

ON THE SPACE OF K -FINITE SOLUTIONS TO INTERTWINING DIFFERENTIAL OPERATORS

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ABSTRACT. In this paper we give Peter–Weyl-type decomposition theorems for the space of K -finite solutions to intertwining differential operators between parabolically induced representations. Our results generalize a result of Kable for conformally invariant systems. The main idea is based on the duality theorem between intertwining differential operators and homomorphisms between generalized Verma modules. As an application we uniformly realize on the solution spaces of intertwining differential operators all small representations of $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ attached to the minimal nilpotent orbit.

1. INTRODUCTION

Let G be a real reductive Lie group, and let P be a parabolic subgroup of G . Given finite-dimensional representations W and E of P , write $\mathcal{W} = G \times_P W$ and $\mathcal{E} = G \times_P E$, the homogeneous vector bundles over G/P with fibers W and E , respectively. The aim of this paper is to understand the representation realized on the kernel (solution space) of an intertwining differential operator $\mathcal{D}: C^\infty(G/P, \mathcal{W}) \rightarrow C^\infty(G/P, \mathcal{E})$ between the spaces of smooth sections for \mathcal{W} and \mathcal{E} .

Realization of a representation on the space of solutions to intertwining differential operators has been studied by a number of people such as Kostant ([25]), Binegar–Zierau ([6, 7]), Ørsted ([29]), Kobayashi–Ørsted ([21], [22], [23]), Kable ([16], [17], [18], [19]), Wang ([39]), Sepanski and his collaborators ([10], [14], [15], [32]), among others. For instance, in [21–23], Kobayashi–Ørsted realized the minimal representation of $O(p, q)$ on the solution space of the Yamabe operator and studied it in great depth from the various perspectives of conformal geometry, branching law, and harmonic analysis, whereas Kable in [17] used the Peter–Weyl theorem to realize the minimal representation of G_2 on the common solution space of a system of differential operators constructed in [2]. In this paper we take the approach of Kable to understand the K -type formula of the representation realized

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on the space of K -finite solutions to intertwining differential operators. For convenience we refer to a general K -type decomposition formula such as (1.3) below as a *Peter–Weyl-type formula* (or *PW-type formula* for short).

In [16] (for linear group G), Kable gave a Peter–Weyl-type formula for the space of K -finite solutions to a system of differential operators that are equivariant under an action of the Lie algebra of G . Such a system of operators is called a *conformally invariant system* ([2, 3]). From the viewpoint of intertwining operators a conformally invariant system is an intertwining differential operator from a line bundle to a vector bundle. In this paper we give a Peter–Weyl-type formula for the space of K -finite solutions to intertwining differential operators from general vector bundles (Theorem 2.37); the case from a line bundle to a vector bundle is also further investigated (Proposition 2.42 and Theorems 1.2 and 2.46). In addition, we also consider the common solution space of a system of intertwining differential operators (Section 2.6).

Our main tools are the Peter–Weyl theorem for the space $C^\infty(G/P, \mathcal{W})_K$ of K -finite sections (2.26) and the duality theorem between intertwining differential operators and homomorphisms between generalized Verma modules (Theorem 2.3). There are three main cases:

- (1) VV case: a vector bundle to a vector bundle (Theorem 2.37),
- (2) LV case: a line bundle to a vector bundle (Theorem 2.46),
- (3) LL case: a line bundle to a line bundle (Theorem 1.2).

We next briefly describe the Peter–Weyl-type formula for the LL case.

Let G and P be as above and fix a maximal compact subgroup K of G . Write $P = MAN$ for a Langlands decomposition of P . Let \mathfrak{g}_0 be the Lie algebra of G , and we denote by \mathfrak{g} and $\mathcal{U}(\mathfrak{g})$ the complexification of \mathfrak{g}_0 and the universal enveloping algebra of \mathfrak{g} , respectively. A similar convention is also employed for the subgroups K and P . Given characters χ_{triv}, χ of M with χ_{triv} the trivial character, and also λ, ν of A , let $\mathbb{C}_{\text{triv}, \lambda}$ (resp., $\mathbb{C}_{\chi, \nu}$) denote the one-dimensional representation $(\chi_{\text{triv}} \otimes (\lambda + \rho), \mathbb{C})$ (resp., $(\chi \otimes (\nu + \rho), \mathbb{C})$) of $P = MAN$ with trivial N action, where ρ is half the sum of the positive roots. We write $\mathcal{L}_{\text{triv}, \lambda}$ and $\mathcal{L}_{\chi, \nu}$ for the line bundles over G/P with fibers $\mathbb{C}_{\text{triv}, \lambda}$ and $\mathbb{C}_{\chi, \nu}$, respectively. We realize the degenerate principal series representation $I_P(\chi_{\text{triv}}, \lambda)$ on the space of smooth sections $C^\infty(G/P, \mathcal{L}_{\text{triv}, \lambda})$ for $\mathcal{L}_{\text{triv}, \lambda}$. The representation $I_P(\chi, \nu)$ is defined similarly. We write $\text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ for the space of intertwining differential operators $\mathcal{D}: C^\infty(G/P, \mathcal{L}_{\text{triv}, \lambda}) \rightarrow C^\infty(G/P, \mathcal{L}_{\chi, \nu})$. It follows from the duality theorem (Theorem 2.3) that any differential operator $\mathcal{D} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ is of the form $\mathcal{D} = R(u)$ for some $u \in \mathcal{U}(\mathfrak{g})$, where R denotes the infinitesimal right translation of $\mathcal{U}(\mathfrak{g})$. To emphasize the element u , we write \mathcal{D}_u for the differential operator such that $\mathcal{D}_u = R(u)$.

Let $\text{Irr}(K)$ and $\text{Irr}(M/M_0)$ be the sets of equivalence classes of irreducible representations of K and the component group M/M_0 of M , respectively. It follows from Lemma 2.17 with (2.23) in Section 2.2 that, for $\mathcal{D}_u \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ and $\xi \in \text{Irr}(M/M_0)$, we have

$$\mathcal{D}_u \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\xi, \lambda), I_P(\chi \otimes \xi, \nu)).$$

We remark that the representation $\xi \in \text{Irr}(M/M_0)$ need not be a character, as G is not necessarily linear (see (1.4) below for the case that $M = M/M_0$).

For $V_\delta := (\delta, V) \in \text{Irr}(K)$ and $u \in \mathcal{U}(\mathfrak{g})$, we define a subspace $\text{Sol}_{(u)}(\delta)$ of V_δ by

$$(1.1) \quad \text{Sol}_{(u)}(\delta) := \{v \in V_\delta : d\delta(\tau(u^\flat))v = 0\}.$$

Here $d\delta$ denotes the differential of δ , τ denotes the conjugation of \mathfrak{g} with respect to \mathfrak{g}_0 , and u^\flat is some element of $\mathcal{U}(\mathfrak{k})$ such that $u^\flat \otimes \mathbb{1}_{-\lambda-\rho} = u \otimes \mathbb{1}_{-\lambda-\rho}$ in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_{-\lambda-\rho}$ (see Lemma 2.24 and (2.25)). It will be shown in Lemma 2.41 that the space $\text{Sol}_{(u)}(\delta)$ is a $K \cap M$ -representation.

Given $\mathcal{D}_u \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ and $\xi \in \text{Irr}(M/M_0)$, we set

$$\text{Sol}_{(u;\lambda)}(\xi)_K := \text{the space of the } K\text{-finite solutions to } \mathcal{D}_u \otimes \text{id}_\xi.$$

With the notation we obtain the following as a specialization of Theorem 2.46. (For some details see Section 2.7.)

Theorem 1.2 (PW-type formula: LL case). *Let $\mathcal{D}_u \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ and $\xi \in \text{Irr}(M/M_0)$. Then the space $\text{Sol}_{(u;\lambda)}(\xi)_K$ of K -finite solutions to $\mathcal{D}_u \otimes \text{id}_\xi$ is decomposed as a K -representation as*

$$(1.3) \quad \text{Sol}_{(u;\lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M}(\text{Sol}_{(u)}(\delta), \xi).$$

As an application of Theorem 1.2, we take G to be $\widetilde{\text{SL}}(3, \mathbb{R})$, the nonlinear double cover of $\text{SL}(3, \mathbb{R})$, and P to be a minimal parabolic \widetilde{B} of $\widetilde{\text{SL}}(3, \mathbb{R})$. Write $\widetilde{B} = \widetilde{M}AN$ for a Langlands decomposition of \widetilde{B} . Here \widetilde{M} is isomorphic to the quaternion group Q_8 , a noncommutative group of order 8. In particular, \widetilde{M} is a discrete subgroup of $\widetilde{\text{SL}}(3, \mathbb{R})$ so that $\widetilde{M} = \widetilde{M}/\widetilde{M}_0$. As $\widetilde{M} \simeq Q_8$, the set $\text{Irr}(\widetilde{M})$ is given by

$$(1.4) \quad \text{Irr}(\widetilde{M}) = \{(+, +), (+, -), (-, +), (-, -), \mathbb{H}\},$$

where (\pm, \pm) are characters and \mathbb{H} is the unique 2-dimensional irreducible representation of $Q_8 \simeq \widetilde{M}$. (For the notation (\pm, \pm) , see Section 4.)

In this setting we consider two cases, namely, the case for infinitesimal character ρ and that for infinitesimal character $\tilde{\rho} := (1/2)\rho$. For each case we take λ in (1.3) to be $\lambda = -\rho$ and $\lambda = -\tilde{\rho}$, respectively. Via the duality theorem we obtain first-order operators $\mathcal{D}_X, \mathcal{D}_Y$, third-order operators $\mathcal{D}_{Y^2X}, \mathcal{D}_{X^2Y}$, and a fourth-order operator $\mathcal{D}_{XY^2X} (= \mathcal{D}_{YX^2Y})$ for the ρ case, and a second-order operator $\mathcal{D}_{X \circ Y}$ is obtained for the $\tilde{\rho}$ case. (For the notation X, Y , and $X \circ Y$, see (4.1) and (6.5).) We remark that all operators are also constructed via the BKZ-construction ([2, 26]). Moreover, the second-order operator $\mathcal{D}_{X \circ Y}$ is a specialization of Kable’s Heisenberg ultrahyperbolic operator ([18, 19]), which is used in [18] to establish a Heisenberg analogue of classic Maxwell’s theorem on harmonic polynomials on Euclidean space.

For the sake of simplicity, we write $\mathcal{D}_u^\sigma = \mathcal{D}_u \otimes \text{id}_\sigma$ for $\sigma \in \text{Irr}(\widetilde{M})$. It is easily observed that the solution space $\text{Sol}_{(X;-\rho)}(\sigma)$ of \mathcal{D}_X^σ (resp., $\text{Sol}_{(Y;-\rho)}(\sigma)$ of \mathcal{D}_Y^σ) is contained in that of $\mathcal{D}_{Y^2X}^\sigma$ and $\mathcal{D}_{X^2Y}^\sigma$ (resp., $\mathcal{D}_{XY^2X}^\sigma$ and $\mathcal{D}_{YX^2Y}^\sigma$). Then, in this paper, we focus on the solution spaces of \mathcal{D}_X^σ and \mathcal{D}_Y^σ for the ρ case. Further the common solution space $\text{Sol}_{(X,Y;-\rho)}(\sigma)$ of \mathcal{D}_X^σ and \mathcal{D}_Y^σ is also investigated.

On the solution spaces of the first-order operators \mathcal{D}_X^σ and \mathcal{D}_Y^σ , we realize a number of irreducible representations studied in [11]. In order to state the results let \widetilde{K} be a maximal compact subgroup of $\widetilde{\text{SL}}(3, \mathbb{R})$. As $\widetilde{K} \simeq \text{SU}(2) \simeq \text{Spin}(3)$, the irreducible representations $\delta \in \text{Irr}(\widetilde{K})$ of \widetilde{K} can be parametrized as

$$\text{Irr}(\widetilde{K}) \simeq \{V_{(\frac{a}{2})} : a \in \mathbb{Z}_{\geq 0}\},$$

where $V_{(\frac{a}{2})}$ is the irreducible representation of \widetilde{K} with $\dim_{\mathbb{C}} V_{(\frac{a}{2})} = a + 1$. Then, for $u = X, Y, (X, Y)$, the classification of $\sigma \in \text{Irr}(\widetilde{M})$ such that $\text{Sol}_{(u; -\rho)}(\sigma) \neq \{0\}$ and the \widetilde{K} -type formula of the space $\text{Sol}_{(u; -\rho)}(\sigma)_{\widetilde{K}}$ of \widetilde{K} -finite solutions to \mathcal{D}_u^σ are obtained as follows.

Theorem 1.5. *For $\sigma \in \text{Irr}(\widetilde{M})$, the following hold:*

- (1) $\text{Sol}_{(X; -\rho)}(\sigma) \neq \{0\} \iff \sigma = (+, +), (+, -)$.
- (2) $\text{Sol}_{(Y; -\rho)}(\sigma) \neq \{0\} \iff \sigma = (+, +), (-, +)$.
- (3) $\text{Sol}_{(X, Y; -\rho)}(\sigma) \neq \{0\} \iff \sigma = (+, +)$.

Moreover, for $\sigma \in \text{Irr}(\widetilde{M})$ such that $\text{Sol}_{(u; -\rho)}(\sigma) \neq \{0\}$, the \widetilde{K} -type formula of $\text{Sol}_{(u; -\rho)}(\sigma)_{\widetilde{K}}$ is determined as follows:

- (a) $u = X$: $\text{Sol}_{(X; -\rho)}((+, +))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a)}$, $\text{Sol}_{(X; -\rho)}((+, -))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a+1)}$,
- (b) $u = Y$: $\text{Sol}_{(Y; -\rho)}((+, +))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a)}$, $\text{Sol}_{(Y; -\rho)}((- , +))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a+1)}$.
- (c) $u = (X, Y)$: $\text{Sol}_{(X, Y; -\rho)}((+, +))_{\widetilde{K}} \simeq V_{(0)}$

We prove Theorem 1.5 at the end of Section 5 (see Theorem 5.14). We remark that although the \widetilde{K} -type formulas for the spaces $\text{Sol}_{(X; -\rho)}((+, +))_{\widetilde{K}}$ and $\text{Sol}_{(Y; -\rho)}((+, +))_{\widetilde{K}}$, and those for $\text{Sol}_{(X; -\rho)}((+, -))_{\widetilde{K}}$ and $\text{Sol}_{(Y; -\rho)}((- , +))_{\widetilde{K}}$, are the same, these spaces are different as $(\mathfrak{g}, \widetilde{K})$ -modules. Moreover, for $u = X, Y$, we set

$$\text{Sol}_{(u/(X, Y); -\rho)}((+, +))_{\widetilde{K}} := \text{Sol}_{(u; -\rho)}((+, +))_{\widetilde{K}} / \text{Sol}_{(X, Y; -\rho)}((+, +))_{\widetilde{K}}.$$

Then the four representations $\text{Sol}_{(X/(X, Y); -\rho)}((+, +))_{\widetilde{K}}$, $\text{Sol}_{(Y/(X, Y); -\rho)}((+, +))_{\widetilde{K}}$, $\text{Sol}_{(X; -\rho)}((+, -))_{\widetilde{K}}$, and $\text{Sol}_{(Y; -\rho)}((- , +))_{\widetilde{K}}$ are all irreducible $(\mathfrak{g}, \widetilde{K})$ -modules. (For the remarks see, for instance, [11].)

For the $\widetilde{\rho}$ case, we successfully realize all small representations of $\widetilde{\text{SL}}(3, \mathbb{R})$ attached to the minimal nilpotent orbit on the solution space $\text{Sol}_{(X \circ Y; -\widetilde{\rho})}(\sigma)$ of $\mathcal{D}_{X \circ Y}^\sigma$, one of which is the so-called Torasso’s representation. Here is the main result for the second-order operator $\mathcal{D}_{X \circ Y}$.

Theorem 1.6. *For $\sigma \in \text{Irr}(\widetilde{M})$, we have*

$$\text{Sol}_{(X \circ Y; -\widetilde{\rho})}(\sigma) \neq \{0\} \iff \sigma = (+, +), \mathbb{H}, (-, -).$$

Moreover, for $\sigma = (+, +), \mathbb{H}, (-, -)$, the \widetilde{K} -type formula of $\text{Sol}_{(X \circ Y; -\widetilde{\rho})}(\sigma)_{\widetilde{K}}$ is obtained as follows:

- (a) $\sigma = (+, +)$: $\text{Sol}_{(X \circ Y; -\widetilde{\rho})}((+, +))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a)}$.
- (b) $\sigma = \mathbb{H}$: $\text{Sol}_{(X \circ Y; -\widetilde{\rho})}(\mathbb{H})_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a+\frac{1}{2})}$.
- (c) $\sigma = (-, -)$: $\text{Sol}_{(X \circ Y; -\widetilde{\rho})}((- , -))_{\widetilde{K}} \simeq \bigoplus_{a=0}^{\infty} V_{(2a+1)}$.

We would like to note that the set of \widetilde{K} -types $\bigoplus_{a=0}^{\infty} V_{(2a+\frac{3}{2})}$ is mysteriously missing. A similar observation was also made in [31] and [38, Ex. 12.4].

The proof of Theorem 1.6 is given in Section 6 (see Theorem 6.16). It is remarked that one can read the dimension $\dim_{\mathbb{C}} \text{Sol}_{(X \circ Y)}(\delta)$ (see (1.1) and (1.3)) from [18, Thm. 5.13], as it is independent from taking the covering group of $\text{SL}(3, \mathbb{R})$. The spherical representation in (a) is recently realized in [12] as the range of a residue operator of $\text{SL}(3, \mathbb{R})$.

The representation obtained in the case of $\sigma = \mathbb{H}$ is Torasso’s representation. As Torasso’s representation is the unique genuine irreducible representation of $\widetilde{\text{SL}}(3, \mathbb{R})$ attached to the minimal nilpotent orbit, it has been widely studied from various points of view. See, for instance, [28, 29, 31, 33–36] as related works. We provide another realization of the genuine representation, which seems more elementary than any other realization in the literature.

In turn to the general theory observe that in order to determine the K -type formula of solution spaces via the isomorphism (1.3), one needs to solve the equation $d\delta(\tau(u^b))v = 0$ in (1.1) on each K -type V_{δ} . For the case of $\widetilde{\text{SL}}(3, \mathbb{R})$, as the maximal compact subgroup \widetilde{K} is isomorphic to $\text{SU}(2)$, such equations can be identified as some recurrence relations that arise from the standard $\mathfrak{sl}(2)$ -computation. In this paper, instead of solving the recurrence relations, we realize each \widetilde{K} -type as the space $\text{Pol}_n[t]$ of polynomials of one variable t with degree $\leq n$, in such a way that one can solve the equations concerned by solving ordinary differential equations such as Euler’s (Gauss) hypergeometric differential equation. In this realization it is revealed that there is a correspondence between the representations realized on the solution spaces of \mathcal{D}_X^{σ} , \mathcal{D}_Y^{σ} , and $\mathcal{D}_{X \circ Y}^{\sigma}$, and polynomial solutions to ordinary differential equations. For instance, as shown in Theorem 1.6, three irreducible representations are realized on the solution space of $\mathcal{D}_{X \circ Y}^{\sigma}$ with $\sigma = (+, +), \mathbb{H}, (-, -)$. We denote these representations by $\Pi_{(0)}$, $\Pi_{(\frac{1}{2})}$, and $\Pi_{(1)}$, where $\Pi_{(\frac{\sigma}{2})}$ denotes the irreducible representation with lowest \widetilde{K} -type $V_{(\frac{\sigma}{2})}$. (For instance, the representation $\Pi_{(\frac{1}{2})}$ is Torasso’s representation.) Then Theorem 1.6, Equation (4.11), and Propositions 6.10 and 6.11 imply that the representations $\Pi_{(0)}$, $\Pi_{(\frac{1}{2})}$, and $\Pi_{(1)}$ correspond to the following subspaces of $\text{Pol}_n[t]$ with appropriate $n \in \mathbb{Z}_{\geq 0}$:

$$\begin{aligned} \Pi_{(0)} &\longleftrightarrow \mathbb{C}u_n(t), \\ \Pi_{(\frac{1}{2})} &\longleftrightarrow \mathbb{C}u_n(t) \oplus \mathbb{C}v_n(t), \\ \Pi_{(1)} &\longleftrightarrow \mathbb{C}v_n(t), \end{aligned}$$

where $u_n(t)$ and $v_n(t)$ are given by

$$u_n(t) = {}_2F_1\left[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4\right] \quad \text{and} \quad v_n(t) = t {}_2F_1\left[-\frac{n-1}{4}, -\frac{n-2}{4}, \frac{5}{4}; t^4\right].$$

The functions $u_n(t)$ and $v_n(t)$ form a fundamental set of solutions to Euler’s (Gauss) hypergeometric differential equation $D[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4]f(t) = 0$ (7.2). We also have a similar correspondence for the ρ case (see Theorem 1.5, Propositions 5.10 and 5.12, and Corollary 5.11).

It is known that the representations $\Pi_{(0)}$, $\Pi_{(\frac{1}{2})}$, and $\Pi_{(1)}$ are all unitary. We hope that we can also report on an explicit construction of the unitary structures in the future.

We now outline the rest of this paper. This paper consists of seven sections with this introduction. First we discuss intertwining differential operators and the Peter–Weyl theorem in Section 2. In this section we start by recalling the duality theorem between intertwining differential operators and homomorphisms between generalized Verma modules. We then study the space of K -finite solutions to an intertwining differential operator via the Peter–Weyl theorem. The main results of this section are Theorems 2.37 and 2.46, which give Peter–Weyl-type formulas of the space of K -finite solutions. The common solution space of a system of intertwining differential operators is also discussed in this section. In the end we illustrate as a recipe our technique to determine the K -type formula for the case of a line bundle to a line bundle (the LL case).

In Section 3, to prepare for the later application to $\widetilde{SL}(3, \mathbb{R})$, we specialize G to be split real and take parabolic P to be a minimal parabolic B . The purpose of this section is to discuss the classification and construction of homomorphisms between Verma modules as a (\mathfrak{g}, B) -module. We give a summary of our technique as a recipe at the end of this section.

Section 4 is a preliminary section for Sections 5 and 6. In this section G and P are taken to be $G = \widetilde{SL}(3, \mathbb{R})$ and $P = \widetilde{B}$, a minimal parabolic of $\widetilde{SL}(3, \mathbb{R})$, and we settle necessary notation and normalizations for the later sections. In particular, we identify the elements of \widetilde{M} for $\widetilde{B} = \widetilde{M}AN$ with these of the corresponding linear group $M \subset \mathrm{SL}(3, \mathbb{R})$ in a canonical way. We also recall the realization of irreducible representations of $\widetilde{K} \simeq \mathrm{SU}(2)$ as the space of polynomials of one variable.

In Sections 5 and 6, by using the results from the previous sections, we study the \widetilde{K} -type formulas of the spaces of \widetilde{K} -finite solutions to several intertwining differential operators. In Section 5, we consider the principal series representation with infinitesimal character ρ and study the solution spaces of first-order differential operators \mathcal{D}_X^σ and \mathcal{D}_Y^σ . The \widetilde{K} -type formulas are achieved in Theorem 5.14, which shows Theorem 1.5. In Section 6, we take the infinitesimal character to be $\widetilde{\rho}$ and consider a second-order operator $\mathcal{D}_{X \circ Y}^\sigma$. We accomplish the \widetilde{K} -type formulas in Theorem 6.16, which concludes Theorem 1.6. In this section there is one proposition whose proof involves some classical facts on the Gauss hypergeometric series ${}_2F_1[a, b, c; x]$. We give the proof in Section 7 after recalling these facts.

2. INTERTWINING DIFFERENTIAL OPERATORS AND THE PETER–WEYL THEOREM

The aim of this section is to discuss intertwining differential operators between parabolically induced representations. More precisely, we first review a well-known duality theorem between intertwining differential operators and homomorphisms between generalized Verma modules. The spaces of K -finite solutions to such differential operators are then studied via the Peter–Weyl theorem.

2.1. Duality theorem between parabolically induced representations and generalized Verma modules. We start by reviewing a well-known duality theorem between the space of intertwining differential operators between parabolically induced representations and that of homomorphisms between generalized Verma modules.

Let G be a real reductive Lie group with Lie algebra \mathfrak{g}_0 . Choose a Cartan involution θ on \mathfrak{g}_0 and write $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ for the corresponding Cartan decomposition with \mathfrak{k}_0 the $+1$ eigenspace and \mathfrak{s}_0 the -1 eigenspace of θ . Let $\mathfrak{a}_0^{\min} \subset \mathfrak{s}_0$ be a

maximal abelian subspace of \mathfrak{s}_0 . Put $\mathfrak{h}_0 := \mathfrak{t}_0^{\min} \oplus \mathfrak{a}_0^{\min}$, where \mathfrak{t}_0^{\min} is a maximal abelian subspace of $\mathfrak{m}_0^{\min} := Z_{\mathfrak{t}_0}(\mathfrak{a}_0^{\min})$.

For real Lie algebra \mathfrak{h}_0 , we express its complexification by \mathfrak{h} (simply omitting the subscript 0) and the universal enveloping algebra of \mathfrak{h} by $\mathcal{U}(\mathfrak{h})$. For instance, \mathfrak{g} , \mathfrak{h} , and \mathfrak{a}^{\min} denote the complexifications of \mathfrak{g}_0 , \mathfrak{h}_0 , and \mathfrak{a}_0^{\min} , respectively. We write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h})$ for the set of roots of \mathfrak{g} with respect to the Cartan subalgebra \mathfrak{h} and $\Sigma \equiv \Sigma(\mathfrak{g}_0, \mathfrak{a}_0^{\min})$ for that of restricted roots of \mathfrak{g}_0 with respect to \mathfrak{a}_0^{\min} . Choose positive systems Δ^+ and Σ^+ of Δ and Σ , respectively, in such a way that Δ^+ and Σ^+ are compatible. We write ρ for half the sum of the positive roots $\alpha \in \Delta^+$.

Let \mathfrak{p}_0^{\min} be the minimal parabolic subalgebra of \mathfrak{g}_0 with Langlands decomposition $\mathfrak{p}_0^{\min} = \mathfrak{m}_0^{\min} \oplus \mathfrak{a}_0^{\min} \oplus \mathfrak{n}_0^{\min}$, where the nilpotent radical \mathfrak{n}_0^{\min} corresponds to Σ^+ . Fix a standard parabolic subalgebra $\mathfrak{p}_0 \supset \mathfrak{p}_0^{\min}$ with Langlands decomposition $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. Let P be a parabolic subgroup of G with Lie algebra \mathfrak{p}_0 . We write $P = MAN$ for the Langlands decomposition of P corresponding to $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$.

For $\mu \in \mathfrak{a}^* \simeq \text{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{C})$, we define a one-dimensional representation \mathbb{C}_μ of A by $a \mapsto e^\mu(a) := e^{\mu(\log a)}$ for $a \in A$. Then, given a finite-dimensional representation $W_\sigma = (\sigma, W)$ of M and weight $\lambda \in \mathfrak{a}^*$, we define an MA -representation $W_{\sigma, \lambda}$ by

$$W_{\sigma, \lambda} := W_\sigma \otimes \mathbb{C}_{\lambda + \rho}.$$

As usual, by letting N act on $W_{\sigma, \lambda}$ trivially, we regard $W_{\sigma, \lambda}$ as a representation of P . We then write $\mathcal{W}_{\sigma, \lambda} = G \times_P W_{\sigma, \lambda}$ for the G -equivariant homogeneous vector bundle over G/P with fiber $W_{\sigma, \lambda}$. We identify the Fréchet space $C^\infty(G/P, \mathcal{W}_{\sigma, \lambda})$ of smooth sections as $C^\infty(G/P, \mathcal{W}_{\sigma, \lambda}) \simeq (C^\infty(G) \otimes W_{\sigma, \lambda})^P$, that is,

$$\begin{aligned} &C^\infty(G/P, \mathcal{W}_{\sigma, \lambda}) \\ &\simeq \left\{ F \in C^\infty(G) \otimes W_{\sigma, \lambda} : F(gman) = \sigma(m)^{-1} e^{-(\lambda + \rho)}(a) F(g) \text{ for all } man \in MAN \right\}, \end{aligned}$$

where G acts by left translation. Then we form a parabolically induced representation

$$I_P(\sigma, \lambda) := \text{Ind}_P^G(\sigma \otimes (\lambda + \rho) \otimes \mathbb{1})$$

of G on $C^\infty(G/P, \mathcal{W}_{\sigma, \lambda})$.

Let P_0 be the identity component of P . Then, via the identification

$$C^\infty(G/P_0, \mathcal{W}_{\sigma, \lambda}) \simeq (C^\infty(G) \otimes W_{\sigma, \lambda})^{P_0} \supset (C^\infty(G) \otimes W_{\sigma, \lambda})^P \simeq C^\infty(G/P, \mathcal{W}_{\sigma, \lambda}),$$

the induced representation $I_P(\sigma, \lambda)$ can be identified as

$$(2.1) \quad I_P(\sigma, \lambda) \simeq I_{P_0}(\sigma, \lambda)^M.$$

For the P -representation $W_{\sigma, \lambda}$, we set

$$W_{\sigma, \lambda}^\vee := W_\sigma^\vee \otimes \mathbb{C}_{-(\lambda + \rho)},$$

where $W_\sigma^\vee = (\sigma^\vee, W^\vee)$ is the contragredient representation of W_σ . As $W_{\sigma, \lambda}^\vee$ is a representation of P , it can be thought of as a $\mathcal{U}(\mathfrak{p})$ -module. We then define a (\mathfrak{g}, P) -module $M_{\mathfrak{p}}(\sigma^\vee, -\lambda)$ (*generalized Verma module*) induced from $W_{\sigma, \lambda}^\vee$ by

$$M_{\mathfrak{p}}(\sigma^\vee, -\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} W_{\sigma, \lambda}^\vee.$$

The parabolic subgroup P acts on $M_{\mathfrak{p}}(\sigma^\vee, -\lambda)$ diagonally via the adjoint action Ad on $\mathcal{U}(\mathfrak{g})$ and the representation $\sigma^\vee \otimes e^{-(\lambda + \rho)} \otimes \mathbb{1}$ on $W_{\sigma, \lambda}^\vee$.

Given a finite-dimensional representation $E_\eta \equiv (\eta, E)$ of M and a weight $\nu \in \mathfrak{a}^*$, we similarly define P -representations $E_{\eta, \nu}$ and $E_{\eta, \nu}^\vee$. Let $\mathcal{E}_{\eta, \nu} \rightarrow G/P$ be the G -equivariant homogeneous vector bundle over G/P with fiber $E_{\eta, \nu}$. We realize the parabolically induced representation $I_P(\eta, \nu)$ of G on $C^\infty(G/P, \mathcal{E}_{\eta, \nu})$. We denote by $\text{Hom}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$ the space of intertwining operators from $I_P(\sigma, \lambda)$ to $I_P(\eta, \nu)$. Then we set

$$\text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)) := \text{Diff}(I_P(\sigma, \lambda), I_P(\eta, \nu)) \cap \text{Hom}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)),$$

where $\text{Diff}(I_P(\sigma, \lambda), I_P(\eta, \nu))$ is the space of differential operators from $I_P(\sigma, \lambda)$ to $I_P(\eta, \nu)$. Via the identification $I_P(\sigma, \lambda) \simeq I_{P_0}(\sigma, \lambda)^M$ and $I_P(\eta, \nu) \simeq I_{P_0}(\eta, \nu)^M$ in (2.1), the space $\text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$ can be identified as

$$(2.2) \quad \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)) \simeq \text{Diff}_G(I_{P_0}(\sigma, \lambda)^M, I_{P_0}(\eta, \nu)^M).$$

Let $R(X)$ denote the infinitesimal right translation of $X \in \mathfrak{g}_0$. We extend it complex linearly to \mathfrak{g} and naturally to $\mathcal{U}(\mathfrak{g})$. For finite-dimensional vector space V , we define

$$\langle \cdot, \cdot \rangle_V : V \times V^\vee \rightarrow \mathbb{C}$$

as the natural pairing of V and V^\vee .

The following duality theorem plays a key role for our construction of intertwining differential operators. For the proof see, for instance, [8, Lem. 2.4], [20, Thm. 2.9], or [24, Prop. 1.2].

Theorem 2.3 (Duality theorem). *There exists a natural linear isomorphism*

$$(2.4) \quad \mathcal{D}_{H \rightarrow D} : \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \xrightarrow{\sim} \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)).$$

For $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$ with $\varphi(x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) = \sum_i u_{x_i} \otimes (w_{x_i}^\vee \otimes \mathbb{1}_{-(\lambda+\rho)})$, where $u_{x_i} \in \mathcal{U}(\mathfrak{g})$ and $w_{x_i}^\vee \otimes \mathbb{1}_{-(\lambda+\rho)} \in W_{\sigma, \lambda}^\vee$, the operator $\mathcal{D}_{H \rightarrow D}(\varphi)$ is given by

$$(2.5) \quad \langle \mathcal{D}_{H \rightarrow D}(\varphi)F, x^\vee \rangle_E = \sum_i \langle R(u_{x_i})F, w_{x_i}^\vee \rangle_W \quad \text{for } F \in I_P(\sigma, \lambda).$$

For $M_{\mathfrak{p}}(\eta^\vee, -\nu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E_{\eta, \nu}^\vee$, we have

$$(2.6) \quad \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \simeq \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee, -\lambda)).$$

Thus Theorem 2.3 implies

$$(2.7) \quad \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \simeq \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)).$$

It follows from $P_0 \subset P$ that

$$(2.8) \quad \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \subset \text{Hom}_{P_0}(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)).$$

Via the identification (2.2), this induces an injection

$$\text{Diff}_G(I_{P_0}(\sigma, \lambda)^M, I_{P_0}(\eta, \nu)^M) \hookrightarrow \text{Diff}_G(I_{P_0}(\sigma, \lambda), I_{P_0}(\eta, \nu))$$

such that the following diagram commutes:

$$(2.9) \quad \begin{array}{ccc} \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) & \subset & \text{Hom}_{P_0}(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \\ \mathcal{D}_{H \rightarrow D}^P \Big\downarrow \wr & & \mathcal{D}_{H \rightarrow D}^{P_0} \Big\downarrow \wr \\ \text{Diff}_G(I_{P_0}(\sigma, \lambda)^M, I_{P_0}(\eta, \nu)^M) & \hookrightarrow & \text{Diff}_G(I_{P_0}(\sigma, \lambda), I_{P_0}(\eta, \nu)) \end{array}$$

Here $\mathcal{D}_{H \rightarrow D}^P$ and $\mathcal{D}_{H \rightarrow D}^{P_0}$ denote the duality isomorphisms (2.4) from the left-hand side and right-hand side of (2.8), respectively. It follows from (2.9) that, for $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$, the differential operator $\mathcal{D}_{H \rightarrow D}^P(\varphi)$ can be given by

$$(2.10) \quad \mathcal{D}_{H \rightarrow D}^P(\varphi) = \mathcal{D}_{H \rightarrow D}^{P_0}(\varphi)|_{I_{P_0}(\sigma, \lambda)^M}.$$

2.2. Tensored operator $\mathcal{D} \otimes \text{id}_\xi$. Loosely speaking, for some representations ξ of M , a differential operator $\mathcal{D} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$ induces another operator $\mathcal{D} \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\sigma \otimes \xi, \lambda), I_P(\eta \otimes \xi, \nu))$. In this subsection we first discuss such operators from the perspective of generalized Verma modules. The actual operators are then considered via the duality theorem.

Let $\text{Rep}(M)_{\text{fin}}$ be the set of finite-dimensional representations of M . As M is not connected in general, we write M_0 for the identity component of M . Let $\text{Rep}(M/M_0)$ denote the set of representations of the component group M/M_0 . Via the surjection $M \rightarrow M/M_0$, we regard $\text{Rep}(M/M_0)$ as a subset of $\text{Rep}(M)_{\text{fin}}$.

First, for $U_\xi := (\xi, U) \in \text{Rep}(M)_{\text{fin}}$, we define a P -isomorphism

$$\psi_{(\sigma^\vee, -\lambda; \xi^\vee)} : W_{\sigma, \lambda}^\vee \otimes U_\xi^\vee \xrightarrow{\sim} (W_\sigma^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\lambda+\rho)}$$

simply by

$$(w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) \otimes z^\vee \mapsto (w^\vee \otimes z^\vee) \otimes \mathbb{1}_{-(\lambda+\rho)}.$$

This induces a P -isomorphism

$$\tilde{\psi}'_{(\sigma^\vee, -\lambda; \xi^\vee)} : \mathcal{U}(\mathfrak{g}) \otimes W_{\sigma, \lambda}^\vee \otimes U_\xi^\vee \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}) \otimes (W_\sigma^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\lambda+\rho)},$$

given by

$$\tilde{\psi}'_{(\sigma^\vee, -\lambda; \xi^\vee)} := \text{id}_{\mathcal{U}(\mathfrak{g})} \otimes \psi_{(\sigma^\vee, -\lambda; \xi^\vee)}.$$

Here $\text{id}_{\mathcal{U}(\mathfrak{g})}$ denotes the identity map on $\mathcal{U}(\mathfrak{g})$. Further, if $U_\xi \in \text{Rep}(M/M_0) \subset \text{Rep}(M)_{\text{fin}}$, then, as $\mathcal{U}(\mathfrak{p})$ acts on U_ξ^\vee trivially, the map $\tilde{\psi}'_{(\sigma^\vee, -\lambda; \xi^\vee)}$ induces a P -isomorphism

$$(2.11) \quad \tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)} : M_{\mathfrak{p}}(\sigma^\vee, -\lambda) \otimes U_\xi^\vee \xrightarrow{\sim} M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) \otimes W_{\sigma, \lambda}^\vee \otimes U_\xi^\vee & \xrightarrow[\tilde{\psi}'_{(\sigma^\vee, -\lambda; \xi^\vee)}]{\sim} & \mathcal{U}(\mathfrak{g}) \otimes (W_\sigma^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\lambda+\rho)} \\ \downarrow & & \downarrow \\ M_{\mathfrak{p}}(\sigma^\vee, -\lambda) \otimes U_\xi^\vee & \xrightarrow[\tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)}]{\sim} & M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda) \end{array}$$

Remark that we have

$$\tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)} = \text{id}_{\mathcal{U}(\mathfrak{g})} \otimes \psi_{(\sigma^\vee, -\lambda; \xi^\vee)}.$$

Regarding $M_{\mathfrak{p}}(\sigma^\vee, -\lambda) \otimes U_\xi^\vee$ as a $\mathcal{U}(\mathfrak{g})$ -module via the action

$$z' \cdot (z \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) \otimes u^\vee) = (z'z) \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) \otimes u^\vee \text{ for } z' \in \mathcal{U}(\mathfrak{g}),$$

the map $\tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)}$ is indeed a (\mathfrak{g}, P) -isomorphism.

Now, for $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$, we define a linear map

$$\varphi_\xi : (E_\eta^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\nu+\rho)} \longrightarrow M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)$$

by

$$(2.12) \quad \varphi_\xi = \tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)} \circ (\varphi \otimes \text{id}_{\xi^\vee}) \circ \psi_{(\eta^\vee, -\nu; \xi^\vee)}^{-1},$$

so that the following diagram commutes:

$$\begin{array}{ccc}
 (E_\eta^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\nu+\rho)} & \xrightarrow{\varphi_\xi} & M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda) \\
 \psi_{(\eta^\vee, -\nu; \xi^\vee)}^{-1} \downarrow \wr & & \wr \uparrow \tilde{\psi}_{(\sigma^\vee, -\lambda; \xi^\vee)} \\
 E_{\eta, \nu}^\vee \otimes U_\xi^\vee & \xrightarrow{\varphi \otimes \text{id}_{\xi^\vee}} & M_{\mathfrak{p}}(\sigma^\vee, -\lambda) \otimes U_\xi^\vee
 \end{array}$$

Here id_{ξ^\vee} denotes the identity map on U_ξ^\vee .

Lemma 2.13. *Let*

$$(\eta, \nu), (\sigma, \lambda) \in \text{Rep}(M)_{\text{fin}} \times \mathfrak{a}^*,$$

and let

$$\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda)).$$

Then, for $\xi \in \text{Rep}(M/M_0)$, we have

$$(2.14) \quad \varphi_\xi \in \text{Hom}_P((E_\eta^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\nu+\rho)}, M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)).$$

Consequently, for any $\xi \in \text{Rep}(M/M_0)$, there exists a natural injection

$$\text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \hookrightarrow \text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee \otimes \xi^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)).$$

Proof. Each composed map in (2.12) respects P -actions. This implies the first assertion. The isomorphism (2.6) and the first assertion conclude the second. \square

Remark 2.15. When $\xi \in \text{Rep}(M/M_0)$ is a character, the injection in Lemma 2.13 is bijective.

Corollary 2.16. *The following conditions on $(\sigma, \eta; \lambda, \nu) \in \text{Rep}(M)_{\text{fin}}^2 \times (\mathfrak{a}^*)^2$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee, -\lambda)) \neq \{0\}$.
- (ii) $\text{Hom}_{\mathfrak{g}, P}(M_{\mathfrak{p}}(\eta^\vee \otimes \xi^\vee, -\nu), M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)) \neq \{0\}$ for any $\xi \in \text{Rep}(M/M_0)$.

Via the duality isomorphism $\mathcal{D}_{H \rightarrow D}$ in (2.4), the differential-operator counterpart of Lemma 2.13 and Corollary 2.16 is given as follows.

Lemma 2.17. *Retain the assumptions from Lemma 2.13. Then, for $\xi \in \text{Rep}(M/M_0)$, we have*

$$\mathcal{D}_{H \rightarrow D}(\varphi_\xi) \in \text{Diff}_G(I_P(\sigma \otimes \xi, \lambda), I_P(\eta \otimes \xi, \nu)).$$

In particular, the following conditions on $(\sigma, \eta; \lambda, \nu) \in \text{Rep}(M)_{\text{fin}}^2 \times (\mathfrak{a}^)^2$ are equivalent:*

- (i) $\text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)) \neq \{0\}$.
- (ii) $\text{Diff}_G(I_P(\sigma \otimes \xi, \lambda), I_P(\eta \otimes \xi, \nu)) \neq \{0\}$ for any $\xi \in \text{Rep}(M/M_0)$.

We now give the operator $\mathcal{D}_{H \rightarrow D}(\varphi_\xi)$ in terms of $\mathcal{D}_{H \rightarrow D}(\varphi) \otimes \text{id}_\xi$. For simplicity put $(E_\eta \otimes U_\xi)^\vee := (E_\eta^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\nu+\rho)}$. As in (2.8), we have

$$(2.18) \quad \text{Hom}_P((E_\eta \otimes U_\xi)^\vee, M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)) \subset \text{Hom}_{P_0}((E_\eta \otimes U_\xi)^\vee, M_{\mathfrak{p}}(\sigma^\vee \otimes \xi^\vee, -\lambda)).$$

This yields the identity

$$(2.19) \quad \mathcal{D}_{H \rightarrow D}^P(\varphi_\xi) = \mathcal{D}_{H \rightarrow D}^{P_0}(\varphi_\xi)|_{I_{P_0}(\sigma \otimes \xi, \lambda)^M}$$

as in (2.9) and (2.10). Thus, to understand $\mathcal{D}_{H \rightarrow D}(\varphi_\xi) = \mathcal{D}_{H \rightarrow D}^P(\varphi_\xi)$, it suffices to consider $\mathcal{D}_{H \rightarrow D}^{P_0}(\varphi_\xi)$.

Observe that since M_0 acts on U_ξ trivially, we have

$$I_{P_0}(\sigma \otimes \xi, \lambda) = I_{P_0}(\sigma, \lambda) \otimes U_\xi,$$

where $I_{P_0}(\sigma, \lambda) \otimes U_\xi$ is regarded as a G -module by letting G act on U_ξ trivially. It then follows from the trivial action of G on U_ξ that

$$\begin{aligned} \mathcal{D}_{H \rightarrow D}^{P_0}(\varphi) \otimes \text{id}_\xi &\in \text{Diff}_G(I_{P_0}(\sigma, \lambda) \otimes U_\xi, I_{P_0}(\eta, \nu) \otimes U_\xi) \\ &= \text{Diff}_G(I_{P_0}(\sigma \otimes \xi, \lambda), I_{P_0}(\eta \otimes \xi, \nu)) \end{aligned}$$

for $\varphi \in \text{Hom}_{P_0}(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$. Further, a direct observation on (2.5) shows that

$$(2.20) \quad \mathcal{D}_{H \rightarrow D}^{P_0}(\varphi_\xi) = \mathcal{D}_{H \rightarrow D}^{P_0}(\varphi) \otimes \text{id}_\xi.$$

In summary the operator $\mathcal{D}_{H \rightarrow D}^P(\varphi_\xi)$ can be given as follows.

Lemma 2.21. *Retain the assumptions from Lemma 2.17. We have*

$$(2.22) \quad \mathcal{D}_{H \rightarrow D}^P(\varphi_\xi) = \left(\mathcal{D}_{H \rightarrow D}^{P_0}(\varphi) \otimes \text{id}_\xi \right) \Big|_{I_{P_0}(\sigma \otimes \xi, \lambda)^M}.$$

Proof. Equations (2.19) and (2.20) yield the lemma. □

By abuse of notation we simply write (2.22) as

$$(2.23) \quad \mathcal{D}_{H \rightarrow D}(\varphi_\xi) = \mathcal{D}_{H \rightarrow D}(\varphi) \otimes \text{id}_\xi.$$

2.3. Peter–Weyl-type formula for the space of K -finite solutions. Let K be the maximal compact subgroup of G with Lie algebra \mathfrak{k}_0 . We next study the K -type decomposition of the space of K -finite solutions to intertwining differential operators. The following lemma plays a key role.

Lemma 2.24. *There exists a $(\mathcal{U}(\mathfrak{k} \cap \mathfrak{m}), K \cap M)$ -isomorphism*

$$\iota: \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k} \cap \mathfrak{m})} W_\sigma^\vee \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} W_{\sigma, \lambda}^\vee, \quad u \otimes w^\vee \mapsto u \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}),$$

where $K \cap M$ acts on $\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k} \cap \mathfrak{m})} W_\sigma^\vee$ and $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} W_{\sigma, \lambda}^\vee$ diagonally.

Proof. This is an immediate generalization of [16, Lem. 2.1]. □

It follows from Lemma 2.24 that, for $u \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} W_{\sigma, \lambda}^\vee$, there exists an element $u^\flat \in \mathcal{U}(\mathfrak{k})$ such that

$$\iota(u^\flat \otimes w^\vee) = u \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}),$$

that is,

$$(2.25) \quad u^\flat \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) = u \otimes (w^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}).$$

We note that the choice of u^\flat is not unique in general; any choice satisfying (2.25) is acceptable.

We identify $C^\infty(G/P, \mathcal{W}_{\sigma, \lambda})$ with the space $C^\infty(K/(K \cap M), \mathcal{W}_{\sigma, \lambda}|_K)$ of smooth sections for the restricted vector bundle $\mathcal{W}_{\sigma, \lambda}|_K \rightarrow K/(K \cap M)$, which is further identified as

$$\begin{aligned} &C^\infty(K/(K \cap M), \mathcal{W}_{\sigma, \lambda}|_K) \\ &\simeq \{F \in C^\infty(K) \otimes W_\sigma : F(km) = \sigma(m)^{-1}F(k) \text{ for all } m \in K \cap M\}. \end{aligned}$$

Let $\text{Irr}(K)$ denote the set of equivalence classes of irreducible representations of K . For $V_\delta = (\delta, V) \in \text{Irr}(K)$, we denote by $(\cdot, \cdot)_\delta$ a K -invariant Hermitian inner

product of V_δ . We take the first argument of $(\cdot, \cdot)_\delta$ to be linear and the second to be conjugate linear. We then identify the contragredient representation $V_\delta^\vee = (\delta^\vee, V^\vee)$ with the conjugate representation $\bar{V}_\delta = (\bar{\delta}, \bar{V})$ via the map $\bar{v} \mapsto (\cdot, v)_\delta$.

Let $I_P(\sigma, \lambda)_K$ denote the (\mathfrak{g}, K) -module consisting of the K -finite vectors of $C^\infty(K/(K \cap M), \mathcal{W}_{\sigma, \lambda|_K})$. It then follows from the Peter–Weyl theorem that, via the identification $V_\delta^\vee \simeq \bar{V}_\delta$, there exists a K -isomorphism

$$(2.26) \quad \Phi_\sigma : \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes (\bar{V}_\delta \otimes W_\sigma)^{K \cap M} \xrightarrow{\sim} I_P(\sigma, \lambda)_K$$

with $\Phi_\sigma := \bigoplus_{\delta \in \text{Irr}(K)} \Phi_{(\sigma, \delta)}$, where the map

$$\Phi_{(\sigma, \delta)} : V_\delta \otimes (\bar{V}_\delta \otimes W_\sigma)^{K \cap M} \longrightarrow I_P(\sigma, \lambda)_K$$

is given by

$$(2.27) \quad \Phi_{(\sigma, \delta)}(v \otimes \bar{v}' \otimes w)(k) := (v, \delta(k)v')_\delta w = (\delta(k^{-1})v, v')_\delta w.$$

We note that the direct sum on the left-hand side of (2.26) is algebraic.

Let $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ be the conjugation of \mathfrak{g} with respect to the real form \mathfrak{g}_0 , that is, $\tau(X + \sqrt{-1}Y) = X - \sqrt{-1}Y$ for $X, Y \in \mathfrak{g}_0$. Given $\delta \in \text{Irr}(K)$, let $d\delta$ denote the infinitesimal representation of \mathfrak{k}_0 . As usual we extend τ and $d\delta$ to the universal enveloping algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k})$, respectively. It then follows from (2.27) and the conjugate-linearity of the second argument of $(\cdot, \cdot)_\delta$ that, for $u \in \mathcal{U}(\mathfrak{k})$, we have

$$(2.28) \quad R(u)\Phi_{(\sigma, \delta)}(v \otimes \bar{v}' \otimes w)(k) = (\delta(k^{-1})v, d\delta(\tau(u))v')_\delta w.$$

Now, for $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$, we write

$$(2.29) \quad \mathcal{D}_\varphi = \mathcal{D}_{H \rightarrow D}(\varphi).$$

By the duality theorem, any $\mathcal{D} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$ is of the form $\mathcal{D} = \mathcal{D}_\varphi$ for some $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$, namely,

$$\text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)) = \{\mathcal{D}_\varphi : \varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))\}.$$

Then, for $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$, we set

$$\begin{aligned} \text{Sol}_{(\varphi; \lambda)}(\sigma) &:= \{F \in I_P(\sigma, \lambda) : \mathcal{D}_\varphi F = 0\}, \\ \text{Sol}_{(\varphi; \lambda)}(\sigma)_K &:= \{F \in I_P(\sigma, \lambda)_K : \mathcal{D}_\varphi F = 0\}, \end{aligned}$$

the spaces of smooth solutions and K -finite solutions to \mathcal{D}_φ , respectively.

As \mathcal{D}_φ is an intertwining operator, the solution space $\text{Sol}_{(\varphi; \lambda)}(\sigma)$ is a representation of G . Similarly, the space $\text{Sol}_{(\varphi; \lambda)}(\sigma)_K$ of K -finite solutions is a (\mathfrak{g}, K) -module. We wish to understand $\text{Sol}_{(\varphi; \lambda)}(\sigma)_K$ via the Peter–Weyl theorem (2.26).

Observe that, for $x^\vee \otimes \mathbb{1}_{-(\nu+\rho)} \in E_{\eta, \nu}^\vee$, there exist $u_{x_i}^b \in \mathcal{U}(\mathfrak{k})$ and $w_{x_i}^\vee \in W_{\sigma, \lambda}^\vee$ such that

$$\varphi(x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) = \sum_i u_{x_i}^b \otimes (w_{x_i}^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}) \in M_{\mathfrak{p}}(\sigma^\vee, -\lambda).$$

(See Lemma 2.24 and (2.25).) Then we define a family of linear maps

$$D_\varphi^{(\delta \otimes \sigma)}(\cdot; \cdot) : (\bar{V}_\delta \otimes W_\sigma) \times E_\eta^\vee \rightarrow V_\delta$$

by

$$(2.30) \quad D_\varphi^{(\delta \otimes \sigma)}(T; x^\vee) := \sum_{i, j} \langle w_j, w_{x_i}^\vee \rangle d\delta(\tau(u_{x_i}^b))v_j,$$

where $T = \sum_j \bar{v}_j \otimes w_j \in \bar{V}_\delta \otimes W_\sigma$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_W$, the natural pairing of W and W^\vee . The following two lemmas on $D_\varphi^{(\delta \otimes \sigma)}(\cdot; \cdot)$ play a key role later.

Lemma 2.31. *Let $x^\vee \in E_\eta^\vee$, and let $T \in \bar{V}_\delta \otimes W_\sigma$. Then, for $F(k) := \Phi_{(\sigma, \delta)}(v \otimes T)(k) \in I_P(\sigma, \lambda)_K$, we have*

$$(2.32) \quad \langle \mathcal{D}_\varphi F(k), x^\vee \rangle_E = (\delta(k^{-1})v, D_\varphi^{(\delta \otimes \sigma)}(T; x^\vee))_\delta.$$

Proof. This is a direct consequence of (2.5) and (2.28). □

Lemma 2.33. *For $m \in K \cap M$, we have*

$$D_\varphi^{(\delta \otimes \sigma)}((\bar{\delta} \otimes \sigma)(m)T; \eta^\vee(m)x^\vee) = \delta(m)D_\varphi^{(\delta \otimes \sigma)}(T; x^\vee).$$

In particular, the linear map $D_\varphi^{(\delta \otimes \sigma)}(\cdot; \cdot): (\bar{V}_\delta \otimes W_\sigma) \times E_\eta^\vee \rightarrow V_\delta$ is $K \cap M$ -equivariant.

Proof. For $T = \sum_j \bar{v}_j \otimes w_j$ and $\varphi(x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) = \sum_i u_{x_i}^b \otimes (w_{x_i}^\vee \otimes \mathbb{1}_{-(\lambda+\rho)})$, we have

$$\begin{aligned} D_\varphi^{(\delta \otimes \sigma)}((\bar{\delta} \otimes \sigma)(m)T; x^\vee) &= \sum_{i,j} \langle \sigma(m)w_j, w_{x_i}^\vee \rangle d\delta(\tau(u_{x_i}^b)) \delta(m)v_j \\ &= \delta(m) \sum_{i,j} \langle w_j, \sigma^\vee(m^{-1})w_{x_i}^\vee \rangle d\delta(\tau(\text{Ad}(m^{-1})u_{x_i}^b))v_j. \end{aligned}$$

On the other hand, it follows from $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$ that

$$\begin{aligned} \varphi(\eta^\vee(m^{-1})x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) &= (\text{Ad} \otimes \sigma^\vee)(m^{-1})\varphi(x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) \\ &= \sum_i \text{Ad}(m^{-1})u_{x_i}^b \otimes (\sigma^\vee(m^{-1})w_{x_i}^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}). \end{aligned}$$

Therefore,

$$(2.34) \quad D_\varphi^{(\delta \otimes \sigma)}((\bar{\delta} \otimes \sigma)(m)T; x^\vee) = \delta(m)D_\varphi^{(\delta \otimes \sigma)}(T; \eta^\vee(m^{-1})x^\vee).$$

Now the lemma follows by replacing x^\vee with $\eta^\vee(m)x^\vee$. □

For $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\sigma^\vee, -\lambda))$ and a subspace $J \subset \bar{V}_\delta \otimes W_\sigma$, we define

$$(2.35) \quad \text{Sol}_{(\varphi)}(J) := \{T \in J : D_\varphi^{(\delta \otimes \sigma)}(T; x^\vee) = 0 \text{ for all } x^\vee \in E_\eta^\vee\}.$$

Lemma 2.36. *For a subspace $J \subset \bar{V}_\delta \otimes W_\sigma$, the space $\text{Sol}_{(\varphi)}(J)$ is a $K \cap M$ -representation.*

Proof. Let $T \in \text{Sol}_{(\varphi)}(J)$, let $m \in K \cap M$, and let $x^\vee \in E_\eta^\vee$. Since $\eta^\vee(m^{-1})x^\vee \in E_\eta^\vee$, we have $D_\varphi^{(\delta \otimes \sigma)}(T; \eta^\vee(m^{-1})x^\vee) = 0$. It then follows from (2.34) that

$$D_\varphi^{(\delta \otimes \sigma)}((\bar{\delta} \otimes \sigma)(m)T; x^\vee) = \delta(m)D_\varphi^{(\delta \otimes \sigma)}(T; \eta^\vee(m^{-1})x^\vee) = 0.$$

This concludes the lemma. □

Theorem 2.37 (PW-type formula: VV case). *Let $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))$. Then the space $\text{Sol}_{(\varphi; \lambda)}(\sigma)_K$ of K -finite solutions to \mathcal{D}_φ has a K -type decomposition*

$$(2.38) \quad \text{Sol}_{(\varphi; \lambda)}(\sigma)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Sol}_{(\varphi)}(\bar{V}_\delta \otimes W_\sigma)^{K \cap M}.$$

Proof. Since $Sol_{(\varphi;\lambda)}(\sigma)_K$ is a K -subrepresentation of $I_P(\sigma, \lambda)_K$, there exists a subspace $S_{\delta,\sigma} \subset (\bar{V}_\delta \otimes W_\sigma)^{K \cap M}$ such that

$$Sol_{(\varphi;\lambda)}(\sigma)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes S_{\delta,\sigma}.$$

Observe that $F \in I_P(\sigma, \lambda)_K$ is in $Sol_{(\varphi;\lambda)}(\sigma)_K$ if and only if $\langle \mathcal{D}_\varphi F, x^\vee \rangle_E = 0$ for all $x^\vee \in E_\eta^\vee$. It then follows from (2.32) that we have $S_{\delta,\sigma} = \text{Sol}_{(\varphi)}((\bar{V}_\delta \otimes W_\sigma)^{K \cap M})$. Since $\text{Sol}_{(\varphi)}((\bar{V}_\delta \otimes W_\sigma)^{K \cap M}) = \text{Sol}_{(\varphi)}(\bar{V}_\delta \otimes W_\sigma)^{K \cap M}$, this proves the theorem. \square

Remark 2.39. In the case that σ is one-dimensional, the isomorphism (2.38) is obtained in [16, Thm. 2.6] in the framework of conformally invariant systems.

2.4. Specialization to the case $(\mathcal{L}_{\chi,\lambda}, \mathcal{E}_{\eta,\nu})$. We now investigate the K -type decomposition (2.38) for the case that the vector bundle $W_{\sigma,\lambda}$ is specialized to a line bundle $\mathcal{L}_{\chi,\lambda}$ with character χ of M . In this case intertwining differential operators $\mathcal{D} \in \text{Diff}_G(I_P(\chi, \lambda), I_P(\eta, \nu))$ are of the form $\mathcal{D} = \mathcal{D}_\varphi$ for some $\varphi \in \text{Hom}_P(E_{\eta,\nu}^\vee, M_{\mathfrak{p}}(\chi^{-1}, -\lambda))$.

Let $\varphi \in \text{Hom}_P(E_{\eta,\nu}^\vee, M_{\mathfrak{p}}(\chi^{-1}, -\lambda))$ and take $x^\vee \otimes \mathbb{1}_{-(\nu+\rho)} \in E_{\eta,\nu}^\vee$. We have $\varphi(x^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) = u_x^b \otimes (\mathbb{1}_{\chi^{-1}} \otimes \mathbb{1}_{-(\lambda+\rho)})$ for some $u_x^b \in \mathcal{U}(\mathfrak{k})$. It then follows from (2.30) that, for $\bar{v} \otimes \mathbb{1}_\chi \in \bar{V}_\delta \otimes \mathbb{C}_\chi$, we have

$$D_\varphi^{(\delta \otimes \chi)}(\bar{v} \otimes \mathbb{1}_\chi; x^\vee) = d\delta(\tau(u_x^b))v.$$

Then we simply write

$$D_\varphi^{(\delta)}(v; x^\vee) = D_\varphi^{(\delta \otimes \chi)}(\bar{v} \otimes \mathbb{1}_\chi; x^\vee).$$

We set

$$(2.40) \quad \text{Sol}_{(\varphi)}(\delta) := \{v \in V_\delta : D_\varphi^{(\delta)}(v; x^\vee) = 0 \text{ for all } x^\vee \in E_\eta^\vee\}.$$

Lemma 2.41. *The space $\text{Sol}_{(\varphi)}(\delta)$ is a $K \cap M$ -representation.*

Proof. As the proof is similar to Lemma 2.36, we omit the proof. \square

Proposition 2.42 (PW-type formula: LV-case 1). *Let $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\chi, \lambda), I_P(\eta, \nu))$. Then there exists a K -isomorphism*

$$Sol_{(\varphi;\lambda)}(\chi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M}(\text{Sol}_{(\varphi)}(\delta), \chi).$$

Proof. We have

$$\begin{aligned} \text{Sol}_{(\varphi)}(\bar{V}_\delta \otimes \mathbb{C}_\chi)^{K \cap M} &= \overline{(\text{Sol}_{(\varphi)}(\delta) \otimes \mathbb{C}_\chi)^{K \cap M}} \\ &\simeq (\text{Sol}_{(\varphi)}(\delta)^\vee \otimes \mathbb{C}_\chi)^{K \cap M} \\ &\simeq \text{Hom}_{K \cap M}(\text{Sol}_{(\varphi)}(\delta), \chi). \end{aligned}$$

Now the proposition follows from Theorem 2.37. \square

2.5. Solution space of tensored operator $\mathcal{D}_\varphi \otimes \text{id}_\xi$. Now take the character χ of M in Section 2.4 to be the trivial character $\chi = \chi_{\text{triv}}$, and let $U_\xi = (\xi, U) \in \text{Rep}(M/M_0)$. Let $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\chi_{\text{triv}}, -\lambda))$ so that we have $\mathcal{D}_\varphi \in \text{Diff}_G(IP(\chi_{\text{triv}}, \lambda), IP(\eta, \nu))$.

First observe that it follows from Lemma 2.13 that

$$\varphi_\xi \in \text{Hom}_P((E_\eta^\vee \otimes U_\xi^\vee) \otimes \mathbb{C}_{-(\nu+\rho)}, M_{\mathfrak{p}}(\xi^\vee, -\lambda)).$$

Equivalently, by Lemma 2.17 and (2.29), we have

$$\mathcal{D}_{\varphi_\xi} = \mathcal{D}_{H \rightarrow D}(\varphi_\xi) \in \text{Diff}_G(IP(\xi, \lambda), IP(\eta \otimes \xi, \nu)).$$

As $\mathcal{D}_{\varphi_\xi} = \mathcal{D}_\varphi \otimes \text{id}_\xi$ by (2.23), the solution spaces $\text{Sol}_{(\varphi_\xi; \lambda)}(\xi)$ and $\text{Sol}_{(\varphi_\xi; \lambda)}(\xi)_K$ of $\mathcal{D}_{\varphi_\xi}$ are given by

$$\begin{aligned} \text{Sol}_{(\varphi_\xi; \lambda)}(\xi) &= \{F \in IP(\xi, \lambda) : (\mathcal{D}_\varphi \otimes \text{id}_\xi)F = 0\}, \\ \text{Sol}_{(\varphi_\xi; \lambda)}(\xi)_K &= \{F \in IP(\xi, \lambda)_K : (\mathcal{D}_\varphi \otimes \text{id}_\xi)F = 0\}. \end{aligned}$$

It follows from Theorem 2.37 that we have

$$(2.43) \quad \text{Sol}_{(\varphi_\xi; \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi)^{K \cap M}.$$

Lemma 2.44. For $\varphi \in \text{Hom}_P(E_{\eta, \nu}^\vee, M_{\mathfrak{p}}(\chi_{\text{triv}}, -\lambda))$ and $U_\xi \in \text{Rep}(M/M_0)$, we have

$$\text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi) = \overline{\text{Sol}_{(\varphi)}(\delta)} \otimes U_\xi.$$

Proof. First observe that the space $\text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi)$ is given by

$$\begin{aligned} \text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi) &= \{T \in \bar{V}_\delta \otimes U_\xi : D_{\varphi_\xi}^{(\delta \otimes \xi)}(T; S) = 0 \text{ for all } S \in E_\eta^\vee \otimes U_\xi^\vee\} \\ \text{(see (2.35)). For } S &= \sum_i x_i^\vee \otimes z_i^\vee \in E_\eta^\vee \otimes U_\xi^\vee \text{ and } \varphi(\sum_i x_i^\vee \otimes \mathbb{1}_{-(\nu+\rho)}) = \sum_i u_{x_i}^b \otimes \mathbb{1}_{-(\lambda+\rho)} \text{ with } u_{x_i}^b \in \mathcal{U}(\mathfrak{k}), \text{ we have} \end{aligned}$$

$$\varphi_\xi(S \otimes \mathbb{1}_{-(\nu+\rho)}) = \sum_i u_{x_i}^b \otimes (z_i^\vee \otimes \mathbb{1}_{-(\lambda+\rho)}).$$

It then follows from (2.30) that, for $T = \sum_j \bar{v}_j \otimes z_j \in \bar{V}_\delta \otimes U_\xi$, the vector $D_{\varphi_\xi}^{(\delta \otimes \xi)}(T; S)$ is given by

$$(2.45) \quad D_{\varphi_\xi}^{(\delta \otimes \xi)}(T; S) = \sum_{i,j} \langle z_j, z_i^\vee \rangle d\delta(\tau(u_{x_i}^b))v_j,$$

where $\langle z_j, z_i^\vee \rangle = \langle z_j, z_i^\vee \rangle_U$. By (2.40), this shows the inclusion $\overline{\text{Sol}_{(\varphi)}(\delta)} \otimes U_\xi \subset \text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi)$.

To show the other inclusion, let $T = \sum_j \bar{v}_j \otimes z_j \in \bar{V}_\delta \otimes U_\xi$. If $\{u_1, \dots, u_p\}$ is a basis of U_ξ with dual basis $\{u_1^\vee, \dots, u_p^\vee\}$ of U_ξ^\vee , then T can be given by $T = \sum_{j,k} \langle z_j, u_k^\vee \rangle \bar{v}_j \otimes u_k$. Then T is in $\text{Sol}_{(\varphi_\xi)}(\bar{V}_\delta \otimes U_\xi)$ if and only if, for every $z^\vee \in U_\xi^\vee$ and $x^\vee \in E_\eta^\vee$,

$$\sum_k \langle u_k, z^\vee \rangle d\delta(\tau(u_x^b))(\sum_j \langle z_j, u_k^\vee \rangle \bar{v}_j) = 0.$$

This implies that, for every k , we have $\sum_j \langle z_j, u_k^\vee \rangle \bar{v}_j \in \overline{\text{Sol}_{(\varphi)}(\delta)}$, and the result follows. \square

Now (2.43) and Lemma 2.44 conclude the following theorem.

Theorem 2.46 (PW-type formula: LV-case 2). *Let $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\eta, \nu))$ and $\xi \in \text{Rep}(M/M_0)$. Then the space $\text{Sol}_{(\varphi_\xi; \lambda)}(\xi)_K$ of K -finite solutions to $\mathcal{D}_\varphi \otimes \text{id}_\xi$ has a K -type decomposition*

$$(2.47) \quad \text{Sol}_{(\varphi_\xi; \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M}(\text{Sol}_{(\varphi)}(\delta), \xi).$$

Remark 2.48. Theorem 2.46 can be thought of as a generalization of the argument given after the proof of Theorem 2.6 of [16].

If G is split real and $P = MAN$ is minimal parabolic, then $M = M/M_0$. Therefore, for $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\eta, \nu))$ and $\sigma \in \text{Rep}(M)$, we have $\mathcal{D}_\varphi \otimes \text{id}_\sigma \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta \otimes \sigma, \nu))$. In this case, as $M \subset K$, Theorem 2.46 is given as follows.

Corollary 2.49. *Suppose that G is split real and $P = MAN$ is minimal parabolic. Let $\mathcal{D}_\varphi \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\eta, \nu))$, and let $\sigma \in \text{Irr}(M)$. Then the space $\text{Sol}_{(\varphi_\sigma; \lambda)}(\sigma)_K$ of K -finite solutions to $\mathcal{D}_\varphi \otimes \text{id}_\sigma$ has a K -type decomposition*

$$\text{Sol}_{(\varphi_\sigma; \lambda)}(\sigma)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_M(\text{Sol}_{(\varphi)}(\delta), \sigma).$$

2.6. Common solution space of a system of intertwining differential operators. In this subsection we discuss the common solution space of a system of intertwining differential operators $\mathcal{D}_{\varphi_j} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta_j, \nu_j))$ with $(\sigma, \lambda), (\eta_j, \nu_j) \in \text{Rep}(M)_{\text{fin}} \times \mathfrak{a}^*$ for $j = 1, \dots, n$.

For each $\mathcal{D}_{\varphi_j} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta_j, \nu_j))$, we have $\text{Sol}_{(\varphi_j; \lambda)}(\sigma) \subset I_P(\sigma, \lambda)$. Thus one can consider the common solution spaces

$$\begin{aligned} \text{Sol}_{(\varphi_1, \dots, \varphi_n; \lambda)}(\sigma) &:= \bigcap_{j=1}^n \text{Sol}_{(\varphi_j; \lambda)}(\sigma) \subset I_P(\sigma, \lambda), \\ \text{Sol}_{(\varphi_1, \dots, \varphi_n; \lambda)}(\sigma)_K &:= \bigcap_{j=1}^n \text{Sol}_{(\varphi_j; \lambda)}(\sigma)_K \subset I_P(\sigma, \lambda)_K. \end{aligned}$$

For $\delta \in \text{Irr}(K)$, we set

$$\text{Sol}_{(\varphi_1, \dots, \varphi_n)}(\bar{V}_\delta \otimes W_\sigma) := \bigcap_{j=1}^n \text{Sol}_{(\varphi_j)}(\bar{V}_\delta \otimes W_\sigma),$$

where $\text{Sol}_{(\varphi_j)}(\bar{V}_\delta \otimes W_\sigma)$ is the subspace of $\bar{V}_\delta \otimes W_\sigma$ defined as in (2.35). It then follows from Theorem 2.37 that the space $\text{Sol}_{(\varphi_1, \dots, \varphi_n; \lambda)}(\sigma)_K$ of K -finite solutions to the system of differential operators $\mathcal{D}_{\varphi_1}, \dots, \mathcal{D}_{\varphi_n}$ can be decomposed as

$$\text{Sol}_{(\varphi_1, \dots, \varphi_n; \lambda)}(\sigma)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Sol}_{(\varphi_1, \dots, \varphi_n)}(\bar{V}_\delta \otimes W_\sigma)^{K \cap M}.$$

Now take σ to be the trivial character $\sigma = \chi_{\text{triv}}$. It then follows from Lemma 2.17 that, for any $\xi \in \text{Rep}(M/M_0)$, we have

$$\mathcal{D}_{\varphi_j} \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\xi, \lambda), I_P(\eta_j \otimes \xi, \nu_j)) \quad \text{for } j = 1, \dots, n.$$

Therefore, for each $\xi \in \text{Rep}(M/M_0)$ and $\delta \in \text{Irr}(K)$, we set

$$\begin{aligned}
 \text{Sol}_{((\varphi_1, \dots, \varphi_n)_\xi; \lambda)}(\xi)_K &:= \bigcap_{j=1}^n \text{Sol}_{((\varphi_j)_\xi; \lambda)}(\xi)_K, \\
 \text{Sol}_{(\varphi_1, \dots, \varphi_n)}(\delta) &:= \bigcap_{i=1}^n \text{Sol}_{(\varphi_i)}(\delta),
 \end{aligned}
 \tag{2.50}$$

where $\text{Sol}_{(\varphi_j)}(\delta)$ is the subspace of V_δ defined as in (2.40). It then follows from Theorem 2.46 that, for $\xi \in \text{Rep}(M/M_0)$, there exists a K -isomorphism

$$\text{Sol}_{((\varphi_1, \dots, \varphi_n)_\xi; \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M}(\text{Sol}_{(\varphi_1, \dots, \varphi_n)}(\delta), \xi).$$

2.7. Recipe for determining the K -type formula. In this subsection, for the later applications in mind, we further take the targeted vector bundle $\mathcal{E}_{\eta, \nu}$ to be a line bundle $\mathcal{L}_{\chi, \nu}$ and summarize as a recipe how to determine the K -type formula of the solution space of the intertwining differential operator \mathcal{D} from $\mathcal{L}_{\chi_{\text{triv}}, \lambda}$ to $\mathcal{L}_{\chi, \nu}$. (See Theorem 1.2.)

To the end, we first observe that in this case, by the duality theorem and a general fact on the space of homomorphisms between generalized Verma modules ([27, Thm. 1.1]), we have $\dim_{\mathbb{C}} \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu)) \leq 1$ for any character χ of M and $\lambda, \nu \in \mathfrak{a}^*$. Further, any map $\varphi \in \text{Hom}_P(\mathbb{C}_{\chi^{-1}, -\nu}, M_{\mathfrak{p}}(\chi_{\text{triv}}, -\lambda))$ is of the form $\varphi(\mathbb{1}_{\chi^{-1}} \otimes \mathbb{1}_{-(\nu+\rho)}) = u \otimes (\mathbb{1}_{\chi_{\text{triv}}} \otimes \mathbb{1}_{-(\lambda+\rho)})$ with $u \in \mathcal{U}(\bar{\mathfrak{n}})$, where $\bar{\mathfrak{n}}$ denotes the opposite nilpotent radical to \mathfrak{n} . Thus the differential operator $\mathcal{D} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ is given by $\mathcal{D} = R(u)$ for some $u \in \mathcal{U}(\bar{\mathfrak{n}})$. We write \mathcal{D}_u for the differential operator on $I_P(\chi_{\text{triv}}, \lambda)$ such that $\mathcal{D}_u = R(u)$. Similarly, we put

$$\begin{aligned}
 \text{Sol}_{(u; \lambda)}(\xi) &:= \{F \in I_P(\xi, \lambda) : (\mathcal{D}_u \otimes \text{id}_\xi)F = 0\}, \\
 \text{Sol}_{(u)}(\delta) &:= \{v \in V_\delta : d\delta(\tau(u^b))v = 0\}.
 \end{aligned}
 \tag{2.51}$$

With the notation the K -type decomposition (2.47) is then given by

$$\text{Sol}_{(u; \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M}(\text{Sol}_{(u)}(\delta), \xi).
 \tag{2.52}$$

Let $\text{Irr}(M/M_0)$ denote the set of equivalence classes of irreducible representations of M/M_0 . The aim of the recipe is to determine the representations $\delta \in \text{Irr}(K)$ and $\xi \in \text{Irr}(M/M_0)$ such that $\text{Hom}_{K \cap M}(\text{Sol}_{(u)}(\delta), \xi) \neq \{0\}$. The recipe consists of five steps.

Recipe for determining the K -type formula. Let $\mathcal{D}_u \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$. The K -type formula for the solution space $\text{Sol}_{(u; \lambda)}(\xi)$ of $\mathcal{D}_u \otimes \text{id}_\xi$ can be obtained as follows.

Step S1. Find $\tau(u^b) \in \mathcal{U}(\mathfrak{k})$ for $u \in \mathcal{U}(\bar{\mathfrak{n}})$. Note that the choice of u^b is not unique; any choice with the property

$$u^b \otimes (\mathbb{1}_{\chi_{\text{triv}}} \otimes \mathbb{1}_{-(\lambda+\rho)}) = u \otimes (\mathbb{1}_{\chi_{\text{triv}}} \otimes \mathbb{1}_{-(\lambda+\rho)})
 \tag{2.53}$$

in $M_{\mathfrak{p}}(\chi_{\text{triv}}, -\lambda)$ is acceptable.

Step S2. Choose a realization of $\delta \in \text{Irr}(K)$ to solve the equation $d\delta(\tau(u^b))v = 0$ (see (2.51)). Find the explicit formula of the operator $d\delta(\tau(u^b)) \in \text{End}(V_\delta)$ if necessary.

Step S3. Solve the equation $d\delta(\tau(u^b))v = 0$ in the realization chosen in Step S2 and determine $\delta \in \text{Irr}(K)$ such that $\text{Sol}_{(u)}(\delta) \neq \{0\}$.

Step S4. For $\delta \in \text{Irr}(K)$ with $\text{Sol}_{(u)}(\delta) \neq \{0\}$, determine the $K \cap M$ -representation on $\text{Sol}_{(u)}(\delta)$.

Step S5. Given $\xi \in \text{Irr}(M/M_0)$, determine $\delta \in \text{Irr}(K)$ with $\text{Sol}_{(u)}(\delta) \neq \{0\}$ such that

$$\text{Hom}_{K \cap M}(\text{Sol}_{(u)}(\delta), \xi) \neq \{0\}.$$

We remark that one can also determine $\dim_{\mathbb{C}} \text{Hom}_{K \cap M}(\text{Sol}_{(u)}(\delta), \xi)$ through the recipe. (See Propositions 5.13 and 6.15.)

Given characters χ_j of M and $\nu_j \in \mathfrak{a}^*$ for $j = 1, \dots, n$, take differential operators $\mathcal{D}_{u_j} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi_j, \nu_j))$. Then the K -type formula for the common solution space $\text{Sol}_{(u_1, \dots, u_n; \lambda)}(\xi)$ of the operators $\mathcal{D}_{u_1} \otimes \text{id}_{\xi}, \dots, \mathcal{D}_{u_n} \otimes \text{id}_{\xi}$ can be achieved as follows.

Step CS1. Perform Steps S1–S3 for each $\mathcal{D}_{u_j} \otimes \text{id}_{\xi}$.

Step CS2. Determine $\delta \in \text{Irr}(K)$ such that $\text{Sol}_{(u_1, \dots, u_n)}(\delta) \neq \{0\}$.

The rest of the steps are the same as Steps S4 and S5 by replacing $\text{Sol}_{(u)}(\delta)$ with $\text{Sol}_{(u_1, \dots, u_n)}(\delta)$.

In Sections 5 and 6, we determine the K -type formulas for certain intertwining differential operators in accordance with the recipe for $G = \widetilde{\text{SL}}(3, \mathbb{R})$.

3. (\mathfrak{g}, B) -HOMOMORPHISMS BETWEEN VERMA MODULES

The duality theorem (Theorem 2.3) shows that the classification and construction of intertwining differential operators between parabolically induced representations are equivalent to those of (\mathfrak{g}, P) -homomorphisms between generalized Verma modules. In this section, for later convenience, we discuss (\mathfrak{g}, B) -homomorphisms between (full) Verma modules, where B is a minimal parabolic subgroup of a split real simple Lie group G .

3.1. Notation. Let G be a split real simple Lie group and fix a minimal parabolic subgroup $B = MAN$. In this case, as M is discrete, we have $\text{Irr}(M) = \text{Irr}(M)_{\text{fin}} = \text{Irr}(M/M_0)$. We denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{a}$, and \mathfrak{n} the complexifications of the Lie algebras of G, B, A , and N , respectively. Then $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$ is a Borel subalgebra of \mathfrak{g} . For $(\sigma, \lambda) \in \text{Irr}(M) \times \mathfrak{a}^*$, we write

$$M(\sigma, \lambda) = M_{\mathfrak{b}}(\sigma, \lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (\sigma \otimes (\lambda - \rho)),$$

a (full) Verma module. By letting B act diagonally, we regard $M(\sigma, \lambda)$ as a (\mathfrak{g}, B) -module. When $M(\sigma, \lambda)$ is regarded just as a $\mathcal{U}(\mathfrak{g})$ -module, we write $M(\sigma, \lambda)|_{\mathfrak{g}}$. When σ is the trivial character χ_{triv} , we also write

$$M(\lambda) = M(\chi_{\text{triv}}, \lambda)|_{\mathfrak{g}}.$$

Let $\text{Irr}(M)_{\text{char}}$ denote the set of characters of M . For any $\chi \in \text{Irr}(M)_{\text{char}}$, we have $M(\chi, \lambda)|_{\mathfrak{g}} = M(\chi_{\text{triv}}, \lambda)|_{\mathfrak{g}}$. Thus $M(\chi, \lambda)|_{\mathfrak{g}} = M(\lambda)$ for $\chi \in \text{Irr}(M)_{\text{char}}$.

For $(\sigma, \lambda) \in \text{Irr}(M) \times \mathfrak{a}^*$, we write

$$I(\sigma, \lambda) = I_B(\sigma, \lambda),$$

where $I_B(\sigma, \lambda)$ is the parabolically induced representation as in Section 2.

By Corollary 2.49, we are initially interested in $\mathcal{D} \in \text{Diff}_G(I(\chi_{\text{triv}}, \lambda_1), I(\sigma, \lambda_2))$ for $\sigma \in \text{Irr}(M)$. In Sections 3.2 and 3.3 below we shall discuss the classification and constructions of the differential operators \mathcal{D} in terms of homomorphisms between Verma modules.

3.2. Classification of homomorphisms between Verma modules. We start with the classification of the parameters $(\sigma, \lambda_1, \lambda_2) \in \text{Irr}(M) \times (\mathfrak{a}^*)^2$ such that $\text{Diff}_G(I(\chi_{\text{triv}}, \lambda_1), I(\sigma, \lambda_2)) \neq \{0\}$. As

$$(3.1) \quad \text{Diff}_G(I(\chi_{\text{triv}}, \lambda_1), I(\sigma, \lambda_2)) \simeq \text{Hom}_{\mathfrak{g}, B}(M(\sigma^\vee, -\lambda_2), M(\chi_{\text{triv}}, -\lambda_1)),$$

it is equivalent to classifying $(\sigma, \lambda_1, \lambda_2) \in \text{Irr}(M) \times (\mathfrak{a}^*)^2$ such that

$$\text{Hom}_{\mathfrak{g}, B}(M(\sigma^\vee, -\lambda_2), M(\chi_{\text{triv}}, -\lambda_1)) \neq \{0\}.$$

Lemma 3.2 below shows that it suffices to consider the case $\sigma = \chi \in \text{Irr}(M)_{\text{char}}$. For simplicity we set $\text{Irr}(M)' := \text{Irr}(M) \setminus \text{Irr}(M)_{\text{char}}$.

Lemma 3.2. *For $(\chi, \lambda, \nu) \in \text{Irr}(M)_{\text{char}} \times (\mathfrak{a}^*)^2$, we have*

$$\text{Hom}_{\mathfrak{g}, B}(M(\sigma, \lambda), M(\chi, \nu)) = \{0\} \quad \text{for any } \sigma \in \text{Irr}(M)'.$$

Proof. Assume that there exists $(\sigma_0, \chi_0, \lambda_0, \nu_0) \in \text{Irr}(M)' \times \text{Irr}(M)_{\text{char}} \times (\mathfrak{a}^*)^2$ such that $\text{Hom}_{\mathfrak{g}, B}(M(\sigma_0, \lambda_0), M(\chi_0, \nu_0)) \neq \{0\}$. If $\varphi \in \text{Hom}_{\mathfrak{g}, B}(M(\sigma_0, \lambda_0), M(\chi_0, \nu_0))$ is a nonzero homomorphism, then the restriction $\varphi|_{1 \otimes \sigma_0 \otimes \mathbb{1}_{\lambda_0 - \rho}}$ is nonzero. Since φ is a B -homomorphism and σ_0 is irreducible, this implies that $\varphi|_{1 \otimes \sigma_0 \otimes \mathbb{1}_{\lambda_0 - \rho}}$ is injective; in particular,

$$(3.3) \quad \dim_{\mathbb{C}} \varphi(1 \otimes \sigma_0 \otimes \mathbb{1}_{\lambda_0 - \rho}) = \dim_{\mathbb{C}} \sigma_0 > 1.$$

However, since the elements of $\varphi(1 \otimes \sigma_0 \otimes \mathbb{1}_{\lambda_0 - \rho})$ are highest weight vectors with weight $\lambda_0 - \rho$ in $M(\nu_0)$, the inequality (3.3) contradicts the fact that

$$\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M(\lambda_0), M(\nu_0)) \leq 1$$

(see, for instance, [13, Thm. 4.2]). Now the lemma follows. □

It follows from (3.1) with $\sigma = \chi \in \text{Irr}(M)_{\text{char}}$ that

$$\begin{aligned} \text{Diff}_G(I(\chi_{\text{triv}}, \lambda_1), I(\chi, \lambda_2)) &\simeq \text{Hom}_{\mathfrak{g}, B}(M(\chi^{-1}, -\lambda_2), M(\chi_{\text{triv}}, -\lambda_1)) \\ &\subset \text{Hom}_{\mathfrak{g}}(M(\chi^{-1}, -\lambda_2)|_{\mathfrak{g}}, M(\chi_{\text{triv}}, -\lambda_1)|_{\mathfrak{g}}) \\ &= \text{Hom}_{\mathfrak{g}}(M(-\lambda_2), M(-\lambda_1)). \end{aligned}$$

Then we next briefly recall a well-known fact on the classification of parameters $(\lambda, \nu) \in \mathfrak{a}^*$ for which $\text{Hom}_{\mathfrak{g}}(M(\lambda), M(\nu)) \neq \{0\}$. For the details, see, for instance, [4, 5], [9, Chap. 7], [13, Chap. 5], and [37].

For the rest of this subsection and Section 3.3, we assume that \mathfrak{g} is a complex simple Lie algebra and we fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . We also fix an inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* . Let Δ denote the set of roots of \mathfrak{g} with respect to \mathfrak{h} . Choose a positive system Δ^+ for Δ and write Π for the set of simple roots for Δ^+ . We write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ for the Borel subalgebra corresponding to Δ^+ . For $\alpha \in \Delta$ and $\lambda \in \mathfrak{h}^*$, we write $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ with $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$, the coroot of α .

We first recall from the literature the notion of a *link* between weights.

Definition 3.4 (Bernstein–Gelfand–Gelfand). Let $\nu, \lambda \in \mathfrak{h}^*$, and let $\beta_1, \dots, \beta_t \in \Delta^+$. Set $\lambda_0 = \lambda$ and $\lambda_i = s_{\beta_i} \cdots s_{\beta_1}(\lambda)$ for $1 \leq i \leq t$. We say that the sequence $(\beta_1, \dots, \beta_t)$ *links* λ to ν if the following two conditions are satisfied:

- (1) $\lambda_t = \nu$;
- (2) $\langle \lambda_{i-1}, \beta_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq t$.

It is well known that the Verma module $M(\nu)$ has a unique irreducible quotient, to be denoted by $L(\nu)$, and also that $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M(\nu), M(\lambda)) \leq 1$. The following celebrated result of BGG–Verma shows when $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M(\nu), M(\lambda)) = 1$.

Theorem 3.5 (BGG–Verma). *The following three conditions on $\nu, \lambda \in \mathfrak{h}^*$ are equivalent:*

- (i) $\dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}}(M(\nu), M(\lambda)) = 1$;
- (ii) $L(\nu)$ is a composition factor of $M(\lambda)$;
- (iii) there exists a sequence $(\beta_1, \dots, \beta_t)$ with $\beta_i \in \Delta^+$ that links λ to ν .

To determine $\chi \in \text{Irr}(M)_{\text{char}}$ such that $\text{Hom}_{\mathfrak{g}, B}(M(\chi^{-1}, -\nu), M(\chi_{\text{triv}}, -\lambda)) \neq \{0\}$ in the setting of Section 3.1, observe that $\varphi \in \text{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda))$ is determined by the image $\varphi(1 \otimes \mathbb{1}_{-\nu-\rho})$ of highest weight vector $1 \otimes \mathbb{1}_{-\nu-\rho}$ of $M(-\nu)$. Let $u_0 \in \mathcal{U}(\bar{\mathfrak{n}})$ such that $\varphi(1 \otimes \mathbb{1}_{-\nu-\rho}) = u_0 \otimes \mathbb{1}_{-\lambda-\rho}$. Since each root space is a one-dimensional representation of M , the group M acts on $\mathbb{C}u_0$ as a character χ_0 . This is the character we look for. In the next subsection we then discuss how to find such u_0 as a construction of a homomorphism between Verma modules.

3.3. Construction of homomorphisms between Verma modules. We set

$$M(\lambda)^{\mathfrak{n}} := \{u \otimes \mathbb{1}_{\lambda-\rho} \in M(\lambda) : X \cdot (u \otimes \mathbb{1}_{\lambda-\rho}) = 0 \text{ for all } X \in \mathfrak{n}\}.$$

As \mathfrak{n} is generated by the root vectors X_α for $\alpha \in \Pi$, it follows that

$$M(\lambda)^{\mathfrak{n}} = \{u \otimes \mathbb{1}_{\lambda-\rho} \in M(\lambda) : X_\alpha \cdot (u \otimes \mathbb{1}_{\lambda-\rho}) = 0 \text{ for all } \alpha \in \Pi\}.$$

The elements $u \otimes \mathbb{1}_{\lambda-\rho} \in M(\lambda)^{\mathfrak{n}}$ are called *singular vectors* (or *highest weight vectors*) of $M(\lambda)$. Since $\text{Hom}_{\mathfrak{g}}(M(\nu), M(\lambda)) \simeq \text{Hom}_{\mathfrak{h}}(\mathbb{C}_{\nu-\rho}, M(\lambda)^{\mathfrak{n}})$, to construct a homomorphism $\varphi \in \text{Hom}_{\mathfrak{g}}(M(\nu), M(\lambda))$, it suffices to find a singular vector $u \otimes \mathbb{1}_{\lambda-\rho} \in M(\lambda)^{\mathfrak{n}}$ with weight $\nu - \rho$, namely, the element $u \otimes \mathbb{1}_{\lambda-\rho} \in M(\lambda)$ satisfying the following two conditions:

- (C1) the element u has weight $\nu - \lambda$;
- (C2) $X_\alpha \cdot (u \otimes \mathbb{1}_{\lambda-\rho}) = 0$ for all $\alpha \in \Pi$.

When ν is given by $\nu = s_\alpha(\lambda)$ with $\langle \lambda, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}$ for some $\alpha \in \Pi$, the singular vector of $M(\lambda)$ with weight $\nu - \rho$ is easy to find. For the proof of the next proposition see, for instance, [13, Prop. 1.4].

Proposition 3.6. *Given $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Pi$, suppose that $k := \langle \lambda, \alpha^\vee \rangle \in 1 + \mathbb{Z}_{\geq 0}$. Then $X_{-\alpha}^k \otimes \mathbb{1}_{\lambda-\rho}$ is a singular vector of $M(\lambda)$ with weight $-k\alpha + (\lambda - \rho)$. Consequently, up to scalar multiple, the map $\varphi \in \text{Hom}_{\mathfrak{g}}(M(s_\alpha(\lambda)), M(\lambda))$ is given by*

$$1 \otimes \mathbb{1}_{s_\alpha(\lambda)-\rho} \mapsto X_{-\alpha}^k \otimes \mathbb{1}_{\lambda-\rho}.$$

For later convenience we finish this subsection with the following simple observation.

Lemma 3.7. For $\lambda, \nu, \mu \in \mathfrak{h}^*$, let $\varphi: M(\lambda) \rightarrow M(\nu)$ and $\psi: M(\nu) \rightarrow M(\mu)$ be homomorphisms between Verma modules such that

$$\begin{aligned} M(\lambda) &\xrightarrow{\varphi} M(\nu) \xrightarrow{\psi} M(\mu) \\ 1 \otimes \mathbb{1}_{\nu-\rho} &\longmapsto u \otimes \mathbb{1}_{\mu-\rho} \\ 1 \otimes \mathbb{1}_{\lambda-\rho} &\longmapsto v \otimes \mathbb{1}_{\nu-\rho} \end{aligned}$$

that is, $\varphi(1 \otimes \mathbb{1}_{\lambda-\rho}) = v \otimes \mathbb{1}_{\nu-\rho}$ and $\psi(1 \otimes \mathbb{1}_{\nu-\rho}) = u \otimes \mathbb{1}_{\mu-\rho}$. Then $(\psi \circ \varphi)(1 \otimes \mathbb{1}_{\lambda-\rho})$ is given by

$$(\psi \circ \varphi)(1 \otimes \mathbb{1}_{\lambda-\rho}) = vu \otimes \mathbb{1}_{\mu-\rho}.$$

Proof. We have

$$(\psi \circ \varphi)(1 \otimes \mathbb{1}_{\lambda-\rho}) = \psi(v \otimes \mathbb{1}_{\nu-\rho}) = v\psi(1 \otimes \mathbb{1}_{\nu-\rho}) = vu \otimes \mathbb{1}_{\mu-\rho}. \quad \square$$

3.4. Recipe of classification and construction of (\mathfrak{g}, B) -homomorphisms between Verma modules. Here, for the sake of convenience, we summarize the classification and construction of the (\mathfrak{g}, B) -homomorphism from $M(\chi^{-1}, -\nu)$ to $M(\chi_{\text{triv}}, -\lambda)$ for fixed $\lambda \in \mathfrak{a}^*$. Via the duality theorem, these are equivalent to those of the intertwining differential operator $\mathcal{D}_u \in \text{Diff}_G(I(\chi_{\text{triv}}, \lambda), I(\chi, \nu))$. In this subsection we use the notation in Section 3.1.

Recipe for (\mathfrak{g}, B) -homomorphisms between Verma modules. Fix $\lambda \in \mathfrak{a}^*$.

Step H1. Classify $\nu \in \mathfrak{a}^*$ with $\nu \neq \lambda$ such that

$$\text{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda)) \neq \{0\}.$$

We remark that, by the BGG–Verma theorem (Theorem 3.5), we have

$$\#\{\nu \in \mathfrak{a}^* : \text{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda)) \neq \{0\}\} < \infty,$$

where $\#S$ denotes the cardinality of a given set S .

Step H2. For each $\nu \in \mathfrak{a}^*$ classified in Step H1, construct a homomorphism

$$\varphi_{(-\nu, -\lambda)} \in \text{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda)),$$

that is, determine $u_{-\nu} \in \mathcal{U}(\bar{\mathfrak{n}})$ such that

$$\varphi_{(-\nu, -\lambda)}(1 \otimes \mathbb{1}_{-\nu-\rho}) \in \mathbb{C}u_{-\nu} \otimes \mathbb{1}_{-\lambda-\rho}.$$

Step H3. For each $\nu \in \mathfrak{a}^*$ classified in Step H1, observe the adjoint action Ad of M on $\mathbb{C}u_{-\nu}$ to determine the character $\chi_{-\nu} \in \text{Irr}(M)_{\text{char}}$ such that

$$(3.8) \quad \varphi_{(-\nu, -\lambda)} \in \text{Hom}_{\mathfrak{g}, B}(M(\chi_{-\nu}^{-1}, -\nu), M(\chi_{\text{triv}}, -\lambda)).$$

4. APPLICATION TO $\widetilde{\text{SL}}(3, \mathbb{R})$

In this section, toward the later applications, we discuss the structure of $\widetilde{\text{SL}}(3, \mathbb{R})$, the nonlinear double cover of $\text{SL}(3, \mathbb{R})$. The characters of \widetilde{M} and the polynomial realization of the irreducible representations of \widetilde{K} are also discussed.

4.1. Notation and normalizations. We start with the notation and normalizations for $\widetilde{\mathrm{SL}}(3, \mathbb{R})$. Let $\widetilde{G} = \widetilde{\mathrm{SL}}(3, \mathbb{R})$ with Lie algebra $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})$. Take the Cartan involution $\theta: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ to be $\theta(U) = -U^t$. We write $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ for the Cartan decomposition of \mathfrak{g}_0 with respect to θ , where \mathfrak{k}_0 and \mathfrak{s}_0 are as usual the $+1$ and -1 eigenspaces of θ , respectively. We have $\mathfrak{k}_0 = \mathfrak{so}(3) \simeq \mathfrak{su}(2)$.

Let \mathfrak{a}_0 be the maximal abelian subspace of \mathfrak{s}_0 defined by $\mathfrak{a}_0 := \text{span}_{\mathbb{R}}\{E_{ii} - E_{i+1, i+1} : i = 1, 2\}$, where E_{ij} are the matrix units. We also define a nilpotent subalgebra \mathfrak{n}_0 of \mathfrak{g}_0 by $\mathfrak{n}_0 := \text{span}_{\mathbb{R}}\{E_{12}, E_{23}, E_{13}\}$. Then $\mathfrak{b}_0 := \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is a minimal parabolic subalgebra of \mathfrak{g}_0 .

Let \widetilde{K} , A , and N be the analytic subgroups of \widetilde{G} with Lie algebras \mathfrak{k}_0 , \mathfrak{a}_0 , and \mathfrak{n}_0 , respectively, so that $\widetilde{G} = \widetilde{K}AN$ is an Iwasawa decomposition of \widetilde{G} . We write $\widetilde{M} = Z_{\widetilde{K}}(\mathfrak{a}_0)$. Then $\widetilde{B} := \widetilde{M}AN$ is a minimal parabolic subgroup of \widetilde{G} with Lie algebra \mathfrak{b}_0 .

For real Lie algebra \mathfrak{h}_0 , we express its complexification by \mathfrak{h} . The complexification $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$ of the minimal parabolic subalgebra $\mathfrak{b}_0 = \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is a Borel subalgebra of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. We write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{a})$ for the set of roots of \mathfrak{g} with respect to \mathfrak{a} and denote by Δ^+ the positive system corresponding to \mathfrak{b} . Let $\Pi = \{\alpha, \beta\}$ be the set of simple roots for the positive system Δ^+ . We fix α and β in such a way that the root spaces \mathfrak{g}_α and \mathfrak{g}_β are given by $\mathfrak{g}_\alpha = \mathbb{C}E_{12}$ and $\mathfrak{g}_\beta = \mathbb{C}E_{23}$. We write ρ for half the sum of the positive roots.

Define $X, Y \in \mathfrak{g}$ by

$$(4.1) \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then X and Y are root vectors of $-\alpha$ and $-\beta$, respectively. The nilpotent radical $\bar{\mathfrak{n}}$ opposite to \mathfrak{n} is then given as the spanned space of $\{X, Y, [X, Y]\}$.

4.2. Characters $\tilde{\chi}_{(\varepsilon, \varepsilon')}$ of \widetilde{M} . As described in Section 3.4, the characters $\tilde{\chi}$ of $\widetilde{M} \subset \widetilde{\mathrm{SL}}(3, \mathbb{R})$ play a key role in constructing intertwining differential operators \mathcal{D} . In this subsection we describe the characters $\tilde{\chi}$ via the characters χ of the linear group $M \subset \mathrm{SL}(3, \mathbb{R})$. To the end we first aim to identify the elements of \widetilde{M} with those of M in a canonical way.

We start with the identifications of $\mathfrak{k}_0 = \mathfrak{so}(3)$ and $\mathfrak{k} = \mathfrak{so}(3, \mathbb{C})$ with $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$, respectively. First observe that $\mathfrak{su}(2)$ is spanned by the three matrices

$$A_1 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

with commutation relations $[A_1, A_2] = 2A_3$, $[A_1, A_3] = -2A_2$, and $[A_2, A_3] = 2A_1$. The Lie algebra $\mathfrak{k}_0 = \mathfrak{so}(3)$ is spanned by

$$B_1 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with commutation relations $[B_1, B_2] = B_3$, $[B_1, B_3] = -B_2$, and $[B_2, B_3] = B_1$. Thus \mathfrak{k}_0 can be identified with $\mathfrak{su}(2)$ via the map

$$\Omega: \mathfrak{k}_0 \xrightarrow{\sim} \mathfrak{su}(2), \quad B_j \mapsto \frac{1}{2}A_j \quad \text{for } j = 1, 2, 3.$$

Let Z_+, Z_-, Z_0 be the elements of \mathfrak{k} defined by

$$(4.2) \quad Z_+ := B_2 - \sqrt{-1}B_3, \quad Z_- := -(B_2 + \sqrt{-1}B_3), \quad Z_0 := [Z_+, Z_-].$$

We set

$$(4.3) \quad E_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $A_2 - \sqrt{-1}A_3 = 2E_+$ and $-(A_2 + \sqrt{-1}A_3) = 2E_-$, one may identify $\mathfrak{k} = \mathfrak{so}(3, \mathbb{C})$ with $\mathfrak{sl}(2, \mathbb{C})$ via the map

$$(4.4) \quad \Omega_{\mathbb{C}}: \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}), \quad Z_j \mapsto E_j \quad \text{for } j = +, -, 0.$$

The subgroup $\widetilde{M} = Z_{\widetilde{K}}(\mathfrak{a}_0)$ is isomorphic to the quaternion group Q_8 , a non-commutative group of order 8. Since \widetilde{K} is isomorphic to $SU(2)$, we realize \widetilde{M} as a subgroup of $SU(2)$ by

$$\widetilde{M} \simeq \{\pm \widetilde{m}_0, \pm \widetilde{m}_1, \pm \widetilde{m}_2, \pm \widetilde{m}_3\},$$

where

$$(4.5) \quad \widetilde{m}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{m}_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \widetilde{m}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widetilde{m}_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Let $M = Z_K(\mathfrak{a}_0)$ with $K = SO(3) \subset SL(3, \mathbb{R})$. We have

$$M = \{m_0, m_1, m_2, m_3\},$$

where

$$m_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The adjoint action Ad of $SU(2)$ on $\mathfrak{su}(2)$ yields a two-to-one covering map $SU(2) \rightarrow SO(3)$. We realize $\text{Ad}(SU(2))$ as a matrix group with respect to the ordered basis $\{A_2, A_1, A_3\}$ of $\mathfrak{su}(2)$ in such a way that $\pm \widetilde{m}_j$ are mapped to m_j for $j = 0, 1, 2, 3$. Lemma 4.6 below shows that the map $\pm \widetilde{m}_j \mapsto m_j$ respects the Lie algebra isomorphism $\Omega_{\mathbb{C}}: \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})$.

Lemma 4.6. *For $Z \in \mathfrak{k}$, we have*

$$\Omega_{\mathbb{C}}(\text{Ad}(m_j)Z) = \text{Ad}(\widetilde{m}_j)\Omega_{\mathbb{C}}(Z) \quad \text{for } j = 0, 1, 2, 3.$$

Proof. By the Lie algebra isomorphism $\Omega_{\mathbb{C}}: \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})$ in (4.4), in order to prove the lemma, it suffices to show that $\Omega_{\mathbb{C}}(\text{Ad}(m_j)Z_k) = \text{Ad}(\widetilde{m}_j)E_k$ for $j = 0, 1, 2, 3$ and $k = +, -, 0$. One can easily check that these identities indeed hold. \square

For $\varepsilon, \varepsilon' \in \{\pm 1\}$, we define a character $\chi_{(\varepsilon, \varepsilon')}: M \rightarrow \{\pm 1\}$ of M by

$$\chi_{(\varepsilon, \varepsilon')}(\text{diag}(a_1, a_2, a_3)) := |a_1|_{\varepsilon} |a_3|_{\varepsilon'},$$

where $|a|_+ := |a|$ and $|a|_- := a$. Via the character $\chi_{(\varepsilon, \varepsilon')}$ of M , we define a character $\widetilde{\chi}_{(\varepsilon, \varepsilon')}: \widetilde{M} \rightarrow \{\pm 1\}$ of \widetilde{M} by

$$\widetilde{\chi}_{(\varepsilon, \varepsilon')}(\pm \widetilde{m}_j) := \chi_{(\varepsilon, \varepsilon')}(m_j) \quad \text{for } j = 0, 1, 2, 3.$$

TABLE 1. Character table for $(\varepsilon, \varepsilon')$

	$\pm\tilde{m}_0$	$\pm\tilde{m}_1$	$\pm\tilde{m}_2$	$\pm\tilde{m}_3$
$(+, +)$	1	1	1	1
$(+, -)$	1	-1	-1	1
$(-, +)$	1	-1	1	-1
$(-, -)$	1	1	-1	-1

We often abbreviate $\tilde{\chi}_{(\varepsilon, \varepsilon')}$ as $(\varepsilon, \varepsilon')$. Table 1 illustrates the character table for $(\varepsilon, \varepsilon') = \tilde{\chi}_{(\varepsilon, \varepsilon')}$. With the characters $(\varepsilon, \varepsilon')$, the set $\text{Irr}(\tilde{M})$ of equivalence classes of irreducible representations of \tilde{M} is given as follows:

$$(4.7) \quad \text{Irr}(\tilde{M}) = \{(+, +), (+, -), (-, +), (-, -), \mathbb{H}\},$$

where \mathbb{H} is the unique genuine two-dimensional representation of $\tilde{M} \simeq Q_8$.

4.3. Polynomial realization of the irreducible representations of \tilde{K} . As indicated in the recipe in Section 2.7, to determine the \tilde{K} -type formula of $\mathcal{S}ol_{(u, \lambda)}(\sigma)_{\tilde{K}}$, a realization of irreducible representations δ of \tilde{K} is chosen. In the present situation that $\tilde{K} \simeq \text{SU}(2)$, we realize $\text{Irr}(\tilde{K})$ as $\text{Irr}(\tilde{K}) \simeq \{(\pi_n, \text{Pol}_n[t]) : n \in \mathbb{Z}_{\geq 0}\}$ with $\text{Pol}_n[t] := \{p(t) \in \text{Pol}[t] : \deg p(t) \leq n\}$, where $\text{Pol}[t]$ is the space of polynomials of one variable t with complex coefficients. The representation π_n of $\text{SU}(2)$ on $\text{Pol}_n[t]$ is defined by

$$(4.8) \quad (\pi_n(g)p)(t) := (ct + d)^n p\left(\frac{at + b}{ct + d}\right) \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

The elements \tilde{m}_j for $j = 1, 2, 3$ of $\tilde{M} \subset \text{SU}(2)$ defined in (4.5) act on $\text{Pol}_n[t]$ as follows:

$$(4.9) \quad \tilde{m}_1 : p(t) \mapsto (\sqrt{-1})^n p(-t); \quad \tilde{m}_2 : p(t) \mapsto t^n p\left(-\frac{1}{t}\right); \quad \tilde{m}_3 : p(t) \mapsto (-\sqrt{-1}t)^n p\left(\frac{1}{t}\right).$$

Let $d\pi_n$ be the differential of the representation π_n . Then (4.3) and (4.8) imply that we have

$$(4.10) \quad d\pi_n(E_+) = -\frac{d}{dt} \quad \text{and} \quad d\pi_n(E_-) = -nt + t^2 \frac{d}{dt}.$$

As usual we extend $d\pi_n$ complex-linearly to $\mathfrak{sl}(2, \mathbb{C})$ and also naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$. We then let $\mathcal{U}(\mathfrak{k})$ act on $\text{Pol}_n[t]$ via the isomorphism $\Omega_{\mathbb{C}} : \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})$ in (4.4). For simplicity we write $d\pi_n(F) = d\pi_n(\Omega_{\mathbb{C}}(F))$ for $F \in \mathcal{U}(\mathfrak{k})$.

By Corollary 2.49 and (2.52) with the realization $\text{Irr}(\tilde{K}) \simeq \{(\pi_n, \text{Pol}_n[t]) : n \in \mathbb{Z}_{\geq 0}\}$, we have

$$(4.11) \quad \mathcal{S}ol_{(u, \lambda)}(\sigma)_{\tilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_n[t] \otimes \text{Hom}_{\tilde{M}}(\text{Sol}_{(u)}(n), \sigma),$$

where

$$\text{Sol}_{(u)}(n) := \{p(t) \in \text{Pol}_n[t] : d\pi_n(\tau(u^b))p(t) = 0\}.$$

Here $\tau: \mathfrak{g} \rightarrow \mathfrak{g}$ is the conjugation with respect to the real form \mathfrak{g}_0 . Since \mathfrak{g}_0 is the split real form of \mathfrak{g} , we have $\tau(u^b) = u^b$. Thus,

$$(4.12) \quad \text{Sol}_{(u)}(n) = \{p(t) \in \text{Pol}_n[t] : d\pi_n(u^b)p(t) = 0\}.$$

By using (4.11) and (4.12), we shall determine the \tilde{K} -type formulas for intertwining differential operators for the cases of $\lambda = -\rho$ and $\lambda = -(1/2)\rho$ in Sections 5 and 6, respectively.

5. THE CASE OF INFINITESIMAL CHARACTER ρ

In this section, in accordance with the recipes given in Sections 2.7 and 3.4, we determine the \tilde{K} -type formula of the solution space $\mathcal{S}ol_{(u;\lambda)}(\sigma)$ with $\lambda = -\rho$ for $\tilde{G} = \tilde{SL}(3, \mathbb{R})$. This is done in Theorem 5.14. We continue with the notation and normalizations from the previous section.

5.1. Classification and construction of intertwining differential operators.

Our first goal is to classify and construct intertwining differential operators, namely, to achieve Steps H1–H3 in the recipe in Section 3.4. As the first step we start by classifying $\nu \in \mathfrak{a}^*$ with $\nu \neq -\rho$ such that $\text{Hom}_{\mathfrak{g}}(M(-\nu), M(\rho)) \neq \{0\}$.

Lemma 5.1. *The following conditions on $\nu \in \mathfrak{a}^*$ with $\nu \neq -\rho$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}}(M(-\nu), M(\rho)) \neq \{0\}$.
- (ii) $\nu = \pm\alpha, \pm\beta, \rho$.

Proof. The BGG–Verma theorem (Theorem 3.5) shows that the following are all homomorphisms (that are not proportional to the identities) obtained from $M(\rho)$, where $\varphi_{(\mu_1, \mu_2)}$ denotes the homomorphism from $M(\mu_1)$ to $M(\mu_2)$:

$$(5.2) \quad \begin{array}{ccccc} & & M(-\beta) & \xrightarrow{\varphi_{(-\beta, \beta)}} & M(\beta) \\ & \nearrow \varphi_{(-\rho, -\beta)} & & & \searrow \varphi_{(\beta, \rho)} \\ M(-\rho) & \xrightarrow{\varphi_{(-\rho, \rho)}} & & & \xrightarrow{\varphi_{(-\beta, \alpha)}} & M(\rho) \\ & \searrow \varphi_{(-\rho, -\alpha)} & & & \nearrow \varphi_{(\alpha, \rho)} \\ & & M(-\alpha) & \xrightarrow{\varphi_{(-\alpha, \alpha)}} & M(\alpha) \end{array}$$

□

The next step is to construct the homomorphism $\varphi_{(-\nu, \rho)} \in \text{Hom}_{\mathfrak{g}}(M(-\nu), M(\rho))$ for $\nu = \pm\alpha, \pm\beta, \rho$. Let X and Y be the root vectors of $-\alpha$ and $-\beta$ defined in (4.1), respectively.

Lemma 5.3. *Up to scalar multiple, the image of $1 \otimes \mathbb{1}_{-\nu-\rho} \in M(-\nu)$ under the map $\varphi_{(-\nu, \rho)} \in \text{Hom}_{\mathfrak{g}}(M(-\nu), M(\rho))$ for $\nu = \pm\alpha, \pm\beta, \rho$ is given as follows:*

$$\begin{aligned} \varphi_{(\beta, \rho)} &: X \otimes \mathbb{1}_0, & \varphi_{(-\beta, \rho)} &: Y^2 X \otimes \mathbb{1}_0, \\ \varphi_{(\alpha, \rho)} &: Y \otimes \mathbb{1}_0, & \varphi_{(-\alpha, \rho)} &: X^2 Y \otimes \mathbb{1}_0, \\ \varphi_{(-\rho, \rho)} &: XY^2 X \otimes \mathbb{1}_0 (= YX^2Y \otimes \mathbb{1}_0). \end{aligned}$$

Proof. Proposition 3.6 gives maps $\varphi_{(\mu_1, \mu_2)}$ in (5.2) with

$$\begin{aligned} (\mu_1, \mu_2) = & (-\rho, -\beta), (-\beta, \beta), (\beta, \rho), \\ & (-\rho, -\alpha), (-\alpha, \alpha), (\alpha, \rho), \end{aligned}$$

namely,

$$\begin{aligned} \varphi_{(-\rho, -\beta)} : X \otimes \mathbb{1}_{-\beta-\rho}, & \quad \varphi_{(-\beta, \beta)} : Y^2 \otimes \mathbb{1}_{\beta-\rho}, & \quad \varphi_{(\beta, \rho)} : X \otimes \mathbb{1}_0, \\ \varphi_{(-\rho, -\alpha)} : Y \otimes \mathbb{1}_{-\alpha-\rho}, & \quad \varphi_{(-\alpha, \alpha)} : X^2 \otimes \mathbb{1}_{\alpha-\rho}, & \quad \varphi_{(\alpha, \rho)} : Y \otimes \mathbb{1}_0. \end{aligned}$$

Lemma 3.7 then yields the lemma. □

As the third step, for each $\nu = \pm\alpha, \pm\beta, \rho$, we next determine a character $\tilde{\chi}_{-\nu} \in \text{Irr}(\widetilde{M})_{\text{char}}$ such that $\text{Hom}_{\mathfrak{g}, \widetilde{B}}(M(\tilde{\chi}_{-\nu}^{-1}, -\nu), M((+, +), \rho)) \neq \{0\}$. Let $u_{-\nu}$ be the element of $\mathcal{U}(\bar{\mathfrak{n}})$ determined in Lemma 5.3 such that $\varphi_{(-\nu, \rho)}(1 \otimes \mathbb{1}_{-\nu-\rho}) = u_{-\nu} \otimes \mathbb{1}_0$. We have

$$(5.4) \quad \begin{aligned} \nu = -\beta : u_{\beta} &= X, & \nu = \beta : u_{-\beta} &= Y^2 X, \\ \nu = -\alpha : u_{\alpha} &= Y, & \nu = \alpha : u_{-\alpha} &= X^2 Y, \\ \nu = \rho : u_{-\rho} &= XY^2 X (= YX^2Y). \end{aligned}$$

Lemma 5.5. *Let $\nu \in \{\pm\alpha, \pm\beta, \rho\}$. Then the character $\tilde{\chi}_{-\nu} \in \text{Irr}(\widetilde{M})_{\text{char}}$ for which we have $\text{Hom}_{\mathfrak{g}, \widetilde{B}}(M(\tilde{\chi}_{-\nu}^{-1}, -\nu), M((+, +), \rho)) \neq \{0\}$ is given as follows:*

- (a) $\nu = \pm\beta : \tilde{\chi}_{-\nu} = (+, -)$.
- (b) $\nu = \pm\alpha : \tilde{\chi}_{-\nu} = (-, +)$.
- (c) $\nu = \rho : \tilde{\chi}_{-\nu} = (+, +)$.

Proof. We wish to check that, via the adjoint action, the subgroup \widetilde{M} acts on $\mathbb{C}u_{-\nu}$ by the proposed character. Since the adjoint action of \widetilde{M} factors through M , it suffices to consider the adjoint action of M on $\mathbb{C}u_{-\nu}$. Now a direct computation yields the lemma. □

For $\varphi_{(-\nu, \rho)}(1 \otimes \mathbb{1}_{-\nu-\rho}) = u_{-\nu} \otimes \mathbb{1}_0$, write $\mathcal{D}_{u_{-\nu}} = R(u_{-\nu})$. We then obtain the following.

Lemma 5.6. *For $\nu = \pm\alpha, \pm\beta, \rho$, let $\tilde{\chi}_{-\nu}$ be the character of \widetilde{M} determined in Lemma 5.5. Then we have*

$$\text{Diff}_{\widetilde{G}}(I((+, +), -\rho), I(\tilde{\chi}_{-\nu}, \nu)) = \mathbb{C}\mathcal{D}_{u_{-\nu}}.$$

Proof. This is an immediate consequence of Lemmas 5.3 and 5.5 and the duality theorem. □

It follows from (5.4) that we have

$$\begin{aligned} \text{Sol}_{(X; -\rho)}(\sigma) &\subset \text{Sol}_{(u_{-\nu}; -\rho)}(\sigma) & \text{for } \nu = \beta, \rho, \\ \text{Sol}_{(Y; -\rho)}(\sigma) &\subset \text{Sol}_{(u_{-\nu}; -\rho)}(\sigma) & \text{for } \nu = \alpha, \rho. \end{aligned}$$

Then, in the next subsection, we consider the \widetilde{K} -type formulas of the solution spaces $\text{Sol}_{(X; -\rho)}(\sigma)_{\widetilde{K}}$ and $\text{Sol}_{(Y; -\rho)}(\sigma)_{\widetilde{K}}$. In addition, the common solution space

$$\text{Sol}_{(X, Y; -\rho)}(\sigma)_{\widetilde{K}} := \text{Sol}_{(X; -\rho)}(\sigma)_{\widetilde{K}} \cap \text{Sol}_{(Y; -\rho)}(\sigma)_{\widetilde{K}}$$

is also considered.

5.2. \tilde{K} -type formulas of solution spaces. We now aim to find the \tilde{K} -type formulas of $\mathcal{S}ol_{(X;-\rho)}(\sigma)_{\tilde{K}}$, $\mathcal{S}ol_{(Y;-\rho)}(\sigma)_{\tilde{K}}$, and $\mathcal{S}ol_{(X,Y;-\rho)}(\sigma)_{\tilde{K}}$. In order to determine them, as in the recipe in Section 2.7, we start by finding $\tau(u^b) = u^b \in \mathfrak{k}_0$ for $u = X, Y \in \mathfrak{n}_0$. Let Z_+ and Z_- be the nilpotent elements in the $\mathfrak{sl}(2)$ -triple of $\mathfrak{k} = \mathfrak{so}(3, \mathbb{C})$ defined in (4.2). We set

$$X^b := X + \theta(X) \quad \text{and} \quad Y^b := Y + \theta(Y),$$

where θ is the Cartan involution defined by $\theta(U) = -U^t$. Clearly, X^b and Y^b are elements of \mathfrak{k}_0 .

Lemma 5.7. *The elements X^b and Y^b satisfy the identity (2.53) with $u = X, Y$, respectively. Further, we have*

$$X^b = \frac{\sqrt{-1}}{2}(Z_+ + Z_-) \quad \text{and} \quad Y^b = \frac{1}{2}(Z_+ - Z_-).$$

Proof. As X and Y are root vectors for $-\alpha$ and $-\beta$ with $\alpha, \beta \in \Pi$, respectively, we have $\theta(X), \theta(Y) \in \bar{\mathfrak{n}}$, which implies the first assertion. The second assertion follows from a direct computation. □

The next step is to choose a realization of $\delta \in \text{Irr}(\tilde{K})$. As described in Section 4.3, we realize $\text{Irr}(\tilde{K})$ as $\text{Irr}(\tilde{K}) \simeq \{(\pi_n, \text{Pol}_n[t]) : n \in \mathbb{Z}_{\geq 0}\}$. The explicit formulas for the operators $d\pi_n(X^b)$ and $d\pi_n(Y^b)$ are given as follows.

Lemma 5.8. *We have*

$$d\pi_n(X^b) = -\frac{\sqrt{-1}}{2}((1-t^2)\frac{d}{dt} + nt) \quad \text{and} \quad d\pi_n(Y^b) = -\frac{1}{2}((1+t^2)\frac{d}{dt} - nt).$$

Proof. It follows from (4.4) and (4.10) that

$$(5.9) \quad d\pi_n(Z_+) = -\frac{d}{dt} \quad \text{and} \quad d\pi_n(Z_-) = -nt + t^2\frac{d}{dt}.$$

Now the proposed identities follow from Lemma 5.7. □

Recall from (4.11) and (4.12) that the space $\mathcal{S}ol_{(u;-\rho)}(\sigma)_{\tilde{K}}$ of \tilde{K} -finite solutions to $\mathcal{D}_u \otimes \text{id}_\sigma$ for $u = X, Y$ is decomposed as

$$\mathcal{S}ol_{(u;-\rho)}(\sigma)_{\tilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_n[t] \otimes \text{Hom}_{\tilde{M}}(\text{Sol}_{(u)}(n), \sigma),$$

where

$$\text{Sol}_{(u)}(n) = \{p(t) \in \text{Pol}_n[t] : d\pi_n(u^b)p(t) = 0\}.$$

We next wish to determine for what $n \in \mathbb{Z}_{\geq 0}$ the solution space $\text{Sol}_{(u)}(n)$ is non-zero.

Proposition 5.10. *For $u = X, Y$, we have $\text{Sol}_{(u)}(n) \neq \{0\}$ if and only if $n \in 2\mathbb{Z}_{\geq 0}$. Moreover, for $n \in 2\mathbb{Z}_{\geq 0}$, we have*

$$\text{Sol}_{(X)}(n) = \mathbb{C}(1-t^2)^{\frac{n}{2}} \quad \text{and} \quad \text{Sol}_{(Y)}(n) = \mathbb{C}(1+t^2)^{\frac{n}{2}}.$$

Proof. We only discuss $\text{Sol}_{(X)}(n)$; the assertion for $\text{Sol}_{(Y)}(n)$ can be drawn similarly. By Lemma 5.8, it suffices to solve the differential equation $((1-t^2)\frac{d}{dt} + nt)p(t) = 0$. By separation of variables, one can easily check that the solution of the differential equation has to be proportional to $(1-t^2)^{\frac{n}{2}}$. The assertion then follows from a simple observation that $(1-t^2)^{\frac{n}{2}} \in \text{Pol}_n[t]$ if and only if $n \in 2\mathbb{Z}_{\geq 0}$. □

As in (2.50), we define

$$\text{Sol}_{(X,Y)}(n) := \text{Sol}_{(X)}(n) \cap \text{Sol}_{(Y)}(n).$$

Corollary 5.11. *We have $\text{Sol}_{(X,Y)}(n) \neq \{0\}$ if and only if $n = 0$. Moreover, we have $\text{Sol}_{(X,Y)}(0) = \mathbb{C} \cdot 1$.*

Proof. This is an immediate consequence of Proposition 5.10. □

We next show the \widetilde{M} -representation on $\text{Sol}_{(X)}(n)$, $\text{Sol}_{(Y)}(n)$, and $\text{Sol}_{(X,Y)}(n)$.

Proposition 5.12. *As an \widetilde{M} -representation, we have the following.*

- (1) $n \equiv 0 \pmod{4} : \text{Sol}_{(X)}(n) \simeq (+, +), \text{Sol}_{(Y)}(n) \simeq (+, +).$
- (2) $n \equiv 2 \pmod{4} : \text{Sol}_{(X)}(n) \simeq (+, -), \text{Sol}_{(Y)}(n) \simeq (-, +).$
- (3) $n = 0 : \text{Sol}_{(X,Y)}(0) \simeq (+, +).$

Proof. For $u = X, Y, (X, Y)$, let $p_{(u)}(t)$ be the polynomial determined in Proposition 5.10 and Corollary 5.11 such that $\text{Sol}_{(u)}(n) = \mathbb{C}p_{(u)}(t)$. The assertions then follow from a direct observation of the transformation laws (4.9) of $\widetilde{m}_j \in \widetilde{M}$ on $p_{(u)}(t)$ with Table 1. □

As the last step of the recipe, we determine $n \in \mathbb{Z}_{\geq 0}$ such that $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(u)}(n), \sigma) \neq \{0\}$ for given $\sigma \in \text{Irr}(\widetilde{M})$.

Proposition 5.13. *For $u = X, Y, (X, Y)$, the following conditions on $\sigma \in \text{Irr}(\widetilde{M})$ are equivalent:*

- (i) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(u)}(n), \sigma) \neq \{0\};$
- (ii) $\dim_{\mathbb{C}} \text{Hom}_{\widetilde{M}}(\text{Sol}_{(u)}(n), \sigma) = 1.$

Further, for each $u = X, Y, (X, Y)$, we have the following:

- (1) *If $\sigma \neq (+, +), (+, -)$, then $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X)}(n), \sigma) = \{0\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover,*
 - (a) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X)}(n), (+, +)) \neq \{0\} \iff n \equiv 0 \pmod{4};$
 - (b) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X)}(n), (+, -)) \neq \{0\} \iff n \equiv 2 \pmod{4}.$
- (2) *If $\sigma \neq (+, +), (-, +)$, then $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(Y)}(n), \sigma) = \{0\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover,*
 - (a) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(Y)}(n), (+, +)) \neq \{0\} \iff n \equiv 0 \pmod{4};$
 - (b) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(Y)}(n), (-, +)) \neq \{0\} \iff n \equiv 2 \pmod{4}.$
- (3) *If $\sigma \neq (+, +)$, then $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X,Y)}(n), \sigma) = \{0\}$ for all $n \in \mathbb{Z}_{\geq 0}$. Moreover,*

$$\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X,Y)}(n), (+, +)) \neq \{0\} \iff n = 0.$$

Proof. The assertions easily follow from Proposition 5.10, Corollary 5.11, and Proposition 5.12. □

Here is a summary of the results that we have obtained in this section.

Theorem 5.14. *For $\sigma \in \text{Irr}(\widetilde{M})$, the following hold:*

- (1) $\text{Sol}_{(X;-\rho)}(\sigma)_{\widetilde{K}} \neq \{0\} \iff \sigma = (+, +), (+, -).$
- (2) $\text{Sol}_{(Y;-\rho)}(\sigma)_{\widetilde{K}} \neq \{0\} \iff \sigma = (+, +), (-, +).$
- (3) $\text{Sol}_{(X,Y;-\rho)}(\sigma)_{\widetilde{K}} \neq \{0\} \iff \sigma = (+, +).$

Moreover, for $\sigma \in \text{Irr}(\widetilde{M})$ such that $\text{Sol}_{(u;-\rho)}(\sigma) \neq \{0\}$, the \widetilde{K} -type formula of $\text{Sol}_{(u;-\rho)}(\sigma)_{\widetilde{K}}$ is determined as follows:

$$\begin{aligned}
 \text{(a) } u = X & : \text{Sol}_{(X;-\rho)}((+,+))_{\widetilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n}[t], \\
 & \text{Sol}_{(X;-\rho)}((+,-))_{\widetilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n+2}[t]. \\
 \text{(b) } u = Y & : \text{Sol}_{(Y;-\rho)}((+,+))_{\widetilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n}[t], \\
 & \text{Sol}_{(Y;-\rho)}((-,+))_{\widetilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n+2}[t]. \\
 \text{(c) } u = (X, Y) & : \text{Sol}_{(X,Y;-\rho)}((+,+))_{\widetilde{K}} \simeq \text{Pol}_0[t].
 \end{aligned}$$

Proof. We only demonstrate the proof for the case $u = X$; the other cases can be shown similarly. By (4.11), we have

$$\text{Sol}_{(X;-\rho)}(\sigma)_{\widetilde{K}} \simeq \bigoplus_{n=0}^{\infty} \text{Pol}_n[t] \otimes \text{Hom}_{\widetilde{M}}(\text{Sol}_{(X)}(n), \sigma).$$

It then follows from Proposition 5.13 that $\text{Sol}_{(X;-\rho)}(\sigma)_{\widetilde{K}} = \{0\}$ for $\sigma \neq (+, +), (+, -)$ and, for $\sigma = (+, +), (+, -)$, we have

$$\text{Sol}_{(X;-\rho)}((+,+))_{\widetilde{K}} \simeq \bigoplus_{n \equiv 0 \pmod{4}} \text{Pol}_n[t]$$

and

$$\text{Sol}_{(X;-\rho)}((+,-))_{\widetilde{K}} \simeq \bigoplus_{n \equiv 2 \pmod{4}} \text{Pol}_n[t].$$

This proves the assertion. □

We now give a proof of Theorem 1.5.

Proof of Theorem 1.5. Since the space $\text{Sol}_{(u;-\rho)}(\sigma)_{\widetilde{K}}$ is dense in $\text{Sol}_{(u;-\rho)}(\sigma)$, we have $\text{Sol}_{(u;-\rho)}(\sigma) \neq \{0\}$ if and only if $\text{Sol}_{(u;-\rho)}(\sigma)_{\widetilde{K}} \neq \{0\}$. Now the assertions follow from Theorem 5.14, as $(\pi_n, \text{Pol}_n[t]) \simeq V_{(\frac{n}{2})}$. □

6. THE CASE OF INFINITESIMAL CHARACTER $\widetilde{\rho}$

The aim of this section is to determine the \widetilde{K} -type formula of the solution space $\text{Sol}_{(u;\lambda)}(\sigma)$ for $\lambda = -(1/2)\rho$. This is achieved in Theorem 6.16. For the rest of this section we write $\widetilde{\rho} = (1/2)\rho$.

6.1. Classification and construction of intertwining differential operators.

As for the case of ρ , we start by classifying $\nu \in \mathfrak{a}^*$ with $\nu \neq -\widetilde{\rho}$ such that $\text{Hom}_{\mathfrak{g}}(M(-\nu), M(\widetilde{\rho})) \neq \{0\}$.

Lemma 6.1. *The following conditions on $\nu \in \mathfrak{a}^*$ with $\nu \neq -\widetilde{\rho}$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}}(M(-\nu), M(\widetilde{\rho})) \neq \{0\}$.
- (ii) $\nu = \widetilde{\rho}$.

Proof. A simple observation of the BGG-Verma theorem (Theorem 3.5) for $\lambda = \widetilde{\rho}$. □

The next step is to construct $\varphi_{(-\tilde{\rho}, \tilde{\rho})} \in \text{Hom}_{\mathfrak{g}}(M(-\tilde{\rho}), M(\tilde{\rho}))$ with $\varphi_{(-\tilde{\rho}, \tilde{\rho})} \neq 0$. To do so, we first prepare some notation. Let $\mathcal{U}_r(\bar{\mathfrak{n}})$ be the subspace of $\mathcal{U}(\bar{\mathfrak{n}})$ that is spanned by products of at most r elements of $\bar{\mathfrak{n}}$. Also, let $S^r(\bar{\mathfrak{n}})$ denote the subspace of the symmetric algebra $S(\bar{\mathfrak{n}})$ of $\bar{\mathfrak{n}}$ spanned by products of r elements of $\bar{\mathfrak{n}}$. Then we have

$$(6.2) \quad \mathcal{U}_r(\bar{\mathfrak{n}}) = \text{sym}^r(S^r(\bar{\mathfrak{n}})) \oplus \mathcal{U}_{r-1}(\bar{\mathfrak{n}}),$$

where $\text{sym}^r: S^r(\bar{\mathfrak{n}}) \rightarrow \mathcal{U}_r(\bar{\mathfrak{n}})$ is the symmetrization map. Recall from (4.1) that X and Y are root vectors of $-\alpha$ and $-\beta$, respectively. With the notation we show the following.

Lemma 6.3. *Up to scalar multiple, the map $\varphi_{(-\tilde{\rho}, \tilde{\rho})} \in \text{Hom}_{\mathfrak{g}}(M(-\tilde{\rho}), M(\tilde{\rho}))$ is given by*

$$1 \otimes \mathbb{1}_{-(\tilde{\rho}+\rho)} \longmapsto (XY + YX) \otimes \mathbb{1}_{-\tilde{\rho}}.$$

Proof. It is first remarked that, in contrast to Lemma 5.3, we cannot apply Proposition 3.6 in the present case, as $-\tilde{\rho} \neq s_\gamma(\tilde{\rho})$ for $\gamma = \alpha, \beta$. We thus consider conditions (C1) and (C2) in Section 3.3. Let $u_0 \otimes \mathbb{1}_{-\tilde{\rho}}$ be the image of $1 \otimes \mathbb{1}_{-(\tilde{\rho}+\rho)}$ under the map $\varphi_{(-\tilde{\rho}, \tilde{\rho})}$. Suppose that u_0 satisfies condition (C1), namely, u_0 has weight $-\rho = -\alpha - \beta$. Then u_0 is a linear combination of XY and YX ; in particular, $u_0 \in \mathcal{U}_2(\bar{\mathfrak{n}})$. By (6.2), this implies that u_0 is proportional to $(XY + YX) + t[X, Y]$ for some $t \in \mathbb{C}$ with $XY + YX \in \text{sym}^2(S^2(\bar{\mathfrak{n}}))$ and $[X, Y] \in \mathcal{U}_1(\bar{\mathfrak{n}})$. A direct calculation shows that $((XY + YX) + t[X, Y]) \otimes \mathbb{1}_{-\tilde{\rho}}$ satisfies condition (C2), that is, it is annihilated by X_α and X_β in $M(\tilde{\rho})$, if and only if $t = 0$. Now the lemma follows. \square

We next determine $\tilde{\chi} \in \text{Irr}(\widetilde{M})_{\text{char}}$ such that $\text{Hom}_{\mathfrak{g}, \tilde{B}}(M(\tilde{\chi}^{-1}, -\tilde{\rho}), M((+, +), \tilde{\rho})) \neq \{0\}$.

Lemma 6.4. *The following conditions on $\tilde{\chi} \in \text{Irr}(\widetilde{M})_{\text{char}}$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}, \tilde{B}}(M(\tilde{\chi}^{-1}, -\tilde{\rho}), M((+, +), \tilde{\rho})) \neq \{0\}$.
- (ii) $\tilde{\chi} = (-, -)$.

Proof. Since the proof is similar to the one for Lemma 5.5, we omit the proof. \square

For simplicity we write

$$(6.5) \quad X \circ Y = XY + YX,$$

so that $\mathcal{D}_{X \circ Y} = R(X)R(Y) + R(Y)R(X)$.

Lemma 6.6. *We have $\text{Diff}_{\tilde{C}}(I((+, +), -\tilde{\rho}), I((-,-), \tilde{\rho})) = \mathbb{C}\mathcal{D}_{X \circ Y}$.*

Proof. This is an immediate consequence of Lemmas 6.3 and 6.4 and the duality theorem. \square

6.2. \tilde{K} -type formulas of the solution spaces $\text{Sol}_{(X \circ Y; -\tilde{\rho})}(\sigma)_{\tilde{K}}$. We now aim to obtain the \tilde{K} -type formula of $\text{Sol}_{(X \circ Y; -\tilde{\rho})}(\sigma)_{\tilde{K}}$. As for the ρ case, we start by finding $(X \circ Y)^b$. We set

$$(X \circ Y)^b := (X^b \circ Y^b) = X^b Y^b + Y^b X^b.$$

Lemma 6.7. *The element $(X \circ Y)^b$ satisfies the identity (2.53) with $u = X \circ Y$. Moreover, we have*

$$(X \circ Y)^b = \frac{\sqrt{-1}}{2}(Z_+^2 - Z_-^2).$$

Proof. A direct computation shows that

$$(X^b \circ Y^b) \otimes (\mathbb{1}_{\text{triv}} \otimes \mathbb{1}_{-\tilde{\rho}}) = (X \circ Y) \otimes (\mathbb{1}_{\text{triv}} \otimes \mathbb{1}_{-\tilde{\rho}})$$

in $M((+, +), \tilde{\rho})$, which shows the first assertion. The second identity easily follows from Lemma 5.7. \square

For the next we find the explicit formula of $d\pi_n((X \circ Y)^b)$.

Lemma 6.8. *We have*

$$(6.9) \quad d\pi_n((X \circ Y)^b) = \frac{\sqrt{-1}}{2}((1 - t^4) \frac{d^2}{dt^2} + 2(n - 1)t^3 \frac{d}{dt} - n(n - 1)t^2).$$

Proof. By Lemma 6.7, we have

$$d\pi_n((X \circ Y)^b) = \frac{\sqrt{-1}}{2}(d\pi_n(Z_+)^2 - d\pi_n(Z_-)^2).$$

The desired identity then follows from (5.9) with a direct computation. \square

We next want to determine for what $n \in \mathbb{Z}_{\geq 0}$ the solution space $\text{Sol}_{(X \circ Y)}(n)$ (see (4.12)) is nonzero. Let ${}_2F_1[a, b, c; x]$ denote the Gauss hypergeometric series. We put

$$u_n(t) := {}_2F_1[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4] \quad \text{and} \quad v_n(t) := t {}_2F_1[-\frac{n-1}{4}, -\frac{n-2}{4}, \frac{5}{4}; t^4].$$

It is remarked that $u_n(t)$ and $v_n(t)$ form a fundamental set of solutions to Euler’s (Gauss) hypergeometric differential equation $D[a, b, c; t^4]f(t) = 0$ with $a = -\frac{n}{4}$, $b = -\frac{n-1}{4}$, and $c = \frac{3}{4}$. (For some details see the Appendix.) In particular, $u_n(t)$ and $v_n(t)$ are linearly independent.

Proposition 6.10. *We have $\text{Sol}_{(X \circ Y)}(n) \neq \{0\}$ if and only if $n \equiv 0, 1, 2 \pmod{4}$. Moreover, the solution space $\text{Sol}_{(X \circ Y)}(n)$ is given as follows:*

- (1) $n \equiv 0 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) = \mathbb{C}u_n(t)$.
- (2) $n \equiv 1 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) = \mathbb{C}u_n(t) \oplus \mathbb{C}v_n(t)$.
- (3) $n \equiv 2 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) = \mathbb{C}v_n(t)$.

Since the proof involves some classical facts on the Gauss hypergeometric series ${}_2F_1[a, b, c; x]$, we give the proof in Section 7.2 of the Appendix. Then, by taking the assertions of Proposition 6.10 as granted, we next show the \widetilde{M} -representations on $\text{Sol}_{(X \circ Y)}(n)$ for $n \equiv 0, 1, 2 \pmod{4}$.

Proposition 6.11. *As an \widetilde{M} -representation, we have the following:*

- (1) $n \equiv 0 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) \simeq (+, +)$.
- (2) $n \equiv 1 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) \simeq \mathbb{H}$.
- (3) $n \equiv 2 \pmod{4}$: $\text{Sol}_{(X \circ Y)}(n) \simeq (-, -)$.

Proof. We discuss (1) and (2) only; assertion (3) can be shown similarly to (1). We start with assertion (1). Suppose that $n \equiv 0 \pmod{4}$. Write $n = 4k$ for some $k \in \mathbb{Z}_{\geq 0}$ so that

$$(6.12) \quad u_{4k}(t) = {}_2F_1[-k, -k + \frac{1}{4}, \frac{3}{4}; t^4].$$

By Proposition 6.10, we have $\text{Sol}_{(X \circ Y)}(4k) = \mathbb{C}u_{4k}(t)$; in particular, \widetilde{M} acts on $\text{Sol}_{(X \circ Y)}(4k)$ as a character $(\varepsilon, \varepsilon')$. As $\widetilde{m}_3 = \widetilde{m}_1 \widetilde{m}_2$, to determine the character $(\varepsilon, \varepsilon')$, it suffices to consider the actions of \widetilde{m}_1 and \widetilde{m}_2 on $\mathbb{C}u_{4k}(t)$. We claim that

both \tilde{m}_1 and \tilde{m}_2 act on $\mathbb{C}u_{4k}(t)$ trivially. First it is easy to see that the action of \tilde{m}_1 on $\mathbb{C}u_{4k}(t)$ is trivial. Indeed, it follows from (4.9) and (6.12) that we have

$$\tilde{m}_1 : u_{4k}(t) \mapsto (\sqrt{-1})^{4k} u_{4k}(-t) = u_{4k}(t).$$

In order to show that the action of \tilde{m}_2 is also trivial, observe that \tilde{m}_2 transforms $u_{4k}(t)$ as

$$\tilde{m}_2 : u_{4k}(t) \mapsto t^{4k} u_{4k}\left(-\frac{1}{t}\right) = t^{4k} u_{4k}\left(\frac{1}{t}\right).$$

Since $\text{Sol}_{(X \circ Y)}(4k) = \mathbb{C}u_{4k}(t)$ is an \widetilde{M} -representation, there exists a constant $c \in \mathbb{C}$ such that $t^{4k} u_{4k}\left(\frac{1}{t}\right) = cu_{4k}(t)$. In particular, we have $u_{4k}(1) = cu_{4k}(1)$. It follows from a general fact (Fact 7.1 in Section 7.1) on the Gauss hypergeometric series ${}_2F_1[a, b, c; x]$ that

$$u_{4k}(1) = {}_2F_1\left[-k, -k + \frac{1}{4}, \frac{3}{4}; 1\right] = \frac{\Gamma(\frac{3}{4})\Gamma(2k + \frac{1}{2})}{\Gamma(k + \frac{3}{4})\Gamma(k + \frac{1}{2})} \neq 0.$$

Thus we have $c = 1$. Therefore \tilde{m}_2 also acts trivially on $\mathbb{C}u_{4k}(t)$.

In order to show assertion (2), suppose that $n \equiv 1 \pmod{4}$. Since there is only a unique irreducible two-dimensional \widetilde{M} -representation (see (4.7)), it suffices to show that the \widetilde{M} -representation $\mathbb{C}u_n(t) \oplus \mathbb{C}v_n(t)$ is irreducible. We show this by contradiction. Assume the contrary, that is, there exists a nontrivial proper \widetilde{M} -invariant subspace $\{0\} \neq V \subsetneq \mathbb{C}u_n(t) \oplus \mathbb{C}v_n(t)$. Then V is of the form $V = \mathbb{C}(ru_n(t) + sv_n(t))$ for some $(r, s) \in \mathbb{C}^2$ with $(r, s) \neq (0, 0)$. As $n \equiv 1 \pmod{4}$, we have $(\sqrt{-1})^n = \sqrt{-1}$. Thus \tilde{m}_1 transforms $u_n(t)$ and $v_n(t)$ as

$$\begin{aligned} u_n(t) &\mapsto (\sqrt{-1})^n u_n(-t) = \sqrt{-1}u_n(t), \\ v_n(t) &\mapsto (\sqrt{-1})^n v_n(-t) = -\sqrt{-1}v_n(t). \end{aligned}$$

Therefore $ru_n(t) + sv_n(t)$ is transformed by \tilde{m}_1 as

$$(6.13) \quad ru_n(t) + sv_n(t) \mapsto \sqrt{-1}(ru_n(t) - sv_n(t)).$$

On the other hand, since V is assumed to be a one-dimensional \widetilde{M} -representation, \widetilde{M} acts on V as a character $(\varepsilon, \varepsilon')$. In particular, \tilde{m}_1 acts on V by ± 1 , that is,

$$(6.14) \quad ru_n(t) + sv_n(t) \mapsto \pm(ru_n(t) + sv_n(t)).$$

Equations (6.13) and (6.14) imply that $(\sqrt{-1} - 1)ru_n(t) - (\sqrt{-1} + 1)sv_n(t) = 0$ or $(\sqrt{-1} + 1)ru_n(t) - (\sqrt{-1} - 1)sv_n(t) = 0$. Since $u_n(t)$ and $v_n(t)$ are linearly independent, this concludes that $(r, s) = (0, 0)$, which contradicts the choice of $(r, s) \neq (0, 0)$. Hence $\mathbb{C}u_n(t) \oplus \mathbb{C}v_n(t)$ is irreducible. \square

Proposition 6.15. *The following conditions on $\sigma \in \text{Irr}(\widetilde{M})$ are equivalent:*

- (i) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X \circ Y)}(n), \sigma) \neq \{0\}$;
- (ii) $\dim_{\mathbb{C}} \text{Hom}_{\widetilde{M}}(\text{Sol}_{(X \circ Y)}(n), \sigma) = 1$;
- (iii) $\sigma = (+, +), \mathbb{H}, (-, -)$.

Moreover we have the following:

- (1) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X \circ Y)}(n), (+, +)) \neq \{0\} \iff n \equiv 0 \pmod{4}$.
- (2) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X \circ Y)}(n), \mathbb{H}) \neq \{0\} \iff n \equiv 1 \pmod{4}$.
- (3) $\text{Hom}_{\widetilde{M}}(\text{Sol}_{(X \circ Y)}(n), (-, -)) \neq \{0\} \iff n \equiv 2 \pmod{4}$.

Proof. The proposition easily follows from Propositions 6.10 and 6.11 with (4.11). □

As a summary of the results in this section, we obtain the following.

Theorem 6.16. *For $\sigma \in \text{Irr}(\widetilde{M})$, we have*

$$\text{Sol}_{(X \circ Y; -\bar{\rho})}(\sigma)_{\widetilde{K}} \neq \{0\} \iff \sigma = (+, +), \mathbb{H}, (-, -).$$

Moreover, for $\sigma = (+, +), \mathbb{H}, (-, -)$, the \widetilde{K} -type formula of $\text{Sol}_{(X \circ Y; -\bar{\rho})}(\sigma)_{\widetilde{K}}$ is obtained as follows:

$$\begin{aligned} \text{(a) } \sigma = (+, +) : \text{Sol}_{(X \circ Y; -\bar{\rho})}((+, +))_{\widetilde{K}} &\simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n}[t]. \\ \text{(b) } \sigma = \mathbb{H} : \text{Sol}_{(X \circ Y; -\bar{\rho})}(\mathbb{H})_{\widetilde{K}} &\simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n+1}[t]. \\ \text{(c) } \sigma = (-, -) : \text{Sol}_{(X \circ Y; -\bar{\rho})}((-, -))_{\widetilde{K}} &\simeq \bigoplus_{n=0}^{\infty} \text{Pol}_{4n+2}[t]. \end{aligned}$$

Proof. Since the proof is similar to the one for Theorem 5.14, we omit the proof. □

Now we give a proof of Theorem 1.6.

Proof of Theorem 1.6. As for Theorem 1.5, the assertions follow from Theorem 6.16. □

7. APPENDIX

The purpose of this short appendix is to give a proof of Proposition 6.10, in which we determine the space $\text{Sol}_{(X \circ Y)}(n)$ of polynomial solutions to the differential equation $d\pi_n((X \circ Y)^p)p(t) = 0$. To do so we observe Euler’s (Gauss) hypergeometric differential equation.

7.1. Gauss hypergeometric series ${}_2F_1[a, b, c; x]$. We start by recalling some well-known facts on the Gauss hypergeometric series

$${}_2F_1[a, b, c; x] := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{x^k}{k!},$$

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a-1) \cdots (a+k-1)$. It is clear that when $c \notin \mathbb{Z}_{\geq 0}$, the series ${}_2F_1[a, b, c; x]$ is a polynomial if and only if either $-a \in \mathbb{Z}_{\geq 0}$ or $-b \in \mathbb{Z}_{\geq 0}$. Moreover, if $-a \in \mathbb{Z}_{\geq 0}$ and $-b, -c \notin \mathbb{Z}_{\geq 0}$, then $\text{deg } {}_2F_1[a, b, c; x] = a$. For the proof of the following identity, see, for instance, [1, Thm. 2.2.2].

Fact 7.1 (Gauss, 1812). For $\text{Re}(c - a - b) > 0$, we have

$${}_2F_1[a, b, c; 1] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

Define a second-order differential operator

$$(7.2) \quad D[a, b, c; x] := x(1-x) \frac{d^2}{dx^2} + (c - (a + b + 1)x) \frac{d}{dx} - ab,$$

so that the equation $D[a, b, c; x]f(x) = 0$ is Euler's (Gauss) hypergeometric differential equation. We put

$$u'_{[a,b,c]}(x) := {}_2F_1[a, b, c; x] \quad \text{and} \quad v'_{[a,b,c]}(x) := x^{1-c} {}_2F_1[a - c + 1, b - c + 1, 2 - c; x].$$

It is well known that if $c \notin \mathbb{Z}$, then $u'_{[a,b,c]}(x)$ and $v'_{[a,b,c]}(x)$ form a fundamental set of solutions to $D[a, b, c; x]f(x) = 0$. (See, for instance, [30, Sect. 3].)

7.2. Proof of Proposition 6.10. Recall from (6.9) that

$$-2\sqrt{-1}d\pi_n((X \circ Y)^b) = (1 - t^4) \frac{d^2}{dt^2} + 2(n - 1)t^3 \frac{d}{dt} - n(n - 1)t^2.$$

We thank Hiroyuki Ochiai for pointing out the suitable change of variables in Lemma 7.3 below.

Lemma 7.3. *We have*

$$\frac{\sqrt{-1}}{32t^2} d\pi_n((X \circ Y)^b) = D\left[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4\right].$$

Proof. This follows from a direct computation with change of variables $x = t^4$. □

We set

$$u_n(t) := u'_{[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}]}(t^4) \quad \text{and} \quad v_n(t) := v'_{[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}]}(t^4),$$

namely,

$$(7.4) \quad u_n(t) = {}_2F_1\left[-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4\right] \quad \text{and} \quad v_n(t) = t {}_2F_1\left[-\frac{n-1}{4}, -\frac{n-2}{4}, \frac{5}{4}; t^4\right].$$

We now give a proof of Proposition 6.10.

Proof of Proposition 6.10. Since $\frac{3}{4} \notin \mathbb{Z}$, it follows from Lemma 7.3 that the functions $u_n(t)$ and $v_n(t)$ form a fundamental set of solutions to the equation $d\pi_n((X \circ Y)^b)g(t) = 0$. Moreover, (7.4) shows that if $n \equiv 3 \pmod{4}$, then neither $u_n(t)$ nor $v_n(t)$ is a polynomial; for $n \equiv 0, 1, 2 \pmod{4}$, we have the following:

- (1) $n \equiv 0 \pmod{4} : u_n(t) \in \text{Pol}_n[t].$
- (2) $n \equiv 1 \pmod{4} : u_n(t), v_n(t) \in \text{Pol}_n[t].$
- (3) $n \equiv 2 \pmod{4} : v_n(t) \in \text{Pol}_n[t].$

This concludes the proposition. □

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