REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 23, Pages 57–87 (January 22, 2019)
https://doi.org/10.1090/ert/521

INVOLUTIONS ON PRO-p-IWAHORI HECKE ALGEBRAS

NORIYUKI ABE

Abstract. The pro-p-Iwahori Hecke algebra has an involution ι defined in terms of the Iwahori-Matsumoto basis. Then for a module π of pro-p-Iwahori Hecke, $\pi^\iota = \pi \circ \iota$ is also a module. We calculate π^ι for simple modules π. We also calculate the dual of π. These calculations will be used for calculating the extensions between simple modules.

1. Introduction

This is the sequel of [Abe16] and the aim of these papers is to calculate the extension of simple modules of pro-p-Iwahori Hecke algebras. The calculation will appear in a sequel where we will use the results of this paper.

Let G be a connected reductive group over a non-Archimedean local field with residue characteristic p. For a field C, we can attach the pro-p-Iwahori Hecke algebra of G. This is the convolution algebra of compactly supported functions which is bi-invariant under the pro-p radical of an Iwahori subgroup. If the characteristic of C is p, then this algebra plays an important role in the representation theory of G over C (cf. [AHHV17]).

The main object of this paper are the anti-involution ζ and the involution ι when the characteristic of C is p. These are defined as follows:

- ζ: Let $W(1)$ be the "pro-p Weyl group" (see subsection 2.1 for the precise definition). Then $\mathcal{H}$ has a basis $\{T_w \mid w \in W(1)\}$ parametrized by $W(1)$ which is called the *Iwahori-Matsumoto basis*. The anti-involution ζ is defined by $\zeta(T_w) = T_{w^{-1}}$.
- ι: We also have another basis of $\mathcal{H}$ denoted by $\{T_w^* \mid w \in W(1)\}$. Then the involution ι is defined by $\iota(T_w) = (-1)^{\ell(w)} T_w^*$ where ℓ is the length function on $W(1)$.

By the multiplication rule of $\mathcal{H}$ in terms of the basis $\{T_w \mid w \in W(1)\}$ (the braid relations and the quadratic relations), these maps respect the multiplication.

Let π be a right $\mathcal{H}$-module. Then we can attach the following two modules:

- $\pi^* = \mathrm{Hom}_C(\pi, C)$ where the action of $X \in \mathcal{H}$ on $f \in \pi^*$ is given by $(fX)(v) = f(v\zeta(X))$ for $v \in \pi$.
- $\pi^\iota = \pi \circ \iota$.

If π is simple, then π^* and π^ι are also simple. We determine these modules (Theorems 3.24, 4.9).

Simple modules are classified in [Abe] based on a parabolic induction. Let P be a parabolic subgroup (which is at a good position with respect to our fixed Iwahori

Received by the editors February 6, 2018, and, in revised form, September 30, 2018.
2010 *Mathematics Subject Classification.* Primary 20C08, 20G25.
The work was supported by JSPS KAKENHI Grant Number 26707001.

subgroup) and let $\mathcal{H}_P$ be a pro-p-Iwahori Hecke algebra attached to the Levi part of P. Then we have the parabolic iundction I_P from the category of $\mathcal{H}_P$-modules to the category of $\mathcal{H}$-modules (see 2.7). The theorem in [Abe] says that simple modules are classified in terms of parabolic inductions and simple supersingular modules. More precisely, any simple module is given by the form

$$I_P(\mathrm{St}_Q^P(\sigma))$$

where P, Q are parabolic subgroups, σ is a simple supersingular module, and $\mathrm{St}_Q^P(\sigma)$ is the generalized Steinberg module (see 2.9) which is defined with parabolic inductions. Moreover, supersingular modules are classified [Oll14, Vig17]. So it is sufficient to calculate the following:

(1) $I_P(\sigma)^\iota$, $I_P(\sigma)^*$,
(2) $\mathrm{St}_Q(\sigma)^\iota$, $\mathrm{St}_Q(\sigma)^*$,
(3) π^ι, π^* for a simple supersingular module π,

where σ is a module of the pro-p-Iwahori Hecke algebra attached to the Levi part of a parabolic subgroup.

We first calculate π^ι. This is done in section 3. We do the calculation for (1) in subsection 3.1 and for (3) in subsection 3.8. The hardest step is for (2) which is calculated from subsections 3.2 to 3.7. In section 4, we calculate π^* using the calculation of π^ι.

In the next section we give notation and recall some results on pro-p-Iwahori Hecke algebras. We use the same notation as [Abe16] and often refer to this paper.

1.1. **Applications.** The results will be applied to the calculation of the extension group $\mathrm{Ext}^1_{\mathcal{H}}(\pi_1, \pi_2)$ for simple modules π_1, π_2 [Abe17]. Again, by the classification theorem, the calculation is divided into three steps.

The results in this paper will be used for the calculation of the extensions between generalized Steinberg modules $\mathrm{Ext}^1_{\mathcal{H}}(\mathrm{St}_{Q_1}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2))$. By the definition, $\mathrm{St}_{Q_1}(\sigma_1)$ is a quotient of a parabolically induced module and in fact we have a resolution of $\mathrm{St}_{Q_1}(\sigma_1)$ via parabolically induced modules. Using this resolution, the calculation of $\mathrm{Ext}^1(\mathrm{St}_{Q_1}(\sigma_1), \mathrm{St}_{Q_2}(\sigma_2))$ is deduced to that of $\mathrm{Ext}^1(I_{Q_1'}(\sigma_1'), \mathrm{St}_{Q_2}(\sigma_2))$ (where Q_1' (resp., σ_1') is a certain parabolic subgroup (resp., simple module) relating with (Q_1, σ_1)).

We also have a similar resolution for $\mathrm{St}_{Q_2}(\sigma_2)$, however this resolution is not useful for calculations $\mathrm{Ext}^1(I_{Q_1'}(\sigma_1'), \mathrm{St}_{Q_2}(\sigma_2))$ since, for the calculations, we need a resolution which has a form $0 \to \mathrm{St}_{Q_2}(\sigma_2) \to I_1 \to I_2 \to \cdots$. Here is a point where we can apply results in this paper. Taking the dual we have $\mathrm{Ext}^1(I_{Q_1'}(\sigma_1'), \mathrm{St}_{Q_2}(\sigma_2)) \simeq \mathrm{Ext}^1(\mathrm{St}_{Q_2}(\sigma_2)^*, I_{Q_1'}(\sigma_1')^*)$ and using results in this paper, we have $\mathrm{St}_{Q_2}(\sigma_2)^* \simeq \mathrm{St}_{Q_2'}(\sigma_2')$ and $I_{Q_1'}(\sigma_1')^* \simeq I'_{Q_1''}(\sigma_1'')$ for certain $Q_2', \sigma_2', Q_1'', \sigma_1''$. Here $I'_{Q_1''}$ is another functor which is defined in a similar way to parabolic inductions (see 3.1). Hence again using the resolution of generalized Steinberg modules, the calculation is deduced to that of $\mathrm{Ext}^1(I_{Q_2''}(\sigma_2''), I'_{Q_1''}(\sigma_1''))$.

2. Preliminaries

2.1. **Pro-p-Iwahori Hecke algebra.** Let F be a non-Archimedean local field, let κ be its residue field, let p be its residue characteristic, and let G be a connected reductive group over F. We denote the group of valued points $G(F)$ by the same letter G and we apply the same notation for other algebraic groups. We can get the data $(W_{\mathrm{aff}}, S_{\mathrm{aff}}, \Omega, W, W(1), Z_\kappa)$ from G as follows. See [Vig16], especially 3.9 and 4.2 for the details.

Fix a maximal split torus S and denote the centralizer of S by Z. Let Z^0 be the unique parahoric subgroup of Z and let $Z(1)$ be its pro-p radical. Then the group $W(1)$ (resp., W) is defined by $W(1) = N_G(Z)/Z(1)$ (resp., $W = N_G(Z)/Z^0$) where $N_G(Z)$ is the normalizer of Z in G. We also have $Z_\kappa = Z^0/Z(1)$. Let G' be the group generated by the unipotent radical of parabolic subgroups [AHHV17, II.1] and let W_{aff} be the image of $G' \cap N_G(Z)$ in W. Then this is a Coxeter group. Fix a set of simple reflections S_{aff}. The group Ω is the stabilizer of S_{aff} in W. Then we get the data $(W_{\mathrm{aff}}, S_{\mathrm{aff}}, \Omega, W, W(1), Z_\kappa)$. We denote the image of $G' \cap N_G(Z)$ in $W(1)$ by $W_{\mathrm{aff}}(1)$.

Attached to this data and a parameter (q, c) as in [Vig16, 4.3], we have the generic algebra $\mathcal{H}$ which we call a pro-p-Iwahori Hecke algebra. In this paper, the parameter c is always the one which comes from the group G, namely the one defined in [Vig16, 4.2]. (In [Vig16] it is denoted by $c_{s(u)}$.)

Consider the apartment attached to S and an alcove surrounded by $\{H_s \mid s \in S_{\mathrm{aff}}\}$ where H_s is the hyperplane pointwisely fixed by $s \in S_{\mathrm{aff}}$. Let $I(1)$ be the pro-p-Iwahori subgroup attached to this alcove. Then with $q_s = \#(I(1)\widetilde{s}I(1)/I(1))$ for $s \in S_{\mathrm{aff}}$ with a lift $\widetilde{s} \in N_G(Z)$, the algebra $\mathcal{H}$ is isomorphic to the Hecke algebra attached to $(G, I(1))$ [Vig16, Proposition 4.4].

We recall a little about $\mathcal{H}$. Let $S_{\mathrm{aff}}(1)$ be the inverse image of S_{aff} in $W(1)$. For $s \in S_{\mathrm{aff}}(1)$, we write q_s for $q_{\bar{s}}$ where $\bar{s} \in S_{\mathrm{aff}}$ is the image of s. The length function on W_{aff} is denoted by ℓ and its inflation to W and $W(1)$ is also denoted by ℓ.

The C-algebra $\mathcal{H}$ is a free C-module and has a basis $\{T_w\}_{w \in W(1)}$. The multiplication is given by

- (Quadratic relations) $T_s^2 = q_s T_{s^2} + c_s T_s$ for $s \in S_{\mathrm{aff}}(1)$.
- (Braid relations) $T_{vw} = T_v T_w$ if $\ell(vw) = \ell(v) + \ell(w)$.

We extend $q\colon S_{\mathrm{aff}} \to C$ to $q\colon W \to C$ as follows. For $w \in W$, take $s_1, \dots, s_l \in S_{\mathrm{aff}}$ and $u \in \Omega$ such that $w = s_1 \cdots s_l u$ and $\ell(w) = l$. Then put $q_w = q_{s_1} \cdots q_{s_l}$. From the definition, we have $q_{w^{-1}} = q_w$. We also put $q_w = q_{\overline{w}}$ for $w \in W(1)$ with the image $\overline{w}$ in W.

The aim of this paper is to study a representation theory of $\mathcal{H}$. *In this paper, modules mean right modules unless otherwise stated.*

2.2. **The algebra $\mathcal{H}[q_s]$ and $\mathcal{H}[q_s^{\pm 1}]$.** For each $s \in S_{\mathrm{aff}}$, let $\mathbf{q}_s$ be an indeterminate such that if $wsw^{-1} \in S_{\mathrm{aff}}$ for $w \in W$, we have $\mathbf{q}_{wsw^{-1}} = \mathbf{q}_s$. Let $C[\mathbf{q}_s]$ be a polynomial ring with this indeterminate. Then with the parameter $s \mapsto \mathbf{q}_s$ and the other data coming from G, we have the algebra. This algebra is denoted by $\mathcal{H}[\mathbf{q}_s]$ and we put $\mathcal{H}[\mathbf{q}_s^{\pm 1}] = \mathcal{H}[\mathbf{q}_s] \otimes_{C[\mathbf{q}_s]} C[\mathbf{q}_s^{\pm 1}]$. Under $\mathbf{q}_s \mapsto \#(I(1)\widetilde{s}I(1)/I(1)) \in C$ where $\widetilde{s} \in N_G(Z)$ is a lift of s, we have $\mathcal{H}[\mathbf{q}_s] \otimes_{C[\mathbf{q}_s]} C \simeq \mathcal{H}$. As an abbreviation, we denote $\mathbf{q}_s$ by just q_s. Consequently, we denote by $\mathcal{H}[q_s]$ (resp., $\mathcal{H}[q_s^{\pm 1}]$).

Since q_s is invertible in $\mathcal{H}[q_s^{\pm 1}]$, we can do some calculations in $\mathcal{H}[q_s^{\pm 1}]$ with q_s^{-1}. If the result can be stated in $\mathcal{H}[q_s]$, then this is an equality in $\mathcal{H}[q_s]$ since $\mathcal{H}[q_s]$ is a subalgebra of $\mathcal{H}[q_s^{\pm 1}]$ and, by specializing, we can get some equality in $\mathcal{H}$. See [Vig16, 4.5] for more details.

2.3. **The root system and the Weyl groups.** Let $W_0 = N_G(Z)/Z$ be the finite Weyl group. Then this is a quotient of W. Recall that we have the alcove defining $I(1)$. Fix a special point $\boldsymbol{x}_0$ from the border of this alcove. Then $W_0 \simeq \mathrm{Stab}_W \boldsymbol{x}_0$

and the inclusion $\mathrm{Stab}_W \boldsymbol{x}_0 \hookrightarrow W$ is a splitting of the canonical projection $W \to W_0$. Throughout this paper, we fix this special point and regard W_0 as a subgroup of W. Set $S_0 = S_{\mathrm{aff}} \cap W_0 \subset W$. This is a set of simple reflections in W_0. For each $w \in W_0$, we fix a representative $n_w \in W(1)$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

The group W_0 is the Weyl group of the root system Σ attached to (G, S). Our fixed alcove and special point give a positive system of Σ, denoted by Σ^+. The set of simple roots is denoted by Δ. As usual, for $\alpha \in \Delta$, let $s_\alpha \in S_0$ be a simple reflection for α.

The kernel of $W(1) \to W_0$ (resp., $W \to W_0$) is denoted by $\Lambda(1)$ (resp., Λ). Then $Z_\kappa \subset \Lambda(1)$ and we have $\Lambda = \Lambda(1)/Z_\kappa$. The group Λ (resp., $\Lambda(1)$) is isomorphic to Z/Z^0 (resp., $Z/Z(1)$). Any element in $W(1)$ can be uniquely written as $n_w \lambda$ where $w \in W_0$ and $\lambda \in \Lambda(1)$. We have $W = W_0 \ltimes \Lambda$.

2.4. **The map ν.** The group W acts on the apartment attached to S and the action of Λ is by the translation. Since the group of translations of the apartment is $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$, we have a group homomorphism $\nu\colon \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$. The compositions $\Lambda(1) \to \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ and $Z \to \Lambda \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ are also denoted by ν. The homomorphism $\nu\colon Z \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$ is characterized by the following: For $t \in S$ and $\chi \in X^*(S)$, we have $\nu(t)(\chi) = -\mathrm{val}(\chi(t))$ where val is the normalized valuation of F. The kernel of $\nu\colon Z \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the maximal compact subgroup $\widetilde{Z}$ of Z. In particular, $\mathrm{Ker}(\Lambda(1) \xrightarrow{\nu} X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}) = \widetilde{Z}/Z(1)$ is a finite group.

We call $\lambda \in \Lambda(1)$ dominant (resp., antidominant) if $\nu(\lambda)$ is dominant (resp., antidominant).

Since the group W_{aff} is a Coxeter system, it has the Bruhat order denoted by $\le$. For $w_1, w_2 \in W_{\mathrm{aff}}$, we write $w_1 < w_2$ if there exists $u \in \Omega$ such that $w_1 u, w_2 u \in W_{\mathrm{aff}}$ and $w_1 u < w_2 u$. Moreover, for $w_1, w_2 \in W(1)$, we write $w_1 < w_2$ if $w_1 \in W_{\mathrm{aff}}(1) w_2$ and $\overline{w}_1 < \overline{w}_2$ where $\overline{w}_1, \overline{w}_2$ are the image of w_1, w_2 in W, respectively. We write $w_1 \le w_2$ if $w_1 < w_2$ or $w_1 = w_2$.

2.5. **Other basis.** From the definition the algebra $\mathcal{H}$ has the basis $\{T_w\}_{w \in W(1)}$. This algebra also has another base which also has an important role in this paper.

The first is denoted by $\{T_w^*\}_{w \in W(1)}$ defined as follows. For $w \in W(1)$, take $s_1, \cdots, s_l \in S_{\mathrm{aff}}(1)$ and $u \in W(1)$ such that $l = \ell(w)$, $\ell(u) = 0$ and $w = s_1 \cdots s_l u$. Set $T_w^* = (T_{s_1} - c_{s_1}) \cdots (T_{s_l} - c_{s_l}) T_u$. Then this does not depend on the choice. It is not so difficult to see that we have $T_w^* \in T_w + \sum_{v < w} C T_v$ and this implies that $\{T_w^*\}_{w \in W(1)}$is a basis. In $\mathcal{H}[q_s^{\pm 1}]$, we have $T_w^* = q_w T_{w^{-1}}^{-1}$.

The second basis is called a Bernstein basis. This basis is attached to a spherical orientation o [Vig16, 5.2]. We do not recall the definition of a spherical orientation, but we remark that there is a natural bijection between spherical orientations and Weyl chambers. The Weyl group W_0 acts on spherical orientation (resp., Weyl chambers) from the right (resp., left). If a spherical orientation o and a Weyl chamber $\mathcal{C}$ corresponds to each other, $o \cdot w$ and $w^{-1}(\mathcal{C})$ corresponds for $w \in W_0$. The spherical orientation corresponding to the dominant (resp., antidominant) chamber is called the dominant (resp., antidominant) spherical orientation which is denoted by o_+ (resp., o_-).

For simplicity, we always assume that our commutative ring C contains a square root of q_s which is denoted by $q_s^{1/2}$ for $s \in S_{\mathrm{aff}}$. For $w = s_1 \cdots s_l u$ where $\ell(w) = l$, $s_1, \dots, s_l \in S_{\mathrm{aff}}$ and $\ell(u) = 0$, $q_w^{1/2} = q_{s_1}^{1/2} \cdots q_{s_l}^{1/2}$ is a square root of q_w.

We recall some properties of the Bernstein basis. The Bernstein basis attached to a spherical orientation o is denoted by $\{E_o(w)\}_{w \in W(1)}$ [Vig16, 5]. Since the definition is complicated, we do not recall it here. Similar to $\{T_w^*\}$, we have

$$E_o(w) \in T_w + \sum_{v<w} CT_v.$$

The basis satisfies the following product formula [Vig16, Theorem 5.25]:

$$E_o(w_1)E_{o\cdot w_1}(w_2) = q_{w_1w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2} E_o(w_1w_2). \tag{2.1}$$

Here $w_1, w_2 \in W(1)$ and $o \cdot w_1$ means $o \cdot \overline{w}_1$ with the image $\overline{w}_1$ of w_1 in W_0.

Remark 2.1. Since we do not assume that q_s is invertible in C, $q_{w_1w_2}^{-1/2} q_{w_1}^{1/2} q_{w_2}^{1/2}$ does not make sense in a usual way. See [Abe16, Remark 2.2].

2.6. **Levi subalgebra.** Since we have a positive system Σ^+, we have a minimal parabolic subgroup B with a Levi part Z. In this paper, *parabolic subgroups are always standard, namely containing* B. Note that such parabolic subgroups correspond to subsets of Δ.

Let P be a parabolic subgroup. Attached to the Levi part of P containing Z, we have the data $(W_{\mathrm{aff},P}, S_{\mathrm{aff},P}, \Omega_P, W_P, W_P(1), Z_\kappa)$ and the parameters (q_P, c_P). Hence we have the algebra $\mathcal{H}$. The parameter c_P is given by the restriction of c, hence we denote it just by c. The parameter q_P is defined as in [Abe, 4.1].

For the objects attached to this data, we add the suffix P. We have the set of simple roots Δ_P, the root system Σ_P and its positive system Σ_P^+, the finite Weyl group $W_{0,P}$, the set of simple reflections $S_{0,P} \subset W_{0,P}$, the length function ℓ_P and the base $\{T_w^P\}_{w \in W_P(1)}$, $\{T_w^{P*}\}_{w \in W_P(1)}$ and $\{E_o^P(w)\}_{w \in W_P(1)}$ of $\mathcal{H}_P$. Note that we have no Λ_P, $\Lambda_P(1)$ and $Z_{\kappa,P}$ since they are equal to Λ, $\Lambda(1)$, and Z_κ.

An element $w = n_v\lambda \in W_P(1)$ where $v \in W_{0,P}$ and $\lambda \in \Lambda(1)$ is called P-positive (resp., P-negative) if $\langle \alpha, \nu(\lambda) \rangle \le 0$ (resp., $\langle \alpha, \nu(\lambda) \rangle \ge 0$) for any $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Let $W_P^+(1)$ (resp., $W_P^-(1)$) be the set of P-positive (resp., P-negative) elements and put $\mathcal{H}_P^{\pm} = \bigoplus_{w \in W_P^{\pm}(1)} CT_w^P$. These are subalgebras of $\mathcal{H}_P$ [Abe, Lemma 4.1].

Proposition 2.2 ([Vig15, Theorem 1.4]). *Let λ_P^+ (resp., λ_P^-) be in the center of $W_P(1)$ such that $\langle \alpha, \nu(\lambda_P^+) \rangle < 0$ (resp., $\langle \alpha, \nu(\lambda_P^-) \rangle > 0$) for all $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. Then $T_{\lambda_P^+}^P = T_{\lambda_P^+}^{P*} = E_{o_{-,P}}^P(\lambda_P^+)$ (resp., $T_{\lambda_P^-}^P = T_{\lambda_P^-}^{P*} = E_{o_{-,P}}^P(\lambda_P^-)$) is in the center of $\mathcal{H}_P$ and we have $\mathcal{H}_P = \mathcal{H}_P^+ E_{o_{-,P}}^P(\lambda_P^+)^{-1}$ (resp., $\mathcal{H}_P = \mathcal{H}_P^- E_{o_{-,P}}^P(\lambda_P^-)^{-1}$).*

Note that such $\lambda_P^{\pm}$ always exists [Abe16, Lemma 2.4].

We define $j_P^{\pm} \colon \mathcal{H}_P^{\pm} \to \mathcal{H}$ and $j_P^{\pm *} \colon \mathcal{H}_P^{\pm} \to \mathcal{H}$ by $j_P^{\pm}(T_w^P) = T_w$ and $j_P^{\pm *}(T_w^{P*}) = T_w^*$ for $w \in W_P^{\pm}(1)$. Then these are algebra homomorphisms.

Remark 2.3. In [Abe] the homomorphism j_P^{-*} is denoted by j_M^- where M is the Levi part of P. Note that the notation j_P^- is used for a different homomorphism in this paper.

Let Q be a parabolic subgroup containing P and let $W_P^{Q+}(1)$ (resp., $W_P^{Q-}(1)$) be the set of $n_w\lambda$ where $\langle \alpha, \nu(\lambda) \rangle \le 0$ (resp., $\langle \alpha, \nu(\lambda) \rangle \ge 0$) for any $\alpha \in \Sigma_Q^+ \setminus \Sigma_P^+$ and

$w \in W_{0,P}$. Put $\mathcal{H}_P^{Q\pm} = \bigoplus_{w \in W_P^{Q\pm}(1)} CT_w^P \subset \mathcal{H}_P$. Then we have homomorphisms $j_P^{Q\pm}, j_P^{Q\pm *}\colon \mathcal{H}_P^{Q\pm} \to \mathcal{H}_Q$ defined in a similar way.

2.7. **Parabolic induction.** Let P be the parabolic subgroup and let σ be an $\mathcal{H}_P$-module. (This is a right module as in subsection 2.1.) Then we define an $\mathcal{H}$-module $I_P(\sigma)$ by

$$I_P(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^{-*})}(\mathcal{H}, \sigma).$$

(This is the same as that defined in [Abe]. We again remark that j_P^{-*} is denoted by j_P^- in [Abe].) Namely, the space of homomorphism φ from $\mathcal{H} \to \sigma$ such that $\varphi(X j_P^{-*}(Z)) = \varphi(X)Z$ for any $X \in \mathcal{H}$ and $Z \in \mathcal{H}_P^-$. The module structure is given by $(\varphi X)(Y) = \varphi(XY)$ where $X, Y \in \mathcal{H}$. We call I_P the parabolic induction.

For $P \subset P_1$, we write

$$I_P^{P_1}(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^{P_1 -}, j_P^{P_1 - *})}(\mathcal{H}_{P_1}, \sigma).$$

Let P be a parabolic subgroup. Set $W_0^P = \{w \in W_0 \mid w(\Delta_P) \subset \Sigma^+\}$. Then the multiplication map $W_0^P \times W_{0,P} \to W_0$ is bijective and for $w_1 \in W_0^P$ and $w_2 \in W_{0,P}$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. We also put ${}^P W_0 = \{w \in W_0 \mid w^{-1}(\Delta_P) \subset \Sigma^+\}$. Then the multiplication map $W_{0,P} \times {}^P W_0 \to W_0$ is bijective and for $w_1 \in W_{0,P}$ and $w_2 \in {}^P W_0$, we have $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. See [Abe16, Proposition 2.9] for the following proposition.

Proposition 2.4. *Let P be a parabolic subgroup and let σ be an $\mathcal{H}_P$-module.*

(1) *The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{n_w}))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.*
(2) *The map $I_P(\sigma) \ni \varphi \mapsto (\varphi(T_{n_w}^*))_{w \in W_0^P} \in \bigoplus_{w \in W_0^P} \sigma$ is bijective.*

Proposition 2.5 ([Abe, Proposition 4.12])**.** *Assume that $q_s = 0$ for any $s \in S_{\mathrm{aff}}$. Let $w \in W_0^P$ and $\lambda \in \Lambda(1)$. Then for $\varphi \in I_P(\sigma)$, we have*

$$(\varphi E_{o_-}(\lambda))(T_{n_w}) = \begin{cases} \varphi(T_{n_w})\sigma(E_{o_-,P}^P(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \in W_P^-(1)), \\ 0, & (n_w^{-1}\cdot\lambda \notin W_P^-(1)). \end{cases}$$

We also define

$$I'_P(\sigma) = \mathrm{Hom}_{(\mathcal{H}_P^-, j_P^-)}(\mathcal{H}, \sigma).$$

The module structure is given in the same way as I_P. We also define $I_P^{P_1\prime}$ in a similar way.

2.8. **Twist by $n_{w_G w_P}$.** For a parabolic subgroup P, let w_P be the longest element in $W_{0,P}$. In particular, w_G is the longest element in W_0. Let P' be a parabolic subgroup corresponding to $-w_G(\Delta_P)$. In other words, $P' = n_{w_G w_P} P^{\mathrm{op}} n_{w_G w_P}^{-1}$ where P^{op} is the opposite parabolic subgroup of P with respect to the Levi part of P containing Z. Set $n = n_{w_G w_P}$. Then the map $P^{\mathrm{op}} \to P'$ defined by $p \mapsto npn^{-1}$ is an isomorphism which preserves the data used to define the pro-p-Iwahori Hecke algebras. Hence $T_w^P \mapsto T_{nwn^{-1}}^{P'}$ gives an isomorphism $\mathcal{H}_P \to \mathcal{H}_{P'}$. This sends T_w^{P*} to $T_{nwn^{-1}}^{P'*}$ and $E_{o_{+,P}\cdot v}^P(w)$ to $E_{o_{+,P'}\cdot nvn^{-1}}^{P'}(nwn^{-1})$ where $v \in W_{0,P}$.

Let σ be an $\mathcal{H}_P$-module. Then we define an $\mathcal{H}_{P'}$-module $n_{w_G w_P}\sigma$ via the pull-back of the above isomorphism. Namely, for $w \in W_{P'}(1)$, we put $(n_{w_G w_P}\sigma)(T_w^{P'}) = \sigma(T^P_{n_{w_G w_P}^{-1} w n_{w_G w_P}})$.

2.9. **The extension and the generalized Steinberg modules.** Let P be a parabolic subgroup and let σ be an $\mathcal{H}_P$-module. For $\alpha \in \Delta$, let P_α be a parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Then we define $\Delta(\sigma) \subset \Delta$ by

$$\begin{aligned}&\Delta(\sigma)\\&= \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0,\ \sigma(T_\lambda^P) = 1 \text{ for any } \lambda \in W_{\mathrm{aff},P_\alpha}(1) \cap \Lambda(1)\} \cup \Delta_P.\end{aligned}$$

Let $P(\sigma)$ be a parabolic subgroup corresponding to $\Delta(\sigma)$.

Proposition 2.6 ([AHV17, Corollary 3.9])**.** *Let σ be an $\mathcal{H}_P$-module and let Q be a parabolic subgroup between P and $P(\sigma)$. Denote the parabolic subgroup corresponding to $\Delta_Q \setminus \Delta_P$ by P_2. Then there exists a unique $\mathcal{H}_Q$-module $e_Q(\sigma)$ acting on the same space as σ such that*

- *$e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ for any $w \in W_P(1)$.*
- *$e_Q(\sigma)(T_w^{Q*}) = 1$ for any $w \in W_{\mathrm{aff},P_2}(1)$.*

Moreover, one of the following conditions gives a characterization of $e_Q(\sigma)$:

1. *For any $w \in W_P^{Q-}(1)$, $e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ (namely, $e_Q(\sigma) \simeq \sigma$ as $(\mathcal{H}_P^{Q-}, j_P^{Q-*})$-modules) and for any $w \in W_{\mathrm{aff},P_2}(1)$, $e_Q(\sigma)(T_w^{Q*}) = 1$.*
2. *For any $w \in W_P^{Q+}(1)$, $e_Q(\sigma)(T_w^{Q*}) = \sigma(T_w^{P*})$ and for any $w \in W_{\mathrm{aff},P_2}(1)$, $e_Q(\sigma)(T_w^{Q*}) = 1$.*

We call $e_Q(\sigma)$ the extension of σ to $\mathcal{H}_Q$. A typical example of the extension is the trivial representation $\mathbf{1} = \mathbf{1}_G$. This is a one-dimensional $\mathcal{H}$-module defined by $\mathbf{1}(T_w) = q_w$, or, equivalently $\mathbf{1}(T_w^*) = 1$. We have $\Delta(\mathbf{1}_P) = \{\alpha \in \Delta \mid \langle \Delta_P, \alpha^\vee \rangle = 0\} \cup \Delta_P$ and if Q is a parabolic subgroup between P and $P(\mathbf{1}_P)$, we have $e_Q(\mathbf{1}_P) = \mathbf{1}_Q$

Let $P(\sigma) \supset P_0 \supset Q_1 \supset Q \supset P$. Then as in [Abe, 4.5], we have $I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \subset I_Q^{P_0}(e_Q(\sigma))$. Define

$$\mathrm{St}_Q^{P_0}(\sigma) = \mathrm{Cok}\left(\bigoplus_{Q_1 \supsetneq Q} I_{Q_1}^{P_0}(e_{Q_1}(\sigma)) \to I_Q^{P_0}(e_Q(\sigma)) \right).$$

When $P_0 = G$, we write $\mathrm{St}_Q(\sigma)$.

In the rest of this subsection, we assume that $P(\sigma) = G$. As we mentioned in the above, for $Q_1 \supset Q \supset P$ we have $I_{Q_1}(e_{Q_1}(\sigma)) \hookrightarrow I_Q(e_Q(\sigma))$. The proof of [Abe, Lemma 4.23] implies the following lemma.

Lemma 2.7. *Assume that $q_s = 0$ for any $s \in S_{\mathrm{aff}}$. The diagram*

$$\begin{tikzcd} I_{Q_1}(e_{Q_1}(\sigma)) \arrow[r, hook] \arrow[d, "\wr"] & I_Q(e_Q(\sigma)) \arrow[d, "\wr"] \\ \bigoplus_{w \in W_0^{Q_1}} \sigma \arrow[r, hook] & \bigoplus_{w \in W_0^{Q}} \sigma \end{tikzcd}$$

is commutative. Here the embedding $\bigoplus_{w \in W_0^{Q_1}} \sigma \hookrightarrow \bigoplus_{w \in W_0^{Q_1}} \sigma$ is induced by $W_0^{Q_1} \hookrightarrow W_0^{Q}$.

Lemma 2.8. *Assume that $q_s = 0$ for any $s \in S_{\mathrm{aff}}$. Let $\varphi \in I_Q(e_Q(\sigma))$. Then for $w \in W_0^Q$ and $\lambda \in \Lambda(1)$, we have*

$$(\varphi E_{o_-}(\lambda))(T_{n_w}) = \begin{cases} \varphi(T_{n_w})\sigma(E^P_{o_-,P}(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \text{ is } P\text{-negative}),\\ 0 & (\text{otherwise}).\end{cases}$$

Proof. Assume that $n_w^{-1}\cdot\lambda$ is P-negative. Then in particular it is Q-negative. By Proposition 2.5, we have

$$(\varphi E_{o_-}(\lambda))(T_{n_w}) = \varphi(T_{n_w})e_Q(\sigma)(E^Q_{o_-,Q}(n_w^{-1}\cdot\lambda)).$$

Since $n_w^{-1}\cdot\lambda$ is P-negative, we have $E^Q_{o_-,Q}(n_w^{-1}\cdot\lambda) \in \mathcal{H}_P^{Q-}$. Hence $E^Q_{o_-,Q}(n_w^{-1}\cdot\lambda) = j_P^{Q-*}(E^P_{o_-,P}(n_w^{-1}\cdot\lambda))$ by [Abe16, Lemma 2.6]. Therefore we have $e_Q(\sigma)(E^Q_{o_-,Q}(n_w^{-1}\cdot\lambda)) = \sigma(E^P_{o_-,P}(n_w^{-1}\cdot\lambda))$. We get the lemma in this case.

Assume that $n_w^{-1}\cdot\lambda$ is not P-negative. Then there exists $\alpha \in \Sigma^+ \setminus \Sigma_P^+$ such that $\langle w(\alpha), \nu(\lambda)\rangle < 0$. Take λ_P^- as in Proposition 2.2 and put $\lambda_0 = n_w\cdot\lambda_P^-$. We have $\langle w(\alpha), \nu(\lambda_0)\rangle = \langle \alpha, \nu(\lambda_P^-)\rangle > 0$. Hence λ and λ_0 are not in the same chamber. Therefore we have $E_{o_-}(\lambda)E_{o_-}(\lambda_0) = 0$ by (2.1) and [Abe16, Lemma 2.11]. Hence we have $(\varphi E_{o_-}(\lambda)E_{o_-}(\lambda_0))(T_{n_w}) = 0$. Since $n_w^{-1}\cdot\lambda_0 = \lambda_P^-$ is P-negative, we have $(\varphi E_{o_-}(\lambda)E_{o_-}(\lambda_0))(T_{n_w}) = (\varphi E_{o_-}(\lambda))(T_{n_w})\sigma(E^P_{o_-,P}(\lambda_P^-))$ as we have already proved. Since $\lambda_P^- \in Z(W_P(1))$, $E^P_{o_-,P}(\lambda_P^-)$ is invertible. Hence we have $(\varphi E_{o_-}(\lambda))(T_{n_w}) = 0$. □

2.10. **Module $\sigma_{\ell-\ell_P}$.** Let P be a parabolic subgroup and let σ be an $\mathcal{H}_P$-module. Define a linear map $\sigma_{\ell-\ell_P}$ by $\sigma_{\ell-\ell_P}(T_w) = (-1)^{\ell(w)-\ell_P(w)}\sigma(T_w)$. Then this defines a new $\mathcal{H}_P$-module $\sigma_{\ell-\ell_P}$ [Abe16, Lemma 4.1].

2.11. **Supersingular modules.** Assume that $q_s = 0$ for any $s \in S_{\mathrm{aff}}$. Let $\mathcal{O}$ be a conjugacy class in $W(1)$ which is contained in $\Lambda(1)$. For a spherical orientation o, set $z_{\mathcal{O}} = \sum_{\lambda\in\mathcal{O}} E_o(\lambda)$. Then this does not depend on o and gives an element of the center of $\mathcal{H}$ [Vig17, Theorem 5.1]. The length of $\lambda \in \mathcal{O}$ does not depend on λ. We denote it by $\ell(\mathcal{O})$.

Definition 2.9. Let π be an $\mathcal{H}$-module. We call π supersingular if there exists $n \in \mathbb{Z}_{>0}$ such that $\pi z_{\mathcal{O}}^n = 0$ for any $\mathcal{O}$ such that $\ell(\mathcal{O}) > 0$.

The simple supersingular $\mathcal{H}$-modules are classified in [Oll14, Vig17]. We recall their results. Assume that C is a field. Let χ be a character of $Z_\kappa \cap W_{\mathrm{aff}}(1)$ and put $S_{\mathrm{aff},\chi} = \{s \in S_{\mathrm{aff}} \mid \chi(c_{\widetilde{s}}) \ne 0\}$ where $\widetilde{s} \in W(1)$ is a lift of $s \in S_{\mathrm{aff}}$. Note that if $\widetilde{s}'$ is another lift, then $\widetilde{s}' = t\widetilde{s}$ for some $t \in Z_\kappa$. Hence $\chi(c_{\widetilde{s}'}) = \chi(t)\chi(c_{\widetilde{s}})$. Therefore the condition does not depend on a choice of a lift. Let $J \subset S_{\mathrm{aff},\chi}$. Then the character $\Xi = \Xi_{J,\chi}$ of $\mathcal{H}_{\mathrm{aff}}$ is defined by

$$\Xi_{J,\chi}(T_t) = \chi(t) \quad (t \in Z_\kappa \cap W_{\mathrm{aff}}(1)),$$

$$\Xi_{J,\chi}(T_{\widetilde{s}}) = \begin{cases} \chi(c_{\widetilde{s}}), & (s \in S_{\mathrm{aff},\chi}\setminus J),\\ 0, & (s \notin S_{\mathrm{aff},\chi}\setminus J),\end{cases}$$

where $\widetilde{s} \in W_{\mathrm{aff}}(1)$ is a lift of s. Let $\Omega(1)_\Xi$ be the stabilizer of Ξ and let V be a simple $C[\Omega(1)_\Xi]$-module such that $V|_{Z_\kappa\cap W_{\mathrm{aff}}(1)}$ is a direct sum of χ. Put $\mathcal{H}_\Xi = \mathcal{H}_{\mathrm{aff}}C[\Omega(1)_\Xi]$. This is a subalgebra of $\mathcal{H}$. For $X \in \mathcal{H}_{\mathrm{aff}}$ and $Y \in C[\Omega(1)_\Xi]$, we define the action of XY on $\Xi\otimes V$ by $x\otimes y \mapsto xX\otimes yY$. Then this defines a well-defined action of $\mathcal{H}_\Xi$ on $\Xi\otimes V$. Set $\pi_{\chi,J,V} = (\Xi\otimes V)\otimes_{\mathcal{H}_\Xi}\mathcal{H}$.

Proposition 2.10 ([Vig17, Theorem 1.6]). *The module $\pi_{\chi,J,V}$ is simple and it is supersingular if and only if the groups generated by J and generated by $S_{\mathrm{aff},\chi} \setminus J$ are both finite. If C is an algebraically closed field, then any simple supersingular modules are given in this way.*

Remark 2.11. This classification result is valid even though the data which defines $\mathcal{H}$ does not come from a reductive group.

2.12. **Simple modules.** Assume that C is an algebraically closed field of characteristic p. We consider the following triple (P, σ, Q):

- P is a parabolic subgroup.
- σ is a simple supersingular $\mathcal{H}_P$-module.
- Q is a parabolic subgroup between P and $P(\sigma)$.

Define

$$I(P, \sigma, Q) = I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma)).$$

Theorem 2.12 ([Abe, Theorem 1.1]). *The module $I(P, \sigma, Q)$ is simple and any simple module has this form. Moreover, (P, σ, Q) is unique up to isomorphism.*

Let χ be a character of $Z_\kappa \cap W_{\mathrm{aff},P}(1)$, $J \subset S_{\mathrm{aff},P,\chi}$ and let V be a simple module of $C[\Omega_P(1)_{\Xi_{J,\chi}}]$ whose restriction to $Z_\kappa \cap W_{\mathrm{aff},P}(1)$ is a direct sum of χ. Assume that the group generated by J and generated by $S_{P,\mathrm{aff},\chi} \setminus J$ are finite. Then we put $I(P; \chi, J, V; Q) = I(P, \pi_{\chi,J,V}, Q)$. This is a simple module.

2.13. **Möbius function.** Let Q be a parabolic subgroup and let μ^Q be the Möbius function associated to $(W_0^Q, \leq)$ where $\leq$ is the Bruhat order. The theorem due to Deodhar [Deo77, Theorem 1.2] says

$$\mu^Q(v, w) = \begin{cases} 0, & (\text{there exists } \alpha \in \Delta_Q \text{ such that } vs_\alpha \leq w), \\ (-1)^{\ell(v)+\ell(w)}, & (\text{otherwise}). \end{cases}$$

Set $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$ for $w \in W_0$. We use the following special value of the Möbius function in this paper.

Lemma 2.13. *Let $w \in W_0^Q$ such that $\Delta_{ww_Q} = \Delta \setminus \Delta_Q$ and let w_c be the longest element of the finite Weyl group of the parabolic subgroup corresponding to $\Delta \setminus \Delta_Q$. Then we have*

$$\mu^Q(w, w_G w_Q) = \begin{cases} (-1)^{\ell(w_c)}, & (w = w_G w_c w_Q), \\ 0, & (w \neq w_G w_c w_Q). \end{cases}$$

We prove this lemma by backward induction on the length of w. For the inductive step, we use the following.

Lemma 2.14. *Let $w \in W_0^Q$ such that $\Delta_{ww_Q} = \Delta \setminus \Delta_Q$, $w \neq w_G w_c w_Q$, and $\alpha \in \Delta$ such that $s_\alpha w w_Q > w w_Q$ and $\Delta_{ww_Q} = \Delta_{s_\alpha w w_Q}$. (Such α exists* [Abe, Lemma 3.15].) *We have*

(1) $s_\alpha w \in W_0^Q$.
(2) $s_\alpha w > w$.

Proof. If $\beta \in \Delta_Q$, then $w_Q(\beta) \in -\Delta_Q$. Hence $s_\alpha w w_Q(w_Q(\beta)) > 0$ since $\Delta_{s_\alpha w w_Q} = \Delta \setminus \Delta_Q$. Therefore $s_\alpha w(\beta) > 0$ for any $\beta \in \Delta_Q$. Namely, we have $s_\alpha w \in W_0^Q$. Therefore we have $s_\alpha w, w \in W_0^Q$, $w_Q \in W_{0,Q}$ and $s_\alpha w w_Q > w w_Q$. By [Deo77, Lemma 3.5], $s_\alpha w > w$. $\square$

Proof of Lemma 2.13. Assume that $w = w_G w_c w_Q$ and there exists $\alpha \in \Delta_Q$ such that $ws \le w_G w_Q$ where $s = s_\alpha$. Then we have $w_G w_c w_Q s \le w_G w_Q$. Hence $w_c w_Q s \ge w_Q$. Let Q_0 be the parabolic subgroup corresponding to $\Delta \setminus \Delta_Q$. Then $w_c \in W_{0,Q_0}$ and $W_{0,Q} \subset {}^{Q_0}W_0$. Therefore $w_Q, w_Q s \in {}^{Q_0}W_0$. Hence by [Deo77, Lemma 3.5], we have $w_Q s \ge w_Q$. This is a contradiction. Hence $\mu^Q(w, w_G w_Q) = (-1)^{\ell(w)+\ell(w_G w_Q)} = (-1)^{\ell(w_c)}$.

If $w \ne w_G w_c w_Q$, then $ww_Q \ne w_G w_c$. Take α as in the previous lemma. Assume that $s_\alpha w = w_G w_c w_Q$. Since $s_\alpha w_G w_c = ww_Q < s_\alpha w w_Q = w_G w_c$, we have $(w_G w_c)^{-1}(\alpha) < 0$. Put $\alpha' = -w_G^{-1}(\alpha) \in \Delta$. Then we have $w_c^{-1}(\alpha') > 0$. Hence $\alpha' \in \Delta_Q$. Put $\beta = -w_Q(\alpha') \in \Delta_Q$ and we prove $ws_\beta \le w_G w_Q$. We have $ws_\beta = s_\alpha w_G w_c w_Q s_\beta = w_G s_{\alpha'} w_c w_Q s_\beta$. Hence it is sufficient to prove that $s_{\alpha'} w_c w_Q s_\beta \ge w_Q$.

We have $\Delta_{w_G s_{\alpha'} w_c} = \Delta_{s_\alpha w_G w_c} = \Delta_{ww_Q} = \Delta \setminus \Delta_Q$. Hence $\Delta_{s_{\alpha'} w_c} = \Delta_Q$. In particular, $s_{\alpha'} w_c \in W_0^Q$. Hence $\ell(s_{\alpha'} w_c w_Q s_\beta) = \ell(s_{\alpha'} w_c) + \ell(w_Q s_\beta)$ as $w_Q, s_\beta \in W_{0,Q}$. Since $\alpha' \in \Delta_Q$, $s_{\alpha'} \in W_{0,Q} \subset W_0^{Q_0}$. Hence we have $\ell(s_{\alpha'} w_c) = \ell(s_{\alpha'}) + \ell(w_c)$. Therefore we get

$$\ell(s_{\alpha'} w_c w_Q s_\beta) = \ell(s_{\alpha'}) + \ell(w_c) + \ell(w_Q s_\beta).$$

Hence $s_{\alpha'} w_c w_Q s_\beta \ge s_{\alpha'} w_Q s_\beta = w_Q$.

Finally assume that $s_\alpha w \ne w_G w_c$ and we prove the lemma by backward induction on $\ell(ww_Q)$. By inductive hypothesis, there exists $\beta \in \Delta_Q$ such that $s_\alpha w s_\beta \le w_G w_Q$. Since $s_\alpha w \in W_0^Q$, we have $s_\alpha w s_\beta > s_\alpha w$. Therefore we have $s_\alpha w s_\beta > s_\alpha w > w$. By property $Z(ws_\beta, s_\alpha w s_\beta, s_\beta)$ [Deo77, §1, Remarks (2)], we have $ws_\beta \le s_\alpha w s_\beta$. Since we have $s_\alpha w s_\beta \le w_G w_Q$, we get $ws_\beta \le w_G w_Q$. $\square$

2.14. **A lemma.** In this subsection we do not assume anything about C, so C is any commutative ring. We prove the following lemma which is used in subsection 3.3. For parabolic subgroups Q_1, Q_2, let $\langle Q_1, Q_2\rangle$ be the parabolic subgroup generated by Q_1 and Q_2. Note that $\Delta_{\langle Q_1,Q_2\rangle} = \Delta_{Q_1} \cup \Delta_{Q_2}$.

Lemma 2.15. *Let P be a parabolic subgroup and let σ be an $\mathcal{H}_P$-module. Assume that $P(\sigma) = G$. For $\mathcal{P}_1, \mathcal{P}_2$ subsets of $\{Q \mid Q \supset P\}$, we have*

$$\left(\sum_{Q_1 \in \mathcal{P}_1} I_{Q_1}(e_{Q_1}(\sigma))\right) \cap \left(\sum_{Q_2 \in \mathcal{P}_2} I_{Q_2}(e_{Q_2}(\sigma))\right) = \sum_{Q_1 \in \mathcal{P}_1, Q_2 \in \mathcal{P}_2} I_{\langle Q_1,Q_2\rangle}(e_{\langle Q_1,Q_2\rangle}(\sigma))$$

in $I_P(\sigma)$.

Proof. Let P_2 be the parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Note that Δ_P is orthogonal to Δ_{P_2} as we assumed. Therefore we have $W_0^P = W_{0,P_2}$ and for any parabolic subgroup Q containing P, we have $W_{0,P_2} = W_0^Q W_{0,Q\cap P_2}$.

Put $I_Q = I_Q(e_Q(\sigma))$. We prove the lemma by induction on $\#\mathcal{P}_1$. Assume that $\#\mathcal{P}_1 = 1$. By [Abe16, Lemma 3.8], it is sufficient to prove that $I_{Q_1} \cap I_{Q_2} = I_{\langle Q_1,Q_2\rangle}$. Obviously we have $I_{Q_1} \cap I_{Q_2} \supset I_{\langle Q_1,Q_2\rangle}$. Let $\varphi \in I_{Q_1} \cap I_{Q_2}$. Then for $w \in W_0^{\langle Q_1,Q_2\rangle}$ and $\alpha \in \Delta_{Q_1} \setminus \Delta_P$, we have $\varphi(T_{n_{ws_\alpha}}) = \varphi(T_{n_w}) e_{Q_1}(\sigma)(T^{Q_1}_{n_{s_\alpha}})$. Since $n_{s_\alpha} \in W_{\mathrm{aff},P_2\cap Q}$, we have $e_{Q_1}(\sigma)(T^{Q_1}_{n_{s_\alpha}}) = q_{s_\alpha}$. Therefore we have $\varphi(T_{n_{ws_\alpha}}) = q_{s_\alpha}\varphi(T_{n_w})$. This also holds for $\alpha \in \Delta_{Q_2} \setminus \Delta_P$. Therefore $\varphi(T_{n_{wv}}) = q_v \varphi(T_{n_w})$ for any v generated by $\{s_\alpha \mid \alpha \in (\Delta_{Q_1} \cup \Delta_{Q_2}) \setminus \Delta_P\}$.

Since $(\Delta_{Q_1}\cup\Delta_{Q_2})\setminus\Delta_P = \Delta_{\langle Q_1,Q_2\rangle\cap P_2}$, the group generated by $\{s_\alpha \mid \alpha\in(\Delta_{Q_1}\cup\Delta_{Q_2})\setminus\Delta_P\}$ is $W_{0,\langle Q_1,Q_2\rangle\cap P_2}$. Hence $\varphi(T_{n_{wv}}) = q_v\varphi(T_{n_w})$ for any $w\in W_0^{\langle Q_1,Q_2\rangle}$ and $v\in W_{0,\langle Q_1,Q_2\rangle\cap P_2}$. Define $\varphi'\in I_{\langle Q_1,Q_2\rangle}(e_{\langle Q_1,Q_2\rangle}(\sigma))$ by $\varphi'(T_{n_w}) = \varphi(T_{n_w})$ for any $w\in W_0^{\langle Q_1,Q_2\rangle}$. (Such an element uniquely exists by Proposition 2.4.) Then φ' also satisfies $\varphi(T_{n_{wv}}) = q_v\varphi(T_{n_w})$ for any $w\in W_0^{\langle Q_1,Q_2\rangle}$ and $v\in W_{0,\langle Q_1,Q_2\rangle\cap P_2}$. Hence $\varphi(T_{n_w}) = \varphi'(T_{n_w})$ for any $w\in W_0^{\langle Q_1,Q_2\rangle}W_{0,\langle Q_1,Q_2\rangle\cap P_2} = W_{0,P_2} = W_0^P$. By Proposition 2.4, $\varphi = \varphi'\in I_{\langle Q_1,Q_2\rangle}$.

Now we prove the general case. Obviously we have

$$\left(\sum_{Q_1\in\mathcal{P}_1} I_{Q_1}\right)\cap\left(\sum_{Q_2\in\mathcal{P}_2} I_{Q_2}\right)\supset\sum_{Q_1\in\mathcal{P}_1,Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle}.$$

We prove the reverse inclusion. Take f from the left hand side. Fix $Q_0\in\mathcal{P}_1$ and put $\mathcal{P}_1' = \mathcal{P}_1\setminus\{Q_0\}$. Take $f_1\in I_{Q_0}$ and $f_2\in\sum_{Q_1\in\mathcal{P}_1'} I_{Q_1}$ such that $f = f_1+f_2$. Then we have

$$f_2\in\left(\sum_{Q_2\in\mathcal{P}_2} I_{Q_2} + I_{Q_0}\right)\cap\sum_{Q_1\in\mathcal{P}_1'} I_{Q_1}.$$

By inductive hypothesis, the right hand side is

$$\sum_{Q_1\in\mathcal{P}_1',Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle} + \sum_{Q_1\in\mathcal{P}_1'} I_{\langle Q_0,Q_1\rangle}.$$

Since $I_{\langle Q_0,Q_1\rangle}\subset I_{Q_0}$, we get

$$f_2\in\sum_{Q_1\in\mathcal{P}_1',Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle} + I_{Q_0}.$$

We have $f_1\in I_{Q_0}$. Therefore

$$f = f_1+f_2\in\sum_{Q_1\in\mathcal{P}_1',Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle} + I_{Q_0}.$$

Take $f_1'\in\sum_{Q_1\in\mathcal{P}_1',Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle}$ and $f_2'\in I_{Q_0}$ such that $f = f_1'+f_2'$. Then $f_1'\in\sum_{Q_2\in\mathcal{P}_2} I_{Q_2}$. By the assumption, $f\in\sum_{Q_2\in\mathcal{P}_2} I_{Q_2}$. Therefore we have $f_2' = f - f_1'\in\sum_{Q_2\in\mathcal{P}_2} I_{Q_2}$. Hence

$$f_2'\in I_{Q_0}\cap\sum_{Q_2\in\mathcal{P}_2} I_{Q_2} = \sum_{Q_2\in\mathcal{P}_2} I_{\langle Q_0,Q_2\rangle}.$$

Here we use the lemma for $\mathcal{P}_1 = \{Q_0\}$. Hence

$$\begin{aligned} f = f_1'+f_2' &\in \sum_{Q_1\in\mathcal{P}_1',Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle} + \sum_{Q_2\in\mathcal{P}_2} I_{\langle Q_0,Q_2\rangle} \\ &= \sum_{Q_1\in\mathcal{P}_1,Q_2\in\mathcal{P}_2} I_{\langle Q_1,Q_2\rangle}. \end{aligned}$$

We get the lemma. $\square$

2.15. **Assumption on C.** In the rest of this paper, we always assume that $p = 0$ in C unless otherwise stated since almost all results in this paper is proved only under this assumption. Since q_s is a power of p, this assumption implies $q_w = 0$ for any $w \in W(1)$ such that $\ell(w) > 0$. When we discuss simple modules, we also assume that C is a field. Such assumptions are written at the top of the subsections or in the statement of the theorems.

3. Twist

We define an involution $\iota = \iota_G \colon \mathcal{H} \to \mathcal{H}$ by $\iota(T_w) = (-1)^{\ell(w)} T_w^*$ [Vig16, Proposition 4.23] and $\pi^\iota = \pi \circ \iota$ for an $\mathcal{H}$-module π. Obviously, π^ι is simple if π is simple. In this section, we calculate π^ι for simple modules π.

3.1. **Parabolic induction.** Let P be a parabolic subgroup. Then by [Abe16, Lemma 4.2], we have $(\sigma^{\iota_P})_{\ell-\ell_P} = (\sigma_{\ell-\ell_P})^{\iota_P}$. We denote this module by $\sigma^{\iota_P}_{\ell-\ell_P}$. We have $I_P(\sigma)^\iota \simeq I'_P(\sigma^{\iota_P}_{\ell-\ell_P})$ [Abe16, Proposition 4.11]. In this subsection, we prove the following proposition.

Proposition 3.1. *There exists a homomorphism $\Phi \colon I_P \to I'_P$ which is characterized by $\Phi(\varphi)(XT_{n_{w_G w_P}}) = \varphi(XT_{n_{w_G w_P}})$ for any $X \in \mathcal{H}$ and $\varphi \in I_P$.*

Remark 3.2. Assume that an $\mathcal{H}_P$-module σ is not zero and $\varphi \in I_P(\sigma)$. Then we have $\Phi(\varphi)(T_{n_{w_G w_P}}) = \varphi(T_{n_{w_G w_P}})$. Since $I_P(\sigma) \to \sigma$ defined by $\varphi \mapsto \varphi(T_{n_{w_G w_P}})$ is surjective by Proposition 2.4, there exists $\varphi \in I_P(\sigma)$ such that $\varphi(T_{n_{w_G w_P}}) \neq 0$. Hence $\Phi(\varphi) \neq 0$. Therefore if σ is not zero, then $\Phi \neq 0$.

The following corollary is the first step to calculate π^ι for a simple $\mathcal{H}$-module π.

Corollary 3.3. *Assume that C is a field. Let σ be an $\mathcal{H}_P$-module. The representation $I_P(\sigma)$ is simple if and only if $I'_P(\sigma)$ is simple. Moreover, if it is the case, then $I_P(\sigma) \simeq I'_P(\sigma)$. Therefore if $I_P(\sigma)$ is simple, then $I_P(\sigma)^\iota \simeq I_P(\sigma^{\iota_P}_{\ell-\ell_P})$.*

Proof. If $I_P(\sigma)$ is simple, then the homomorphism in Proposition 3.1 is injective. We prove that $\dim I_P(\sigma) = \dim I'_P(\sigma) < \infty$. Since $I_P(\sigma)$ is simple, σ is also simple. Hence it is finite-dimensional. By $I_P(\sigma) \simeq \bigoplus_{w \in W_0^P} \sigma$ (Proposition 2.4), we have $\dim I_P(\sigma) = \# W_0^P \dim \sigma$. We also have $\dim I'_P(\sigma) = \# W_0^P \dim \sigma$ by [Abe16, Proposition 4.12]. □

For the proof of Proposition 3.1, by [Abe16, Proposition 4.13], it is sufficient to prove the following lemma.

Lemma 3.4. *Put $P' = n_{w_G w_P} P^{\mathrm{op}} n_{w_G w_P}^{-1}$. The map $\varphi \mapsto (X \mapsto \varphi(XT_{n_{w_G w_P}}))$ gives a homomorphism*

$$I_P(\sigma) \to \operatorname{Hom}_{(\mathcal{H}^+_{P'}, j^+_{P'})}(\mathcal{H}, n_{w_G w_P}\sigma).$$

Proof. Set $n = n_{w_G w_P}$ and we prove $I_P(\sigma) \ni \varphi \mapsto \varphi(T_n) \in n\sigma$ is an $(\mathcal{H}^+_{P'}, j^+_{P'})$-module homomorphism. Let $w \in W_{P'}(1)$ be a P'-positive element. By (4.1) in [Abe16], we have $j^+_{P'}(E^{P'}_{o_{+,P'}}(w))T_n = T_n j^-_P(E^P_{o_{+,P}}(n^{-1}wn))$. Hence

$$\begin{aligned}(\varphi j^+_{P'}(E^{P'}_{o_{+,P'}}(w)))(T_n) &= \varphi(j^+_{P'}(E^{P'}_{o_{+,P'}}(w))T_n)\\ &= \varphi(T_n j^-_P(E^P_{o_{+,P}}(n^{-1}wn))).\end{aligned}$$

We prove the following claim. From this claim, $(\varphi j^+_{P'}(E^{P'}_{o_{+,P'}}(w)))(T_n)$ only depends on $\varphi(T_n)$ and w.

Claim. Let $\varphi \in I_P(\sigma)$ and $X \in \mathcal{H}$. Then $\varphi(T_n X)$ only depends on $\varphi(T_n)$ and X.

We introduce a basis defined by

$$E_-(n_w\lambda) = q_{n_w\lambda}^{1/2} q_{n_w}^{-1/2} q_\lambda^{-1/2} T_{n_w}^* E_{o_-}(\lambda)$$

for $w \in W_0$ and $\lambda \in \Lambda(1)$. By [Abe, Lemma 4.2], $\{E_-(w) \mid w \in W(1)\}$ is a C-basis of $\mathcal{H}$.

To prove the claim, we may assume $X = E_-(n_w\lambda)$ for $w \in W_0$ and $\lambda \in \Lambda(1)$. Take $\lambda_P^- \in \Lambda(1)$ as in Proposition 2.2 such that $\lambda\lambda_P^-$ is P-negative. Then we have

$$\varphi(T_n E_-(n_w\lambda)) = \varphi(T_n E_-(n_w\lambda) E_{o_-}(\lambda_P^-)) \sigma(E_{o_{-,P}}^P(\lambda_P^-))^{-1}$$

by [Abe16, Lemma 2.6]. If $\ell(n_w\lambda) + \ell(\lambda_P^-) > \ell(n_w\lambda\lambda_P^-)$, then $E_-(n_w\lambda)E_{o_-}(\lambda_P^-) = 0$. Hence we have $\varphi(T_n E_-(n_w\lambda)) = 0$, so we get the claim. If $\ell(n_w\lambda) + \ell(\lambda_P^-) = \ell(n_w\lambda\lambda_P^-)$, then $E_-(n_w\lambda)E_{o_-}(\lambda_P^-) = E_-(n_w\lambda\lambda_P^-)$. Let $w_1 \in W_0^P$ and $w_2 \in W_{0,P}$ such that $w = w_1 w_2$. Then $n_{w_2}\lambda\lambda_P^- \in W_P(1)$ is P-negative. Hence $\ell(n_{w_1} n_{w_2}\lambda\lambda_P^-) = \ell(n_{w_1}) + \ell(n_{w_2}\lambda\lambda_P^-)$ by [Abe16, Lemma 2.18]. Therefore we have $E_-(n_w\lambda\lambda_P^-) = T_{n_{w_1}}^* E_-(n_{w_2}\lambda\lambda_P^-)$. If $w_1 \neq 1$, then $w_1 \notin W_{0,P}$. Hence in a reduced expression of w_1, a simple reflection s_α for some $\alpha \in \Delta \setminus \Delta_P$ appears. Therefore, there exists $x \in W_{0,P}$, $\alpha \in \Delta \setminus \Delta_P$, and $y \in W_0$ such that $w_1 = x s_\alpha y$ and $\ell(w_1) = \ell(x) + \ell(s_\alpha) + \ell(y)$. Since $x \in W_{0,P}$, $x(\alpha) \in \Sigma^+ \setminus \Sigma_P^+$. Hence $w_P x(\alpha) > 0$. Therefore $w_G w_P x(\alpha) < 0$. Hence $n n_x n_{s_\alpha} < n n_x$. Therefore we have

$$\begin{aligned}
\ell(n) + \ell(w_1) &= \ell(w_G w_P) + \ell(x) + \ell(s_\alpha) + \ell(y) \\
&\geq \ell(w_G w_P x) + \ell(s_\alpha) + \ell(y) \\
&> \ell(w_G w_P x s_\alpha) + \ell(y) \\
&\geq \ell(w_G w_P x s_\alpha y) = \ell(n n_{w_1}).
\end{aligned}$$

Hence $T_n T_{n_{w_1}}^* = E_{o_+ \cdot n^{-1}}(n) E_{o_+}(n_{w_1}) = 0$ by [Vig16, Example 5.22] and (2.1). Therefore, if $w_1 \neq 1$, namely, $w \notin W_{0,P}$, then we have $\varphi(T_n E_-(n_w\lambda\lambda_P^-)) = \varphi(T_n T_{n_{w_1}}^* E_-(n_{w_2}\lambda\lambda_P^-)) = 0$ again. If $w \in W_{0,P}$, then $n_w\lambda\lambda_P^-$ is a P-negative element. Hence $E_-^P(n_w\lambda\lambda_P^-) \in \mathcal{H}_P^-$ and we have $E_-(n_w\lambda\lambda_P^-) = j_P^{-*}(E_-^P(n_w\lambda\lambda_P^-))$ by [Abe, Lemma 4.6]. Therefore we have

$$\begin{aligned}
\varphi(T_n E_-(n_w\lambda)) &= \varphi(T_n E_-(n_w\lambda\lambda_P^-)) \sigma(E_{o_{-,P}}^P(\lambda_P^-))^{-1} \\
&= \varphi(T_n) \sigma(E_-^P(n_w\lambda\lambda_P^-) E_{o_{-,P}}^P(\lambda_P^-)^{-1}) \\
&= \varphi(T_n) \sigma(E_-^P(n_w\lambda)).
\end{aligned}$$

The claim is proved.

Let $\varphi_0 \in I_P(\sigma)$ be such that $\varphi_0(T_n) = \varphi(T_n)$ and $\varphi_0(T_{n_v}) = 0$ for $v \in W_0^P \setminus \{w_G w_P\}$. Then, as a consequence of the claim, we have

$$(\varphi j_{P'}^+(E_{o_{+,P'}}^{P'}(w)))(T_n) = (\varphi_0 j_{P'}^+(E_{o_{+,P'}}^{P'}(w)))(T_n).$$

By the proof of [Abe, Proposition 4.14], we have

$$(\varphi_0 j_{P'}^+(E_{o_{+,P'}}^{P'}(w)))(T_n) = \varphi_0(T_n)(n\sigma)(E_{o_{+,P'}}^{P'}(w)).$$

Since $\varphi_0(T_n) = \varphi(T_n)$, we have

$$\varphi_0(T_n)(n\sigma)(E^{P'}_{o_+,P'}(w)) = \varphi(T_n)(n\sigma)(E^{P'}_{o_+,P'}(w)).$$

Hence

$$(\varphi j^+_{P'}(E^{P'}_{o_+,P'}(w)))(T_n) = \varphi(T_n)(n\sigma)(E^{P'}_{o_+,P'}(w)).$$

We get the lemma. $\square$

Here is the compatibility with the transitivity of I_P and I'_P [Abe16, Proposition 4.12].

Lemma 3.5. *Let $Q \supset P$ be a parabolic subgroup. Then the following three maps are equal:*

(1) $I_P \to I'_P$.
(2) $I_P = I_Q \circ I_P^Q \to I'_Q \circ I_P^Q \to I'_Q \circ I_P^{Q\prime} = I'_P$.
(3) $I_P = I_Q \circ I_P^Q \to I_Q \circ I_P^{Q\prime} \to I'_Q \circ I_P^{Q\prime} = I'_P$.

Proof. Let σ be an $\mathcal{H}_P$-module. In each case, let $\varphi \in I_P(\sigma)$ and $\psi_i \in I'_P(\sigma)$ be the image of φ by the map in (i) for $i = 1, 2, 3$. Then ψ_1 is characterized by $\varphi(XT_{n_{w_G w_P}}) = \psi_1(XT_{n_{w_G w_P}})$ for any $X \in \mathcal{H}$. We denote the corresponding element to $\varphi \in I_P(\sigma)$ (resp., $\psi_i \in I'_P(\sigma)$) by $\varphi' \in (I_Q \circ I_P^Q)(\sigma)$ (resp., $\psi'_i \in (I'_Q \circ I_P^{Q\prime})(\sigma)$).

We consider ψ_2. We have $\psi_2(XT_{n_{w_G w_P}}) = \psi'_2(XT_{n_{w_G w_P}})(1)$. Since $\ell(w_G w_P) = \ell(w_G w_Q) + \ell(w_Q w_P)$, we have $T_{n_G w_P} = T_{n_{w_G w_Q}} T_{n_{w_Q w_P}}$. Hence $\psi'_2(XT_{n_{w_G w_P}})(1) = \psi'_2(XT_{n_{w_G w_Q}} T_{n_{w_Q w_P}})(1)$. Since $T^Q_{n_{w_Q w_P}} \in \mathcal{H}_Q^-$ and $j_Q^-(T^Q_{n_{w_Q w_P}}) = T_{n_{w_Q w_P}}$ [Abe16, Lemma 2.6], we have $\psi'_2(XT_{n_{w_G w_Q}} T_{n_{w_Q w_P}})(1) = \psi'_2(XT_{n_{w_G w_Q}})(T^Q_{n_{w_Q w_P}})$. Let $\psi''_2 \in (I'_Q \circ I_P^Q)(\sigma)$ be the image of φ. Then $\psi''_2(XT_{n_{w_G w_Q}})(Y) = \varphi'(XT_{n_{w_G w_Q}})(Y)$ and $\psi''_2(X)(YT^Q_{n_{w_Q w_P}}) = \psi'_2(X)(YT^Q_{n_{w_Q w_P}})$ for $X \in \mathcal{H}$ and $Y \in \mathcal{H}_Q$. Therefore we have $\psi_2(XT_{n_{w_G w_P}}) = \psi''_2(XT_{n_{w_G w_P}})(T^Q_{n_{w_Q w_P}}) = \varphi'(XT_{n_{w_G w_P}})(T^Q_{n_{w_Q w_P}})$. Again, since $T^Q_{n_{w_Q w_P}} \in \mathcal{H}_Q^-$ and $j_Q^{-*}(T^Q_{n_{w_Q w_P}}) = T_{n_{w_Q w_P}}$ [Abe16, Lemma 2.6], we have $\varphi'(XT_{n_{w_G w_Q}})(T^Q_{n_{w_Q w_P}}) = \varphi'(XT_{n_{w_G w_Q}} T_{n_{w_Q w_P}})(1) = \varphi'(XT_{n_{w_G w_P}})(1) = \varphi(XT_{n_{w_G w_P}})$. Hence the map in (2) satisfies the characterization of the map in (1).

The proof for (3) is similar. Let $\psi''_3 \in (I_Q \circ I_P^{Q\prime})(\sigma)$ be the image of φ. Then we have

$$\begin{aligned}
\psi_3(XT_{n_{w_G w_P}}) &= \psi'_3(XT_{n_{w_G w_P}})(1) \\
&= \psi'_3(XT_{n_{w_G w_Q}} T_{n_{w_Q w_P}})(1) \\
&= \psi'_3(XT_{n_{w_G w_Q}})(T^Q_{n_{w_Q w_P}}) \\
&= \psi''_3(XT_{n_{w_G w_Q}})(T^Q_{n_{w_Q w_P}}) \\
&= \varphi'(XT_{n_{w_G w_Q}})(T^Q_{n_{w_Q w_P}}) \\
&= \varphi'(XT_{n_{w_G w_Q}} T_{n_{w_Q w_P}})(1) \\
&= \varphi'(XT_{n_{w_G w_P}})(1) \\
&= \varphi(XT_{n_{w_G w_P}}).
\end{aligned}$$

This finishes the proof. $\square$

3.2. **Steinberg modules.** Next we consider the twist of the generalized Steinberg modules. Until subsection 3.7, we keep the following settings:

- P is a parabolic subgroup such that Δ_P is orthogonal to $\Delta \setminus \Delta_P$.
- Let P_2 be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$.
- σ is an $\mathcal{H}_P$-module such that $P(\sigma) = G$.

We prove the following proposition.

Theorem 3.6. *Let Q be a parabolic subgroup containing P and let Q^c be the parabolic subgroup corresponding to $\Delta_P \cup (\Delta \setminus \Delta_Q)$. Then we have $\mathrm{St}_Q(\sigma)^\iota \simeq \mathrm{St}_{Q^c}(\sigma^{\iota_P}_{\ell-\ell_P})$.*

The proof of this theorem continues until subsection 3.7. In this subsection, we prove this proposition for $Q = P$.

Lemma 3.7. *Theorem 3.6 is true if $Q = P$, namely we have $\mathrm{St}_P(\sigma)^\iota \simeq e_G(\sigma^{\iota_P}_{\ell-\ell_P})$.*

Proof. We use [Abe16, Proposition 3.12]. Let $w \in W_P(1)$ and assume that w is P-positive. Then $(\mathrm{St}_P(\sigma))^\iota(T_w^*) = (-1)^{\ell(w)}\mathrm{St}_P(\sigma)(T_w) = (-1)^{\ell(w)}\sigma(T_w^P) = (-1)^{\ell(w)-\ell_P(w)}\sigma^{\iota_P}(T_w^{P*}) = \sigma^{\iota_P}_{\ell-\ell_P}(T_w^{P*})$. On the other hand, for $w \in W_{\mathrm{aff},P_2}(1)$, we have $(\mathrm{St}_P(\sigma))^\iota(T_w^*) = (-1)^{\ell(w)}\mathrm{St}_P(\sigma)(T_w) = 1$. Therefore, by a characterization of the extension, we have $(\mathrm{St}_P(\sigma))^\iota \simeq e_G(\sigma^{\iota_P}_{\ell-\ell_P}) = \mathrm{St}_G(\sigma^{\iota_P}_{\ell-\ell_P})$. Since $P^c = G$, we get the lemma. □

3.3. **An exact sequence.** We express $\mathrm{St}_{Q^c}(\sigma)$ as the kernel of a certain homomorphism. As a consequence, we deduce Theorem 3.6 from the exactness of a certain sequence (Lemma 3.9).

Let Q be a parabolic subgroup containing P. Then we have an exact sequence

$$0 \to \sum_{Q\supset R\supsetneq P} I_R^Q(e_R(\sigma)) \to I_P^Q(\sigma) \to \mathrm{St}_P^Q(\sigma) \to 0.$$

Applying I_Q and using the transitivity [Vig15, Proposition 4.10], we have an exact sequence

$$0 \to \sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma)) \to I_P(\sigma) \to I_Q(\mathrm{St}_P^Q(\sigma)) \to 0.$$

Let Q_1 be a parabolic subgroup containing Q. Then we have

$$\sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma)) \subset \sum_{Q_1\supset R\supsetneq P} I_R(e_R(\sigma)).$$

Hence we have the homomorphism $I_Q(\mathrm{St}_P^Q(\sigma)) \to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))$ which makes the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & \sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma)) \arrow[r] \arrow[d, hook] & I_P(\sigma) \arrow[r] \arrow[d, equal] & I_Q(\mathrm{St}_P^Q(\sigma)) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \sum_{Q_1\supset R\supsetneq P} I_R(e_R(\sigma)) \arrow[r] & I_P(\sigma) \arrow[r] & I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma)) \arrow[r] & 0.
\end{tikzcd}$$

Lemma 3.8. *With the above homomorphisms, we have an exact sequence*

$$0 \to \mathrm{St}_{Q^c}(\sigma) \to I_Q(\mathrm{St}_P^Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma)).$$

We will prove this lemma at the end of this subsection. Applying ι to the exact sequence in the lemma, we have

$$0 \to \mathrm{St}_{Q^c}(\sigma)^\iota \to I_Q(\mathrm{St}_P^Q(\sigma))^\iota \to \bigoplus_{Q_1 \supsetneq Q} I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))^\iota.$$

Using Lemma 3.7 and [Abe16, Lemma 4.9, Proposition 4.11], we have $I_Q(\mathrm{St}_P^Q(\sigma))^\iota = I'_Q((\mathrm{St}_P^Q(\sigma))^{\iota_Q}_{\ell-\ell_Q}) = I'_Q(e_Q(\sigma^{\iota_P}_{\ell_Q-\ell_P})_{\ell-\ell_Q}) = I'_Q(e_Q(\sigma^{\iota_P}_{\ell-\ell_P}))$. We get the following exact sequence

$$0 \to (\mathrm{St}_{Q^c}\sigma)^\iota \to I'_Q(e_Q(\sigma^{\iota_P}_{\ell-\ell_P})) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma^{\iota_P}_{\ell-\ell_P})).$$

Therefore Theorem 3.6 follows from the following lemma.

Lemma 3.9. *The following sequence is exact:*

$$\bigoplus_{Q_1 \supsetneq Q} I_{Q_1}(e_{Q_1}(\sigma)) \to I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)).$$

Here $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))$ is given in Proposition 3.1 *and $I_{Q_1}(e_{Q_1}(\sigma)) \to I_Q(e_Q(\sigma))$ is the natural embedding.*

The proof of this lemma continues until subsection 3.7. Here is the outline of the proof.

- We determine the kernel of the second map in Lemma 3.10. This implies the exactness at $I_Q(e_Q(\sigma))$.
- The kernel of the third map is given in Proposition 3.13. This follows from an explicit description of the map (Proposition 3.13).
- In subsection 3.6, we prove that the kernel of the third map contains the image of the second map using a result in [Abe16] and the reverse inclusion is in Lemma 3.16. This gives the exactness at $I'_Q(e_Q(\sigma))$.

As the end of this subsection, we prove Lemma 3.8. For a family of parabolic subgroups $\{P_\lambda\}$, we denote the parabolic subgroup generated by $\{P_\lambda\}$ by $\langle P_\lambda \rangle_\lambda$. In other words, $\langle P_\lambda \rangle_\lambda$ is the parabolic subgroup corresponding to $\bigcup_\lambda \Delta_{P_\lambda}$.

Since this lemma is true over any commutative ring, we assume that C is any commutative ring in the proof.

Proof of Lemma 3.8. Let $Q_1 \supsetneq Q$. By the exact sequence before Lemma 3.8, the kernel of $I_P(\sigma) \to I_Q(\mathrm{St}_P^Q(\sigma)) \to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))$ is $\sum_{Q_1 \supset R \supsetneq P} I_R(e_R(\sigma))$. Since

$\mathrm{Ker}(I_P(\sigma) \to I_Q(\mathrm{St}_P^Q(\sigma))) = \sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma))$, we have

$$\begin{aligned}
&\mathrm{Ker}\left(I_Q(\mathrm{St}_P^Q(\sigma)) \to \bigoplus_{Q_1\supsetneq Q} I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))\right)\\
&\simeq \mathrm{Ker}\left(I_P(\sigma) \to \bigoplus_{Q_1\supsetneq Q} I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))\right) \Big/ \mathrm{Ker}(I_P(\sigma)\to I_Q(\mathrm{St}_P^Q(\sigma)))\\
&= \left(\bigcap_{Q_1\supsetneq Q} \mathrm{Ker}(I_P(\sigma)\to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma)))\right) \Big/ \mathrm{Ker}(I_P(\sigma)\to I_Q(\mathrm{St}_P^Q(\sigma)))\\
&= \bigcap_{Q_1\supsetneq Q}\sum_{Q_1\supset R\supsetneq P} I_R(e_R(\sigma)) \Big/ \sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma))\,.
\end{aligned}$$

Set $A = \bigcap_{Q_1\supsetneq Q}\sum_{Q_1\supset R\supsetneq P} I_R(e_R(\sigma))$ and $B = \sum_{Q\supset R\supsetneq P} I_R(e_R(\sigma))$. We prove the following which gives the lemma:

(1) $I_{Q^c}(e_{Q^c}(\sigma)) + B = A$.
(2) $I_{Q^c}(e_{Q^c}(\sigma)) \cap B = \sum_{R\supsetneq Q^c} I_R(e_R(\sigma))$.

First we prove (1). We prove $I_{Q^c}(e_{Q^c}(\sigma)) + B \subset A$. Let $Q_1 \supsetneq Q$. Take $\alpha \in \Delta_{Q_1}\setminus\Delta_Q$ and let R be a parabolic subgroup corresponding to $\Delta_P\cup\{\alpha\}$. Since $\alpha\in\Delta_{Q^c}$ and $\Delta_P\subset\Delta_{Q^c}$, we have $\Delta_R\subset\Delta_{Q^c}$. Hence $I_{Q^c}(e_{Q^c}(\sigma))\subset I_R(e_R(\sigma))\subset A$. Obviously we have $B\subset A$.

We prove the reverse inclusion. By Lemma 2.15, we have

$$A = \sum_{(R_{Q_1})_{Q_1\supsetneq Q}} I_{\langle R_{Q_1}\rangle_{Q_1}}(e_{\langle R_{Q_1}\rangle_{Q_1}}(\sigma)),$$

where R_{Q_1} satisfies $Q_1\supset R_{Q_1}\supsetneq P$ and $\langle R_{Q_1}\rangle_{Q_1}$ is the group generated by $\{R_{Q_1}\mid Q_1\supsetneq Q\}$. Hence it is sufficient to prove that each $I_{\langle R_{Q_1}\rangle_{Q_1}}(e_{\langle R_{Q_1}\rangle_{Q_1}}(\sigma))$ is contained in $I_{Q^c}(e_{Q^c}(\sigma))+B$. If $Q\supset R_{Q_0}$ for some $Q_0\supsetneq Q$, then for such Q_0, we have $I_{\langle R_{Q_1}\rangle_{Q_1}}(e_{\langle R_{Q_1}\rangle_{Q_1}}(\sigma))\subset I_{R_{Q_0}}(e_{R_{Q_0}}(\sigma))\subset B$. Assume that $Q\not\supset R_{Q_1}$ for any $Q_1\supsetneq Q$, and we prove $\Delta_{Q^c}\subset\bigcup_{Q_1}\Delta_{R_{Q_1}}$. Let $\alpha\in\Delta_{Q^c} = (\Delta\setminus\Delta_Q)\cup\Delta_P$. If $\alpha\in\Delta_P$, then we have $\alpha\in\bigcup_{Q_1}\Delta_{R_{Q_1}}$ since $P\subset R_{Q_1}$. Assume that $\alpha\in\Delta\setminus\Delta_Q$. Let Q_α be the parabolic subgroup corresponding to $\Delta_Q\cup\{\alpha\}$. Then by the assumption, $\Delta_{R_{Q_\alpha}}$ is contained in $\Delta_{Q_\alpha} = \Delta_Q\cup\{\alpha\}$ and is not contained in Δ_Q. Hence $\alpha\in\Delta_{R_{Q_\alpha}}$. Therefore $\alpha\in\Delta_{R_{Q_\alpha}}\subset\bigcup_{Q_1\supsetneq Q}\Delta_{R_{Q_1}} = \Delta_{\langle R_{Q_1}\rangle_{Q_1}}$ by taking $Q_1 = Q_\alpha$. Hence we have $Q^c\subset\langle R_{Q_1}\rangle_{Q_1}$. Therefore $I_{\langle R_{Q_1}\rangle_{Q_1}}(e_{\langle R_{Q_1}\rangle_{Q_1}}(\sigma))\subset I_{Q^c}(e_{Q^c}(\sigma))$. We get (1).

We prove (2). By Lemma 2.15, we have

$$I_{Q^c}(e_{Q^c}(\sigma))\cap B = \sum_{Q\supset R\supsetneq P} I_{\langle R,Q^c\rangle}(e_{\langle R,Q^c\rangle}(\sigma)).$$

First we prove $I_{Q^c}(e_{Q^c}(\sigma))\cap B\subset\sum_{R_1\supsetneq Q^c} I_{R_1}(e_{R_1}(\sigma))$, namely, for each R such that $Q\supset R\supsetneq P$ we have $I_{\langle R,Q^c\rangle}(e_{\langle R,Q^c\rangle}(\sigma))\subset\sum_{R_1\supsetneq Q^c} I_{R_1}R(e_{R_1}(\sigma))$. For such R, we can take $\alpha\in\Delta_R\setminus\Delta_P$. Since $\alpha\in\Delta_Q\setminus\Delta_P$, $\alpha\notin\Delta_{Q^c}$. Hence $\Delta_{\langle R,Q^c\rangle} = \Delta_R\cup\Delta_{Q^c}\supset\{\alpha\}\cup\Delta_{Q^c}\supsetneq\Delta_{Q^c}$. Therefore $\langle R,Q^c\rangle\supsetneq Q^c$. Hence $I_{\langle R,Q^c\rangle}(e_{\langle R,Q^c\rangle}(\sigma))\subset\sum_{R_1\supsetneq Q^c} I_{R_1}(e_{R_1}(\sigma))$ by taking $R_1 = \langle R,Q^c\rangle$.

We prove $I_R(e_R(\sigma)) \subset I_{Q^c}(e_{Q^c}(\sigma)) \cap B$ for any R such that $R \supsetneq Q^c$. We can take $\alpha \in \Delta_R \setminus \Delta_{Q^c}$. Let P_α be the parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$. Since $\alpha \notin \Delta_{Q^c}$, we have $\alpha \in \Delta_Q$. Therefore $Q \supset P_\alpha \supsetneq P$. Hence $\Delta_R \supset \Delta_{Q^c} \cup \{\alpha\} = \Delta_{\langle P_\alpha, Q^c\rangle}$. Therefore $R \supset \langle P_\alpha, Q^c\rangle$. Hence $I_R(e_R(\sigma)) \subset I_{\langle P_\alpha, Q^c\rangle}(e_{\langle P_\alpha, Q^c\rangle}(\sigma)) \subset I_{Q^c}(e_{Q^c}(\sigma)) \cap B$. We get (2) and the proof of the lemma is finished. □

3.4. **The kernel of $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))$.** Recall that we put $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$ for $w \in W_0$. We determine $\mathrm{Ker}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$, namely we prove the following lemma.

Lemma 3.10. *Set $A = \{w \in W_0^Q \mid \Delta_w = \Delta_Q\}$. Then we have*

$$\begin{aligned}&\mathrm{Ker}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))\\ &= \{\varphi \in I_Q(e_Q(\sigma)) \mid \varphi(XT_{n_w}) = 0 \text{ for any } X \in \mathcal{H} \text{ and } w \in A\}\\ &= \{\varphi \in I_Q(e_Q(\sigma)) \mid \varphi(T_{n_w}) = 0 \text{ for any } w \in A\} = \sum_{Q_1 \supsetneq Q} I_{Q_1}(e_{Q_1}(\sigma)).\end{aligned}$$

The last equality follows from Lemma 2.7 and the following lemma.

Lemma 3.11. *We have $\bigcap_{Q_1 \supsetneq Q}(W_0^Q \setminus W_0^{Q_1}) = \{w \in W_0^Q \mid \Delta_w = \Delta_Q\}$.*

Proof. Let $w \in W_0^Q$. Then $w(\Delta_Q) \subset \Sigma^+$ and we have $\Delta_w = \Delta_Q$ if and only if for any $Q_1 \supsetneq Q$, $w(\Delta_{Q_1}) \not\subset \Sigma^+$. Since we have $w(\Delta_{Q_1}) \not\subset \Sigma^+$ if and only if $w \notin W_0^{Q_1}$, we get the lemma. □

Let $\varphi \in I_Q(e_Q(\sigma))$ such that $\varphi(T_{n_w}) = 0$ for any $w \in A$ and take $X \in \mathcal{H}$. The last equality implies that the set of such φ is stable under $\mathcal{H}$. Hence φX also satisfies the same condition. Therefore $\varphi(XT_{n_w}) = 0$ for any $w \in A$. Namely, we get the second equality.

Let ψ be the image of φ under $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))$. Then ψ is characterized by $\varphi(XT_{n_{w_G w_Q}}) = \psi(XT_{n_{w_G w_Q}})$. Therefore we have

$$\begin{aligned}&\mathrm{Ker}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))\\ &= \{\varphi \in I_Q(e_Q(\sigma)) \mid \varphi(XT_{n_{w_G w_Q}}) = 0 \text{ for any } X \in \mathcal{H}\}.\end{aligned}$$

Therefore, to prove the lemma, it is sufficient to prove the following lemma.

Lemma 3.12. *Let $\varphi \in I_Q(e_Q(\sigma))$. Assume that $\varphi(XT_{n_{w_G w_Q}}) = 0$ for any $X \in \mathcal{H}$. Then we have $\varphi(XT_{n_w}) = 0$ for any $X \in \mathcal{H}$ and $w \in W_0^Q$ such that $\Delta_w = \Delta_Q$.*

Proof. We prove the lemma by backward induction on $\ell(w)$. If $w \neq w_G w_Q$, then there exists $\alpha \in \Delta$ such that $s_\alpha w > w$, $\Delta_w = \Delta_{s_\alpha w}$, and $w^{-1}(\alpha)$ is not simple [Abe, Lemma 3.15]. Set $s = s_\alpha$. Since $\Delta_{sw} = \Delta_w = \Delta_Q$, we have $sw \in W_0^Q$. If $w^{-1}(\alpha) \in \Sigma_Q^+$, then since $sw \in W_0^Q$, we have $-\alpha = sw(w^{-1}(\alpha)) \in \Sigma^+$. This is a contradiction. Hence

$$w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_Q^+.$$

Take $\lambda_P^- \in Z(W_P(1))$ as in Proposition 2.2 . Put $\lambda = n_w \cdot (\lambda_P^-)^2$. We prove the following.

Claim. $E_-(\lambda n_s^{-1})(T_{n_s} - c_{n_s}) = E_{o_-}(\lambda)$ in $\mathcal{H}$.

We calculate the left hand side in $\mathcal{H}[q_s^{\pm 1}]$. We use notation in [Abe, Lemma 2.10]. Since $w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_Q^+ \subset \Sigma^+ \setminus \Sigma_P^+$, we have $\langle \alpha, \nu(\lambda)\rangle = \langle w^{-1}(\alpha), \nu((\lambda_P^-)^2)\rangle > 0$. Therefore we have $\ell(\lambda n_s^{-1}) = \ell(\lambda) - 1$ by [Abe16, Lemma 2.17]. Hence $q_{\lambda n_s^{-1}} = q_\lambda q_{n_s}^{-1}$. Therefore we have

$$\begin{aligned} E_-(\lambda n_s^{-1}) &= E_-(n_s^{-1}(n_s \cdot \lambda)) \\ &= q_{\lambda n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda) \\ &= q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda). \end{aligned}$$

Hence by [Abe, Lemma 2.10], we have

$$\begin{aligned} &E_-(\lambda n_s^{-1}) T_{n_s} \\ &= q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda) T_{n_s} \\ &= q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* T_{n_s} \theta(\lambda) + \sum_{k=0}^{\langle \alpha, \nu(\lambda) - 1} q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda \mu_{n_s}(k)) c_{n_s,k}. \end{aligned} \tag{3.1}$$

We have

$$q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* T_{n_s} \theta(\lambda) = q_\lambda^{1/2} \theta(\lambda) = E_{o_-}(\lambda)$$

and if $k = 0$, since $q_{\lambda n_s^{-1}} = q_\lambda q_{n_s}^{-1}$, we have

$$\begin{aligned} q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda \mu_{n_s}(k)) c_{n_s,k}. &= q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda) c_{n_s} \\ &= q_{\lambda n_s^{-1}}^{1/2} q_{n_s}^{-1/2} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda) c_{n_s} \\ &= E_-(n_s^{-1}(n_s \cdot \lambda)) c_{n_s} = E_-(\lambda n_s^{-1}) c_{n_s}. \end{aligned}$$

We prove that if $1 \le k \le \langle \alpha, \nu(\lambda)\rangle - 1$, then $q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda \mu_{n_s}(k)) c_{n_s,k} = 0$ in $\mathcal{H}$. We have

$$\begin{aligned} &q_\lambda^{1/2} q_{n_s}^{-1} T_{n_s^{-1}}^* \theta(n_s \cdot \lambda \mu_{n_s}(k)) c_{n_s,k} \\ &= q_\lambda^{1/2} q_{n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))}^{-1/2} q_{n_s}^{-1/2} E_-(n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))) c_{n_s,k}. \end{aligned}$$

Hence (3.1) is an expansion of $E_-(\lambda n_s^{-1}) T_{n_s}$ with respect to the basis $\{E_-(w) \mid w \in W(1)\}$. Since this is a basis of $\mathcal{H}[q_s]$ as a $C[q_s]$-module, each coefficient is in $C[q_s]$. Hence $q_\lambda^{1/2} q_{n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))}^{-1/2} q_{n_s}^{-1/2} \in C[q_s]$. Namely, for each $s \in S_{\mathrm{aff}}$ there exists $k_s \in \mathbb{Z}_{\ge 0}/{\sim}$ (where the equivalence relation $\sim$ is defined by the adjoint action of W on S_{aff}) such that $q_\lambda^{1/2} q_{n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))}^{-1/2} q_{n_s}^{-1/2} = \prod_{s \in S_{\mathrm{aff}}/\sim} q_s^{k_s}$. We have $\sum_s k_s = (1/2)(\ell(\lambda) - \ell(n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))) - \ell(n_s))$. We calculate

$$\begin{aligned} \ell(\lambda) - \ell(n_s^{-1}(n_s \cdot \lambda \mu_{n_s}(k))) - \ell(n_s) &\ge \ell(\lambda) - \ell(n_s^{-1}) - \ell(n_s \cdot \lambda \mu_{n_s}(k)) - \ell(n_s) \\ &= \ell(\lambda) - \ell(n_s \cdot \lambda \mu_{n_s}(k)) - 2. \end{aligned}$$

By [Abe, Lemma 2.12], $\ell(\lambda) - \ell(n_s \cdot \lambda \mu_{n_s}(k)) \ge 2\min\{k, \langle \alpha, \nu(\lambda)\rangle - k\}$. If the equality holds, again by [Abe, Lemma 2.12], there exists $v \in W_0$ such that $v\nu(\lambda)$ is dominant and $v(\alpha)$ is simple.

Assume that $v(\nu(\lambda))$ is dominant for $v \in W_0$. We have $\nu(\lambda) = w(\nu((\lambda_P^-)^2))$ and since $\nu((\lambda_P^-)^2)$ is dominant, we have $vw \in \mathrm{Stab}_{W_0}(\nu((\lambda_P^-)^2))$. By the condition of λ_P^-, the stabilizer of $\nu((\lambda_P^-)^2)$ is $W_{0,P}$. Hence $vw \in W_{0,P}$. Since $w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_Q^+$, we have $w^{-1}(\alpha) \in \Sigma_{P_2}^+$. Any element in Σ_{P_2} is fixed by elements in $W_{0,P}$. Hence

$vw(w^{-1}(\alpha)) = w^{-1}(\alpha)$. Therefore $v(\alpha) = w^{-1}(\alpha)$. This is not simple by the condition on α.

Hence we always have

$$\ell(\lambda) - \ell(n_s^{-1}(n_s \cdot \lambda\mu_{n_s}(k))) - \ell(n_s) > 0$$

for $1 \le k \le \langle \alpha, \nu(\lambda)\rangle - 1$. Hence $\sum_s k_s > 0$. Therefore there exists s such that $k_s > 0$. Hence $\prod_s q_s^{k_s} = 0$ in $\mathcal{H}$. We get $q_\lambda^{1/2} q_{n_s}^{-1} T^*_{n_s^{-1}} \theta(n_s \cdot \lambda\mu_{n_s}(k)) c_{n_s,k} = 0$. Therefore we have

$$E_-(\lambda n_s^{-1})T_{n_s} = E_{o_-}(\lambda) + E_-(\lambda n_s^{-1})c_{n_s}.$$

This gives the claim.

We return to the proof of the lemma. Since $n_w^{-1} \cdot \lambda = (\lambda_P^-)^2$ is P-negative, by Lemma 2.8, we have

$$\varphi(XE_{o_-}(\lambda)T_{n_w}) = \varphi(XT_{n_w})\sigma(E^P_{o_-,P}(n_w^{-1} \cdot \lambda)).$$

Since $\sigma(E^P_{o_-,P}(n_w^{-1}\cdot\lambda)) = \sigma(E^P_{o_-,P}((\lambda_0^-)^2))$ is invertible, it is sufficient to prove that $\varphi(XE_{o_-}(\lambda)T_{n_w}) = 0$. By the claim, we have

$$\varphi(XE_{o_-}(\lambda)T_{n_w}) = \varphi(XE_-(\lambda n_s^{-1})T_{n_s}T_{n_w}) - \varphi(XE_-(\lambda n_s^{-1})c_{n_s}T_{n_w}).$$

By inductive hypothesis, $\varphi(XE_-(\lambda n_s^{-1})T_{n_s}T_{n_w}) = \varphi(XE_-(\lambda n_s^{-1})T_{n_{sw}}) = 0$. We have

$$\varphi(XE_{o_-}(\lambda n_s^{-1})c_{n_s}T_{n_w}) = \varphi(X((\lambda n_s^{-1})\cdot c_{n_s})E_-(\lambda n_s^{-1})T_{n_w})$$

Set $\lambda' = n_w \cdot \lambda_P^-$. We have $\ell(\lambda n_s^{-1}) = \ell(\lambda) - 1 = \ell((\lambda')^2) - 1 = 2\ell(\lambda') - 1$ as $\langle \alpha, \nu(\lambda)\rangle > 0$. Since $\langle \alpha, \nu(\lambda')\rangle = \langle w^{-1}(\alpha), \nu(\lambda_P^-)\rangle > 0$, we have $\ell(\lambda') - 1 = \ell(\lambda' n_s^{-1})$. By [Abe16, Lemma 2.15], we have $\ell(\lambda') = \ell(n_s \cdot \lambda')$. Hence $\ell(\lambda n_s^{-1}) = \ell(\lambda' n_s^{-1}) + \ell(n_s \cdot \lambda')$. Therefore

$$E_-(\lambda n_s^{-1}) = E_-(\lambda' n_s^{-1}(n_s \cdot \lambda')) = E_-(\lambda' n_s^{-1})E_{o_-}(n_s \cdot \lambda')$$

by the definition of $E_-(\lambda n_s^{-1})$. Hence

$$\begin{aligned}&\varphi(X((\lambda n_s^{-1}) \cdot c_{n_s})E_-(\lambda n_s^{-1})T_{n_w})\\&= \varphi(X((\lambda n_s^{-1})\cdot c_{n_s})E_-(\lambda' n_s^{-1})E_{o_-}(n_s \cdot \lambda')T_{n_w}).\end{aligned}$$

Since $w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_Q^+ \subset \Sigma^+ \setminus \Sigma_P^+$, we have $\langle w^{-1}(\alpha), \nu(n_w^{-1}n_s \cdot \lambda')\rangle = -\langle \alpha, \nu(\lambda')\rangle = -\langle w^{-1}(\alpha), \nu(\lambda_P^-)\rangle < 0$. Therefore $n_w^{-1}n_s \cdot \lambda'$ is not Q-negative. Hence $\varphi(X((\lambda n_s^{-1})\cdot c_{n_s})E_-(\lambda' n_s^{-1})E_{o_-}(n_s \cdot \lambda')T_{n_w}) = 0$ by Proposition 2.5. □

3.5. **The homomorphism $I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma))$.** Let $Q_1 \supset Q \supset P$ be parabolic subgroups. Recall that we have the homomorphism $I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma))$. This is defined by $I_Q(\mathrm{St}_P^Q(\sigma^{\iota_P}_{\ell-\ell_P})) \to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma^{\iota_P}_{\ell-\ell_P}))$ with ι. We give the following description of this homomorphism.

Proposition 3.13. *Let $\varphi \in I'_Q(e_Q(\sigma))$ and $\varphi' \in I'_{Q_1}(e_{Q_1}(\sigma))$ be the image of φ. Then for $w \in W_0^{Q_1}$, we have $\varphi'(T^*_{n_w}) = (-1)^{\ell(w_{Q_1}w_Q)}\varphi(T^*_{n_{ww_{Q_1}w_Q}})$. In particular, combining with* [Abe16, Proposition 4.12]*, we have*

$$\begin{aligned}&\mathrm{Ker}(I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma)))\\&= \{\varphi \in I'_Q(e_Q(\sigma)) \mid \varphi(T^*_{n_w}) = 0 \textit{ for any } w \in W_0^{Q_1}w_{Q_1}w_Q\}.\end{aligned}$$

First we describe the homomorphism $I_Q(\mathrm{St}_P^Q(\sigma)) \to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))$. Recall that the kernel of $I_P^Q(\sigma) \ni \varphi \mapsto \varphi(T_{n_{w_Q w_P}}) \in \sigma$ is $\sum_{Q\supset P_1 \supsetneq P} I_{P_1}^Q(e_{P_1}(\sigma))$ and hence it gives an identification $\sigma \simeq \mathrm{St}_P^Q(\sigma)$ as vector spaces by Lemma 2.7.

Lemma 3.14. *The homomorphism $I_Q(\mathrm{St}_P^Q(\sigma)) \to I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma))$ is given by $\varphi \mapsto (X \mapsto \varphi(XT_{n_{w_{Q_1} w_Q}}))$. (Here we identify $\mathrm{St}_P^Q(\sigma)$ and $\mathrm{St}_P^{Q_1}(\sigma)$ with σ.)*

Proof. Since $I_P^Q(\sigma) \to \mathrm{St}_P^Q(\sigma)$ is given by $\varphi \mapsto \varphi(T^Q_{n_{w_Q w_P}})$ (under the identification $\sigma = \mathrm{St}_P^Q(\sigma)$), $I_P(\sigma) \to I_Q(\mathrm{St}_P^Q(\sigma))$ is given by $\varphi \mapsto (X \mapsto \varphi(XT_{n_{w_Q w_P}}))$. Now recall the following commutative diagram which defines the homomorphism in the lemma:

$$\begin{array}{ccc} I_P(\sigma) & \longrightarrow & I_Q(\mathrm{St}_P^Q(\sigma)) \\ \| & & \downarrow \\ I_P(\sigma) & \longrightarrow & I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma)). \end{array}$$

Let $\varphi \in I_Q(\mathrm{St}_P^Q(\sigma))$ and take $\widetilde{\varphi} \in I_P(\sigma)$ which is a lift of φ. Then we have $\varphi(X) = \widetilde{\varphi}(XT_{n_{w_Q w_P}})$. Let φ' be the image of φ. Then from the above commutative diagram we have $\varphi'(X) = \widetilde{\varphi}(XT_{n_{w_{Q_1} w_P}})$. Since $n_{w_{Q_1} w_P} = n_{w_{Q_1} w_Q} n_{w_Q w_P}$, we have $\widetilde{\varphi}(XT_{n_{w_{Q_1} w_P}}) = \widetilde{\varphi}(XT_{n_{w_{Q_1} w_Q}} T_{n_{w_Q w_P}}) = \varphi(XT_{n_{w_{Q_1} w_Q}})$. $\square$

Proof of Proposition 3.13. Let $\varphi \in I'_Q(e_Q(\sigma))$ and $\varphi' \in I'_{Q_1}(e_{Q_1}(\sigma))$ its image. Then $\varphi \circ \iota \in I_Q(\mathrm{St}_P^Q(\sigma^{\iota_P}_{\ell-\ell_P}))$ and $\varphi' \circ \iota \in I_{Q_1}(\mathrm{St}_P^{Q_1}(\sigma^{\iota_P}_{\ell-\ell_P}))$. By the above lemma, we have $\varphi' \circ \iota(X) = \varphi \circ \iota(XT_{n_{w_{Q_1} w_Q}})$. Hence $\varphi'(\iota(X)) = \varphi(\iota(XT_{n_{w_{Q_1} w_Q}})) = (-1)^{\ell(w_{Q_1} w_Q)} \varphi(\iota(X) T^*_{n_{w_{Q_1} w_Q}})$ for any $X \in \mathcal{H}$. Therefore, for any $X \in \mathcal{H}$, we have $\varphi'(X) = (-1)^{\ell(w_{Q_1} w_Q)} \varphi(XT^*_{n_{w_{Q_1} w_Q}})$ for any $X \in \mathcal{H}$. $\square$

By Proposition 3.13, $\varphi \in \mathrm{Ker}(I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)))$ if and only if $\varphi(T^*_{n_w}) = 0$ for any $w \in \bigcup_{Q_1 \supsetneq Q} W_0^{Q_1} w_{Q_1} w_Q$. We get the following description of the kernel appearing in Lemma 3.9.

Lemma 3.15. *Let $w \in W_0^Q$. We have $\Delta_{ww_Q} \neq \Delta \setminus \Delta_Q$ if and only if for some $Q_1 \supsetneq Q$, $w \in W_0^{Q_1} w_{Q_1} w_Q$. Hence we have*

$$\mathrm{Ker}\left(I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)) \right)$$
$$= \{\varphi \in I'_Q(e_Q(\sigma)) \mid \varphi(T^*_{n_w}) = 0 \textit{ for any } w \in W_0^Q \textit{ such that } \Delta_{ww_Q} \neq \Delta \setminus \Delta_Q\}.$$

Proof. Let $w \in W_0^Q$ and assume that for some $Q_1 \supsetneq Q$ and $v \in W_0^{Q_1}$, we have $w = vw_{Q_1} w_Q$. Then for $\alpha \in \Delta_{Q_1} \setminus \Delta_Q$, $w_{Q_1}(\alpha) \in \Sigma^-_{Q_1}$. Since $v \in W_0^{Q_1}$, we have $vw_{Q_1}(\alpha) < 0$. Hence $ww_Q(\alpha) < 0$. Therefore $\alpha \notin \Delta_{ww_Q}$. Hence $\Delta_{ww_Q} \neq \Delta \setminus \Delta_Q$.

Assume that $\Delta_{ww_Q} \neq \Delta \setminus \Delta_Q$. Since $w \in W_0^Q$, for any $\alpha \in \Delta_Q$ we have $ww_Q(\alpha) < 0$. Hence $\alpha \notin \Delta_{ww_Q}$. Therefore $\Delta_Q \subset \Delta \setminus \Delta_{ww_Q}$, namely we have $\Delta_{ww_Q} \subset \Delta \setminus \Delta_Q$. Hence $\Delta_{ww_Q} \not\supset \Delta \setminus \Delta_Q$. Take $\alpha \in (\Delta \setminus \Delta_Q) \setminus \Delta_{ww_Q}$. Let Q_1 be a parabolic subgroup corresponding to $\Delta_Q \cup \{\alpha\}$. Then we have $\Delta_{Q_1} \subset \Delta \setminus \Delta_{ww_Q}$. If $\beta \in \Delta_{Q_1}$, then $w_{Q_1}(\beta) \in -\Delta_{Q_1} \subset -(\Delta \setminus \Delta_{ww_Q})$. Hence $ww_Q w_{Q_1}(\beta) > 0$. Therefore $ww_Q w_{Q_1} \in W_0^{Q_1}$. We have $w \in W_0^{Q_1} w_{Q_1} w_Q$. $\square$

3.6. **Complex.** In this subsection, we prove that the sequence in Lemma 3.9 is a complex, namely the composition

$$I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma))$$

is zero for any parabolic subgroup $Q_1 \supsetneq Q$.

We have the following diagram:

$$\begin{tikzcd}
I_Q(e_Q(\sigma)) \arrow[rr] \arrow[d, "\wr"] & & I'_Q(e_Q(\sigma)) \arrow[r] \arrow[d, "\wr"] & I'_{Q_1}(e_{Q_1}(\sigma)) \\
I_{Q_1}I_Q^{Q_1}(e_Q(\sigma)) \arrow[r] & I'_{Q_1}(I_Q^{Q_1}(e_Q(\sigma))) \arrow[r] & I'_{Q_1}(I_Q^{Q_1\prime}(e_Q(\sigma))). \arrow[ur] &
\end{tikzcd}$$

This is commutative by Lemma 3.5. The sequence

$$I'_{Q_1}(I_Q^{Q_1}(e_Q(\sigma))) \to I'_{Q_1}(I_Q^{Q_1\prime}(e_Q(\sigma))) \to I'_{Q_1}(e_{Q_1}(\sigma))$$

comes from

$$I_Q^{Q_1}(e_Q(\sigma)) \to I_Q^{Q_1\prime}(e_Q(\sigma)) \to e_{Q_1}(\sigma).$$

Let $R_Q^{Q_1}$ be the right adjoint functor of $I_Q^{Q_1}$ [Vig15, Proposition 4.1]. Since we have $R_Q^{Q_1}(e_{Q_1}(\sigma)) = 0$ by [Abe16, Lemma 5.17], this composition is zero.

3.7. **Exactness.** Now we finish the proof of Lemma 3.9 by proving the following lemma.

Lemma 3.16. *We have*

$$\operatorname{Im}\bigl(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma))\bigr) \supset \operatorname{Ker}\left(I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma))\right).$$

We start with the following lemma.

Lemma 3.17. *Let $\psi \in I'_Q(e_Q(\sigma))$ and $w \in W_0^Q$. Then we have $\psi(T^*_{n_w}) = \sum_{v \in W_0^Q, v \le w} \psi(T_{n_v})$.*

Proof. The same argument as the proof of [AHHV17, IV.9 Proposition] implies

$$\psi(T^*_{n_w}) = \sum_{v \le w} \psi(T_{n_v}).$$

If $v \notin W_0^Q$, then there exists $v_1 \in W_0^Q$ and $v_2 \in W_{0,Q} \setminus \{1\}$ such that $v = v_1 v_2$. We have $\ell(v_1 v_2) = \ell(v_1) + \ell(v_2)$. Hence $\psi(T_{n_v}) = \psi(T_{n_{v_1}} T_{n_{v_2}}) = \psi(T_{n_{v_1}}) e_Q(\sigma)(T^Q_{n_{v_2}}) = 0$ by the definition of $e_Q(\sigma)$. We get the lemma. □

Let w_c be the longest element of the finite Weyl group of the parabolic subgroup corresponding to $\Delta \setminus \Delta_Q$. Then $\Delta_{w_G w_c} = \Delta \setminus \Delta_Q$ and $w = w_G w_c$ is maximal in $\{w \in W_0 \mid \Delta_w = \Delta \setminus \Delta_Q\}$ [Abe, Remark 3.16].

We assume that $\psi \in \operatorname{Ker}(I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)))$. Let μ^Q be the Möbius function associated to $(W_0^Q, \le)$. Then Lemma 3.17 and the definition of the Möbius function gives

$$\psi(T_{n_w}) = \sum_{v \le w, v \in W_0^Q} \mu^Q(v, w) \psi(T^*_{n_v}).$$

By Lemma 2.13, we have the following.

Lemma 3.18. *If $\psi \in \operatorname{Ker}(I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)))$, then $\psi(T_{n_{w_G w_Q}}) = (-1)^{\ell(w_c)}\psi(T^*_{n_{w_G w_c w_Q}})$.*

Lemma 3.19. *Consider a linear map $I'_Q(e_Q(\sigma)) \to \sigma$ defined by $\psi \mapsto \psi(T^*_{n_{w_G w_c w_Q}})$. Then the composition $I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)) \to \sigma$ is surjective.*

Proof. Let $\psi \in I'_Q(e_Q(\sigma))$ be the image of $\varphi \in I_Q(e_Q(\sigma))$. Then the characterization of the homomorphism (Proposition 3.1) gives $\varphi(T_{n_{w_G w_Q}}) = \psi(T_{n_{w_G w_Q}})$. Since ψ is also in the kernel of $I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma))$ by 3.6, we have $\psi(T^*_{n_{w_G w_c w_Q}}) = (-1)^{\ell(w_c)}\psi(T_{n_{w_G w_Q}})$. Hence $\psi(T^*_{n_{w_G w_c w_Q}}) = (-1)^{\ell(w_c)}\varphi(T_{n_{w_G w_Q}})$. The lemma follows from the surectivity of the map $I_Q(e_Q(\sigma)) \ni \varphi \mapsto \varphi(T_{n_{w_G w_Q}}) \in \sigma$ (Proposition 2.4). □

The following lemma ends the proof of Lemma 3.16, hence that of Theorem 3.6.

Lemma 3.20. *For $w \in W_0^Q$ such that $\Delta_{w w_Q} = \Delta \setminus \Delta_Q$ and $x \in \sigma$, there exists $\psi \in \operatorname{Im}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$ such that for any $v \in W_0^Q$ we have*

$$\psi(T^*_{n_v}) = \begin{cases} x, & (v = w), \\ 0, & (v \ne w). \end{cases}$$

We need one lemma.

Lemma 3.21. *Let $w \in W_0^P$ and $\lambda \in \Lambda(1)$. Then for $\varphi \in I'_P(\sigma)$, we have*

$$(\varphi E_{o_+}(\lambda))(T^*_{n_w}) = \begin{cases} \varphi(T^*_{n_w})\sigma(E^P_{o_+,P}(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \in W_P^-(\lambda)), \\ 0, & (n_w^{-1}\cdot\lambda \notin W_P^-(\lambda)). \end{cases}$$

Proof. Set $\varphi^\iota = \varphi\circ\iota$ and $\sigma' = \sigma^{\iota_P}_{\ell-\ell_P}$. Then we have $\varphi^\iota \in I_P(\sigma')$ [Abe16, Proposition 4.11]. Hence, by Proposition 2.5, we have

$$\varphi^\iota(E_{o_-}(\lambda)T_{n_w}) = \begin{cases} \varphi^\iota(T_{n_w})\sigma'(E^P_{o_-}(n_w^{-1}\cdot\lambda)), & (n_w^{-1}\cdot\lambda \in W_P^-(\lambda)), \\ 0, & (n_w^{-1}\cdot\lambda \notin W_P^-(\lambda)). \end{cases}$$

The left hand side is

$$\varphi^\iota(E_{o_-}(\lambda)T_{n_w}) = \varphi(\iota(E_{o_-}(\lambda))\iota(T_{n_w})) = \varphi((-1)^{\ell(\lambda)}E_{o_+}(\lambda)(-1)^{\ell(n_w)}T^*_{n_w})$$

by [Vig16, Lemma 5.31]. Therefore if $n_w^{-1}\cdot\lambda \notin W_P^-(\lambda)$, then $\varphi(E_{o_+}(\lambda)T^*_{n_w}) = 0$.

If $n_w^{-1}\cdot\lambda \in W_P^-(\lambda)$, then

$$\begin{aligned} \sigma'(E^P_{o_-}(n_w^{-1}\cdot\lambda)) &= (-1)^{\ell(n_w^{-1}\cdot\lambda)-\ell_P(n_w^{-1}\cdot\lambda)}\sigma(\iota_P(E^P_{o_-}(n_w^{-1}\cdot\lambda))) \\ &= (-1)^{\ell(n_w^{-1}\cdot\lambda)}\sigma(E^P_{o_+}(n_w^{-1}\cdot\lambda)) \end{aligned}$$

again by [Vig16, Lemma 5.31] and [Abe16, Lemma 4.5]. We also have $\varphi^\iota(T_{n_w}) = (-1)^{\ell(n_w)}\varphi(T^*_{n_w})$. Hence we get

$$\varphi((-1)^{\ell(\lambda)}E_{o_+}(\lambda)(-1)^{\ell(n_w)}T^*_{n_w}) = (-1)^{\ell(n_w)}(-1)^{\ell(n_w^{-1}\cdot\lambda)}\varphi(T^*_{n_w})\sigma(E^P_{o_+}(n_w^{-1}\cdot\lambda)).$$

Since $\ell(n_w^{-1}\cdot\lambda) = \ell(\lambda)$ [Abe16, Lemma 2.15], we get the lemma. □

Proof of Lemma 3.20. We prove the lemma by induction on the length of ww_Q. Assume that $ww_Q = w_G w_c$, namely $w = w_G w_c w_Q$. Let $\lambda_P^- \in Z(W_P(1))$ as in Proposition 2.2. Notice that $n_{w_Q}^{-1} \cdot \lambda_P^-$ is P-negative since w_Q (in fact, any element in W_0) preserves $\Sigma^+ \setminus \Sigma_P^+ = \Sigma_{P_2}^+$. We also have that $n_{w_Q}^{-1} \cdot \lambda_P^- \in Z(W_P(1))$ since n_{w_Q} normalizes $W_P(1)$. Hence $e_Q(\sigma)(T^{Q*}_{n_{w_Q}^{-1}\cdot\lambda_P^-}) = \sigma(T^{P*}_{n_{w_Q}^{-1}\cdot\lambda_P^-})$ is invertible. Take $\psi_0 \in \operatorname{Im}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$ such that $\psi_0(T^*_{n_{w_G w_c w_Q}}) = xe_Q(\sigma)(T^{Q*}_{n_{w_Q}^{-1}\cdot\lambda_P^-})^{-1}$. Put $\lambda = n_{w_G w_c}\cdot\lambda_P^-$ and set $\psi = \psi_0 E_{o_+}(\lambda)$. Let $v \in W_0^Q$. If $v \ne w_G w_c w_Q$, then since v and $w_G w_c w_Q$ are in W_0^Q, we have $v \notin w_G w_c w_Q W_{0,Q}$. Hence $(w_G w_c)^{-1} v \notin W_{0,Q}$. Therefore there exists $\alpha \in \Sigma^+ \setminus \Sigma_Q^+$ such that $(w_G w_c)^{-1} v(\alpha) < 0$. Since $\Sigma^+ \setminus \Sigma_Q^+ \subset \Sigma_{P_2}$ and, Σ_{P_2} is stabilized by W_0, we have $(w_G w_c)^{-1} v(\alpha) \in \Sigma_{P_2}^- = \Sigma^- \setminus \Sigma_P^-$. Hence $\langle (w_G w_c)^{-1} v(\alpha), \nu(\lambda_P^-)\rangle < 0$. The left hand side is $\langle \alpha, \nu(n_v^{-1}\cdot\lambda)\rangle$. Hence $n_v^{-1}\cdot\lambda$ is not Q-negative. Therefore $\psi(T_{n_v}) = (\psi_0 E_{o_+}(\lambda))(T_{n_v}) = 0$ if $v \ne w_G w_c w_Q$ by the above lemma.

Assume that $v = w_G w_c w_Q$. Then $n_v^{-1}\cdot\lambda = n_{w_Q}^{-1}\cdot\lambda_P^-$. Since λ_P^- is dominant, it is Q-negative. The set of Q-negative elements is stable under the conjugate action of $W_Q(1)$. Hence $n_{w_Q}^{-1}\cdot\lambda_P^-$ is also Q-negative. Therefore we have $\psi(T^*_{n_{w_G w_c w_Q}}) = (\psi_0 E_{o_+}(\lambda))(T^*_{n_{w_G w_c w_Q}}) = \psi_0(T^*_{n_{w_G w_c w_Q}}) e_Q(\sigma)(T^{Q*}_{n_{w_Q}^{-1}\cdot\lambda_P^-}) = x$ by the previous lemma. We get the lemma when $w = w_G w_c w_Q$.

Assume that $w \ne w_G w_c w_Q$ and take α such that $s_\alpha w w_Q > w w_Q$ and $\Delta_{s_\alpha w w_Q} = \Delta_{w w_Q}$ as in Lemma 2.14. Then $s_\alpha w > w$ by Lemma 2.14. Set $s = s_\alpha$. By inductive hypothesis, there exists $\psi_0 \in \operatorname{Im}(I_Q(e_Q(\sigma)) \to I'_Q(e_Q(\sigma)))$ such that for $v \in W_0^Q \setminus \{s_\alpha w\}$, $\psi_0(T^*_{n_v}) = 0$ and $\psi_0(T^*_{n_{s_\alpha w}}) = x$. We prove that $\psi = \psi_0 T_{n_s}$ satisfies the condition of the lemma. First we have

$$\psi(T^*_{n_w}) = \psi_0(T_{n_s}T^*_{n_w}) = \psi_0((T^*_{n_s} - c_{n_s})T^*_{n_w}) = \psi_0(T^*_{n_s}T^*_{n_w}) - \psi_0(c_{n_s}T^*_{n_w}).$$

Since $sw > w$, we have $T^*_{n_s}T^*_{n_w} = T^*_{n_{sw}}$. Hence $\psi_0(T^*_{n_s}T^*_{n_w}) = x$. Since $\psi_0(T^*_{n_w}) = 0$, we have $\psi_0(c_{n_s}T^*_{n_w}) = \psi_0(T^*_{n_w})e_Q(\sigma)(n_w^{-1}\cdot c_{n_s}) = 0$. Hence $\psi(T^*_{n_w}) = x$.

Assume that $v \ne w$. If $\ell(s) + \ell(v) > \ell(sv)$, then $T_{n_s}T^*_{n_v} = T_{n_s}T^*_{n_s}T^*_{n_{sv}} = 0$. Hence $\psi(T^*_{n_v}) = 0$. If $\ell(s) + \ell(v) = \ell(sv)$, then

$$\psi(T^*_{n_v}) = \psi_0(T_{n_s}T^*_{n_v}) = \psi_0((T^*_{n_s} - c_{n_s})T^*_{n_v}) = \psi_0(T^*_{n_{sv}}) - \psi_0(c_{n_s}T^*_{n_v}).$$

Since $v \ne w$, $sv \ne sw$. Hence $\psi_0(T^*_{n_{sv}}) = 0$. Since $sv > v$ and $s(sw) < sw$, we have $v \ne sw$. Hence $\psi_0(c_{n_s}T^*_{n_v}) = \psi_0(T^*_{n_v})e_Q(\sigma)(n_v^{-1}\cdot c_{n_s}) = 0$. Therefore we get $\psi(T^*_{n_v}) = 0$. □

3.8. **Supersingular modules.**

Proposition 3.22. *If π is supsersingular, then π^ι is also supsersingular.*

Proof. For each $W(1)$-orbit $\mathcal{O} \subset \Lambda(1)$ such that $\ell(\mathcal{O}) > 0$, we have

$$\iota(z_\mathcal{O}) = \sum_{\lambda\in\mathcal{O}} \iota(E_{o_-}(\lambda)) = \sum_{\lambda\in\mathcal{O}} (-1)^{\ell(\mathcal{O})} E_{o_+}(\lambda) = (-1)^{\ell(\mathcal{O})} z_\mathcal{O}$$

by [Vig16, Lemma 5.31]. Hence $\pi(z_\mathcal{O}^n) = 0$ implies $\pi^\iota(z_\mathcal{O}^n) = 0$. □

Proposition 3.23. *Assume that C is a field. Let $\pi = \pi_{\chi,J,V}$ be a simple supersingular representation. Then $\pi^\iota = \pi_{\chi,S_{\mathrm{aff},\chi}\setminus J,V}$.*

Proof. Let $\Xi = \Xi_{\chi,J}$ be the character of $\mathcal{H}_{\mathrm{aff}}$ parametrized by (χ, J). The representation π is given by $\pi = (V \otimes \Xi_{\chi,J}) \otimes_{\mathcal{H}_{\mathrm{aff}} C[\Omega(1)_\Xi]} \mathcal{H}$. The homomorphism ι preserves $\mathcal{H}_{\mathrm{aff}}$ and $C[\Omega(1)_\Xi]$. (On $C[\Omega(1)]$, ι is identity.) Hence we get $\pi^\iota = (V^\iota \otimes \Xi^\iota) \otimes_{\mathcal{H}_{\mathrm{aff}} C[\Omega(1)_\Xi]} \otimes \mathcal{H}$. Since ι is trivial on $C[\Omega(1)_\Xi]$, $V^\iota = V$. Let (χ', J') be the pair such that Ξ^ι is parametrized by (χ', J'). The character χ' is a direct summand of $V^\iota|_{Z_\kappa \cap W_{\mathrm{aff}}(1)}$ and since $V^\iota = V$, we have $V^\iota|_{Z_\kappa \cap W_{\mathrm{aff}}(1)} = V|_{Z_\kappa \cap W_{\mathrm{aff}}(1)}$, since $V|_{Z_\kappa \cap W_{\mathrm{aff}}(1)}$ is a direct sum of χ, $\chi' = \chi$. The subset $J \subset S_{\mathrm{aff},\chi}$ satisfies

$$\Xi(T_{\widetilde{s}}) = \begin{cases} 0, & (s \in J), \\ \chi(c_{\widetilde{s}}), & (s \in S_{\mathrm{aff},\chi} \setminus J), \end{cases}$$

where $\widetilde{s} \in W_{\mathrm{aff}}(1)$ is a lift of s. We have $\Xi(\iota(T_{\widetilde{s}})) = -\Xi(T_{\widetilde{s}} - c_{\widetilde{s}}) = -\Xi(T_{\widetilde{s}}) + \chi(c_{\widetilde{s}})$. Therefore we have

$$\Xi^\iota(T_{\widetilde{s}}) = \begin{cases} \chi(c_{\widetilde{s}}), & (s \in J), \\ 0, & (s \in S_{\mathrm{aff},\chi} \setminus J). \end{cases}$$

We have $J' = \{s \in S_{\mathrm{aff},\chi} \mid \Xi^\iota(T_{n_s}) = 0\}$. Hence $J' = S_{\mathrm{aff},\chi} \setminus J$. □

3.9. **Simple modules.** Assume that C is an algebraically closed field. Summarizing the results in this section, we have the following. We need notation. By [Abe16, Remark 4.6], $T_w^P \mapsto (-1)^{\ell(w) - \ell_P(w)} T_w^P$ is an algebra homomorphism of $\mathcal{H}_P$. This preserves the subalgebra $C[\Omega_P(1)]$. Let Ξ be a character of $\mathcal{H}_{\mathrm{aff},P}$. Then the above homomorphism also preserves $C[\Omega_P(1)_\Xi]$ since the homomorphism is trivial on $\mathcal{H}_{\mathrm{aff},P}$ by [Abe16, Lemma 4.7]. For a $C[\Omega_P(1)_\Xi]$-module V, let $V_{\ell - \ell_P}$ be the pull-back of V by this homomorphism.

Theorem 3.24. *Let $I(P; \chi, J, V; Q)$ be a simple representation. Then we have $I(P; \chi, J, V; Q)^\iota = I(P; \chi, S_{\mathrm{aff},P,\chi} \setminus J, V_{\ell - \ell_p}; Q^c)$ where $\Delta_{Q^c} = \Delta_P \cup (\Delta(\sigma) \setminus \Delta_Q)$.*

Proof. Since $I(P; \chi, J, V; Q) = I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\pi_{\chi,J,V}))$ is simple, by Corollary 3.3, we have $I(P; \chi, J, V; Q)^\iota = I_{P(\sigma)}((\mathrm{St}_Q^{P(\sigma)}(\pi_{\chi,J,V}))^{\iota_{P(\sigma)}}_{\ell - \ell_{P(\sigma)}})$. By Theorem 3.6, Proposition 3.23, and [Abe16, Lemma 4.9], we have

$$\begin{aligned} I(P; \chi, J, V; Q)^\iota &\simeq I_{P(\sigma)}(\mathrm{St}_{Q^c}^{P(\sigma)}((\pi_{\chi,J,V})^{\iota_P}_{\ell - \ell_P})) \\ &\simeq I_{P(\sigma)}(\mathrm{St}_{Q^c}^{P(\sigma)}((\pi_{\chi,S_{\mathrm{aff},P,\chi} \setminus J, V})_{\ell - \ell_P})). \end{aligned}$$

Let Ξ be a character of $\mathcal{H}_{\mathrm{aff},P}$ defined by the pair χ and $S_{\mathrm{aff},P,\chi} \setminus J$. Put $\mathcal{H}_{P,\Xi} = \mathcal{H}_{\mathrm{aff},P} C[\Omega_P(1)_\Xi]$. Then $\pi_{\chi,S_{\mathrm{aff},P,\chi} \setminus J, V} = (\Xi \otimes V) \otimes_{\mathcal{H}_{P,\Xi}} \mathcal{H}_P$. Let $f \colon \mathcal{H}_P \to \mathcal{H}_P$ be an algebra homomorphism defined by $f(T_w^P) = (-1)^{\ell(w) - \ell_P(w)} T_w^P$. Then f preserves $\mathcal{H}_{\mathrm{aff},P}$ and $C[\Omega_P(1)_\Xi]$ and we have $(\pi_{\chi,S_{\mathrm{aff},P,\chi} \setminus J, V})_{\ell - \ell_P} = \pi_{\chi,S_{\mathrm{aff},P,\chi} \setminus J, V} \circ f = ((\Xi \circ f) \otimes (V \circ f)) \otimes_{\mathcal{H}_{P,\Xi}} \mathcal{H}_P$. By the definition, $V \circ f = V_{\ell - \ell_P}$. By [Abe16, Lemma 4.7], f is identity on $\mathcal{H}_{\mathrm{aff},P}$. Hence $\Xi \circ f = \Xi$. Hence $(\pi_{\chi,S_{\mathrm{aff},P,\chi} \setminus J, V})_{\ell - \ell_P} = (\Xi \otimes V_{\ell - \ell_P}) \otimes_{\mathcal{H}_{P,\Xi}} \mathcal{H}$ and we get the theorem. □

3.10. **Structure of I'_P.** Assume that C is an algebraically closed field.

Proposition 3.25. *Let P be a parabolic subgroup and let σ be a simple supersingular representation of $\mathcal{H}_P$. Then for each parabolic subgroup Q between P and $P(\sigma)$, there exists a submodule $\pi_Q \subset I'_P(\sigma)$ such that*

1. *if $Q_1 \subset Q_2$, then $\pi_{Q_1} \subset \pi_{Q_2}$.*
2. *$\pi_Q / \sum_{Q_1 \subsetneq Q} \pi_{Q_1} = I(P, \sigma, Q)$.*

Compare with $I_Q(e_Q(\sigma)) \subset I_P(\sigma)$. In other words, the structure of $I'_P(\sigma)$ is "opposite to" that of $I_P(\sigma)$.

Proof. First assume that $P(\sigma) = G$. Put $\sigma' = \sigma^{\iota_P}_{\ell-\ell_P}$. Then $I'_P(\sigma) = I_P(\sigma')^\iota$. Set $\pi_Q = I_{Q^c}(e_{Q^c}(\sigma'))^\iota \subset I'_P(\sigma)$ where $\Delta_{Q^c} = (\Delta \setminus \Delta_Q) \cup \Delta_P$. Then the first condition is satisfied. Since $Q_1 \subset Q_2$ if and only if $Q_1^c \supset Q_2^c$, we have

$$\begin{aligned}
\pi_Q / \sum_{Q_1 \subsetneq Q} \pi_{Q_1} &= \left(I_{Q^c}(e_{Q^c}(\sigma')) / \sum_{Q_1 \subsetneq Q} I_{Q_1^c}(e_{Q_1^c}(\sigma')) \right)^\iota \\
&= \left(I_{Q^c}(e_{Q^c}(\sigma')) / \sum_{Q_1 \supsetneq Q^c} I_{Q_1}(e_{Q_1}(\sigma')) \right)^\iota \\
&= (\mathrm{St}_{Q^c}(\sigma'))^\iota = \mathrm{St}_Q(\sigma)
\end{aligned}$$

by Theorem 3.6. By the assumption $P(\sigma) = G$, we have $\mathrm{St}_Q(\sigma) = I(P, \sigma, Q)$. We get the proposition in this case.

In general, applying the proposition for $I_P^{P(\sigma)\prime}(\sigma)$, we get $\pi'_Q \subset I_P^{P(\sigma)\prime}(\sigma)$ for each $P(\sigma) \supset Q \supset P$. Put $\pi_Q = I'_{P(\sigma)}(\pi'_Q)$. The first condition is obvious. For the second condition, we have

$$\begin{aligned}
\pi_Q / \sum_{Q_1 \subsetneq Q} \pi_{Q_1} &= I'_{P(\sigma)}(\pi'_Q) / \sum_{Q_1 \subsetneq Q} I'_{P(\sigma)}(\pi'_{Q_1}) \\
&\simeq I'_{P(\sigma)} \left(\pi'_Q / \sum_{Q_1 \subsetneq Q} \pi'_{Q_1} \right) \\
&\simeq I'_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma)).
\end{aligned}$$

Since $I(P, \sigma, Q) = I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma))$ is simple, by Corollary 3.3, we have $I(P, \sigma, Q) \simeq I'_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}(\sigma))$. Now we get the proposition. $\square$

4. Dual

We have an antiautomorphism $\zeta = \zeta_G \colon \mathcal{H} \to \mathcal{H}$ defined by $\zeta(T_w) = T_{w^{-1}}$. Hence for a representation π, its linear dual $\pi^* = \mathrm{Hom}_C(\pi, C)$ has a structure of a right $\mathcal{H}$-module defined by $(fX)(v) = f(v\zeta(X))$ for $f \in \pi^*, v \in \pi$ and $X \in \mathcal{H}$. Since any simple representation is finite-dimensional, if π is simple, then π^* is again simple. In this section, we compute π^*.

Lemma 4.1. *We have* $\zeta(T_w^*) = T_{w^{-1}}^*$.

Proof. In $\mathcal{H}[q_s^{\pm 1}]$, we have $\zeta(T_w^*) = \zeta(q_w T_{w^{-1}}^{-1}) = q_w T_w^{-1} = T_{w^{-1}}^*$. $\square$

4.1. **Parabolic inductions.** In this subsection, we calculate $I_P(\sigma)^*$. Let $P' = n_{w_G w_P} P^{\mathrm{op}} n_{w_G w_P}^{-1}$. Then we have $I_P(\sigma) \simeq n_{w_G w_P}\sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}$ by [Abe16, Proposition 2.21]. Hence we have

$$\begin{aligned}
\mathrm{Hom}_C(I_P(\sigma), C) &\simeq \mathrm{Hom}_C(n_{w_G w_P}\sigma \otimes_{(\mathcal{H}_{P'}^+, j_{P'}^+)} \mathcal{H}, C) \\
&\simeq \mathrm{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}, \mathrm{Hom}_C(n_{w_G w_P}\sigma, C)).
\end{aligned}$$

Therefore $I_P(\sigma)^* \simeq \mathrm{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}, \mathrm{Hom}_C(n_{w_G w_P}\sigma, C))$ and here the action on the right hand side is twisted by ζ. Let $\varphi \in \mathrm{Hom}_{(\mathcal{H}_{P'}^+, j_{P'}^+)}(\mathcal{H}, \mathrm{Hom}_C(n_{w_G w_P}\sigma, C))$ and set $\varphi^\zeta = \varphi \circ \zeta$. Let $w \in W_{P'}(1)$ which is P'-negative. Then w^{-1} is P'-positive. Hence for $X \in \mathcal{H}$ and $x \in n_{w_G w_P}\sigma$, we have $\varphi^\zeta(XT_w)(x) = \varphi(\zeta(XT_w))(x) = \varphi(T_{w^{-1}}\zeta(X))(x) = (T_{w^{-1}}^{P'}\varphi(\zeta(X)))(x) = \varphi(\zeta(X))(xT_{w^{-1}}^{P'}) = \varphi^\zeta(X)(x\zeta_P(T_w^{P'}))$. (Here we regard $\mathrm{Hom}_C(n_{w_G w_P}\sigma, C)$ as a left $\mathcal{H}_{P'}$-module.) Therefore

$$\varphi^\zeta \in \mathrm{Hom}_{(\mathcal{H}_{P'}^-, j_{P'}^-)}(\mathcal{H}, (n_{w_G w_P}\sigma)^*).$$

For $X, Y \in \mathcal{H}$, we have $(\varphi Y)^\zeta(X) = (\varphi Y)(\zeta(X)) = \varphi(\zeta(X)\zeta(Y)) = \varphi(\zeta(YX)) = \varphi^\zeta(YX) = (\varphi^\zeta Y)(X)$. Hence $\varphi \mapsto \varphi^\zeta$ induces

$$I_P(\sigma)^* \simeq \mathrm{Hom}_{(\mathcal{H}_{P'}^-, j_{P'}^-)}(\mathcal{H}, (n_{w_G w_P}\sigma)^*) = I'_{P'}(n_{w_G w_P}\sigma^*).$$

Proposition 4.2. *We have* $I_P(\sigma)^* \simeq I'_{P'}(n_{w_G w_P}\sigma^*)$.

The same calculation shows the following.

Proposition 4.3. *We have* $I'_P(\sigma)^* \simeq I_{P'}(n_{w_G w_P}\sigma^*)$.

Remark 4.4. These propositions are true for any commutative ring C.

4.2. **Steinberg modules.** Let P be a parabolic subgroup, let σ be an $\mathcal{H}_P$-module such that $P(\sigma) = G$ and let P_2 be the parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. We calculate $(\mathrm{St}_Q\sigma)^*$.

Proposition 4.5. *Let Q be a parabolic subgroup containing P and put $Q' = n_{w_G w_Q} Q^{\mathrm{op}} n_{w_G w_Q}^{-1}$. Then* $(\mathrm{St}_Q\sigma)^* \simeq \mathrm{St}_{Q'}\sigma^*$.

We start with the case of $Q = G$.

Lemma 4.6. *We have* $e_G(\sigma)^* \simeq e_G(\sigma^*)$.

Proof. Let $f \in e_G(\sigma)^*$ and $x \in e_G(\sigma)$. For $w \in W_P(1)$, we have

$$\begin{aligned}(fe_G(\sigma)^*(T_w^*))(x) &= f(xe_G(\sigma)(\zeta(T_w^*))) = f(xe_G(\sigma)(T_{w^{-1}}^*))\\ &= f(x\sigma(T_{w^{-1}}^{P*})) = f(x\sigma(\zeta_P(T_w^{P*}))) = (f\sigma^*(T_w^{P*}))(x).\end{aligned}$$

Hence $e_G(\sigma)^*(T_w^*) = \sigma^*(T_w^{P*})$. For $w \in W_{\mathrm{aff},P_2}(1)$, we have $(fe_G(\sigma)^*(T_w^*))(x) = f(xe_G(\sigma)(\zeta(T_w^*))) = f(xe_G(\sigma)(T_{w^{-1}}^*)) = f(x)$. Hence $e_G(\sigma)^*(T_w^*) = 1$. Therefore by the characterization of $e_G(\sigma^*)$, we have the lemma. ☐

Proof of Proposition 4.5. By Lemma 3.9, we have the following exact sequence:

$$0 \to \mathrm{St}_Q(\sigma) \to I'_Q(e_Q(\sigma)) \to \bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma)).$$

Taking the dual, we get an exact sequence

$$\bigoplus_{Q_1 \supsetneq Q} I'_{Q_1}(e_{Q_1}(\sigma))^* \to I'_Q(e_Q(\sigma))^* \to \mathrm{St}_Q(\sigma)^* \to 0.$$

Put $Q' = n_{w_G w_Q} Q^{\mathrm{op}} n_{w_G w_Q}^{-1}$. Then by Proposition 4.3, we have $I'_Q(e_Q(\sigma))^* = I_{Q'}(n_{w_G w_Q} e_Q(\sigma)^*) = I_{Q'}(e_{Q'}(\sigma)^*) = I_{Q'}(e_{Q'}(\sigma^*))$ by Lemma 4.6 and [Abe16, Lemma 2.27]. Put $Q'_1 = n_{w_G w_{Q_1}} Q_1^{\mathrm{op}} n_{w_G w_{Q_1}}^{-1}$. Then

$$\bigoplus_{Q_1 \supsetneq Q} I_{Q'_1}(e_{Q'_1}(\sigma^*)) \to I_{Q'}(e_{Q'}(\sigma^*)) \to \mathrm{St}_Q(\sigma)^* \to 0.$$

Since $\Delta_{Q'} = -w_G(\Delta_Q)$ and $Q'_1 = -w_G(\Delta_{Q_1})$, we have $Q_1 \supsetneq Q$ if and only if $Q'_1 \supsetneq Q'$. Hence

$$\bigoplus_{Q_2 \supsetneq Q'} I_{Q_2}(e_{Q_2}(\sigma^*)) \to I_{Q'}(e_{Q'}(\sigma^*)) \to \mathrm{St}_Q(\sigma)^* \to 0.$$

By the lemma below, we get $\mathrm{St}_Q(\sigma)^* = \mathrm{St}_{Q'}(\sigma^*)$. □

Lemma 4.7. *Let $Q_1 \supset Q$ be parabolic subgroups. Put $Q' = n_{w_G w_Q} Q^{\mathrm{op}} n_{w_G w_Q}^{-1}$ and $Q'_1 = n_{w_G w_{Q_1}} Q_1^{\mathrm{op}} n_{w_G w_{Q_1}}^{-1}$. Then the homomorphism induced by $I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma))$ with the dual is the inclusion $I_{Q'_1}(e_{Q'_1}(\sigma^*)) \hookrightarrow I_{Q'}(e_{Q'}(\sigma^*))$ times $(-1)^{\ell(w_{Q_1} w_Q)}$.*

Proof. By Proposition 3.13, the homomorphism $I'_Q(e_Q(\sigma)) \to I'_{Q_1}(e_{Q_1}(\sigma))$ is given by $\varphi \mapsto (X \mapsto (-1)^{\ell(w_{Q_1} w_Q)}\varphi(X T^*_{n_{w_{Q_1} w_Q}}))$. We recall that the isomorphism $I'_Q(e_Q(\sigma)) \simeq e_{Q'}(\sigma) \otimes_{(\mathcal{H}^+_{Q'}, j^+_{Q'})} \mathcal{H}$ is given by $\varphi \mapsto \sum_{w \in {}^{Q'}W_0} \varphi(T^*_{n_{w^{-1} w_G w_Q}}) \otimes T_{n_w}$ [Abe16, Lemma 2.22]. Let $\varphi' \in I'_{Q_1}(e_{Q_1}(\sigma))$ be the image of φ. Then the image of φ' in $e_{Q'_1}(\sigma) \otimes_{(\mathcal{H}^+_{Q'_1}, j^+_{Q'_1})} \mathcal{H}$ is

$$\begin{aligned} &\sum_{w \in {}^{Q'_1}W_0} \varphi'(T^*_{n_{w^{-1} w_G w_{Q_1}}}) \otimes T_{n_w} \\ &= (-1)^{\ell(w_{Q_1} w_Q)} \sum_{w \in {}^{Q'_1}W_0} \varphi(T^*_{n_{w^{-1} w_G w_{Q_1}}} T^*_{n_{w_{Q_1} w_Q}}) \otimes T_{n_w}. \end{aligned}$$

Since $w_{Q_1} w_Q \in W_{0,Q_1}$ and $w^{-1} w_G w_{Q_1} \in W_0^{Q_1}$ [Abe16, Lemma 2.22], we have $\ell(w^{-1} w_G w_{Q_1}) + \ell(w_{Q_1} w_Q) = \ell(w^{-1} w_G w_Q)$. Hence we have $T^*_{n_{w^{-1} w_G w_{Q_1}}} T^*_{n_{w_{Q_1} w_Q}} = T^*_{n_{w^{-1} w_G w_Q}}$. Therefore

$$\sum_{w \in {}^{Q'_1}W_0} \varphi'(T^*_{n_{w^{-1} w_G w_{Q_1}}}) \otimes T_{n_w} = (-1)^{\ell(w_{Q_1} w_Q)} \sum_{w \in {}^{Q'_1}W_0} \varphi(T^*_{n_{w^{-1} w_G w_Q}}) \otimes T_{n_w}.$$

Let P_2 be a parabolic subgroup corresponding to $\Delta \setminus \Delta_P$. Let $w \in {}^{Q'}W_0$ but $w \notin {}^{Q'_1}W_0$. Then there exists a simple reflection $s \in W_{0,Q'_1}$ such that $sw < w$. Since $w \in {}^{Q'}W_0 \subset W_{0,P_2}$, we have $s \in S_{0,P_2}$. Hence for any $x \in e_{Q'_1}(\sigma)$, we have $x \otimes T_{n_w} = x \otimes T_{n_s} T_{n_{sw}} = x e_{Q'_1}(\sigma)(T^{Q'_1}_{n_s}) \otimes T_{n_{sw}} = 0$ since $e_{Q'_1}(\sigma)(T^{Q'_1}_{n_s}) = 0$. Hence

$$\begin{aligned} &(-1)^{\ell(w_{Q_1} w_Q)} \sum_{w \in {}^{Q'_1}W_0} \varphi(T^*_{n_{w^{-1} w_G w_Q}}) \otimes T_{n_w} \\ &= (-1)^{\ell(w_{Q_1} w_Q)} \sum_{w \in {}^{Q'}W_0} \varphi(T^*_{n_{w^{-1} w_G w_Q}}) \otimes T_{n_w}. \end{aligned}$$

Therefore the homomorphism

$$e_{Q'}(\sigma) \otimes_{(\mathcal{H}^+_{Q'}, j^+_{Q'})} \mathcal{H} \to e_{Q'_1}(\sigma) \otimes_{(\mathcal{H}^+_{Q'_1}, j^+_{Q'_1})} \mathcal{H}$$

is given by $x \otimes X \mapsto (-1)^{\ell(w_{Q_1} w_Q)} x \otimes X$. (Here we identify $x \in e_{Q'}(\sigma)$ with $x \in e_{Q'_1}(\sigma)$.)

The isomorphism

$$(e_{Q'}(\sigma) \otimes_{(\mathcal{H}^+_{Q'}, j^{+*}_{Q'})} \mathcal{H})^* \simeq \mathrm{Hom}_{(\mathcal{H}^-_{Q'}, j^{-*}_{Q'})}(\mathcal{H}, e_Q(\sigma)^*) = I_Q(e_Q(\sigma)^*)$$

is given by $f \mapsto (X \mapsto (x \mapsto f(x \otimes \zeta(X))))$ and the opposite is given by $f' \mapsto ((x \otimes X) \mapsto f'(\zeta(X))(x))$. (Here we identify $e_Q(\sigma)^*$ with $e_Q(\sigma^*)$.) Hence the maps

$$I_{Q_1}(e_{Q_1}(\sigma)^*) \simeq (e_{Q_1'}(\sigma) \otimes_{(\mathcal{H}^+_{Q_1'}, j^+_{Q_1'})} \mathcal{H})^* \to (e_{Q'}(\sigma) \otimes_{(\mathcal{H}^+_{Q'}, j^+_{Q'})} \mathcal{H})^* \simeq I_Q(e_Q(\sigma)^*)$$

send $f \in I'_{Q_1}(e_{Q_1}(\sigma)^*)$ to

$$(x \otimes X) \mapsto f(\zeta(X))(x) \in (e_{Q_1'}(\sigma) \otimes_{(\mathcal{H}^+_{Q_1'}, j^+_{Q_1'})} \mathcal{H})^*,$$
$$(x \otimes X) \mapsto (-1)^{\ell(w_{Q_1} w_Q)} f(\zeta(X))(x) \in (e_{Q'}(\sigma) \otimes_{(\mathcal{H}^+_{Q'}, j^+_{Q'})} \mathcal{H})^*,$$
$$X \mapsto (x \mapsto (-1)^{\ell(w_{Q_1} w_Q)} f(X)(x)) \in I_Q(e_Q(\sigma)^*).$$

Namely, it is equal to the the natural embedding times $(-1)^{\ell(w_{Q_1} w_Q)}$. $\square$

4.3. **Supersingular modules.** Assume that C is a field.

Theorem 4.8. *Let (χ, J, V) be as in subsection 2.11. Then we have $\pi^*_{\chi,J,V} \simeq \pi_{\chi^{-1},J,V^*}$.*

Proof. Let Ξ be a character of $\mathcal{H}_{\mathrm{aff}}$ determined by (χ, J). By the proof of [Vig17, Proposition 6.17], $\Xi \otimes V \subset \pi_{\chi,J,V}|_{\mathcal{H}_\Xi}$ is a direct summand. Hence $(\Xi \otimes V)^* \subset (\pi_{\chi,J,V})^*|_{\mathcal{H}_\Xi}$. Since Ξ and V are finite-dimensional, we have $(\Xi \otimes V)^* = \Xi^* \otimes V^*$. Therefore we have a non-zero homomorphism $(\Xi^* \otimes V^*) \otimes_{\mathcal{H}_\Xi} \mathcal{H} \to \pi^*_{\chi,J,V}$. The restriction of V^* to $Z_\kappa \cap W_{\mathrm{aff},P}(1)$ is the direct sum of $\chi^* = \chi^{-1}$ since $V|_{Z_\kappa \cap W_{\mathrm{aff},P}(1)}$ is a direct sum of χ. For $s \in S_{\mathrm{aff},\chi} = S_{\mathrm{aff},\chi^{-1}}$, $\Xi^*(T_{\widetilde{s}}) = \Xi(\zeta(T_{\widetilde{s}})) = \Xi(T_{\widetilde{s}^{-1}})$ where $\widetilde{s}$ is a lift of s. This is 0 or $\chi(c_{\widetilde{s}^{-1}})$ and 0 if and only if $s \in J$. Hence the subset of $S_{\mathrm{aff},\chi^{-1}} = S_{\mathrm{aff},\chi}$ attached to Ξ^* is J. Therefore $(\Xi^* \otimes V^*) \otimes_{\mathcal{H}_\Xi} \mathcal{H} = \pi_{\chi^{-1},J,V^*}$. Hence we get a non-zero homomorphism $\pi_{\chi^{-1},J,V^*} \to \pi^*_{\chi,J,V}$. Since this is a non-zero homomorphism between simple modules, this is an isomorphism. $\square$

4.4. **Simple modules.** Assume that C is a field. Combining Propositions 4.2 and 4.5 and Theorem 4.8, we get the following theorem.

Theorem 4.9. *Set $P' = n_{w_G w_P} P^{\mathrm{op}} n_{w_G w_P}^{-1}$ and $Q' = n_{w_G w_Q} Q^{\mathrm{op}} n_{w_G w_Q}^{-1}$. Let (χ', J', V') be a triple for $\mathcal{H}_{P'}$ defined by the pull-back of the triple (χ^{-1}, J, V^*) by $n_{w_G w_P}$. Then we have $I(P; \chi, J, V; Q)^* = I(P'; \chi', J', V'; Q')$.*

We use the following lemma.

Lemma 4.10. *Let P be a parabolic subgroup and let σ be an $\mathcal{H}_P$-module. Then we have $P(n_{w_G w_P} \sigma) = n_{w_G w_{P(\sigma)}} P(\sigma)^{\mathrm{op}} n_{w_G w_{P(\sigma)}}^{-1}$.*

Remark 4.11. By [Abe16, Lemma 2.27], we have $n_{w_G w_{P(\sigma)}} \sigma = n_{w_G w_P} \sigma$.

Proof. Let P_α be the parabolic subgroup corresponding to $\Delta_P \cup \{\alpha\}$ where $\alpha \in \Delta \setminus \Delta_P$. Set $n = n_{w_G w_P}$ and $P' = n P^{\mathrm{op}} n^{-1}$. Let $\alpha \in \Delta(n\sigma) \setminus \Delta_{P'}$ and we prove that $\alpha \in -w_G(\Delta(\sigma) \setminus \Delta_P)$. Since $\Delta_{P'} = -w_G(\Delta_P)$, $\langle \alpha, \Delta_{P'} \rangle = 0$ implies $\langle -w_G(\alpha), \Delta_P \rangle = 0$. If for any $\lambda \in \Lambda(1) \cap W_{\mathrm{aff},P_\alpha}(1)$ satisfies $(n\sigma)(T^{P'}_\lambda) = 1$, then we have $\sigma(T^P_{n^{-1}\lambda n}) = 1$. Hence $\sigma(T^P_\lambda) = 1$ for any $\lambda \in n^{-1}(\Lambda(1) \cap W_{\mathrm{aff},P_\alpha}(1))n = \Lambda(1) \cap W_{\mathrm{aff},P_{-(w_G w_P)^{-1}(\alpha)}}(1)$. Since $\langle -w_G(\alpha), \Delta_P \rangle = 0$, we have $-(w_G w_P)^{-1}(\alpha) = -w_P w_G(\alpha) = -w_G(\alpha)$. Hence $-w_G(\alpha) \in \Delta(\sigma)$. Therefore we have $\Delta(n\sigma) \setminus \Delta_{P'} \subset -w_G(\Delta(\sigma) \setminus \Delta_P)$. Counting the numbers, we have $\Delta(n\sigma) \setminus \Delta_{P'} = -w_G(\Delta(\sigma) \setminus \Delta_P)$. Since $-w_G(\Delta_P) = w_G w_P(\Delta_P) = \Delta_{P'}$, we have $\Delta(n\sigma) = -w_G(\Delta(\sigma))$. Therefore we get $\Delta(n\sigma) = w_G w_{P(\sigma)}(\Delta(\sigma))$. The left hand side corresponds to $P(n\sigma)$ and the right hand side corresponds to $nP(\sigma)^{\mathrm{op}}n^{-1}$. Hence we get the lemma. $\square$

Proof of Theorem 4.9. Set $\sigma = \pi_{\chi,J,V}$ and $P(\sigma)' = n_{w_G w_{P(\sigma)}} P(\sigma)^{\mathrm{op}} n_{w_G w_{P(\sigma)}}^{-1}$. By Proposition 4.2,

$$\begin{aligned} I(P;\chi,J,V;Q)^* &= I_{P(\sigma)}(\mathrm{St}_Q^{P(\sigma)}\sigma)^* \\ &\simeq I'_{P(\sigma)'}(n_{w_G w_{P(\sigma)}}(\mathrm{St}_Q^{P(\sigma)}\sigma)^*). \end{aligned}$$

We have

$$I'_{P(\sigma)'}(n_{w_G w_{P(\sigma)}}(\mathrm{St}_Q^{P(\sigma)}\sigma)^*) = I_{P(\sigma)'}(n_{w_G w_{P(\sigma)}}(\mathrm{St}_Q^{P(\sigma)}\sigma)^*)$$

by Corollary 3.3. Proposition 4.5 implies

$$I_{P(\sigma)'}(n_{w_G w_{P(\sigma)}}(\mathrm{St}_Q^{P(\sigma)}\sigma)^*) \simeq I_{P(\sigma)'}(n_{w_G w_{P(\sigma)}}(\mathrm{St}^{P(\sigma)}_{n_{w_{P(\sigma)}w_Q} Q^{\mathrm{op}} n^{-1}_{w_{P(\sigma)}w_Q}}\sigma^*)).$$

The adjoint action of $n_{w_G w_{P(\sigma)}}$ induces an isomorphism $\mathcal{H}_{P(\sigma)} \simeq \mathcal{H}_{P(\sigma)'}$. For a parabolic subgroup Q_1 between $P(\sigma)$ and P, let Q_2 be a parabolic subgroup corresponding to $w_G w_{P(\sigma)}(\Delta_{Q_1})$. Then the adjoint action of $n_{w_G w_{P(\sigma)}}$ induces an isomorphism $\mathcal{H}_{Q_1} \simeq \mathcal{H}_{Q_2}$ and sends $\mathcal{H}_{Q_1}^{P(\sigma)-}$ to $\mathcal{H}_{Q_2}^{P(\sigma)'-}$. Moreover, it is compatible with homomorphisms $j_{Q_1}^{P(\sigma)-*}$ and $j_{Q_2}^{P(\sigma)'-*}$. Hence, by the definition of parabolic inductions and the twist $n_{w_G w_{P(\sigma)}}$, we have $n_{w_G w_{P(\sigma)}} I_{Q_1}^{P(\sigma)}(e_{Q_1}(\sigma)) \simeq I_{Q_2}^{P(\sigma)'}(n_{w_G w_{P(\sigma)}} e_{Q_1}(\sigma))$.

Since $\Delta_{P(\sigma)} \setminus \Delta_P$ is orthogonal to Δ_P, $w_{P(\sigma)} w_P(\Delta_P) = \Delta_P$. Therefore we have $w_G w_{P(\sigma)}(\Delta_P) = w_G w_P(\Delta_P) = \Delta_{P'}$. Hence the isomorphism $\mathcal{H}_{Q_1} \simeq \mathcal{H}_{Q_2}$ induced by the adjoint action of $n_{w_G w_{P(\sigma)}}$ sends $\mathcal{H}_{Q_1}^{P-}$ to $\mathcal{H}_{Q_2}^{P'-}$. It is also compatible with homomorphisms $j_{Q_1}^{P-*}$ and $j_{Q_2}^{P'-*}$. Therefore the restriction of $n_{w_G w_{P(\sigma)}} e_{Q_1}(\sigma)$ to $\mathcal{H}_{Q_2}^{P'-}$ (which is regarded as a subalgebra of $\mathcal{H}_{Q_2}$ by $j_{Q_2}^{P'-*}$) is isomorphic to $n_{w_G w_{P(\sigma)}}\sigma$. We denote the parabolic subgroup corresponding to $\Delta_{Q_1} \setminus \Delta_P$ (resp., $\Delta_{Q_2} \setminus \Delta_{P'}$) by R_1 (resp., R_2). Then since $w_G w_{P(\sigma)}(\Delta_{Q_1}) = \Delta_{Q_2}$ and $w_G w_{P(\sigma)}(\Delta_P) = \Delta_{P'}$, we have $w_G w_{P(\sigma)}(\Delta_{R_1}) = \Delta_{R_2}$. Hence $n_{w_G w_{P(\sigma)}} W_{\mathrm{aff},R_1}(1) n_{w_G w_{P(\sigma)}}^{-1} = W_{\mathrm{aff},R_2}(1)$. Therefore the action of $T_w^{Q_2*}$ for $w \in W_{R_2,\mathrm{aff}}(1)$ is trivial on $n_{w_G w_{P(\sigma)}} e_{Q_1}(\sigma)$ since $n_{w_G w_{P(\sigma)}}^{-1} w n_{w_G w_{P(\sigma)}} \in W_{R_1,\mathrm{aff}}(1)$ and $T_v^{Q_1*}$ is trivial on $e_{Q_1}(\sigma)$ for $v \in W_{R_1,\mathrm{aff}}(1)$. Therefore, by the characterization of the extension, we have $n_{w_G w_{P(\sigma)}} e_{Q_1}(\sigma) \simeq e_{Q_2}(n_{w_G w_{P(\sigma)}}\sigma)$. Hence, combining the formula in the previous paragraph, we get $n_{w_G w_{P(\sigma)}} \mathrm{St}_{Q_1}^{P(\sigma)}(\sigma) \simeq \mathrm{St}_{Q_2}^{P(\sigma)'}(n_{w_G w_{P(\sigma)}}\sigma)$.

Now set $Q_1 = n_{w_{P(\sigma)}w_Q} Q^{\mathrm{op}} n_{w_{P(\sigma)}w_Q}$. Then $\Delta_{Q_1} = w_{P(\sigma)} w_Q(\Delta_Q)$. Hence $\Delta_{Q_2} = w_G w_Q(\Delta_Q) = \Delta_{Q'}$. Therefore we have

$$I(P;\chi,J,V;Q)^* \simeq I_{P(\sigma)'}(\mathrm{St}_{Q'}^{P(\sigma)'} n_{w_G w_{P(\sigma)}}\sigma^*).$$

By [Abe16, Lemma 2.27], $\sigma^* = n_{w_{P(\sigma)}w_P}\sigma^*$. Hence we get

$$I(P,\sigma,Q)^* \simeq I_{P(\sigma)'}(\mathrm{St}_{Q'}^{P(\sigma)'} n_{w_G w_P}\sigma^*).$$

We get the theorem by Theorem 4.8. $\square$

Acknowledgment

Most of this work was done during the author's pleasant stay at Institut de mathématiques de Jussieu.

References

[Abe] N. Abe, *Modulo p parabolic induction of pro-p-Iwahori Hecke algebra*, J. Reine Angew. Math., DOI:10.1515/crelle-2016-0043.

[Abe16] N. Abe, *Parabolic inductions for pro-p-Iwahori Hecke algebras*, arXiv:1612.01312.

[Abe17] N. Abe, *Extension between simple modules of pro-p-Iwahori Hecke algebras*, arXiv:1705.00728.

[AHHV17] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, *A classification of irreducible admissible* mod p *representations of p-adic reductive groups*, J. Amer. Math. Soc. **30** (2017), no. 2, 495–559, DOI 10.1090/jams/862. MR3600042

[AHV17] Marie-France Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Deo77] Vinay V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), no. 2, 187–198, DOI 10.1007/BF01390109. MR0435249

[Oll14] Rachel Ollivier, *Compatibility between Satake and Bernstein isomorphisms in characteristic p*, Algebra Number Theory **8** (2014), no. 5, 1071–1111, DOI 10.2140/ant.2014.8.1071. MR3263136

[Vig15] Marie-France Vignéras, *The pro-p Iwahori Hecke algebra of a reductive p-adic group, V (parabolic induction)*, Pacific J. Math. **279** (2015), no. 1-2, 499–529, DOI 10.2140/pjm.2015.279.499. MR3437789

[Vig16] Marie-France Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group I*, Compos. Math. **152** (2016), no. 4, 693–753, DOI 10.1112/S0010437X15007666. MR3484112

[Vig17] Marie-France Vigneras, *The pro-p-Iwahori Hecke algebra of a reductive p-adic group III (spherical Hecke algebras and supersingular modules)*, J. Inst. Math. Jussieu **16** (2017), no. 3, 571–608, DOI 10.1017/S1474748015000146. MR3646282

Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

Email address: abenori@math.sci.hokudai.ac.jp