INVOLUTIONS ON PRO-\(p\)-IWAHORI HECKE ALGEBRAS

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Abstract. The pro-\(p\)-Iwahori Hecke algebra has an involution \(\iota\) defined in terms of the Iwahori-Matsumoto basis. Then for a module \(\pi\) of pro-\(p\)-Iwahori Hecke, \(\pi^\iota = \pi \circ \iota\) is also a module. We calculate \(\pi^\iota\) for simple modules \(\pi\). We also calculate the dual of \(\pi\). These calculations will be used for calculating the extensions between simple modules.

1. Introduction

This is the sequel of [Abe16] and the aim of these papers is to calculate the extension of simple modules of pro-\(p\)-Iwahori Hecke algebras. The calculation will appear in a sequel where we will use the results of this paper.

Let \(G\) be a connected reductive group over a non-Archimedean local field with residue characteristic \(p\). For a field \(C\), we can attach the pro-\(p\)-Iwahori Hecke algebra of \(G\). This is the convolution algebra of compactly supported functions which is bi-invariant under the pro-\(p\) radical of an Iwahori subgroup. If the characteristic of \(C\) is \(p\), then this algebra plays an important role in the representation theory of \(G\) over \(C\) (cf. [AHHV17]).

The main object of this paper are the anti-involution \(\zeta\) and the involution \(\iota\) when the characteristic of \(C\) is \(p\). These are defined as follows:

\(\bullet\) \(\zeta\): Let \(W(1)\) be the “pro-\(p\) Weyl group” (see subsection 2.1 for the precise definition). Then \(\mathcal{H}\) has a basis \(\{T_w \mid w \in W(1)\}\) parametrized by \(W(1)\) which is called the Iwahori-Matsumoto basis. The anti-involution \(\zeta\) is defined by \(\zeta(T_w) = T_{w^{-1}}\).

\(\bullet\) \(\iota\): We also have another basis of \(\mathcal{H}\) denoted by \(\{T_w^* \mid w \in W(1)\}\). Then the involution \(\iota\) is defined by \(\iota(T_w) = (-1)^{\ell(w)}T_w^*\) where \(\ell\) is the length function on \(W(1)\).

By the multiplication rule of \(\mathcal{H}\) in terms of the basis \(\{T_w \mid w \in W(1)\}\) (the braid relations and the quadratic relations), these maps respect the multiplication.

Let \(\pi\) be a right \(\mathcal{H}\)-module. Then we can attach the following two modules:

\(\bullet\) \(\pi^* = \text{Hom}_C(\pi, C)\) where the action of \(X \in \mathcal{H}\) on \(f \in \pi^*\) is given by \((fX)(v) = f(v\zeta(X))\) for \(v \in \pi\).

\(\bullet\) \(\pi^\iota = \pi \circ \iota\).

If \(\pi\) is simple, then \(\pi^*\) and \(\pi^\iota\) are also simple. We determine these modules (Theorems 3.24, 4.9).

Simple modules are classified in [Abe] based on a parabolic induction. Let \(P\) be a parabolic subgroup (which is at a good position with respect to our fixed Iwahori
subgroup) and let $\mathcal{H}_P$ be a pro-$p$-Iwahori Hecke algebra attached to the Levi part of $P$. Then we have the parabolic induction $I_P$ from the category of $\mathcal{H}_P$-modules to the category of $\mathcal{H}$-modules (see [2,7]). The theorem in [Abe] says that simple modules are classified in terms of parabolic inductions and simple supersingular modules. More precisely, any simple module is given by the form

$$I_P(\text{St}^P_Q(\sigma))$$

where $P, Q$ are parabolic subgroups, $\sigma$ is a simple supersingular module, and $\text{St}^P_Q(\sigma)$ is the generalized Steinberg module (see [2,9]) which is defined with parabolic inductions. Moreover, supersingular modules are classified [Oll14, Vig17]. So it is sufficient to calculate the following:

1. $I_P(\sigma)^\vee$, $I_P(\sigma)^*$,
2. $\text{St}_Q(\sigma)^\vee$, $\text{St}_Q(\sigma)^*$,
3. $\pi^\vee$, $\pi^*$ for a simple supersingular module $\pi$,

where $\sigma$ is a module of the pro-$p$-Iwahori Hecke algebra attached to the Levi part of a parabolic subgroup.

We first calculate $\pi^\vee$. This is done in section 3. We do the calculation for (1) in subsection 3.2 and for (3) in subsection 3.8. The hardest step is for (2) which is calculated from subsections 3.2 to 3.7. In section 4 we calculate $\pi^*$ using the calculation of $\pi^\vee$.

In the next section we give notation and recall some results on pro-$p$-Iwahori Hecke algebras. We use the same notation as [Abe16] and often refer to this paper.

1.1. Applications. The results will be applied to the calculation of the extension group $\text{Ext}^1_{\mathcal{H}}(\pi_1, \pi_2)$ for simple modules $\pi_1, \pi_2$ [Abe17]. Again, by the classification theorem, the calculation is divided into three steps.

The results in this paper will be used for the calculation of the extensions between generalized Steinberg modules $\text{Ext}^1_{\mathcal{H}}(\text{St}_{Q_1}(\sigma_1), \text{St}_{Q_2}(\sigma_2))$. By the definition, $\text{St}_{Q_1}(\sigma_1)$ is a quotient of a parabolically induced module and in fact we have a resolution of $\text{St}_{Q_1}(\sigma_1)$ via parabolically induced modules. Using this resolution, the calculation of $\text{Ext}^1(\text{St}_{Q_1}(\sigma_1), \text{St}_{Q_2}(\sigma_2))$ is deduced to that of $\text{Ext}^1(I_{Q_1'}(\sigma_1'), \text{St}_{Q_2}(\sigma_2))$ (where $Q_1'$ (resp., $\sigma_1'$) is a certain parabolic subgroup (resp., simple module) relating with $(Q_1, \sigma_1)$).

We also have a similar resolution for $\text{St}_{Q_2}(\sigma_2)$, however this resolution is not useful for calculations $\text{Ext}^1(I_{Q_1'}(\sigma_1'), \text{St}_{Q_2}(\sigma_2))$ since, for the calculations, we need a resolution which has a form $0 \rightarrow \text{St}_{Q_2}(\sigma_2) \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$. Here is a point where we can apply results in this paper. Taking the dual we have $\text{Ext}^1(I_{Q_1'}(\sigma_1'), \text{St}_{Q_2}(\sigma_2)) \simeq \text{Ext}^1(\text{St}_{Q_2}(\sigma_2)^*, I_{Q_1'}(\sigma_1')^*)$ and using results in this paper, we have $\text{St}_{Q_2}(\sigma_2)^* \simeq \text{St}_{Q_2}(\sigma_2')$ and $I_{Q_2'}(\sigma_2')^* \simeq I_{Q_1''}(\sigma_1'')$ for certain $Q_2', \sigma_2', Q_1'', \sigma_1''$. Here $I_{Q_1''}$ is another functor which is defined in a similar way to parabolic inductions (see [3,1]). Hence again using the resolution of generalized Steinberg modules, the calculation is deduced to that of $\text{Ext}^1(I_{Q_2'}(\sigma_2'), I_{Q_1''}(\sigma_1'))$.

2. Preliminaries

2.1. Pro-$p$-Iwahori Hecke algebra. Let $F$ be a non-Archimedean local field, let $\kappa$ be its residue field, let $p$ be its residue characteristic, and let $G$ be a connected reductive group over $F$. We denote the group of valued points $G(F)$ by the same letter $G$ and we apply the same notation for other algebraic groups. We can get
the data \((W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_n)\) from \(G\) as follows. See [Vig16], especially 3.9 and 4.2 for the details.

Fix a maximal split torus \(S\) and denote the centralizer of \(S\) by \(Z\). Let \(Z^0\) be the unique parahoric subgroup of \(Z\) and let \(Z(1)\) be its pro-\(p\) radical. Then the group \(W(1)\) (resp., \(W\)) is defined by \(W(1) = N_G(Z)/Z(1)\) (resp., \(W = N_G(Z)/Z^0\)) where \(N_G(Z)\) is the normalizer of \(Z\) in \(G\). We also have \(Z_n = Z^0/Z(1)\). Let \(G'\) be the group generated by the unipotent radical of parabolic subgroups [AHHV17, II.1] and let \(W_{\text{aff}}\) be the image of \(G' \cap N_G(Z)\) in \(W\). Then this is a Coxeter group. Fix a set of simple reflections \(S_{\text{aff}}\). The group \(\Omega\) is the stabilizer of \(S_{\text{aff}}\) in \(W\). Then we get the data \((W_{\text{aff}}, S_{\text{aff}}, \Omega, W, W(1), Z_n)\). We denote the image of \(G' \cap N_G(Z)\) in \(W(1)\) by \(W_{\text{aff}}(1)\).

Attached to this data and a parameter \((q, c)\) as in [Vig16, 4.3], we have the generic algebra \(\mathcal{H}\) which we call a pro-\(p\)-Iwahori Hecke algebra. In this paper, the parameter \(c\) is always the one which comes from the group \(G\), namely the one defined in [Vig16, 4.2]. (In [Vig16] it is denoted by \(c_{s(u)}\).)

Consider the apartment attached to \(S\) and an alcove surrounded by \(\{H_s \mid s \in S_{\text{aff}}\}\) where \(H_s\) is the hyperplane pointwisely fixed by \(s \in S_{\text{aff}}\). Let \(I(1)\) be the pro-\(p\)-Iwahori subgroup attached to this alcove. Then with \(q_s = \#(I(1)\tilde{s}I(1)/I(1))\) for \(s \in S_{\text{aff}}\) with a lift \(\tilde{s} \in N_G(Z)\), the algebra \(\mathcal{H}\) is isomorphic to the Hecke algebra attached to \((G, I(1))\) [Vig16, Proposition 4.4].

We recall a little about \(\mathcal{H}\). Let \(S_{\text{aff}}(1)\) be the inverse image of \(S_{\text{aff}}\) in \(W(1)\). For \(s \in S_{\text{aff}}(1)\), we write \(q_s\) for \(q_{\tilde{s}}\) where \(\tilde{s} \in S_{\text{aff}}\) is the image of \(s\). The length function on \(W_{\text{aff}}\) is denoted by \(\ell\) and its inflation to \(W\) and \(W(1)\) is also denoted by \(\ell\).

The \(C\)-algebra \(\mathcal{H}\) is a free \(C\)-module and has a basis \(\{T_w\}_{w \in W(1)}\). The multiplication is given by

- (Quadratic relations) \(T_s^2 = q_s T_s + c_s T_s\) for \(s \in S_{\text{aff}}(1)\).
- (Braid relations) \(T_v T_w = T_w T_v\) if \(\ell(vw) = \ell(v) + \ell(w)\).

We extend \(q: S_{\text{aff}} \to C\) to \(q: W \to C\) as follows. For \(w \in W\), take \(s_1, \ldots, s_l \in S_{\text{aff}}\) and \(u \in \Omega\) such that \(w = s_1 \cdots s_l u\) and \(\ell(w) = l\). Then put \(q_w = q_{s_1} \cdots q_{s_l}\). From the definition, we have \(q_w^{-1} = q_w\). We also put \(q_w = q_{\overline{w}}\) for \(w \in W(1)\) with the image \(\overline{w}\) in \(W\).

The aim of this paper is to study a representation theory of \(\mathcal{H}\). In this paper, modules mean right modules unless otherwise stated.

2.2. The algebra \(\mathcal{H}[q_s]\) and \(\mathcal{H}[q_s^{\pm 1}]\). For each \(s \in S_{\text{aff}}\), let \(q_s\) be an indeterminate such that if \(wsu^{-1} \in S_{\text{aff}}\) for \(w \in W\), we have \(q_{wsu^{-1}} = q_s\). Let \(C[q_s]\) be a polynomial ring with this indeterminate. Then with the parameter \(s \mapsto q_s\) and the other data coming from \(G\), we have the algebra. This algebra is denoted by \(\mathcal{H}[q_s]\) and we put \(\mathcal{H}[q_s^{\pm 1}] = \mathcal{H}[q_s] \otimes_{C[q_s]} C[q_s^{\pm 1}]\). Under \(q_s \mapsto \#(I(1)\tilde{s}I(1)/I(1)) \in C\) where \(\tilde{s} \in N_G(Z)\) is a lift of \(s\), we have \(\mathcal{H}[q_s] \otimes_{C[q_s]} C \simeq \mathcal{H}\). As an abbreviation, we denote \(q_s\) by just \(q_s\). Consequently, we denote by \(\mathcal{H}[q_s]\) (resp., \(\mathcal{H}[q_s^{\pm 1}]\)).

Since \(q_s\) is invertible in \(\mathcal{H}[q_s^{\pm 1}]\), we can do some calculations in \(\mathcal{H}[q_s^{\pm 1}]\) with \(q_s^{-1}\). If the result can be stated in \(\mathcal{H}[q_s]\), then this is an equality in \(\mathcal{H}[q_s]\) since \(\mathcal{H}[q_s]\) is a subalgebra of \(\mathcal{H}[q_s^{\pm 1}]\) and, by specializing, we can get some equality in \(\mathcal{H}\). See [Vig16, 4.5] for more details.

2.3. The root system and the Weyl groups. Let \(W_0 = N_G(Z)/Z\) be the finite Weyl group. Then this is a quotient of \(W\). Recall that we have the alcove defining \(I(1)\). Fix a special point \(x_0\) from the border of this alcove. Then \(W_0 \simeq \text{Stab}_W x_0\).
and the inclusion \( \text{Stab}_W x_0 \hookrightarrow W \) is a splitting of the canonical projection \( W \rightarrow W_0 \). Throughout this paper, we fix this special point and regard \( W_0 \) as a subgroup of \( W \). Set \( S_0 = S_{\text{aff}} \cap W_0 \subset W \). This is a set of simple reflections in \( W_0 \). For each \( w \in W_0 \), we fix a representative \( n_w \in W(1) \) such that \( n_{w_1 w_2} = n_{w_1} n_{w_2} \) if \( \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \).

The group \( W_0 \) is the Weyl group of the root system \( \Sigma \) attached to \((G,S)\). Our fixed alcove and special point give a positive system of \( \Sigma \), denoted by \( \Sigma^+ \). The set of simple roots is denoted by \( \Delta \). As usual, for \( \alpha \in \Delta \), let \( s_\alpha \in S_0 \) be a simple reflection for \( \alpha \).

The kernel of \( W(1) \rightarrow W_0 \) (resp., \( W \rightarrow W_0 \)) is denoted by \( \Lambda(1) \) (resp., \( \Lambda \)). Then, \( \Sigma_\kappa \subset \Lambda(1) \) and we have \( \Lambda = \Lambda(1)/\Sigma_\kappa \). The group \( \Lambda \) (resp., \( \Lambda(1) \)) is isomorphic to \( Z/Z^0 \) (resp., \( Z/Z(1) \)). Any element in \( W(1) \) can be uniquely written as \( n_w \lambda \) where \( w \in W_0 \) and \( \lambda \in \Lambda \).

### 2.4. The map \( \nu \)

The group \( W \) acts on the apartment attached to \( S \) and the action of \( \Lambda \) is by the translation. Since the group of translations of the apartment is \( X_*(S) \otimes Z \mathbb{R} \), we have a group homomorphism \( \nu : \Lambda \rightarrow X_*(S) \otimes Z \mathbb{R} \). The compositions \( \Lambda(1) \rightarrow \Lambda \rightarrow X_*(S) \otimes Z \mathbb{R} \) and \( Z \rightarrow \Lambda \rightarrow X_*(S) \otimes Z \mathbb{R} \) are also denoted by \( \nu \). The homomorphism \( \nu : Z \rightarrow X_*(S) \otimes Z \mathbb{R} \simeq \text{Hom}_Z(X^*(S), \mathbb{R}) \) is characterized by the following: For \( t \in S \) and \( \chi \in X^*(S) \), we have \( \nu(t)(\chi) = -\text{val}(\chi(t)) \) where \( \text{val} \) is the normalized valuation of \( F \). The kernel of \( \nu : Z \rightarrow X_*(S) \otimes Z \mathbb{R} \) is equal to the maximal compact subgroup \( \bar{Z} \) of \( Z \). In particular, \( \text{Ker}(\Lambda(1) \rightarrow X_*(S) \otimes Z \mathbb{R}) = \bar{Z}/Z(1) \) is a finite group.

We call \( \lambda \in \Lambda(1) \) dominant (resp., antidominant) if \( \nu(\lambda) \) is dominant (resp., antidominant).

Since the group \( W_{\text{aff}} \) is a Coxeter system, it has the Bruhat order denoted by \( \leq \). For \( w_1, w_2 \in W_{\text{aff}} \), we write \( w_1 < w_2 \) if there exists \( u \in \Omega \) such that \( w_1 u, w_2 u \in W_{\text{aff}} \) and \( w_1 u < w_2 u \). Moreover, for \( w_1, w_2 \in W(1) \), we write \( w_1 < w_2 \) if \( w_1 \in W(1)w_2 \) and \( \bar{w}_1 < \bar{w}_2 \) where \( \bar{w}_1, \bar{w}_2 \) are the image of \( w_1, w_2 \) in \( W \), respectively. We write \( w_1 \leq w_2 \) if \( w_1 < w_2 \) or \( w_1 = w_2 \).

### 2.5. Other basis

From the definition the algebra \( \mathcal{H} \) has the basis \( \{T_w\}_{w \in W(1)} \). This algebra also has another base which also has an important role in this paper.

The first is denoted by \( \{T_w^*\}_{w \in W(1)} \) defined as follows. For \( w \in W(1) \), take \( s_1, \ldots, s_l \in S_{\text{aff}}(1) \) and \( u \in W(1) \) such that \( l = \ell(w) \), \( \ell(u) = 0 \) and \( w = s_1 \cdots s_l u \). Set \( T_w^* = (T_{s_1} - c_{s_1}) \cdots (T_{s_l} - c_{s_l}) T_u \). Then this does not depend on the choice. It is not so difficult to see that we have \( T_w^* \in T_w + \sum_{\nu < w} \text{CT}_v \) and this implies that \( \{T_w^*\}_{w \in W(1)} \) is a basis. In \( \mathcal{H}[q^{\pm 1}] \), we have \( T_w^* = q_w T_{w^{-1}} \).

The second basis is called a Bernstein basis. This basis is attached to a spherical orientation \( o [\text{Vig16} 5.2] \). We do not recall the definition of a spherical orientation, but we remark that there is a natural bijection between spherical orientations and Weyl chambers. The Weyl group \( W_0 \) acts on spherical orientation (resp., Weyl chambers) from the right (resp., left). If a spherical orientation \( o \) and a Weyl chamber \( C \) corresponds to each other, \( o \cdot w \) and \( w^{-1}(C) \) corresponds for \( w \in W_0 \). The spherical orientation corresponding to the dominant (resp., antidominant) chamber is called the dominant (resp., antidominant) spherical orientation which is denoted by \( o_+ \) (resp., \( o_- \)).
For simplicity, we always assume that our commutative ring $C$ contains a square root of $q_s$ which is denoted by $q_s^{1/2}$ for $s \in S_{\text{aff}}$. For $w = s_1 \cdots s_l u$ where $\ell(w) = l$, $s_1, \ldots, s_l \in S_{\text{aff}}$ and $\ell(u) = 0$, $q_w^{1/2} = q_{s_1}^{1/2} \cdots q_{s_l}^{1/2}$ is a square root of $q_w$.

We recall some properties of the Bernstein basis. The Bernstein basis attached to a spherical orientation $o$ is denoted by $\{E_o(w)\}_{w \in W(1)}$ [Vig16, 5]. Since the definition is complicated, we do not recall it here. Similar to $\{T^*_w\}$, we have

$$E_o(w) \in T_w + \sum_{v < w} CT_v.$$ The basis satisfies the following product formula [Vig16, Theorem 5.25]:

$$(2.1) \quad E_o(w_1)E_{o-w_1}(w_2) = q_{w_1w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2} E_o(w_1w_2).$$

Here $w_1, w_2 \in W(1)$ and $o - w_1$ means $o \cdot w_1$ with the image $\overline{w}_1$ of $w_1$ in $W_0$.

**Remark 2.1.** Since we do not assume that $q_s$ is invertible in $C$, $q_{w_1w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2}$ does not make sense in a usual way. See [Abe16, Remark 2.2].

2.6. **Levi subalgebra.** Since we have a positive system $\Sigma^+$, we have a minimal parabolic subgroup $B$ with a Levi part $Z$. In this paper, parabolic subgroups are always standard, namely containing $B$. Note that such parabolic subgroups correspond to subsets of $\Delta$.

Let $P$ be a parabolic subgroup. Attached to the Levi part of $P$ containing $Z$, we have the data $(W_{\text{aff}, P}, S_{\text{aff}, P}, \Omega_P, W_P, W_P(1), Z_\kappa)$ and the parameters $(q_P, c_P)$. Hence we have the algebra $\mathcal{H}$. The parameter $c_P$ is given by the restriction of $c$, hence we denote it just by $c$. The parameter $q_P$ is defined as in [Abe16, 4.1].

For the objects attached to this data, we add the suffix $P$. We have the set of simple roots $\Delta_P$, the root system $\Sigma_P$ and its positive system $\Sigma_P^+$, the finite Weyl group $W_0, P$, the set of simple reflections $S_{\text{aff}, P} \subset W_0, P$, the length function $\ell_P$ and the base $\{T^P_w\}_{w \in W_P(1)}$, $\{T^P_{\pm \nu}\}_{w \in W_P(1)}$ and $\{E^P_o(w)\}_{w \in W_P(1)}$ of $\mathcal{H}_P$. Note that we have no $\Lambda_P, \Lambda_P(1)$ and $Z_\kappa, P$ since they are equal to $\Lambda, \Lambda(1)$, and $Z_\kappa$.

An element $w = n_w \lambda \in W_P(1)$ where $v \in W_{0, P}$ and $\lambda \in \Lambda(1)$ is called $P$-positive (resp., $P$-negative) if $\langle \alpha, \nu(\lambda) \rangle \leq 0$ (resp., $\langle \alpha, \nu(\lambda) \rangle \geq 0$) for any $\alpha \in \Sigma^+ \setminus \Sigma^+_P$. Let $W_P^+(1)$ (resp., $W_P^-(1)$) be the set of $P$-positive (resp., $P$-negative) elements and put $\mathcal{H}_P^+ = \bigoplus_{w \in W_P^+(1)} CT^P_w$. These are subalgebras of $\mathcal{H}_P$ [Abe16, Lemma 4.1].

**Proposition 2.2** (Vig15, Theorem 1.4). Let $\lambda_P^\pm$ (resp., $\lambda_P^\mp$) be in the center of $W_P(1)$ such that $\langle \alpha, \nu(\lambda_P^\pm) \rangle < 0$ (resp., $\langle \alpha, \nu(\lambda_P^\mp) \rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma^+_P$. Then $T^P_{\pm \nu} = T^P_{\pm \nu} = E^P_{\pm \nu}(\lambda_P^\pm)$ (resp., $T^P_{\pm \nu} = T^P_{\pm \nu} = E^P_{\pm \nu}(\lambda_P^\mp)$) is in the center of $\mathcal{H}_P$ and we have $\mathcal{H}_P = \mathcal{H}_P^+ E^P_{\pm \nu}(\lambda_P^\pm)^{-1}$ (resp., $\mathcal{H}_P = \mathcal{H}_P^+ E^P_{\pm \nu}(\lambda_P^\mp)^{-1}$).

Note that such $\lambda_P^\pm$ always exists [Abe16, Lemma 2.4].

We define $j_P^+: \mathcal{H}_P^+ \to \mathcal{H}$ and $j_P^{\pm*}: \mathcal{H}_P^\pm \to \mathcal{H}$ by $j_P^+(T^P_w) = T_w$ and $j_P^{\pm*}(T^P_{\pm \nu}) = T^*_w$ for $w \in W_P^+(1)$. Then these are algebra homomorphisms.

**Remark 2.3.** In [Abe] the homomorphism $j_P^{\pm*}$ is denoted by $j_P^-$ where $M$ is the Levi part of $P$. Note that the notation $j_P^-$ is used for a different homomorphism in this paper.

Let $Q$ be a parabolic subgroup containing $P$ and let $W_P^{Q+}(1)$ (resp., $W_P^{Q-}(1)$) be the set of $n_w \lambda$ where $\langle \alpha, \nu(\lambda) \rangle \leq 0$ (resp., $\langle \alpha, \nu(\lambda) \rangle \geq 0$) for any $\alpha \in \Sigma^+_Q \setminus \Sigma^+_P$ and
Let $P$ be the parabolic subgroup and let $\sigma$ be an $\mathcal{H}_P$-module. (This is a right module as in subsection 2.1.) Then we define an $\mathcal{H}$-module $I_P(\sigma)$ by

$$I_P(\sigma) = \text{Hom}_{(\mathcal{H}_P, J_P^{-1})}(\mathcal{H}, \sigma).$$

(This is the same as that defined in [Abe].) We again remark that $j_P^*$ is denoted by $j_P^*$ in [Abe].) Namely, the space of homomorphism $\varphi$ from $\mathcal{H} \rightarrow \sigma$ such that $\varphi(X_j^*(Z)) = \varphi(X)Z$ for any $X \in \mathcal{H}$ and $Z \in \mathcal{H}_P$. The module structure is given by $(\varphi X)(Y) = \varphi(XY)$ where $X, Y \in \mathcal{H}$. We call $I_P$ the parabolic induction.

For $P \subset P$, we write

$$I_P^P(\sigma) = \text{Hom}_{(\mathcal{H}_P, J_P^{-1})}(\mathcal{H}_P, \sigma).$$

Let $P$ be a parabolic subgroup. Set $W_0^P = \{ w \in W_0 \mid w(\Delta_P) \subset \Sigma^+ \}$. Then the multiplication map $W_0^P \times W_0^P \rightarrow W_0$ is bijective and for $w_1 \in W_0^P$ and $w_2 \in W_0^P$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. We also put $P W_0 = \{ w \in W_0 \mid w^{-1}(\Delta_P) \subset \Sigma^+ \}$. Then the multiplication map $W_0^P \times P W_0 \rightarrow W_0$ is bijective and for $w_1 \in W_0^P$ and $w_2 \in P W_0$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. See [Abe16] Proposition 2.9 for the following proposition.

**Proposition 2.4.** Let $P$ be a parabolic subgroup and let $\sigma$ be an $\mathcal{H}_P$-module.

1. The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{w_0}))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.
2. The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{w_0}^*))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.

**Proposition 2.5 ([Abe] Proposition 4.12).** Assume that $q_s = 0$ for any $s \in S_{\text{aff}}$. Let $w \in W_0^P$ and $\lambda \in \Lambda(1)$. Then for $\varphi \in I_P(\sigma)$, we have

$$(\varphi E_{\alpha^-}(\lambda))(T_{w_0}) = \begin{cases} \varphi(T_{w_0})\sigma(E_{\alpha^-}^P(n_w^{-1} \cdot \lambda)), & (n_w^{-1} \cdot \lambda) \in W_P(1), \\ 0, & (n_w^{-1} \cdot \lambda) \notin W_P(1). \end{cases}$$

We also define $I_P^P(\sigma) = \text{Hom}_{(\mathcal{H}_P, J_P^{-1})}(\mathcal{H}, \sigma)$. The module structure is given in the same way as $I_P$. We also define $I_P^{P'}$ in a similar way.

**2.8. Twist by $n_{w_G w_P}$.** For a parabolic subgroup $P$, let $w_P$ be the longest element in $W_0$. In particular, $w_G$ is the longest element in $W_0$. Let $P'$ be a parabolic subgroup corresponding to $-w_G(\Delta_P)$. In other words, $P' = n_{w_G w_P} P^{\text{op}} n_{w_G w_P}$, where $P^{\text{op}}$ is the opposite parabolic subgroup of $P$ with respect to the Levi part of $P$ containing $Z$. Set $n = n_{w_G w_P}$. Then the map $P^{\text{op}} \rightarrow P'$ defined by $p \mapsto np^{-1}$ is an isomorphism which preserves the data used to define the pro-$p$-Iwahori Hecke algebras. Hence $T_w^{P'} \rightarrow T_{w_0 w_0^{-1}}^{P'}$ gives an isomorphism $\mathcal{H}_{P'} \rightarrow \mathcal{H}_{P'}$. This sends $T_w^{P'}$ to $T_{w_0 w_0^{-1}}^{P'}$ and $E_{\alpha^-}^P(w) \rightarrow E_{\alpha^-}^{P'}(w)$ to $E_{\alpha^-}^{P'}(n w_0^{-1} w) \in W_0$. Let $\sigma$ be an $\mathcal{H}_{P'}$-module. Then we define an $\mathcal{H}_{P'}$-module $n_{w_G w_P} \sigma$ via the pullback of the above isomorphism. Namely, for $w \in W_{P'}$, we put $(n_{w_G w_P} \sigma)(T_{w_0}^{P'}) = \sigma(T_{w_0}^{P'})$. Then we have homomorphisms $j_P^{\pm} : \mathcal{H}_P^{\pm} \rightarrow \mathcal{H}_Q^{\pm}$ defined in a similar way.

$w \in W_{0, P}$. Put $\mathcal{H}_P^{\pm} = \bigoplus_{w \in W_0^{\pm}(1)} CT_w^P \subset \mathcal{H}_P$. Then we have homomorphisms $j_P^{\pm} : \mathcal{H}_P^{\pm} \rightarrow \mathcal{H}_Q^{\pm}$ defined in a similar way.
2.9. The extension and the generalized Steinberg modules. Let $P$ be a parabolic subgroup and let $\sigma$ be an $\mathcal{H}_P$-module. For $\alpha \in \Delta$, let $P_\alpha$ be a parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Then we define $\Delta(\sigma) \subset \Delta$ by

$$\Delta(\sigma) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0, \ \sigma(T_P^\lambda) = 1 \text{ for any } \lambda \in W_{aff,P_\alpha}(1) \cap \Lambda(1) \} \cup \Delta_P.$$ 

Let $P(\sigma)$ be a parabolic subgroup corresponding to $\Delta(\sigma)$.

**Proposition 2.6** ([AHV17] Corollary 3.9). Let $\sigma$ be an $\mathcal{H}_P$-module and let $Q$ be a parabolic subgroup between $P$ and $P(\sigma)$. Denote the parabolic subgroup corresponding to $\Delta_Q \setminus \Delta_P$ by $P_2$. Then there exists a unique $\mathcal{H}_Q$-module $e_Q(\sigma)$ acting on the same space as $\sigma$ such that

- $e_Q(\sigma)(T_Q^{\ast}) = \sigma(T_P^{\ast})$ for any $w \in W_P(1)$.
- $e_Q(\sigma)(T_Q^{\ast}) = 1$ for any $w \in W_{aff,P_2}(1)$.

Moreover, one of the following conditions gives a characterization of $e_Q(\sigma)$:

1. For any $w \in W_Q^{-}(1)$, $e_Q(\sigma)(T_Q^{\ast}) = \sigma(T_P^{\ast})$ (namely, $e_Q(\sigma) \simeq \sigma$ as $(\mathcal{H}_P^{-}, j_P^{-})$-modules) and for any $w \in W_{aff,P_2}(1)$, $e_Q(\sigma)(T_Q^{\ast}) = 1$.
2. For any $w \in W_Q^{+}(1)$, $e_Q(\sigma)(T_Q^{\ast}) = \sigma(T_P^{\ast})$ and for any $w \in W_{aff,P_2}(1)$, $e_Q(\sigma)(T_Q^{\ast}) = 1$.

We call $e_Q(\sigma)$ the extension of $\sigma$ to $\mathcal{H}_Q$. A typical example of the extension is the trivial representation $\mathbf{1} = \mathbf{1}_G$. This is a one-dimensional $\mathcal{H}$-module defined by $1(T_w) = q_w$, or, equivalently $1(T_w^\ast) = 1$. We have $\Delta(1_P) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0\} \cup \Delta_P$ and if $Q$ is a parabolic subgroup between $P$ and $P(1_P)$, we have $e_Q(1_P) = 1_Q$.

Let $P(\sigma) \supset P_0 \supset Q_1 \supset Q \supset P$. Then as in [Abde 4.5], we have $I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \subset I_{Q_0}^{P_0}(e_Q(\sigma))$. Define

$$\text{St}_{Q_0}^{P_0}(\sigma) = \text{Cok} \left( \bigoplus_{Q_1 \supset Q} I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \rightarrow I_{Q_0}^{P_0}(e_Q(\sigma)) \right).$$

When $P_0 = G$, we write $\text{St}_Q(\sigma)$.

In the rest of this subsection, we assume that $P(\sigma) = G$. As we mentioned in the above, for $Q_1 \supset Q \supset P$ we have $I_{Q_1}(e_{Q_1}(\sigma)) \hookrightarrow I_Q(e_Q(\sigma))$. The proof of [Abde Lemma 4.23] implies the following lemma.

**Lemma 2.7.** Assume that $q_s = 0$ for any $s \in S_{aff}$. The diagram

$$\begin{CD}
I_{Q_1}(e_{Q_1}(\sigma)) @>>> I_Q(e_Q(\sigma)) \\
@VVV \oplus_{w \in W_0^{Q_1}} \sigma \Rightarrow \oplus_{w \in W_0^Q} \sigma
\end{CD}$$

is commutative. Here the embedding $\oplus_{w \in W_0^{Q_1}} \sigma \hookrightarrow \oplus_{w \in W_0^Q} \sigma$ is induced by $W_0^{Q_1} \hookrightarrow W_0^Q$.
Lemma 2.8. Assume that $q_s = 0$ for any $s \in S_{\text{aff}}$. Let $\varphi \in I_Q(e_Q(\sigma))$. Then for $w \in W_0^Q$ and $\lambda \in \Lambda(1)$, we have

$$(\varphi E_{\alpha, \lambda})(T_{n_w}) = \begin{cases} \varphi(T_{n_w})\sigma(E^P_{\alpha, \lambda}(n_w^{-1} \cdot \lambda)), & (n_w^{-1} \cdot \lambda \text{ is } P\text{-negative}), \\ 0, & \text{(otherwise)}. \end{cases}$$

Proof. Assume that $n_w^{-1} \cdot \lambda$ is $P$-negative. Then in particular it is $Q$-negative. By Proposition 2.5 we have

$$(\varphi E_{\alpha, \lambda})(T_{n_w}) = \varphi(T_{n_w})e_Q(\sigma)(E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda)).$$

Since $n_w^{-1} \cdot \lambda$ is $P$-negative, we have $E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda) \in \mathcal{H}_P^Q$. Hence $E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda) = j_P^{-Q}(E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda))$ by [Abe16 Lemma 2.6]. Therefore we have $e_Q(\sigma)(E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda)) = \sigma(E_{\alpha, \lambda}^Q(n_w^{-1} \cdot \lambda))$. We get the lemma in this case.

Assume that $n_w^{-1} \cdot \lambda$ is not $P$-negative. Then there exists $\alpha \in \Sigma^+ \setminus \Sigma_P^+$ such that $\langle w(\alpha), \nu(\lambda) \rangle < 0$. Take $\lambda^-_P$ as in Proposition 2.2 and put $\lambda_0 = n_w \cdot \lambda^-_P$. We have $\langle w(\alpha), \nu(\lambda_0) \rangle = \langle \alpha, \nu(\lambda^-_P) \rangle > 0$. Hence $\lambda$ and $\lambda_0$ are not in the same chamber. Therefore we have $E_{\alpha, \lambda}E_{\alpha, \lambda^-_P}(\lambda^-_P) = 0$ by (2.1) and [Abe16 Lemma 2.11]. Hence we have $\varphi E_{\alpha, \lambda}(E_{\alpha, \lambda^-_P}(\lambda^-_P))(T_{n_w}) = 0$. Since $n_w^{-1} \cdot \lambda_0 = \lambda^-_P$ is $P$-negative, we have $\varphi E_{\alpha, \lambda}(E_{\alpha, \lambda}(\lambda_0))(T_{n_w}) = (\varphi E_{\alpha, \lambda})(T_{n_w})\sigma(E_{\alpha, \lambda}(\lambda^-_P))$ as we have already proved. Since $\lambda^-_P \in Z(W_P(1))$, $E_{\alpha, \lambda}(\lambda^-_P)$ is invertible. Hence we have $(\varphi E_{\alpha, \lambda})(T_{n_w}) = 0$. □

2.10. Module $\sigma_{\ell - \ell_P}$. Let $P$ be a parabolic subgroup and let $\sigma$ be an $\mathcal{H}_P$-module. Define a linear map $\sigma_{\ell - \ell_P}(T_w) = (-1)^{\ell(w) - \ell_P(w)}\sigma(T_w)$. Then this defines a new $\mathcal{H}_P$-module $\sigma_{\ell - \ell_P}$ [Abe16 Lemma 4.1].

2.11. Supersingular modules. Assume that $q_s = 0$ for any $s \in S_{\text{aff}}$. Let $\mathcal{O}$ be a conjugacy class in $W(1)$ which is contained in $\Lambda(1)$. For a spherical orientation $o$, set $z_o = \sum_{\lambda \in \mathcal{O}} E_o(\lambda)$. Then this does not depend on $o$ and gives an element of the center of $\mathcal{H}$ [Vig17 Theorem 5.1]. The length of $\lambda \in \mathcal{O}$ does not depend on $\lambda$. We denote it by $\ell(\mathcal{O})$.

Definition 2.9. Let $\pi$ be an $\mathcal{H}$-module. We call $\pi$ supersingular if there exists $n \in \mathbb{Z}_{>0}$ such that $\pi z_o^n = 0$ for any $\mathcal{O}$ such that $\ell(\mathcal{O}) > 0$.

The simple supersingular $\mathcal{H}$-modules are classified in [Oll14, Vig17]. We recall their results. Assume that $C$ is a field. Let $\chi$ be a character of $Z_K \cap W_{\text{aff}}(1)$ and put $S_{\text{aff}, \chi} = \{ s \in S_{\text{aff}} \mid \chi(c_s) \neq 0 \}$ where $s \in W(1)$ is a lift of $s \in S_{\text{aff}}$. Note that if $s$ is another lift, then $s' = ts$ for some $t \in Z_K$. Hence $\chi(c_{s'}) = \chi(t)\chi(c_s)$. Therefore the condition does not depend on a choice of a lift. Let $J \subset S_{\text{aff}, \chi}$. Then the character $\Xi = \Xi_{J, \chi}$ of $\mathcal{H}_{\text{aff}}$ is defined by

$$\Xi_{J, \chi}(T_t) = \chi(t) \quad (t \in Z_K \cap W_{\text{aff}}(1)),
\Xi_{J, \chi}(T_{s}) = \begin{cases} \chi(c_s), & (s \in S_{\text{aff}, \chi} \setminus J), \\ 0, & (s \notin S_{\text{aff}, \chi} \setminus J), \end{cases}$$

where $s \in W_{\text{aff}}(1)$ is a lift of $s$. Let $\Xi(1)_{\Xi}$ be the stabilizer of $\Xi$ and let $V$ be a simple $C[\Xi(1)_{\Xi}]$-module such that $V|_{Z_K \cap W_{\text{aff}}(1)}$ is a direct sum of $\chi$. Put $\mathcal{H}_{\Xi} = \mathcal{H}_{\text{aff}}C[\Xi(1)_{\Xi}]$. This is a subalgebra of $\mathcal{H}$. For $X \in \mathcal{H}_{\text{aff}}$ and $Y \in C[\Xi(1)_{\Xi}]$, we define the action of $XY$ on $\Xi \otimes V$ by $x \otimes y \mapsto xX \otimes yY$. Then this defines a well-defined action of $\mathcal{H}_{\Xi}$ on $\Xi \otimes V$. Set $\pi_{\chi, J, V} = (\Xi \otimes V) \otimes_{\mathcal{H}_{\Xi}} \mathcal{H}$. 


Proposition 2.10 ([Vig17, Theorem 1.6]). The module $\pi_{\chi,J,V}$ is simple and it is supersingular if and only if the groups generated by $J$ and generated by $S_{\text{aff},\chi} \setminus J$ are both finite. If $C$ is an algebraically closed field, then any simple supersingular modules are given in this way.

Remark 2.11. This classification result is valid even though the data which defines $\mathcal{H}$ does not come from a reductive group.

2.12. Simple modules. Assume that $C$ is an algebraically closed field of characteristic $p$. We consider the following triple $(P, \sigma, Q)$:

- $P$ is a parabolic subgroup.
- $\sigma$ is a simple supersingular $\mathcal{H}_P$-module.
- $Q$ is a parabolic subgroup between $P$ and $P(\sigma)$.

Define

$$I(P, \sigma, Q) = I_{P(\sigma)}(\text{St}_{Q}^{P(\sigma)}(\sigma)).$$

Theorem 2.12 ([Abe, Theorem 1.1]). The module $I(P, \sigma, Q)$ is simple and any simple module has this form. Moreover, $(P, \sigma, Q)$ is unique up to isomorphism.

Let $\chi$ be a character of $Z_n \cap W_{\text{aff},P}(1)$, $J \subset S_{\text{aff},P,\chi}$ and let $V$ be a simple module of $C[\Omega_{P}(1)_{\Xi}]$ whose restriction to $Z_n \cap W_{\text{aff},P}(1)$ is a direct sum of $\chi$. Assume that the group generated by $J$ and generated by $S_{P,\text{aff},\chi} \setminus J$ are finite. Then we put $I(P;\chi,J,V;Q) = I(P,\pi_{\chi,J,V},Q)$. This is a simple module.

2.13. Möbius function. Let $Q$ be a parabolic subgroup and let $\mu^Q$ be the Möbius function associated to $(W_0^Q, \leq)$ where $\leq$ is the Bruhat order. The theorem due to Deodhar [Deo77, Theorem 1.2] says

$$\mu^Q(v, w) = \begin{cases} 0, & \text{(there exists } \alpha \in \Delta_Q \text{ such that } vs_\alpha \leq w), \\ (-1)^{\ell(v)+\ell(w)}, & \text{(otherwise).} \end{cases}$$

Set $\Delta_w = \{ \alpha \in \Delta \mid w(\alpha) > 0 \}$ for $w \in W_0$. We use the following special value of the Möbius function in this paper.

Lemma 2.13. Let $w \in W_0^Q$ such that $\Delta_{wwQ} = \Delta \setminus \Delta_Q$ and let $w_0$ be the longest element of the finite Weyl group of the parabolic subgroup corresponding to $\Delta \setminus \Delta_Q$. Then we have

$$\mu^Q(w, w_Gw_Q) = \begin{cases} (-1)^{\ell(w_0)}, & (w = w_Gw_0w_Q), \\ 0, & (w \neq w_Gw_0w_Q). \end{cases}$$

We prove this lemma by backward induction on the length of $w$. For the inductive step, we use the following.

Lemma 2.14. Let $w \in W_0^Q$ such that $\Delta_{wwQ} = \Delta \setminus \Delta_Q$, $w \neq w_Gw_0w_Q$, and $\alpha \in \Delta$ such that $s_\alpha w w_Q > ww_Q$ and $\Delta_{wwQ} = \Delta_{s_\alpha w w_Q}$. (Such $\alpha$ exists [Abe, Lemma 3.15].) We have

(1) $s_\alpha w \in W_0^Q$.
(2) $s_\alpha w > w$.

Proof. If $\beta \in \Delta_Q$, then $w_Q(\beta) \in -\Delta_Q$. Hence $s_\alpha w w_Q(w_Q(\beta)) > 0$ since $\Delta_{s_\alpha w w_Q} = \Delta \setminus \Delta_Q$. Therefore $s_\alpha w(\beta) > 0$ for any $\beta \in \Delta_Q$. Namely, we have $s_\alpha w \in W_0^Q$. Therefore we have $s_\alpha w, w \in W_0^Q$, $w_Q \in W_{0,Q}$ and $s_\alpha w w_Q > ww_Q$. By [Deo77, Lemma 3.5], $s_\alpha w > w$. \qed
Proof of Lemma 2.13 Assume that \( w = w_G w_c w_Q \) and there exists \( \alpha \in \Delta_Q \) such that \( ws \leq w_G w_Q \) where \( s = s_\alpha \). Then we have \( w_G w_c w_Q s \leq w_G w_Q \). Hence \( w_c w_Q s \geq w_Q \). Let \( Q_0 \) be the parabolic subgroup corresponding to \( \Delta \setminus \Delta_Q \). Then \( w_c \in W_{0,Q_0} \) and \( W_{0,Q} \subset Q_0 W_0 \). Therefore \( w_Q, w_Q s \in Q_0 W_0 \). Hence by [Deo77, Lemma 3.5], we have \( w_Q s \geq w_Q \). This is a contradiction. Hence \( \mu^Q(w, w_G w_Q) = (-1)\ell(w) + \ell(w_G w_Q) = (-1)\ell(w) \).

If \( w \neq w_G w_c w_Q \), then \( w w_Q \neq w_G w_c \). Take \( \alpha \) as in the previous lemma. Assume that \( s_\alpha w = w_G w_c w_Q \). Since \( s_\alpha w_G w_c = w w_Q < s_\alpha w w_Q = w_G w_c \), we have \( (w_G w_c)^{-1}(\alpha) < 0 \). Put \( \alpha' = -w_G^{-1}(\alpha) \in \Delta \). Then we have \( w_c^{-1}(\alpha') > 0 \). Hence \( \alpha' \in \Delta_Q \). Put \( \beta = -w_Q(\alpha') \in \Delta_Q \) and we prove \( w_\beta \leq w_G w_c \). We have \( w_\beta = s_\alpha w_G w_c w_Q s_\beta = w_G s_\alpha w_c w_Q s_\beta \). Hence it is sufficient to prove that \( s_\alpha w_c w_Q s_\beta \geq w_Q \).

We have \( \Delta_{w_G s_\alpha} = w_G s_\alpha \). In particular, \( s_\alpha w \in W_0^Q \). Hence \( \ell(s_\alpha w_c w_Q s_\beta) = \ell(s_\alpha w_c) + \ell(w_Q s_\beta) \) as \( w_Q, s_\beta \in W_0 Q \). Since \( \alpha' \in \Delta_Q \), \( s_\alpha' \in W_{0,Q} \subset W_0^Q \). Hence we have \( \ell(s_\alpha w_c) = \ell(s_\alpha') + \ell(w_c) \). Therefore we get \( \ell(s_\alpha w_c w_Q s_\beta) = \ell(s_\alpha') + \ell(w_c) + \ell(w_Q s_\beta) \).

Hence \( s_\alpha w_c w_Q s_\beta \geq s_\alpha w_Q s_\beta = w_Q \).

Finally assume that \( s_\alpha w \neq w_G w_c \) and we prove the lemma by backward induction on \( \ell(w w_Q) \). By inductive hypothesis, there exists \( \beta \in \Delta_Q \) such that \( s_\alpha w s_\beta \leq w_G w_Q \). Since \( s_\alpha w \in W_0^Q \), we have \( s_\alpha w s_\beta > s_\alpha w \). Therefore we have \( s_\alpha w s_\beta > s_\alpha w > w \). By property \( Z(w_\beta, s_\alpha w_\beta, s_\beta) \) \( \Delta_\beta Q \), \( \Delta Q \), \( Q_1 \), \( Q_2 \) be the parabolic subgroup generated by \( Q_1 \) and \( Q_2 \). Note that \( \Delta_{Q_1 Q_2} = \Delta Q_1 \cup \Delta Q_2 \).

Lemma 2.15. Let \( P \) be a parabolic subgroup and let \( \sigma \) be an \( H_P \)-module. Assume that \( P(\sigma) = G \). For \( \Pi_1, \Pi_2 \) subsets of \( \{ Q \mid Q \supset P \} \), we have

\[
\left( \sum_{Q_1 \in \Pi_1} I_{Q_1}(e_{Q_1}(\sigma)) \right) \cap \left( \sum_{Q_2 \in \Pi_2} I_{Q_2}(e_{Q_2}(\sigma)) \right) = \sum_{Q_1 \in \Pi_1, Q_2 \in \Pi_2} I_{(Q_1, Q_2)}(e_{(Q_1, Q_2)}(\sigma))
\]

in \( I_P(\sigma) \).

Proof. Let \( \Pi_2 \) be the parabolic subgroup corresponding to \( \Delta \setminus \Delta_P \). Note that \( \Delta_P \) is orthogonal to \( \Delta_{P_2} \) as we assumed. Therefore we have \( W_0^P = W_0, P_2 \) for any parabolic subgroup \( Q \) containing \( P \), we have \( W_0, P_2 = W_0^Q W_{0,Q \cap P_2} \).

Put \( I_Q = I_Q(e_Q(\sigma)) \). We prove the lemma by induction on \#\( \Pi_1 \). Assume that \#\( \Pi_1 = 1 \). By [Abe16, Lemma 3.8], it is sufficient to prove that \( I_Q \cap I_Q = I_{(Q_1, Q_2)} \). Obviously we have \( I_Q \cap I_Q \supset I_{(Q_1, Q_2)} \). Let \( \varphi \in I_{Q_1} \cap I_{Q_2} \). Then for \( w \in W_{0(Q_1 Q_2)} \), we have \( \varphi(T_{n_{w \alpha}}) = \varphi(T_{n_w}) e_{Q_1}(\sigma)(T_{n_{w \alpha}})^{Q_1} \). Since \( n_{w \alpha} \in W_{aff, P_2 \cap Q_2} \), we have \( e_{Q_1}(\sigma)(T_{n_{w \alpha}})^{Q_1} = q_{s_\alpha} \). Therefore we have \( \varphi(T_{n_{w \alpha}}) = q_{s_\alpha} \varphi(T_{n_w}) \). This also holds for \( \alpha \in \Delta Q_2 \setminus \Delta_P \). Therefore \( \varphi(T_{n_w}) = q_v \varphi(T_{n_w}) \) for any \( v \) generated by \( \{ s_\alpha \mid \alpha \in (\Delta Q_1 \cup \Delta Q_2) \setminus \Delta_P \} \).
Since $(\Delta_{Q_1} \cup \Delta_{Q_2}) \setminus \Delta_P = \Delta_{\langle Q_1, Q_2 \rangle \cap P_2}$, the group generated by $\{s_\alpha \mid \alpha \in (\Delta_{Q_1} \cup \Delta_{Q_2}) \setminus \Delta_P\}$ is $W_{\langle Q_1, Q_2 \rangle \cap P_2}$. Hence $\varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}}) = q_v \varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}})$ for any $w \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$ and $v \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$. Define $\varphi' \in I_{\langle Q_1, Q_2 \rangle}(e_{\langle Q_1, Q_2 \rangle}(\sigma))$ by $\varphi'(T_w) = \varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}})$ for any $w \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$. (Such an element uniquely exists by Proposition 2.4.) Then $\varphi'$ also satisfies $\varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}}) = q_v \varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}})$ for any $w \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$ and $v \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$. Hence $\varphi(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}}) = \varphi'(T_{w_{\langle Q_1, Q_2 \rangle \cap P_2}})$ for any $w \in W_{\langle Q_1, Q_2 \rangle \cap P_2}$. By Proposition 2.4, $\varphi = \varphi' \in I_{\langle Q_1, Q_2 \rangle}$.

Now we prove the general case. Obviously we have

$$\left( \sum_{Q_1 \in P_1} I_{Q_1} \right) \cap \left( \sum_{Q_2 \in P_2} I_{Q_2} \right) \supseteq \sum_{Q_1 \in P_1, Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle}.$$  

We prove the reverse inclusion. Take $f$ from the left hand side. Fix $Q_0 \in P_1$ and put $P_1' = P_1 \setminus \{Q_0\}$. Take $f_1 \in I_{Q_0}$ and $f_2 \in \sum_{Q_1 \in P_1'} I_{Q_1}$ such that $f = f_1 + f_2$. Then we have

$$f_2 \in \left( \sum_{Q_2 \in P_2} I_{Q_2} + I_{Q_0} \right) \cap \sum_{Q_1 \in P_1'} I_{Q_1}.$$  

By inductive hypothesis, the right hand side is

$$\sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle} + \sum_{Q_1 \in P_1'} I_{\langle Q_0, Q_1 \rangle}.$$  

Since $I_{\langle Q_0, Q_1 \rangle} \subseteq I_{Q_0}$, we get

$$f_2 \in \sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle} + I_{Q_0}.$$  

We have $f_1 \in I_{Q_0}$. Therefore

$$f = f_1 + f_2 \in \sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle} + I_{Q_0}.$$  

Take $f_1' \in \sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle}$ and $f_2' \in I_{Q_0}$ such that $f = f_1' + f_2'$. Then $f_1' \in \sum_{Q_1 \in P_1'} I_{Q_1}$ and $f_2' \in I_{Q_0}$. By the assumption, $f \in \sum_{Q_2 \in P_2} I_{Q_2}$. Therefore we have $f_2' = f - f_1' \in \sum_{Q_2 \in P_2} I_{Q_2}$. Hence

$$f_2' \in I_{Q_0} \cap \sum_{Q_2 \in P_2} I_{Q_2} = \sum_{Q_2 \in P_2} I_{\langle Q_0, Q_2 \rangle}.$$  

Here we use the lemma for $P_1 = \{Q_0\}$. Hence

$$f = f_1' + f_2' \subseteq \sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle} + \sum_{Q_2 \in P_2} I_{\langle Q_0, Q_2 \rangle}$$

$$= \sum_{Q_1 \in P_1', Q_2 \in P_2} I_{\langle Q_1, Q_2 \rangle}.$$  

We get the lemma. \qed
2.15. Assumption on $C$. In the rest of this paper, we always assume that $p = 0$ in $C$ unless otherwise stated since almost all results in this paper is proved only under this assumption. Since $q_p$ is a power of $p$, this assumption implies $q_w = 0$ for any $w \in W(1)$ such that $\ell(w) > 0$. When we discuss simple modules, we also assume that $C$ is a field. Such assumptions are written at the top of the subsections or in the statement of the theorems.

3. Twist

We define an involution $\iota = \iota_G: \mathcal{H} \to \mathcal{H}$ by $\iota(T_w) = (-1)^{\ell(w)}T_w$ [Vig16 Proposition 4.23] and $\pi^\iota = \pi \circ \iota$ for an $\mathcal{H}$-module $\pi$. Obviously, $\pi^\iota$ is simple if $\pi$ is simple. In this section, we calculate $\pi^\iota$ for simple modules $\pi$.

3.1. Parabolic induction. Let $P$ be a parabolic subgroup. Then by [Abe16 Lemma 4.2], we have $(\sigma^P)_{\ell - \ell_P} = (\sigma_{\ell - \ell_P})^P$. We denote this module by $\sigma_{\ell - \ell_P}^P$. We have $I_P(\sigma)^\iota \simeq I'_P(\sigma_{\ell - \ell_P}^P)$ [Abe16 Proposition 4.11]. In this subsection, we prove the following proposition.

**Proposition 3.1.** There exists a homomorphism $\Phi: I_P \to I'_P$ which is characterized by $\Phi(\varphi)(XT_{n_{wGwP}}) = \varphi(XT_{n_{wGwP}})$ for any $X \in \mathcal{H}$ and $\varphi \in I_P$.

**Remark 3.2.** Assume that an $\mathcal{H}_P$-module $\sigma$ is not zero and $\varphi \in I_P(\sigma)$. Then we have $\Phi(\varphi)(T_{n_{wGwP}}) = \varphi(T_{n_{wGwP}})$. Since $I_P(\sigma) \to \sigma$ defined by $\varphi \mapsto \varphi(T_{n_{wGwP}})$ is surjective by Proposition 2.4, there exists $\varphi \in I_P(\sigma)$ such that $\varphi(T_{n_{wGwP}}) \neq 0$. Hence $\Phi(\varphi) \neq 0$. Therefore if $\sigma$ is not zero, then $\Phi \neq 0$.

The following corollary is the first step to calculate $\pi^\iota$ for a simple $\mathcal{H}$-module $\pi$.

**Corollary 3.3.** Assume that $\sigma$ is a field. Let $\sigma$ be an $\mathcal{H}_P$-module. The representation $I_P(\sigma)^\iota$ is simple if and only if $I_P(\sigma)^\iota$ is simple. Moreover, if it is the case, then $I_P(\sigma)^\iota \simeq I'_P(\sigma)$. Therefore if $I_P(\sigma)^\iota$ is simple, then $I_P(\sigma)^\iota \simeq I'_P(\sigma_{\ell - \ell_P})$.

**Proof.** If $I_P(\sigma)$ is simple, then the homomorphism in Proposition 3.1 is injective. We prove that $\dim I_P(\sigma) = \dim I'_P(\sigma) < \infty$. Since $I_P(\sigma)$ is simple, $\sigma$ is also simple. Hence it is finite-dimensional. By $I_P(\sigma) \simeq \bigoplus_{w \in W_0^P} \sigma$ (Proposition 2.4), we have $\dim I_P(\sigma) = \#W_0^P \dim \sigma$. We also have $\dim I'_P(\sigma) = \#W_0^P \dim \sigma$ by [Abe16 Proposition 4.12].

For the proof of Proposition 3.1 by [Abe16 Proposition 4.13], it is sufficient to prove the following lemma.

**Lemma 3.4.** Put $P' = n_{wGwP}P^{P_{\text{op}}P_{\text{op}}^{-1}}_{wGwP}$. The map $\varphi \mapsto (X \mapsto \varphi(XT_{n_{wGwP}}))$ gives a homomorphism

$$I_P(\sigma) \to \text{Hom}_{(\mathcal{H}_P, j_{P'}^+)}(\mathcal{H}, n_{wGwP} \sigma).$$

**Proof.** Set $n = n_{wGwP}$ and we prove $I_P(\sigma) \ni \varphi \mapsto \varphi(T_n) \in n \sigma$ is an $(\mathcal{H}_P, j_{P'}^+)$-module homomorphism. Let $w \in W_P(1)$ be a $P'$-positive element. By (4.1) in [Abe16], we have $j_{P'}^+(E_{0+,P'}^{P'}(w))T_n = T_n j_{P'}(E_{0+,P'}^P(n^{-1}wn))$. Hence

$$\varphi j_{P'}^+(E_{0+,P'}^P(w))(T_n) = \varphi(T_n j_{P'}(E_{0+,P'}^P(n^{-1}wn))) = \varphi(T_n j_{P'}(E_{0+,P'}^P(wn))).$$

We prove the following claim. From this claim, $(\varphi j_{P'}^+(E_{0+,P'}^P(w)))(T_n)$ only depends on $\varphi(T_n)$ and $w$. 

Claim. Let $\varphi \in I_P(\sigma)$ and $X \in \mathcal{H}$. Then $\varphi(T_n X)$ only depends on $\varphi(T_n)$ and $X$.

We introduce a basis defined by

$$E_-(n w \lambda) = q_{n w \lambda}^{1/2} q_{n w \lambda}^{-1/2} q_{\lambda}^{-1/2} T_{n w}^* E_{o_-(\lambda)}$$

for $w \in W_0$ and $\lambda \in \Lambda(1)$. By [Abe] Lemma 4.2, $\{E_-(w) \mid w \in W(1)\}$ is a $C$-basis of $\mathcal{H}$.

To prove the claim, we may assume $X = E_-(n w \lambda)$ for $w \in W_0$ and $\lambda \in \Lambda(1)$. Take $\lambda \lambda P \in \Lambda(1)$ as in Proposition 2.22 such that $\lambda \lambda P$ is $P$-negative. Then we have

$$\varphi(T_n E_-(n w \lambda)) = \varphi(T_n E_-(n w \lambda) E_{o_-(\lambda \lambda P)}) \sigma(E_{o_-,P}(\lambda \lambda P))^{-1}$$

by [Abe] Lemma 2.6. If $\ell(n w \lambda) + \ell(\lambda \lambda P) > \ell(n w \lambda \lambda \lambda P)$, then $E_-(n w \lambda) E_{o_-(\lambda \lambda P)} = 0$. Hence we have $\varphi(T_n E_-(n w \lambda)) = 0$, so we get the claim. If $\ell(n w \lambda) + \ell(\lambda \lambda P) = \ell(n w \lambda \lambda \lambda P)$, then $E_-(n w \lambda) E_{o_-(\lambda \lambda P)} = E_-(n w \lambda \lambda \lambda P)$. Let $w_1 \in W^P_0$ and $w_2 \in W_{0,P}$ such that $w = w_1 w_2$. Then $n w_2 \lambda \lambda P \in W_P(1)$ is $P$-negative. Hence $\ell(n w_2 \lambda \lambda P) = \ell(n w_1) + \ell(n w_2 \lambda \lambda P)$ by [Abe] Lemma 2.18. Therefore we have $E_-(n w \lambda \lambda P) = T_{n w_1}^* E_-(n w \lambda \lambda \lambda P)$. If $w_1 \neq 1$, then $w_1 \notin W_{0,P}$. Hence in a reduced expression of $w_1$, a simple reflection $s_\alpha$ for some $\alpha \in \Delta \setminus \Delta_P$ appears. Therefore, there exists $x \in W_{0,P}, \alpha \in \Delta \setminus \Delta_P$, and $y \in W_0$ such that $w_1 = x s_\alpha y$ and $\ell(w_1) = \ell(x) + \ell(s_\alpha) + \ell(y)$. Since $x \in W_{0,P}, x(\alpha) \in \Sigma^+ \setminus \Sigma_P^+$. Hence $wp x(\alpha) > 0$. Therefore $w_G w_P y(\alpha) < 0$. Hence $n n_x s_\alpha < n n_x$. Therefore we have

$$\ell(n) + \ell(w_1) = \ell(w_G w_P) + \ell(x) + \ell(s_\alpha) + \ell(y)$$

$$\geq \ell(w_G w_P x s_\alpha) + \ell(y)$$

$$\geq \ell(w_G w_P x s_\alpha y) = \ell(n n_w).$$

Hence $T_{n w_1}^* = E_{o_+,n^{-1}}(n) E_{o_+}(n w_1) = 0$ by [Vig16] Example 5.22 and (2.1). Therefore, if $w_1 \neq 1$, namely, $w \notin W_{0,P}$, then we have $\varphi(T_n E_-(n w \lambda \lambda \lambda P)) = \varphi(T_n T_{n w_1}^* E_-(n w_2 \lambda \lambda \lambda P)) = 0$ again. If $w \in W_{0,P}$, then $n w \lambda \lambda \lambda P$ is a $P$-negative element. Hence $E_{o_+}^P(n w \lambda \lambda P) \in \mathcal{H}^P$ and we have $E_-(n w \lambda \lambda P) = j_P^P(E_{o_+}^P(n w \lambda \lambda P))$ by [Abe] Lemma 4.6. Therefore we have

$$\varphi(T_n E_-(n w \lambda)) = \varphi(T_n E_-(n w \lambda \lambda \lambda P)) \sigma(E_{o_-,P}^P(\lambda \lambda P))^{-1}$$

$$= \varphi(T_n) \sigma(E_{o_+}^P(n w \lambda \lambda \lambda P) E_{o_-,P}^P(\lambda \lambda P))^{-1}$$

$$= \varphi(T_n) \sigma(E_{o_+}^P(n w \lambda)).$$

The claim is proved.

Let $\varphi_0 \in I_P(\sigma)$ be such that $\varphi_0(T_n) = \varphi(T_n)$ and $\varphi_0(T_{n_0}) = 0$ for $v \in W_0^P \setminus \{w_G w_P\}$. Then, as a consequence of the claim, we have

$$(\varphi j_P^+ (E_{o_+}^P(v))(w))(T_n) = (\varphi_0 j_P^+ (E_{o_+}^P(v))(w))(T_n).$$

By the proof of [Abe] Proposition 4.14, we have

$$(\varphi_0 j_P^+ (E_{o_+}^P(v))(w))(T_n) = \varphi_0(T_n)(n\sigma)(E_{o_+}^P(v)(w)).$$
Since \( \varphi_0(T_n) = \varphi(T_n) \), we have
\[
\varphi_0(T_n)(n\sigma)(E_{\alpha_+}^{P'}(w)) = \varphi(T_n)(n\sigma)(E_{\alpha_+}^{P'}(w)).
\]
Hence
\[
(\varphi j_{P'}^+(E_{\alpha_+}^{P'}(w)))(T_n) = \varphi(T_n)(n\sigma)(E_{\alpha_+}^{P'}(w)).
\]
We get the lemma.

Here is the compatibility with the transitivity of \( I_P \) and \( I'_P \) \cite[Proposition 4.12]{Abe16}.

**Lemma 3.5.** Let \( Q \supset P \) be a parabolic subgroup. Then the following three maps are equal:

1. \( I_P \rightarrow I'_P \).
2. \( I'_P = I_Q \circ I'_P \rightarrow I'_Q \circ I'_P \rightarrow I'_Q \circ I'^Q_P = I'_P \).
3. \( I_P = I_Q \circ I'_P \rightarrow I_Q \circ I'^Q_P \rightarrow I'_Q \circ I'^Q_P = I'_P \).

**Proof.** Let \( \sigma \) be an \( H_P \)-module. In each case, let \( \varphi \in I_P(\sigma) \) and \( \psi_i \in I'_P(\sigma) \) be the image of \( \varphi \) by the map in (i) for \( i = 1, 2, 3 \). Then \( \psi_i \) is characterized by \( \varphi(XT_{n_Gw_{Q_P}}) = \psi_1(XT_{n_Gw_{Q_P}}) \) for any \( X \in H \). We denote the corresponding element to \( \varphi \in I_P(\sigma) \) (resp., \( \psi_i \in I'_P(\sigma) \)) by \( \varphi' \in (I_Q \circ I'_P)(\sigma) \) (resp., \( \psi'_i \in (I'_Q \circ I'^Q_P)(\sigma) \)).

We consider \( \psi_2 \). We have \( \psi_2(XT_{n_Gw_{Q_P}}) = \psi'_2(XT_{n_Gw_{Q_{P'}}})(1) \). Since \( \ell(w_Gw_P) = \ell(w_Gw_Q) + \ell(w_Qw_P) \), we have \( T_{n_Gw_P} = T_{n_Gw_Q}T_{n_Qw_{P'}} \). Hence \( \psi'_2(XT_{n_Gw_{Q_{P'}}})(1) = \psi'_2(XT_{n_Gw_Q}T_{n_Qw_{P'}})(1) \).

Since \( T_{n_Gw_{Q_{P'}}} \in H_{Q'} \) and \( j_Q^{-1}(T_{n_Qw_{P'}}) = T_{n_Qw_{P'}} \) \cite[Lemma 2.6]{Abe16}, we have \( \psi'_2(XT_{n_Gw_Q}T_{n_Qw_{P'}})(1) = \psi'_2(XT_{n_Gw_Q})(T_{n_Qw_{P'}})(1) \).

Let \( \psi''_2 \) be the image of \( \varphi \). Then \( \psi''_2(XT_{n_Gw_Q})(Y) = \varphi'(XT_{n_Gw_Q})(Y) \) and \( \psi''_2(X)(YT_{n_Gw_Q}) = \psi'_2(X)(YT_{n_Gw_Q}) \) for \( X \in H \) and \( Y \in H_Q \). Therefore we have \( \psi_2(XT_{n_Gw_{Q_{P'}}})(T_{n_Qw_{P'}})(1) = \varphi'(XT_{n_Gw_{Q_{P'}}})(T_{n_Qw_{P'}})(1) \).

Again, since \( T_{n_Qw_{P'}} \in H_{Q'} \) and \( j_Q^{*}(T_{n_Qw_{P'}}) = T_{n_Qw_{P'}} \) \cite[Lemma 2.6]{Abe16}, we have \( \varphi'(XT_{n_Gw_{Q_{P'}}})(T_{n_Qw_{P'}})(1) = \varphi'(XT_{n_Gw_Q}T_{n_Qw_{P'}})(1) = \varphi'(XT_{n_Gw_{Q_{P'}}})(1) = \varphi(XT_{n_Gw_{Q_{P'}}}) \).

Hence the map in (2) satisfies the characterization of the map in (1).

The proof for (3) is similar. Let \( \psi''_3 \in (I_Q \circ I'^Q_P)(\sigma) \) be the image of \( \varphi \). Then we have
\[
\psi_3(XT_{n_Gw_{Q_{P'}}}) = \psi'_3(XT_{n_Gw_{Q_{P'}}})(1) = \psi''_3(XT_{n_Gw_Q}T_{n_Qw_{P'}})(1) = \psi'_3(XT_{n_Gw_Q})(T_{n_Qw_{P'}}) = \varphi'(XT_{n_Gw_{Q_{P'}}})(T_{n_Qw_{P'}}) = \varphi'(XT_{n_Gw_Q}T_{n_Qw_{P'}})(1) = \varphi'(XT_{n_Gw_{Q_{P'}}})(1) = \varphi(XT_{n_Gw_{Q_{P'}}}).
\]

This finishes the proof.
3.2. Steinberg modules. Next we consider the twist of the generalized Steinberg modules. Until subsection 3.7, we keep the following settings:

- $P$ is a parabolic subgroup such that $\Delta_P$ is orthogonal to $\Delta \setminus \Delta_P$.
- Let $P_2$ be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$.
- $\sigma$ is an $H_P$-module such that $P(\sigma) = G$.

We prove the following proposition.

**Theorem 3.6.** Let $Q$ be a parabolic subgroup containing $P$ and let $Q^c$ be the parabolic subgroup corresponding to $\Delta_P \cup (\Delta \setminus \Delta_Q)$. Then we have $\text{St}_Q(\sigma)^t \simeq \text{St}_{Q^c}(\sigma^{I^P}_{\ell-\ell_P})$.

The proof of this theorem continues until subsection 3.7. In this subsection, we prove this proposition for $Q = P$.

**Lemma 3.7.** **Theorem 3.6** is true if $Q = P$, namely we have $\text{St}_P(\sigma)^t \simeq e_G(\sigma^{I^P}_{\ell-\ell_P})$.

**Proof.** We use [Abe16 Proposition 3.12]. Let $w \in W_P(1)$ and assume that $w$ is $P$-positive. Then $(\text{St}_P(\sigma)^t)(T^*_w) = (-1)^{\ell(w)\text{St}_P(\sigma)}(T_w) = (-1)^{\ell(w)}\sigma(T^*_w) = (-1)^{\ell(w)-\ell_P(\omega)}\sigma^tP(T^*_w) = \sigma^{I^P}_{\ell-\ell_P}(T^*_w)$. On the other hand, for $w \in W_{\text{aff}, P_2}(1)$, we have $(\text{St}_P(\sigma)^t)(T^*_w) = (-1)^{\ell(w)}\text{St}_P(\sigma)(T_w) = 1$. Therefore, by a characterization of the extension, we have $(\text{St}_P(\sigma)^t) \simeq e_G(\sigma^{I^P}_{\ell-\ell_P}) = \text{St}_G(\sigma^{I^P}_{\ell-\ell_P})$. Since $P^c = G$, we get the lemma.

3.3. An exact sequence. We express $\text{St}_{Q^c}(\sigma)$ as the kernel of a certain homomorphism. As a consequence, we deduce Theorem 3.6 from the exactness of a certain sequence (Lemma 3.9).

Let $Q$ be a parabolic subgroup containing $P$. Then we have an exact sequence

$$0 \to \sum_{Q \supset R \supset P} I^Q_R(e_R(\sigma)) \to I^Q_P(\sigma) \to \text{St}^Q_P(\sigma) \to 0.$$

Applying $I_Q$ and using the transitivity [Vig15 Proposition 4.10], we have an exact sequence

$$0 \to \sum_{Q \supset R \supset P} I_R(e_R(\sigma)) \to I_P(\sigma) \to I_Q(\text{St}^Q_P(\sigma)) \to 0.$$

Let $Q_1$ be a parabolic subgroup containing $Q$. Then we have

$$\sum_{Q \supset R \supset P} I_R(e_R(\sigma)) \subset \sum_{Q_1 \supset R \supset P} I_R(e_R(\sigma)).$$

Hence we have the homomorphism $I_Q(\text{St}^Q_P(\sigma)) \to I_{Q_1}(\text{St}^{Q_1}_P(\sigma))$ which makes the following diagram commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \sum_{Q \supset R \supset P} I_R(e_R(\sigma)) & \longrightarrow & I_P(\sigma) & \longrightarrow & I_Q(\text{St}^Q_P(\sigma)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sum_{Q_1 \supset R \supset P} I_R(e_R(\sigma)) & \longrightarrow & I_P(\sigma) & \longrightarrow & I_{Q_1}(\text{St}^{Q_1}_P(\sigma)) & \longrightarrow & 0.
\end{array}
$$
Lemma 3.8. With the above homomorphisms, we have an exact sequence

$$0 \to \text{St}_{Q^c}(\sigma) \to I_Q(\text{St}_{P}^Q(\sigma)) \to \bigoplus_{Q_1 \geq Q} I_{Q_1}(\text{St}_{P}^{Q_1}(\sigma)).$$

We will prove this lemma at the end of this subsection. Applying $\iota$ to the exact sequence in the lemma, we have

$$0 \to \text{St}_{Q^c}(\sigma)^t \to I_Q(\text{St}_{P}^Q(\sigma))^t \to \bigoplus_{Q_1 \geq Q} I_{Q_1}(\text{St}_{P}^{Q_1}(\sigma))^t.$$

Using Lemma 3.7 and [Abe16, Lemma 4.9, Proposition 4.11], we have

$$I_{Q_1}(\text{St}_{P}^{Q_1}(\sigma))^t = I'_{Q_1}(e_{Q_1}(\sigma_{t_{-t_{Q}}}(\sigma))) = I'_{Q_1}(e_{Q_1}(\sigma_{t_{-t_{Q}}})).$$

Therefore Theorem 3.6 follows from the following lemma.

Lemma 3.9. The following sequence is exact:

$$\bigoplus_{Q_1 \geq Q} I_{Q_1}(e_{Q_1}(\sigma)) \to I_Q(e_{Q}(\sigma)) \to I'_{Q}(e_{Q}(\sigma)) \to \bigoplus_{Q_1 \geq Q} I'_{Q_1}(e_{Q_1}(\sigma)).$$

Here $I_Q(e_{Q}(\sigma)) \to I'_{Q}(e_{Q}(\sigma))$ is given in Proposition 3.1 and $I_{Q_1}(e_{Q_1}(\sigma)) \to I_Q(e_{Q}(\sigma))$ is the natural embedding.

The proof of this lemma continues until subsection 3.7. Here is the outline of the proof.

- We determine the kernel of the second map in Lemma 3.10. This implies the exactness at $I_Q(e_{Q}(\sigma))$.
- The kernel of the third map is given in Proposition 3.13. This follows from an explicit description of the map (Proposition 3.13).
- In subsection 3.6, we prove that the kernel of the third map contains the image of the second map using a result in [Abe16] and the reverse inclusion is in Lemma 3.16. This gives the exactness at $I'_{Q}(e_{Q}(\sigma))$.

As the end of this subsection, we prove Lemma 3.8. For a family of parabgolic subgroups $\{P_{\lambda}\}$, we denote the parabolic subgroup generated by $\{P_{\lambda}\}$ by $\langle P_{\lambda} \rangle$. In other words, $\langle P_{\lambda} \rangle_{\lambda}$ is the parabolic subgroup corresponding to $\bigcup_{\lambda} \Delta_{P_{\lambda}}$. Since this lemma is true over any commutative ring, we assume that $C$ is any commutative ring in the proof.

Proof of Lemma 3.8. Let $Q_1 \geq Q$. By the exact sequence before Lemma 3.8, the kernel of $I_P(\sigma) \to I_Q(\text{St}_{P}^Q(\sigma)) \to I_{Q_1}(\text{St}_{P}^{Q_1}(\sigma))$ is $\sum_{Q_1 \geq R \geq P} I_R(e_{R}(\sigma))$. Since
First we prove (1). We prove the reverse inclusion. By Lemma 2.15, we have $\Delta_Q \supseteq \{\alpha\}$, and we prove $\Delta_R = \bigcup_{Q \supseteq R} \Delta_Q$. Hence $\Delta_R \subseteq \Delta_{Q^c}$. Since $\Delta_R \subseteq \Delta_{Q^c}$. Hence $\Delta_R \subseteq \Delta_{Q^c}$, and we prove $\Delta_R(\Delta_R \subseteq \Delta_{Q^c}) \subseteq I_R(e_R(\sigma)) \subseteq I_R(e_R(\sigma))$. We get (1).

We prove the reverse inclusion. By Lemma 2.15, we have

$$A = \sum_{(R_{Q_1}, Q_1 \supseteq Q)} I_{(R_{Q_1}, Q_1)}(e_{(R_{Q_1}, Q_1)}(\sigma)),$$

where $R_{Q_1}$ satisfies $Q_1 \supset R_{Q_1} \supseteq P$ and $\langle R_{Q_1} \rangle_{Q_1}$ is the group generated by $\{R_{Q_1} \mid Q_1 \supseteq Q\}$. Hence it is sufficient to prove that each $I_{(R_{Q_1}, Q_1)}(e_{(R_{Q_1}, Q_1)}(\sigma))$ is contained in $I_Q(e_Q(\sigma)) + B$. If $Q \supseteq R_{Q_0}$, for some $Q_0 \supseteq Q$, then for such $Q_0$, we have $I_{(R_{Q_1}, Q_1)}(e_{(R_{Q_1}, Q_1)}(\sigma)) \subseteq I_{R_{Q_0}}(e_{R_{Q_0}}(\sigma)) \subseteq B$. Assume that $Q \supseteq R_{Q_1}$ for any $Q_1 \supseteq Q$, and we prove $\Delta_{Q^c} \subseteq \bigcup_{Q_1 \supseteq Q} \Delta_{R_{Q_1}}$. Let $\alpha \in \Delta_{Q^c} = (\Delta_Q \setminus \Delta_{Q^c}) \cup \Delta_P$. If $\alpha \in \Delta_P$, then we have $\alpha \in \bigcup_{Q_1 \supseteq Q} \Delta_{R_{Q_1}}$. Assume that $\alpha \in \Delta_Q \setminus \Delta_{Q^c}$. Let $Q_\alpha$ be the parabolic subgroup corresponding to $\Delta_Q \cup \{\alpha\}$. Then by the assumption, $\Delta_{R_{Q_\alpha}}$ is contained in $\Delta_{Q_\alpha} = \Delta_Q \cup \{\alpha\}$ and is not contained in $\Delta_{Q^c}$. Hence $\alpha \in \Delta_{R_{Q_\alpha}}$. Therefore $\alpha \in \Delta_{R_{Q_\alpha}} \subseteq \bigcup_{Q_1 \supseteq Q} \Delta_{R_{Q_1}} = \Delta_{(R_{Q_1}, Q_1)}$ by taking $Q_1 = Q_\alpha$. Hence we have $Q_\alpha \subseteq \langle R_{Q_1} \rangle_{Q_1}$. Therefore $I_{(R_{Q_1}, Q_1)}(e_{(R_{Q_1}, Q_1)}(\sigma)) \subseteq I_{Q^c}(e_{Q^c}(\sigma))$. We get (1).

We prove (2). By Lemma 2.15, we have

$$I_{Q^c}(e_{Q^c}(\sigma)) \cap B = \sum_{Q \supseteq R \supseteq P} I_{(R, Q^c)}(e_{(R, Q^c)}(\sigma)).$$

First we prove $I_{Q^c}(e_{Q^c}(\sigma)) \cap B \subseteq \sum_{R \supseteq Q_1 Q} I_{R_1}(e_{R_1}(\sigma))$, namely, for each $R$ such that $Q \supseteq R \supseteq P$ we have $I_{(R, Q^c)}(e_{(R, Q^c)}(\sigma)) \subseteq \sum_{R_1 \supseteq Q_1} I_{R_1}(e_{R_1}(\sigma))$. For such $R$, we can take $\alpha \in \Delta_R \setminus \Delta_P$. Since $\alpha \in \Delta_Q \setminus \Delta_P$, $\alpha \notin \Delta_{Q^c}$. Hence $\Delta_{(R, Q^c)} = \Delta_R \cup \Delta_{Q^c} \supseteq \{\alpha\} \cup \Delta_{Q^c} \subseteq \Delta_{Q^c}$. Therefore $(R, Q^c) \supseteq Q^c$. Hence $I_{(R, Q^c)}(e_{(R, Q^c)}(\sigma)) \subseteq \sum_{R_1 \supseteq Q^c} I_{R_1}(e_{R_1}(\sigma))$ by taking $R_1 = \langle R, Q^c \rangle$. 

We now prove $\sum_{Q \supseteq R \supseteq P} I_{(R, Q^c)}(e_{(R, Q^c)}(\sigma)) \subseteq \sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma))$. We have $\sum_{Q \supseteq R \supseteq P} I_{(R, Q^c)}(e_{(R, Q^c)}(\sigma)) \subseteq \sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma))$ by taking $Q^c = \{\alpha\}$. 

We now prove $\sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma)) \subseteq \sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma))$. We have $\sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma)) \subseteq \sum_{Q \supseteq R \supseteq P} I_{(R, Q)}(e_{(R, Q)}(\sigma))$ by taking $Q = \{\alpha\}$.
We prove $I_R(e_R(\sigma)) \subset I_Q(e_Q(\sigma)) \cap B$ for any $R$ such that $R \supseteq Q^c$. We can take $\alpha \in \Delta_R \setminus \Delta_Q$. Let $P_\alpha$ be the parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Since $\alpha \notin \Delta_Q$, we have $\alpha \in \Delta_Q$. Therefore $Q \supset P_\alpha \supset P$. Hence $\Delta_R \supset \Delta_Q^c \cup \{\alpha\} = \Delta(P_\alpha, Q^c)$. Therefore $R \supset \langle P_\alpha, Q^c \rangle$. Hence $I_R(e_R(\sigma)) \subset I_{\langle P_\alpha, Q^c \rangle}(e_{\langle P_\alpha, Q^c \rangle}(\sigma)) \subset I_Q(e_Q(\sigma)) \cap B$. We get (2) and the proof of the lemma is finished. \hfill \square

3.4. **The kernel of $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))$.** Recall that we put $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$ for $w \in W_0$. We determine $\ker(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$, namely we prove the following lemma.

**Lemma 3.10.** Set $A = \{w \in W_0^Q \mid \Delta_w = \Delta_Q\}$. Then we have

$$\ker(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$$

is stable under $H$. Therefore $\varphi(XT_{n_w}) = 0$ for any $X \in H$ and $w \in A$. We get the lemma.

Let $\varphi \in I_Q(e_Q(\sigma))$ such that $\varphi(T_{n_w}) = 0$ for any $w \in A$ and take $X \in H$. The last equality implies that the set of such $\varphi$ is stable under $H$. Hence $\varphi X$ also satisfies the same condition. Therefore $\varphi(XT_{n_w}) = 0$ for any $w \in A$. Namely, we get the second equality.

Let $\varphi$ be the image of $\varphi$ under $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))$. Then $\varphi$ is characterized by $\varphi(XT_{n_{wGw\sigma}Q}) = \varphi(XT_{n_{wGw\sigma}Q})$. Therefore we have

$$\ker(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$$

is stable under $H$. Therefore, to prove the lemma, it is sufficient to prove the following lemma.

**Lemma 3.12.** Let $\varphi \in I_Q(e_Q(\sigma))$. Assume that $\varphi(XT_{n_{wGw\sigma}Q}) = 0$ for any $X \in H$. Then we have $\varphi(XT_{n_w}) = 0$ for any $X \in H$ and $w \in W_0^Q$ such that $\Delta_w = \Delta_Q$.

**Proof.** We prove the lemma by backward induction on $\ell(w)$. If $w \neq w_{Gw\sigma}Q$, then there exists $\alpha \in \Delta$ such that $s_\alpha w > w$, $\Delta_w = \Delta_{s_\alpha w}$, and $w^{-1}(\alpha)$ is not simple [Abe Lemma 3.15]. Set $s = s_\alpha$. Since $\Delta_{sw} = \Delta_w = \Delta_Q$, we have $sw \in W_0^Q$. If $w^{-1}(\alpha) \in \Sigma_Q^+$, then since $sw \in W_0^Q$, we have $-\alpha = sw(w^{-1}(\alpha)) \in \Sigma^+$. This is a contradiction. Hence

$$w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_Q^+.$$ 

Take $\lambda^-_p \in Z(W_p(1))$ as in Proposition [2.2]. Put $\lambda = n_w \cdot (\lambda^-_p)^2$. We prove the following.

**Claim.** $E_-(\lambda n_s^{-1})(T_{n_s} - c_{n_s}) = E_{o_-}(\lambda)$ in $H$. \hfill \square
We calculate the left hand side in $H[q_s^{±1}]$. We use notation in [Abe Lemma 2.10]. Since $w^{-1}(α) ∈ \Sigma^+ \setminus \Sigma_Q \subset \Sigma^+ \setminus \Sigma_P$, we have $⟨α, ν(λ)⟩ = (w^{-1}(α), ν((λ_P)^{-2})) > 0$. Therefore we have $ℓ(λ n_s^{-1}) = ℓ(λ) - 1$ by [Abe16 Lemma 2.17]. Hence $q_λ n_s^{-1} = q_λ q_{n_s}^{-1}$. Therefore we have

\[ E_-(λ n_s^{-1}) = E_-(n_s^{-1}(n_s \cdot λ)) = q_{λ n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} \theta(n_s \cdot λ) = q_{λ n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} \theta(n_s \cdot λ). \]

Hence by [Abe Lemma 2.10], we have

\[ E_-(λ n_s^{-1})T_{n_s} = q_{λ n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} \theta(n_s \cdot λ) + \sum_{k=0}^{⟨α, ν(λ)-1⟩} q_{λ n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} \theta(n_s \cdot λ μ_{n_s}(k)) c_{n_s,k}. \]

We have

\[ q_{λ n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} \theta(λ) = q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ) = E_{α-}(λ) \]

and if $k = 0$, since $q_{λ n_s^{-1}} = q_λ q_{n_s}^{-1}$, we have

\[ q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ μ_{n_s}(k)) c_{n_s,k} = q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ) c_{n_s} = q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ) c_{n_s} = E_-(n_s^{-1}(n_s \cdot λ)) c_{n_s} = E_-(λ n_s^{-1}) c_{n_s}. \]

We prove that if $1 ≤ k ≤ ⟨α, ν(λ)-1⟩$, then $q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ μ_{n_s}(k)) c_{n_s,k} = 0$ in $H$. We have

\[ q_{λ}^{1/2} q_{n_s}^{-1} T_{n_s}^{n_s} \theta(λ μ_{n_s}(k)) c_{n_s,k} = q_{λ}^{1/2} q_{n_s}^{-1/2} T_{n_s}^{n_s} E_-(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) c_{n_s,k} = q_{λ}^{1/2} q_{n_s}^{-1} E_-(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) c_{n_s,k}. \]

Hence (3.1) is an expansion of $E_-(λ n_s^{-1})T_{n_s}$ with respect to the basis $\{E_-(w) \mid w ∈ W(1)\}$. Since this is a basis of $H[q_s]$ as a $C[q_s]$-module, each coefficient is in $C[q_s]$. Hence $q_{λ}^{1/2} q_{n_s}^{-1} q_{n_s}^{-1}(n_s \cdot λ μ_{n_s}(k)) q_{n_s}^{-1/2} E_-(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) c_{n_s,k}.$

Namely, for each $s ∈ S_{aff}$ there exists $k_s ∈ \mathbb{Z}_{≥0}/\sim$ (where the equivalence relation $\sim$ is defined by the adjoint action of $W$ on $S_{aff}$) such that $q_{λ}^{1/2} q_{n_s}^{-1/2}(n_s \cdot λ μ_{n_s}(k)) q_{n_s}^{-1/2} E_-(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) c_{n_s,k}.$

We have

\[ \sum_s k_s = (1/2)(ℓ(λ) - ℓ(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) - ℓ(n_s)) \]

\[ = ℓ(λ) - ℓ(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) - ℓ(n_s) ≥ ℓ(λ) - ℓ(n_s^{-1}(n_s \cdot λ μ_{n_s}(k))) - ℓ(n_s) = ℓ(λ) - ℓ(n_s \cdot λ μ_{n_s}(k)) - 2. \]

By [Abe Lemma 2.12], $ℓ(λ) - ℓ(n_s \cdot λ μ_{n_s}(k)) ≥ 2 \min\{k, ⟨α, ν(λ)-1⟩ - k\}$. If the equality holds, again by [Abe Lemma 2.12], there exists $v ∈ W(0)$ such that $vν(λ)$ is dominant and $v(α)$ is simple.

Assume that $v(ν(λ))$ is dominant for $v ∈ W_0$. We have $v(ν(λ)) = w(ν((λ_P)^{-2}))$ and since $ν((λ_P)^{-2})$ is dominant, we have $vw ∈ \text{Stab}_{W_0}(ν((λ_P)^{-2}))$. By the condition of $λ_P$, the stabilizer of $ν((λ_P)^{-2})$ is $W_{0,P}$. Hence $vw ∈ W_{0,P}$. Since $w^{-1}(α) ∈ \Sigma^+ \setminus \Sigma_Q^+$, we have $w^{-1}(α) ∈ \Sigma_{P_2}^+$. Any element in $Σ_{P_2}$ is fixed by elements in $W_{0,P}$. Hence
Lemma 2.8, we have $\varphi$. Therefore $v(\alpha) = w^{-1}(\alpha)$. This is not simple by the condition on $\alpha$.

Hence we always have

$$\ell(\lambda) - \ell(n_s^{-1}(n_s \cdot \lambda\mu_n_s(k))) - \ell(n_s) > 0$$

for $1 \leq k \leq \langle \alpha, \nu(\lambda) \rangle - 1$. Hence $\sum_s k_s > 0$. Therefore there exists $s$ such that $k_s > 0$. Hence $\prod_s q_{s_k} = 0$ in $\mathcal{H}$. We get $q_1^{1/2} q_n^{-1} T_{n_s}^* \theta(n_s \cdot \lambda\mu_n_s(k)) c_{n_s,k} = 0$. Therefore we have

$$E_{-}(\lambda n_s^{-1}) T_{n_s} = E_{o_{-}}(\lambda) + E_{-}(\lambda n_s^{-1}) c_{n_s}.$$ 

This gives the claim.

We return to the proof of the lemma. Since $n_w^{-1} \cdot \lambda = (\lambda P)^{-1}$ is $P$-negative, by Lemma 2.8 we have

$$\varphi(X E_{o_{-}}(\lambda) T_{n_w}) = \varphi(X T_{n_w}) \sigma(E_{o_{-}}(\lambda n_w^{-1} \cdot \lambda)).$$

Since $\sigma(E_{o_{-}}(\lambda n_w^{-1} \cdot \lambda)) = \sigma(E_{o_{-}}((\lambda P)^{-1}))$ is invertible, it is sufficient to prove that $\varphi(X E_{o_{-}}(\lambda) T_{n_w}) = 0$. By the claim, we have

$$\varphi(X E_{o_{-}}(\lambda) T_{n_w}) = \varphi(X E_{-}(\lambda n_s^{-1}) T_{n_s} T_{n_w}) - \varphi(X E_{-}(\lambda n_s^{-1}) c_{n_s} T_{n_w}).$$

By inductive hypothesis, $\varphi(X E_{-}(\lambda n_s^{-1}) T_{n_s} T_{n_w}) = \varphi(X E_{-}(\lambda n_s^{-1}) T_{n_{s_w}}) = 0$. We have

$$\varphi(X E_{o_{-}}(\lambda n_s^{-1}) c_{n_s} T_{n_w}) = \varphi(X ((\lambda n_s^{-1}) \cdot c_{n_s}) E_{-}(\lambda n_s^{-1}) T_{n_w})$$

Set $\lambda' = n_w \cdot \lambda P$. We have $\ell(n_s^{-1}) = \ell(\lambda) - 1 = \ell((\lambda P)^{-1}) - 1 = 2\ell(\lambda') - 1$ as $\langle \alpha, \nu(\lambda) \rangle > 0$. Since $\langle \alpha, \nu(\lambda') \rangle = \langle w^{-1}(\alpha), \nu(\lambda P) \rangle > 0$, we have $\ell(\lambda') - 1 = \ell(\lambda' n_s^{-1})$.

By Lemma 2.15, we have $\ell(\lambda') = \ell(n_s \cdot \lambda')$. Hence $\ell(\lambda n_s^{-1}) = \ell(\lambda' n_s^{-1}) + \ell(n_s \cdot \lambda')$. Therefore

$$E_{-}(\lambda n_s^{-1}) = E_{-}(\lambda' n_s^{-1} (n_s \cdot \lambda')) = E_{-}(\lambda' n_s^{-1}) E_{o_{-}}(n_s \cdot \lambda')$$

by the definition of $E_{-}(\lambda n_s^{-1})$. Hence

$$\varphi(X ((\lambda n_s^{-1}) \cdot c_{n_s}) E_{-}(\lambda n_s^{-1}) T_{n_w}) = \varphi(X ((\lambda n_s^{-1}) \cdot c_{n_s}) E_{-}(\lambda n_s^{-1}) E_{o_{-}}(n_s \cdot \lambda') T_{n_w}) = 0$$

by Proposition 2.8.

The homomorphism $I'_Q(e_Q(\sigma)) \to I'_Q(e_{Q_1}(\sigma))$. Let $Q_1 \supset Q \supset P$ be parabolic subgroups. Recall that we have the homomorphism $I'_Q(e_Q(\sigma)) \to I'_Q(e_{Q_1}(\sigma))$. This is defined by $I_Q(Su(P(\sigma_{\ell'} P)))) \to I_{Q_1}(Su(P(\sigma_{\ell'} P))))$ with $\iota$. We give the following description of this homomorphism.

**Proposition 3.13.** Let $\varphi \in I'_Q(e_Q(\sigma))$ and $\varphi' \in I'_Q(e_{Q_1}(\sigma))$ be the image of $\varphi$. Then for $w \in W_{0}^{Q_1}$, we have $\varphi'(T_{n_w}^*) = (-1)^{\ell(w Q_1, w Q)} \varphi(T_{n_w}^*)$. In particular, combining with [Abel16 Proposition 4.12], we have

$$\text{Ker}(I'_Q(e_Q(\sigma)) \to I'_Q(e_{Q_1}(\sigma)))$$

$$= \{ \varphi \in I'_Q(e_Q(\sigma)) \mid \varphi(T_{n_w}^*) = 0 \text{ for any } w \in W_{0}^{Q_1} \}. $$
First we describe the homomorphism \( I_Q(\text{St}_P^Q(\sigma)) \to I_{Q_1}(\text{St}_P^{Q_1}(\sigma)) \). Recall that the kernel of \( I_P^Q(\sigma) \ni \varphi \mapsto \varphi(T_{\text{w}Q\text{w}P}) \in \sigma \) is \( \sum_{Q \supseteq P_1 \supseteq P} I_{P_1}^Q(e_{P_1}(\sigma)) \) and hence it gives an identification \( \sigma \simeq \text{St}_P^Q(\sigma) \) as vector spaces by Lemma [2.7]

**Lemma 3.14.** The homomorphism \( I_Q(\text{St}_P^Q(\sigma)) \to I_{Q_1}(\text{St}_P^{Q_1}(\sigma)) \) is given by \( \varphi \mapsto (X \mapsto \varphi(XT_{n_{\text{w}Q\text{w}P}})) \). (Here we identify \( \text{St}_P^Q(\sigma) \) and \( \text{St}_P^{Q_1}(\sigma) \) with \( \sigma \).)

**Proof.** Since \( I_P^Q(\sigma) \to \text{St}_P^Q(\sigma) \) is given by \( \varphi \mapsto \varphi(T_{\text{w}Q\text{w}P}) \) (under the identification \( \sigma = \text{St}_P^Q(\sigma) \)), \( I_P(\sigma) \to I_Q(\text{St}_P^Q(\sigma)) \) is given by \( \varphi \mapsto (X \mapsto \varphi(XT_{n_{\text{w}Q\text{w}P}})) \). Now recall the following commutative diagram which defines the homomorphism in the lemma:

\[
\begin{array}{ccc}
I_P(\sigma) & \rightarrow & I_Q(\text{St}_P^Q(\sigma)) \\
\downarrow & & \downarrow \\
I_P(\sigma) & \rightarrow & I_{Q_1}(\text{St}_P^{Q_1}(\sigma)).
\end{array}
\]

Let \( \varphi \in I_Q(\text{St}_P^Q(\sigma)) \) and take \( \tilde{\varphi} \in I_P(\sigma) \) which is a lift of \( \varphi \). Then we have \( \varphi(X) = \tilde{\varphi}(XT_{n_{\text{w}Q\text{w}P}}) \). Let \( \varphi' \) be the image of \( \varphi \). Then from the above commutative diagram we have \( \varphi'(X) = \tilde{\varphi}(XT_{n_{\text{w}Q\text{w}P}}) \). Since \( n_{\text{w}Q\text{w}P} = n_{\text{w}Q},n_{\text{w}Q\text{w}P} \), we have \( \tilde{\varphi}(XT_{n_{\text{w}Q\text{w}P}}) = \tilde{\varphi}(XT_{n_{\text{w}Q\text{w}P}}) = \varphi(XT_{n_{\text{w}Q\text{w}P}}) \). Hence \( \varphi'(X) = \varphi(XT_{n_{\text{w}Q\text{w}P}}) \) for any \( X \in \mathcal{H} \). Therefore, for any \( X \in \mathcal{H} \), we have \( \varphi'(X) = \varphi(XT_{n_{\text{w}Q\text{w}P}}) \) for any \( X \in \mathcal{H} \).

**Proof of Proposition 3.13.** Let \( \varphi \in I_Q(e_Q(\sigma)) \) and \( \varphi' \in I_{Q_1}(e_{Q_1}(\sigma)) \) its image. Then \( \varphi \circ \iota \in I_Q(\text{St}_P^Q(\sigma e_{\text{w}P})) \) and \( \varphi' \circ \iota \in I_{Q_1}(\text{St}_P^{Q_1}(\sigma e_{\text{w}P})) \). By the above lemma, we have \( \varphi' \circ \iota(X) = \varphi \circ \iota(XT_{n_{\text{w}P}}) \). Hence \( \varphi'(\iota(X)) = \varphi'(\iota(XT_{n_{\text{w}P}})) = (-1)^{t_{\text{w}Q\text{w}P}} \varphi(\iota(XT_{n_{\text{w}Q\text{w}P}})) \) for any \( X \in \mathcal{H} \). Therefore, for any \( X \in \mathcal{H} \), we have \( \varphi'(X) = \varphi(XT_{n_{\text{w}Q\text{w}P}}) \) for any \( X \in \mathcal{H} \).

By Proposition 3.13, \( \varphi \in \text{Ker}(I_Q^Q(e_Q(\sigma)) \rightarrow \bigoplus_{Q_1 \supseteq P} I_{Q_1}(e_{Q_1}(\sigma))) \) if and only if \( \varphi(T_{n_{\text{w}P}}) = 0 \) for any \( w \in \bigcup_{Q_1 \supseteq P} W_0^Q \). We get the following description of the kernel appearing in Lemma [3.9]

**Lemma 3.15.** Let \( w \in W_0^Q \). We have \( \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_Q \) if and only if for some \( Q_1 \supseteq Q \), \( w \in W_0^{Q_1} \). Hence we have

\[
\text{Ker} \left( I_Q^Q(e_Q(\sigma)) \rightarrow \bigoplus_{Q_1 \supseteq Q} I_{Q_1}(e_{Q_1}(\sigma)) \right)
\]

\[= \{ \varphi \in I_Q^Q(e_Q(\sigma)) | \varphi(T_{n_{\text{w}P}}) = 0 \ \text{for any} \ w \in W_0^Q \ \text{such that} \ \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_Q \}. \]

**Proof.** Let \( w \in W_0^Q \) and assume that for some \( Q_1 \supseteq Q \) and \( v \in W_0^{Q_1} \), we have \( w = vQ_1, wQ_1 \). Then for \( \alpha \in \Delta_{Q_1} \setminus \Delta_Q, wQ_1(\alpha) \in \Sigma_{Q_1} \). Since \( v \in W_0^{Q_1} \), we have \( wQ_1(\alpha) < 0 \). Hence \( wQ_1(\alpha) < 0 \). Therefore \( \alpha \notin \Delta_{\text{w}Q} \). Hence \( \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_Q \).

Assume that \( \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_Q \). Since \( w \in W_0^Q \), for any \( \alpha \in \Delta_Q \) we have \( wQ_1(\alpha) < 0 \). Hence \( \alpha \notin \Delta_{\text{w}Q} \). Therefore \( \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_{\text{w}Q} \). Hence \( \Delta_{\text{w}Q} \neq \Delta \setminus \Delta_Q \). Take \( \alpha \in (\Delta \setminus \Delta_Q) \setminus \Delta_{\text{w}Q} \). Let \( Q_1 \) be a parabolic subgroup corresponding to \( \Delta_Q \cup \{ \alpha \} \). Then we have \( \Delta_{Q_1} \subset \Delta \setminus \Delta_{\text{w}Q} \). If \( \beta \in \Delta_{Q_1} \), then \( wQ_1(\beta) \in -\Delta_{Q_1} \subset -\Delta_{\text{w}Q} \). Hence \( wQ_1(\beta) > 0 \). Therefore \( wQ_1 \in W_0^{Q_1} \). We have \( w \in W_0^{Q_1}wQ_1 \).
3.6. **Complex.** In this subsection, we prove that the sequence in Lemma 3.9 is a complex, namely the composition

\[ I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma)) \]

is zero for any parabolic subgroup \( Q_1 \supseteq Q \).

We have the following diagram:

\[
\begin{array}{ccc}
I_Q(e_Q(\sigma)) & \longrightarrow & I'_Q(e_Q(\sigma)) \\
\uparrow & & \uparrow \\
I_Q(I_{Q_1}(e_Q(\sigma))) & \longrightarrow & I'_{Q_1}(I_{Q_1}(e_Q(\sigma))) \\
\end{array}
\]

This is commutative by Lemma 3.5. The sequence

\[ I'_{Q_1}(I_{Q_1}(e_Q(\sigma))) \to I'_{Q_1}(I_{Q_1}(e_Q(\sigma))) \to I'_{Q_1}(e_{Q_1}(\sigma)) \]

comes from

\[ I_{Q_1}(e_Q(\sigma)) \to I_{Q_1}(e_Q(\sigma)) \to e_{Q_1}(\sigma). \]

Let \( R^Q_{Q_1} \) be the right adjoint functor of \( I^{Q_1}_Q \) [Vig15, Proposition 4.1]. Since we have \( R^Q_{Q_1}(e_{Q_1}(\sigma)) = 0 \) by [Abe16, Lemma 5.17, this composition is zero.

3.7. **Exactness.** Now we finish the proof of Lemma 3.9 by proving the following lemma.

**Lemma 3.16.** We have

\[ \text{Im} \left( I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)) \right) \supset \text{Ker} \left( I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supseteq Q} I'_{Q_1}(e_{Q_1}(\sigma)) \right). \]

We start with the following lemma.

**Lemma 3.17.** Let \( \psi \in I'_Q(e_Q(\sigma)) \) and \( w \in W^Q_0 \). Then we have \( \psi(T^*_{n_w}) = \sum_{v \leq w} \psi(T_{n_v}). \)

**Proof.** The same argument as the proof of [AHHV17, IV.9 Proposition] implies

\[ \psi(T^*_{n_w}) = \sum_{v \leq w} \psi(T_{n_v}). \]

If \( v \notin W^Q_0 \), then there exists \( v_1 \in W^Q_0 \) and \( v_2 \in W_{0,Q} \) such that \( v = v_1v_2 \). We have \( \ell(v_1v_2) = \ell(v_1) + \ell(v_2) \). Hence \( \psi(T_{n_w}) = \psi(T_{n_{v_1}}T_{n_{v_2}}) = \psi(T_{n_{v_1}}) e_Q(\sigma)(T^Q_{n_{v_2}}) = 0 \) by the definition of \( e_Q(\sigma) \). We get the lemma. \( \square \)

Let \( w_c \) be the longest element of the finite Weyl group of the parabolic subgroup corresponding to \( \Delta \setminus \Delta_Q \). Then \( \Delta_{w_c} = \Delta \setminus \Delta_Q \) and \( w = w_{G\cdot w_c} \) is maximal in \( \{ w \in W_0 | \Delta_w = \Delta \setminus \Delta_Q \} \) [Abe, Remark 3.16].

We assume that \( \psi \in \text{Ker}(I_Q^Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supseteq Q} I^Q_{Q_1}(e_{Q_1}(\sigma))) \). Let \( \mu^Q \) be the Möbius function associated to \( (W^Q_0, \leq) \). Then Lemma 3.17 and the definition of the Möbius function gives

\[ \psi(T^*_{n_w}) = \sum_{v \leq w, v \in W^Q_0} \mu^Q(v, w) \psi(T^*_{n_v}). \]

By Lemma 2.13 we have the following.
Lemma 3.18. If $\psi \in \text{Ker}(I_Q'(e_Q(\sigma)) \to \bigoplus_{Q} I_Q'(e_Q(\sigma)))$, then $\psi(T_{wGwQ}) = (-1)^{\ell(w_c)} \psi(T_{wGwQ}^s)$.

Lemma 3.19. Consider a linear map $I_Q'(e_Q(\sigma)) \to \sigma$ defined by $\psi \mapsto \psi(T_{wGwQ}^s)$. Then the composition $I_Q'(e_Q(\sigma)) \to I_Q'^{(w_c)}\sigma \to \sigma$ is surjective.

Proof. Let $\psi \in I_Q'(e_Q(\sigma))$ be the image of $\phi \in I_Q(e_Q(\sigma))$. Then the characterization of the homomorphism (Proposition 3.1) gives $\phi(T_{wGwQ}^s) = \psi(T_{wGwQ}^s)$.

Since $\psi$ is also in the kernel of $I_Q'(e_Q(\sigma)) \to \bigoplus_{Q} I_Q'(e_Q(\sigma))$ by 3.6 we have $\psi(T_{wGwQ}^s) = (-1)^{\ell(w_c)} \psi(T_{wGwQ}^s)$. Hence $\psi(T_{wGwQ}^s) = (-1)^{\ell(w_c)} \phi(T_{wGwQ}^s)$.

The lemma follows from the surjectivity of the map $I_Q'(e_Q(\sigma)) \ni \phi \mapsto \phi(T_{wGwQ}^s) \in \sigma$ (Proposition 2.4).

The following lemma ends the proof of Lemma 3.16 hence that of Theorem 3.6

Lemma 3.20. For $w \in W^Q_0$ such that $\Delta_{wGQ} = \Delta \setminus \Delta_Q$ and $x \in \sigma$, there exists $\psi \in \text{Im}(I_Q(e_Q(\sigma)) \to I_Q'(e_Q(\sigma)))$ such that for any $v \in W^Q_0$ we have

$$
\psi(T_{wGwQ}^s) = \begin{cases} x, & (v = w), \\ 0, & (v \neq w). 
\end{cases}
$$

We need one lemma.

Lemma 3.21. Let $w \in W^P_0$ and $\lambda \in \Lambda(1)$. Then for $\phi \in I_P'(\sigma)$, we have

$$(\phi E_{\sigma+}(\lambda))(T_{wGw}^s) = \begin{cases} \phi(T_{wGw}^s)\sigma(E_{\sigma+}(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \in W^{-p}_P(\lambda)), \\ \phi(T_{wGw}^s)\sigma(E_{\sigma-(n_w^{-1}\cdot\lambda)}), & (n_w^{-1}\cdot\lambda \notin W^{-p}_P(\lambda)). 
\end{cases}$$

Proof. Set $\phi' = \phi \circ \iota$ and $\sigma' = \sigma_{\iota^p}^{i^p}$. Then we have $\phi' \in I_P'(\sigma')$ [Abe16 Proposition 4.11]. Hence, by Proposition 2.11 we have

$$
\phi'(E_{\sigma-}(\lambda))(T_{wGw}^s) = \begin{cases} \phi'(T_{wGw}^s)\sigma'(E_{\sigma+}(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \in W^-P(\lambda)), \\ \phi'(T_{wGw}^s)\sigma'(E_{\sigma-(n_w^{-1}\cdot\lambda)}), & (n_w^{-1}\cdot\lambda \notin W^-P(\lambda)). 
\end{cases}
$$

The left hand side is

$$
\phi'(E_{\sigma-}(\lambda))(T_{wGw}^s) = \phi'(\iota(E_{\sigma-}(\lambda))(T_{wGw}^s)) = \phi'((-1)^{\ell(\lambda)}E_{\sigma+}(\lambda)(-1)^{\ell(n_w)}T_{wGw}^s)
$$

by [Vig16 Lemma 5.31]. Therefore if $n_w^{-1}\cdot\lambda \notin W^-P(\lambda)$, then $\phi(E_{\sigma+}(\lambda))T_{wGw}^s = 0$.

If $n_w^{-1}\cdot\lambda \in W^-P(\lambda)$, then

$$
\sigma'(E_{\sigma+}(n_w^{-1}\cdot\lambda)) = (-1)^{\ell(n_w^{-1}\cdot\lambda)}E_{\sigma+}(\lambda)(-1)^{\ell(n_w)}T_{wGw}^s
$$

again by [Vig16 Lemma 5.31] and [Abe16 Lemma 4.5]. We also have $\phi'(T_{wGw}^s) = (-1)^{\ell(n_w)}\phi(T_{wGw}^s)$. Hence we get

$$
\phi((-1)^{\ell(\lambda)}E_{\sigma+}(\lambda)(-1)^{\ell(n_w)}T_{wGw}^s) = (-1)^{\ell(n_w)}(\phi(E_{\sigma+}(\lambda))T_{wGw}^s) = \phi(T_{wGw}^s)(E_{\sigma+}(n_w^{-1}\cdot\lambda))
$$

Since $\ell(n_w^{-1}\cdot\lambda) = \ell(\lambda)$ [Abe16 Lemma 2.15], we get the lemma.
Proof of Lemma 3.20 We prove the lemma by induction on the length of $wwQ$. Assume that $wwQ = wGw_c$, namely $w = wGw_cw_Q$. Let $\lambda_P \in Z(W_P(1))$ as in Proposition 2.2. Notice that $n_{wQ}^{-1} \cdot \lambda_P$ is $P$-negative since $wQ$ (in fact, any element in $W_0$) preserves $\Sigma^+ \ \backslash \Sigma_P^+ = \Sigma_{P_2}^+$. We also have that $n_{wQ}^{-1} \cdot \lambda_P \in Z(W_P(1))$ since $n_{wQ}$ normalizes $W_P(1)$. Hence $e_Q(\sigma(T_{n_{wQ}^{-1} \cdot \lambda_P}^{Q*})) = \sigma(T_{n_{wQ}^{-1} \cdot \lambda_P}^{P*})$ is invertible. Take $\psi \in \text{Im}(I_Q(e_Q(\sigma)) \rightarrow I_Q'(e_Q(\sigma)))$ such that $\psi(T_{n_{wQ}^{-1} \cdot \lambda_P}^{Q*}) = xe_Q(\sigma(T_{n_{wQ}^{-1} \cdot \lambda_P}^{Q*})^{-1}$.

Put $\lambda = n_{wGw_c} \cdot \lambda_P$ and set $\psi = \varphi E_{o+}(\lambda)$. Let $v \in W_0^Q$. If $v \neq wGw_cw_Q$, then since $v$ and $wGw_cw_Q$ are in $W_0^Q$, we have $v \notin wGw_cw_QW_0^Q$. Hence $(wGw_c)^{-1}v \notin W_0^Q$. Therefore there exists $\alpha \in \Sigma^+ \ \backslash \Sigma_Q^+$ such that $(wGw_c)^{-1}v(\alpha) < 0$. Since $\Sigma^+ \ \backslash \Sigma_Q^+ \subset \Sigma_{P_2}$ and, $\Sigma_{P_2}$ is stabilized by $W_0$, we have $(wGw_c)^{-1}v(\alpha) \in \Sigma_{P_2}^- = \Sigma^- \ \backslash \Sigma_P$. Hence $\langle (wGw_c)^{-1}v(\alpha), v(\lambda_P) \rangle < 0$. The left hand side is $(\alpha, v(\lambda_P))$. Hence $n_{wQ}^{-1} \cdot \lambda$ is not $Q$-negative. Therefore $\psi(T_{n_\alpha}^\sigma = 0)$.

Assume that $v = wGw_cw_Q$. Then $n_{wQ}^{-1} \cdot \lambda = n_{wQ}^{-1} \cdot \lambda_P$. Since $\lambda_P$ is $Q$-negative, the set of $Q$-negative elements is stable under the conjugate action of $W_0^Q(1)$. Hence $n_{wQ}^{-1} \cdot \lambda$ is also $Q$-negative. Therefore we have $\psi(T_{n_{wGw_cw_Q}^{-1}}^\sigma = 0)$.

Assume that $w \neq wGw_cw_Q$ and take $s$ such that $s_{s \alpha}wQ > wQ$ and $\Delta_{s \alpha}wQ = \Delta_{wQ}$ as in Lemma 2.14. Then $s_{s \alpha}w > w$ by Lemma 2.14. Set $s = s_{s \alpha}$. By inductive hypothesis, there exists $\psi \in \text{Im}(I_Q(e_Q(\sigma)) \rightarrow I_Q'(e_Q(\sigma)))$ such that for $v \in W_0^Q \backslash \{s_{s \alpha}w\}$, $\psi(T_{n_{s \alpha}w}^\sigma = 0)$ and $\psi(T_{n_{s \alpha}w}^\sigma = x)$. We prove that $\psi = \varphi T_{n_\alpha}$ satisfies the condition of the lemma. First we have $\psi(T_{n_{s \alpha}w}^\sigma = 0)$.

Since $sw > w$, we have $T_{n_{s \alpha}w}^\sigma = T_{n_{s \alpha}w}^\sigma$. Hence $\psi(T_{n_{s \alpha}w}^\sigma = 0)$. Since $\psi(T_{n_{s \alpha}w}^\sigma = 0)$, we have $\psi(c_n^\sigma T_{n_{s \alpha}w}^\sigma) = \psi(T_{n_{s \alpha}w}^\sigma e_Q(\sigma)(n_{wQ}^{-1} \cdot c_n) = 0).$ Hence $\psi(T_{n_{s \alpha}w}^\sigma = 0)$. Assume that $v \neq w$. If $\ell(s) + \ell(v) > \ell(sv)$, then $T_{n_{s \alpha}w}^\sigma = T_{n_{s \alpha}w}^\sigma T_{n_{s \alpha}}^\sigma = 0$. Hence $\psi(T_{n_{s \alpha}w}^\sigma = 0)$. If $\ell(s) + \ell(v) = \ell(sv)$, then $\psi(T_{n_{s \alpha}w}^\sigma = \psi(T_{n_{s \alpha}w}^\sigma = 0)$.

Since $v \neq w, sv \neq sw$. Hence $\psi(T_{n_{s \alpha}w}^\sigma = 0)$. Since $sv > v$ and $s(sv) < sw$, we have $v \neq sw$. Hence $\psi(c_n T_{n_{s \alpha}w}^\sigma = \psi(T_{n_{s \alpha}w}^\sigma e_Q(\sigma)(n_{wQ}^{-1} \cdot c_n) = 0).$ Therefore we get $\psi(T_{n_{s \alpha}w}^\sigma = 0)$.}

3.8. Supersingular modules.

Proposition 3.22. If $\pi$ is supersingular, then $\pi^\dagger$ is also supersingular.

Proof. For each $W(1)$-orbit $\mathcal{O} \subset \Lambda(1)$ such that $\ell(\mathcal{O}) > 0$, we have $\ell(z_\mathcal{O}) = \sum_{\lambda \in \mathcal{O}} \ell(E_{o+}(\lambda)) = \sum_{\lambda \in \mathcal{O}} (-1)^{\ell(\mathcal{O})} E_{o+}(\lambda) = (-1)^{\ell(\mathcal{O})} z_\mathcal{O}$ by [Vig16] Lemma 5.31. Hence $\pi^\dagger(z_\mathcal{O}^\sigma = 0)$ implies $\pi^\dagger(z_\mathcal{O}^\sigma = 0).$
Proof. Let $\Xi = \Xi_{\chi,J}$ be the character of $\mathcal{H}_{\text{aff}}$ parametrized by $(\chi, J)$. The representation $\pi$ is given by $\pi = (V \otimes \Xi_{\chi,J}) \otimes_{\mathcal{H}_{\text{aff}}} C[\Omega(1)_{\Xi}] \mathcal{H}$. The homomorphism $\iota$ preserves $\mathcal{H}_{\text{aff}}$ and $C[\Omega(1)_{\Xi}]$. (On $C[\Omega(1)]$, $\iota$ is identity.) Hence we get $\pi^t = (V^t \otimes \Xi^t) \otimes_{\mathcal{H}_{\text{aff}}} C[\Omega(1)_{\Xi}] \otimes \mathcal{H}$. Since $\iota$ is trivial on $C[\Omega(1)_{\Xi}]$, $V^t = V$. Let $(\chi', J')$ be the pair such that $\Xi'$ is parametrized by $(\chi', J')$. The character $\chi'$ is a direct summand of $V^t|_{Z_n \cap W_{\text{aff}}(1)}$ and since $V^t = V$, we have $V^t|_{Z_n \cap W_{\text{aff}}(1)} = V|_{Z_n \cap W_{\text{aff}}(1)}$, since $V|_{Z_n \cap W_{\text{aff}}(1)}$ is a direct sum of $\chi$, $\chi' = \chi$. The subset $J \subseteq S_{\text{aff}, \chi}$ satisfies

$$\Xi(T_{\tilde{s}}) = \begin{cases} 0, & (s \in J), \\ \chi(c_{\tilde{s}}), & (s \in S_{\text{aff}, \chi} \backslash J), \end{cases}$$

where $\tilde{s} \in W_{\text{aff}}(1)$ is a lift of $s$. We have $\Xi(\iota(T_{\tilde{s}})) = -\Xi(T_{\tilde{s}} - c_{\tilde{s}}) = -\Xi(T_{\tilde{s}}) + \chi(c_{\tilde{s}})$. Therefore we have

$$\Xi'(T_{\tilde{s}}) = \begin{cases} \chi(c_{\tilde{s}}), & (s \in J), \\ 0, & (s \in S_{\text{aff}, \chi} \backslash J). \end{cases}$$

We have $J' = \{s \in S_{\text{aff}, \chi} \mid \Xi'(T_{\tilde{n}}) = 0\}$. Hence $J' = S_{\text{aff}, \chi} \backslash J$. \hfill $\square$

3.9. Simple modules. Assume that $C$ is an algebraically closed field. Summarizing the results in this section, we have the following. We need notation. By [Abe16] Remark 4.6, $T_w^P \mapsto (-1)^{\ell(w)-\ell_P(w)}T_w^P$ is an algebra homomorphism of $\mathcal{H}_P$. This preserves the subalgebra $C[\Omega_P(1)]$. Let $\Xi$ be a character of $\mathcal{H}_{\text{aff}, P}$. Then the above homomorphism also preserves $C[\Omega_P(1)_{\Xi}]$ since the homomorphism is trivial on $\mathcal{H}_{\text{aff}, P}$ by [Abe16] Lemma 4.7. For a $C[\Omega_P(1)]$-module $V$, let $V_{\ell-\ell_P}$ be the pull-back of $V$ by this homomorphism.

Theorem 3.24. Let $I(P; \chi, J, V; Q)$ be a simple representation. Then we have $I(P; \chi, J, V; Q)^t = I(P; \chi, S_{\text{aff}}, P, \chi \backslash J, V_{\ell-\ell_P}; Q^t)$ where $\Delta_{Q^t} = \Delta_P \cup (\Delta(\sigma) \setminus \Delta_Q)$.

Proof. Since $I(P; \chi, J, V; Q) = I(P; \chi, S_{\text{aff}}, P, \chi \backslash J, V_{\ell-\ell_P}; Q^t)$ is simple, by Corollary [83], we have $I(P; \chi, J, V; Q)^t = I(P; \chi, S_{\text{aff}}, P, \chi \backslash J, V_{\ell-\ell_P}; Q^t)^t$. By Theorem 3.6 Proposition 3.23 and [Abe16] Lemma 4.9, we have

$$I(P; \chi, J, V; Q)^t \simeq I(P; \chi, S_{\text{aff}}, P, \chi \backslash J, V_{\ell-\ell_P}) \simeq I(P; \chi, S_{\text{aff}}, P, \chi \backslash J, V_{\ell-\ell_P}).$$

Let $\Xi$ be a character of $\mathcal{H}_{\text{aff}, P}$ defined by the pair $\chi$ and $S_{\text{aff}, P, \chi \backslash J}$. Put $\mathcal{H}_{P, \Xi} = \mathcal{H}_{\text{aff}, P}C[\Omega_P(1)_{\Xi}]$. Then $\pi_{\chi, S_{\text{aff}}, P, \chi \backslash J, V} = (\Xi \otimes V) \otimes_{\mathcal{H}_{P, \Xi}} \mathcal{H}_P$. Let $f: \mathcal{H}_P \to \mathcal{H}_{P, \Xi}$ be an algebra homomorphism defined by $f(T_w^P) = (-1)^{\ell(w)-\ell_P(w)}T_w^P$. Then $f$ preserves $\mathcal{H}_{\text{aff}, P}$ and $C[\Omega_P(1)_{\Xi}]$ and we have $(\pi_{\chi, S_{\text{aff}}, P, \chi \backslash J, V})_{\ell-\ell_P} = \pi_{\chi, S_{\text{aff}}, P, \chi \backslash J, V} \circ f = ((\Xi \circ f) \otimes (V \circ f)) \otimes_{\mathcal{H}_{P, \Xi}} \mathcal{H}_P$. By the definition, $V \circ f = V_{\ell-\ell_P}$. By [Abe16] Lemma 4.7, $f$ is identity on $\mathcal{H}_{\text{aff}, P}$. Hence $\Xi \circ f = \Xi$. Hence $(\pi_{\chi, S_{\text{aff}}, P, \chi \backslash J, V})_{\ell-\ell_P} = (\Xi \otimes V_{\ell-\ell_P}) \otimes_{\mathcal{H}_{P, \Xi}} \mathcal{H}$ and we get the theorem. \hfill $\square$

3.10. Structure of $I'_P$. Assume that $C$ is an algebraically closed field.

Proposition 3.25. Let $P$ be a parabolic subgroup and let $\sigma$ be a simple supersingular representation of $\mathcal{H}_P$. Then for each parabolic subgroup $Q$ between $P$ and $P(\sigma)$, there exists a submodule $\pi_Q \subseteq I'_P(\sigma)$ such that

1. if $Q_1 \subseteq Q_2$, then $\pi_{Q_1} \subseteq \pi_{Q_2}$.
2. $\pi_Q / \sum_{Q_1 \subseteq Q} \pi_{Q_1} = I(P, \sigma, Q).$
Compare with \( I_Q(e_Q(\sigma)) \subset I_P(\sigma) \). In other words, the structure of \( I'_P(\sigma) \) is “opposite to” that of \( I_P(\sigma) \).

**Proof.** First assume that \( P(\sigma) = G \). Put \( \sigma' = \sigma_{\ell-P} \). Then \( I'_P(\sigma) = I_P(\sigma') \). Set \( \pi_Q = I_Q^-(e_Q^-(\sigma')) \subset I'_P(\sigma) \) where \( \Delta_Q^- = (\Delta \setminus \Delta_Q) \cup \Delta_P \). Then the first condition is satisfied. Since \( Q_1 \subset Q_2 \) if and only if \( Q_1^c \supset Q_2^c \), we have

\[
\pi_Q / \sum_{Q_1 \subset Q} \pi_{Q_1} = \left( I_Q^-(e_Q^-(\sigma')) / \sum_{Q_1 \subset Q} I_{Q_1}^-(e_{Q_1}^- (\sigma')) \right) \\
= \left( I_Q^-(e_Q^-(\sigma')) / \sum_{Q_1 \subset Q^c} I_{Q_1}^- (e_{Q_1}^- (\sigma')) \right) \\
= (\text{St}_{Q^c}^- (\sigma'))^c = \text{St}_{Q}^- (\sigma)
\]

by Theorem \[3.6\] By the assumption \( P(\sigma) = G \), we have \( \text{St}_Q^- (\sigma) = I(P, \sigma, Q) \). We get the proposition in this case.

In general, applying the proposition for \( I^P_\sigma (\sigma') \), we get \( \pi'_Q \subset I^P_\sigma (\sigma') \) for each \( P(\sigma) \supset Q \supset P \). Put \( \pi_Q = I'_P(\sigma) (\pi_Q) \). The first condition is obvious. For the second condition, we have

\[
\pi_Q / \sum_{Q_1 \subset Q} \pi_{Q_1} = I'_P(\sigma) (\pi_Q) / \sum_{Q_1 \subset Q} I'_P(\sigma) (\pi_{Q_1}) \\
\simeq I'_P(\sigma) \left( \pi'_Q / \sum_{Q_1 \subset Q} \pi'_{Q_1} \right) \\
\simeq I'_P(\sigma) (\text{St}^{P(\sigma)} (\sigma)).
\]

Since \( I(P, \sigma, Q) = I_P(\sigma) (\text{St}^{P(\sigma)} (\sigma)) \) is simple, by Corollary \[3.3\] we have \( I(P, \sigma, Q) \simeq I'_P(\sigma) (\text{St}^{P(\sigma)} (\sigma)) \). Now we get the proposition. \( \square \)

4. **Dual**

We have an antiautomorphism \( \zeta = \zeta_G : \mathcal{H} \to \mathcal{H} \) defined by \( \zeta(T_x) = T_{x^{-1}} \). Hence for a representation \( \pi \), its linear dual \( \pi^* = \text{Hom}_C (\pi, C) \) has a structure of a right \( \mathcal{H} \)-module defined by \( (fX)(v) = f(\zeta(X)) \) for \( f \in \pi^*, v \in \pi \) and \( X \in \mathcal{H} \). Since any simple representation is finite-dimensional, if \( \pi \) is simple, then \( \pi^* \) is again simple. In this section, we compute \( \pi^* \).

**Lemma 4.1.** We have \( \zeta(T_{w^s}) = T_{w^{-1}} \).

**Proof.** In \( \mathcal{H}[q^\pm 1] \), we have \( \zeta(T_{w^s}) = \zeta(q_w T_{w^{-1}}) = q_w T_{w^{-1}} = T_{w^{-1}} \). \( \square \)

4.1. **Parabolic inductions.** In this subsection, we calculate \( I_P(\sigma)^* \). Let \( P' = n_{w_G w_P} P_w p_n \). Then we have \( I_P(\sigma) \simeq n_{w_G w_P} \sigma \otimes (\mathcal{H}_{P', j}^+) \mathcal{H} \) by \[Abe16\] Proposition 2.21]. Hence we have

\[
\text{Hom}_C(I_P(\sigma), C) \simeq \text{Hom}_C(n_{w_G w_P} \sigma \otimes (\mathcal{H}_{P', j}^+) \mathcal{H}, C) \\
\simeq \text{Hom}_{(\mathcal{H}_{P', j}^+, j_P)}(\mathcal{H}, \text{Hom}_C(n_{w_G w_P} \sigma, C)).
\]
Therefore $I_P(\sigma)^* \simeq \text{Hom}(H_{P^t}^+(\mathcal{H}, \text{Hom}_C(n_{wGwP}^*\sigma, C)))$ and here the action on the right hand side is twisted by $\zeta$. Let $\varphi \in \text{Hom}(H_{P^t}^+(\mathcal{H}, \text{Hom}_C(n_{wGwP}^*\sigma, C)))$ and set $\varphi^\sigma = \varphi \circ \zeta$. Let $w \in W_P(1)$ which is $P'$-negative. Then $w^{-1}$ is $P'$-positive. Hence for $X \in \mathcal{H}$ and $x \in n_{wGwP}^*\sigma$, we have $\varphi^\sigma(XT_w)(x) = \varphi(\zeta(XT_w))(x) = \varphi(T_{w^{-1}}\zeta(X))(x) = (T_{w^{-1}}\varphi(\zeta(X)))(x) = \varphi(\zeta(X))(xT_{w^{-1}}) = \varphi^\sigma(x)\varphi(T_{w')}(x)$. (Here we regard $\text{Hom}_C(n_{wGwP}^*\sigma, C)$ as a left $\mathcal{H}_{P^t}$-module.) Therefore $\varphi^\sigma \in \text{Hom}(H_{P^t}^+(\mathcal{H}, (n_{wGwP}^*)^*)^*)$.

For $X, Y \in \mathcal{H}$, we have $(\varphi^\sigma Y)^\sigma(X) = (\varphi Y)(\zeta(X)) = \varphi(\zeta(X)\zeta(Y)) = \varphi(\zeta(YY)) = \varphi^\sigma(YX) = (\varphi^\sigma Y)(X)$. Hence $\varphi \mapsto \varphi^\sigma$ induces $I_P(\sigma)^* \simeq \text{Hom}(H_{P^t}^+(\mathcal{H}, (n_{wGwP}^*)^*)^*) = I'_{P^t}(n_{wGwP}^*)^*$.

**Proposition 4.2.** We have $I_P(\sigma)^* \simeq I'_{P^t}(n_{wGwP}^*)^*$.

The same calculation shows the following.

**Proposition 4.3.** We have $I_P(\sigma)^* \simeq I'_{P^t}(n_{wGwP}^*)^*$.

**Remark 4.4.** These propositions are true for any commutative ring $C$.

4.2. **Steinberg modules.** Let $P$ be a parabolic subgroup, let $\sigma$ be an $\mathcal{H}_P$-module such that $P(\sigma) = G$ and let $P_2$ be the parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. We calculate $(\text{St}_Q^\sigma)^*).

**Proposition 4.5.** Let $Q$ be a parabolic subgroup containing $P$ and put $Q' = n_{wQwQ}^*Q_{wQwQ}^*$. Then $(\text{St}_Q^\sigma)^* \simeq \text{St}_{Q'}^{\sigma^*}$.

We start with the case of $Q = G$.

**Lemma 4.6.** We have $e_G(\sigma)^* \simeq e_G(\sigma^*)$.

**Proof.** Let $f \in e_G(\sigma)^*$ and $x \in e_G(\sigma)$. For $w \in W_P(1)$, we have

$$(fe_G(\sigma)^*(T_w^*))((x) = f(xe_G(\sigma)(\zeta(T_w^*))) = f(xe_G(\sigma)(T_w^{-*}))$$

$$= f(x\sigma(T_{w^{-*}})) = f(x\sigma(T_{w^{-*}})) = (f\sigma(T_{w^{-*}}))(x).$$

Hence $e_G(\sigma)^*(T_w^*) = \sigma^*(T_w^*)$. For $w \in W_{ab,P_2}(1)$, we have $(fe_G(\sigma)^*(T_w^*))((x) = f(xe_G(\sigma)(T_w^*))) = f(xe_G(\sigma)(T_w^{-*})) = f(x)$. Hence $e_G(\sigma)^*(T_w^*) = 1$. Therefore by the characterization of $e_G(\sigma^*)$, we have the lemma. \hfill $\square$

**Proof of Proposition 4.5.** By Lemma 3.9, we have the following exact sequence:

$$0 \to \text{St}_Q^\sigma \to I_Q'^{\sigma}(e_Q(\sigma)) \to \bigoplus_{Q \supsetneq Q} I_Q(\sigma)_1(e_Q(\sigma)).$$

Taking the dual, we get an exact sequence

$$\bigoplus_{Q \supsetneq Q} I_Q(\sigma)_1(e_Q(\sigma))^* \to I_Q(\sigma)_1(e_Q(\sigma))^* \to \text{St}_Q^\sigma \to 0.$$

Put $Q' = n_{wQwQ}^*Q_{wQwQ}^*$. Then by Proposition 4.3, we have $I_Q(\sigma)_1(e_Q(\sigma))^* = I_Q'^*(n_{wQwQ}^*e_Q(\sigma)^*) = I_Q'^*(e_Q(\sigma)^*) = I_Q(\sigma)_1(e_Q(\sigma)^*)$ by Lemma 4.6 and Abe16 Lemma 2.27]. Put $Q_1 = n_{wQwQ}^*Q_1^*Q_{wQwQ}^*$. Then

$$\bigoplus_{Q \supsetneq Q} I_Q(\sigma)_1(e_Q(\sigma))^* \to I_Q(\sigma)_1(e_Q(\sigma)^*) \to \text{St}_Q^\sigma \to 0.$$
Since $\Delta_{Q'} = -w_Q(\Delta_Q)$ and $Q'_1 = -w_Q(\Delta_{Q'_1})$, we have $Q_1 \supseteq Q$ if and only if $Q'_1 \supseteq Q'$.

Hence
\[ \bigoplus_{Q \supseteq Q'} I_{Q'_1}(e_{Q'_1}(\sigma^*)) \to I_{Q'}(e_{Q'}(\sigma^*)) \to St_Q(\sigma)^* \to 0. \]

By the lemma below, we get $St_Q(\sigma)^* = St_{Q'}(\sigma^*)$. \hfill \Box

**Lemma 4.7.** Let $Q_1 \supseteq Q$ be parabolic subgroups. Put $Q' = n_{w_Gw_Q}Q'_{W_Gw_Q}$ and $Q'_1 = n_{w_Gw_{Q'_1}}Q'_{W_Gw_Q}$. Then the homomorphism induced by $I_Q(e_Q(\sigma)) \to I_{Q'_1}(e_{Q'_1}(\sigma))$ with the dual is the inclusion $I_{Q'_1}(e_{Q'_1}(\sigma^*)) \to I_{Q'}(e_{Q'}(\sigma^*))$ times $(-1)^{\ell(w_Qw_{Q'})}$.

**Proof.** By Proposition 3.13 the homomorphism $I_Q(e_Q(\sigma)) \to I_{Q'_1}(e_{Q'_1}(\sigma))$ is given by $\varphi \mapsto (X \mapsto (-1)^{\ell(w_Qw_{Q'})}\varphi(XT_{w_{Q'_1}w_Q}^*))$. We recall that the isomorphism $I_Q(e_Q(\sigma)) \cong e_Q(\sigma) \otimes \langle H_{Q'_1}, J_{Q'_1} \rangle \mathcal{H}$ is given by $\varphi \mapsto \sum_{w \in q'_1 w_0} \varphi(T_{w_{Q'_1}w_Q}^*) \otimes T_{w_{Q'_1}w_Q}$. Notice that $Q_{Q'_1}$.

Let $\varphi' \in I_{Q'_1}(e_{Q'_1}(\sigma))$ be the image of $\varphi$. Then the image of $\varphi'$ in $e_{Q'_1}(\sigma) \otimes \langle H_{Q'_1}, J_{Q'_1} \rangle \mathcal{H}$ is
\[ \sum_{w \in q'_1 w_0} \varphi'(T_{w_{Q'_1}w_Q}^*) \otimes T_{w_{Q'_1}w_Q} = (-1)^{\ell(w_Qw_{Q'})} \sum_{w \in q'_1 w_0} \varphi(T_{w_{Q'_1}w_Q}^*) T_{w_{Q'_1}w_Q}^* \otimes T_{w_{Q'_1}w_Q}. \]

Since $w_{Q'_1}w_Q \in W_{Q_1}Q_1$ and $w_{Q'_1}w_Q \in W_{Q'_1}^{Q'}$, we have $\ell(w_{Q'_1}w_Q) = \ell(w_{Q'_1}w_Q) + \ell(w_{Q'_1}w_Q) = \ell(w_{Q'_1}w_Q)$. Hence we have $T_{w_{Q'_1}w_Q}^* T_{w_{Q'_1}w_Q}^* = T_{w_{Q'_1}w_Q}^*$. Therefore
\[ \sum_{w \in q'_1 w_0} \varphi'(T_{w_{Q'_1}w_Q}^*) \otimes T_{w_{Q'_1}w_Q} = (-1)^{\ell(w_Qw_{Q'})} \sum_{w \in q'_1 w_0} \varphi(T_{w_{Q'_1}w_Q}^*) \otimes T_{w_{Q'_1}w_Q}. \]

Let $P_2$ be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Let $w \in Q'W_0$ but $w \notin q'_1 w_0$. Then there exists a simple reflection $s \in W_{Q'_1}Q_1$ such that $sw < w$. Since $w \in Q'W_0 \subset W_{Q_1}Q_1$, we have $s \in S_{Q'_1}$. Hence for any $x \in e_{Q'_1}(\sigma)$, we have $x \otimes T_{n_w} = x \otimes T_{n_w} = x e_{Q'_1}(\sigma)(T_{n_{Q'_1}^*}) \otimes T_{n_{Q'_1}} = 0$ since $e_{Q'_1}(\sigma)(T_{n_{Q'_1}^*}) = 0$. Hence
\[ (-1)^{\ell(w_Qw_{Q'})} \sum_{w \in q'_1 w_0} \varphi(T_{w_{Q'_1}w_Q}^*) \otimes T_{n_w} = (-1)^{\ell(w_Qw_{Q'})} \sum_{w \in q'_1 w_0} \varphi(T_{w_{Q'_1}w_Q}^*) \otimes T_{n_w}. \]

Therefore the homomorphism
\[ e_{Q'}(\sigma) \otimes \langle H_{Q'_1}, J_{Q'_1} \rangle \mathcal{H} \to e_{Q'_1}(\sigma) \otimes \langle H_{Q'_1}, J_{Q'_1} \rangle \mathcal{H} \]

is given by $x \otimes X \mapsto (-1)^{\ell(w_{Q'_1}w_Q)}x \otimes X$. (Here we identify $x \in e_{Q'_{W_G}w_Q}$ with $x \in e_{Q'_1}(\sigma)$.)

The isomorphism
\[ (e_{Q'}(\sigma) \otimes \langle H_{Q'_1}, J_{Q'_1} \rangle \mathcal{H})^* \cong \text{Hom}_{\langle H_{Q'_1}, J_{Q'_1} \rangle} \langle \mathcal{H}, e_{Q'}(\sigma)^* \rangle = I_Q(e_Q(\sigma)^*) \]
is given by \( f \mapsto (X \mapsto (x \mapsto f(x \otimes \zeta(X)))) \) and the opposite is given by \( f' \mapsto ((x \otimes X) \mapsto f'(\zeta(X)(x))) \). (Here we identify \( e_Q(\sigma)^* \) with \( e_Q(\sigma^*) \).) Hence the maps
\[
I_{Q_i}(e_Q(\sigma)^*) \simeq (e_Q(\sigma) \otimes (H^{+}_{Q_i} \times H^{+}_{Q_i}))(\mathcal{H})^* \rightarrow (e_Q(\sigma) \otimes (H^{+}_{Q'} \times H^{+}_{Q'}))(\mathcal{H})^* \simeq I_Q(e_Q(\sigma)^*)
\]
send \( f \in I'_{Q_i}(e_Q(\sigma)^*) \) to
\[
(x \otimes X) \mapsto f(\zeta(X))(x) \in (e_Q(\sigma) \otimes (H^{+}_{Q_i} \times H^{+}_{Q_i}))(\mathcal{H})^*,
\]
\[
(x \otimes X) \mapsto -1 \ell(w_{Q_1}w_Q)f(\zeta(X))(x) \in (e_Q(\sigma) \otimes (H^{+}_{Q'} \times H^{+}_{Q'}))(\mathcal{H})^*,
\]
\[
X \mapsto (x \mapsto -1 \ell(w_{Q_1}w_Q)f(X)(x)) \in I_Q(e_Q(\sigma)^*).
\]
Namely, it is equal to the the natural embedding times \((-1)^{\ell(w_{Q_1}w_Q)}\).

\[\square\]

4.3. **Supersingular modules.** Assume that \( C \) is a field.

**Theorem 4.8.** Let \((\chi, J, V)\) be as in subsection 2.11. Then we have \( \pi_{\chi, J,V}^{*} \cong \pi_{\chi^{-1}, J, V^*} \).

**Proof.** Let \( \Xi \) be a character of \( H_{\text{aff}} \) determined by \((\chi, J)\). By the proof of [Vig17 Proposition 6.17], \( \Xi \otimes V \subset \pi_{\chi, J,V}^{-1} \otimes_{\mathbb{H}_{\Xi}} \mathbb{H} \) is a direct summand. Hence \((\Xi \otimes V)^* \subset (\pi_{\chi, J,V})^* \otimes_{\mathbb{H}_{\Xi}} \mathbb{H} \). Since \( \Xi \) and \( V \) are finite-dimensional, we have \((\Xi \otimes V)^* \cong \Xi^* \otimes V^* \). Therefore we have a non-zero homomorphism \((\Xi^* \otimes V^*) \otimes_{\mathbb{H}_{\Xi}} \mathbb{H} \rightarrow \pi_{\chi, J,V}^* \). The restriction of \( V^* \) to \( Z_{\chi} \cap W_{\text{aff}, P(1)} \) is the direct sum of \( \chi^* = \chi^{-1} \) since \( V|_{Z_{\chi} \cap W_{\text{aff}, P(1)}} \) is a direct sum of \( \chi \). For \( s \in S_{\chi, \chi} = S_{\text{aff}, \chi^{-1}} \), \( \Xi^*(T_{\bar{s}}) = \Xi(\zeta(T_{\bar{s}})) = \Xi(T_{\bar{s}}^{-1}) \) where \( \bar{s} \) is a lift of \( s \). This is 0 or \( \chi(c_{\bar{s}} - 1) \) and 0 if and only if \( s \in J \). Hence the subset of \( S_{\chi, \chi} = S_{\text{aff}, \chi} \) attached to \( \Xi^* \) is \( J \). Therefore \( (\Xi^* \otimes V^*) \otimes_{\mathbb{H}_{\Xi}} \mathbb{H} = \pi_{\chi^{-1}, J, V^*} \). Hence we get a non-zero homomorphism \( \pi_{\chi^{-1}, J, V^*} \rightarrow \pi_{\chi^*, J, V} \). Since this is a non-zero homomorphism between simple modules, this is an isomorphism. \[\square\]

4.4. **Simple modules.** Assume that \( C \) is a field. Combining Propositions 4.2 and 4.5 and Theorem 4.8 we get the following theorem.

**Theorem 4.9.** Set \( P' = n_{w_{G\text{wp}}}P_{\text{aff}, n_{w_{G\text{wp}}}^{-1}} \) and \( Q' = n_{w_{G\text{wp}}}Q_{\text{aff}, n_{w_{G\text{wp}}}^{-1}} \). Let \((\chi', J', V')\) be a triple for \( \mathcal{H}_{P'} \) defined by the pull-back of the triple \((\chi^{-1}, J, V^*)\) by \( n_{w_{G\text{wp}}} \). Then we have \( I(P; \chi, J, V; Q) = I(P'; \chi', J', V'; Q') \).

We use the following lemma.

**Lemma 4.10.** Let \( P \) be a parabolic subgroup and let \( \sigma \) be an \( \mathcal{H}_{P} \)-module. Then we have \( P(n_{w_{G\text{wp}}} \sigma) = n_{w_{G\text{wp}}}P(\sigma)_{\text{aff}, n_{w_{G\text{wp}}}^{-1}} \).

**Remark 4.11.** By [Abel16 Lemma 2.27], we have \( n_{w_{G\text{wp}}(\sigma)} \).
we get $\Delta(n\sigma) = w_G w_{P(\sigma)}(\Delta(\sigma))$. The left hand side corresponds to $P(n\sigma)$ and the right hand side corresponds to $nP(\sigma)^{op}n^{-1}$. Hence we get the lemma.

Proof of Theorem 4.5 Set $\sigma = \pi_X, J, V$ and $P(\sigma') = n_{w_G w_{P(\sigma)}} P(\sigma)^{op} n_{w_G w_{P(\sigma)}}^{-1}$. By Proposition 4.2

$$I(P; X, J, V; Q)^* = I_P(\sigma) (\text{St}_P^P(\sigma))^* 
\simeq I_P(\sigma'')(n_{w_G w_{P(\sigma)}} (\text{St}_Q^P(\sigma))^*).$$

We have

$$I_P(\sigma'')(n_{w_G w_{P(\sigma)}} (\text{St}_Q^P(\sigma))^*) = I_P(\sigma')(n_{w_G w_{P(\sigma)}} (\text{St}_Q^P(\sigma))^*)$$

by Corollary 3.3. Proposition 4.5 implies

$$I_P(\sigma')(n_{w_G w_{P(\sigma)}} (\text{St}_Q^P(\sigma))^*) \simeq I_P(\sigma')(n_{w_G w_{P(\sigma)}} (\text{St}_Q^{P(\sigma)}(\sigma)^*).$$

The adjoint action of $n_{w_G w_{P(\sigma)}}$ induces an isomorphism $H_{P(\sigma)} \simeq H_{P(\sigma')}.$. For a parabolic subgroup $Q_1$ between $P(\sigma)$ and $P$, let $Q_2$ be a parabolic subgroup corresponding to $w_G w_{P(\sigma)}(\Delta Q_1)$. Then the adjoint action of $n_{w_G w_{P(\sigma)}}$ induces an isomorphism $H_{Q_1} \simeq H_{Q_2}$ and sends $H_{P(\sigma)}^{P(\sigma)}$ to $H_{Q_2}^{P(\sigma')}$. Moreover, it is compatible with homomorphisms $j_{Q_1}^{P(\sigma)}$ and $j_{Q_2}^{P(\sigma')}$. Hence, by the definition of parabolic inductions and the twist $n_{w_G w_{P(\sigma)}}$, we have $n_{w_G w_{P(\sigma)}} I_{Q_1}^{P(\sigma)}(\epsilon_Q(\sigma)) \simeq I_{Q_2}^{P(\sigma')}(n_{w_G w_{P(\sigma)}} e_{Q_1}(\sigma)).$

Since $\Delta P(\sigma) \setminus \Delta P$ is orthogonal to $\Delta P$, $w_P(\sigma) w_P(\Delta P) = \Delta P$. Therefore we have $w_G w_{P(\sigma)}(\Delta P) = w_G w_{P(\sigma)^{op}}(\Delta P)$. Hence the isomorphism $H_{Q_1} \simeq H_{Q_2}$ induced by the adjoint action of $n_{w_G w_{P(\sigma)}}$ sends $H_{Q_1}^{P(\sigma)}$ to $H_{Q_2}^{P(\sigma')}$. It is also compatible with homomorphisms $j_{Q_1}^{P(\sigma)}$ and $j_{Q_2}^{P(\sigma')}$. Therefore the restriction of $n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)$ to $H_{Q_2}^{P(\sigma')}$ (which is regarded as a subalgebra of $H_{Q_2}$ by $j_{Q_2}^{P(\sigma')}$, $j_{Q_2}^{P(\sigma)^{op}}$ is isomorphic to $n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)$. We denote the parabolic subgroup corresponding to $\Delta Q_1 \setminus \Delta P$ (resp., $\Delta Q_2 \setminus \Delta P$) by $R_1$ (resp., $R_2$). Then since $w_G w_{P(\sigma)}(\Delta Q_1) = \Delta Q_2$ and $w_G w_{P(\sigma)}(\Delta P) = \Delta P$, we have $w_G w_{P(\sigma)}(\Delta R_1) = \Delta R_2$. Hence $n_{w_G w_{P(\sigma)}} W_{aff, R_1}(1) n_{w_G w_{P(\sigma)}}^{-1} = W_{aff, R_2}(1)$. Therefore the action of $T^{Q_2*}_w$ for $w \in W_{R_2, aff}(1)$ is trivial on $n_{w_G w_{P(\sigma)}} \epsilon_{Q_1}(\sigma)$ since $n_{w_G w_{P(\sigma)}} w_{w_G w_{P(\sigma)}} \epsilon_{Q_1}(\sigma)$ in $W_{R_1, aff}(1)$ and $T^{Q_1*}_v$ is trivial on $\epsilon_{Q_1}(\sigma)$ for $v \in W_{R_1, aff}(1)$. Therefore, by the characterization of the extension, we have $n_{w_G w_{P(\sigma)}} \epsilon_{Q_1}(\sigma) \simeq e_{Q_2} n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)$. Hence, combining the formula in the previous paragraph, we get

$$n_{w_G w_{P(\sigma)}} (\text{St}_{Q_2}^P(\sigma)) \simeq \text{St}_{Q_2}^{P(\sigma)}(n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)).$$

Now set $Q_1 = n_{w_{P(\sigma)} w_Q} Q^{op} n_{w_{P(\sigma)} w_Q}$. Then $\Delta Q_1 = w_{P(\sigma)} w_Q(\Delta Q)$. Hence $\Delta Q_2 = w_G w_Q(\Delta Q) = \Delta Q'$. Therefore we have

$$I(P; X, J, V; Q)^* \simeq I_P(\sigma'') (\text{St}_{Q'}^P(\sigma)^* n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)).$$

By [Abe16] Lemma 2.27, $\sigma^* = n_{w_{P(\sigma)} w_{P(\sigma)}} \epsilon_Q(\sigma)$. Hence we get

$$I(P, \sigma, Q)^* \simeq I_P(\sigma'') (\text{St}_{Q'}^P(\sigma)^* n_{w_G w_{P(\sigma)}} \epsilon_Q(\sigma)).$$

We get the theorem by Theorem 4.8.\qed
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