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HECKE MODULES BASED ON INVOLUTIONS IN EXTENDED WEYL GROUPS

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Abstract. Let X be the group of weights of a maximal torus of a simply connected semisimple group over $\mathbf{C}$ and let W be the Weyl group. The semidirect product $W((\mathbf{Q}\otimes X)/X)$ is called an extended Weyl group. There is a natural $\mathbf{C}(v)$-algebra $\mathbf{H}$ called the extended Hecke algebra with basis indexed by the extended Weyl group which contains the usual Hecke algebra as a subalgebra. We construct an $\mathbf{H}$-module with basis indexed by the involutions in the extended Weyl group. This generalizes a construction of the author and Vogan.

Introduction and statement of results

0.1. Let $\mathbf{k}$ be an algebraically closed field. Let G be a connected reductive group over $\mathbf{k}$. Let T be a maximal torus of G and let U be the unipotent radical of a Borel subgroup of G containing T. Let N be the normalizer of T and let $W=N/T$ be the Weyl group; let $w\mapsto |w|$ be the length function on W, let $S=\{w\in W;|w|=1\}$, and let $\kappa:N\to W$ be the obvious map. The obvious action of W on T is denoted by $w:t\mapsto w(t)$. Let $Y=\mathrm{Hom}(\mathbf{k}^*,T)$, $X=\mathrm{Hom}(T,\mathbf{k}^*)$ and let $\langle,\rangle:Y\times X\to\mathbf{Z}$ be the obvious pairing. We regard Y,X as groups with operation written as addition. Let K be a field of characteristic zero and let $X_K=K\otimes X=\mathrm{Hom}(Y,K)$. Let $\bar X=X_K/X=(K/\mathbf{Z})\otimes X$. The obvious pairing $\langle,\rangle:Y\times X_K\to K$ restricts to a pairing $Y\times X\to\mathbf{Z}$ and hence it induces a pairing $\lfloor,\rfloor:Y\times\bar X\to K/\mathbf{Z}$. We define an action of W on Y by $w:y\mapsto y'$, where $y'(z)=w(y(z))$ for $z\in\mathbf{k}^*$. We define an action of W on X_K by the equality $\langle w(y),w(x)\rangle=\langle y,x\rangle$ for all $y\in Y,x\in X_K,w\in W$. This action preserves X and hence it induces a W-action on $\bar X$. Let $\check R\subset Y$ be the set of coroots, let $\check R^+\subset\check R$ be the set of positive coroots determined by U, let $\check R^-=\check R-\check R^+$. For $s\in S$ we denote by $\check\alpha_s\in Y$ the simple coroot such that $s(\check\alpha_s)=-\check\alpha_s$. For $\lambda\in\bar X,s\in S$ we write $s\in W_\lambda$ if $\lfloor\check\alpha_s,\lambda\rfloor=0$; we write $s\notin W_\lambda$ if $\lfloor\check\alpha_s,\lambda\rfloor\ne 0$. Note that if $s\in W_\lambda$, then $s\lambda=\lambda$. For $s\in S$ let T_s be the image of $\check\alpha_s:\mathbf{k}^*\to T$.

0.2. Let $W_2=\{w\in W;w^2=1\}$. For any integer $m\ge 1$ we set

$$\bar X_m=\{\lambda\in\bar X;m^2\lambda=\lambda\},$$

$$\tilde X_m=\{(w,\lambda)\in W_2\times\bar X;w(\lambda)=-m\lambda\}.$$

We write $W\bar X$ instead of $W\times\bar X$ with the group structure

$$(w,\lambda)(w',\lambda')=(ww',w'^{-1}(\lambda)+\lambda').$$

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We call $W\bar{X}$ the *extended Weyl group*. Then

$$\tilde{X}_1 = \{(w,\lambda) \in W_2 \times \bar{X}; w(\lambda) = -\lambda\} = \{(w,\lambda) \in W\bar{X}; (w,\lambda)^2 = (1,0)\}$$

is exactly the set of involutions in the extended Weyl group $W\bar{X}$.

More generally, if $m \geq 1$, then $\{(w,\lambda) \in W \times \bar{X}; \lambda \in \bar{X}_m\}$ is a subgroup of $W\bar{X}$ denoted by $W\bar{X}_m$ and $(w,\lambda) \mapsto (w,\lambda)^* := (w, m\lambda)$ is an involutive automorphism of $W\bar{X}_m$. Moreover, $\tilde{X}_m$ is the set of $*$-twisted involutions of $W\bar{X}_m$, that is, the set of all $(w,\lambda) \in W\bar{X}_m$ such that $(w,\lambda)(w,\lambda)^* = (1,0)$.

If $m \geq 1$ and $(w,\lambda) \in \tilde{X}_m$, then $\lambda \in \bar{X}_m$. Note that if $(w,\lambda) \in \tilde{X}_m$ and $s \in S$, then $(sws, s\lambda) \in \tilde{X}_m$; if in addition $sw = ws$, then $(w, s\lambda) \in \tilde{X}_m$. If we have both $sw = ws$ and $s\lambda = \lambda$, then $(sw, \lambda) \in \tilde{X}_m$.

Let p be a prime number and let $q > 1$ be a power of p. We set $Q = q^2$. We assume that the characteristic of $\mathbf{k}$ is either 0 or p. Then $\bar{X}_q, \tilde{X}_q$ are defined.

We fix a square root $\sqrt{-1}$ of -1 in $\mathbf{C}$. For $\lambda \in \bar{X}_q$, $s \in S$, we define $[\lambda, s] \in \{1, -1\}$ as follows. We have $\langle \check{\alpha}_s, \lambda \rangle = e/(Q-1)$ with $e \in \mathbf{Z}$. When $p \neq 2$ we set $[\lambda, s] = 1$ if $e \in 2\mathbf{Z}$ and $[\lambda, s] = \sqrt{-1}$ if $e \in \mathbf{Z} - 2\mathbf{Z}$; when $p = 2$ we set $[\lambda, s] = 1$.

0.3. For $w \in W_2, s \in S$ such that $sw = ws$ we define, following [L5, 1.18], a number $(w : s) \in \{-1, 0, 1\}$ as follows. Assume first that G is almost simple, simply laced. In [L5, 1.5, 1.7], a root system with a set of coroots $\check{R}_w \subset \check{R}$ and a set of simple coroots $\check{\Pi}_w$ for $\check{R}_w$ was associated to w; we have $\check{\alpha}_s \in \check{\Pi}_w$. This root system is simply laced and has no component of type $A_l, l > 1$. If the component containing $\check{\alpha}_s$ is not of type A_1, there is a unique sequence $\check{\alpha}_1, \check{\alpha}_2, \dots, \check{\alpha}_e$ in $\check{\Pi}_w$ such that $\check{\alpha}_i, \check{\alpha}_{i+1}$ are joined in the Dynkin diagram of $\check{R}_w$ for $i = 1, 2, \dots, e-1$, $\check{\alpha}_1 = \check{\alpha}_s$ and $\check{\alpha}_e$ corresponds to a branch point of the Dynkin diagram of $\check{R}_w$; if the component containing $\check{\alpha}_s$ is of type A_1 we define $\check{\alpha}_1, \check{\alpha}_2, \dots, \check{\alpha}_e$ as the sequence with one term $\check{\alpha}_s$ (so that $e = 1$). We define $(w : s) = (-1)^e$ if $|sw| < |w|$ and $(w : s) = (-1)^{e+1}$ if $|sw| > |w|$. Next we assume that G is almost simple, simply connected, not simply laced. Then G can be regarded as a fixed point set of an automorphism of a simply connected, almost simple, simply laced group G' (as in [L5, 1.14]) with Weyl group W', a Coxeter group with a length preserving automorphism $W' \to W'$ with fixed point set W. When s is regarded as an element of W', it is a product of k commuting simple reflections $s'_1, s'_2, \dots, s'_k$ of W'; here $k \in \{1, 2, 3\}$. If $k \neq 2$, we define $(w : s)$ for W to be $(w : s_i)$ for G', where i is any element of $\{1, \dots, k\}$. If $k = 2$ we have either $ws_1 = s_1 w$, $ws_2 = s_2 w$ (in which case $(w : s)$ for G is defined to be $(w : s_1) = (w : s_2)$ for G') or $ws_1 = s_2 w$, $ws_2 = s_1 w$ (in which case $(w : s)$ for G is defined to be 0). We now drop the assumption that G is almost simple. Let G'' be the simply connected cover of an almost simple factor of the adjoint group of G with Weyl group $W'' \subset W$ such that $s \in W''$ and let w'' be the W'-component of w. Then $(w : s)$ for G is defined to be $(w'' : s)$ for G'' (which is defined as above).

For p, q as in §0.2, $(w,\lambda) \in \tilde{X}_q$, $s \in S$ such that $sw = ws$, we set

$$\delta_{w,\lambda;s} = \exp(2\pi\sqrt{-1}((q-e)/2)(1 - (w : s))\langle \check{\alpha}_s, \lambda \rangle)$$

if $p \neq 2$, $e = |w| - |sw| = \pm 1$ and $\delta_{w,\lambda;s} = 1$ if $p = 2$. (Note that $\exp(2\pi\sqrt{-1}x)$ is well defined for $x \in \mathbf{Q}/\mathbf{Z}$.) If G is simply laced, then $\delta_{w,\lambda;s} = 1$ (since $(w : s) = \pm 1$). In general we have $\delta_{w,\lambda;s} = \pm 1$. Indeed, we can assume that $p \neq 2$. It is enough to show that $(q-e)\langle\check\alpha_s,\lambda\rangle=0$. From our assumption we have

$$\lfloor\check\alpha_s,\lambda\rfloor=\lfloor w\check\alpha_s,w\lambda\rfloor=\lfloor -e\check\alpha_s,-q\lambda\rfloor=qe\lfloor\check\alpha_s,\lambda\rfloor=qe^{-1}\lfloor\check\alpha_s,\lambda\rfloor$$

and hence $(q-e)\lfloor\check\alpha_s,\lambda\rfloor=0$; our claim follows.

The following assumption will be made in parts of the paper (it will simplify some proofs).

(a) For $s\in S$, $\check\alpha_s;\mathbf{k}^*\to T_s$ is an isomorphism.
This is certainly satisfied if G is simply connected.

Here is one of the main results of this paper.

Theorem 0.4. *Let q,p be as in §0.2. Assume that §0.3(a) holds. Let M_q be the $\mathbf{C}$-vector space with basis $\{a_{w,\lambda};(w,\lambda)\in\tilde X_q\}$. If $p\ne 2$ let $z\in\mathbf{Z}$ be such that $2z\notin(q^2-1)\mathbf{Z}$; if $p=2$ let $z\in\mathbf{Z}$ be arbitrary. There is a unique action of the braid group of W on M_q in which the generators $\{\mathcal{T}_s;s\in S\}$ of the braid group applied to the basis elements of M_q are as follows. (We set $\Delta=1$ if $s\in W_\lambda$ and $\Delta=0$ if $s\notin W_\lambda$.)*

(a) $\mathcal{T}_s a_{w,\lambda}=a_{sws,\lambda}$ *if* $sw\ne ws,|sw|>|w|,\Delta=1$;
(b) $\mathcal{T}_s a_{w,\lambda}=a_{sws,\lambda}+(q-q^{-1})a_{w,\lambda}$ *if* $sw\ne ws,|sw|<|w|,\Delta=1$;
(c) $\mathcal{T}_s a_{w,\lambda}=a_{w,\lambda}+(q+1)a_{sw,\lambda}$ *if* $sw=ws,|sw|>|w|,\Delta=1$;
(d) $\mathcal{T}_s a_{w,\lambda}=(1-q^{-1})a_{sw,\lambda}+(q-q^{-1}-1)a_{w,\lambda}$ *if* $sw=ws,|sw|<|w|,\Delta=1$;
(e) $\mathcal{T}_s a_{w,\lambda}=[\lambda,s]a_{sws,s\lambda}$ *if* $sw\ne ws,|sw|>|w|,\Delta=0$;
(f) $\mathcal{T}_s a_{w,\lambda}=[\lambda,s]^{-1}a_{sws,s\lambda}$ *if* $sw\ne ws,|sw|<|w|,\Delta=0$;
(g) $\mathcal{T}_s a_{w,\lambda}=\delta_{w,s\lambda;s}a_{w,s\lambda}$ *if* $sw=ws,|sw|>|w|,\Delta=0$;
(h) $\mathcal{T}_s a_{w,\lambda}=-\delta_{w,s\lambda;s}\exp(2\pi\sqrt{-1}(w:s)z\langle\check\alpha_s,\lambda\rangle)a_{w,s\lambda}$ *if* $sw=ws,|sw|<|w|,\Delta=1$.

Note that the subspace of M_q spanned by $\{a_{w,0};w\in W_2\}$ is stable under the braid group action; the resulting braid group action on that subspace involves only the cases where $\Delta=1$ and in fact is the representation of the Hecke algebra of W with parameter q introduced in [LV]. Thus the theorem is a generalization of a part of [LV]. In the general case we can define operators $1_\lambda:M_q\to M_q$ (for $\lambda\in\bar X_q$) by $1_\lambda a_{w,\lambda'}=\delta_{\lambda,\lambda'}a_{w,\lambda'}$ for all $(w,\lambda')\in\tilde X_q$. The operators $\mathcal{T}_s$ and 1_λ on M_q satisfy the relations of an "extended Hecke algebra", isomorphic to the endomorphism algebra of the representation of $G(F_q)$ induced by the trivial representation of $U(F_q)$ (assuming that $\mathbf{k}$ is an algebraic closure of a finite field F_q and G is split over F_q). This endomorphism algebra was studied by Yokonuma [Y] and a description of it in terms of generators like $\mathcal{T}_s,1_\lambda$ was given in [L2]. The proof of the theorem is given in §4, in terms of $G(F_q),U(F_q)$ as above. Namely, we show that M_q can be interpreted as the vector space spanned by the double cosets $\Gamma_1\backslash\Gamma/\Gamma_2$ regarded naturally as a module over the algebra spanned as a vector space by the double cosets $\Gamma_1\backslash\Gamma/\Gamma_1$ for suitable finite groups $\Gamma_1\subset\Gamma\supset\Gamma_2$. (In our case we have $\Gamma=G(F_{q^2})$, $\Gamma_1=U(F_{q^2})$, $\Gamma_2=G(F_q)$.) A key role in our proof is played by a certain non-standard lifting (introduced in [L5]) to N for the involutions in W. (The usual lifting, due to Tits [T], is not suitable for the purposes of this paper.)

0.5. We now assume that $\mathbf{k}=\mathbf{C}$. Let v be an indeterminate and let $\mathbf{M}$ be the $\mathbf{C}(v)$-vector space with basis $\{\mathbf{a}_{w,\lambda};(w,\lambda)\in\tilde X_1\}$. For any

(a) $(w,\lambda)\in\tilde X_1$ and $s\in S$ such that $|sw|>|w|$ we set

$$\delta'_{w,\lambda;s}=\exp(2\pi\sqrt{-1}(1-(w:s))\lfloor\check\alpha_s,\lambda\rfloor).$$

We note that for w, λ, s as in (a) we have

$$\lfloor \check{\alpha}_s, \lambda \rfloor = \lfloor w\check{\alpha}_s, w\lambda \rfloor = \lfloor \check{\alpha}_s, -\lambda \rfloor = -\lfloor \check{\alpha}_s, \lambda \rfloor$$

and hence

(b) $2\lfloor \check{\alpha}_s, \lambda \rfloor = 0$ so that $\delta'_{w,\lambda;s}$ is well defined and is in $\{1, -1\}$.

The following result is a generic version of Theorem 0.4 in which q is replaced by v^2 and M_q is replaced by $\mathbf{M}$.

Theorem 0.6. *We assume that* $\mathbf{k} = \mathbf{C}$ *and that* §0.3(a) *holds. There is a unique action of the braid group of* W *on* $\mathbf{M}$ *in which the generators* $\{\mathcal{T}_s; s \in S\}$ *of the braid group applied to the basis elements of* $\mathbf{M}$ *are as follows.* (*We write* $\Delta = 1$ *if* $s \in W_\lambda$ *and* $\Delta = 0$ *if* $s \notin W_\lambda$.)

(a) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda}$ *if* $sw \neq ws, |sw| > |w|$;

(b) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda} + \Delta(v^2 - v^{-2})\mathbf{a}_{w,\lambda}$ *if* $sw \neq ws, |sw| < |w|$;

(c) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \delta'_{w,s\lambda;s}\mathbf{a}_{w,s\lambda} + \Delta(v + v^{-1})\mathbf{a}_{sw,\lambda}$ *if* $sw = ws, |sw| > |w|$;

(d) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \Delta(v - v^{-1})\mathbf{a}_{sw,\lambda} + \Delta(v^2 - v^{-2})\mathbf{a}_{w,\lambda} - \mathbf{a}_{w,s\lambda}$ *if* $sw = ws, |sw| < |w|$.

This can be deduced from Theorem 0.4 (see §4).

We can interpret the theorem as providing an $\mathbf{H}$-module structure on $\mathbf{M}$ where $\mathbf{H}$ is the extended Hecke algebra (see §4.5). The subspace of $\mathbf{M}$ spanned by $\{\mathbf{a}_{w,0}; w \in W_2\}$ is stable under the operators $\mathcal{T}_s$ and this defines a representation of the generic Hecke algebra of W which was defined in [LV].

0.7. The action in Theorem 0.6 can be specialized to $v = 1$. It becomes the braid group action on the $\mathbf{C}$-vector space with basis $\{\mathbf{a}_{w,\lambda}; (w, \lambda) \in \tilde{X}_1\}$ in which the generators $\mathcal{T}_s$ of the braid group act as follows. (Notation and assumptions are from Theorem 0.6.)

(a) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda}$ if $sw \neq ws$;

(b) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \delta'_{w,s\lambda;s}\mathbf{a}_{w,s\lambda} + 2\Delta\mathbf{a}_{sw,\lambda}$ if $sw = ws, |sw| > |w|$;

(c) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = -\mathbf{a}_{w,s\lambda}$ if $sw = ws, |sw| < |w|$.

This is actually a W-action since $\mathcal{T}_s^2$ acts as 1.

0.8. Let m be an integer ≥ 1 and let $\mathbf{M}_m$ be the $\mathbf{C}(v)$-vector space with basis $\{\mathbf{a}_{w,\lambda}; (w, \lambda) \in \tilde{X}_m\}$. In the following result (a variant of Theorems 0.4 and 0.6) the assumption §0.3(a) is not used.

Theorem 0.9. *There is a unique action of the braid group of* W *on* $\mathbf{M}_m$ *in which the generators* $\{\mathcal{T}_s; s \in S\}$ *of the braid group applied to the basis elements of* $\mathbf{M}_m$ *are as follows.* (*We write* $\Delta = 1$ *if* $s \in W_\lambda$ *and* $\Delta = 0$ *if* $s \notin W_\lambda$.)

(a) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda}$ *if* $sw \neq ws, |sw| > |w|$;

(b) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda} + \Delta(v^2 - v^{-2})\mathbf{a}_{w,\lambda}$ *if* $sw \neq ws, |sw| < |w|$;

(c) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{w,s\lambda} + \Delta(v + v^{-1})\mathbf{a}_{sw,\lambda}$ *if* $sw = ws, |sw| > |w|$;

(d) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \Delta(v - v^{-1})\mathbf{a}_{sw,\lambda} + \Delta(v^2 - v^{-2} - 1)\mathbf{a}_{w,\lambda} + (1 - \Delta)\mathbf{a}_{w,s\lambda}$ *if* $sw = ws, |sw| < |w|$.

The proof is given in §3. It relies on results in [LV] and [L4].

0.10. The action in Theorem 0.9 can be specialized to $v = 1$. It becomes the braid group action on the $\mathbf{C}$-vector space with basis $\{\mathbf{a}_{w,\lambda}; (w, \lambda) \in \tilde{X}_m\}$ in which the generators $\mathcal{T}_s$ of the braid group act as follows. (Notation and assumptions are from Theorem 0.9.)

(a) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda}$ if $sw \neq ws$;
(b) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{w,s\lambda} + 2\Delta \mathbf{a}_{sw,\lambda}$ if $sw = ws, |sw| > |w|$;
(c) $\mathcal{T}_s \mathbf{a}_{w,\lambda} = \mathbf{a}_{w,s\lambda} - 2\Delta \mathbf{a}_{w,\lambda}$ if $sw = ws, |sw| < |w|$.
This is actually a W-action.

0.11. *Notation.* If $X \subset X'$ are sets and $\iota : X' \to X'$ satisfies $\iota(X) \subset X$ we write $X^\iota = \{x \in X; \iota(x) = x\}$.

1. The algebra $\mathcal{F}$

1.1. Let p, q, Q be as in §0.2. We now assume that $\mathbf{k}$ is an algebraic closure of the finite field F_q with $\sharp(F_q) = q$. We fix a pinning $(x_s : \mathbf{k} \to G, y_s : \mathbf{k} \to G; s \in S)$ corresponding to T, U. (We have $x_s(\mathbf{k}) \subset U$.) Let $W \to N$, $w \mapsto \dot{w}$ be the Tits cross section of $\kappa : N \to W$ associated to this pinning; see [T]. We fix an F_q-rational structure on G with Frobenius map $\phi : G \to G$ such that $\phi(t) = t^q$ for all $t \in T$ and $\phi(x_s(z)) = x_s(z^q), \phi(y_s(z)) = y_s(z^q)$ for all $z \in \mathbf{k}$. We have $\phi(\dot{w}) = \dot{w}$ for any $w \in W$ and $\phi(U) = U$. Let F_Q be the subfield of $\mathbf{k}$ with $\sharp(F_Q) = Q$. We set $\Phi = \phi^2$. We set $\epsilon = -1 \in \mathbf{k}^*$.

For $s \in S$, $z \in \mathbf{k}^*$ we set $z_s = \check{\alpha}_s(z) \in T_s$. In particular, $\epsilon_s \in T_s$ is defined and we have $\dot{s}^2 = \epsilon_s$.

1.2. Let $\mathcal{X} = G/U$. Now G acts on $\mathcal{X}$ by $g : xU \mapsto gxU$ and on $\mathcal{X}^2$ by $g : (xU, yU) \mapsto (gxU, gyU)$. We have $\mathcal{X}^2 = \bigsqcup_{n \in N} O_n$, where $O_n = \{(xU, yU) \in \mathcal{X}^2; x^{-1}y \in UnU\}$. Now ϕ, Φ induce endomorphisms of $\mathcal{X}$ and $\mathcal{X}^2$ denoted again by ϕ, Φ. For $n \in N$, we have $\phi(O_n) = O_{\phi(n)}$ and hence $\Phi(O_n) = O_{\Phi(n)}$. Thus we have $(\mathcal{X}^2)^\Phi = \bigsqcup_{n \in N^\Phi} O_n^\Phi$ and O_n^Φ $(n \in N^\Phi)$ are exactly the orbits of G^Φ on $(\mathcal{X}^2)^\Phi$.

1.3. Let

$$\mathcal{F} = \{f : (\mathcal{X}^2)^\Phi \to \mathbf{C}; f \text{ constant on the orbits of } G^\Phi\}.$$

This is a $\mathbf{C}$-vector space with basis $\{k_n; n \in N^\Phi\}$ where k_n is 1 on O_n^Φ and is 0 on $(\mathcal{X}^2)^\Phi - O_n^\Phi$. Now $\mathcal{F}$ is an associative algebra with 1 under convolution:

$$(f_1 f_2)(xU, zU) = \sum_{yU \in \mathcal{X}^\Phi} f_1(xU, yU) f_2(yU, zU);$$

here $f_1 \in \mathcal{F}, f_2 \in \mathcal{F}$, $(xU, zU) \in (\mathcal{X}^2)^\Phi$.

The following two lemmas are well known; they are also used in [Y].

Lemma 1.4. *Assume that $n, n' \in N$, $\kappa(n) = w, \kappa(n') = w'$ satisfy $|ww'| = |w| + |w'|$.*

(a) *If $(xU, yU) \in O_n, (yU, zU) \in O_{n'}$, then $(xU, zU) \in O_{nn'}$.*

(b) *If $(xU, zU) \in O_{nn'}$, then there is a unique $yU \in X$ such that $(xU, yU) \in O_n, (yU, zU) \in O_{n'}$.*

Lemma 1.5. *Assume that $s \in S$. Assume that* §0.3(a) *holds.*

(a) *If $(xU, x'U) \in O_{\dot{s}}, (x'U, zU) \in O_{\dot{s}^{-1}}$, then $(xU, zU) \in O_1$ or $(xU, zU) \in \bigsqcup_{y \in T_s} O_{\dot{s}y}$.*

(b) *If $(xU, zU) \in O_1$, then $\{x'U \in X; (xU, x'U) \in O_{\dot{s}}, (x'U, zU) \in O_{\dot{s}^{-1}}\}$ is an affine line.*

(c) *If $(xU, zU) \in O_{\dot{s}y}$ with $y \in T_s$, then $\{x'U \in X; (xU, x'U) \in O_{\dot{s}}, (x'U, zU) \in O_{\dot{s}^{-1}}\}$ is a point.*

The following result can be deduced from Lemmas 1.4, 1.5.

Lemma 1.6. *Assume that $s \in S$, $n \in N$, $\kappa(n) = w$ satisfy $|ws| < |w|$. Assume that* §0.3(a) *holds.*

(a) *If $(xU, x'U) \in O_n, (x'U, x''U) \in O_{\dot{s}^{-1}}$, then $(xU, x''U) \in O_{n\dot{s}^{-1}}$ or $(xU, x''U) \in \bigsqcup_{\tau \in T_s} O_{n\tau}$.*

(b) *If $(xU, x''U) \in O_{n\dot{s}^{-1}}$, then $\{x'U \in X; (xU, x'U) \in O_n, (x'U, x''U) \in O_{\dot{s}^{-1}}\}$ is an affine line.*

(c) *If $(xU, x''U) \in O_{n\tau}$ with $y \in T_s$, then*

$$\{x'U \in X; (xU, x'U) \in O_n, (x'U, x''U) \in O_{\dot{s}^{-1}}\}$$

is a point.

1.7. Assume that §0.3(a) holds. From Lemma 1.4 we deduce that for $n, n' \in N^\Phi$ such that $|\kappa(nn')| = |\kappa(n)| + |\kappa(n')|$ we have

(a) $$k_n k_{n'} = k_{nn'}$$

in $\mathcal{F}$. In particular, k_1 is the unit element of $\mathcal{F}$. From Lemma 1.5 we deduce as in [Y] that for $s \in S$ we have

(b) $$k_{\dot{s}} k_{\dot{s}} = Q k_{\epsilon_s} + \sum_{y \in T_s^\Phi} k_{\dot{s}} k_y.$$

It follows that for $s \in S, w \in W, n \in N^\Phi$ such that $|sw| < |w|$, $\kappa(n) = w$ we have

(c) $$k_{\dot{s}} k_n = Q k_{\dot{s}n} + \sum_{y \in T_s^\Phi} k_{yn}$$

and for $s \in S, w \in W, n \in N^\Phi$ such that $|ws| < |w|$, $\kappa(n) = w$ we have

(d) $$k_n k_{\dot{s}^{-1}} = Q k_{n\dot{s}^{-1}} + \sum_{y \in T_s^\Phi} k_{ny}.$$

From (a), (c), and (d) we deduce that for $s \in S, w \in W, n \in N^\Phi$ such that $sw = ws, |sw| < |w|$, $\kappa(n) = w$ we have

(e) $$k_{\dot{s}} k_n k_{\dot{s}^{-1}} = Q k_{\dot{s}n\dot{s}^{-1}} + Q \sum_{y \in T_s^\Phi} k_{\dot{s}ny} + \sum_{y \in T_s^\Phi, y' \in T_s^\Phi} k_{yny'}.$$

1.8. We set $\mathfrak{s} = \mathrm{Hom}(T^\Phi, \mathbf{C}^*)$. Here T^Φ is as in 0.11. Now W acts on $\mathfrak{s}$ by $w : \nu \mapsto w\nu$ where $(w\nu)(t) = \nu(w^{-1}(t))$ for $t \in T^\Phi$. For $\nu \in \mathfrak{s}$ we set

(a) $$1_\nu = |T^\Phi|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau) k_\tau \in \mathcal{F}.$$

We have

(b) $$\sum_{\nu \in \mathfrak{s}} 1_\nu = k_1 = 1.$$

Indeed,

$$\sum_{\nu \in \mathfrak{s}} 1_\nu = |T^\Phi|^{-1} \sum_{\tau \in T^\Phi} \sum_{\nu \in \mathfrak{s}} \nu(\tau) k_\tau = \sum_{\tau \in T^\Phi} \delta_{\tau,1} k_\tau = k_1.$$

For ν, ν' in $\mathfrak{s}$ we have

(c) $$1_\nu 1_{\nu'} = \delta_{\nu,\nu'} 1_\nu.$$

Indeed,

$$\begin{aligned}1_\nu 1_{\nu'} &= |T^\Phi|^{-2} \sum_{\tau \in T^\Phi, \tau' \in T^\Phi} \nu(\tau)\nu'(\tau')k_{\tau\tau'} \\ &= |T^\Phi|^{-2} \sum_{\tau \in T^\Phi, \tau'' \in T^\Phi} \nu(\tau)\nu'(\tau''\tau^{-1})k_{\tau''} \\ &= \delta_{\nu,\nu'}|T^\Phi|^{-1} \sum_{\tau'' \in T^\Phi} \nu'(\tau'')k_{\tau''} = \delta_{\nu,\nu'}1_\nu.\end{aligned}$$

For $\nu \in \mathfrak{s}, n \in N^\Phi, w = \kappa(n) \in W$ we have

(d) $$k_n 1_\nu = 1_{w\nu} 1_\nu.$$

Indeed,

$$\begin{aligned}k_n 1_\nu &= |T^\Phi|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau)k_{n\tau} = |T^\Phi|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau)k_{w(\tau)n} \\ &= |T^\Phi|^{-1} \sum_{\tau' \in T^\Phi} \nu(w^{-1}(\tau'))k_{\tau' n} = 1_{w\nu}k_n.\end{aligned}$$

For $t \in T^\Phi, \nu \in \mathfrak{s}$ we have

(e) $$k_t 1_\nu = \nu(t^{-1})1_\nu.$$

Indeed,

$$\begin{aligned}k_t 1_\nu &= |T^\Phi|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau)k_{t\tau} = |T^\Phi|^{-1} \sum_{\tau' \in T^\Phi} \nu(t^{-1}\tau')k_{\tau'} \\ &= \nu(t^{-1})|T^\Phi|^{-1} \sum_{\tau' \in T^\Phi} \nu(\tau')k_{\tau'} = \nu(t^{-1})1_\nu.\end{aligned}$$

For $\nu \in \mathfrak{s}, s \in S$ we write $s \in W_\nu$ if $\nu(\check\alpha_s(z)) = 1$ for all $z \in F_Q^*$ or equivalently if $\nu|_{T_s^\Phi} = 1$; we write $s \notin W_\nu$ if $\nu|_{T_s^\Phi}$ is not identically 1.

For $\nu \in \mathfrak{s}, \check\alpha \in \check R$ we define $[\nu, \check\alpha]$ as follows. If $\nu(\check\alpha(\epsilon)) = 1$ we set $[\nu, \check\alpha] = 1$; if $\nu(\check\alpha(\epsilon)) = -1$ we set $[\nu, \check\alpha] = \sqrt{-1}$. (Since $\check\alpha(\epsilon)^2 = 1$ we must have $\nu(\check\alpha(\epsilon)) \in \{1, -1\}$.) If $p = 2$ we have $\check\alpha(\epsilon) = 1$ and hence $[\nu, \check\alpha] = 1$. We have $[\nu, \check\alpha]^2 = \nu(\check\alpha(\epsilon))$.

For $s \in S$ we set

(f) $$\mathcal{T}_s = q^{-1}k_{\dot s} \sum_{\nu \in \mathfrak{s}} [\nu, \check\alpha_s]1_\nu \in \mathcal{F}.$$

We show

(g) $$\mathcal{T}_s\mathcal{T}_s = 1 + (q - q^{-1}) \sum_{\nu \in \mathfrak{s}; s \in W_\nu} \mathcal{T}_s 1_\nu.$$

Indeed, we have

$$\begin{aligned}
\mathcal{T}_s\mathcal{T}_s &= Q^{-1}\sum_{\nu\in\mathfrak{s},\nu'\in\mathfrak{s}}[\nu,\check{\alpha}_s][\nu',\check{\alpha}_s]k_{\dot{s}}1_\nu k_{\dot{s}}1_{\nu'}\\
&= Q^{-1}\sum_{\nu\in\mathfrak{s},\nu'\in\mathfrak{s}}[\nu,\check{\alpha}_s][\nu',\check{\alpha}_s]k_{\dot{s}}k_{\dot{s}}1_{s\nu}1_{\nu'}\\
&= Q^{-1}\sum_{\nu'\in\mathfrak{s}}[s\nu',\check{\alpha}_s][\nu',\check{\alpha}_s]k_{\dot{s}}k_{\dot{s}}1_{\nu'}\\
&= Q^{-1}\sum_{\nu\in\mathfrak{s}}\nu(\epsilon_s)k_{\dot{s}}k_{\dot{s}}1_\nu\\
&= \sum_{\nu\in\mathfrak{s}}\nu(\epsilon_s)k_{\epsilon_s}1_\nu + Q^{-1}\sum_{\nu\in\mathfrak{s},y\in T_s^\Phi}\nu(\epsilon_s)k_{\dot{s}}k_y1_\nu\\
&= \sum_{\nu\in\mathfrak{s}}1_\nu + Q^{-1}\sum_{\nu\in\mathfrak{s},y\in T_s^\Phi}\nu(\epsilon_s)\nu(y^{-1})k_{\dot{s}}1_\nu\\
&= 1 + Q^{-1}(Q-1)\sum_{\nu\in\mathfrak{s},\nu|_{T_s^\Phi}=1}k_{\dot{s}}1_\nu.
\end{aligned}$$

It remains to use that if $\nu|_{T_s^\Phi}=1$, then $\nu(\epsilon_s)=1$ and hence $[\nu,\check{\alpha}_s]=1$.

Now (g) implies that $\mathcal{T}_s^{-1}\in\mathcal{F}$ is well defined and we have

(h) $$\mathcal{T}_s^{-1}=\mathcal{T}_s-(q-q^{-1})\sum_{\nu\in\mathfrak{s};s\in W_\nu}1_\nu.$$

From (h) we see that for any $\nu\in\mathfrak{s}$:

(i) $$\mathcal{T}_s^{-1}1_\nu=\mathcal{T}_s1_\nu-\Delta(q-q^{-1})1_\nu,,$$

where $\Delta=1$ if $s\in W_\nu$ and $\Delta=0$ if $s\notin W_\nu$.

For any $\nu\in\mathfrak{s}$ we show

(j) $$1_\nu\mathcal{T}_s=\mathcal{T}_s1_{s\nu}.$$

Indeed, we have

$$1_\nu\mathcal{T}_s=q^{-1}1_\nu k_{\dot{s}}\sum_{\nu'\in\mathfrak{s}}[\nu',\check{\alpha}_s]1_{\nu'}=q^{-1}\sum_{\nu'\in\mathfrak{s}}k_{\dot{s}}[\nu',\check{\alpha}_s]1_{s\nu}1_{\nu'}=q^{-1}k_{\dot{s}}[\nu,\check{\alpha}_s]1_{s\nu},$$

$$\mathcal{T}_s1_{s\nu}=q^{-1}k_{\dot{s}}\sum_{\nu'\in\mathfrak{s}}[\nu',\check{\alpha}_s]1_{\nu'}1_{\sigma\nu}=q^{-1}k_{\dot{s}}[\nu,\check{\alpha}_s]1_{\sigma\nu}.$$

1.9. For any $w\in W$ we set

$$\mathcal{T}_w=q^{-|w|}k_{\dot{w}}\sum_{\nu\in\mathfrak{s}}\prod_{\check{\alpha}\in\check{R}^+;w^{-1}(\check{\alpha})\in\check{R}^-}[\nu,w^{-1}\check{\alpha}]1_\nu\in\mathcal{F}.$$

When $w=s\in S$, this definition agrees with the earlier definition of $\mathcal{T}_s$. For $s\in S$, $w\in W$ such that $|ws|>|w|$ we show

(a) $$\mathcal{T}_{ws}=\mathcal{T}_w\mathcal{T}_s.$$

Since $|ws| > |w|$, we have $w(\check\alpha_s) \in R^+$ and $\{\check\alpha \in \check R^+; (ws)^{-1}(\check\alpha) \in \check R^-\} = \{\check\alpha \in R^+; w^{-1}(\check\alpha) \in \check R^-\} \sqcup \{w(\check\alpha_s)\}$. Hence we have

$$\begin{aligned}\mathcal{T}_{ws} &= q^{-|ws|}k_{\dot w\dot s}\sum_{\nu\in\mathfrak{s}}\prod_{\check\alpha\in\check R^+,(ws)^{-1})(\check\alpha)\in\check R^-}[\nu,(ws)^{-1}\check\alpha]1_\nu\\ &= q^{-|ws|}k_{\dot w\dot s}\sum_{\nu\in\mathfrak{s}}[\nu,(ws)^{-1}(w(\check\alpha_s))]\prod_{\check\alpha\in\check R^+;w^{-1}(\check\alpha)\in\check R^-}[\nu,(ws)^{-1}(\check\alpha)]1_\nu\\ &= q^{-|ws|}k_{\dot w\dot s}\sum_{\nu\in\mathfrak{s}}[\nu,\check\alpha_s]\prod_{\check\alpha\in\check R^+;w^{-1}(\check\alpha)\in\check R^-}[\nu,(ws)^{-1}(\check\alpha)]1_\nu.\end{aligned}$$

We have

$$\begin{aligned}\mathcal{T}_w\mathcal{T}_s &= q^{-|w|}q^{-1}k_{\dot w}\sum_{\nu\in\mathfrak{s},\nu'\in\mathfrak{s}}\prod_{\check\alpha\in\check R^+,w^{-1}(\check\alpha)\in\check R^-}[\nu,w^{-1}(\check\alpha)][\nu',\check\alpha_s]1_\nu k_{\dot s}1_{\nu'}\\ &= q^{-|ws|}k_{\dot w}k_{\dot s}\sum_{\nu\in\mathfrak{s},\nu'\in\mathfrak{s}}[\nu',\check\alpha_s]\prod_{\check\alpha\in\check R^+,w^{-1}(\check\alpha)\in\check R^-}[\nu,w^{-1}(\check\alpha)]1_{s\nu}1_{\nu'}\\ &= q^{-|ws|}k_{\dot w\dot s}\sum_{\nu\in\mathfrak{s}}[\nu,\check\alpha_s]\prod_{\check\alpha\in\check R^+,w^{-1}(\check\alpha)\in\check R^-}[\nu,(ws)^{-1}(\check\alpha)]1_\nu.\end{aligned}$$

This proves (a).

From (a) we deduce:

(b) $$\mathcal{T}_{ww'} = \mathcal{T}_w\mathcal{T}_{w'} \text{ if } w,w' \text{ in } W \text{ satisfy } |ww'| = |w| + |w'|.$$

Using §1.8(j) we see that

(c) $$1_\nu\mathcal{T}_w = \mathcal{T}_w1_{w^{-1}\nu} \text{ for } w\in W, \nu\in\mathfrak{s}.$$

We note that

(d) $\{\mathcal{T}_w1_\nu; w\in W,\nu\in\mathfrak{s}\}$ *is a* $\mathbf{C}$*-basis of* $\mathcal{F}$.

This follows from the fact that (up to a non-zero scalar) $\mathcal{T}_w1_\nu$ is equal to

$$\sum_{\tau\in T^\Phi}\nu(\tau)k_{\dot w\tau}.$$

2. The $\mathcal{F}$-module $\mathcal{F}'$

2.1. In this section we assume that §0.3(a) holds. We preserve the setup of §1.1. We define $\phi' : N \to N$ by $\phi'(n) = \phi(n)^{-1}$. We define $\psi : \mathcal{X}^2 \to \mathcal{X}^2$ by $\psi(xU, yU) = (\phi(y)U, \phi(x)U)$. This is a Frobenius map for an F_q-rational structure on $\mathcal{X}^2$. The G-action on $\mathcal{X}^2$ in §1.2 is compatible with this F_q-rational structure on $\mathcal{X}^2$ and with the F_q-rational structure on G given by ϕ. It follows that any G-orbit O_n on $\mathcal{X}^2$ such that $\psi(O_n) = O_n$ satisfies the condition that $O_n^\psi \neq \emptyset$ and that G^ϕ acts transitively on O_n^ψ. (We use Lang's theorem [La] and the connectedness of the stabilizers of the G-action on O_n.) For $n \in N$ we have $\psi(O_n) = O_{\phi'(n)}$; thus $\psi(O_n) = O_n$ precisely when $n \in N^{\phi'}$. Thus we have $(\mathcal{X}^2)^\psi = \bigsqcup_{n\in N^{\phi'}} O_n^\psi$ and O_n^ψ (for various $n \in N^{\phi'}$) are precisely the G^ϕ-orbits in $(\mathcal{X}^2)^\psi$. Let

$$\mathcal{F}' = \{h : (\mathcal{X}^2)^\psi \to \mathbf{C}; h \text{ is constant on the orbits of } G^\phi\}.$$

This is a **C**-vector space with basis $\{\theta_m; m \in N^{\phi'}\}$, where θ_m is 1 on O^ψ_m and is 0 on $(\mathcal{X}^2)^\psi - O^\psi_m$. Now $\mathcal{F}'$ is an $\mathcal{F}$-module under convolution

$$(fh)(xU, \phi(x)U) = \sum_{yU \in \mathcal{X}^\Phi} f(xU, yU)h(yU, \phi(y)U);$$

here $f \in \mathcal{F}, h \in \mathcal{F}', (xU, \phi(x)U) \in (\mathcal{X}^2)^\psi$. (In this $\mathcal{F}$-module, multiplication by the unit element of $\mathcal{F}$ is the identity map of $\mathcal{F}'$.)

2.2. Now $\phi' : N \to N$ is an F_q-structure on N not necessarily compatible with the group structure of N. But it is compatible with the $T \times T$-action on N given by $(t_1, t_2) : n \mapsto t_1 n t_2^{-1}$ and the F_q-rational structure on $T \times T$ with Frobenius map $(t_1, t_2) \mapsto (\phi(t_2), \phi(t_1))$. Hence any $T \times T$-orbit of the action on N which is stable under $\phi' : N \to N$ must have a ϕ'-fixed point. Such an orbit is of the form $\kappa^{-1}(w)$ with $w \in W$ satisfying $w^{-1} = w$, that is, $w \in W_2$. Using Lang's theorem and the connectedness of the stabilizers of the $T \times T$-action on $\kappa^{-1}(w)$, we see that for $w \in W_2$, $\kappa^{-1}(w) \cap N^{\phi'}$ is non-empty and is exactly one orbit for the subgroup $\{(t_1, t_2) \in T \times T; (t_1, t_2) = (\phi(t_2), \phi(t_1))\}$ of $T \times T$. Thus,

(a) $N^{\phi'} = \bigsqcup_{w \in W_2} N(w)$, *where for any* $w \in W_2$, $N(w) := \kappa^{-1}(w) \cap N^{\phi'}$ *is non-empty and is a single orbit for the action of* T^Φ *on* $N^{\phi'}$ *given by* $t : n \mapsto tn\phi(t)^{-1}$.

For $w \in W_2$ we have $N(w) = \{\dot{w}t; t \in T, w(t^q)t\dot{w}^2 = 1\}$. Let $T(w) = \{t \in T; w(t^q)t = 1\}$. Clearly,

(b) $N(w)$ *is a single orbit under right translation by* $T(w)$.

We note:

(c) *For* $w \in W_2$, $z \in W$ *we have* $\dot{z}N(w)\dot{z}^{-1} = N(zwz^{-1})$.

It is enough to show that $\dot{z}N^{\phi'}\dot{z}^{-1} = N^{\phi'}$. More generally, if $n \in N^\Phi$, then $nN^{\phi'}\phi(n)^{-1} = N^{\phi'}$. This is easily verified.

For $w \in W_2$, we define a homomorphism $e_w : T^\Phi \to T(w)$ by $\tau \mapsto w(\tau)\tau^{-q}$. We show:

(d) e_w *is surjective.*

Let $t \in T(w)$. By Lang's theorem we have $t = w(\tau)\tau^{-q}$ for some $\tau \in T$. Since $t \in T(w)$ we have automatically $\tau \in T^\Phi$ and (d) follows.

For $w \in I$, $s \in S$ such that $sw = ws$ we show:

(e) *If* $|sw| > |w|$, *then* $\{c_s; c \in F_Q, c^{q+1} = 1\} \subset T(w)$; *if* $|sw| < |w|$, *then* $\{c_s; c \in F_Q, c^{q-1} = 1\} \subset T(w)$.

Assume first that $|sw| > |w|$ and that $c^{q+1} = 1$. We have $w(c_s) = c_s$ and hence $w(c_s^q)c_s = c_s^{q+1} = 1$. Next we assume that $|sw| < |w|$ and that $c^{q-1} = 1$. We have $w(c_s) = c_s^{-1}$ and hence $w(c_s^q)c_s = c_s^{-q+1} = 1$. This proves (e).

2.3. For $n \in N^\Phi, m \in N^{\phi'}$ we have $k_n\theta_m = \sum_{m' \in N_*} \mathcal{N}_{n,m,m'}\theta_{m'}$, where

$$\mathcal{N}_{n,m,m'} = \sharp\{yU \in X^\Phi; (xU, yU) \in O^\Phi_n, (yU, \phi(y)U) \in O^\psi_m\}.$$

We have also

$$\mathcal{N}_{n,m,m'} = \sharp Z^\psi_{xU,\phi(x)U},$$

where

$$Z_{xU,\phi(x)U} = \{(yU, y'U) \in O_m; (xU, yU) \in O_n, (y'U, \phi(x)U) \in O_{\phi(n)^{-1}}\}$$

with $(xU, \phi(x)U)$ fixed in $O^\psi_{m'}$ (note that $Z_{xU,\phi(x)U}$ is ψ-stable).

Lemma 2.4. *Assume that* $n = t \in T^\Phi$, $m \in N^{\phi'}$. *We have* $k_t\theta_m = \theta_{tm\phi(t)^{-1}}$.

If $m' \in N^{\phi'}$ satisfies $\mathcal{N}_{n,m,m'} \neq 0$, then from Lemma 1.4 (applied twice) we see that $Z_{xU,\phi(x)U}$ is a point and $m' = tm\phi(t)^{-1}$; moreover we have $\mathcal{N}_{n,m,m'} = 1$. The result follows.

Lemma 2.5. *Assume that $s \in S$, $w \in I$, $m \in N(w)$, $sw \neq ws$, $|ws| > |w|$. Recall that $\dot{s}m\dot{s}^{-1} \in N(sws)$. We have*

$$k_{\dot{s}}\theta_m = \theta_{\dot{s}m\dot{s}^{-1}}.$$

In this case we have $|sws| = |w| + 2$. If $m' \in N^{\phi'}$ satisfies $\mathcal{N}_{n,m,m'} \neq 0$, then from Lemma 1.4 (applied twice) we see that $Z_{xU,\phi(x)U}$ (in §2.3 with $n = \dot{s}$) is a point and $m' = \dot{s}m\phi(\dot{s})^{-1}$; moreover we have $\mathcal{N}_{n,m,m'} = 1$. The result follows.

Lemma 2.6. *Assume that $s \in S$, $w \in I$, $m \in N(w)$, $sw = ws$, $|ws| > |w|$. Write $m = \dot{w}t$ where $t \in T$ satisfies $w(t^q)t\dot{w}^2 = 1$.*

(a) *We have $\dot{w}s(t) = \dot{s}m\dot{s}^{-1} \in N(w)$. We have $s(t)^{-1}t\epsilon_s = \dot{s}m^{-1}\dot{s}m \in T_s$, $(\dot{s}m^{-1}\dot{s}m)^{q+1} = 1$.*

(b) *For $y \in T_s$ we have $\dot{s}\dot{w}ty = \dot{s}my \in N(sw)$ if and only if $y^{q-1} = s(t)^{-1}t\epsilon_s = \dot{s}m^{-1}\dot{s}m$. There are exactly $q - 1$ such y; they are all automatically in T_s^{Φ}.*

(c) *We have*

$$k_{\dot{s}}\theta_m = q\theta_{\dot{s}m\dot{s}^{-1}} + \sum_{y \in T_s; y^{q-1} = \dot{s}m^{-1}\dot{s}m} \theta_{\dot{s}my}.$$

The equalities in (a) are easily checked; the inclusion $\dot{s}n\dot{s}^{-1} \in N(w)$ follows from §2.2(c). We have $s(t)^{-1}t\epsilon_s \in T_s$. To prove (a) it remains to show that $(s(t)^{-1}t\epsilon_s)^{q+1} = 1$. We have $\dot{s}\dot{w}^2 = \dot{w}^2\dot{s}$ and hence $\dot{w}^2 = \dot{s}\dot{w}^2\dot{s}^{-1} = s(\dot{w}^2) = \dot{w}^2\check{\alpha}_s(\alpha_s\dot{w}^{-2})$. Thus we have $\check{\alpha}_s(\alpha_s(\dot{w}^{-2})) = 1$, that is, $\check{\alpha}_s(\alpha_s(w(t^q)t)) = 1$. Since $w(\alpha_s) = \alpha_s$ it follows that $\check{\alpha}_s(\alpha_s(t^{q+1})) = 1$ and hence $(\check{\alpha}_s(-\alpha_s(t)))^{q+1} = 1$. Thus (a) holds.

From our assumptions we have that $w(y') = y'$ and $s(y') = y'^{-1}$ for any $y' \in T_s$; since $s(t)t^{-1} \in T_s$, it follows that $w(s(t)t^{-1}) = s(t)t^{-1}$. Moreover we have $w(\dot{s}^2) = \dot{s}^2$. Hence for $y \in T_s$ we have

$$sw(t^qy^q)ty(\dot{s}\dot{w})^2 = s(w(t^q)t\dot{w}^2)sw(y^q)s(t)^{-1}ty\dot{s}^2 = y^{-q}s(t)^{-1}ty\dot{s}^2.$$

This equals 1 if and only if $y^{q-1} = s(t)^{-1}t\dot{s}^2$. This proves the first sentence of (b). The second sentence of (b) follows from (a).

We prove (c). For $m' \in N^{\phi'}$ and $(xU, \phi(x)U) \in O^{\psi}_{m'}$ fixed, the variety $Z_{xU,\phi(x)U}$ in §2.3 (with $n = \dot{s}$) can be identified with

$$Z'_{xU,\phi(x)U} = \{x'U \in X; (xU, x'U) \in O_{\dot{s}m}, (x'U, \phi(x)U) \in O_{\dot{s}^{-1}}\}.$$

(We use Lemma 1.4 and the equality $|sw| = |w| + 1$.) By Lemma 1.6, $Z'_{xU,\phi(x)U}$ is an affine line if $m' = \dot{s}m\dot{s}^{-1}$, is a point if $m' = \dot{s}my$ for some $y \in T_s$, and is empty otherwise. Hence $\sharp(Z^{\psi}_{xU,\phi(x)U})$ is q if $m' = \dot{s}m\dot{s}^{-1}$, is 1 if $m' = \dot{s}my$ for some $y \in T_s$, and is 0 otherwise. Now (c) follows from (a), (b).

Lemma 2.7. *Assume that $s \in S$, $w \in I$, $m \in N(w)$ and that $sw = ws$, $|ws| < |w|$. Write $m = \dot{w}t$, where $t \in T$ satisfies $w(t^q)t\dot{w}^2 = 1$.*

(a) *For $y \in T_s$ we have $\dot{s}m\dot{s}^{-1}y \in N(w)$ if and only if $y^{q-1} = 1$.*

(b) *We have $s(t)t^{-1}\epsilon_s = m^{-1}\dot{s}m\dot{s} \in T_s^{\phi}$.*

(c) *For $y \in T_s$ we have $\dot{s}my \in N(sw)$ if and only if $y^{q+1} = s(t)t^{-1}\epsilon_s = m^{-1}\dot{s}m\dot{s}$. There are exactly $q + 1$ such y; they are all automatically in T_s^{Φ}.*

(d) *We have*

$$k_{\dot{s}}\theta_m = q\sum_{y\in T_s;y^{q+1}=m^{-1}\dot{s}m\dot{s}}\theta\dot{s}my + \theta_{\dot{s}m\dot{s}^{-1}} + (q+1)\sum_{y\in T_s;y^{q-1}=1,y\neq 1}\theta_{\dot{s}m\dot{s}^{-1}y}.$$

We prove (a). We have

$$\phi(\dot{s}m\dot{s}^{-1}y)\dot{s}m\dot{s}^{-1}y = \dot{s}\phi(m)\dot{s}^{-1}y^q\dot{s}m\dot{s}^{-1}y = \dot{s}m^{-1}y^{-q}m\dot{s}^{-1}y$$
$$= \dot{s}w(y^{-q})\dot{s}^{-1}y = \dot{s}y^q\dot{s}^{-1}y = y^{-q}y = y^{1-q}.$$

This proves (a).

The equality in (b) is easily checked. We have $s(t)t^{-1}\epsilon_s \in T_s$. To prove (b) it remains to show that $(s(t)t^{-1}\epsilon_s)^{q-1} = 1$. We have $\dot{s}^{-1}\dot{w}^2 = \dot{w}^2\dot{s}^{-1}$ and hence $\dot{w}^2 = \dot{s}^{-1}\dot{w}^2\dot{s} = s(\dot{w}^2) = \dot{w}^2\check{\alpha}_s(\alpha_s\dot{w}^{-2})$. Thus we have $\check{\alpha}_s(\alpha_s(\dot{w}^{-2})) = 1$, that is, $\check{\alpha}_s(\alpha_s(w(t^q)t)) = 1$. Since $w(\alpha_s) = \alpha_s^{-1}$ it follows that $\check{\alpha}_s(\alpha_s(t^{-q+1})) = 1$ and hence $(\check{\alpha}_s(-\alpha_s(t)))^{-q+1} = 1$. Thus (b) holds.

We prove (c). We have

$$\phi(\dot{s}my)\dot{s}my = \dot{s}\phi(m)y^q\dot{s}my = \dot{s}m^{-1}y^q\dot{s}my = \dot{s}t^{-1}\dot{w}^{-1}y^q\dot{s}\dot{w}ty$$
$$= \dot{s}t^{-1}\dot{w}^{-1}y^q\dot{w}\dot{s}ty = \dot{s}t^{-1}w(y^q)\dot{s}ty = \dot{s}t^{-1}y^{-q}\dot{s}ty$$
$$= s(t^{-1}y^{-q})\epsilon_s ty = y^{q+1}s(t^{-1})t\epsilon_s.$$

This proves the first sentence of (c). The second sentence of (c) follows from (b).

We prove (d). For $m' \in N^{\phi'}$ and $(xU,\phi(x)U) \in O^{\psi}_{m'}$ fixed, the variety $Z_{xU,\phi(x)U}$ in §2.3 (with $n = \dot{s}$) is

(i) an affine line if $m' = \dot{s}my$ for some $y \in T_s$ such that $\dot{s}my \in N(sw)$,

(ii) an affine line minus a point if $m' = \dot{s}m\dot{s}^{-1}y$ with $y \in T_s - \{1\}$,

(iii) a union of two affine lines with one point in common if $m' = \dot{s}m\dot{s}^{-1}$.

This is a geometric reinterpretation (and refinement) of the formula 1.7(e), in which the number of Φ-fixed points on these varieties enter; this number is Q in case (i), is $Q-1$ in case (ii), and is $2Q-1$ in case (iii). It is enough to show that the number of ψ-fixed points on $Z_{xU,\phi(x)U}$ is q in case (i), is $q+1$ in case (ii), and is 1 in case (iii). This is verified directly by calculation in each case. (In case (iii), ψ interchanges the two lines, keeping fixed the point common to the two lines.) We give the details of the calculation assuming that $G = SL_2(\mathbf{k})$, T is the diagonal matrices, TU is the upper triangular matrices, $\dot{s} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and ϕ raises each matrix entry to the qth power. We have $N^{\phi'} = \{M_a; a \in F_Q^*; a^q + a = 0\} \sqcup \{M'_a; a \in F_Q^*; a^{q+1} = 1\}$, where $M_a = \left(\begin{smallmatrix} 0 & -a^{-1} \\ a & 0 \end{smallmatrix}\right)$, $M'_a = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. We must show:

if $x \in G$, $x^{-1}\phi(x) = M_a$, *then* $\sharp(yU \in G/U; y^{-1}\phi(y) \in UM_bU, x^{-1}y \in U\dot{s}U) = 1 + \delta_{a,b}q$ *(here* $a^q + a = 0, b^q + b = 0$*)*;

if $x \in G$, $x^{-1}\phi(x) = M_{a'}$, *then* $\sharp(yU \in G/U; y^{-1}\phi(y) \in UM_bU, x^{-1}y \in U\dot{s}U) = q$ *(here* $a'^{q+1} + a' = 0$, $b^{q+1} = 0$*)*.

Setting $y = xD$ we see that we must show that if $b^q + b = 0$, then:

if $a^q + a = 0$, then $\sharp(DU \in (U\dot{s}U)/U; D^{-1}M_a\phi(D) \in UM_bU) = 1 + (1-\delta_{a,b})q$;

if $a'^{q+1} = 1$, then $\sharp(DU \in (U\dot{s}U)/U; D^{-1}M_{a'}\phi(D) \in UM_bU) = q$.

Equivalently, we must show that if $b^q + b = 0$, then:

(e) if $a^q + a = 0$, then $\sharp(d \in F_Q; d^{q+1}a - a^{-1} = b) = 1 + (1-\delta_{a,b})q$;

(f) if $a'^{q+1} = 1$, then $\sharp(d \in F_Q; -a'd^q + a'^{-1}d = b) = q$.

If $a=b$, the equation in (e) is $d^{q+1}=0$ which has one solution, namely $d=0$. If $a\ne b$ the equation in (e) is $d^{q+1}=ba^{-1}+a^{-2}$. Here $(ba^{-1}+a^{-2})^q=ba^{-1}+a^{-2}\ne 0$. Hence the equation in (e) has exactly $q+1$ solutions. Setting $d'=a'^{-1}d$, the equation in (f) is $-d'^q+d'=b$ and this has exactly d solutions in F_Q since $b^q+b=0$. This completes the proof.

2.8. Let $T(w)^*=\mathrm{Hom}(T(w),\mathbf{C}^*)$. Since e_w is surjective (see §2.2(d)), the map $T(w)^*\to\mathfrak{s}$, $\zeta\mapsto\zeta e_w$ is an injective homomorphism. Let $\mathfrak{s}_w$ be the image of this homomorphism. We have $\mathfrak{s}_w=\{\nu\in\mathfrak{s};w(\nu)\nu^q=1\}$. Note that if $w\in W_2,z\in W$, then $z(\mathfrak{s}_w)=\mathfrak{s}_{zwz^{-1}}$.

For $\nu\in\mathfrak{s}_w$ we denote by $\underline{\nu}_w$ the element of $T(w)^*$ such that $\nu=\underline{\nu}_w e_w$. We set $\mathfrak{K}_w=\ker(e_w)$.

For any $w\in I$, $n\in N(w)$, and $\nu\in\mathfrak{s}_w$ we define $a'_{n,\nu}\in\mathcal{F}'$ by

$$a'_{n,\nu}=\sum_{t\in T(w)}\underline{\nu}_w(t)\theta_{nt}=|\mathfrak{K}_w|^{-1}\sum_{\tau\in T^\Phi}\nu(\tau)\theta_{\tau n\tau^{-q}}.$$

To verify the last equality we note that the sum over $t\in T(w)$ is equal to $|\mathfrak{K}_w|^{-1}\sum_{\tau\in T^\Phi}\underline{\nu}_w(e_w(\tau))\theta_{ne_w(\tau)}$. We show:

(a) *If $w\in I,n\in N(w),\tau\in T^\Phi,t\in T(w)$, and $\nu\in\mathfrak{s}_w$, then $a'_{nt,\nu}=\underline{\nu}_w(t^{-1})a'_{n,\nu}$ and $a'_{\tau n\tau^{-q},\nu}=\nu(\tau^{-1})a'_{n,\nu}$. In particular, the line spanned by $a'_{n,\nu}$ depends only on w,ν and not on n.*

Indeed, we have

$$a'_{nt,\nu}=\sum_{t'\in T(w)}\underline{\nu}_w(t')\theta_{ntt'}=\sum_{t''\in T(w)}\underline{\nu}_w(t''t^{-1})\theta_{nt''}=\underline{\nu}_w(t^{-1})a'_{n,\nu},$$

$$a'_{\tau n\tau^{-q},\nu}$$
$$=|\mathfrak{K}_w|^{-1}\sum_{\tau'\in T^\Phi}\nu(\tau')\theta_{\tau'\tau n\tau\tau'^{-q}}|\mathfrak{K}_w|^{-1}\sum_{\tau_1\in T^\Phi}\nu(\tau_1\tau^{-1})\theta_{\tau_1 n\tau_1^{-q}}=\nu(\tau^{-1})a'_{n,\nu}.$$

This proves (a).

From §2.1, §2.2(a), (b), we see that:

(b) *if $\{t_w;w\in W_2\}$ is a collection of elements in T such that $\dot{w}t_w\in N(w)$ for all $w\in W_2$, then $\{a'_{\dot{w}t_w,\nu};w\in W_2,\nu\in\mathfrak{s}_w\}$ is a $\mathbf{C}$-basis of $\mathcal{F}'$.*

For $\nu\in\mathfrak{s},w\in I,n\in N(w),\nu'\in\mathfrak{s}_w$ we show:

(c) $$1_\nu a'_{n,\nu'}=\delta_{\nu,\nu'}a'_{n,\nu'}.$$

Indeed, we have

$$1_\nu a'_{n,\nu'}=|\mathfrak{K}_w|^{-1}|T^\Phi|^{-1}\sum_{\tau\in T^\Phi}\nu(\tau)k_\tau\sum_{\tau'\in T^\Phi}\nu'(\tau')\theta_{\tau' n\tau'^{-q}}$$
$$=|\mathfrak{K}_w|^{-1}|T^\Phi|^{-1}\sum_{\tau\in T^\Phi}\nu(\tau)\sum_{\tau'\in T^\Phi}\nu'(\tau')\theta_{\tau\tau' n\tau'^{-q}t^{-q}}.$$

Setting $\tau\tau'=\tau_1$ we obtain

$$1_\nu a'_{n,\nu'}=|\mathfrak{K}_w|^{-1}|T^\Phi|^{-1}\sum_{\tau_1\in T^\Phi}\nu'(\tau_1)\sum_{\tau\in T^\Phi}\nu(\tau)\nu'(\tau^{-1})\theta_{\tau_1 n\tau_1^{-1}-q}$$
$$=\delta_{\nu,\nu'}|\mathfrak{K}_w|^{-1}\sum_{\tau_1\in T^\Phi}\nu'(\tau_1)\theta_{\tau_1 n\tau_1^{-1}-q}=\delta_{\nu,\nu'}a'_{n,\nu'}.$$

This proves (c).

For $s \in S$, $w \in W, n \in N(w), \nu \in \mathfrak{s}_w$, we have (using (c)):

(d) $$\mathcal{T}_s a'_{n,\nu} = q^{-1}[\nu, \check{\alpha}_s] k_{\dot{s}} a_{n,\nu}.$$

Lemma 2.9. *Let $s \in S, w \in I, n \in N(w), \nu \in \mathfrak{s}_w$. Note that $s\nu \in \mathfrak{s}_{sws}$. Assume that $sw \neq ws, |sw| > |w|$. We have*

$$\mathcal{T}_s a'_{n,\nu} = q^{-1}[\nu, \check{\alpha}_s] a'_{\dot{s}n\dot{s}^{-1}, s\nu}.$$

Using §2.8(d) we see that it is enough to show

$$k_{\dot{s}} a'_{n,\nu} = a'_{\dot{s}n\dot{s}^{-1}, s\nu}.$$

Using Lemma 2.5 and the equality $|\mathfrak{K}_w| = |\mathfrak{K}_{sws}|$ we see that

$$k_{\dot{s}} a'_{n,\nu} = |\mathfrak{K}_w|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau) k_{\dot{s}} \theta_{\tau n \tau^{-q}}$$

$$= |\mathfrak{K}_w|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau) \theta_{\dot{s}\tau n \tau^{-q} \dot{s}^{-1}}$$

$$= |\mathfrak{K}_{sws}|^{-1} \sum_{\tau' \in T^\Phi} \nu(s(\tau')) \theta_{\tau' \dot{s} n \dot{s}^{-1} \tau'^{-q}} = a'_{\dot{s}n\dot{s}^{-1}, s\nu}.$$

The lemma is proved.

Lemma 2.10. *Let $s \in S, w \in I, n \in N(w), \nu \in \mathfrak{s}_w$. Assume that $sw = ws$, $|sw| > |w|$. If $s \in W_\nu$ we set $\Delta = 1$; if $s \notin W_\nu$ we set $\Delta = 0$. Note that we have $s\nu \in \mathfrak{s}_w$; moreover, if $\Delta = 1$, then $s\nu = \nu \in \mathfrak{s}_{sw}$. We set $z = \dot{s}n^{-1}\dot{s}n \in T_s$; see Lemma 2.6(a). We have $z^{q+1} = 1$; see Lemma 2.6(a). We have*

$$\mathcal{T}_s a'_{n,\nu} = a'_{\dot{s}n\dot{s}^{-1}, s\nu} \text{ if } \Delta = 0,$$

$$\mathcal{T}_s a'_{n,\nu} = a'_{n,\nu} + (q^{-1} + 1) a'_{\dot{s}nu, \nu} \text{ if } \Delta = 1,$$

where $u \in T_s^\Phi$ is such that $u^{q-1} = z$ (see Lemma 2.6(b)).

Using Lemma 2.6(c) and §2.8(d) we have $\mathcal{T}_s a'_{n,\nu} = A + B$, where

$$A = |\mathfrak{K}_w|^{-1} \sum_{\tau \in T^\Phi} \nu(\tau) \theta_{\dot{s}\tau n t^{-q} \dot{s}^{-1}},$$

and

$$B = q^{-1} |\mathfrak{K}_w|^{-1} \sum_{\tau \in T^\Phi, y \in T_s; y^{q-1} = \dot{s}\tau^q n^{-1} \tau^{-1} \dot{s}\tau n \tau^{-q}} \nu(\tau) \theta_{\dot{s}\tau n \tau^{-q} y}.$$

We have used that $\nu(\epsilon_s) = 1$ (and hence $[\nu, \check{\alpha}_s] = 1$). Indeed, we have $\nu(\epsilon_s) = \underline{\nu}_w(e_w(\epsilon_s)) = \underline{\nu}_w(w(\epsilon_s)\epsilon_s) = \underline{\nu}_w(1) = 1$ since $w(\epsilon_s) = \epsilon_s$.

In the sum A we set $\tau' = s(\tau)$. We get

$$A = |\mathfrak{K}_w|^{-1} \sum_{\tau' \in T^\Phi} (s\nu)(\tau') \theta_{t' \dot{s} n \dot{s}^{-1} t'^{-q}} = a'_{\dot{s}n\dot{s}^{-1}, s\nu}.$$

We now show that if $\Delta = 1$, then

$$a'_{\dot{s}n\dot{s}^{-1}, s\nu} = a'_{n,\nu}.$$

We write $n = \dot{w}t$ with $t \in T$. We have $\dot{s}n\dot{s}^{-1} = \dot{s}\dot{w}t\dot{s}^{-1} = \dot{w}\dot{s}t\dot{s}^{-1} = nt^{-1}s(t)$. By Lemma 2.6(a) we have $(t^{-1}s(t))^{q+1} = 1$. Since $t^{-1}s(t) \in T_s$ we have $t_1^{-1}s(t) = t_1^{q-1}$ with $t_1 \in T_s^\Phi$. Thus we have $\dot{s}n\dot{s}^{-1} = nt_1^{q-1}$ and hence $a'_{\dot{s}n\dot{s}^{-1},\nu} = a'_{nt_1^{q-1},\nu} = a'_{t_1^{-1}nt_1^q,\nu} = a'_{n,\nu}$ since $\nu(t_1) = 1$. This proves our claim.

We now consider the sum B. In that sum we have

$$\dot{s}\tau^q n^{-1}\tau^{-1}\dot{s}\tau n\tau^{-q} = s(\tau^q)\dot{s}n^{-1}\dot{s}s(\tau)^{-1}\tau n\tau^{-q}$$
$$= \dot{s}n^{-1}\dot{s}ns(\tau)^{-1}\tau\tau^{-q}s(\tau^q) = z(\tau s(\tau)^{-1})^{1-q}.$$

Thus we have

$$B = q^{-1}|\mathfrak{K}_w|^{-1}\sum_{(\tau,y)\in\mathcal{Y}}\nu(\tau)\theta_{\dot{s}nw(\tau)\tau^{-q}y},$$

where $\mathcal{Y} = \{(\tau,y)\in T^\Phi\times T_s; y^{q-1} = z(\tau s(\tau)^{-1})^{1-q}\}$. Let $\mathcal{Y}' = \{(\tau',u)\in T^\Phi\times (T_s^\Phi); u^{q-1} = z\}$. The map $\xi:\mathcal{Y}'\to\mathcal{Y}$, $(\tau',u)\mapsto(s(\tau'),s(\tau')^q\tau'^{-q}u)$ is a well defined bijection. Now the sum B can be written in terms of this bijection as follows:

$$B = q^{-1}|\mathfrak{K}_w|^{-1}\sum_{(\tau',u)\in\mathcal{Y}'}\nu(s(\tau'))\theta_{\dot{s}nw(s(\tau'))\tau'^{-q}u}.$$

We have a free action of T_s^Φ on $\mathcal{Y}'$ given by $e:(\tau',u)\mapsto(\tau's(e),ue^{-q-1})$. Note that the quantity $\theta_{\dot{s}nw(s(\tau'))\tau'^{-q}u)}$ is constant on the orbits of this action. Hence if $\mathcal{Y}'_0$ is a set of representatives for the T_s^Φ-orbits on $\mathcal{Y}'$ we have

$$B = q^{-1}|\mathfrak{K}_w|^{-1}\sum_{(\tau',y)\in\mathcal{Y}'_0,e\in T_s^\Phi}(s\nu)(\tau')\nu(e)\theta_{\tau'\dot{s}n\tau'^{-q}u}.$$

Note that $\sum_{e\in T_s^\Phi}\nu(e) = \delta(q^2-1)$. In particular, if $\Delta = 0$ we have $B = 0$. We now assume that $\Delta = 1$. For any $u\in T_s^\Phi$ such that $u^{q-1} = z$ we set

$$B_u = q^{-1}|\mathfrak{K}_w|^{-1}\sum_{\tau'\in T^\Phi}(s\nu)(\tau')\theta_{\tau'\dot{s}n\tau'^{-q}u}.$$

We have $B = \sum_{u\in T_s^\Phi;u^{q-1}=z}B_u$. For any u as above and any $e\in T_s^\Phi$ we have $B'_{ue^{-1-q}} = B'_u$ since $\nu(e) = 1$. If u,u' in T_s^Φ are such that $u^{q-1} = u'^{q-1} = z$, we have $u' = u\tilde{e}$, where $\tilde{e}\in T_s^\Phi$ satisfies $\tilde{e}^{q-1} = 1$. Hence we have $\tilde{e} = e^{-q-1}$ for some $e\in T_s^\Phi$ so that $u' = ue^{-q-1}$. Thus we have $B_{u'} = B_u$. We see that $B = (q-1)B_u$ where $u\in T_s^\Phi$ is such that $u^{q-1} = z$. We have $B_u = q^{-1}|\mathfrak{K}_{sw}||\mathfrak{K}_w|^{-1}a'_{\dot{s}nu,\nu}$. It remains to show that $(q-1)|\mathfrak{K}_{sw}||\mathfrak{K}_w|^{-1} = q+1$ or equivalently, that $|T(sw)||T(w)|^{-1} = (q-1)(q+1)^{-1}$. This follows from the following fact: there exists c,c' in $\mathbf{N}$ such that $|T(w)| = (q-1)^c(q+1)^{c'}$, $|T(sw)| = (q-1)^{c+1}(q+1)^{c'-1}$. The lemma is proved.

Lemma 2.11. *Let $s\in S, w\in I, n\in N(w), \nu\in\mathfrak{s}_w$. Assume that $sw = ws$, $|sw| < |w|$, $s\notin W_\nu$. Note that $s\nu\in\mathfrak{s}_w$. We have*

$$\mathcal{T}_s a'_{n,\nu} = -a'_{\dot{s}n\dot{s}^{-1},s\nu}.$$

Using Lemma 2.7(d) and §2.8(d) we have $\mathcal{T}_s a'_{n,\nu} = A + B$ where

$$A = q^{-1}|\mathfrak{K}_w|^{-1}\sum_{\tau\in T^\Phi,y\in T_s;y^{q-1}=1}c_y\nu(\tau)\theta_{\dot{s}\tau n\tau^{-q}\dot{s}^{-1}y},$$

where $c_y = q+1$ if $y\neq 1$, $c_y = 1$ if $y = 1$ and

$$B = |\mathfrak{K}_w|^{-1}\sum_{\tau\in T^\Phi,y\in T_s;y^{q+1}=\tau^q n^{-1}\tau^{-1}\dot{s}\tau n\tau^{-q}\dot{s}}\nu(\tau)\theta_{\dot{s}\tau n\tau^{-q}y}.$$

We have used that, as in the proof of Lemma 2.10, we have $\nu(\epsilon_s)=1$ (and hence $[\nu,\check{\alpha}_s]=1$). In the sum A we set $\tau'=s(\tau)$. We get

$$A=q^{-1}|\mathfrak{K}_w|^{-1}\sum_{\tau'\in T^\Phi,y\in T_s;y^{q-1}=1}c_y(s\nu)(\tau')\theta_{\tau'\dot{s}n\dot{s}^{-1}\tau'^{-q}y}.$$

For $y\in T_s$ such that $y^{q-1}=1$ we can find $y'\in T_s$ such that $y'^{q+1}=y$ (there are $q+1$ such y') and we have automatically $y'\in T^\Phi$. Thus we have

$$A=q^{-1}|\mathfrak{K}_w|^{-1}(q+1)^{-1}\sum_{\tau'\in T^\Phi,y'\in T_s^\Phi}c_{y'^{-q-1}}(s\nu)(\tau')\theta_{\tau'\dot{s}n\dot{s}^{-1}\tau'^{-q}y'^{-q-1}}$$
$$=q^{-1}|\mathfrak{K}_w|^{-1}(q+1)^{-1}\sum_{\tau'\in T^\Phi,y'\in T_s^\Phi}c_{y'^{-q-1}}(s\nu)(\tau')\theta_{y'\tau'\dot{s}n\dot{s}^{-1}y'^{-q}\tau'^{-q}}.$$

With the change of variable $\tau'y'=\tau''$ we obtain

$$A=q^{-1}|\mathfrak{K}_w|^{-1}(q+1)^{-1}\sum_{\tau''\in T^\Phi,y'\in T_s^\Phi}c_{y'^{-q-1}}(s\nu)(\tau'')\nu(y')\theta_{\tau''\dot{s}n\dot{s}^{-1}\tau''^{-q}}.$$

(We have used that $s(y')=y'^{-1}$.) Using our assumption that $s\notin W_\nu$, we have

$$\sum_{y'\in T_s^\Phi}c_{y'^{-q-1}}\nu(y')$$
$$=\sum_{y'\in T_s^\Phi;y'^{q+1}=1}\nu(y')+(q+1)\sum_{y'\in T_s^\Phi;y'^{q+1}\neq 1}\nu(y')$$
$$=(q+1)\sum_{y'\in T_s^\Phi}\nu(y')-q\sum_{y'\in T_s^\Phi;y'^{q+1}=1}\nu(y')$$
$$=-q\sum_{y'\in T_s^\Phi;y'^{q+1}=1}\nu(y')=-q\sum_{y'\in T_s^\Phi;y'^{q+1}=1}\underline{\nu}_w(y'^{-q-1})$$
$$=-q\sharp(y'\in T_s^\Phi;y'^{q+1}=1)=-q(q+1).$$

It follows that

$$A=q^{-1}|\mathfrak{K}_w|^{-1}(q+1)^{-1}(-q)(q+1)\sum_{\tau''\in T^\Phi}\nu(\tau'')\theta_{\tau''\dot{s}n\dot{s}^{-1}\tau''^{-q}}=-a'_{\dot{s}n\dot{s}^{-1},s\nu}.$$

It remains to prove that $B=0$. We set $z=n^{-1}\dot{s}n\dot{s}\in T_s$; see Lemma 2.7(b). In the sum B we have

$$\tau^q n^{-1}\tau^{-1}\dot{s}\tau n\tau^{-q}\dot{s}=\tau^q n^{-1}\tau^{-1}s(\tau)\dot{s}n\dot{s}s(\tau)^{-q}$$
$$=\tau^q\tau s(\tau)^{-1}n^{-1}\dot{s}n\dot{s}s(\tau)^{-q}=z\tau^q\tau s(\tau)^{-1}s(\tau)^{-q}=z(\tau s(\tau)^{-1})^{q+1}.$$

Thus we have

$$B=q^{-1}|\mathfrak{K}_w|^{-1}q\sum_{(\tau,y)\in\mathcal{Z}}\nu(\tau)\theta_{\dot{s}\tau n\tau^{-q}y},$$

where $\mathcal{Z}=\{(\tau,y)\in T^\Phi\times T_s;y^{q+1}=z(\tau s(\tau)^{-1})^{q+1}\}$. The group T_s^Φ acts freely on $\mathcal{Z}$ by $e:(\tau,y)\mapsto(\tau e,ye^{q+1})$. (We must show that the equation $y^{q+1}=z(\tau s(\tau)^{-1})^{q+1}$ implies $(ye^{q+1})^{q+1}=z(\tau es(\tau e)^{-1})^{q+1}$; it is enough to show that $e^{(q+1)^2}=e^{2(q+1)}$ and this follows from $e^{q^2-1}=1$.) We show that the last sum restricted to any T_s^Φ-orbit is zero. Since $\theta_{\dot{s}\tau n\tau^{-q}y}$ is constant on any T_s^Φ-orbit it is enough to show that $\sum_{e\in T_s^\Phi}\nu(e)=0$; this follows from our assumption that $s\notin W_\nu$. We deduce that $B=0$. The lemma is proved.

2.12. For $w \in I$ let $||w||$ be the dimension of the -1 eigenspace of the linear map induced by w on the real vector space $\mathbf{R} \otimes Y$. We have $|w| = ||w|| \mod 2$. For $w \in N(w), \nu \in \mathfrak{s}_w$ we set

$$\tilde{a}_{n,\nu} = q^{-(|w|+||w||)/2} a'_{n,\nu} \in \mathcal{F}'.$$

We have the following result.

Lemma 2.13. *Let $s \in S, w \in W_2, n \in N(w), \nu \in \mathfrak{s}_w$. Write $n = \dot{w}t$ where $t \in T$. We have:*

(a) $\mathcal{T}_s \tilde{a}_{n,\nu} = [\nu, \check{\alpha}_s]\tilde{a}_{\dot{s}n\dot{s}^{-1},s\nu}$ *if $sw \neq ws, |sw| > |w|$;*

(b) $\mathcal{T}_s \tilde{a}_{n,\nu} = \tilde{a}_{\dot{s}n\dot{s}^{-1},s\nu}$ *if $sw = ws, |sw| > |w|, s \notin W_\nu$;*

(c) $\mathcal{T}_s \tilde{a}_{n,\nu} = \tilde{a}_{n,\nu} + (q+1)\tilde{a}_{\dot{s}nu,\nu}$ *(where $u \in T_s^\Phi$ is such that $u^{q-1} = \dot{s}n^{-1}\dot{s}n = s(t)^{-1}t\epsilon_s$; see Lemma* 2.6(a), (b)) *if $sw = ws$, $|sw| > |w|, s \in W_\nu$;*

(d) $\mathcal{T}_s \tilde{a}_{n,\nu} = -\tilde{a}_{\dot{s}n\dot{s}^{-1},s\nu}$ *if $sw = ws$, $|sw| < |w|$, $s \notin W_\nu$.*

(a) is a reformulation of Lemma 2.9; (b), (c) are reformulations of Lemma 2.10; (d) is a reformulation of Lemma 2.11.

3. Proof of Theorem 0.4

3.1. We preserve the setup of §1.1. Let L be the subgroup of Y generated by $\{\check{\alpha}_s; s \in S\}$. Let S' be a *halving* of S, that is, a subset S' of S such that $s_1s_2 = s_2s_1$ whenever s_1, s_2 in S are both in S' or both in $S - S'$. (Such S' always exists.) Let $W_2 \to Y$, $w \mapsto r_w$, and $W_2 \to L/2L$, $w \mapsto b_w = b_w^{S'}$ be the maps defined in [L5, 0.2, 0.3]. From [L5, 0.2, 0.3] and from the proof of [L5, 1.14(a)] we have:

(i) $r_1 = 0$, $r_s = \check{\alpha}_s$ for any $s \in S$, $b_1 = 0$, $b_s = \check{\alpha}_s$ for any $s \in S'$, $b_s = 0$ for any $s \in S - S'$;

(ii) for any $w \in W_2, s \in S$ such that $sw \neq ws$ we have $s(r_w) = r_{sws}$, $s(b_w) = b_{sws} + \check{\alpha}_s$;

(iii) for any $w \in W_2, s \in S$ such that $sw = ws$ we have $r_{sw} = r_w + \mathcal{N}\check{\alpha}_s$, $b_{sw} = b_w + l\check{\alpha}_s$ where $l \in \{0,1\}$, $\mathcal{N} \in \{-1,0,1\}$.

(iv) for any $w \in W_2, s \in S$ such that $sw = ws, |sw| > |w|$ we have $s(r_w) = r_w$;

(v) for any $w \in W_2, s \in S$ such that $sw = ws$ we have $s(b_w) = b_w + (1-\mathcal{N})\check{\alpha}_s$ where $\mathcal{N}$ is as in (iii).

Moreover, by [L5, 0.5],

(vi) if $c \in F_Q, c^{q-1} = \epsilon$, the element $n_{w,c} = \dot{w}r_w(c)b_w(\epsilon) \in \kappa^{-1}(w)$ belongs to $N(w)$.

Here $r_w(c) \in T$, $b_w(\epsilon) \in T$ are obtained by evaluating a homomorphism $\mathbf{k}^* \to Y$ at c or ϵ. Note that $b_w(\epsilon) = b_w(\epsilon)^{-1}$. From [L5, 1.18] we deduce:

(vii) in the setup of (iii) we have $\mathcal{N} = (w : s)$.

The following equality complements (iv):

(viii) for any $w \in W_2, s \in S$ such that $sw = ws, |sw| < |w|$ we have $s(r_w) = r_w + 2(w : s)\check{\alpha}_s$.

Indeed, using (iii), (iv), (vii) we have

$s(r_w) = s(r_{sw} - (w : s)\check{\alpha}_s) = r_{sw} + (w : s)\check{\alpha}_s = r_w + 2(w : s)\check{\alpha}_s$.

For any $w \in W_2$, any $c \in F_Q$ such that $c^{q-1} = \epsilon$, and any $\nu \in \mathfrak{s}_w$ we set

$$a_{w,c,\nu} = \tilde{a}_{n_{w,c},\nu}.$$

This is well defined by (vi). By §2.8(b),

(a) *for any c as above, $\{a_{w,c,\nu}; w \in W_2, \nu \in \mathfrak{s}_w\}$ is a $\mathbf{C}$-basis of $\mathcal{F}'$.*

In the remainder of this section we assume that §0.3(a) holds. We have the following result.

Proposition 3.2. *Let $s\in S, w\in W_2, \nu\in\mathfrak{s}_w$. Let c be as in* §3.1(vi). *We have*
(a) $\mathcal{T}_s a_{w,c,\nu}=a_{sws,c,s\nu}$ *if* $sw\neq ws, |sw|>|w|, s\in W_\nu$;
(b) $\mathcal{T}_s a_{w,c,\nu}=a_{sws,c,s\nu}+(q-q^{-1})a_{w,c,\nu}$ *if* $sw\neq ws, |sw|<|w|, s\in W_\nu$;
(c) $\mathcal{T}_s a_{w,c,\nu}=a_{w,c,\nu}+(q+1)a_{sw,c,\nu}$ *if* $sw=ws, |sw|>|w|, s\in W_\nu$;
(d) $\mathcal{T}_s a_{w,c,\nu}=(1-q^{-1})a_{sw,c,\nu}+(q-q^{-1}-1)a_{w,c,\nu}$ *if* $sw=ws, |sw|<|w|, s\in W_\nu$;
(e) $\mathcal{T}_s a_{w,c,\nu}=[\nu,\check\alpha_s]a_{sws,c,s\nu}$ *if* $sw\neq ws, |sw|>|w|, s\notin W_\nu$;
(f) $\mathcal{T}_s a_{w,c,\nu}=[\nu,\check\alpha_s]^{-1}a_{sws,c,s\nu}$ *if* $sw\neq ws, |sw|<|w|, s\notin W_\nu$;
(g) $\mathcal{T}_s a_{w,c,\nu}=\underline{s\nu}_w(\epsilon_s^{1-(w:s)})a_{w,c,s\nu}$ *if* $sw=ws, |sw|>|w|, s\notin W_\nu$;
(h) $\mathcal{T}_s a_{w,c,\nu}=-\underline{s\nu}_w(\epsilon_s^{1-(w:s)})\underline{s\nu}_w(c_s^{-2(w:s)})a_{w,c,s\nu}$ *if* $sw=ws, |sw|<|w|, s\notin W_\nu$.

This will be deduced in §§3.3–3.8 from Lemma 2.13 with $n=n_{w,c}$ as in §3.1(vi), using the equality $\tilde a_{n't,\nu'}=\underline{\nu'}_{w'}(t^{-1})\tilde a_{n',\nu'}$ where $w'\in W_2, n'\in N(w), \nu'\in\mathfrak{s}_{w'}, t\in T(w')$, which follows from §2.8(a).

3.3. Assume that we are in the setup of Proposition 3.2(a) or Proposition 3.2(e). Using Lemma 2.13(a) and §3.1(ii) we obtain

$$\begin{aligned}\mathcal{T}_s a_{w,c,\nu}&=[\nu,\check\alpha_s]\tilde a_{\dot s\dot w r_w(c)b_w(\epsilon)\dot s^{-1},s\nu}\\&=[\nu,\check\alpha_s]\tilde a_{\dot s\dot w\dot s\dot s^{-1}r_w(c)b_w(\epsilon)\dot s^{-1},s\nu}\\&=[\nu,\check\alpha_s]\tilde a_{n_{sws,c}r_{sws}(c)^{-1}b_{sws}(\epsilon)^{-1}ds^{-1}r_w(c)b_w(\epsilon)\dot s^{-1},s\nu}\\&=[\nu,\check\alpha_s]\underline{s\nu}_{sws}(\dot s b_w(\epsilon)r_w(c)^{-1}\dot s r_{sws}(c)b_{sws}(\epsilon))a_{sws,c,s\nu}\\&=[\nu,\check\alpha_s]\underline{s\nu}_{sws}(b_{sws}(\epsilon)\epsilon_s r_{sws}(c)^{-1}\epsilon_s r_{sws}(c)b_{sws}(\epsilon))a_{sws,c,s\nu}=[\nu,\check\alpha_s]a_{sws,c,s\nu}.\end{aligned}$$

This proves Proppposition 3.2(e). Now Proposition 3.2 follows also since in that case we have $[\nu,\check\alpha_s]=1$. (It is enough to show that $\nu(\epsilon_s)=1$. This follows from $s\in W_\nu$.) This proves Proposition 3.2(a).

3.4. Assume that we are in the setup of Proposition 3.2(g). Using Lemma 2.13(b), §3.1(iv), (v), (vii), we obtain

$$\begin{aligned}\mathcal{T}_s a_{w,c,\nu}&=\tilde a_{\dot s\dot w r_w(c)b_w(\epsilon)\dot s^{-1},s\nu}=\tilde a_{\dot w s(r_w(c)b_w(\epsilon)),s\nu}\\&=\tilde a_{\dot w r_w(c)b_w(\epsilon)\epsilon_s^{1-(w:s)},s\nu}=\underline{s\nu}_w(\epsilon_s^{1-(w:s)})a_{w,c,s\nu}.\end{aligned}$$

This proves Proposition 3.2(g).

3.5. Assume that we are in the setup of Lemma 2.13(c) with $n=n_{w,c}$. Using §3.1(iv), (v), (vii), we have

$$\begin{aligned}u^{q-1}&=s(r_w(c)b_w(e))^{-1}r_w(c)b_w(e)\epsilon_s=s(b_w(e))^{-1}b_w(e)\epsilon_s\\&=\epsilon_s^{1-(w:s)}\epsilon_s=\epsilon_s^{(w:s)}.\end{aligned}\tag{a}$$

For $l\in\{0,1\}$ we show:

$$\underline{\nu}_{sw}(c_s^{(w:s)}\epsilon_s^l u^{-1})=1.\tag{b}$$

Since ν is 1 on T_s^Φ, $\underline{\nu}_{sw}$ must be trivial on $e_{sw}(T_s^\Phi)$, that is, on the image of $T_s^\Phi\to T_s^\Phi$, $t\mapsto t^{q+1}$ which is the same as $\{t'\in T_s; t'^{q-1}=1\}$. Since $c_s^{(w:s)}\epsilon_s^l u^{-1}\in T_s$, it is enough to show that

$$(c_s^{(w:s)}\epsilon_s^l u^{-1})^{q-1}=1.\tag{c}$$

Using (a) and the equations $c^{q-1}=\epsilon$, $\epsilon^{q-1}=1$, we see that the left-hand side of (c) is $\epsilon_s^{(w:s)}\epsilon_s^{-(w:s)}=1$. This completes the proof of (b).

We now assume that we are in the setup of Proposition 3.2(c) (which is the same as the setup of Lemma 2.13(c) with $n=n_{w,c}$). From Lemma 2.13(c) we deduce using (b) and §3.1(iii) that for some $l\in\{0,1\}$ we have

$$\begin{aligned}\mathcal{T}_s a_{w,c,\nu}-a_{w,c,\nu}&=(q+1)\tilde a_{\dot s\dot w r_w(c)b_w(\epsilon)u,\nu}\\&=(q+1)\tilde a_{\dot s\dot w r_{sw}(c)b_{sw}(\epsilon)r_{sw}(c)^{-1}b_{sw}(\epsilon)r_w(c)b_w(\epsilon)u,\nu}\\&=(q+1)\underline{\nu}_{sw}(r_{sw}(c)b_{sw}(\epsilon)r_w(c)^{-1}b_w(\epsilon)u^{-1})a_{sw,c,\nu}\\&=(q+1)\underline{\nu}_{sw}(c_s^{(w:s)}\epsilon_s^l u^{-1})a_{sw,c,\nu}=(q+1)a_{sw,c,\nu}.\end{aligned}$$

This completes the proof of Proposition 3.2(c).

3.6. Assume that we are in the setup of Proposition 3.2(h). From Lemma 2.13(d) we deduce using §3.1(viii):

$$\begin{aligned}&\mathcal{T}_s a_{w,c,\nu}=-\tilde a_{\dot s\dot w r_w(c)b_w(\epsilon)\dot s^{-1},s\nu}=-\tilde a_{\dot w s(r_w(c)b_w(\epsilon)),s\nu}\\&=-\tilde a_{\dot w r_w(c)b_w(\epsilon)c_s^{2(w:s)}\epsilon_s^{1-(w:s)},s\nu}=\underline{s\nu}_w(c_s^{-2(w:s)}\epsilon_s^{1-(w:s)})a_{w,c,s\nu}\\&=\underline{s\nu}_w(\epsilon_s^{1-(w:s)}\underline{s\nu}_w(c_s^{-2(w:s)})a_{w,c,s\nu}.\end{aligned}$$

This proves Proposition 3.2(h).

3.7. Assume that $sw\ne ws$, $|sw|<|w|$. Then Proposition 3.2(a), (e) are applicable with $sws,s\nu$ instead of w,ν so that

$$\mathcal{T}_s a_{sws,c,s\nu}=[s\nu,\check\alpha_s]a_{w,c,\nu}.$$

We apply $\mathcal{T}_s^{-1}$ to both sides; we obtain

$$\mathcal{T}_s^{-1}a_{w,c,\nu}=\mathcal{T}_s^{-1}1_\nu a_{w,c,\nu}=[s\nu,\check\alpha_s]^{-1}a_{sws,c,s\nu}.$$

Using §1.8(i) we deduce

$$\mathcal{T}_s a_{w,c,\nu}-\delta(q-q^{-1})a_{w,c,\nu}=[s\nu,\check\alpha_s]^{-1}a_{sws,c,s\nu},$$

where $\Delta=1$ if $s\in W_\nu$, $\Delta=0$ if $s\notin W_\nu$. This proves Proposition 3.2(b), (f). (We use that $[s\nu,\check\alpha_s]=[\nu,\check\alpha_s]$ is 1 when $s\in W_\nu$ since $\nu(\epsilon_s)=1$ in that case.)

3.8. Assume that s,w,ν are as in Proposition 3.2(d). Then Proposition 3.2(c) is applicable to sw,ν instead of w,ν and gives:

(a) $$\mathcal{T}_s a_{sw,c,\nu}=a_{sw,c,\nu}+(q+1)a_{w,c,\nu}.$$

We apply $\mathcal{T}_s$ to (a). We obtain

$$\mathcal{T}_s\mathcal{T}_s a_{sw,c,\nu}=\mathcal{T}_s a_{sw,c,\nu}+(q+1)\mathcal{T}_s a_{w,c,\nu}.$$

Using §1.8(h) we deduce

$$a_{sw,c,\nu}+(q-q^{-1})\mathcal{T}_s a_{sw,c,\nu}=\mathcal{T}_s a_{sw,c,\nu}+(q+1)\mathcal{T}_s a_{w,c,\nu}$$

and hence, using (a):

$$a_{sw,c,\nu}+(q-q^{-1}-1)(a_{sw,c,\nu}+(q+1)a_{w,c,\nu})=(q+1)\mathcal{T}_s a_{w,c,\nu}.$$

Dividing by $q+1$ we get Proposition 3.2(d). This completes the proof of Proposition 3.2.

3.9. We choose a generator γ of the cyclic group F_Q^* so that we have an isomorphism

(a) $$\mathbf{Z}/(Q-1)\mathbf{Z} \xrightarrow{\sim} F_Q^*$$

which takes 1 to γ.

Let $z \in \mathbf{Z}$ be as in §0.2. Let $c = \gamma^{z(q+1)/2} \in F_Q^*$. (If $p = 2$ so that $(q+1)/2$ is not an integer, this is interpreted as a square root of $\gamma^{z(q+1)}$ which is uniquely defined.) If $p \neq 2$ we have $c^{q-1} = \gamma^{z(q^2-1)/2} = \epsilon$ by the choice of z. If $p = 2$, then $(c^{q-1})^2 = (c^2)^{q-1} = \gamma^{z(q^2-1)} = 1$ and hence $c^{q-1} = 1 = \epsilon$. Thus in any case we have $c^{q-1} = \epsilon$.

We have an isomorphism of groups $F_Q^* \otimes Y \xrightarrow{\sim} T^\Phi$, $z \otimes y \mapsto y(z)$. Using (a) this can be viewed as an isomorphism of groups $(\mathbf{Z}/(Q-1)\mathbf{Z}) \otimes Y \xrightarrow{\sim} T^\Phi$; it takes $n \otimes y$ to $y(\gamma^n)$. We have a pairing

$$(,) : ((\mathbf{Z}/(Q-1)\mathbf{Z}) \otimes Y) \times \bar{X}_q \to \mathbf{C}^*$$

given by

$$(d \otimes y, \frac{a}{Q-1} \otimes x) = \exp(2\pi\sqrt{-1}\frac{da}{Q-1}\lfloor y, x\rfloor),$$

where $y \in Y, x \in X$, $a \in \mathbf{Z}$, $d \in \mathbf{Z}$. This pairing identifies $\bar{X}_q$ with $\mathrm{Hom}((\mathbf{Z}/(Q-1)\mathbf{Z}) \otimes Y, \mathbf{C}^*) = \mathrm{Hom}(T^\Phi, \mathbf{C}^*) = \mathfrak{s}$. This identification is compatible with the natural W-actions on $\bar{X}_q$ and $\mathfrak{s}$; it induces an identification $\tilde{X}_q = \{(w, \nu); w \in W_2, \nu \in \mathfrak{s}_w\}$. Thus, the basis §3.1(a) of $\mathcal{F}'$ can be naturally indexed by the elements of $\tilde{X}_q$. We shall interpret the quantities

$$[\nu, \check{\alpha}_s], \underline{s\nu}_w(\epsilon_s^{1-(w:s)}), \underline{s\nu}_w(c_s^{-2(w:s)})$$

which appear in Proposition 3.2 in terms of the corresponding parameter in $\tilde{X}_q$. Assume that $(w, \nu) \in W_2 \times \mathfrak{s}$ (with $\nu \in \mathfrak{s}_w$) corresponds to $(w, \lambda) \in \tilde{X}_q$. Then for any $s \in S$ we have

(b) $$\nu(\check{\alpha}_s(\gamma)) = \exp(2\pi\sqrt{-1}\lfloor\check{\alpha}_s, \lambda\rfloor).$$

We show:

(c) If $sw = ws, |sw| < |w|, s \notin W_\nu$, then

$$\underline{s\nu}_w(c_s^{-2(w:s)}) = \exp(2\pi\sqrt{-1}(w:s)z\lfloor\check{\alpha}_s, \lambda\rfloor).$$

Let $\tilde{c} = \gamma^z$. We have $\tilde{c}_s^{q+1} = c_s^2$ and hence

$$\underline{s\nu}_w(c_s^{-2(w:s)}) = \underline{s\nu}_w((\tilde{c}_s^{-(w:s)})^{q+1}) = \underline{s\nu}_w(e_w(\tilde{c}_s^{-(w:s)})) = (s\nu)(\tilde{c}_s^{-(w:s)}) = \nu(\tilde{c}_s^{(w:s)}).$$

It remains to show:

$$\nu(\check{\alpha}_s(\gamma^z) = \exp(2\pi\sqrt{-1}z\lfloor\check{\alpha}_s, \lambda\rfloor).$$

This clearly follows from (b).

We show:

(d) If $sw = ws$, then $\underline{s\nu}_w(\epsilon_s^{1-(w:s)}) = \delta_{w,s\lambda,s}$.

If $p = 2$, both sides are 1. Thus we can assume that $p \neq 2$. We must show that

$$\underline{s\nu}_w(\epsilon_s^{1-(w:s)}) = \exp(2\pi\sqrt{-1}((q-e)/2)(1-(w:s))\lfloor\check{\alpha}_s, s\lambda\rfloor),$$

where $e = |w| - |sw| = \pm 1$. It is enough to show that

$$\underline{s\nu}_w(\epsilon_s) = \exp(2\pi\sqrt{-1}((q-e)/2)\lfloor\check{\alpha}_s, s\lambda\rfloor).$$

We have $\epsilon_s = (\gamma_s^{(q-e)/2})^{q+e} = e_w(\gamma_s^{(q-e)/2})$ so that

$$\underline{s\nu}_w(\epsilon_s) = \underline{s\nu}_w(e_w(\gamma_s^{(q-e)/2})) = (s\nu)(\gamma_s^{(q-e)/2}).$$

Thus it is enough to show that

$$(s\nu)(\check{\alpha}_s(\gamma)) = \exp(2\pi\sqrt{-1}\lfloor\check{\alpha}_s, s\lambda\rfloor).$$

This clearly follows from (b).

We show:

(e) If $s \in S$, then $[\lambda, s] = [\nu, \check{\alpha}_s]$.

If $p = 2$ both sides are 1. Thus we can assume that $p \neq 2$. We must show that we have $[\lambda, s] = 1$ if and only if $[\nu, \check{\alpha}_s] = 1$ or that $\exp(2\pi\sqrt{-1}(1/2)(Q-1)\lfloor\check{\alpha}_s, s\lambda\rfloor) = 1$ if and only if $\nu(\check{\alpha}_s(\epsilon)) = 1$ or (using (b)) that $\nu(\check{\alpha}_s(\gamma))^{(1/2)(Q-1)} = 1$ if and only if $\nu(\check{\alpha}_s(\epsilon)) = 1$. This follows from the equality $\gamma^{(1/2)(Q-1)} = \epsilon$.

From (b) and the definitions we see that:

(f) If $s \in S$, then we have $s \in W_\lambda$ if and only if $s \in W_\nu$.

We now see that Proposition 3.2 implies the truth of Theorem 0.4 in the special case where $\mathbf{k}$ is as in §1.1. But then Theorem 0.4 follows immediately for any $\mathbf{k}$ as in §0.1 such that the characteristic of $\mathbf{k}$ is 0 or p. This completes the proof of Theorem 0.4.

4. The generic case

4.1. In this section we assume that $\mathbf{k} = \mathbf{C}$ and that §0.3(a) holds. We have $\bar{X}_1 = \bar{X}$. Hence $\tilde{X}_1 = \{(w, \lambda) \in W_2 \times \bar{X}; w(\lambda) = -\lambda\}$.

Until the end of §4.2, we fix a W-orbit $\mathcal{O}$ in $\bar{X}$ which is contained in the image of $X_{\mathbf{Q}}$ under $X_K \to \bar{X}$. We can find an integer $\mathfrak{e} \geq 1$ such that $\mathfrak{e}\lfloor y, \lambda\rfloor = 0$ for any $y \in Y$ and any $\lambda \in \mathcal{O}$. We can write $\mathfrak{e} = \prod_{p \in \mathfrak{P}} p^{c_p}$ where $\mathfrak{P}$ is a finite set of prime numbers and $c_p \geq 1$ are integers. Let $\mathfrak{P}'$ be the set of prime numbers which do not divide $2\mathfrak{e}$. Note that $\mathfrak{P} \cap \mathfrak{P}' = \emptyset$. Hence if $p \in \mathfrak{P}, p' \in \mathfrak{P}'$, then p' is a unit in the ring $\mathbf{Z}/p^{c_p}\mathbf{Z}$ and hence for some integer $a_p \geq 1$ independent of p' we have ${p'}^{a_p} = 1$ in $\mathbf{Z}/p^{c_p}\mathbf{Z}$, that is, p^{c_p} divides ${p'}^{a_p} - 1$. Let $\mathcal{S}$ be the set of all integers $z \geq 1$ such that z is divisible by $\prod_{\pi \in \mathfrak{P}} a_p$. Then for any $p \in \mathfrak{P}, p' \in \mathfrak{P}'$ and any $z \in \mathcal{S}$, p^{c_p} divides ${p'}^z - 1$. Hence for any $p' \in \mathfrak{P}'$ and any $z \in \mathcal{S}$, $\mathfrak{e}$ divides ${p'}^z - 1$. Let $\mathfrak{Q}$ be the set of all numbers of the form ${p'}^z$ with $p' \in \mathfrak{P}', z \in \mathcal{S}$. Then we have $(q-1)\lfloor y, \lambda\rfloor = 0$ for any $q \in \mathfrak{Q}$, any $y \in Y$, and any $\lambda \in \mathcal{O}$. Hence

(a) $(q-1)\lambda = 0$ for any $q \in \mathfrak{Q}$ and any $\lambda \in \mathcal{O}$.

It follows that

(b) if $(w, \lambda) \in \tilde{X}_1$ and $\lambda \in \mathcal{O}$, then $(w, \lambda) \in \tilde{X}_q$ for any $q \in \mathfrak{Q}$.

Indeed, we have $w(\lambda) = -\lambda$ and we must show that $w(\lambda) = -q\lambda$. It is enough to show that $q\lambda = \lambda$ and this follows from (a).

4.2. Let $\tilde{\mathfrak{Q}}$ be the set of squares of the numbers in $\mathfrak{Q}$. We have $\tilde{\mathfrak{Q}} \subset \mathfrak{Q}$. We now fix $q \in \tilde{\mathfrak{Q}}$. We have $q = {q'}^2$ with $q' \in \mathfrak{Q}$. Note that $q = 4t + 1$ for some $t \in \mathbf{N}$. Let $(w, \lambda) \in \tilde{X}_1$ with $\lambda \in \mathcal{O}$ (so that $(w, \lambda) \in \tilde{X}_{q'}$ and $(w, \lambda) \in \tilde{X}_q$ by §4.1(b)) and let $s \in S$. We show:

(a) $[\lambda, s]$ defined as in §0.2 in terms of q is equal to 1.

Since $(w, \lambda) \in \tilde{X}_{q'}$ we have $\lfloor\check{\alpha}_s, \lambda\rfloor = e'/({q'}^2 - 1)$ with $e' \in \mathbf{Z}$. Hence $\lfloor\check{\alpha}_s, \lambda\rfloor = e/(q^2 - 1)$ with $e = e'({q'}^2 + 1)$. Since e is even we see that (a) holds.

We show:

(b) If $sw = ws$, $|sw| > |w|$, then $\delta_{w,\lambda;s}$ defined as in §0.3 in terms of q is equal to $\delta'_{w,\lambda;s}$ defined as in §0.5.

It is enough to show that $\exp(2\pi\sqrt{-1}((q+1)/2)\lfloor\check{\alpha}_s,\lambda\rfloor) = \exp(2\pi\sqrt{-1}\lfloor\check{\alpha}_s,\lambda\rfloor)$ or that $(-1+(q+1)/2)\lfloor\check{\alpha}_s,\lambda\rfloor = 0$, or that $2t\lfloor\check{\alpha}_s,\lambda\rfloor = 0$. This follows from §0.5(b).

We show:

(c) If $sw = ws$, $|sw| < |w|$, then $\delta_{w,\lambda;s}$ defined as in §0.3 in terms of q is equal to 1.

It is enough to show that

$$\exp(2\pi\sqrt{-1}((q-1)/2)\lfloor\check{\alpha}_s,\lambda\rfloor) = 1$$

or that $((q-1)/2)\lfloor\check{\alpha}_s,\lambda\rfloor = 0$. Since $\lambda \in \bar{X}_{q'}$ we have $(q'-1)\lfloor\check{\alpha}_s,\lambda\rfloor = 0$ by the argument at the end of §0.3. We have $(q-1)/2 = (q'-1)(q'+1)/2$ where $q'+1 \in 2\mathbf{Z}$ and hence

$$((q-1)/2)\lfloor\check{\alpha}_s,\lambda\rfloor = ((q'+1)/2)(q'-1)\lfloor\check{\alpha}_s,\lambda\rfloor = 0.$$

This proves (c).

Proposition 4.3. *Let* $\mathbf{q}$ *be an indeterminate and let* $\tilde{\mathbf{M}}$ *denote the* $\mathbf{C}(\mathbf{q})$*-vector space with basis* $\{\tilde{\mathbf{a}}_{w,\lambda}; (w,\lambda) \in \tilde{X}_1\}$. *There is a unique action of the braid group of* W *on* $\tilde{\mathbf{M}}$ *in which the generators* $\{\mathcal{T}_s; s \in S\}$ *of the braid group applied to the basis elements of* $\tilde{\mathbf{M}}$ *are as follows.* (*We write* $\Delta = 1$ *if* $s \in W_\lambda$ *and* $\Delta = 0$ *if* $s \notin W_\lambda$.)

(a) $\mathcal{T}_s\tilde{\mathbf{a}}_{w,\lambda} = \tilde{\mathbf{a}}_{sws,\lambda}$ *if* $sw \neq ws, |sw| > |w|$;

(b) $\mathcal{T}_s\tilde{\mathbf{a}}_{w,\lambda} = \tilde{\mathbf{a}}_{sws,s\lambda} + \Delta(\mathbf{q}-\mathbf{q}^{-1})\tilde{\mathbf{a}}_{w,\lambda}$ *if* $sw \neq ws, |sw| < |w|$;

(c) $\mathcal{T}_s\tilde{\mathbf{a}}_{w,\lambda} = \delta'_{w,s\lambda;s}\tilde{\mathbf{a}}_{w,s\lambda} + \Delta(\mathbf{q}+1)\tilde{\mathbf{a}}_{sw,\lambda}$ *if* $sw = ws, |sw| > |w|$;

(d) $\mathcal{T}_s\tilde{\mathbf{a}}_{w,\lambda} = \Delta(1-\mathbf{q}^{-1})\tilde{\mathbf{a}}_{sw,\lambda} + \Delta(\mathbf{q}-\mathbf{q}^{-1})\tilde{\mathbf{a}}_{w,\lambda} - \tilde{\mathbf{a}}_{w,s\lambda}$ *if* $sw = ws, |sw| < |w|$.

Here $\delta'_{w,s\lambda;s} = \pm 1$ *is as in* §0.5. (*It is* 1 *in the simply laced case; it is also* 1 *if* $\Delta = 1$.)

It is enough to prove the proposition with $\tilde{\mathbf{M}}$ replaced by the $\mathbf{C}(\mathbf{q})$-vector space $\tilde{\mathbf{M}}_{\mathcal{O}}$ with basis $\{\tilde{\mathbf{a}}_{w,\lambda}; (w,\lambda) \in \tilde{X}_1, \lambda \in \mathcal{O}\}$, where $\mathcal{O}$ is any W-orbit in $\bar{X}$.

Assume first that $\mathcal{O}$ is as in §4.1 and let $\mathfrak{e}, \mathfrak{Q}, \tilde{\mathfrak{Q}}$ be as in §4.2. Let $\tilde{\mathfrak{Q}}' = \{q \in \tilde{\mathfrak{Q}}; 2\mathfrak{e} < q^2 - 1\}$. Clearly, $\tilde{\mathfrak{Q}}'$ is an infinite set.

Let $M_{\mathcal{O}}$ be the $\mathbf{C}$-vector space with basis $\{\tilde{\mathbf{a}}_{w,\lambda}; (w,\lambda) \in \tilde{X}_1; \lambda \in \mathcal{O}\}$. By §4.1(b) we can identify $M_{\mathcal{O}}$ with a subspace of M_q (for any $q \in \mathfrak{Q}$) by $\tilde{\mathbf{a}}_{w,\lambda} \mapsto a_{w,\lambda}$. This subspace of M_q is stable under the operators $\mathcal{T}_s, s \in S$ attached in Theorem 0.4 to $z = \mathfrak{e}$, provided that $q \in \tilde{\mathfrak{Q}}'$. (Note for $q \in \tilde{\mathfrak{Q}}'$ we have $2z \notin (q^2-1)\mathbf{Z}$ since $0 < 2fe < q^2 - 1$.) Hence $\mathcal{T}_s : M_q \to M_q$ can be regarded as an operator $\mathcal{T}_s^{(q)} : M_{\mathcal{O}} \to M_{\mathcal{O}}$ for any $q \in \tilde{\mathfrak{Q}}'$. This operator is given by a matrix in the basis of $M_{\mathcal{O}}$ given by Laurent polynomials in q with integer coefficients independent of q. (This follows from the formulas 0.4(a)–(h), from §4.2(a), (b), (c) and from the equality $\exp(2\pi\sqrt{-1}(w:s)\mathfrak{e}\langle\check{\alpha}_s,\lambda\rangle) = 1$ for $\lambda \in \mathcal{O}$.) Since q runs through an infinite set, we deduce that the braid group relations satisfied by the $\mathcal{T}_s^{(q)}$ remain valid when q is replaced by the indeterminate $\mathbf{q}$. We see that if we identify $\tilde{\mathbf{M}}_{\mathcal{O}} = \mathbf{C}(\mathbf{q}) \otimes M_{\mathcal{O}}$, then there is a unique action of the braid group of W on $\tilde{\mathbf{M}}_{\mathcal{O}}$ in which the generators $\{\mathcal{T}_s; s \in S\}$ of the braid group applied to the basis elements of $\tilde{\mathbf{M}}_{\mathcal{O}}$ are as in (a)–(d) above.

We now consider a W-orbit $\mathcal{O}$ in $\bar{X}$ which is not necessarily as in §4.1. We choose $\xi_0 \in X_K$ such that the image of x_0 in $\bar{X}$ belongs to $\mathcal{O}$. Let $\mathfrak{H}$ be the collection of affine hyperplanes

$\{\xi \in X_K; \langle\check{\alpha},\xi\rangle = e\}$ for various $\check{\alpha} \in \check{R}, e \in \mathbf{Z}$;

$\{\xi \in X_K; w(\xi) = \xi + x\}$ for various $w \in W - \{1\}, x \in X$;

$\{\xi \in X_K; w(\xi) = -\xi + x\}$ for various $w \in W_2, x \in X$ such that $w+1$ is not identically zero on X.

We can find $\xi'_0 \in X_{\mathbf{Q}}$ such that a hyperplane in $\mathfrak{H}$ contains ξ_0 if and only if it contains ξ'_0. Let $\mathcal{O}'$ be the W-orbit of the image of ξ'_0 in $\bar{X}$. There is a unique W-equivariant bijection $j : \mathcal{O}' \xrightarrow{\sim} \mathcal{O}$ under which the image of ξ'_0 in $\bar{X}$ corresponds to the image of ξ_0 in $\bar{X}$. We define an isomorphism $\tilde{\mathbf{M}}_{\mathcal{O}'} \xrightarrow{\sim} \tilde{\mathbf{M}}_{\mathcal{O}}$ by $\tilde{\mathbf{a}}_{w,\lambda'} \mapsto \tilde{\mathbf{a}}_{w,j(\lambda')}$. This isomorphism is compatible with the operators $\mathcal{T}_s$ on these two vector spaces. Since these operators satisfy the braid group relations on $\tilde{\mathbf{M}}'_{\mathcal{O}}$ (by the first part of the proof) they will satisfy the braid group relations on $\tilde{\mathbf{M}}_{\mathcal{O}}$. This completes the proof of the proposition.

4.4. Let v be an indeterminate such that $v^2 = \mathbf{q}$. Let $\mathbf{M} = \mathbf{C}(v) \otimes_{\mathbf{C}(\mathbf{q})} \tilde{\mathbf{M}}$. We consider the basis $\{\mathbf{a}_{w,\lambda}; (w,\lambda) \in \tilde{X}_1\}$ defined by $\mathbf{a}_{w,\lambda} = v^{||w||}\tilde{\mathbf{a}}_{w,\lambda}$ where $||w||$ is as in §2.12. The linear maps $\mathcal{T}_s : \tilde{\mathbf{M}} \to \tilde{\mathbf{M}}$ with $s \in S$ extend to linear maps $\mathcal{T}_s : \mathbf{M} \to \mathbf{M}$ which satisfy the equalities in Theorem 0.6. Thus Theorem 0.6 is a consequence of Proposition 4.3.

4.5. Let $\mathbf{H}$ be the $\mathbf{C}(v)$-vector space with basis $\{\mathcal{T}_{w,\lambda}; (w,\lambda) \in W\bar{X}\}$. There is a unique structure of associative $\mathbf{C}(v)$-algebra (without 1 in general) on $\mathbf{H}$ such that (a), (b) below hold.

(a) $$\mathcal{T}_{w,\lambda}\mathcal{T}_{w',\lambda'} = \delta_{w^{-1}(\lambda),\lambda'}\mathcal{T}_{ww',\lambda'}$$

if $(w,\lambda) \in W\bar{X}, (w',\lambda') \in W\bar{X}$, $|ww'| = |w| + |w'|$;

(b) $$\mathcal{T}_{s,\lambda}\mathcal{T}_{s,\lambda'} = \delta_{\lambda,\lambda'}\mathcal{T}_{1,\lambda'} + \Delta\delta_{s(\lambda),\lambda'}(v^2 - v^{-2})\mathcal{T}_{s,\lambda'}$$

if $s \in S, \lambda \in \bar{X}, \lambda' \in \bar{X}$ (here $\Delta = 1$ if $s \in W_\lambda$ and $\Delta = 0$ if $s \notin W_\lambda$). We call $\mathbf{H}$ the *extended Hecke algebra*. This algebra has been studied in [L2], [L4] (at least when $K = \mathbf{Q}$). It is similar but not the same to an algebra studied in [MS].

For any $w \in W$ we define a linear map $\mathcal{T}_w : \mathbf{M} \to \mathbf{M}$ by $\mathcal{T}_w = \mathcal{T}_{s_1}\mathcal{T}_{s_2}\dots\mathcal{T}_{s_k}$, where $s_1, s_2, \dots, s_k$ are elements of S such that $w = s_1s_2\dots s_k$, $|w| = k$. By Theorem 0.6, this is independent of the choice of $s_1, \dots, s_k$. For $\lambda \in \bar{X}$ we define a linear map $1_\lambda : \mathbf{M} \to \mathbf{M}$ by $1_\lambda(\mathbf{a}_{w,\lambda'}) = \delta_{\lambda,\lambda'}\mathbf{a}_{w,\lambda'}$ for any $(w,\lambda') \in \tilde{X}_1$. For $(w,\lambda) \in W\bar{X}$ we define a linear map $\mathcal{T}_{w,\lambda} : \mathbf{M} \to \mathbf{M}$ as the composition $\mathcal{T}_w 1_\lambda$. These maps define an $\mathbf{H}$-module structure on $\mathbf{M}$. (This follows from Theorem 0.6; the relation (b) on $\mathbf{M}$ can be deduced from the analogous relation in M_q.) From (b) we deduce that $\mathcal{T}_s^{-1} : \mathbf{M} \to \mathbf{M}$ is well defined and we have

(c) $$\mathcal{T}_s^{-1} = \mathcal{T}_s - (v^2 - v^2)^{-1} \sum_{\lambda \in \bar{X}; s \in W_\lambda} 1_\lambda.$$

(The last sum may be infinite but at most one term in the sum applied to a given basis element of $\mathbf{M}$ can be non-zero.) It follows that for any $w \in W$, $\mathcal{T}_w : \mathbf{M} \to \mathbf{M}$ is invertible. Its inverse satisfies $\mathcal{T}_{w_1w_2}^{-1} = \mathcal{T}_{w_2}^{-1}\mathcal{T}_{w_1}^{-1} : \mathbf{M} \to \mathbf{M}$ for any w_1, w_2 in W such that $|w_1w_2| = |w_1| + |w_2|$.

For any W-orbit $\mathcal{O}$ in $\bar{X}$ we denote by $\mathbf{H}_{\mathcal{O}}$ the subspace of $\mathbf{H}$ spanned by

$$\{\mathcal{T}_{w,\lambda}; (w,\lambda) \in W \times \mathcal{O}\}.$$

This is a subalgebra of $\mathbf{H}$, this time with unit, namely $\sum_{\lambda \in \mathcal{O}} \mathcal{T}_{1,\lambda}$.

For any $w \in W$ we set $\mathcal{T}_w = \sum_{\lambda \in \mathcal{O}} \mathcal{T}_{w,\lambda} \in \mathbf{H}_{\mathcal{O}}$; for any $\lambda \in \mathcal{O}$ we set $1_\lambda = \mathcal{T}_{1,\lambda} \in \mathbf{H}_{\mathcal{O}}$. We see that the elements $\mathcal{T}_w, 1_\lambda$ exist separately in $\mathbf{H}_{\mathcal{O}}$, not only in the combination $\mathcal{T}_{w,\lambda} = \mathcal{T}_w 1_\lambda$.

We denote by $\mathbf{M}_{\mathcal{O}}$ the subspace of $\mathbf{M}$ spanned by $\{\mathbf{a}_{w,\lambda}; (w,\lambda) \in \tilde{X}_1, \lambda \in \mathcal{O}\}$. Note that the $\mathbf{H}$-module structure on $\mathbf{M}$ restricts to an $\mathbf{H}_{\mathcal{O}}$-module structure on $\mathbf{M}_{\mathcal{O}}$.

5. On the structure of the H-module M

5.1. In this section we assume that $\mathbf{k} = \mathbf{C}$. For $\lambda \in \bar{X}$ let $\check{R}_\lambda = \{\check{\alpha} \in \check{R}; \lfloor \check{\alpha}, \lambda \rfloor = 0\}$, $\check{R}_\lambda^+ = \check{R}_\lambda \cap \check{R}^+$. Then $\check{R}_\lambda$ is the set of coroots of a root system and $\check{R}_\lambda^+$ is a set of positive coroots for it. Let $\check{R}_\lambda^- = \check{R}_\lambda - \check{R}_\lambda^+$. Let $\check{\Pi}_\lambda$ be the set of simple coroots for $\check{R}_\lambda$ contained in $\check{R}_\lambda^+$. For each $\beta \in \check{R}$ let $s_\beta : Y \to Y$ be the reflection in W such that $s_\beta(\beta) = -\beta$. Let W_λ be the subgroup of W generated by $\{s_\beta; \beta \in \check{R}_\lambda\}$. This is a Coxeter group with generators $\{s_\beta; \beta \in \check{\Pi}_\lambda\}$ and with length function $w \mapsto |w|_\lambda = \sharp(\beta \in \check{R}_\lambda^+; w(\beta) \in \check{R}_\lambda^-)$. Note that for $s \in S$ the condition that $s \in W_\lambda$ coincides with the condition denoted in the same way in §0.1; this follows from [L4, 1.2(c)].

If $w \in W$, then there is a unique element $z \in wW_\lambda$ such that $z(\check{R}_\lambda^+) \subset \check{R}^+$; we have $|z| < |zu|$ for any $u \in W_\lambda - \{1\}$; we write $z = \min(wW_\lambda)$. (See [L4, 1.2(e)].)

We now fix an integer $m \geq 1$. We fix a W-orbit $\mathcal{O}$ in $\bar{X}_m$. For any λ, λ' in $\mathcal{O}$ we set

$$[\lambda', \lambda] = \{z \in W; \lambda' = z(\lambda), z = \min(zW_\lambda)\} = \{z \in W; \lambda' = z(\lambda), z(\check{R}_\lambda^+) = \check{R}_{\lambda'}^+\}.$$

Clearly,

(a) $[\lambda, \lambda'] = [\lambda', \lambda]^{-1}$; moreover, if $\lambda, \lambda', \lambda''$ are in $\mathcal{O}$, then $[\lambda'', \lambda'][\lambda', \lambda] \subset [\lambda'', \lambda]$.
Hence the group structure on W makes
(b) $\Xi := \{(\lambda', z, \lambda) \in \mathcal{O} \times W \times \mathcal{O}; z \in [\lambda', \lambda]\}$
into a groupoid; see [L4, 1.2(f)].

5.2. If $\lambda \in \bar{X}$, then $\check{R}_\lambda \subset \check{R}_{-m\lambda}$. If $(w, \lambda) \in \tilde{X}_m$, then $\sharp(\check{R}_\lambda) = \sharp(\check{R}_{-m\lambda})$ so that $\check{R}_\lambda = \check{R}_{-m\lambda}$ and $W_\lambda = W_{-m\lambda}$. We show:

(a) *If $\lambda \in \bar{X}_m$ and $z \in [-m\lambda, m]$, then $z(\check{R}_\lambda^+) = \check{R}_\lambda^+$ so that $\iota_z : u \mapsto zuz^{-1}$ is a Coxeter group automorphism of W_λ.*

We have $z(\check{R}_\lambda) = \check{R}_{z\lambda} = \check{R}_{-m\lambda} = \check{R}_\lambda$; moreover since $z(\check{R}_\lambda^+) \subset \check{R}^+$ we have $z(\check{R}_\lambda^+) = \check{R}_\lambda^+$. This proves (a).

Let $\tilde{X}_m^0 = \{(z, \lambda) \in W_2 \times \bar{X}; z \in [-m\lambda, \lambda]\}$. Note that $\tilde{X}_m^0 \subset \tilde{X}_m$. For $(z, \lambda) \in \tilde{X}_m^0$ let $I_{(z,\lambda)} = \{u \in W_\lambda; \iota_z(u)u = 1\}$ be the set of ι_z-twisted involutions of W_λ. If $u \in I_{z,\lambda}$, then $(zu, \lambda) \in \tilde{X}_m$; indeed we have $(zu)^2 = 1$ and $zu(\lambda) = z(\lambda) = -m\lambda$. Conversely,

(b) if $(w, \lambda) \in \tilde{X}_m$ we have $(w, \lambda) = (zu, \lambda)$ for a well defined $(z, \lambda) \in \tilde{X}_m^0$ and $u \in I_{z,\lambda}$.

Indeed, let $z = \min(wW_\lambda)$. Since $w(l) = -m\lambda$ we have also $z(\lambda) = -m\lambda$ and hence $z \in [-m\lambda, \lambda]$. We have $w = zu$ where $u \in W_\lambda$. We have $w = w^{-1} = u^{-1}z^{-1} = z^{-1}zuz^{-1} = z^{-1}\iota_z(u)$. Since $\iota_z(u) \in W_\lambda$ (see (a)) we have $w \in z^{-1}W_\lambda$. Since $z(\check{R}_\lambda^+) = \check{R}_\lambda^+$ we must have also $z^{-1}(\check{R}_\lambda^+) = \check{R}_\lambda^+$ so that $z^{-1} = \min(wW_\lambda)$. It follows that $z = z^{-1}$ so that $(z, \lambda) \in \tilde{X}_m^0$. Since $1 = w^2 = (zu)^2$ we see that $\iota_z(u)u = 1$ so that $u \in I_{z,\lambda}$. This proves (b).

We see that

(c) we have a bijection $\bigsqcup_{(z,\lambda) \in \tilde{X}_m^0} I_{z,\lambda} \xrightarrow{\sim} \tilde{X}_m$ given by $(z, \lambda, u) \mapsto (zu, \lambda)$ where $(z, \lambda) \in \tilde{X}_m^0, u \in I_{z,\lambda}$.

5.3. Let Ξ be as in §5.1(b). Let $\Xi^0 = \{(z,\lambda) \in \tilde{X}^0_m; \lambda \in \mathcal{O}\}$.

We can view Ξ^0_m as a subset of Ξ by $(z,\lambda) \mapsto (-m\lambda, z, \lambda)$. This subset is the fixed point set of the antiautomorphism

$$(\lambda', z, \lambda) \mapsto (\lambda', z, \lambda)^* := (-m\lambda, z^{-1}, -m\lambda')$$

of the groupoid Ξ (the composition of the inversion $(\lambda', z, \lambda) \mapsto (\lambda, z^{-1}, \lambda')$ with the involutive automorphism $(\lambda', z, \lambda) \mapsto (-m\lambda', z, -m\lambda)$ of the groupoid Ξ). Hence this subset can be viewed as the set of $*$-twisted "involutions" of this groupoid.

Until the end of §5.8 we assume that $m = 1$. From Theorem 0.6 we deduce

(a) If $(w,\lambda) \in \tilde{X}_1$, $s \in S$, and $s \notin W_\lambda$, then $\mathcal{T}_s(\mathbf{a}_{w,\lambda}) = \pm \mathbf{a}_{sws, s\lambda}$.

Note also that in $\mathbf{H}_{\mathcal{O}}$, for $s \in S, w \in W, \lambda \in \mathcal{O}$ we have

(b) $\mathcal{T}_s\mathcal{T}_w 1_\lambda = \mathcal{T}_{sw}1_\lambda$ if $s \notin W_{w(\lambda)}$; $\mathcal{T}_w\mathcal{T}_s 1_\lambda = \mathcal{T}_{ws}1_\lambda$ if $s \notin W_\lambda$.

Lemma 5.4. *Let $\lambda \in \mathcal{O}$. Let $(w,\lambda) \in \tilde{X}_1$, $z \in [\lambda,\lambda]$. Then $(zwz^{-1},\lambda) \in \tilde{X}_1$ and $\mathcal{T}_z\mathbf{a}_{w,\lambda} = \pm\mathbf{a}_{zwz^{-1},\lambda}$.*

The proof is similar to that of [L4, 1.4(c)]. We have $w(\lambda) = -\lambda$ and hence $zwz^{-1}(\lambda) = -\lambda$ since $z(\lambda) = \lambda$. Thus $(zwz^{-1},\lambda) \in \tilde{X}_1$.

We write $z = s_k s_{k-1} \dots s_1$ where $s_1, \dots, s_k$ are in S, $|z| = k$. As in the proof of [L4, 1.4(c)] we have $s_1 \notin W_\lambda$, $s_1s_2s_1 \notin W_\lambda$, $\dots$ $s_1s_2\dots s_k \dots s_2s_1 \notin W_\lambda$. We have $\mathcal{T}_{s_1}\mathbf{a}_{w,\lambda} = \pm\mathbf{a}_{s_1ws_1,s_1\lambda}$ since $s_1 \notin W_\lambda$; see §5.3(a). We have $\mathcal{T}_{s_2}\mathbf{a}_{s_1ws_1,s_1\lambda} = \pm\mathbf{a}_{s_2s_1ws_1s_2,s_2s_1\lambda}$ since $s_2 \notin W_{s_1\lambda}$; see §5.3(a). Continuing in this way we get

$$\mathcal{T}_{s_k}\mathbf{a}_{s_{k-1}\dots s_1ws_1\dots s_{k-1},s_{k-1}\dots s_1\lambda} = \pm\mathbf{a}_{s_k\dots s_1ws_1\dots s_k,s_k\dots s_1\lambda}.$$

Combining these equalities we get

$$\mathcal{T}_z\mathbf{a}_{w,\lambda} = \mathcal{T}_{s_k}\dots\mathcal{T}_{s_1}\mathbf{a}_{w,\lambda} = \pm\mathbf{a}_{s_k\dots s_1ws_1\dots s_k,s_k\dots s_1\lambda} = \pm\mathbf{a}_{zwz^{-1},z\lambda} = \pm\mathbf{a}_{zwz^{-1},\lambda}.$$

The lemma is proved.

The following result is a generalization of the lemma above.

Lemma 5.5. *Let $(w,\lambda) \in \tilde{X}_1$, $z \in [\lambda',\lambda]$ where λ,λ' are in $\mathcal{O}$. Then $(zwz^{-1},\lambda') \in \tilde{X}_1$ and $\mathcal{T}_z\mathbf{a}_{w,\lambda} = \pm\mathbf{a}_{zwz^{-1},\lambda'}$.*

The proof is similar to that of [L4, 1.4(d)]. We have $w(\lambda) = -\lambda$ and hence $zwz^{-1}(\lambda') = -\lambda'$ since $z^{-1}(\lambda') = \lambda'$. Thus $(zwz^{-1},\lambda') \in \tilde{X}_1$.

Since λ,λ' are in the same W-orbit, we can find $r \geq 0$ and $s_1, s_2, \dots, s_r$ in S such that, setting

$$\lambda_0 = \lambda, \lambda_1 = s_1\lambda, \lambda_2 = s_2s_1\lambda, \dots, \lambda_r = s_r\dots s_2s_1\lambda,$$

we have $\lambda_0 \neq \lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_r = \lambda'$. For $j = 1,\dots,r$, we have $s_j \notin W_{\lambda_{j-1}}$ since $s_j(\lambda_{j-1}) = \lambda_j \neq \lambda_{j-1}$ and hence s_j has minimal length in $s_jW_{\lambda_{j-1}}$ and $s_j \in [\lambda_j, \lambda_{j-1}]$. It follows that $s_r\dots s_2s_1 \in [\lambda_r,\lambda_0] = [\lambda',\lambda]$ (we use §5.1(a)). We define $\tilde{z} \in W$ by $z = s_r\dots s_2s_1\tilde{z}$. Then $\tilde{z} \in [\lambda,\lambda]$ (we use again §5.1(a)). For $j \in [1,r]$ we have $s_j \notin W_{s_{j-1}\dots s_1\lambda}$ (since $\lambda_j \neq \lambda_{j-1}$) and hence, using §5.3(a) we have

$$\mathcal{T}_{s_j}\mathbf{a}_{s_{j-1}\dots s_1\tilde{z}w\tilde{z}^{-1}s_1\dots s_{j-1},s_{j-1}\dots s_1\lambda} = \pm\mathbf{a}_{s_js_{j-1}\dots s_1\tilde{z}w\tilde{z}^{-1}s_1\dots s_{j-1}s_j,s_js_{j-1}\dots s_1\lambda}.$$

Applying this repeatedly we deduce

$$\mathcal{T}_{s_r}\dots\mathcal{T}_{s_2}\mathcal{T}_{s_1}\mathbf{a}_{\tilde{z}w\tilde{z}^{-1},\tilde{z}\lambda} = \pm\mathbf{a}_{s_r\dots s_2s_1\tilde{z}w\tilde{z}^{-1}s_1s_2\dots s_r,s_r\dots s_2s_1\tilde{z}\lambda} = \pm\mathbf{a}_{zwz^{-1},z\lambda}.$$

We now apply Lemma 5.4 with z replaced by $\tilde z$; we see that $\mathcal{T}_{\tilde z}\mathbf{a}_{w,\lambda}=\pm\mathbf{a}_{\tilde z w\tilde z^{-1},\lambda}$. Substituting this in the previous equation we obtain

(a) $$\mathcal{T}_{s_r}\dots\mathcal{T}_{s_2}\mathcal{T}_{s_1}\mathcal{T}_{\tilde z}\mathbf{a}_{w,\lambda}=\pm\mathbf{a}_{zwz^{-1},z\lambda}.$$

For $j\in[1,r]$ we have $s_j\notin W_{s_{j-1}\dots s_1\lambda}$ (as above) and hence, using §5.3(b) we have

$$\mathcal{T}_{s_j}\mathcal{T}_{s_{j-1}\dots s_1\tilde z}\mathbf{a}_{w,\lambda}=\mathcal{T}_{s_js_{j-1}\dots s_1\tilde z}\mathbf{a}_{w,\lambda}.$$

Applying this repeatedly we deduce

$$\mathcal{T}_{s_r}\dots\mathcal{T}_{s_2}\mathcal{T}_{s_1}\mathcal{T}_{\tilde z}\mathbf{a}_{w,\lambda}=\mathcal{T}_{s_r\dots s_2s_1\tilde z}\mathbf{a}_{w,\lambda}=\mathcal{T}_{z}\mathbf{a}_{w,\lambda}.$$

Combining this with (a) gives

$$\mathcal{T}_{z}\mathbf{a}_{w,\lambda}=\pm\mathbf{a}_{zwz^{-1},z\lambda}.$$

The lemma is proved.

Lemma 5.6. *Let $(z,\lambda)\in\Xi^0$ and let $u\in W_\lambda$. Let $\alpha\in\check\Pi_\lambda$. We set $\sigma=\sigma_\alpha$; note that $|\sigma|_\lambda=1$. Recall that $u\mapsto\iota_z(u)=zuz^{-1}$ is an involutive Coxeter group automorphism of W_λ. For any $u\in W_\lambda$ we have*

(a) $\mathcal{T}_\sigma\mathbf{a}_{zu,\lambda}=e_1\mathbf{a}_{z\iota_z(\sigma)u\sigma,\lambda}$ *if $u\sigma\ne\iota_z(\sigma)u,|u\sigma|_\lambda>|u|_\lambda$;*

(b) $\mathcal{T}_\sigma\mathbf{a}_{zu,\lambda}=e_2\mathbf{a}_{z\iota_z(s)u\sigma,\lambda}+e_3(v^2-v^{-2})\mathbf{a}_{zu,\lambda}$ *if $u\sigma\ne\iota_z(\sigma)u,|u\sigma|_\lambda<|u|_\lambda$;*

(c) $\mathcal{T}_\sigma\mathbf{a}_{zu,\lambda}=e_4\mathbf{a}_{zu,\lambda}+e_5(v+v^{-1})\mathbf{a}_{zu\sigma,\lambda}$ *if $u\sigma=\iota_z(\sigma)u$, $|u\sigma|_\lambda>|u|_\lambda$;*

(d) $\mathcal{T}_\sigma\mathbf{a}_{zu,\lambda}=e_6(v-v^{-1})\mathbf{a}_{zu\sigma,\lambda}+e_7(v^2-v^{-2}-1)\mathbf{a}_{zu,\lambda}$ *if $u\sigma=\iota_z(\sigma)u$, $|u\sigma|_\lambda<|u|_\lambda$,*

where $e_1,\dots,e_7\in\{1,-1\}$.

As in the proof of [L4, 1.4(f)] we can find $s_1,s_2,\dots,s_r$ in S such that $\sigma=s_1s_2\dots s_{r-1}s_rs_{r-1}\dots s_1$, $|\sigma_\alpha|=2r-1$, $s_1s_2\dots s_{j-1}s_js_{j-1}\dots s_1\notin W_\lambda$ for $j=1,2,\dots r-1$. We argue by induction on $r\ge1$. When $r=1$ the result follows from Theorem 0.6. (Note that $z\iota_z(\sigma)u\sigma=\sigma zu\sigma$, the condition $u\sigma=\iota_z(\sigma)u$ is equivalent to $zu\sigma=\sigma zu$ and if $|\sigma|=1$ the condition $|u\sigma|_\lambda>|u|_\lambda$ is equivalent to $|u\sigma|>|u|$.) Assume now that $r\ge2$. We set $s=s_1$, $\lambda'=s\lambda$, $\beta=s(\alpha)\in R^+_{\lambda'}$, $u'=sus$, $z'=szs$, $\sigma'=s_\beta=s\sigma s$. We have $(z',\lambda')\in\Xi^0_{\mathcal{O}}$, $u'\in W_{\lambda'}$ and $\sigma'\in W_{\lambda'}$, $|\sigma'|_{\lambda'}=1$, $|\sigma'|=|\sigma|-2$. Moreover, we have $s\notin W_\lambda$. By the induction hypothesis we have

(a′) $\mathcal{T}_{\sigma'}\mathbf{a}_{z'u',\lambda'}=e'_1\mathbf{a}_{\sigma'z'u'\sigma',\lambda'}$ if $u'\sigma'\ne z'\sigma'z'u',|u'\sigma'|_{\lambda'}>|u'|_{\lambda'}$;

(b′) $\mathcal{T}_{\sigma'}\mathbf{a}_{z'u',\lambda'}=e'_2\mathbf{a}_{\sigma'z'u'\sigma',\lambda'}+e'_3(v^2-v^{-2})\mathbf{a}_{z'u',\lambda'}$ if $u'\sigma'\ne z'\sigma'z'u',|u'\sigma'|_{\lambda'}<|u'|_{\lambda'}$;

(c′) $\mathcal{T}_{\sigma'}\mathbf{a}_{z'u',\lambda'}=e'_4\mathbf{a}_{z'u',\lambda'}+e'_5(v+v^{-1})\mathbf{a}_{z'u'\sigma',\lambda'}$ if $u'\sigma'=z'\sigma'z'u',|u'\sigma'|_{\lambda'}>|u'|_{\lambda'}$;

(d′) $\mathcal{T}_{\sigma'}\mathbf{a}_{z'u',\lambda'}=e'_6(v-v^{-1})\mathbf{a}_{z'u'\sigma',\lambda'}+e'_7(v^2-v^{-2}-1)\mathbf{a}_{z'u',\lambda'}$ if $u'\sigma'=z'\sigma'z'u'$, $|u'\sigma'|_{\lambda'}<|u'|_{\lambda'}$,

where $e'_1,\dots,e'_7\in\{1,-1\}$. By §5.3(a), §5.3(b) we have

$$\mathcal{T}_s\mathcal{T}_{\sigma'}\mathbf{a}_{z'u',\lambda'}=\mathcal{T}_\sigma\mathcal{T}_s\mathbf{a}_{z'u',\lambda'}=\pm\mathcal{T}_\sigma\mathbf{a}_{zu,\lambda}.$$

Moreover, by §5.3(a) we have

$$\mathcal{T}_s\mathbf{a}_{z'u',\lambda'}=\mathbf{a}_{zu,\lambda},\mathcal{T}_s\mathbf{a}_{\sigma'z'u'\sigma',\lambda'}=\mathbf{a}_{\sigma zu\sigma,\lambda},$$

$\mathcal{T}_s\mathbf{a}_{z'u'\sigma',\lambda'}=\mathbf{a}_{zu\sigma,\lambda}$. Hence (a)–(d) for σ,z,u follow from (a′)–(d′) by applying $\mathcal{T}_s$ to both sides. Here we use that the condition that $z'u'\sigma'=\sigma'z'u'$ is equivalent to the condition $zu\sigma=\sigma zu$ and the inequality $|u'\sigma'|_{\lambda'}>|u'|_{\lambda'}$ is equivalent to the inequality $|u\sigma|_\lambda>|u|_\lambda$ (conjugation by s is a Coxeter group isomorphism $W_{\lambda'}\to W_\lambda$). The lemma is proved.

5.7. For any $\lambda \in \mathcal{O}$ let $\mathbf{H}_\lambda$ be the $\mathbf{C}(v)$-subspace of $\mathbf{H}_\mathcal{O}$ spanned by $\{\mathcal{T}_u 1_\lambda; u \in W_\lambda\}$. This is a subalgebra of $\mathbf{H}_\mathcal{O}$ with unit 1_λ; it can be identified with the Hecke algebra of the Coxeter group W_λ (see [L4, 1.4(g), (h)]) so that the standard generators of the last algebra correspond to the elements $\mathcal{T}_{s_\alpha} 1_\lambda$ of $\mathbf{H}_\lambda$ with $\alpha \in \check{\Pi}_\lambda$.

For $(z, \lambda) \in \Xi^0$ let $\mathbf{M}_{z,\lambda}$ be the subspace of $\mathbf{M}$ spanned by $\{\mathbf{a}_{zu,\lambda}; u \in I_{z,\lambda}\}$. From Lemma 5.6 we see that $\mathbf{M}_{z,\lambda}$ is an $\mathbf{H}_\lambda$-module and that the action of the generators of $\mathbf{H}_\lambda$ on $\mathbf{M}_{z,\lambda}$ is given by a formula which is the same (except for the appearance of certain signs e_j) as the formula for the action of the generators of the Hecke algebra of W_λ on the module based on the twisted involutions in W_λ constructed in [LV].

5.8. We have a direct sum decomposition $\mathbf{H}_\mathcal{O} = \bigoplus_{(\lambda', z, \lambda) \in \Xi} \mathcal{T}_z \mathbf{H}_\lambda$; moreover, $\{\mathcal{T}_z \mathcal{T}_u 1_\lambda; (\lambda', z, \lambda) \in \Xi, u \in W_\lambda\}$ is a basis of $\mathbf{H}_\mathcal{O}$ compatible with this decomposition and it coincides with the basis $\{\mathcal{T}_w 1_\lambda; (w, \lambda) \in \tilde{X}_1, \lambda \in \mathcal{O}\}$ of $\mathbf{H}_\mathcal{O}$. (See [L4, 1.4(d)].) Similarly, by §5.2(b), we have a direct sum decomposition $\mathbf{M}_\mathcal{O} = \bigoplus_{(\tilde{z}, \tilde{\lambda}) \in \Xi^0} \mathbf{M}_{\tilde{z}, \tilde{\lambda}}$ where $\mathbf{M}_{\tilde{z}, \tilde{\lambda}}$ is as in §5.7. From Lemmas 5.5 and 5.6 we see that the direct sum decompositions of $\mathbf{H}_\mathcal{O}$ and $\mathbf{M}_\mathcal{O}$ are compatible in the following sense:

$$(\mathcal{T}_z \mathbf{H}_\lambda) \mathbf{M}_{\tilde{z}, \tilde{\lambda}} \subset \delta_{\tilde{\lambda}, \lambda} \mathbf{M}_{z \tilde{z} z^{-1}, z(\tilde{\lambda})}.$$

Moreover the action of the basis element $\mathcal{T}_z \mathcal{T}_u 1_\lambda = (\mathcal{T}_z 1_\lambda)(\mathcal{T}_u 1_\lambda)$ of $\mathbf{H}_\mathcal{O}$ on a basis element $\mathbf{a}_{\tilde{z}u', \tilde{\lambda}}$ of $\mathbf{M}_\mathcal{O}$ is particularly simple: the operator $\mathcal{T}_z 1_\lambda$ applied to a basis element $\mathbf{a}_{\tilde{z}u', \tilde{\lambda}}$ is $\pm \delta_{\tilde{\lambda}, \lambda}$ times another basis element; the operator $\mathcal{T}_u 1_\lambda$ applied to a basis element $\mathbf{a}_{\tilde{z}u', \tilde{\lambda}}$ is as in §5.7 if $\tilde{\lambda} = \lambda$ and is zero if $\tilde{\lambda} \neq \lambda$.

5.9. Results similar to those in Lemmas 5.4–5.6 and §§5.7, 5.8 hold for $M_\mathcal{O}$ when $m = q$ with (p, q) as in §0.2 and $\mathcal{O} \subset \bar{X}_m$ except that in this case the $\pm$ signs in Lemmas 5.4–5.6 and §§5.7, 5.8 have to be replaced by roots of 1 of possibly higher order.

6. Proof of Theorem 0.9

6.1. We now fix an integer $m \geq 1$. Recall from §0.8 that $\mathbf{M}_m$ is the $\mathbf{C}(v)$-vector space with basis $\{\mathbf{a}_{w,\lambda}; (w, \lambda) \in \tilde{X}_m\}$. We fix a W-orbit $\mathcal{O}$ in $\bar{X}_m$. Let $\mathbf{M}_\mathcal{O}$ be the subspace of $\mathbf{M}_m$ spanned by $\{\mathbf{a}_{w,\lambda}; (w, \lambda) \in \tilde{X}_m, \lambda \in \mathcal{O}\}$.

For any $\lambda \in \mathcal{O}$ let $\mathbf{H}_\lambda$ be as in §5.7. For $(z, \lambda) \in \Xi^0$ let $\mathbf{M}_{z,\lambda}$ be the subspace of $\mathbf{M}_\mathcal{O}$ spanned by $\{\mathbf{a}_{zu,\lambda}; u \in I_{z,\lambda}\}$. By [LV] applied to the Coxeter group W_λ with the involutive automorphism ι_z, there is a well defined $\mathbf{H}_\lambda$-module structure $(h, \xi) \mapsto h \circ \xi$ on $\mathbf{M}_{z,\lambda}$ such that for any $u \in W_\lambda$ and any $\sigma = s_\alpha$, $\alpha \in \check{\Pi}_\lambda$ we have

(a) $(\mathcal{T}_\sigma 1_\lambda) \circ \mathbf{a}_{zu,\lambda} = \mathbf{a}_{z\iota_z(\sigma)u\sigma,\lambda}$ if $u\sigma \neq \iota_z(\sigma)u, |u\sigma|_\lambda > |u|_\lambda$;

(b) $(\mathcal{T}_\sigma 1_\lambda) \circ \mathbf{a}_{zu,\lambda} = \mathbf{a}_{z\iota_z(s)u\sigma,\lambda} + (v^2 - v^{-2})\mathbf{a}_{zu,\lambda}$ if $u\sigma \neq \iota_z(\sigma)u, |u\sigma|_\lambda < |u|_\lambda$;

(c) $(\mathcal{T}_\sigma 1_\lambda) \circ \mathbf{a}_{zu,\lambda} = \mathbf{a}_{zu,\lambda} + (v + v^{-1})\mathbf{a}_{zu\sigma,\lambda}$ if $u\sigma = \iota_z(\sigma)u$, $|u\sigma|_\lambda > |u|_\lambda$;

(d) $(\mathcal{T}_\sigma 1_\lambda) \circ \mathbf{a}_{zu,\lambda} = (v - v^{-1})\mathbf{a}_{zu\sigma,\lambda} + (v^2 - v^{-2} - 1)\mathbf{a}_{zu,\lambda}$ if $u\sigma = \iota_z(\sigma)u$, $|u\sigma|_\lambda < |u|_\lambda$.

6.2. By [L4, 1.4(d)], the basis $\{\mathcal{T}_w 1_\lambda; (w, \lambda) \in \tilde{X}_1, \lambda \in \mathcal{O}\}$ of $\mathbf{H}_\mathcal{O}$ coincides with $\{\mathcal{T}_u \mathcal{T}_z 1_\lambda; (\lambda', z, \lambda) \in \Xi, u \in W_{\lambda'}\}$. We define a bilinear multiplication $\mathbf{H}_\mathcal{O} \times \mathbf{M}_\mathcal{O} \to \mathbf{M}_\mathcal{O}$ (denoted by $(h, \xi) \mapsto h \bullet \xi$) by the rule

$$(\mathcal{T}_u \mathcal{T}_z 1_\lambda) \bullet \mathbf{a}_{\tilde{z}\tilde{u}, \tilde{\lambda}} = 0$$

if $\lambda \neq \tilde\lambda$, while if $\lambda = \tilde\lambda$,

$$(\mathcal{T}_u\mathcal{T}_z 1_\lambda) \bullet \mathbf{a}_{\tilde z\tilde u,\tilde\lambda} = (\mathcal{T}_u 1_{\lambda'}) \circ \mathbf{a}_{(z\tilde z z^{-1})(z\tilde u z^{-1}),\lambda'}$$

for $(\lambda', z, \lambda) \in \Xi, u \in W_{\lambda'}, (\tilde z, \tilde\lambda) \in \Xi^0, \tilde u \in W_{\tilde\lambda}$, where $\circ$ is as in §6.1 with λ replaced by λ'. (We have $(z\tilde z z^{-1}, \lambda') \in \Xi^0$ and $z\tilde u z^{-1} \in W_{\lambda'}$.) We show:

(a) *this is an* $\mathbf{H}_{\mathcal{O}}$*-module structure.*

It is enough to show that for

$$(\lambda', z, \lambda) \in \Xi, (\lambda'_1, z_1, \lambda_1) \in \Xi, u \in W_{\lambda'}, u_1 \in W_{\lambda'_1}, (\tilde z, \tilde\lambda) \in \Xi^0, \tilde u \in W_{\tilde\lambda},$$

with $\lambda' = \lambda_1, \lambda = \tilde\lambda$ we have

$$(\mathcal{T}_{u_1}\mathcal{T}_{z_1} 1_{\lambda_1}) \bullet ((\mathcal{T}_u\mathcal{T}_z 1_\lambda) \bullet \mathbf{a}_{\tilde z\tilde u,\tilde\lambda}) = (\mathcal{T}_{u_1}\mathcal{T}_{z_1 u z_1^{-1}}\mathcal{T}_{z_1 z} 1_\lambda) \bullet \mathbf{a}_{\tilde z\tilde u,\tilde\lambda}$$

or that

$$\begin{aligned}&(\mathcal{T}_{u_1} 1_{\lambda'_1}) \circ ((\mathcal{T}_{z_1 u z_1^{-1}}\mathcal{T}_{z_1 z z_1^{-1}} 1_{z_1\lambda}) \bullet \mathbf{a}_{(z_1\tilde z z_1^{-1})(z_1\tilde u z_1^{-1}),z_1\lambda})\\ &= \sum_{u_2\in W_{\lambda'_1}} (\mathcal{T}_{u_2}\mathcal{T}_{z_1 z} 1_\lambda) \bullet \mathbf{a}_{\tilde z\tilde u,\tilde\lambda},\end{aligned}$$

where we have written $\mathcal{T}_{u_1}\mathcal{T}_{z_1 u z_1^{-1}} 1_{\lambda'_1} = \sum_{u_2\in W_{\lambda'_1}} \gamma_{u_2}\mathcal{T}_{u_2} 1_{\lambda'_1}$, $\gamma_{u_2} \in \mathbf{C}(v)$. (We have used [L4, 1.4(d), (e)].) We have

$$\begin{aligned}&(\mathcal{T}_{z_1 u z_1^{-1}}\mathcal{T}_{z_1 z z_1^{-1}} 1_{z_1\lambda}) \bullet \mathbf{a}_{z_1\tilde z z_1^{-1})(z_1\tilde u z_1^{-1}),z_1\lambda}\\ &= (\mathcal{T}_{z_1 u z_1^{-1}} 1_{\lambda'_1}) \circ \mathbf{a}_{(z_1 z\tilde z z^{-1} z_1^{-1})(z_1 z\tilde u z^{-1} z_1^{-1}),\lambda'_1}.\end{aligned}$$

We have

$$\begin{aligned}&\sum_{u_2\in W_{\lambda'_1}} (\mathcal{T}_{u_2}\mathcal{T}_{z_1 z} 1_\lambda) \bullet \mathbf{a}_{\tilde z\tilde u,\tilde\lambda}\\ &= \sum_{u_2\in W_{\lambda'_1}} (\mathcal{T}_{u_2} 1_{\lambda'_1}) \circ \mathbf{a}_{(z_1 z\tilde z z^{-1} z_1^{-1})(z_1 z\tilde u z^{-1} z_1^{-1}),\lambda'_1}.\end{aligned}$$

Thus it is enough to prove

$$\begin{aligned}&(\mathcal{T}_{u_1} 1_{\lambda'_1}) \circ ((\mathcal{T}_{z_1 u z_1^{-1}} 1_{\lambda'_1}) \circ \mathbf{a}_{(z_1 z\tilde z z^{-1} z_1^{-1})(z_1 z\tilde u z^{-1} z_1^{-1}),\lambda'_1})\\ &= \sum_{u_2\in W_{\lambda'_1}} (\mathcal{T}_{u_2} 1_{\lambda'_1}) \circ \mathbf{a}_{(z_1 z\tilde z z^{-1} z_1^{-1})(z_1 z\tilde u z^{-1} z_1^{-1}),\lambda'_1}.\end{aligned}$$

This follows from the fact that $\circ$ defines a module structure. This proves (a).

6.3. Let $\mathbf{H}_m$ be the $\mathbf{C}(v)$-vector space with basis $\{\mathcal{T}_{w,\lambda}; (w, \lambda) \in W \times \bar X_m\}$. Note that $\mathbf{H}_m$ is a subalgebra of $\mathbf{H}$. There is a unique $\mathbf{H}_m$-module structure $(h, \xi) \mapsto h\bullet\xi$ on $\mathbf{M}_m$ (see §0.8) such that for any two orbits $\mathcal{O}, \mathcal{O}'$ in $\bar X_m$ and any $h \in \mathbf{H}_{\mathcal{O}}, \xi \in \mathbf{M}_{\mathcal{O}'}$ we have $h \bullet \xi = 0$ if $\mathcal{O} \neq \mathcal{O}'$ and $h \bullet \xi$ is as in §6.2 if $\mathcal{O} = \mathcal{O}'$.

6.4. We now prove Theorem 0.9. It is enough to show that Theorem 0.9(a)–(b) hold when $\mathcal{T}_s$ is replaced by $\mathcal{T}_s 1_\lambda \in \mathbf{H}_m$ acting on $\mathbf{M}_m$ via the $\mathbf{H}_m$-module structure on $\mathbf{M}_m$. We can write $w = zu$ where $(z, \lambda) \in \Xi^0$ and $u \in W_\lambda$. If $s \in W_\lambda$, then $s = \sigma$ as in §6.1 and the desired formulas follow from §6.1. If $s \notin W_\lambda$, then s has minimal length in sW_λ and hence $s \in [s(\lambda), \lambda]$. Then by definition we have $(\mathcal{T}_s 1_\lambda) \bullet \mathbf{a}_{w,\lambda} = \mathbf{a}_{sws,s\lambda}$ and the desired formulas hold again. This proves Theorem 0.9.

6.5. In [L4], an affine analogue of $\mathbf{H}$ is considered; it has a basis indexed by the semidirect product $\tilde{W}\bar{X}$ where $\tilde{W}$ is an affine Weyl group acting on $\bar{X}$ via its quotient W. The analogue of Theorem 0.9 continues to hold in this case (with the same proof).

7. Bar operator

7.1. Let m be an integer ≥ 1. In this section we construct a bar operator on $\mathbf{M}_m$ generalizing a definition in [LV]. To do this we will use the method of [L3].

For $s \in S$ the operator $\mathcal{T}_s : \mathbf{M}_m \to \mathbf{M}_m$ in Theorem 0.9 has an inverse $\mathcal{T}_s^{-1}$. For $w \in W$ we set $\mathcal{T}_w = \mathcal{T}_{s_1} \dots \mathcal{T}_{s_k} : \mathbf{M}_m \to \mathbf{M}_m$, $\mathcal{T}_w^{-1} = \mathcal{T}_{s_k}^{-1} \dots \mathcal{T}_{s_1}^{-1} : \mathbf{M}_m \to \mathbf{M}_m$, where $w = s_1 s_2 \dots s_k$ with $s_1, \dots, s_k$ in S, $|w| = k$.

Let $c \mapsto \bar{c}$ be the field automorphism of $\mathbf{C}(v)$ which is the identity on $\mathbf{C}$ and maps v to v^{-1}. For $(w, \lambda) \in \tilde{X}_m$ we write $E(w, \lambda) = (-1)^{|u|}$ where

(a) $w = zu$, $(z, \lambda) \in \tilde{X}_m^0$, $u \in I_{z,\lambda} \subset W_\lambda$;

see §5.2(b).

We show:

(b) If $(w, \lambda) \in \tilde{X}_m, s \in S$, then $E(sws, s\lambda) = E(w, \lambda)$;

(c) if $(w, \lambda) \in \tilde{X}_m, s \in S$ are such that $sw = ws$ and $s \in W_\lambda$, then $E(ws, \lambda) = -E(w, \lambda)$.

We write $w = zu$ as in (a). Assume first that $s \in W_\lambda$. We have $sws = z\iota_z(s)us$ and $\iota_z(s) \in W_{z(\lambda)} = W_\lambda = W_{s\lambda}$ and hence $\iota_z(s)us \in I_{z,\lambda}$ and $E(sws, s\lambda) = (-1)^{|\iota_z(s)us|} = (-1)^{|u|} = E(w, \lambda)$. If $sw = ws$, we have $ws = zus$ and $us \in I_{z,\lambda}$ and hence $E(ws, \lambda) = (-1)^{|us|} = -(-1)^{|u|} = -E(w, \lambda)$. Next we assume that $s \notin W_\lambda$; then $s \in [\lambda, \lambda]$ (see §5.1) and hence $(szs, s\lambda) \in \tilde{X}_m^0$. Moreover, $sws = szus = (szs)(sus)$ and $sus \in W_{s\lambda}$ and more precisely $sus \in I_{szs,s\lambda}$. Hence $E(sws, s\lambda) = (-1)^{|sus|} = (-1)^{|u|} = E(w, \lambda)$. This proves (b) and (c).

Clearly, there is a unique $\mathbf{C}$-linear map $B : \mathbf{M}_m \to \mathbf{M}_m$ such that for any $(w, \lambda) \in \tilde{X}_m$ and any $f \in \mathbf{C}(v)$ we have

$$B(f\mathbf{a}_{w,\lambda}) = \bar{f}E(w, \lambda)\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda}.$$

We state the main result of this section.

Proposition 7.2.

(a) *For any $s \in S$ and any $\xi \in \mathbf{M}_m$ we have $B(\mathcal{T}_s\xi) = \mathcal{T}_s^{-1}B(\xi)$.*

(b) *The square of the map* $^-\colon \mathbf{M}_m \to \mathbf{M}_m$ *is equal to* 1.

To prove (a) it is enough to show that for any $(w, \lambda) \in \tilde{X}_m$ and any $s \in S$ we have

(c) $B(\mathcal{T}_s\mathbf{a}_{w,\lambda}) = E(w, \lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda}$.

We set $\Delta = 1$ if $s \in W_\lambda$ and $\Delta = 0$ if $s \notin W_\lambda$.

Assume that $sw \neq ws$, $|sw| > |w|$. We have

$$B(\mathcal{T}_s\mathbf{a}_{w,\lambda}) = B(\mathbf{a}_{sws,s\lambda}) = E(sws, s\lambda)\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda},$$
$$E(w, \lambda))\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda} = E(w, \lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathcal{T}_s^{-1}\mathcal{T}_s\mathbf{a}_{w,-m\lambda}$$
$$= E(sws, s\lambda)\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda}.$$

Hence (c) holds in this case.

Assume that $sw \neq ws$, $|sw| < |w|$. We must show that

$$B(\mathbf{a}_{sws,s\lambda} + \Delta(v^2 - v^{-2})\mathbf{a}_{w,\lambda}) = E(w, \lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$E(sws,s\lambda)\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda}+\Delta(v^{-2}-v^2)E(w,\lambda)\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda}$$
$$=E(w,\lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,s\lambda}+\Delta(v^{-2}-v^2)\mathcal{T}_s^{-1}\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}$$
$$=\mathcal{T}_s^{-1}\mathcal{T}_s^{-1}\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-\lambda},$$

or that

$$\mathcal{T}_s\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-s\lambda}+\delta(v^{-2}-v^2)\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}=\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda}+(v^2-v^{-2})\Delta\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda}$$
$$+\Delta(v^{-2}-v^2)\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}=\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}.$$

Here we substitute $\mathcal{T}_{sws}^{-1}\mathbf{a}_{sws,-ms\lambda}=\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}$. It remains to note that

$$\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}+(v^2-v^{-2})\Delta\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}$$
$$+\Delta(v^{-2}-v^2)\mathcal{T}_s\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}=\mathcal{T}_{sws}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}.$$

This proves (c) in our case.

Assume that $sw=ws$, $|sw|>|w|$. We must show that

$$B(\mathbf{a}_{w,s\lambda}+\Delta(v+v^{-1})\mathbf{a}_{sw,\lambda})=E(w,\lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$E(w,\lambda)\mathcal{T}_w^{-1}\mathbf{a}_{w,-ms\lambda}+\Delta(v+v^{-1})E(sw,\lambda)\mathcal{T}_{sw}^{-1}\mathbf{a}_{sw,-m\lambda}$$
$$=E(w,\lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathcal{T}_w^{-1}\mathbf{a}_{w,-ms\lambda}-\Delta(v+v^{-1})\mathcal{T}_w^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{sw,-m\lambda}$$
$$=\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathbf{a}_{w,-ms\lambda}-\Delta(v+v^{-1})\mathcal{T}_s^{-1}\mathbf{a}_{sw,-m\lambda}=\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathcal{T}_s\mathbf{a}_{w,-ms\lambda}-\Delta(v+v^{-1})\mathbf{a}_{sw,-m\lambda}=\mathbf{a}_{w,-m\lambda},$$

or that

$$\mathcal{T}_s\mathbf{a}_{w,-ms\lambda}=\mathbf{a}_{w,-m\lambda}+\Delta(v+v^{-1})\mathbf{a}_{sw,-m\lambda}.$$

This follows from the definitions. This proves (c) in our case.

Assume that $sw=ws$, $|sw|<|w|$. We must show that

$$B(\Delta(v-v^{-1})\mathbf{a}_{sw,\lambda}+\Delta(v^2-v^{-2}-1)\mathbf{a}_{w,\lambda}+(1-\Delta)\mathbf{a}_{w,s\lambda})$$
$$=E(w,\lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\Delta(v^{-1}-v)E(sw,\lambda)\mathcal{T}_{sw}^{-1}\mathbf{a}_{sw,-m\lambda}+\Delta(v^{-2}-v^2-1)E(w,\lambda)\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda}$$
$$+(1-\Delta)E(w,s\lambda)\mathcal{T}_w^{-1}\mathbf{a}_{w,-ms\lambda}=E(w,\lambda)\mathcal{T}_s^{-1}\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\Delta(v^{-1}-v)\mathcal{T}_{sw}^{-1}\mathbf{a}_{sw,-m\lambda}-\Delta(v^{-2}-v^2-1)\mathcal{T}_{sw}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}$$
$$-(1-\Delta)\mathcal{T}_{sw}^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-ms\lambda}=-\mathcal{T}_{sw}^{-1}\mathcal{T}_s^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\Delta(v^{-1}-v)\mathbf{a}_{sw,\lambda}-\Delta(v^{-2}-v^2-1)\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda}-(1-\Delta)\mathcal{T}_s^{-1}\mathbf{a}_{w,-ms\lambda}$$
$$=-\mathcal{T}_s^{-1}\mathcal{T}_s^{-1}\mathbf{a}_{w,-m\lambda},$$

or that

$$\Delta(v^{-1}-v)\mathcal{T}_s\mathcal{T}_s\mathbf{a}_{sw,-m\lambda}-\Delta(v^{-2}-v^2-1)\mathcal{T}_s\mathbf{a}_{w,-m\lambda}-(1-\Delta)\mathcal{T}_s\mathbf{a}_{w,-ms\lambda}=-\mathbf{a}_{w,-m\lambda}$$

or that

$$\Delta(v^{-1}-v)\mathbf{a}_{sw,-m\lambda}+\Delta(v^{-1}-v)(v^2-v^{-2}-1)\mathcal{T}_s\mathbf{a}_{sw,-m\lambda}$$
$$-\Delta(v^{-2}-v^2-1)\mathcal{T}_s\mathbf{a}_{w,-m\lambda}-(1-\Delta)\mathcal{T}_s\mathbf{a}_{w,-ms\lambda}=-\mathbf{a}_{w,-m\lambda}.$$

When $\Delta=0$ this is just $\mathcal{T}_s\mathbf{a}_{w,-ms\lambda}=\mathbf{a}_{w,-m\lambda}$ which follows from the definitions. When $\Delta=1$ we see that it is enough to observe the following obvious equality:

$$(v^{-1}-v)\mathbf{a}_{sw,-m\lambda}+(v^{-1}-v)(v^2-v^{-2})(\mathbf{a}_{sw,-m\lambda}+(v+v^{-1})\mathbf{a}_{w,-m\lambda})$$
$$+(v^2-v^{-2}+1)((v-v^{-1})\mathbf{a}_{sw,-m\lambda}+(v^2-v^{-2}-1))\mathbf{a}_{w,-m\lambda}=-\mathbf{a}_{w,-m\lambda}.$$

This completes the proof of (c) and hence that of (a).

We prove (b). We first show that for $(w,\lambda)\in\tilde{X}_m$ and $s\in S$ we have

(d) $$B(\mathcal{T}_s^{-1}\mathbf{a}_{w,\lambda})=\mathcal{T}_sB(\mathbf{a}_{w,\lambda}).$$

Indeed, the left-hand side equals $B(\mathcal{T}_s\mathbf{a}_{w,\lambda})+B((v^2-v^{-2})\mathbf{a}_{w,\lambda})$ which by (a) equals $\mathcal{T}_s^{-1}B(\mathbf{a}_{w,\lambda})+(v^{-2}-v^2)B(\mathbf{a}_{w,\lambda})$ and this equals $\mathcal{T}_sB(\mathbf{a}_{w,\lambda})$. Using (d) repeatedly we see that $B(\mathcal{T}_{w'}^{-1}\mathbf{a}_{w,\lambda})=\mathcal{T}_{w'}B(\mathbf{a}_{w,\lambda})$ for any $w'\in W$. To prove (b) it is enough to prove that for any $(w,\lambda)\in\tilde{X}_m$ we have

$$B(B(\mathbf{a}_{w,\lambda}))=\mathbf{a}_{w,\lambda},$$

that is,

$$B(\mathcal{T}_w^{-1}\mathbf{a}_{w,-m\lambda})=E(w,\lambda)\mathbf{a}_{w,\lambda}.$$

The left-hand side is equal to $\mathcal{T}_wB(\mathbf{a}_{w,-m\lambda})$ and hence to

$$E(w,\lambda)\mathcal{T}_w\mathcal{T}_w^{-1}\mathbf{a}_{w,\lambda}=E(w,\lambda)\mathbf{a}_{w,\lambda}.$$

This completes the proof of (b).

7.3. Let $(z,\lambda)\in\tilde{X}_m^0$. We show:

(a) $B(\mathbf{a}_{z,\lambda})=\mathbf{a}_{z,\lambda}$.

We must show that $\mathcal{T}_z^{-1}\mathbf{a}_{z,-m\lambda}=\mathbf{a}_{z,\lambda}$ or that $\mathcal{T}_z\mathbf{a}_{z,\lambda}=\mathbf{a}_{z,-m\lambda}$. This follows the definition of the $\mathbf{H}_m$-module structure on $\mathbf{M}_m$ since $zzz^{-1}=z, z(\lambda)=-m\lambda$.

7.4. Let $\mathcal{L}$ be the $\mathbf{Z}[v^{-1}]$-submodule of $\mathbf{M}_m$ with basis $\{\mathbf{a}_{w,\lambda}; (w,\lambda) \in \tilde{X}_m\}$. From Proposition 7.2 one can deduce (a), (b) below by standard arguments (see, for example, [L1, 24.2.1]).

(a) For any $(w,\lambda) \in \tilde{X}_m$ there is a unique element $\hat{\mathbf{a}}_{w,\lambda} \in \mathbf{M}_m$ such that

(i) $\hat{\mathbf{a}}_{w,\lambda} \in \mathcal{L}$, $\hat{\mathbf{a}}_{w,\lambda} - \mathbf{a}_{w,\lambda} \in v^{-1}\mathbf{Z}[v^{-1}]$,

(ii) $B(\hat{\mathbf{a}}_{w,\lambda}) = \hat{\mathbf{a}}_{w,\lambda}$.

Moreover,

(b) $\{\hat{\mathbf{a}}_{w,\lambda}; (w,\lambda) \in \tilde{X}_m\}$ is a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}$ and a $\mathbf{C}(v)$-basis of $\mathbf{M}_m$.

For example if $(z,\lambda) \in \tilde{X}^0_m$, then $\hat{\mathbf{a}}_{z,\lambda} = \mathbf{a}_{z,\lambda}$.

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