}} \cap G_{\phi}$ . It follows that

$$M_0 \cap (G_{\underline{0}} \cap G_{\phi})^0 = M_{\phi}^0 = Z.$$

The last equality is because  $e$  is distinguished in  $\mathfrak{m}$ . Now  $H \cap M_0$  is the normalizer of  $Z$  in  $M_0 \cap (G_{\phi} \cap G_{\underline{0}})^0$ , that is, the normalizer of  $Z$  in  $Z$ . We see that  $H \cap M_0 = Z$ . This proves (c).

We show:

(d)  $H' = M_0H$ .

Since  $H \subset H'$ ,  $M_0 \subset H'$ , we have  $M_0H \subset H'$ . Now let  $g \in H'$ . We show that  $g \in M_0H$ . Let  $\phi' = (\text{Ad}(g)e, \text{Ad}(g)h, \text{Ad}(g)f)$ . We have

$$\text{Ad}(g)e \in \overset{\circ}{\mathfrak{m}}_{\eta}, \text{Ad}(g)h \in \mathfrak{m}_0, \text{Ad}(g)f \in \mathfrak{m}_{-\eta}.$$

Since both  $\text{Ad}(g)e, e$  are in  $\overset{\circ}{\mathfrak{m}}_{\eta}$ , we can find  $g_1 \in M_0$  such that  $\text{Ad}(g_1)\text{Ad}(g)e = e$ . Replacing  $g$  by  $g_1g$  we can assume that we have  $\text{Ad}(g)e = e$ . Using [L4, 3.3] for  $J^M$ , we can find  $g_2 \in M_0$  such that

$$(\text{Ad}(g_2)\text{Ad}(g)e, \text{Ad}(g_2)\text{Ad}(g)h, \text{Ad}(g_2)\text{Ad}(g)f) = (e, h, f).$$

We have  $g_2g \in G_{\phi}$ . Replacing  $g$  by  $g_2g$  we can assume that  $g \in G_{\underline{0}} \cap G_{\phi}$ . Using the surjectivity of the map in (a) we see that:

$$G_{\underline{0}} \cap G_{\phi} \subset M_{\phi}(G_{\underline{0}} \cap G_{\phi})^0.$$

Thus we can write  $g$  in the form  $g_3g'$  with  $g_3 \in M_\phi, g' \in (G_\underline{0} \cap G_\phi)^0$ . Replacing  $g$  by  $g_3^{-1}g$  we see that we can assume that  $g \in (G_\underline{0} \cap G_\phi)^0$ . Since  $\text{Ad}(g)M = M$ , we see that  $\text{Ad}(g)Z = Z$ . Thus  $g \in H$ . This proves (d).

From (b),(c),(d) we see that:

(e) *The inclusion  $H \subset H'$  induces an isomorphism  $H/Z \xrightarrow{\sim} H'/M_0$ . In particular,  $M_0$  is the identity component of  $H'$ .*

10.10. Let  $g \in H'$  (notation of 10.9). Then  $\text{Ad}(g)$  restricts to an isomorphism  $\mathfrak{m}_\eta \xrightarrow{\sim} \mathfrak{m}_\eta$ . Let  $\tilde{C}' = \text{Ad}(g)^*\tilde{C}$ , a simple perverse sheaf in  $\mathcal{Q}(\mathfrak{m}_\eta)$ . We show:

(a)  $\tilde{C}' \cong \tilde{C}$ .

Using 10.9(e) we can assume that  $g \in H$  (notation of 10.9). Since  $\tilde{C}, \tilde{C}'$  are intersection cohomology complexes attached to  $M_0$ -equivariant irreducible local systems on  $\mathfrak{m}_\eta$ , they correspond to irreducible representations of

$$(M_0 \cap G(e))/(M_0 \cap G(e))^0 = M_\phi/M_\phi^0.$$

Hence it is enough to show that  $\text{Ad}(g)$  induces the identity automorphism of  $M_\phi/M_\phi^0$ . Using 10.9(a), we see that it is enough to show that  $\text{Ad}(g)$  induces the identity automorphism of  $(G_\underline{0} \cap G_\phi)/(G_\underline{0} \cap G_\phi)^0$ . This is obvious since  $g \in (G_\underline{0} \cap G_\phi)^0$ . This proves (a).

10.11. Let  $\varpi, \varpi' \in \mathbf{E}'$ . Recall that  $P_0 = e^{\epsilon \mathfrak{p}_0^{\varpi}} \subset G_\underline{0}, P'_0 = e^{\epsilon \mathfrak{p}'_0} \subset G_\underline{0}$  are parabolic subgroups of  $G_\underline{0}$  with a common Levi subgroup  $M_0$ . Let  $U_0 = U_{P_0}, U'_0 = U_{P'_0}$ . Let  $X$  be the set of all  $g \in G_\underline{0}$  such that  $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$  and  $\mathfrak{p}_*^{\varpi'}$  have a common splitting (as in 6.3). Note that  $X$  is a union of  $(P'_0, P_0)$ -double cosets in  $G_\underline{0}$ . We show:

(a) *We have  $H' \subset X$  (notation of 10.9). Let  $j : H'/M_0 \rightarrow P'_0 \backslash X / P_0$  be the map induced by the inclusion  $H' \rightarrow X$ . Then  $j$  is a bijection.*

If  $g \in H'$ , then  $g \in G_\underline{0}$  and  $\text{Ad}(g)\mathfrak{m}_* = \mathfrak{m}_*$  hence  $\mathfrak{m}_*$  is a common splitting of  $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$  and  $\mathfrak{p}_*^{\varpi'}$ . Thus we have  $g \in X$ , so that the inclusion  $H' \subset X$  holds. Let  $g \in X$ . Then  $g \in G_\underline{0}$  and  $\text{Ad}(g)\mathfrak{p}_*^{\varpi}, \mathfrak{p}_*^{\varpi'}$  have a common splitting  $\mathfrak{m}'_*$ . Then  $\text{Ad}(g)\mathfrak{m}_*, \mathfrak{m}'_*$  are splittings of  $\text{Ad}(g)\mathfrak{p}_*^{\varpi}$  hence, by [LY, 2.7(a)], we have  $\text{Ad}(gug^{-1})\text{Ad}(g)\mathfrak{m}_* = \mathfrak{m}'_*$  for some  $u \in U_0$ . Moreover,  $\mathfrak{m}_*, \mathfrak{m}'_*$  are splittings of  $\mathfrak{p}_*^{\varpi'}$  hence, by [LY, 2.7(a)], we have  $\text{Ad}(u')\mathfrak{m}_* = \mathfrak{m}'_*$  for some  $u' \in U'_0$ . It follows that  $\text{Ad}(gu)\mathfrak{m}_* = \text{Ad}(u')\mathfrak{m}_*$  hence  $u'^{-1}gu \in H'$ . Since  $U_0 \subset P_0, U'_0 \subset P'_0$ , we see that  $j$  is surjective. It remains to show that  $j$  is injective. Let  $g, g'$  be elements of  $H'$  such that  $g' = p'_0gp_0$  for some  $p_0 \in P_0, p'_0 \in P'_0$ . We must only show that  $g' \in gM_0$ . Let  $NM_0$  be the normalizer of  $M_0$  in  $G_\underline{0}$ . It is enough to show that the obvious map  $NM_0/M_0 \rightarrow P'_0 \backslash G_\underline{0} / P_0$  is injective. This is a well-known property of parabolic subgroups and their Levi subgroups in a connected reductive group. This completes the proof of (a).

Let  $\mathcal{W} = H/Z$  (notation of 10.9). Let  $w \in \mathcal{W}$  and let  $g \in H$  be a representative of  $w$ . Now  $\text{Ad}(g)$  restricts to an automorphism of  $Z$  which depends only on  $w$ ; this induces an isomorphism  $Y_Z \xrightarrow{\sim} Y_Z$  and, by extension of scalars, a vector space isomorphism  $\mathbf{E} \xrightarrow{\sim} \mathbf{E}$  denoted by  $\varpi_1 \mapsto w\varpi_1$ . For any  $(\alpha, n, i) \in X_Z \times \mathbf{Z} \times \mathbf{Z}/m$ ,  $\text{Ad}(g)$  defines an isomorphism  $\mathfrak{g}_i^{\alpha, n} \xrightarrow{\sim} \mathfrak{g}_i^{w\alpha, n}$  where  $w\alpha \in X_Z$  is given by  $w\alpha(z) = \alpha(w^{-1}(z))$ ; hence for any  $i, (\alpha, n) \mapsto (w\alpha, n)$  is a bijection  $\mathcal{R}_i \xrightarrow{\sim} \mathcal{R}_i$ . Moreover, for any  $N \in \mathbf{Z}$  and any  $(\alpha, n) \in \mathcal{R}_N^*$ ,  $w : \mathbf{E} \rightarrow \mathbf{E}$  restricts to a bijection from the affine hyperplane  $\mathfrak{H}_{\alpha, n, N}$  to the affine hyperplane  $\mathfrak{H}_{w\alpha, n, N}$ . It follows that  $w : \mathbf{E} \rightarrow \mathbf{E}$  restricts to a bijection  $\mathbf{E}' \xrightarrow{\sim} \mathbf{E}'$ .

We show:

(b) For any  $\varpi \in \mathbf{E}'$  we have  $\text{Ad}(g)({}^\epsilon \mathfrak{p}_*^{\varpi}) = {}^\epsilon \mathfrak{p}_*^{w\varpi}$ .

From 10.2(h) we have

$$\begin{aligned} \text{Ad}(g)({}^\epsilon \mathfrak{p}_N^{\varpi}) &= \bigoplus_{(\alpha,n) \in \mathcal{R}_N; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} \text{Ad}(g) \mathfrak{g}_N^{\alpha,n} \\ &= \bigoplus_{(\alpha,n) \in \mathcal{R}_N; \langle \varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{w\alpha,n}), \\ {}^\epsilon \mathfrak{p}_N^{w\varpi} &= \bigoplus_{(\alpha,n) \in \mathcal{R}_N; \langle w\varpi : \alpha \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{\alpha,n}) \\ &= \bigoplus_{(\alpha',n) \in \mathcal{R}_N; \langle w\varpi : w\alpha' \rangle \geq 2N/\eta - n} (\mathfrak{g}_N^{w\alpha',n}). \end{aligned}$$

It remains to use that  $\langle \varpi : \alpha \rangle = \langle w\varpi : w\alpha \rangle$ .

10.12. For  $\varpi_1, \varpi_2$  in  $\mathbf{E}$  we set

$$(a) \quad [\varpi_1 | \varpi_2] = (1 - v^2)^{-\dim Z} \sum_{w \in \mathcal{W}} v^{\tau(\varpi_2, w\varpi_1)} \in \mathbf{Q}(v).$$

(Here  $v$  is an indeterminate and  $\tau(\varpi_2, w\varpi_1) \in \mathbf{Z}$  is as in 10.8.) When  $\varpi_1, \varpi_2$  are in  $\mathbf{E}'$  we have:

$$(b) \quad \sum_{j \in \mathbf{Z}} d_j(\mathfrak{g}_\delta; \tilde{I}_{\varpi_1}, D(\tilde{I}_{\varpi_2})) v^{-j} = [\varpi_1 | \varpi_2].$$

This can be deduced from [LY, 6.4] as follows. The set  $X$  in [LY, 6.3] is described in our case in 10.11(a) in terms of the group  $H'/M_0$  which, in turn, is identified in 10.9(e) with  $\mathcal{W} = H/Z$ ; the integers  $\tau(g)$  in 6.3 are identified with the integers  $\tau(\varpi_2, w\varpi_1) \in \mathbf{Z}$  by 10.11(b). Finally, the set  $X'$  in [LY, 6.4] coincides with  $X$  in [LY, 6.4] by 10.10(a).

From the definitions we have  $\tau(\varpi_2, w\varpi_1) = \tau(w^{-1}\varpi_2, \varpi_1) = \tau(\varpi_1, w^{-1}\varpi_2)$  for any  $w \in \mathcal{W}$ . It follows that

$$(c) \quad [\varpi_1 | \varpi_2] = [\varpi_2 | \varpi_1].$$

Let  $\mathbf{c}_1$  (resp.  $\mathbf{c}_2$ ) be the equivalence class for  $\sim$  in  $\mathring{\mathbf{E}}$  that contains  $\varpi_1$  (resp.  $\varpi_2$ ). Using 10.7(a) we see that the right hand side of (b) depends only on  $\mathbf{c}_1, \mathbf{c}_2$  and not on the specific elements  $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}', \varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}'$ . Hence for  $\mathbf{c}_1, \mathbf{c}_2$  in  $\mathring{\mathbf{E}}$  we can set  $[\mathbf{c}_1 | \mathbf{c}_2] = [\varpi_1 | \varpi_2] \in \mathbf{Q}(v)$  for any  $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}', \varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}'$ .

10.13. Let  $\mathfrak{B}$  (resp.  ${}^\xi \mathfrak{B}$ ) be the set of (isomorphism classes of) simple perverse sheaves in  $\mathcal{Q}(\mathfrak{g}_\delta)$  (resp.  ${}^\xi \mathcal{Q}(\mathfrak{g}_\delta)$ ). For any  $G_0$ -orbit  $\mathcal{O}$  in  $\mathfrak{g}_\delta^{nil}$  let  $\mathfrak{B}_\mathcal{O}$  be the set of all  $B \in \mathfrak{B}$  such that the support of  $B$  is equal to the closure of  $\mathcal{O}$ . We have  $\mathfrak{B} = \sqcup_{\mathcal{O}} \mathfrak{B}_\mathcal{O}$ . We define a map  $\kappa : \mathfrak{B} \rightarrow \mathbf{N}$  by  $\kappa(B) = \dim \mathcal{O}$  where  $B \in \mathfrak{B}_\mathcal{O}$ .

10.14. Let  $\mathbf{V}'$  be the  $\mathbf{Q}(v)$ -vector space with basis  $\{\tilde{T}_{\mathbf{c}}; \mathbf{c} \in \mathring{\mathbf{E}}\}$ . On  $\mathbf{V}'$  we have a unique pairing  $(: ) : \mathbf{V}' \times \mathbf{V}' \rightarrow \mathbf{Q}(v)$  which is  $\mathbf{Q}(v)$ -linear in the first argument,  $\mathbf{Q}(v)$ -antilinear in the second argument (for  $f \mapsto \bar{f}$ ) and such that for  $\mathbf{c}_1, \mathbf{c}_2$  in  $\mathring{\mathbf{E}}$  we have  $(\tilde{T}_{\mathbf{c}_1} : \tilde{T}_{\mathbf{c}_2}) = [\mathbf{c}_1 | \mathbf{c}_2]$  (see 10.12).

Setting

$$\begin{aligned} \mathfrak{R}_l &= \{x \in \mathbf{V}'; (x : x') = 0 \quad \forall x' \in \mathbf{V}'\}, \\ \mathfrak{R}_r &= \{x \in \mathbf{V}'; (x' : x) = 0 \quad \forall x' \in \mathbf{V}'\}, \end{aligned}$$

we state the following.

**Lemma 10.15.** *We have  $\mathfrak{R}_l = \mathfrak{R}_r$ .*

The proof is given in 10.17.

10.16. We define a  $\mathbf{Q}$ -linear map

$$\tilde{\gamma} : \mathbf{V}' \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$$

by  $\tilde{T}_{\mathbf{c}} \mapsto \tilde{T}_{\varpi}$ , where  $\varpi$  is an element of  $\mathfrak{c} \cap \mathbf{E}'$ . Now  $\tilde{\gamma}$  is well-defined by 10.7(a) and is surjective by [LY, 8.4(b)], 10.4(b). We define a pairing

$$(\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_{\delta})) \times (\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})) \rightarrow \mathbf{Q}((v))$$

(denoted by  $(:)$ ) by requiring that it is  $\mathbf{Q}(v)$ -linear in the first argument,  $\mathbf{Q}(v)$ -antilinear in the second argument (for  $f \mapsto \bar{f}$ ) and that its restriction

$$\mathcal{K}(\mathfrak{g}_{\delta}) \times \mathcal{K}(\mathfrak{g}_{\delta}) \rightarrow \mathbf{Q}((v))$$

is the same as the restriction of the pairing in [LY, 4.4(c)]. This restricts to a pairing

$$(\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})) \times (\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})) \rightarrow \mathbf{Q}((v))$$

(denoted again by  $(:)$ ). We show that for  $b, b'$  in  $\mathbf{V}'$  we have

$$(a) \quad (\tilde{\gamma}(b) : \tilde{\gamma}(b')) = (b : b')$$

We can assume that  $b = \tilde{T}_{\mathbf{c}}, b' = \tilde{T}_{\mathbf{c}'}$  with  $\mathbf{c}, \mathbf{c}'$  in  $\overset{\circ}{\mathbf{E}}$ . We must show that  $\{\tilde{T}_{\varpi}, D(\tilde{T}_{\varpi'})\} = [\varpi | \varpi']$  where  $\varpi \in \mathfrak{c} \cap \mathbf{E}', \varpi' \in \mathfrak{c}' \cap \mathbf{E}'$ . This follows from 10.12(b). This proves (a).

10.17. Let

$$\begin{aligned} {}'\mathfrak{R}_l &= \{z \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta}); (z : z') = 0 \quad \forall z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})\}, \\ {}'\mathfrak{R}_r &= \{z \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta}); (z' : z) = 0 \quad \forall z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})\}. \end{aligned}$$

We show:

$$(a) \quad {}'\mathfrak{R}_l = {}'\mathfrak{R}_r = 0.$$

Now  $\mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  has a  $\mathbf{Q}(v)$ -basis formed by  ${}^{\xi}\mathfrak{B} = \{B_1, B_2, \dots, B_r\}$ . From [LY, 0.12] we see that  $(B_j : B_{j'}) \in \delta_{j,j'} + v\mathbf{N}[[v]]$  for  $j, j'$  in  $[1, r]$ .

Assume that  $\beta = \sum_{j \in [1, r]} f_j B_j \in {}'\mathfrak{R}_l$ , where  $f_j \in \mathbf{Q}(v)$  are not all zero. We must show that this is a contradiction. We can assume that  $f_j \in \mathbf{Q}[v]$  for all  $j$  and  $f_{j_0} - c_0 \in v\mathbf{Q}[v]$  for some  $j_0 \in [1, r]$  and some  $c_0 \in \mathbf{Q} - \{0\}$ . Then  $0 = (\beta : B_{j_0}) \in c_0 + v\mathbf{Q}[[v]]$ , a contradiction. This proves that  $'\mathfrak{R}_l = 0$ .

Next we assume that  $\beta = \sum_{j \in [1, r]} f_j B_j \in {}'\mathfrak{R}_r$ , where  $f_j \in \mathbf{Q}(v)$  are not all zero. We must show that this is a contradiction. We can assume that  $\bar{f}_j \in \mathbf{Q}[v]$  for all  $j$  and  $\bar{f}_{j_0} - c_0 \in v\mathbf{Q}[v]$  for some  $j_0 \in [1, r]$  and some  $c_0 \in \mathbf{Q} - \{0\}$ . Then  $0 = (B_{j_0} : \beta) \in c_0 + v\mathbf{Q}[[v]]$ , a contradiction. This proves that  $'\mathfrak{R}_r = 0$ . This proves (a).

We show:

$$(b) \quad \mathfrak{R}_l = \tilde{\gamma}^{-1}({}'\mathfrak{R}_l).$$

Let  $x \in \mathfrak{R}_l$ . From 10.16(a) we see that  $(\tilde{\gamma}(x) : \tilde{\gamma}(x')) = 0$  for any  $x' \in \mathbf{V}'$ . Since  $\tilde{\gamma}$  is surjective, it follows that  $(\tilde{\gamma}(x) : z') = 0$  for any  $z' \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$ . Thus,  $\tilde{\gamma}(x) \in {}'\mathfrak{R}_l$ . Conversely, assume that  $x \in \mathbf{V}'$  and  $\tilde{\gamma}(x) \in {}'\mathfrak{R}_l$ . From 10.16(a) we see that for any  $x' \in \mathbf{V}'$  we have  $(x : x') = (\tilde{\gamma}(x) : \tilde{\gamma}(x')) = 0$ . Thus  $x \in \mathfrak{R}_l$ . This proves (b).

An entirely similar proof shows that:

(c)  $\mathfrak{R}_r = \tilde{\gamma}^{-1}(\mathfrak{R}_r)$ .

From (a),(b),(c) we see that  $\mathfrak{R}_l = \mathfrak{R}_r = \tilde{\gamma}^{-1}(0)$ . This proves Lemma 10.15.

10.18. **Definition of  $\mathbf{V}$ .** We define  $\mathbf{V} = \mathbf{V}'/\mathfrak{R}_l = \mathbf{V}'/\mathfrak{R}_r$  (see Lemma 10.15). Note that  $(:)$  on  $\mathbf{V}'$  induces a pairing  $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{Q}(v)$  (denoted again by  $(:)$ ) which is  $\mathbf{Q}(v)$ -linear in the first argument,  $\mathbf{Q}(v)$ -antilinear in the second argument (for  $f \mapsto \bar{f}$ ).

10.19. From the proof of Lemma 10.15 we see that  $\tilde{\gamma}$  induces a  $\mathbf{Q}(v)$ -linear isomorphism

(a) 
$$\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$$

and that for  $b, b'$  in  $\mathbf{V}$  we have

(b) 
$$(\gamma(b) : \gamma(b')) = (b : b').$$

From (a) we deduce the following result.

**Proposition 10.20.** *The number of simple perverse sheaves (up to isomorphism) in  ${}^\xi \mathcal{Q}(\mathfrak{g}_\delta)$  is equal to  $\dim_{\mathbf{Q}(v)} \mathbf{V}$ .*

10.21. We define a  $\mathbf{Q}$ -linear involution  $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$  by  $\overline{f\tilde{T}_c} = \bar{f}\tilde{T}_c$  for any  $f \in \mathbf{Q}(v)$ ,  $\mathbf{c} \in \mathring{\mathbf{E}}$ ; here  $\bar{f}$  is as in [LY, 0.12]. We show:

(a) *For any  $x, x'$  in  $\mathbf{V}'$  we have  $(x : x') = (\bar{x}' : \bar{x})$ .*

We can assume that  $x = \tilde{T}_c, x' = \tilde{T}_{c'}$  for some  $\mathbf{c}, \mathbf{c}'$  in  $\mathring{\mathbf{E}}$ . We must show that

$$[\mathbf{c}|\mathbf{c}'] = [\mathbf{c}'|\mathbf{c}],$$

which follows directly from 10.12(c). This proves (a).

We show:

(b)  $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$  preserves  $\mathfrak{R}_l = \mathfrak{R}_r$  (see Lemma 10.15) hence it induces an  $\mathbf{Q}(v)$ -semilinear involution  $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$  (with respect to  $f \mapsto \bar{f}$ ).

Assume that  $x \in \mathfrak{R}_l$ . We have  $(x : x') = 0$  for all  $x' \in \mathbf{V}$ . Hence, by (a), we have  $(\bar{x}' : \bar{x}) = 0$  for all  $x' \in \mathbf{V}$ . Since  $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$  is surjective, it follows that  $\bar{x} \in \mathfrak{R}_r$ . This proves (b).

10.22. Now let  $\eta_1 \in \mathbf{Z} - \{0\}$  be such that  $\underline{\eta}_1 = \delta$ . Recall that in 10.1 we have fixed  $\xi \in \underline{\mathfrak{T}}_\eta$  and a representative  $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$  for  $\xi$ . Let  $\dot{\xi}_1 = (M, M_0, \mathfrak{m}, \mathfrak{m}_{(*)}, \tilde{C}) \in \mathfrak{T}_{\eta_1}$  be as in [LY, 3.9] and let  $\xi_1 \in \underline{\mathfrak{T}}_{\eta_1}$  be the  $G_0$ -orbit of  $\dot{\xi}_1$ . Note that the  $\mathbf{Q}$ -vector space  $\mathbf{E}$  defined as in 10.1 in terms of  $\dot{\xi}$  is the same as  $\mathbf{E}$  defined in terms of  $\dot{\xi}_1$ . Moreover, the subset  $\mathring{\mathbf{E}}$ , the equivalence relation  $\sim$  on it, and the set  $\mathring{\mathbf{E}}$  defined as in 10.7 in terms of  $\dot{\xi}$  are the same as the analogous objects defined in terms of  $\dot{\xi}_1$ . Also, the pairings  $\tau : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{Z}$  and  $[?|?] : \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{Q}(v)$  defined in 10.8 and 10.12 in terms of  $\dot{\xi}$  are the same as those defined in terms of  $\dot{\xi}_1$ .

The subset  $\mathbf{E}'$  of  $\mathbf{E}$  defined as in 10.1 in terms of  $\dot{\xi}$  is not in general the same as the analogous subset  $\mathbf{E}'_1$  of  $\mathbf{E}$  defined in terms of  $\dot{\xi}_1$ . However, for  $\mathbf{c}_1, \mathbf{c}_2$  in  $\mathring{\mathbf{E}}$ , the quantity  $[\mathbf{c}_1|\mathbf{c}_2] \in \mathbf{Q}(v)$  defined in 10.12 in terms of  $\dot{\xi}$  is the same as that defined in terms of  $\dot{\xi}_1$ . (It is equal to  $[\varpi_1|\varpi_2]$  for any  $\varpi_1 \in \mathbf{c}_1 \cap \mathbf{E}' \cap \mathbf{E}'_1, \varpi_2 \in \mathbf{c}_2 \cap \mathbf{E}' \cap \mathbf{E}'_1$ . Hence the vector space  $\mathbf{V}'$ , the pairing  $(:)$  on it and its quotient  $\mathbf{V}$  defined in 10.14

and 10.18 in terms of  $\dot{\xi}$  are the same as those defined in terms of  $\dot{\xi}_1$ . The involutions  $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$ ,  $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$  defined in 10.21 in terms of  $\dot{\xi}$  are the same as those defined in terms of  $\dot{\xi}_1$ .

11. THE  $\mathcal{A}$ -LATTICE  $\mathbf{V}_{\mathcal{A}}$

In this section we give a combinatorial definition of an  $\mathcal{A}$ -lattice  $\mathbf{V}_{\mathcal{A}}$  in the  $\mathbf{Q}(v)$ -vector space  $\mathbf{V}$  and a signed basis  $\mathbf{B}'$  of it. It turns out that the  $\mathbf{Z}/2$ -orbits on  $\mathbf{B}'$  for the  $\mathbf{Z}/2$ -action  $b \mapsto -b$  are in natural bijection with the simple perverse sheaves in the block  ${}^{\xi}\mathcal{Q}(\mathfrak{g}_{\delta})$ .

11.1. In this section (except in 11.5, 11.6, 11.13) we preserve the setup and notation of 10.1, 10.2 and assume that  $\varpi \in \mathbf{E}''$  (see 10.3). Let

$$\mathfrak{g}^{\phi} = \{y \in \mathfrak{g}; [y, e] = 0, [y, h] = 0, [y, f] = 0\}.$$

Let  $\mathfrak{z}$  be the center of  $\mathfrak{m}$ . Note that  $\mathfrak{m} \cap \mathfrak{g}^{\phi} = \mathfrak{z}$  (since  $e$  is distinguished in  $\mathfrak{m}$ ) and  $\mathfrak{z}$  is a Cartan subalgebra of the reductive Lie algebra  $\mathfrak{g}^{\phi}$ . (This has already been proved at level of groups in [LY, 3.6].) For any  $\alpha \in X_Z$  let

$$(\mathfrak{g}_{\underline{0}}^{\phi})^{\alpha} = \mathfrak{g}_{\underline{0}}^{\alpha} \cap \mathfrak{g}^{\phi}.$$

Let

$$\mathfrak{g}_{\underline{0}, \varpi}^{\phi} = \bigoplus_{\alpha \in X_Z; \langle \varpi, \alpha \rangle = 0} (\mathfrak{g}_{\underline{0}}^{\phi})^{\alpha}.$$

This is a Levi subalgebra (containing  $\mathfrak{z}$ ) of a parabolic subalgebra of  $\mathfrak{g}_{\underline{0}}^{\phi}$ . Let  $\mathcal{B}$  be the variety of Borel subalgebras of  $\mathfrak{g}_{\underline{0}, \varpi}^{\phi}$ , let  $d(\varpi) = \dim \mathcal{B}$  and let

$$a_{\varpi} = \sum_j v^{-2s_j} v^{d(\varpi)} \in \mathcal{A},$$

where  $\rho_{\mathcal{B}; \bar{\mathbf{Q}}_l} = \bigoplus_j \bar{\mathbf{Q}}_l[-2s_j]$ .

*Erratum to [L4].* On page 202, line 1 of 16.8, replace “algebraic group  $M$ ” by “algebraic group  $M$  with a given Lie algebra homomorphism  $\phi$  from  $\mathfrak{s}$  to the Lie algebra of  $M$ ”.

On page 202, line 4 of 16.8, replace “Borel subgroups of  $M$ ” by “Borel subgroups of the connected centralizer of  $\phi(\mathfrak{s})$  in  $M$ ”.

11.2. The subset

(a)  $\{a_{\varpi}^{-1} \tilde{T}_{\mathbf{c}} \in \mathbf{V}'; \mathbf{c} \in \mathring{\mathbf{E}}, \varpi \in \mathbf{E}'' \cap \mathbf{c}\}$

of  $\mathbf{V}'$  (see 10.14) is finite. Indeed, when  $\mathbf{c}$  is fixed, the subgroup  $\mathfrak{g}_{\underline{0}, \varpi}^{\phi}$  of  $\mathfrak{g}_{\underline{0}}^{\phi}$  takes only finitely many values for  $\varpi$  in  $\mathbf{E}'' \cap \mathbf{c}$  hence  $a_{\varpi}$  takes only finitely many values.

Let  $\mathbf{V}'_{\mathcal{A}}$  be the  $\mathcal{A}$ -submodule of  $\mathbf{V}'$  generated by (a). Let  $\mathbf{V}_{\mathcal{A}}$  be the image of  $\mathbf{V}'_{\mathcal{A}}$  under the obvious linear map  $\mathbf{V}' \rightarrow \mathbf{V}$ . The following result will be proved in 11.8.

(b)  $\mathbf{V}_{\mathcal{A}}$  is a free  $\mathcal{A}$ -module such that the obvious  $\mathbf{Q}(v)$ -linear map  $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathbf{V}_{\mathcal{A}} \rightarrow \mathbf{V}$  is an isomorphism.



11.3. Let  $\mathfrak{h} = \epsilon \tilde{\mathfrak{I}}^{\varpi}$ ,  $\mathfrak{h}^{\phi} = \mathfrak{h} \cap \mathfrak{g}^{\phi}$ . We show:

(a) 
$$\mathfrak{g}_{\underline{0}, \varpi}^{\phi} = \mathfrak{h}^{\phi}.$$

From 10.3(b) we have

$$\mathfrak{h} = \bigoplus_{N \in \mathbf{Z}, (\alpha, n) \in \tilde{\mathcal{R}}_N; n=2N/\eta, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_N^{\alpha, n}).$$

Using this and the definitions we have

$$\begin{aligned} \{y \in \mathfrak{h}; [y, h] = 0\} &= \bigoplus_{N \in \mathbf{Z}, (\alpha, n) \in \tilde{\mathcal{R}}_N; n=2N/\eta=0, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_N^{\alpha, n}) \\ &= \bigoplus_{(\alpha, n) \in \mathcal{R}_{\underline{0}}; n=0, \langle \varpi: \alpha \rangle = 0} (\mathfrak{g}_{\underline{0}}^{\alpha, n}) \end{aligned}$$

and (a) follows.

11.4. Recall that  $\varpi \in \mathbf{E}''$ . Let  $\mathfrak{p}_* = \epsilon \mathfrak{p}_*^{\varpi}$ ,  $\mathfrak{u}_* = \epsilon \mathfrak{u}_*^{\varpi}$ . We can find  $\lambda' \in Y_Z$  such that  $\langle \lambda' : \alpha \rangle \neq 0$  for any  $i$  and any  $(\alpha, n) \in \mathcal{R}_i^*$ . Let  $\varpi' = \varpi + \frac{1}{b} \lambda' \in \mathbf{E}$ . Let  $\mathfrak{p}'_* = \epsilon \mathfrak{p}'_*^{\varpi'}$ ,  $\mathfrak{u}'_* = \epsilon \mathfrak{u}'_*^{\varpi'}$  and  $\mathfrak{m}'_* = \epsilon \tilde{\mathfrak{I}}_*^{\varpi'}$ . Assume that  $b$  is sufficiently large; then  $\varpi' \in \mathbf{E}'$  hence by 10.2 we have  $\mathfrak{m}'_* = \mathfrak{m}_*$ . The same argument as in 10.6(a) shows that  $\mathfrak{p}'_N \subset \mathfrak{p}_N$  for all  $N$ . Since  $b$  is large and  $\varpi' = \varpi + \frac{1}{b} \lambda'$ , we see that  $\varpi', \varpi$  are very close, so that

(a) 
$$\varpi' \sim \varpi.$$

For  $N \in \mathbf{Z}$  let  $\mathfrak{q}_N$  be the image of  $\mathfrak{p}'_N$  under the obvious projection  $\mathfrak{p}_N \rightarrow \mathfrak{h}_N$ . From 10.5(a),(b), we see that  $\mathfrak{q} = \bigoplus_N \mathfrak{q}_N$  is a parabolic subalgebra of  $\mathfrak{h}$  and  $\mathfrak{m}$  is a Levi subalgebra of  $\mathfrak{q}$ . Moreover, if  $u_N$  is the image of  $u'_N$  under  $\mathfrak{p}_N$ , then  $u = \bigoplus_N u_N$  is the nilradical of  $\mathfrak{q}$ .

From 10.3(b) we see that the  $\mathbf{Z}$ -grading of  $\mathfrak{h}$  is  $\eta$ -rigid and that  $e \in \mathfrak{h}_{\eta}$ , so that  $\mathring{\mathfrak{m}}_{\eta} \subset \mathring{\mathfrak{h}}_{\eta}$ . Let  $A_{\varpi} \in \mathcal{Q}(\mathfrak{h}_{\eta})$  be the simple perverse sheaf on  $\mathfrak{h}_{\eta}$  such that the support of  $A_{\varpi}$  is  $\mathfrak{h}_{\eta}$  and  $A_{\varpi}|_{\mathring{\mathfrak{m}}_{\eta}}$  is equal up to shift to  $\tilde{C}|_{\mathring{\mathfrak{m}}_{\eta}}$ .

Applying 1.8(b) and the transitivity formula 4.2(a) we deduce

(b) 
$$\begin{aligned} I_{\varpi'} &= \epsilon \widetilde{\text{Ind}}_{\mathfrak{p}'_{\eta}}^{\mathfrak{g}s}(\tilde{C}) = \epsilon \widetilde{\text{Ind}}_{\mathfrak{p}'_{\eta}}^{\mathfrak{g}s}(\text{ind}_{\mathfrak{q}'_{\eta}}^{\mathfrak{h}_{\eta}}(\tilde{C}))[\dim u'_0 + \dim u'_{\eta} - \dim u_0 - \dim u_{\eta}] \\ &\cong \bigoplus_j \epsilon \widetilde{\text{Ind}}_{\mathfrak{p}'_{\eta}}^{\mathfrak{g}s}(A_{\varpi})[-2s_j][\dim \mathfrak{m}_{\eta} - \dim \mathfrak{h}_{\eta} + \dim u'_0 + \dim u'_{\eta} \\ &\quad - \dim u_0 - \dim u_{\eta}] \\ &= \bigoplus_j \epsilon \widetilde{\text{Ind}}_{\mathfrak{p}'_{\eta}}^{\mathfrak{g}s}(A_{\varpi})[-2s_j][d(\varpi)], \end{aligned}$$

where we have used the equality:

(c) 
$$\dim \mathfrak{m}_{\eta} - \dim \mathfrak{h}_{\eta} + \dim u'_0 + \dim u'_{\eta} - \dim u_0 - \dim u_{\eta} = d(\varpi).$$

We now prove (c). Let  $u'$  be the nilradical of the parabolic subalgebra of  $\mathfrak{h}$  that contains  $\mathfrak{m}$  and is opposed to  $\mathfrak{q}$ . We have  $u' = \bigoplus_N u'_N$  where  $u'_N = u' \cap \mathfrak{h}_N$ . Now  $u_0, u'_0$  are nilradicals of two opposite parabolic subalgebras of  $\mathfrak{h}_0$  hence  $\dim u_0 = \dim u'_0$ . We have  $\dim \mathfrak{h}_{\eta} = \dim u_{\eta} + \dim u'_{\eta} + \dim \mathfrak{m}_{\eta}$ ,  $\dim u'_0 - \dim u_0 = \dim u_0$ ,  $\dim u'_{\eta} - \dim u_{\eta} = \dim u_{\eta}$ . Hence the left-hand side of (c) equals  $\dim u_0 - \dim u'_{\eta} = \dim u'_0 - \dim u'_{\eta}$ . Now  $u'$  is normalized by  $\mathfrak{m}$  hence by the Lie subalgebra  $\mathfrak{s}$  of  $\mathfrak{m}$  spanned by  $e, h, f$ . Note that  $u'_0$  (resp.  $u'_{\eta}$ ) is the 0-(resp. 2-) eigenspace of  $\text{ad}(h) : u' \rightarrow u'$ . By the representation theory of  $\mathfrak{s}$ , the map  $\text{ad}(e) : u'_0 \rightarrow u'_{\eta}$  is surjective and its kernel is exactly the space of  $\mathfrak{s}$ -invariants in  $u'$  that is  $u' \cap \mathfrak{h}^{\phi}$ . We see that  $\dim u'_0 - \dim u'_{\eta} = \dim(u' \cap \mathfrak{h}^{\phi})$ . Now  $u' \cap \mathfrak{h}^{\phi}$  is the nilradical of a

parabolic subalgebra of  $\mathfrak{h}^\phi$  with Levi subgroup  $\mathfrak{m} \cap \mathfrak{h}^\phi = \mathfrak{z}$  (see 11.1) hence is the nilradical of a Borel subalgebra of  $\mathfrak{h}^\phi$  (which equals  $\mathfrak{g}_{0,\varpi}^\phi$  by 11.3(a)). By definition, the dimension of this nilradical is equal to  $d(\varpi)$ . This proves (c) and hence also (b).

Now (b) implies the following equality in  ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ :

$$I_{\varpi'} = a_{\varpi}(\epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi})).$$

Using this and (a) we see that for any  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  and any  $\varpi \in \mathbf{E}'' \cap \mathbf{c}$  we have

$$(d) \quad a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}}) = \epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi}) \in \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi\mathcal{K}(\mathfrak{g}_\delta).$$

Since  $\epsilon\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_{\varpi}) \in {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ , we see that:

(e) For any  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  and any  $\varpi \in \mathbf{E}'' \cap \mathbf{c}$  we have  $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}}) \in {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ .

The following result will be proved in 11.7.

(f) The  $\mathcal{A}$ -module  ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$  is generated by the elements  $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}})$  for various  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  and  $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ .

From (f) we deduce:

(g) The isomorphism  $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$  in Proposition 10.19(a) restricts to an isomorphism of  $\mathcal{A}$ -modules  $\gamma_{\mathcal{A}} : \mathbf{V}_{\mathcal{A}} \xrightarrow{\sim} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ .

11.5. Let  $\mathfrak{p}_*$  be an  $\epsilon$ -spiral with a splitting  $\mathfrak{h}_*$  and with nilradical  $\mathfrak{u}_*$ ; let  $A \in \mathcal{Q}(\mathfrak{h}_\eta)$  be a simple perverse sheaf. We show:

(a) Let  $\mathcal{X}$  be the collection of all  $B \in \mathfrak{B}$  such that some shift of  $B$  is a direct summand of  $\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ . Then the map  $\mathcal{X} \rightarrow \mathfrak{T}_\eta$ ,  $B \mapsto \psi(B)$  is constant.

We can find a parabolic subalgebra  $\mathfrak{q}$  of  $\mathfrak{h}$  and a Levi subalgebra  $\mathfrak{m}'$  of  $\mathfrak{q}$  such that  $\mathfrak{q} = \oplus_N \mathfrak{q}_N$ ,  $\mathfrak{m}' = \oplus_N \mathfrak{m}'_N$  where  $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{h}_N$ ,  $\mathfrak{m}'_N = \mathfrak{m}' \cap \mathfrak{h}_N$  and a cuspidal perverse sheaf  $C$  in  $\mathcal{Q}(\mathfrak{m}'_\eta)$  such that some shift of  $A$  is a direct summand of  $\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{h}_\eta}(C)$ . Let  $M' = e^{\mathfrak{m}'}$ ,  $M'_0 = e^{\mathfrak{m}'_0}$ . Setting  $\mathfrak{p}'_N = \mathfrak{u}_N + \mathfrak{q}_N$  for any  $N \in \mathbf{Z}$ , we see from [LY, 2.8(a)] that  $\mathfrak{p}'_*$  is an  $\epsilon$ -spiral and from [LY, 2.8(b)] that  $\mathfrak{m}'_*$  is a splitting of  $\mathfrak{p}'_*$ . We see that  $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C) \in \mathfrak{T}_\eta$ . Let  $\xi'$  be the element of  $\mathfrak{T}_\eta$  determined by  $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C)$ . If  $B \in \mathcal{X}$ , then (by [LY, 4.2]) some shift of  $B$  is a direct summand of  $\widetilde{\text{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C)$  hence  $\psi(B) = \xi'$ . This proves (a).

We say that  $\widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$  (as in (a)) has type  $\xi' \in \mathfrak{T}_\eta$  if  $\psi(B) = \xi'$  for any  $B \in \mathcal{X}$ .

11.6. Recall from 8.4(a) that the  $\mathcal{A}$ -module  $\mathcal{K}(\mathfrak{g}_\delta)$  is generated by the classes of  $\epsilon$ -quasi-monomial objects of  $\mathcal{Q}(\mathfrak{g}_\delta)$ . Using this and 11.5(a), we deduce that in the direct sum decomposition  $\mathcal{K}(\mathfrak{g}_\delta) = \oplus_{\xi \in \mathfrak{T}_\eta} {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$  (see [LY, 6.7]), any summand  ${}^\xi\mathcal{K}(\mathfrak{g}_\delta)$  is generated as an  $\mathcal{A}$ -module by the classes of  $\eta$ -quasi-monomial objects in  $\mathcal{Q}(\mathfrak{g}_\delta)$  of type  $\xi$ .

11.7. We prove 11.4(f). (Thus we are again in the setup of 11.1.) Using 11.6, we see that it is enough to show that if  $A'$  is an  $\eta$ -quasi-monomial object in  $\mathcal{Q}(\mathfrak{g}_\delta)$  of type  $\xi$ , then the class of  $A'$  in  $\mathcal{K}(\mathfrak{g}_\delta)$  is of the form  $a_{\varpi}^{-1}\tilde{\gamma}(\tilde{T}_{\mathbf{c}})$  for some  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  and  $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ . We can find:

(a)  $\mathfrak{p}_*, \mathfrak{h}_*, A, \mathfrak{q}_*, \mathfrak{p}'_*, (M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C) \in \mathfrak{T}_\eta$  (representing  $\xi$ ) as in 11.5 such that  $A' = \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ ; moreover, we can assume that the  $\mathbf{Z}$ -grading of  $\mathfrak{h} = \oplus_N \mathfrak{h}_N$  is

$\eta$ -rigid and  $\overset{\circ}{\mathfrak{m}}'_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$ . Replacing the data (a) by a  $G_0$ -conjugate we can assume in addition that  $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, C)$  is equal to  $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$  in 10.1.

Let  $H = e^{\mathfrak{h}}$ . Since  $\mathfrak{h}_*$  is  $\eta$ -rigid, there exists  $\iota' \in Y_H$  such that:

- (i)  $\iota'_k = \mathfrak{h}_{k\eta/2}$  if  $k \in \mathbf{Z}$ ,  $k\eta/2, \iota'_k = 0$  if  $k \in \mathbf{Z}$ ,  $k\eta/2 \notin \mathbf{Z}$  and
- (ii)  $\iota' = \iota_{\phi'}$  for some  $\phi' = (e', h', f') \in J^H$  such that  $e' \in \overset{\circ}{\mathfrak{h}}_\eta$ ,  $h' \in \mathfrak{h}_0$ ,  $f' \in \mathfrak{h}_{-\eta}$ .

Since  $\overset{\circ}{\mathfrak{m}}_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$  and  $e \in \overset{\circ}{\mathfrak{m}}_\eta$ , we see that  $e, e'$  are in the same  $M_0$ -orbit. Hence we can find  $g \in M_0$  such that  $\text{Ad}(g)$  conjugates  $e', f', h', \iota'$  to  $e, f, h, \iota$ . Applying  $\text{Ad}(g)$  (which preserves  $\mathfrak{h}_k$ ) to (i) and (ii) we see that we can assume that  $\iota' = \iota$ ,  $\phi' = \phi$ .

Recall from 10.3 that  $\tilde{I}_N^\phi = \iota'_{2N/\eta} \mathfrak{g}_N$  for  $N \in \mathbf{Z}$  such that  $2N/\eta \in \mathbf{Z}$ . Using (i) with  $\iota' = \iota$  we see that  $\mathfrak{h}_N \subset \tilde{I}_N^\phi$  for any  $N \in \mathbf{Z}$  such that  $2N/\eta \in \mathbf{Z}$ . Using 10.4(c),(d) we see that for some  $\varpi \in \mathbf{E}''$  we have  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^{\varpi}$ ,  $\mathfrak{h}_* = {}^\epsilon \tilde{I}_*^{\varpi}$ . Using now 11.4(d), we see that  $A' = a_{\varpi}^{-1} \tilde{\gamma}(\tilde{I}_{\mathbf{c}})$ , where  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  contains  $\varpi$ . This completes the proof of 11.4(f), hence that of 11.4(g).

11.8. We can now prove 11.2(b). Using 11.4(g), 11.2(b) is reduced to the following obvious statement:  ${}^\xi \mathcal{K}(\mathfrak{g}_\delta)$  is a free  $\mathcal{A}$ -module.

11.9. We define a  $\mathbf{Q}$ -linear map

$$\bar{\cdot} : \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta) \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta)$$

by  $\overline{fB} = \bar{f}B$  for any  $f \in \mathbf{Q}(v)$  and any  $B \in \mathfrak{B}$  (see 10.13); here  $\bar{f}$  is as in 0.12. This restricts to a  $\mathbf{Q}$ -linear map

$$\bar{\cdot} : \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi \mathcal{K}(\mathfrak{g}_\delta) \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$$

and to a  $\mathbf{Z}$ -linear map  ${}^\xi \mathcal{K}(\mathfrak{g}_\delta) \rightarrow {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$ . We show:

- (a)  $\overline{I_\varpi} = I_\varpi$  for any  $\varpi \in \mathbf{E}'$ .

In  ${}^\xi \mathcal{Q}(\mathfrak{g}_\delta)$  we have  $I_\varpi = \sum_{B \in {}^\epsilon \mathfrak{B}} f_B B$  where  $f_B \in \mathcal{A}$ . It is enough to prove that  $\overline{f_B} = f_B$  for all  $B$ .

We set  $\mathfrak{p}_* = {}^\epsilon \mathfrak{p}_*^{\varpi}$ . Let  $\sigma$  be an automorphism of order 2 of  $\tilde{\mathbf{Q}}_l$  such that  $\sigma(z) = z^{-1}$  for any root of 1 in  $\tilde{\mathbf{Q}}_l$ . Applying  $\sigma$  to  $K \in \mathcal{Q}(\mathfrak{m}_\eta)$  (resp.  $K \in \mathcal{Q}(\mathfrak{g}_\delta)$ ) we obtain  $K^\sigma \in \mathcal{Q}(\mathfrak{m}_\eta)$  (resp.  $K^\sigma \in \mathcal{Q}(\mathfrak{g}_\delta)$ ). Note that  $K \mapsto K^\sigma$  commutes with shifts; moreover, we have

$$(b) \quad ({}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}))^\sigma = {}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}^\sigma).$$

Moreover, if  $K$  is a simple perverse sheaf in  $\mathcal{Q}(\mathfrak{m}_\eta)$  or in  $\mathcal{Q}(\mathfrak{g}_\delta)$ , we have

$$(c) \quad K^\sigma \cong D(K),$$

since  $K$  restricted to an open dense subset of its support is a local system with finite monodromy. By (b),(c) and [LY, 4.1(d)] we have

$$D({}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C})) = {}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(\tilde{C})) = {}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}^\sigma) = ({}^\epsilon \widetilde{\text{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\tilde{C}))^\sigma,$$

hence

$$(D(I_\varpi))^\sigma = I_\varpi$$

and  $\sum_B \overline{f_B} D(B^\sigma) = \sum_B f_B B$ . Using this and (c) for  $K = B$ , we see that  $\overline{f_B} = f_B$  for all  $B$ ; this proves (a).

11.10. We show:

(a)  $\bar{\cdot} : \mathbf{V}' \rightarrow \mathbf{V}'$  (see 10.21) restricts to an involution  $\mathbf{V}'_{\mathcal{A}} \rightarrow \mathbf{V}'_{\mathcal{A}}$  and  $\bar{\cdot} : \mathbf{V} \rightarrow \mathbf{V}$  (see 10.21) restricts to an involution  $\mathbf{V}_{\mathcal{A}} \rightarrow \mathbf{V}_{\mathcal{A}}$  (these restrictions are denoted again by  $\bar{\cdot}$ ).

It is enough to note that for any  $\varpi \in \mathbf{E}''$  we have  $\overline{a_{\varpi}} = a_{\varpi}$ .

We show:

(b)  $\tilde{\gamma} : \mathbf{V}' \rightarrow \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  is compatible with the maps  $\bar{\cdot}$  in the two sides.

This follows from 11.9(a).

We have the following result.

**Proposition 11.11.**

(a) Let  $\mathbf{B}'$  be the set of all  $b \in \mathbf{V}_{\mathcal{A}}$  such that  $\bar{b} = b$  and  $(b : b) \in 1 + v\mathbf{Z}[[v]]$ . Then  $\mathbf{B}'$  is a signed basis of the  $\mathcal{A}$ -module  $\mathbf{V}_{\mathcal{A}}$  (that is, the union of a basis with  $(-1)$  times that basis).

(b) For  $b \in \mathbf{B}'$  we have  $(b : b) \in 1 + v\mathbf{N}[[v]]$ .

(c) There is a unique  $\mathcal{A}$ -basis  $\mathbf{B}$  of  $\mathbf{V}_{\mathcal{A}}$  such that  $\mathbf{B} \subset \mathbf{B}'$  and for any  $\mathbf{c} \in \mathring{\mathbf{E}}$  and any  $\varpi \in \mathbf{E}'' \cap \mathbf{c}$ , the image of  $a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}}$  in  $\mathbf{V}_{\mathcal{A}}$  is an  $\mathbf{N}[v, v^{-1}]$ -linear combination of elements in  $\mathbf{B}$ .

It is enough to prove the analogous statements where  $\mathbf{V}_{\mathcal{A}}$  is identified via  $\gamma$  with  ${}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  with  $(:)$  as in 4.4(c) and with  $\bar{\cdot}$  as in 11.9 (we use 10.19(b), 11.10(b)). Let  ${}^{\xi}\mathfrak{B} = \{B_1, B_2, \dots, B_r\}$  (see 10.13). From 0.12 we have  $(B_j : B_{j'}) \in \delta_{j,j'} + h_{j,j'}$ , where  $h_{j,j'} \in v\mathbf{N}[[v]]$  for all  $j, j'$  in  $[1, r]$ . From the definition (see 11.9) we have  $\overline{B_j} = B_j$  for  $j = 1, \dots, r$ . Now let  $b \in {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  be such that  $\bar{b} = b$  and  $(b : b) \in 1 + v\mathbf{Z}[[v]]$ . To prove (a), it is enough to show that  $b = \pm B_j$  for some  $j$ . We can write  $b = \sum_{j=1}^r f_j B_j$ , where  $f_j \in \mathcal{A}$  satisfy  $\bar{f}_j = f_j$  and  $\sum_{j,j' \in [1,r]} \bar{f}_j f_{j'} (\delta_{j,j'} + h_{j,j'}) \in 1 + v\mathbf{Z}[[v]]$  hence  $\sum_{j,j' \in [1,r]} f_j f_{j'} (\delta_{j,j'} + h_{j,j'}) \in 1 + v\mathbf{Z}[[v]]$ . We can find  $c \in \mathbf{Z}$  such that  $f_j = f_{j,c} v^c \pmod{v^{c+1}\mathbf{Z}[v]}$  where  $f_{j,c} \in \mathbf{Z}$  for all  $j$  and  $f_{j,c} \neq 0$  for some  $j$ . We have  $\sum_{j \in [1,r]} f_{j,c}^2 v^{2c} + v^{2c+1} f' = 1 + v f''$  where  $f', f'' \in \mathbf{Z}[v]$ . Moreover,  $\sum_{j \in [1,r]} f_{j,c}^2 > 0$ . It follows that  $c = 0$  and  $\sum_{j \in [1,r]} f_{j,0}^2 = 1$  so that there exists  $j_0 \in [1, r]$  such that  $f_{j_0,0} = \pm 1$  and  $f_{j,0} = 0$  for  $j \neq j_0$ . We have  $f_j = \pm \delta_{j,j_0} \pmod{v\mathbf{Z}[v]}$  for all  $j$ . Since  $\bar{f}_j = f_j$  we deduce that  $f_j = \pm \delta_{j,j_0}$  for all  $j$ . Thus  $b = \pm B_{j_0}$ . This completes the proof of (a). At the same time we have proved (b). Clearly,  $\{B_1, B_2, \dots, B_r\}$  has the positivity property in (c) (with  $\mathbf{V}_{\mathcal{A}}$  identified with  ${}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  and with  $a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}}$  identified with  $\gamma(a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}})$ ). Since any  $B_j$  appears with  $> 0$  coefficient in some  $\gamma(a_{\varpi}^{-1}\tilde{T}_{\mathbf{c}})$ , we see that  $\{B_1, B_2, \dots, B_r\}$  is the only basis contained in  $\{\pm B_1, \pm B_2, \dots, \pm B_r\}$  with the positivity property in (c). This completes the proof of the proposition.

11.12. From the proof of 11.11 we see that  $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi}\mathcal{K}(\mathfrak{g}_{\delta})$  (see Proposition 10.19(a)) restricts to a bijection

$$(a) \quad \mathbf{B} \xrightarrow{\sim} {}^{\xi}\mathfrak{B}.$$

For any  $G_0$ -orbit  $\mathcal{O}$  in  $\mathfrak{g}_{\delta}^{nil}$  let  $\mathbf{B}_{\mathcal{O}}$  be the set of all  $b \in \mathbf{B}$  such that  $\gamma(b) \in \mathfrak{B}_{\mathcal{O}}$  (see 10.13). We have a partition  $\mathbf{B} = \sqcup_{\mathcal{O}} \mathbf{B}_{\mathcal{O}}$  where  $\mathcal{O}$  runs over the  $G_0$ -orbits in  $\mathfrak{g}_{\delta}^{nil}$ .

11.13. We consider the setup of 10.22. We show:

(a) *The  $\mathcal{A}$ -submodule  $\mathbf{V}_{\mathcal{A}}$  of  $\mathbf{V}$  defined in 11.2 in terms of  $\dot{\xi}$  is the same as that defined in terms of  $\dot{\xi}_1$ .*

It is not clear how to prove this using the definition in 11.2 since  $\mathbf{E}''$  defined in terms of  $\dot{\xi}$  (see 10.3) is not necessarily the same as that defined in terms of  $\dot{\xi}_1$ . Instead we will argue indirectly. Using 11.4(g) it is enough to show:

(b) *The isomorphism  $\gamma : \mathbf{V} \xrightarrow{\sim} \mathbf{Q}(v) \otimes_{\mathcal{A}} {}^{\xi} \mathcal{K}(\mathfrak{g}_{\delta})$  defined in 10.19 in terms of  $\dot{\xi}$  is equal to the analogous isomorphism defined in terms of  $\dot{\xi}_1$ .*

Thus it is enough to show that if  $\varpi \in \mathbf{E}' \cap \mathbf{E}'_1$ , then  $\tilde{I}_{\varpi}$  defined in 10.2 in terms of  $\dot{\xi}$  is the same as that defined in terms of  $\dot{\xi}_1$ . Using the definitions we see that it is enough to show that

$$\begin{aligned} \dot{\eta}_{\mathfrak{p}_{\eta}}^{(|\eta|/2)(\varpi+\iota)} &= \dot{\eta}_1 \mathfrak{p}_{\eta_1}^{(|\eta_1|/2)(\varpi+\iota)}, \\ \dot{\eta}_{\mathfrak{p}_0}^{(|\eta|/2)(\varpi+\iota)} &= \dot{\eta}_1 \mathfrak{p}_0^{(|\eta_1|/2)(\varpi+\iota)}. \end{aligned}$$

or that

$$\begin{aligned} \bigoplus_{\kappa \in \mathbf{Q}; \kappa \geq |\eta|} \binom{(|\eta|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa \in \mathbf{Q}; \kappa \geq |\eta_1|} \binom{(|\eta_1|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta}, \\ \bigoplus_{\kappa \in \mathbf{Q}; \kappa \geq 0} \binom{(|\eta|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa \in \mathbf{Q}; \kappa \geq 0} \binom{(|\eta_1|/2)(\varpi+\iota)}{\kappa} \mathfrak{g}_{\delta}, \end{aligned}$$

or that

$$\begin{aligned} \bigoplus_{\kappa \in \mathbf{Q}; \kappa/|\eta| \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta|} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa \in \mathbf{Q}; \kappa/|\eta_1| \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta_1|} \mathfrak{g}_{\delta}, \\ \bigoplus_{\kappa \in \mathbf{Q}; \kappa/|\eta| \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta|} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa \in \mathbf{Q}; \kappa/|\eta_1| \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa/|\eta_1|} \mathfrak{g}_{\delta}, \end{aligned}$$

or, setting  $\kappa' = \kappa/|\eta|$ ,  $\kappa'' = \kappa/|\eta_1|$ , that

$$\begin{aligned} \bigoplus_{\kappa' \in \mathbf{Q}; \kappa' \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa'} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa'' \in \mathbf{Q}; \kappa'' \geq 1} \binom{(1/2)(\varpi+\iota)}{\kappa''} \mathfrak{g}_{\delta}, \\ \bigoplus_{\kappa' \in \mathbf{Q}; \kappa' \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa'} \mathfrak{g}_{\delta} &= \bigoplus_{\kappa'' \in \mathbf{Q}; \kappa'' \geq 0} \binom{(1/2)(\varpi+\iota)}{\kappa''} \mathfrak{g}_{\delta}, \end{aligned}$$

which are obvious. This proves (a).

Using (b) and 11.12(b) we see that the basis  $\mathbf{B}$  of  $\mathbf{V}_{\mathcal{A}}$  defined in 11.11 in terms of  $\dot{\xi}$  is the same as that defined in terms of  $\dot{\xi}_1$ .

## 12. PURITY PROPERTIES

In this section we show that for any irreducible local system  $\mathcal{L}$  on a  $G_{\underline{0}}$ -orbit in  $\mathfrak{g}_{\delta}^{nil}$  the cohomology sheaves of  $\mathcal{L}^{\sharp} \in \mathcal{D}(\mathfrak{g}_{\delta})$  satisfy a strong purity property. This generalizes the analogous result in the  $\mathbf{Z}$ -graded case in [L4].

12.1. In this section we assume that  $p > 0$  and that  $\mathbf{k}$  is an algebraic closure of a finite field  $\mathbf{F}_q$  with  $q$  elements (here  $q$  is a power of  $p$ ). Replacing  $q$  by larger powers of  $p$  if necessary, we can assume that  $m$  divides  $q - 1$  and that we can find an  $\mathbf{F}_q$ -rational structure on  $G$  with Frobenius map  $F : G \rightarrow G$  such that  $\vartheta : G \rightarrow G$  (see 0.5) commutes with  $F : G \rightarrow G$ . Then  $G_{\underline{0}}$  is defined over  $\mathbf{F}_q$  and  $\mathfrak{g}$  inherits from  $G$  an  $\mathbf{F}_q$ -rational structure with Frobenius map  $F : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying  $F(\mathfrak{g}_i) = \mathfrak{g}_i$  for all  $i$ . Again by replacing  $q$  by larger powers of  $p$  if necessary, we may assume that all  $G_{\underline{0}}$ -orbits in  $\mathfrak{g}_{\delta}^{nil}$  are defined over  $\mathbf{F}_q$  and that for any irreducible  $G_{\underline{0}}$ -equivariant local system  $\mathcal{L}$  on a  $G_{\underline{0}}$ -orbit  $\mathcal{O}$  in  $\mathfrak{g}_{\delta}^{nil}$  we have  $F^* \mathcal{L} \cong \mathcal{L}$ .

We now fix a  $G_{\underline{0}}$ -orbit  $\mathcal{O}$  in  $\mathfrak{g}_{\delta}^{nil}$  with closure  $\bar{\mathcal{O}}$  and an irreducible  $G_{\underline{0}}$ -equivariant local system  $\mathcal{L}$  on  $\mathcal{O}$ . We fix an isomorphism  $\tilde{F} : F^* \mathcal{L} \rightarrow \mathcal{L}$  which induces the identity map on the stalk of  $\mathcal{L}$  at some point of  $\mathcal{O}^F$ . Then  $\tilde{F}$  induces an isomorphism (denoted again by  $\tilde{F}$ )  $F^* \mathcal{L}^{\sharp} \rightarrow \mathcal{L}^{\sharp}$ . Given a finite-dimensional  $\bar{\mathbf{Q}}_l$ -vector space  $V$

with an endomorphism  $\tilde{F} : V \rightarrow V$ , we say that  $\tilde{F} : V \rightarrow V$  is  $a$ -pure (for an integer  $a$ ) if the eigenvalues of  $\tilde{F}$  are algebraic numbers all of whose complex conjugates have absolute value  $q^{a/2}$ . Sometimes we will just say that  $V$  is  $a$ -pure (where this is understood to refer to  $\tilde{F}$ ).

12.2. We show:

(a) For any  $x \in \bar{\mathcal{O}}^F$  and any  $a \in \mathbf{Z}$ , the induced linear map  $\tilde{F} : \mathcal{H}_x^a(\mathcal{L}^\sharp) \rightarrow \mathcal{H}_x^a(\mathcal{L}^\sharp)$  is  $a$ -pure.

Using [LY, 2.3(b)], we can find  $\phi = (e, h, f) \in J_\delta(x)$  such that  $h, f$  are  $\mathbf{F}_q$ -rational. Recall that  $x = e$ . Let  $\iota = \iota_\phi \in Y_G$  be as in 1.1. Let  $\mathfrak{z}(f)$  be the centralizer of  $f$  in  $\mathfrak{g}$  and let  $\Sigma = e + \mathfrak{z}(f) \subset \mathfrak{g}$ . According to Slodowy:

(b) The affine space  $\Sigma$  is a transversal slice at  $x$  to the  $G$ -orbit of  $e$  in  $\mathfrak{g}$  and the  $\mathbf{k}^*$ -action  $t \mapsto t^{-2} \text{Ad}(\iota(t))$  on  $\mathfrak{g}$  keeps  $e$  fixed, leaves  $\Sigma$  stable and defines a contraction of  $\Sigma$  to  $x$ .

Let  $\tilde{\Sigma} = \Sigma \cap \mathfrak{g}_\delta = e + (\mathfrak{z}(f) \cap \mathfrak{g}_\delta)$ . Then:

(c)  $\tilde{\Sigma}$  is a transversal slice to the  $G_{\underline{0}}$ -orbit of  $e$  in  $\mathfrak{g}_\delta$ ; the  $\mathbf{k}^*$ -action in (b) leaves stable  $\tilde{\Sigma}$  and is a contraction of  $\tilde{\Sigma}$  to  $e$ .

(We have a direct sum decomposition  $\mathfrak{g}_\delta = (\mathfrak{z}(f) \cap \mathfrak{g}_\delta) \oplus [e, \mathfrak{g}_0]$  obtained by taking the  $\zeta^\delta$ -eigenspace of  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$  in the two sides of the direct sum decomposition  $\mathfrak{g} = \mathfrak{z}(f) \oplus [e, \mathfrak{g}]$ ; note that both  $\mathfrak{z}(f)$  and  $[e, \mathfrak{g}]$  are  $\theta$ -stable.)

Let  $\mathcal{L}'$  be the restriction of  $\mathcal{L}$  to  $\mathcal{O} \cap \tilde{\Sigma}$  (a smooth irreducible subvariety of  $\tilde{\Sigma}$ ). Note that  $\mathcal{O} \cap \tilde{\Sigma}, \bar{\mathcal{O}} \cap \tilde{\Sigma}$  are stable under the  $\mathbf{k}^*$ -action in (c) and  $\mathcal{L}'$  is equivariant for that action. By (c), we have  $\mathcal{H}_x(\mathcal{L}^\sharp) = \mathcal{H}_x(\mathcal{L}'^\sharp)$ . It remains to show that  $\mathcal{H}_x(\mathcal{L}'^\sharp)$  is  $a$ -pure. This can be reduced to Deligne's hard Lefschetz theorem by an argument in Lemma 4.5(b) in [KL2] applied to  $\bar{\mathcal{O}} \cap \tilde{\Sigma} \subset \tilde{\Sigma}$  with the contraction in (c) and to  $\mathcal{L}'$  instead of  $\underline{\mathcal{Q}}_i$ . Note that in [KL2, 4.5(b)] an inductive purity assumption was made which is in fact unnecessary, by Gabber purity theorem. This completes the proof of (a).

*Erratum to [L4]. On page 209, line -5, replace  $t^{-n} \text{Ad}(\iota'(t))$  by  $t^{-2} \text{Ad}(\iota'(t))$ .*

### 13. AN INNER PRODUCT

This section is an adaptation of [L4, §19] from the  $\mathbf{Z}$ -graded case to the  $\mathbf{Z}/m$ -graded case; we express the matrix whose entries are the values of the  $(\cdot)$ -pairing at two elements of  $\mathfrak{B}$  as a product of three matrices. As an application we show (see 13.8(a)) that if  $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$  in  $\mathcal{I}(\mathfrak{g}_\delta)$  are such that some cohomology sheaf of  $\mathcal{L}^\sharp|_{\mathcal{O}'}$  contains  $\mathcal{L}'$ , then  $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$  are in the same block. Another application (to odd vanishing) is given in Section 14.

13.1. We fix  $(\mathcal{O}, \mathcal{L}), (\mathcal{O}', \mathcal{L}')$  in  $\mathcal{I}(\mathfrak{g}_\delta)$  and we form  $A = \mathcal{L}^\sharp, A' = \mathcal{L}'^\sharp$  in  $\mathcal{D}(\mathfrak{g}_\delta)$ . We want to compute  $d_j(\mathfrak{g}_\delta; A, A')$  (see [LY, 0.12]) for a fixed  $j \in \mathbf{Z}$ . We can arrange the  $G_{\underline{0}}$ -orbits in  $\mathfrak{g}_\delta^{nil}$  in an order  $\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_\beta$  such that  $\mathcal{O}_{\leq s} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \dots \cup \mathcal{O}_s$  is closed in  $\mathfrak{g}_\delta$  for  $s = 0, 1, \dots, \beta$ . We choose a smooth irreducible variety  $\Gamma$  with a free action of  $G_{\underline{0}}$  such that  $H^r(\Gamma, \underline{\mathcal{Q}}_i) = 0$  for  $r = 1, 2, \dots, \mathbf{m}$  where  $\mathbf{m}$  is a large integer (compared to  $j$ ). We assume that  $\mathbf{D} := \dim \Gamma$  is large (compared to  $j$ ). We have  $H_c^{2\mathbf{D}-i}(\Gamma, \underline{\mathcal{Q}}_i) = 0$  for  $i = 1, 2, \dots, \mathbf{m}$ . We form  $X = G_{\underline{0}} \backslash (\Gamma \times \mathfrak{g}_\delta)$ . Let  $\tilde{\mathcal{L}}, \tilde{\mathcal{L}}'$  be the local systems on the smooth subvarieties  $G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}), G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}')$  of  $X$  defined by  $\mathcal{L}, \mathcal{L}'$  and let  $\tilde{A} = \tilde{\mathcal{L}}^\sharp, \tilde{A}' = \tilde{\mathcal{L}}'^\sharp$  be the corresponding intersection cohomology complexes in  $\mathcal{D}(X)$ . Then  $\tilde{A} \otimes \tilde{A}' \in \mathcal{D}(X)$  is well defined; its restriction to various

subvarieties of  $X$  will be denoted by the same symbol. For  $s = 0, 1, \dots, \beta$  we form  $X_s = G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}_s)$ ,  $X_{\leq s} = G_{\underline{0}} \backslash (\Gamma \times \mathcal{O}_{\leq s})$ . We set  $X_{\leq -1} = \emptyset$ . The partition of  $X_{\leq s}$  into  $X_{\leq s-1}$  and  $X_s$  (for  $s = 0, 1, \dots, \beta$ ) gives rise to a long exact sequence

$$(a) \quad \begin{aligned} \dots &\xrightarrow{\xi_a} H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}') \\ &\rightarrow H_c^a(X_{\leq s-1}, \tilde{A} \otimes \tilde{A}') \xrightarrow{\xi_{a+1}} H_c^{a+1}(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow \dots \end{aligned}$$

13.2. In the following proposition we encounter two kinds of integers; some like  $j, m, \dim G_{\underline{0}}, \dim \mathfrak{g}_{\delta}, \beta$  are regarded as “small” (they belong to a fixed finite set of integers), others like  $2\mathbf{D}$  and  $\mathbf{m}$  are regarded as very large (we are free to choose them so). We will also encounter integers  $a$  such that  $2\mathbf{D} - a$  is a “small” integer (we then write  $a \sim 2\mathbf{D}$ ).

**Proposition 13.3.**

(a) Assume that  $\mathbf{k}$  is as in 12.1. Then  $H_c^a(X_s, \tilde{A} \otimes \tilde{A}')$  is  $a$ -pure (see 12.1) for  $s = 0, 1, \dots, \beta$  and  $a \sim 2\mathbf{D}$ .

(b) We choose  $x_s \in \mathcal{O}_s$ . If  $a \sim 2\mathbf{D}$ , then

$$\begin{aligned} &\dim H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \\ &= \sum_{r+r_1+r_2=a} \dim(H_c^r(G_{\underline{0}}(x_s)^0 \backslash \Gamma, \tilde{\mathbf{Q}}_l) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0}, \end{aligned}$$

where the upper script refers to invariants under the finite group  $G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0$ .

(c) Assume that  $\mathbf{k}$  is as in 12.1. Then  $H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}')$  is  $a$ -pure (see 12.1) for  $s = 0, 1, \dots, \beta$  and  $a \sim 2\mathbf{D}$ .

(d) The exact sequence 13.1(a) gives rise to short exact sequences

$$0 \rightarrow H_c^a(X_s, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s}, \tilde{A} \otimes \tilde{A}') \rightarrow H_c^a(X_{\leq s-1}, \tilde{A} \otimes \tilde{A}') \rightarrow 0$$

for  $s = 0, 1, \dots, \beta$  and  $a \sim 2\mathbf{D}$ .

(e) For  $a \sim 2\mathbf{D}$  we have  $\dim H_c^a(X, \tilde{A} \otimes \tilde{A}') = \sum_{s=0}^{\beta} \dim H_c^a(X_s, \tilde{A} \otimes \tilde{A}')$ .

(f) For  $a \sim 2\mathbf{D}$  we have

$$\begin{aligned} &\dim H_c^a(X, \tilde{A} \otimes \tilde{A}') \\ &= \sum_{s=0}^{\beta} \sum_{r+r_1+r_2=a} \dim(H_c^r(G_{\underline{0}}(x_s)^0 \backslash \Gamma, \tilde{\mathbf{Q}}_l) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_{\underline{0}}(x_s)/G_{\underline{0}}(x_s)^0}. \end{aligned}$$

The proof is almost a copy of the proof of [L4, 19.4]. By general principles we can assume that  $\mathbf{k}$  is as in 12.1. We shall use Deligne’s theory of weights. We first prove (a) and (b). We write  $x$  instead of  $x_s$ . We may assume that  $x$  is an  $\mathbf{F}_q$ -rational point. We have a natural spectral sequence

$$(g) \quad E_2^{r,r'} = H_c^r(X_s, \mathcal{H}^{r'}(\tilde{A} \otimes \tilde{A}')) \implies H_c^{r+r'}(X_s, \tilde{A} \otimes \tilde{A}').$$

We show that

$$(h) \quad E_2^{r,r'} \text{ is } (r+r')\text{-pure if } r+r' \sim 2\mathbf{D}.$$

We have  $X_s = G_{\underline{0}}(x) \backslash \Gamma$  and

$$E_2^{r,r'} = (H_c^r(G_{\underline{0}}(x)^0 \backslash \Gamma, \tilde{\mathbf{Q}}_l) \otimes \mathcal{H}_x^{r'}(A \otimes A'))^{G_{\underline{0}}(x)/G_{\underline{0}}(x)^0}.$$

Here  $A \otimes A' \in \mathcal{D}(\mathfrak{g}_\delta)$ . We may assume that  $r'$  is “small” (otherwise,  $E_2^{r,r'} = 0$ ). We then have  $r \sim 2\mathbf{D}$ . Now

$$\mathcal{H}_x^{r'}(A \otimes A') = \oplus_{r_1+r_2=r'} \mathcal{H}_x^{r_1}(A) \otimes \mathcal{H}_x^{r_2}(A')$$

is  $r'$ -pure by 12.2(a). Moreover,  $H_c^r(G_0(x)^0 \setminus \Gamma, \bar{\mathbf{Q}}_l)$  is  $r$ -pure for  $r \sim 2\mathbf{D}$  (a known property of the classifying space of  $G_0(x)^0$ ) and (h) follows.

From (h) it follows that  $E_\infty^{r,r'}$  of the spectral sequence (g) is  $(r + r')$ -pure if  $r + r' \sim 2\mathbf{D}$  and (a) follows. From (h) it also follows that  $E_2^{r,r'} = E_\infty^{r,r'}$  if  $r + r' \sim 2\mathbf{D}$  (many differentials must be zero since they respect weights) and (b) follows.

Now (c) follows from (a) using the exact sequence 13.2(a) and induction on  $s$ . From (c) and (a) we see that the homomorphism  $\xi_{a+1}$  in 13.2(a) is between pure spaces of different weight. Since  $\xi_{a+1}$  preserve weights, it must be 0. Similarly,  $\xi_a = 0$  in 13.2(a) hence (d) holds. Now (e) follows from (d) since the support of  $\bar{A} \otimes \bar{A}'$  is contained in  $X_{\leq \beta}$ ; (f) follows from (b),(e). This completes the proof of the proposition.

13.4. Using the definitions we see that 13.3(f) implies:

(a)

$$d_j(\mathfrak{g}_\delta; A, A') = \sum_s \sum_{-r_0+r_1+r_2=j-2p_s} \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \mathcal{H}_{x_s}^{r_1} A \otimes \mathcal{H}_{x_s}^{r_2} A')^{G_0(x_s)/G_0(x_s)^0},$$

where

(b) 
$$p_s = \dim G_0 - \dim G_0(x_s) = \dim \mathcal{O}_s$$

and  $H_{r_0}^{G_0(x_s)^0}(\text{pt})$  denotes equivariant homology of a point. (See [L6] for the definition of equivariant homology.)

13.5. Given  $(\mathcal{O}, \mathcal{L}), (\tilde{\mathcal{O}}, \tilde{\mathcal{L}})$  in  $\mathcal{I}(\mathfrak{g}_\delta)$  we define  $\mu(\tilde{\mathcal{L}}, \mathcal{L}) \in \mathbf{Z}[v^{-1}]$  by

(a) 
$$\mu(\tilde{\mathcal{L}}, \mathcal{L}) = \sum_a \mu(a; \tilde{\mathcal{L}}, \mathcal{L}) v^{-a},$$

where  $\mu(a; \tilde{\mathcal{L}}, \mathcal{L})$  is the number of times  $\tilde{\mathcal{L}}$  appears in a decomposition of the local system  $\mathcal{H}^a(\mathcal{L}^\sharp)|_{\tilde{\mathcal{O}}}$  as a direct sum of irreducible local systems. Note that  $\mu(\tilde{\mathcal{L}}, \mathcal{L})$  is zero unless  $\tilde{\mathcal{O}}$  is contained in the closure of  $\mathcal{O}$ .

If  $\mathcal{E}$  is an irreducible  $G_0$ -equivariant local system on  $\mathcal{O}_s$ , we denote by  $\rho_{\mathcal{E}}$  the irreducible  $G_0(x_s)/G_0(x_s)^0$ -module corresponding to  $\mathcal{E}$ . With this notation we can rewrite 13.4(a) as follows:

$$d_j(\mathfrak{g}_\delta; A, A') = \sum_s \sum_{-r_0+r_1+r_2=j-2p_s} \sum_{\mathcal{E}, \mathcal{E}'} \mu(r_1; \mathcal{E}, \mathcal{L}) \mu(r_2; \mathcal{E}', \mathcal{L}') \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \rho_{\mathcal{E}} \otimes \rho_{\mathcal{E}'})^{G_0(x_s)/G_0(x_s)^0},$$

where  $\mathcal{E}, \mathcal{E}'$  run over the isomorphism classes of irreducible  $G_0$ -equivariant local systems on  $\mathcal{O}_s$ . This may be written in terms of power series in  $\mathbf{Q}((v))$  as follows:

(b) 
$$\{A, A'\} = \sum_{s=0}^{\beta} \sum_{\mathcal{E}, \mathcal{E}'} \mu(\mathcal{E}, \mathcal{L}) \Xi(\mathcal{E}, \mathcal{E}') \mu(\mathcal{E}', \mathcal{L}'),$$



where

$$(c) \quad \Xi(\mathcal{E}, \mathcal{E}') = \sum_{r_0 \geq 0} \dim(H_{r_0}^{G_0(x_s)^0}(\text{pt}) \otimes \rho_{\mathcal{E}} \otimes \rho_{\mathcal{E}'})^{G_0(x_s)/G_0(x_s)^0} v^{r_0 - 2p_s} \in \mathbf{Q}((v)).$$

13.6. Let  $B_1, B_2 \in \mathfrak{B}$ . We write  $B_1 = \mathcal{L}_1^\sharp[\dim \mathcal{O}_1] \in \mathcal{D}(\mathfrak{g}_\delta)$ ,  $B_2 = \mathcal{L}_2^\sharp[\dim \mathcal{O}_2] \in \mathcal{D}(\mathfrak{g}_\delta)$  with  $(\mathcal{O}_1, \mathcal{L}_1), (\mathcal{O}_2, \mathcal{L}_2)$  in  $\mathcal{I}(\mathfrak{g}_\delta)$ . We set

$$\begin{aligned} P_{B_2, B_1} &= \mu(\mathcal{L}_2, \mathcal{L}_1) \in \mathbf{N}[v^{-1}], \text{ (see 13.5);} \\ \tilde{P}_{B_2, B_1} &= P_{D(B_2), D(B_1)} \in \mathbf{N}[v^{-1}]; \\ \Lambda_{B_2, B_1} &= \Xi(\mathcal{L}_2, \mathcal{L}_1) \in \mathbf{Q}((v)), \text{ (see 13.5) if } \mathcal{O}_1 = \mathcal{O}_2; \\ \Lambda_{B_2, B_1} &= 0 \text{ if } \mathcal{O}_1 \neq \mathcal{O}_2; \\ \tilde{\Lambda}_{B_2, B_1} &= \Lambda_{B_2, D(B_1)} \in \mathbf{Q}((v)). \end{aligned}$$

Note that:

- (a)  $P_{B_2, B_1} \neq 0 \implies \dim \mathcal{O}_2 \leq \dim \mathcal{O}_1$ ;  $\tilde{P}_{B_2, B_1} \neq 0 \implies \dim \mathcal{O}_2 \leq \dim \mathcal{O}_1$ ;
- (b) if  $\dim \mathcal{O}_2 = \dim \mathcal{O}_1$ , then  $P_{B_2, B_1} = \delta_{B_2, B_1}$ ,  $\tilde{P}_{B_2, B_1} = \delta_{B_2, B_1}$ ;
- (c)  $\tilde{\Lambda}_{B_2, B_1} = 0$  if  $\mathcal{O}_1 \neq \mathcal{O}_2$ .

Then 13.5(b) can be rewritten as follows:

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \Lambda_{B_1, B_2} P_{B_2, B''},$$

where  $B = A[\dim \mathcal{O}]$ ,  $B'' = A'[\dim \mathcal{O}']$ , or as

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \Lambda_{B_1, D(B_2)} P_{D(B_2), D(B')},$$

where  $B = A[\dim \mathcal{O}]$ ,  $D(B') = A'[\dim \mathcal{O}']$ , or as

$$\{A, A'\} = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'},$$

where  $B = A[\dim \mathcal{O}]$ ,  $D(B') = A'[\dim \mathcal{O}']$ . We have

$$\{A, A'\} = v^{-\kappa(B) - \kappa(B')} \{B, D(B')\} = v^{-\kappa(B) - \kappa(B')} (B : B'),$$

hence

$$(d) \quad v^{-\kappa(B) - \kappa(B')} (B : B') = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'}$$

for any  $B, B'$  in  $\mathfrak{B}$ . (Here  $\kappa$  is as in 10.13.) We show:

(e) *The following three matrices with entries in  $\mathbf{Q}((v))$  (indexed by  $\mathfrak{B} \times \mathfrak{B}$ ) are invertible:*

- (i) *the matrix  $((B : B'))$ ;*
- (ii) *the matrix  $\mathcal{M} := (v^{-\kappa(B) - \kappa(B')} (B : B'))$ ;*
- (iii) *the matrix  $\mathcal{T} := (\tilde{\Lambda}_{B, B'})$ .*

The matrix in (i) is invertible since  $(B : B') \in \delta_{B, B'} + v\mathbf{N}[[v]]$  for all  $B, B'$ ; see [LY, 0.12]. This implies immediately that  $\mathcal{M}$  is invertible. Now, by (d), we have  $\mathcal{M} = \mathcal{S}\mathcal{T}\mathcal{S}'$  where  $\mathcal{S}$  (resp.  $\mathcal{S}'$ ) is the matrix indexed by  $\mathfrak{B} \times \mathfrak{B}$  whose  $(B, B')$ -entry is  $P_{B', B}$  (resp.  $\tilde{P}_{B, B'}$ ). Since  $\mathcal{M}$  is invertible, it follows that  $\mathcal{T}$  is invertible. This proves (e).

13.7. We show:

(a) *If  $B, B'$  in  $\mathfrak{B}$  satisfy  $\psi(B) \neq \psi(B')$  ( $\psi$  as in [LY, 6.6]), then  $P_{B,B'} = 0$ ,  $\tilde{P}_{B,B'} = 0$ ,  $\tilde{\Lambda}_{B,B'} = 0$ .*

We can find a function  $\mathfrak{B} \rightarrow \mathbf{Z}$ ,  $B \mapsto c_B$  such that for  $B, B'$  in  $\mathfrak{B}$  we have  $c_B = c_{B'}$  if and only if  $\psi(B) = \psi(B')$ . From 13.6 we deduce

$$\begin{aligned} & v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')} (B : B') \\ &= \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} v^{c_B - c_{B_1}} P_{B_1, B} v^{c_{B_1} - c_{B_2}} \tilde{\Lambda}_{B_1, B_2} v^{c_{B_2} - c_{B'}} \tilde{P}_{B_2, B'}. \end{aligned}$$

When  $B, B'$  vary, this again can be expressed as the decomposition of the matrix  $\tilde{\mathcal{M}} := (v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')} (B : B'))$  (indexed by  $\mathfrak{B} \times \mathfrak{B}$ ) as a product of three matrices  $\tilde{\mathcal{S}}\tilde{\mathcal{T}}\tilde{\mathcal{S}}'$ , where  $\tilde{\mathcal{S}}$  (resp.  $\tilde{\mathcal{S}}'$ ) is the matrix indexed by  $\mathfrak{B} \times \mathfrak{B}$  whose  $(B, B')$ -entry is  $v^{c_B - c_{B'}} P_{B', B}$  (resp.  $v^{c_B - c_{B'}} \tilde{P}_{B, B'}$ ) and  $\tilde{\mathcal{T}}$  is the matrix indexed by  $\mathfrak{B} \times \mathfrak{B}$  whose  $(B, B')$ -entry is  $v^{c_B - c_{B'}} \tilde{\Lambda}_{B, B'}$ . From [LY, 7.9(a)] we know that  $(B : B') = 0$  unless  $\psi(B) = \psi(B')$ . Hence  $v^{c_B - c_{B'}} (B : B') = (B : B')$  for all  $B, B'$  so that  $\tilde{\mathcal{M}} = \mathcal{M}$ . Thus we have

$$\tilde{\mathcal{S}}\tilde{\mathcal{T}}\tilde{\mathcal{S}}' = \mathcal{S}\mathcal{T}\mathcal{S}'.$$

Now by 13.6(c),(e), the matrix  $\mathcal{T}$  (and hence the matrix  $\tilde{\mathcal{T}}$ ) belongs to a subgroup of  $GL_N$  ( $N = \sharp(\mathfrak{B})$ ) of the form  $GL_{N_1} \times \cdots \times GL_{N_k}$  where  $N_1, \dots, N_k$  are the sizes of the various subsets  $\mathfrak{B}_{\mathcal{O}}$ ; moreover, by 13.6(a),(b), the matrix  $\mathcal{S}$  (hence the matrix  $\tilde{\mathcal{S}}$ ) belongs to the unipotent radical of a parabolic subgroup of  $GL_N$  with Levi subgroup equal to the subgroup of  $GL_N$  considered above, while the matrix  $\mathcal{S}'$  (hence the matrix  $\tilde{\mathcal{S}}'$ ) belongs to the unipotent radical of the opposite parabolic subgroup with the same Levi subgroup. This forces the equalities  $\tilde{\mathcal{S}} = \mathcal{S}$ ,  $\tilde{\mathcal{T}} = \mathcal{T}$ ,  $\tilde{\mathcal{S}}' = \mathcal{S}'$ . Now the equality  $\tilde{\mathcal{S}} = \mathcal{S}$  implies  $v^{c_B - c_{B'}} P_{B', B} = P_{B', B}$  for all  $B', B$  in  $\mathfrak{B}$ . Thus, if  $\psi(B) \neq \psi(B')$  (so that  $c_B \neq c_{B'}$ ), we must have  $P_{B', B} = 0$ . Similarly, from  $\tilde{\mathcal{T}} = \mathcal{T}$  we see that, if  $\psi(B) \neq \psi(B')$ , then  $\tilde{\Lambda}_{B, B'} = 0$  and from  $\tilde{\mathcal{S}}' = \mathcal{S}'$  we see that, if  $\psi(B) \neq \psi(B')$ , then  $\tilde{P}_{B, B'} = 0$ . This proves (a).

13.8. We now fix  $\xi \in \underline{\Sigma}_\eta$ . We define four matrices  ${}^\xi\mathcal{M}, {}^\xi\mathcal{S}, {}^\xi\mathcal{T}, {}^\xi\mathcal{S}'$  indexed by  ${}^\xi\mathfrak{B} \times {}^\xi\mathfrak{B}$  as follows. For  $B, B'$  in  ${}^\xi\mathfrak{B}$ , the  $(B, B')$ -entry of  ${}^\xi\mathcal{M}$  is  $v^{-\kappa(B) - \kappa(B')} (B : B')$ ; the  $(B, B')$ -entry of  ${}^\xi\mathcal{S}$  is  $P_{B', B}$ ; the  $(B, B')$ -entry of  ${}^\xi\mathcal{T}$  is  $\tilde{\Lambda}_{B, B'}$ ; the  $(B, B')$ -entry of  ${}^\xi\mathcal{S}'$  is  $\tilde{P}_{B, B'}$ . Using 13.7(a) we deduce from 13.6(d) the equality of matrices

$$(a) \quad {}^\xi\mathcal{M} = ({}^\xi\mathcal{S})({}^\xi\mathcal{T})({}^\xi\mathcal{S}').$$

As in the proof of 13.7(a) the last equality determines uniquely the matrices  ${}^\xi\mathcal{S}, {}^\xi\mathcal{T}, {}^\xi\mathcal{S}'$  if the matrix  ${}^\xi\mathcal{M}$  is known; in fact, it provides an algorithm for computing the entries of these three matrices (and in particular for the entries  $P_{B', B}$  in terms of the entries of  ${}^\xi\mathcal{M}$ . Now under the bijection  $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi\mathfrak{B}$  (see 11.12(a)) the matrix  ${}^\xi\mathcal{M}$  becomes a matrix indexed by  $\mathbf{B} \times \mathbf{B}$  whose  $(b, b')$ -entry is  $v^{-\kappa(B) - \kappa(B')} (b, b')$ ; these entries are explicitly computable from the combinatorial description of  $(:)$  on  $\mathbf{V}$ . We see that:

(b) *The quantities  $P_{B', B}$  are computable by an algorithm provided that the bijection  $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi\mathfrak{B}$  is known.*

This can be used in several examples to compute the  $P_{B', B}$  explicitly. The algorithm in (b) seems to depend on the choice of  $\eta$  such that  $\underline{\eta} = \delta$ ; but in fact, by the results in [LY, 3.9], 10.22, 11.13, it does not depend on this choice.

In the case where  $m = 1$ , there is another algorithm to compute the quantities  $P_{B',B}$ ; see [L7, Theorem 24.8]. It again displays the matrix  ${}^\xi\mathcal{S}$  as the first of the three factors in a matrix decomposition like (a), but with the matrix  ${}^\xi\mathcal{M}$  being replaced by a matrix indexed by a pair of irreducible representation of a Weyl group and with entries determined by a prescription quite different from that used in this paper. In that case the bijection  $\gamma : \mathbf{B} \xrightarrow{\sim} {}^\xi\mathfrak{B}$  is replaced by the “generalized Springer correspondence”. It would be interesting to understand better the connection between these two approaches to the quantities  $P_{B',B}$ .

14. ODD VANISHING

In this section we show that for any irreducible local system  $\mathcal{L}$  on a  $G_0$ -orbit in  $\mathfrak{g}_\delta^{nil}$  the cohomology sheaves of  $\mathcal{L}^\sharp \in \mathcal{D}(\mathfrak{g}_\delta)$  are zero in odd degrees. (See Theorem 14.10.) In the case where  $m \gg 0$ , this follows from the analogous result in the  $\mathbf{Z}$ -graded case in [L4]. In the case where  $m = 1$  this follows from [L7, Theorem 24.8(a)]. In the case where  $m > 1, \delta = \underline{1}, \mathcal{L} = \bar{\mathbf{Q}}_l$  and  $G, \mathfrak{g} = \oplus_i \mathfrak{g}_i$  are as in [LY, 0.3], this follows from [L8, Theorem 11.3] and from the known odd vanishing result for affine Schubert varieties.

14.1. We preserve the setup of 10.1, 10.2. For  $\varpi \in \mathbf{E}'$  let  $h(\varpi) \in \mathbf{Z}$  be as in 10.2. For  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  we set  $h(\mathbf{c}) = h(\varpi)$  where  $\varpi$  is any element of  $\mathbf{c} \cap \mathbf{E}'$ ; this is well defined, by 10.7(a). For  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  we set  $T_{\mathbf{c}} = v^{-h(\mathbf{c})}\tilde{T}_{\mathbf{c}} \in \mathbf{V}'$ . Note that  $\{T_{\mathbf{c}}; \mathbf{c} \in \overset{\circ}{\mathbf{E}}\}$  is a  $\mathbf{Q}(v)$ -basis of  $\mathbf{V}'$ . Let  $f \mapsto f^\heartsuit$  be the field involution of  $\mathbf{Q}(v)$  which carries  $v$  to  $-v$ ; this extends to a field involution of  $\mathbf{Q}((v))$  (denoted again by  $f \mapsto f^\heartsuit$ ) given by  $\sum_i c_i v^i \mapsto \sum_i c_i (-v)^i$ , where  $c_i \in \mathbf{Q}$ .

Let  $b \mapsto b^\heartsuit$  be the  $\mathbf{Q}$ -linear involution  $\mathbf{V}' \rightarrow \mathbf{V}'$  such that  $(fT_{\mathbf{c}})^\heartsuit = f^\heartsuit T_{\mathbf{c}}$  for any  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  and  $f \in \mathbf{Q}(v)$ .

**Lemma 14.2** (14.2). *For any  $b, b'$  in  $\mathbf{V}'$  we have  $(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit$ .*

It is enough to show that for any  $\mathbf{c}, \mathbf{c}'$  in  $\overset{\circ}{\mathbf{E}}$  and any  $f, f'$  in  $\mathbf{Q}(v)$  we have

$$(f^\heartsuit T_{\mathbf{c}} : f'^\heartsuit T_{\mathbf{c}'}) = (f T_{\mathbf{c}} : f' T_{\mathbf{c}'})^\heartsuit.$$

We can assume that  $f = f' = 1$ . It is enough to show that

(a) 
$$(T_{\mathbf{c}} : T_{\mathbf{c}'}) \in \mathbf{Q}(v^2),$$

or that

$$v^{-h(\varpi_1)+h(\varpi_2)}[\varpi_1 | \varpi_2] \in \mathbf{Q}(v^2)$$

for any  $\varpi_1, \varpi_2$  in  $\mathbf{E}'$  (notation of 10.12(a)) or that

(b) 
$$v^{-h(\varpi_1)+h(\varpi_2)} \sum_{w \in \mathcal{W}} v^{\tau(\varpi_2, w\varpi_1)} \in \mathbf{Q}(v^2).$$

From 10.8(b) we see that for  $\varpi, \varpi'$  in  $\mathbf{E}'$  we have

(b) 
$$\tau(\varpi, \varpi') = h(\varpi) + h(\varpi') \pmod{2}.$$

From 10.11(b) we see that for  $\varpi \in \mathbf{E}', w \in \mathcal{W}$  and  $N \in \mathbf{Z}$  we have

$$\dim {}^\epsilon \mathfrak{p}_N^{w\varpi} = \dim {}^\epsilon \mathfrak{p}_N^\varpi, \text{ hence } \dim {}^\epsilon \mathfrak{u}_N^{w\varpi} = \dim {}^\epsilon \mathfrak{u}_N^\varpi,$$

so that

$$h(w\varpi) = h(\varpi).$$

Using this and (b) we see that for  $\varpi, \varpi'$  in  $\mathbf{E}'$  we have

$$\sum_{w \in \mathcal{W}} v^{\tau(\varpi', w\varpi)} \in v^{h(\varpi') + h(\varpi)} \mathbf{N}[v^2, v^{-2}].$$

Thus, (b) holds. The lemma is proved.

14.3. From 14.2 we deduce that  $\heartsuit : \mathbf{V}' \rightarrow \mathbf{V}'$  carries  $\mathfrak{R}_l = \mathfrak{R}_r$  onto  $\mathfrak{R}_l = \mathfrak{R}_r$ ; hence it induces a  $\mathbf{Q}$ -linear involution  $\mathbf{V} \rightarrow \mathbf{V}$  (denoted again by  $\heartsuit$ ). From 14.2 we deduce:

(a) For any  $b, b'$  in  $\mathbf{V}$  we have  $(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit$ .

14.4. Let  $\mathbf{c} \in \mathring{\mathbf{E}}, \varpi \in \mathbf{E}'' \cap \mathbf{c}$  and let  $a_\varpi = \sum_j v^{-2s_j} v^{d(\varpi)} \in \mathcal{A}$  be as in 11.1. Note that

$$(a_\varpi)^\heartsuit = \sum_j v^{-2s_j} (-v)^{d(\varpi)} = (-1)^{d(\varpi)} a_\varpi.$$

Hence we have

$$(a_\varpi^{-1} \tilde{T}_\mathbf{c})^\heartsuit = (a_\varpi^{-1} v^{h(\mathbf{c})} T_\mathbf{c})^\heartsuit = (-1)^{d(\varpi) + h(\mathbf{c})} a_\varpi^{-1} v^{h(\mathbf{c})} T_\mathbf{c} = (-1)^{d(\varpi) + h(\mathbf{c})} a_\varpi^{-1} \tilde{T}_\mathbf{c}.$$

It follows that:

(a)  $\heartsuit : \mathbf{V}' \rightarrow \mathbf{V}'$  carries  $\mathbf{V}'_{\mathcal{A}}$  onto  $\mathbf{V}'_{\mathcal{A}}$  and  $\heartsuit : \mathbf{V} \rightarrow \mathbf{V}$  carries  $\mathbf{V}_{\mathcal{A}}$  onto  $\mathbf{V}_{\mathcal{A}}$ .

14.5. We show:

a) For any  $b \in \mathbf{V}'$  we have  $\overline{b^\heartsuit} = (\overline{b})^\heartsuit$ . Hence for any  $b \in \mathbf{V}$  we have  $\overline{b^\heartsuit} = (\overline{b})^\heartsuit$ .

We can assume that  $f = fT_\mathbf{c}$  where  $\mathbf{c} \in \mathring{\mathbf{E}}$  and  $f \in \mathbf{Q}(v)$ . Note that  $\overline{f^\heartsuit} = (\overline{f})^\heartsuit$ . Hence we can assume that  $f = 1$ . We have

$$\overline{T_\mathbf{c}} = \overline{v^{-h(\mathbf{c})} \tilde{T}_\mathbf{c}} = v^{h(\mathbf{c})} \tilde{T}_\mathbf{c} = v^{2h(\mathbf{c})} T_\mathbf{c},$$

hence

$$(\overline{T_\mathbf{c}})^\heartsuit = v^{2h(\mathbf{c})} T_\mathbf{c}.$$

We have  $\overline{T_\mathbf{c}^\heartsuit} = \overline{T_\mathbf{c}} = v^{2h(\mathbf{c})} T_\mathbf{c}$ . This proves (a).

14.6. We show:

(a)  $b \mapsto b^\heartsuit$  is a bijection  $\mathbf{B}' \xrightarrow{\sim} \mathbf{B}'$ .

Let  $b \in \mathbf{B}'$ . From  $b \in \mathbf{V}_{\mathcal{A}}$  we see using 14.4(a) that  $b^\heartsuit \in \mathbf{V}_{\mathcal{A}}$ . From  $\bar{b} = b$  we see using 14.5(a) that  $\overline{b^\heartsuit} = (\overline{b})^\heartsuit = b^\heartsuit$ . From  $(b : b) \in \mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]])$  we see using 14.3(a) that

$$(b^\heartsuit : b'^\heartsuit) = (b : b')^\heartsuit \in (\mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]]))^\heartsuit = \mathbf{Q}(v) \cap (1 + v\mathbf{Z}[[v]]) \subset 1 + v\mathbf{Z}[[v]].$$

Using this and the definitions, we see that  $b^\heartsuit \in \mathbf{B}'$ . Thus the map  $b \mapsto b^\heartsuit, \mathbf{B}' \rightarrow \mathbf{B}'$  is well defined. Since this map has square 1, it is a bijection. This proves (a).

From (a) we deduce:

(b) If  $b \in \mathbf{B}$ , then  $b^\heartsuit = s_b \tilde{b}$  for a well-defined  $s_b \in \{1, -1\}$  and a well-defined  $\tilde{b} \in \mathbf{B}$ .

The following result makes (b) more precise.

**Lemma 14.7** (14.7). *Let  $\mathcal{O}$  be a  $G_0$ -orbit in  $\mathfrak{g}_\delta^{nil}$  and let  $b \in \mathbf{B}_{\mathcal{O}}$ . We have  $b^\heartsuit = (-1)^{\dim \mathcal{O}} b$ .*

We argue by induction on  $\dim \mathcal{O}$ ; we can assume that the result holds when  $\mathcal{O}$  is replaced by an orbit of dimension  $< \dim \mathcal{O}$  (if any). We have  $\gamma(b) = \mathcal{L}^\sharp[\dim \mathcal{O}]$  where  $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ . Let  $x \in \mathcal{O}$ ; we associate to  $x$  an  $\epsilon$ -spiral  $\mathfrak{p}_*^\phi$  and a splitting  $\tilde{\mathcal{I}}_*^\phi$  of it as in [LY, 2.9]. Recall that  $\tilde{\mathcal{I}}_\eta^\phi \subset \mathcal{O}$  and that  $\mathcal{L}_1 := \mathcal{L}|_{\tilde{\mathcal{I}}_\eta^\phi}$  is an irreducible  $\tilde{L}_0^\phi$ -equivariant local system on  $\tilde{\mathcal{I}}_\eta^\phi$ ; thus,  $\mathcal{L}_1^\sharp \in \mathcal{D}(\tilde{\mathcal{I}}_\eta^\phi)$  is defined. Let  $I = {}^\epsilon \text{Ind}_{\mathfrak{p}_\eta^\phi}^{\mathfrak{g}_\delta}(\mathcal{L}_1^\sharp) \in \mathcal{Q}(\mathfrak{g}_\delta)$ ; we have clearly  $I \in {}^\xi \mathcal{K}(\mathfrak{g}_\delta)$ . By [LY, 2.9(b),(c)], in  ${}^\xi \mathcal{K}(\mathfrak{g}_\delta)$  we have

$$I = \mathcal{L}^\sharp + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} \gamma(b'),$$

where  $f_{b'} \in \mathcal{A}$ . Define  $I' \in \mathbf{V}_{\mathcal{A}}$  by  $\gamma(I') = I$ . We have

$$(a) \quad I' = v^{-\dim \mathcal{O}} b + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} b'.$$

By [L4, 17.2, 17.3], in  $\mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)$  we have

$$(b) \quad h\mathcal{L}_1^\sharp = \sum_{j \in J} h_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathcal{I}}_\eta^\phi}(C(j)),$$

where  $h = h^\heartsuit \in \mathcal{A} - \{0\}$ ,  $J$  is a finite set and for  $j \in J$ ,  $\mathfrak{q}(j)$  is a parabolic subalgebra of  $\tilde{\mathcal{I}}_\eta^\phi$  with Levi subalgebra  $\mathfrak{m}(j)$  such that  $\mathfrak{q}(j), \mathfrak{m}(j)$  are compatible with the  $\mathbf{Z}$ -grading of  $\tilde{\mathcal{I}}_\eta^\phi$ ,  $C(j) \in \mathcal{Q}(\mathfrak{m}(j)_\eta)$  is a cuspidal perverse sheaf and  $h_j \in \mathcal{A}$ . Moreover, since  $\mathcal{L}_1^\sharp$  belongs to the block of  $\mathcal{Q}(\tilde{\mathcal{I}}_\eta^\phi)$  given by  $\xi$ , we can assume that  $\mathfrak{m}(j) = \mathfrak{m}$  and  $C(j) = \tilde{C}$  for all  $j$ . Thus (b) can be written in the form

$$(c) \quad h\mathcal{L}_1^\sharp = F_0 + F_1,$$

where

$$F_0 = \sum_{j \in J} h'_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathcal{I}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta]),$$

$$F_1 = \sum_{j \in J} h''_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathcal{I}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta]),$$

$$h'_j + h''_j = h_j v^{\dim \mathfrak{m}_\eta}, h'_j \in \mathbf{Z}[v^2, v^{-2}], h''_j \in v\mathbf{Z}[v^2, v^{-2}].$$

Let  $\mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}$  be the  $\mathbf{Z}[v^2, v^{-2}]$ -submodule of  $\mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}$  with basis  $\mathcal{L}'^\sharp$  for various  $(\mathcal{O}', \mathcal{L}')$  in  $\mathcal{I}(\tilde{\mathcal{I}}_\eta^\phi)$ . By [L4, 21.1(c)], for any  $j \in J$ , we have

$$\text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathcal{I}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta]) \in \mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}.$$

Hence  $F_0 \in \mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}, F_1 \in v\mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}$ . We have clearly  $h\mathcal{L}_1^\sharp \in \mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev}$ . Since  $v\mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev} \cap \mathcal{K}(\tilde{\mathcal{I}}_\eta^\phi)^{ev} = 0$ , we deduce from (c) that  $h\mathcal{L}_1^\sharp = F_0$ ; that is,

$$h\mathcal{L}_1^\sharp = \sum_{j \in J} h'_j \text{ind}_{\mathfrak{q}(j)_\eta}^{\tilde{\mathcal{I}}_\eta^\phi}(\tilde{C}[-\dim \mathfrak{m}_\eta]),$$

where  $h'_j \in \mathbf{Z}[v^2, v^{-2}]$ . Applying  ${}^e \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}$  and using the transitivity property 4.2 from [LY] (and also 10.4(a)) we obtain

$$hI = \sum_{j \in J} h'_j I_{\varpi_j},$$

where  $\varpi_j \in \mathbf{E}'$  for  $j \in J$ . Since  $\gamma^{-1}(\sum_{j \in J} h'_j I_{\varpi_j})$  is fixed by  $\heartsuit$ , it follows that  $hI'$  is fixed by  $\heartsuit$ . Since  $h \neq 0$  and  $h^\heartsuit = h$ , we deduce that  $I'^\heartsuit = I'$ . Using (a) and the induction hypothesis, we see that

$$\begin{aligned} & (v^{-\dim \mathcal{O} b})^\heartsuit + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O} b' \in \mathbf{B}_{\mathcal{O}'}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'}^\heartsuit (-1)^{\dim \mathcal{O}' b'} \\ &= v^{-\dim \mathcal{O} b} + \sum_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O} b' \in \mathbf{B}_{\mathcal{O}'}} \sum_{b' \in \mathbf{B}_{\mathcal{O}'}} f_{b'} b'. \end{aligned}$$

If using this and 14.6(b), we deduce that  $(-v)^{-\dim \mathcal{O}} s_b \tilde{b} - v^{-\dim \mathcal{O} b}$  is a  $\mathcal{A}$ -linear combination of elements in  $\cup_{\mathcal{O}'; \dim \mathcal{O}' < \dim \mathcal{O}} \mathbf{B}_{\mathcal{O}'}$ .

Since  $b \in \mathbf{B}_{\mathcal{O}}$  and  $\tilde{b} \in \mathbf{B}$  this forces  $\tilde{b} = b$  and  $s_b = (-1)^{\dim \mathcal{O}}$ . This completes the proof of the lemma.

14.8. We show:

(a) Let  $\mathcal{O}, \mathcal{O}'$  be  $G_{\mathbb{Q}}$ -orbits in  $\mathfrak{g}_\delta^{nil}$  and let  $B, B'$  in  $\mathfrak{B}$  be such that the support of  $B$  (resp.  $B'$ ) is the closure of  $\mathcal{O}$  (resp.  $\mathcal{O}'$ ). We have  $(B : B')^\heartsuit = (-1)^{\dim \mathcal{O} + \dim \mathcal{O}'} (B : B')$ .

If  $\psi(B) \neq \psi(B')$ , then  $(B : B') = 0$  by [LY, 7.9(a)] and (a) holds. Assume now that  $\psi(B) = \psi(B')$ . We can assume that  $\psi(B) = \psi(B') = \xi$ . It is enough to prove that, if  $b \in \mathbf{B}_{\mathcal{O}}$ ,  $b' \in \mathbf{B}_{\mathcal{O}'}$ , then  $(b : b')^\heartsuit = (-1)^{\dim \mathcal{O} + \dim \mathcal{O}'} (b : b')$ . Using 14.2 and 14.7, we have

$$(b : b')^\heartsuit = (b^\heartsuit : b'^\heartsuit) = ((-1)^{\dim \mathcal{O} b}, (-1)^{\dim \mathcal{O}' b'})$$

and (a) is proved.

14.9. We show:

(a) For any  $B, B'$  in  $\mathfrak{B}$  we have  $P_{B, B'} \in \mathbf{N}[v^{-2}]$ ,  $\tilde{P}_{B, B'} \in \mathbf{N}[v^{-2}]$ ,  $\tilde{\Lambda}_{B, B'} \in \mathbf{Q}((v^2))$  (notation of 13.6).

The proof is similar to that of 13.7(a). With the notation in 14.8(a) we apply  $\heartsuit$  to

$$v^{-\dim \mathcal{O} - \dim \mathcal{O}'} (B : B') = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B} \tilde{\Lambda}_{B_1, B_2} \tilde{P}_{B_2, B'}$$

(see 13.6(d)); using 14.8 we obtain

$$v^{-\dim \mathcal{O} - \dim \mathcal{O}'} (B : B')^\heartsuit = \sum_{B_1 \in \mathfrak{B}, B_2 \in \mathfrak{B}} P_{B_1, B}^\heartsuit \tilde{\Lambda}_{B_1, B_2}^\heartsuit \tilde{P}_{B_2, B'}^\heartsuit.$$

When  $B, B'$  vary, this can be expressed as the decomposition of the matrix  $\mathcal{M} = (v^{c_B - c_{B'}} v^{-\kappa(B) - \kappa(B')}) (B : B')$  (indexed by  $\mathfrak{B} \times \mathfrak{B}$ ) as a product of three matrices  $\mathcal{S}^\heartsuit \mathcal{T}^\heartsuit \mathcal{S}'^\heartsuit$  where  $\mathcal{S}^\heartsuit$  (resp.  $\mathcal{S}'^\heartsuit$ ) is the matrix indexed by  $\mathfrak{B} \times \mathfrak{B}$  whose  $(B, B')$ -entry is  $P_{B', B}^\heartsuit$  (resp.  $\tilde{P}_{B, B'}^\heartsuit$ ) and  $\mathcal{T}^\heartsuit$  is the matrix indexed by  $\mathfrak{B} \times \mathfrak{B}$  whose  $(B, B')$ -entry is  $\tilde{\Lambda}_{B, B'}^\heartsuit$ . Recall from 13.6 that we have also  $\mathcal{M} = \mathcal{S} \mathcal{T} \mathcal{S}'$  (notation of 13.6). Thus we have

$$\mathcal{S}^\heartsuit \mathcal{T}^\heartsuit \mathcal{S}'^\heartsuit = \mathcal{S} \mathcal{T} \mathcal{S}'.$$

Now by 13.6(c),(e), the matrix  $\mathcal{T}$  (and hence the matrix  $\mathcal{T}^\heartsuit$ ) belongs to a subgroup of  $GL_N$  ( $N = \sharp(\mathfrak{B})$ ) of the form  $GL_{N_1} \times \dots \times GL_{N_k}$  where  $N_1, \dots, N_k$  are the sizes of the various subsets  $\mathfrak{B}_O$ ; moreover, by 13.6(a),(b), the matrix  $\mathcal{S}$  (hence the matrix  $\mathcal{S}^\heartsuit$ ) belongs to the unipotent radical of a parabolic subgroup of  $GL_N$  with Levi subgroup equal to the subgroup of  $GL_N$  considered above, while the matrix  $\mathcal{S}'$  (hence the matrix  $\mathcal{S}'^\heartsuit$ ) belongs to the unipotent radical of the opposite parabolic subgroup with the same Levi subgroup. This forces the equalities  $\mathcal{S}^\heartsuit = \mathcal{S}$ ,  $\mathcal{T}^\heartsuit = \mathcal{T}$ ,  $\mathcal{S}'^\heartsuit = \mathcal{S}'$ . Now the equality  $\mathcal{S}^\heartsuit = \mathcal{S}$  implies  $P_{B',B}^\heartsuit = P_{B',B}$  for all  $B', B$  in  $\mathfrak{B}$ . Similarly, from  $\mathcal{T}^\heartsuit = \mathcal{T}$  we see that  $\tilde{\Lambda}_{B,B'}^\heartsuit = \tilde{\Lambda}_{B,B'}$  for all  $B, B'$  in  $\mathfrak{B}$  and from  $\mathcal{S}'^\heartsuit = \mathcal{S}'$  we see that  $\tilde{P}_{B,B'}^\heartsuit = \tilde{P}_{B,B'}$ . This proves (a).

**Theorem 14.10.**

(a) Let  $(O, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$  and let  $A = \mathcal{L}^\sharp \in \mathcal{Q}(\mathfrak{g}_\delta)$ . We have  $\mathcal{H}^a A = 0$  for any odd integer  $a$ .

(b) Let  $\varpi \in \mathbf{E}'$ . We have  $\mathcal{H}^a(I_\varpi) = 0$  for any odd integer  $a$ .

(c) Let  $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$  be as in 4.1(a) of [LY] with  $\epsilon = \dot{\eta}$ , and let  $(O', \mathcal{L}') \in \mathcal{I}(\mathfrak{l}_\eta)$  (see 1.2). We form  $A' = \mathcal{L}'^\sharp \in \mathcal{Q}(\mathfrak{l}_\eta)$ . We have  $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')) = 0$  for any odd integer  $a$ .

(d) In the setup of (c),  ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$  is a direct sum of complexes of the form  $\mathcal{L}^\sharp[s]$  for various  $(O, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$  and various even integers  $s$ .

(a) follows from 14.9(a).

We prove (b). Let  $\mathbf{c} \in \overset{\circ}{\mathbf{E}}$  be such that  $\varpi \in \mathbf{c}$ . In  $\mathbf{V}$  we have  $T_{\mathbf{c}} = \bigoplus_{b \in \mathbf{B}} f_b v^{-\kappa(b)} b$ , where  $f_b \in \mathcal{A}$ . Applying  ${}^\heartsuit$  and using  $T_{\mathbf{c}}^\heartsuit = T_{\mathbf{c}}$  and  $(v^{-\kappa(b)} b)^\heartsuit = v^{-\kappa(b)} b$  for  $b \in \mathbf{B}$  (see 14.7) we see that

$$\bigoplus_{b \in \mathbf{B}} f_b^\heartsuit v^{-\kappa(b)} b = \bigoplus_{b \in \mathbf{B}} f_b v^{-\kappa(b)} b.$$

Hence  $f_b^\heartsuit = f_b$ , that is,  $f_b \in \mathbf{Z}[v^2, v^{-2}]$ . Thus,

(e)  $I_\varpi$  is isomorphic to a direct sum of complexes of the form  $B[-\kappa(B)][2s]$  with  $B \in \mathfrak{B}$  and  $s \in \mathbf{Z}$ .

Hence  $\mathcal{H}^a I_\varpi$  is isomorphic to a direct sum of sheaves of the form  $\mathcal{H}^{a+2s}(B[-\kappa(B)])$  with  $B \in \mathfrak{B}$ . Hence the desired result follows from (a).

We prove (c). By 1.5(a) of [LY] we can find  $\mathfrak{q}_*$ ,  $(\hat{\mathfrak{p}}_*, \hat{L}, \hat{P}_0, \hat{\mathfrak{l}}, \hat{\mathfrak{l}}_*, \hat{\mathfrak{u}}_*)$  as in 4.2 of [LY] with  $\epsilon = \dot{\eta}$  and a cuspidal perverse sheaf  $C$  in  $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$  such that  $A'[s']$  is a direct summand of  $\text{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(C[-\dim \mathfrak{l}_\eta])$  for some  $s' \in \mathbf{Z}$  hence, by 4.2(a) of [LY],  ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')[s']$  is a direct summand of  ${}^\epsilon \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta])$ ; moreover, by [L4, 21.1(c)], we have  $s' = 2s''$  for some  $s'' \in \mathbf{Z}$ . Hence  $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A'))$  is a direct summand of

$$\mathcal{H}^{a-2s''}({}^\epsilon \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta])).$$

We can assume that  $\hat{\mathfrak{p}}_\eta$  is an  $\epsilon$ -spiral with splitting  $\mathfrak{m}_*$  (in 10.1) and that  $C = \tilde{C}$  (in 10.1). Then  ${}^\epsilon \text{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C[-\dim \mathfrak{l}_\eta]) = I_\varpi$  for some  $\varpi \in \mathbf{E}'$  and  $\mathcal{H}^a({}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A'))$  is a direct summand of  $\mathcal{H}^{a-2s''} I_\varpi$ . The desired result follows from (b).

We prove (d). As in the proof of (c),  ${}^\epsilon \text{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$  is a direct summand of  $I_\varpi[-2s'']$  for some  $\varpi \in \mathbf{E}'$  and  $s'' \in \mathbf{Z}$ . This, together with (e) gives the desired result. The theorem is proved.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

*E-mail address:* gyuri@math.mit.edu

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06511

*E-mail address:* zhiweiyun@gmail.com